Q1 For Beta Distribution :-

MLE of $Pr(b|a)$ : $\hat{\theta} = \underset{\theta}{\operatorname{argmax}} L_B(\theta : D) = \frac{N(a,b)}{N(a)}$

$= \frac{1}{3}$

MAP of $Pr(b|a)$ : $\hat{\theta} = \operatorname{argmax} Pr(\theta | D) = \frac{N(a,b) + \alpha - 1}{N(a) + \alpha + \beta - 2}$

$= \frac{1+3-1}{1+3+3-2}$

$= \frac{3}{7}$

Bayesian estimation of $Pr(b|a)$ : $\hat{\theta} = E\{\theta | D\} = \frac{N(a,b) + \alpha}{N(a) + \alpha + \beta}$

$= \frac{1+3}{3+3+3}$

$= \frac{4}{9}$

Q2 $y_i = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$

MLE for $\beta_1$ & $\beta_2$ when $\mu = 0$ & standard dev $= \sigma$

$\varepsilon_i = y_i - \alpha - \beta_1 x_{i1} - \beta_2 x_{i2}$

$L(\varepsilon_i) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\varepsilon_i)^2}{2\sigma^2}}$ (for normal distribution)

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$$L(\varepsilon_1, \varepsilon_2 \ldots \varepsilon_n) = \prod_{i=1}^{n} \frac{e^{\frac{-\Sigma \varepsilon_i^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$\log L = -\frac{\Sigma \varepsilon_i^2}{2\sigma^2} - n\log(\sigma\sqrt{2\pi})$$

$$LL = \log L = -\frac{\Sigma(y_i - \alpha - \beta_1 x_{i1} - \beta_2 x_{i2})^2}{2\sigma^2} - n\log(\sigma\sqrt{2\pi}) \quad \text{——— (A)}$$

differentiate wrt $\alpha$ :-

$$\frac{dLL}{d\alpha} = +\frac{\Sigma(y_i - \alpha - \beta_1 x_{i1} - \beta_2 x_{i2})}{\sigma^2} = 0$$

$$\Rightarrow \Sigma y_i - n\alpha - \beta_1 \Sigma x_{i1} - \beta_2 \Sigma x_{i2} = 0$$

$$\boxed{\alpha = \frac{\Sigma y_i - \beta_1 \Sigma x_{i1} - \beta_2 \Sigma x_{i2}}{n}}$$

$$\alpha = \frac{\Sigma y_i}{n} - \beta_1 \frac{\Sigma x_{i1}}{n} - \beta_2 \frac{\Sigma x_{i2}}{n}$$

$$\alpha = \hat{y} - \beta_1 \hat{x}_1 - \beta_2 \hat{x}_2$$

Putting in A we get :-

$$LL = -\frac{\Sigma(y_i - \hat{y} + \beta_1 \hat{x}_1 + \beta_2 \hat{x}_2 - \beta_1 x_{i1} - \beta_2 x_{i2})^2}{2\sigma^2} - n\log(\sigma\sqrt{2\pi})$$

$$= -\frac{\Sigma[(y_i - \hat{y}) - \beta_1(x_{i1} - \hat{x}_1) - \beta_2(x_{i2} - \hat{x}_2)]^2}{2\sigma^2} - n\log(\sigma\sqrt{2\pi})$$

differentiate wrt $\beta_1$ :-

$$\frac{dLL}{d\beta_1} = -\frac{\sum\left[(y_i-\hat{y}) - \beta_1(x_{i1}-\hat{x}_1) - \beta_2(x_{i2}-\hat{x}_2)\right](x_{i1}-\hat{x}_1)}{\sigma^2} = 0$$

$$\sum(y_i-\hat{y})(x_{i1}-\hat{x}) = \beta_1\sum(x_{i1}-\hat{x}_1)^2 + \beta_2\sum(x_{i2}-\hat{x}_2)(x_{i1}-\hat{x}_1)$$

Similarly,

$\frac{dLL}{d\beta_2} = 0$ we get

$$\sum(y_i-\hat{y})(x_{i2}-\hat{x}_2) = \beta_1\sum(x_{i1}-\hat{x}_1)(x_{i2}-\hat{x}_2) + \beta_2\sum(x_{i2}-\hat{x}_2)^2.$$

