Q1 (a)

$$P(M \mid H=T) = P(M) \times P(B \mid M) \times P(H \mid B)$$

$$= \alpha \left[P(M) \times P(B=T \mid M) \times P(H \mid B=T) + P(M) \times P(B=F \mid M) \times P(H \mid B=F) \right]$$

$$= \alpha \left[0.2 \times 0.2 \times 0.8 + 0.2 \times 0.8 \times 0.8 \right]$$

$$= \alpha \left[0.032 + 0.096 \right]$$

$$= \alpha \, (0.128)$$

$$P(\neg M \mid H=T) = P(\neg M) \times P(B \mid \neg M) \times P(H \mid B)$$

$$= \alpha \left[P(\neg M) \times P(B=T \mid \neg M) \times P(H \mid B=T) + P(\neg M) \times P(B=F \mid \neg M) \times P(H \mid B=F) \right]$$

$$= \alpha \left[0.8 \times 0.85 \times 0.8 + 0.8 \times 0.95 \times 0.6 \right]$$

$$= \alpha \left[0.032 + 0.456 \right]$$

$$= \alpha \left[0.488 \right]$$

$$P(M \mid H=T) + P(\neg M \mid H=T) = 1$$

$$\alpha \left[0.128 + 0.488 \right] = 1$$

$$\alpha = 1.62$$

$$P(M \mid H=T) = \alpha \times 0.128$$

$$= 1.62 \times 0.128$$

$$\boxed{P(M \mid H=T) = 0.207}$$

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(b) $P(B \mid H=T, S=T) = \alpha\, P(M, B, H, S)$

$$= \alpha \left[P(H \mid B) \times P(B \mid M) \times P(M) \times P(S \mid M) \right]$$

$$= \alpha \left[P(H=T \mid B) \times P(B=T \mid M=T) \times P(M=T) \times P(S=T \mid M=T) \right.$$

$$\left. + P(H=T \mid B) \times P(B=T \mid M=F) \times P(M=F) \times P(S=T \mid M=F) \right]$$

$$= \alpha \left[0.2 \times 0.2 \times 0.8 \times 0.8 + 0.8 \times 0.05 \times 0.8 \times 0.2 \right]$$

$$= \alpha \left[0.032 \right]$$

$$P(\neg B \mid H=T, S=T) = \alpha \left[P(H=T \mid B=F) \times P(B=F \mid M=T) \times P(M=T) \times P(S=T \mid M=T) + \right.$$

$$\left. P(H=T \mid B=F) \times P(B=F \mid M=F) \times P(M=F) \times P(S=T \mid M=F) \right]$$

$$= \alpha \left[0.2 \times 0.8 \times 0.6 \times 0.8 + 0.8 \times 0.95 \times 0.6 \times 0.2 \right]$$

$$= \alpha \left[0.1672 \right]$$

$$P(B \mid H=T, S=T) + P(\neg B \mid H=T, S=T) = 1$$

$$\alpha \left[0.032 + 0.1672 \right] = 1$$

$$\alpha = 5.16$$

$$\therefore \quad P(B \mid H=T, S=T) = \alpha\,(0.032)$$

$$= 5.16 \times 0.032$$

$$\boxed{P(B \mid H=T, S=T) = 0.165}$$

Q1(c) $P(B|H=T, S=T, C=T) = \alpha\, P(M, B, H, S, C)$

$= \alpha \left[P(M) P(B|M) P(H|B) P(S|M) P(C|S,B) \right]$

$= \alpha \left[P(M=T) P(B|M=T) P(H|B) P(S|M=T) P(C|S,B) + \right.$

$\left. P(M=F) P(B|M=F) P(H|B) P(S|M=F) P(C|S,B) \right]$

$= \alpha \left[0.2 \times 0.2 \times 0.8 \times 0.8 \times 0.8 + \right.$

$\left. 0.8 \times 0.05 \times 0.8 \times 0.2 \times 0.8 \right]$

$= \alpha \left[0.025 \right]$ ——— (1)

$P(\neg B|H=T, S=T, C=T) = \alpha \left[P(M=T) P(\neg B|M=T) P(H|B=F) P(S|M=T) P(C|S,B=F) + \right.$

$\left. P(M=F) P(\neg B|M=F) P(H|B=F) P(S|M=F) P(C|S,B=F) \right]$

$= \alpha \left[0.2 \times 0.8 \times 0.6 \times 0.8 \times 0.8 + \right.$

$\left. 0.8 \times 0.95 \times 0.6 \times 0.2 \times 0.8 \right)$

$= \alpha \left[0.13 \right]$ ——— (2)

$P(B|H,S,C) + P(\neg B|H,S,C) = 1$

$\alpha \left[0.025 + 0.13 \right] = 1$

$\alpha = 6.45$

from eq^n (1)

$P(B|H=T, S=T, C=T) = \alpha \left[0.025 \right]$

$= 6.45 \left[0.025 \right]$

$= 0.161$

Q2 Density function $f(x|\lambda) = \frac{1}{\lambda} e^{-x/\lambda}$

(a) Likelihood $= \frac{1}{\lambda} e^{-x_1/\lambda} \times \frac{1}{\lambda} e^{-x_2/\lambda} \times \frac{1}{\lambda} e^{-x_3/\lambda} \times \cdots \times \frac{1}{\lambda} e^{-x_n/\lambda}$

($\because$ each is independent of each other)

$$L = \left(\frac{1}{\lambda}\right)^n e^{-\sum_{i=1}^{n} x_i/\lambda}$$

$$\log L = n \log\left(\frac{1}{\lambda}\right) - \frac{\sum_{i=1}^{n} x_i}{\lambda}$$

$$= -n \log\lambda - \frac{\sum x_i}{\lambda} \quad \text{--- (1)}$$

To find MLE wrt λ, differentiate eqn wrt λ. :-

$$\frac{1}{L}\frac{dL}{d\lambda} = -\frac{n}{\lambda} + \frac{\sum x_i}{\lambda^2} = 0$$

$$\frac{\sum x_i}{\lambda^{\cancel{2}}} = \frac{n}{\cancel{\lambda}}$$

$$\boxed{\lambda = \frac{\sum x_i}{n} = \overline{X}}$$

(b) Student t-distribution.

(c) $f(x|\lambda) = \frac{1}{\lambda} e^{-x/\lambda}$

$$E[X] = \int_0^\infty \frac{x}{\lambda} \cdot e^{-x/\lambda} \, dx$$

integrating using integration by parts.

$u = \frac{x}{\lambda}$, $dv = e^{-x/\lambda}$

$\int dv = \int e^{-x/\lambda} dx$

$v = -e^{-x/\lambda} . \lambda$

$$= -\frac{x}{\lambda} . \lambda e^{-x/\lambda} \Big]_0^\infty + \int_0^\infty \left[\frac{1}{\lambda} . e^{-x/\lambda} . \lambda \right] dx.$$

$$= -x.e^{-x/\lambda} \Big]_0^\infty + \int e^{-x/\lambda} dx.$$

$$= -xe^{-x/\lambda} \Big]_0^\infty - \lambda e^{-x/\lambda} \Big]_0^\infty$$

$$= 0 - \left[\lambda e^{-\infty/\lambda} - \lambda e^0 \right]$$

$$\boxed{E[X] = \lambda}$$

$$E[\hat{\lambda}] = E\left[\frac{\Sigma x_i}{n}\right] = \frac{1}{n} E[\Sigma x_i] = \frac{\Sigma E[x]}{n}$$

$$= \frac{\lambda + \lambda