This can be written in form:. $XX^T B = XY$

where :-

$$X = \begin{bmatrix} \sum(x_{i1}-\hat{x}_1)^2 & \sum(x_{i2}-\hat{x}_2)(x_{i1}-\hat{x}_1) \\ \sum(x_{i1}-\hat{x}_1)(x_{i2}-\hat{x}_2) & \sum(x_{i2}-\hat{x}_2)^2 \end{bmatrix}$$

$$B = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$Y = \begin{bmatrix} \sum(y_i-\hat{y})(x_{i1}-\hat{x}_1) \\ \sum(y_i-\hat{y})(x_{i2}-\hat{x}_2) \end{bmatrix}$$

This can also be written as $B = (XX^T)^{-1}(XY)$

& $B$'s solution is obtained.

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Q3 after learning over $D_1$,

P(A)

| A | |
|---|---|
| $a_1$ | $x_{11}+2$ |
| $a_2$ | $x_{12}+3$ |
| $a_3$ | $x_{13}+2$ |
| $a_4$ | $x_{14}+1$ |

P(B|A)

| B/A | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $b_1$ | $y_{11}+2$ | $y_{12}+0$ | $y_{13}+1$ | $y_{14}+0$ |
| $b_2$ | $y_{21}+0$ | $y_{22}+3$ | $y_{23}+0$ | $y_{24}+1$ |
| $b_3$ | $y_{31}+0$ | $y_{32}+0$ | $y_{33}+0$ | $y_{34}+0$ |
| $b_4$ | $y_{41}+0$ | $y_{42}+0$ | $y_{43}+1$ | $y_{44}+0$ |

P(C|B)

| C/B | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $c_1$ | $z_{11}+2$ | $z_{12}+1$ | $z_{13}+0$ | $z_{14}+0$ |
| $c_2$ | $z_{21}+1$ | $z_{22}+2$ | $z_{23}+0$ | $z_{24}+1$ |
| $c_3$ | $z_{31}+0$ | $z_{32}+1$ | $z_{33}+0$ | $z_{34}+0$ |

after learning $D_2$,

P(A)

| A | |
|---|---|
| $a_1$ | $x_{11}+4$ |
| $a_2$ | $x_{12}+7$ |
| $a_3$ | $x_{13}+3$ |
| $a_4$ | $x_{14}+1$ |

P(B|A)

| B/A | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $b_1$ | $y_{11}+2$ | $y_{12}+1$ | $y_{13}+1$ | $y_{14}+0$ |
| $b_2$ | $y_{21}+1$ | $y_{22}+6$ | $y_{23}+0$ | $y_{24}+1$ |
| $b_3$ | $y_{31}+0$ | $y_{32}+0$ | $y_{33}+0$ | $y_{34}+0$ |
| $b_4$ | $y_{41}+1$ | $y_{42}+0$ | $y_{43}+2$ | $y_{44}+1$ |

P(C|B)

| C/B | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $c_1$ | $z_{11}+3$ | $z_{12}+2$ | $z_{13}+0$ | $z_{14}+1$ |
| $c_2$ | $z_{21}+0$ | $z_{22}+3$ | $z_{23}+0$ | $z_{24}+2$ |
| $c_3$ | $z_{31}+0$ | $z_{32}+3$ | $z_{33}+0$ | $z_{34}+1$ |

After learning $D_3$,

$P(A)$

| A | |
|---|---|
| $a_1$ | $x_{11}+7$ |
| $a_2$ | $x_{12}+9$ |
| $a_3$ | $x_{13}+5$ |
| $a_4$ | $x_{14}+2$ |

$P(B|A)$

| B/A | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| $b_1$ | $y_{11}+2$ | $y_{12}+2$ | $y_{13}+2$ | $y_{14}+0$ |
| $b_2$ | $y_{21}+1$ | $y_{22}+7$ | $y_{23}+0$ | $y_{24}+2$ |
| $b_3$ | $y_{31}+1$ | $y_{32}+0$ | $y_{33}+0$ | $y_{34}+1$ |
| $b_4$ | $y_{41}+3$ | $y_{42}+0$ | $y_{43}+2$ | $y_{44}+1$ |

$P(C|B)$

| C/B | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $c_1$ | $z_{11}+4$ | $z_{12}+3$ | $z_{13}+1$ | $z_{14}+1$ |
| $c_2$ | $z_{21}+0$ | $z_{22}+3$ | $z_{23}+1$ | $z_{24}+3$ |
| $c_3$ | $z_{31}+0$ | $z_{32}+4$ | $z_{33}+0$ | $z_{34}+2$ |

$P(B|C)$ when $B=b_1$ & $C=c_1$ :-

$$P(b_1|c_1) = \frac{z_{11}'}{z_{11}'+z_{12}'+z_{13}'+z_{14}'} \quad \text{for } D_1, D_2 \ \& \ D_3$$

Similarly $P(B|C)$ can be calculated for all other $b$ & $c$.

$$P(C|A) = \sum_{b} P(A) \times P(B|A) \times P(C|B) \qquad \text{for all } b_i = b_1, b_2, b_3, b_4$$

for $A=a_1$ & $C=c_1$

$$P(c_1|a_1) = P(a_1)\,P(b_1|a_1)\,P(c_1|b_1) + P(a_1)\,P(b_2|a_1)\,P(c_1|b_2) + P(a_1)\,P(b_3|a_1)\,P(c_1|b_3) + P(a_1)\,P(b_4|a_1)\,P(c_1|b_4)$$

$$= x_{11}'\left[\frac{y_{11}'}{x_{11}'}\right]\left[\frac{z_{11}'}{y_{11}'+y_{12}'+y_{13}'+y_{14}'}\right] + x_{11}'\left[\frac{y_{21}'}{x_{11}'}\right]\left[\frac{z_{12}'}{y_{21}'+y_{22}'+y_{23}'+y_{24}'}\right]$$
$$x_{11}'\left[\frac{y_{31}'}{x_{11}'}\right]\left[\frac{z_{13}'}{y_{31}'+y_{32}'+y_{33}'+y_{34}'}\right] + x_{11}'\left[\frac{y_{41}'}{x_{11}'}\right]\left[\frac{z_{14}'}{y_{41}'+y_{42}'+y_{43}'+y_{44}'}\right]$$

Q4 Taking smoking when random no < 0.4

lung disease ———— 0.4 – 0.6

heart disease ———— > 0.6 :.

initially :- $\begin{bmatrix} Sm & LD & HD & SB \\ f & f & f & T \end{bmatrix}$ Given

Round 1 :-

$P(\text{Smoking} \mid SB) = \alpha P[Sm \mid MB(Sm)]$

$= \alpha P[LD \mid Sm] \; P(HD \mid Sm)$

$= \alpha \; 0.9 \times 0.9$

$= \alpha \times 0.81$

$P(\neg Sm \mid SB) = \alpha P[Sm \mid MB(Sm)]$

$= \alpha \; 0.2 \times 0.4$

$= \alpha \times 0.08$

$\alpha = \frac{1}{0.89} = 1.12$

$$\begin{array}{ccc} & F & T \\ Sm & \begin{bmatrix} 0.9 & 0.08 \end{bmatrix} \end{array}$$

Next random no is 0.51 < 0.9 :. smoking is F.

→ 
| Sm | LD | HD | SB |
|---|---|---|---|
| f | f | f | T |

Round 2

Random no. = 0.6 :. LD

$P[LD \mid MB(LD)] = \alpha\, P(SB \mid LD, HD)\; P(LD \mid Sm)$

$= \alpha\; 0.1 \times 0.9$

$= \alpha \times 0.09$

$P[\neg LD \mid MB(LD)] = \alpha \times 0.7 \times 0.1$

$= \alpha \times 0.07$

$\alpha = \dfrac{1}{0.16} = 6.25$

$$P(LD) \begin{array}{cc} F & T \\ \left[\, 0.56 \right. & \left. 0.43 \,\right] \end{array}$$

Next random no. is $0.7 > 0.56$ $\therefore$ $LD = T$

$\longrightarrow$

| Sm | LD | HD | SB |
|---|---|---|---|
| F | T | F | T |

Round 3 :-

Next random no. is 0.3 , $\therefore$ Smoking

$P[Sm \mid MB(Sm)] = \alpha\, P[LD \mid Sm]\; P[HD \mid Sm)$

$= \alpha \times 0.1 \times 0.9$

$= \alpha \times 0.09$

$P(\neg Sm \mid MB[Sm)] = \alpha \times 0.8 \times 0.4$

$= \alpha \times 0.32$

$\alpha = \dfrac{1}{0.41} = 2.43$

$$P(Sm) \begin{array}{cc} F & T \\ \left[\, 0.21 \right. & \left. 0.7 \,\right] \end{array}$$ next random no is $0.56 > 0.21$ $\therefore$ Smoking $= T$

$\longrightarrow$

| Sm | LD | HD | SB |
|---|---|---|---|
| T | T | F | T |

Round 4 :-

next random no is 0.67 ∴ heart disease

$P(HD \mid MB(HD)) = \alpha P(HD \mid Sm) \, P(SB \mid HD, LD)$

$= \alpha \times 0.4 \times 0.7 = \alpha \, 0.28$

$P(\neg HD \mid MB(HD)] = \alpha \, 0.6 \times 0.9 = \alpha \times 0.54$

$\alpha = \frac{1}{0.82} = 1.21$

$$P \begin{array}{cc} F & T \\ [\,0.34 & 0.68\,] \end{array}$$

Next random no is $0.65 > 0.34$ ∴ HD = T

→

| Sm | LD | HD | SB |
|---|---|---|---|
| T | T | T | T |

Round 5 .

next Random no. is 0.47 ∴ lung disease.

$P(LD \mid MB(LD)) = \alpha P(SB \mid LD, HD) \, P(LD \mid Sm)$

$= \alpha \, 0.9 \times 0.8$

$= \alpha \, 0.72$

$P(\neg LD \mid MB) = \alpha \times 0.8 \times 0.2 = \alpha \times 0.16$

$\alpha = \frac{1}{0.88} = 1.13$

$$P \begin{array}{cc} F & T \\ [\,0.18 & 0.81\,] \end{array}$$

next r.No is $0.34 \nless 0.81$ ∴ LD = T

→

| Sm | LD | HD | SB |
|---|---|---|---|
| T | T | T | T |

Round 6 :-

Next Random no is 0.87 ∴ HD

$P(HD \mid MB(HD)] = \alpha P(SB \mid HD, LD)\ P(HD \mid Sm)$

$= \alpha (0.9) \times 0.6 = \alpha\, 0.54$

$P(\neg HD \mid MB(HD)] = \alpha \times 0.7 \times 0.4 = \alpha\, 0.28$

$P\begin{array}{cc} F & T \\ [\,0.33 & 0.65\,] \end{array}$ next no. is 0.85 < 0.65 ∴ HD = T

→

| Sm | LD | HD | SB |
|---|---|---|---|
| T | T | T | T |

Round 7 :-

next Random no. is 0.43 ∴ lung Disease

$P(LD \mid MB(LD) = \alpha P(SB \mid LD, HD)\ P(LD \mid Sm)$

$= \alpha\, 0.9 \times 0.8 = \alpha\, 0.72$

$P(\neg LD \mid MB(LD)] = \alpha\, 0.8 \times 0.2 = \alpha\, 0.16$

$P\begin{array}{cc} F & T \\ [\,0.18 & 0.81\,] \end{array}$ next Random no is 0.56 < 0.81

∴ LD = T

→

| Sm | LD | HD | SB |
|---|---|---|---|
| T | T | T | T |

Round 8

Next Random no. is $0.76$ $\therefore$ HD.

$P(HD \mid MB(HD)) = \alpha \times 0.9 \times 0.6$

$= \alpha\, 0.54$

$P(\neg HD \mid MB(HD)) = \alpha \times 0.7 \times 0.4$

$= \alpha \times 0.28$

$$P\begin{array}{cc} F & T \\ [\,0.34 & 0.65\,] \end{array}$$

Next Random no. is $0.67 > 0.65$ $\therefore$ HD $= F$

→

| Sm | HD | LD | SB |
|---|---|---|---|
| T | F | T | T |
| T | F | T | T |

Round 9

Next Random no. is $0.83$ $\therefore$ HD

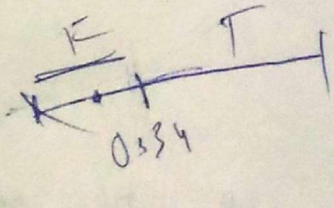
$P(HD \mid MB(HD)) = \alpha \times 0.7 \times 0.4$

$= \alpha \times 0.28$

$P(\neg HD \mid MB(HD)) = \alpha \times 0.9 \times 0.6$

$= \alpha \times 0.54$

$$P\begin{array}{cc} F & T \\ [\,0.34 & 0.65\,] \end{array}$$




Next random no is $0.2 < 0.34$ $\therefore$ HD $= F$

→

| Sm | HD | LD | SB |
|---|---|---|---|
| T | F | T | T |

Round 10

Next random no is 0.3, $\therefore$ smoking

$P(Sm \mid MB(Sm)) = \alpha \times 0.8 \times 0.4$

$= \alpha\, 0.32$

$P(\neg Sm \mid MB(Sm)) = \alpha \times 0.9 \times 0.9$

$\alpha\, 0.81$

$$P\begin{array}{cc} F & T \\ \left[\, 0.71 \right., & \left. 0.2 \,\right] \end{array}$$

next random no. is $0.4 > 0.2$

$\therefore Sm = F$

→ | Sm | LD | HD | SB |
|---|---|---|---|
| F | T | F | T |

Probability until round 6 :- $\begin{array}{cc} T & F \\ \left[\dfrac{4}{5}\right., & \left.\dfrac{1}{5}\right] \end{array}$

Probability until round 10 :- $\left[\dfrac{6}{7}, \dfrac{1}{7}\right]$

Q5

Calculating missing probabilities, for 3rd row

$$P(x_1 | y = y_2, z = z_1) = P(X = x_1 | y = y_2)$$

$$= R_{x_1, y_2}$$

$$P(x_2 | y = y_2, z = z_1) = 1 - R_{x_1, y_2}$$

for 5th row,

$$P(z_1 | x = x_2, y = y_1) = P(z_1 | y_1)$$

$$= L_{z_1, y_1}$$

$$P(z_2 | x = x_2, y = y_1) = 1 - L_{z_1, y_1}$$

for 9th row,

$$P(Y = y_1 | x = x_1, z = z_1) = \frac{P(y_1, x_1, z_1)}{\sum P(Y, X, Z)}$$

$$= \frac{P(x_1 | y_1)\, P(z_1 | y_1)\, P(y_1)}{\sum_{y = 1,2} P(x_i | y_1) P(z_1 | y_1)\, P(y_1)}$$

$$= \frac{R_{x_1, y_1} \cdot L_{z_1, y_1} \cdot \pi_y}{R_{x_1, y_1} \cdot L_{z_1, y_1} \cdot \pi_y + R_{x_1, y_2} \cdot R_{z_1, y_2} (1 - \pi_y)}$$

$$P(Y = y_2 | x = x_1, z = z_1) = \frac{R_{x_1, y_2} \cdot R_{z_1 y_2} (1 - \pi_y)}{R_{x_1, y_1} \cdot L_{z_1, y_1} \cdot \pi_y + R_{x_1, y_2} \cdot R_{z_1 y_2} (1 - \pi_y)}$$

for hard expectation maximization,

$Y = y_1$ , if $(y_1 | X = x_1, Z = z_1) > (y_2 | X = x_1, Z = z_2)$ else

same for 3rd & 5th row.

for soft EM,

we choose both & assign observed probabilities for 3rd & 5th , we do the same obtained probabilities.

We have to calculate new probabilities using this table. We calculate MLE as we are not given prior probabilities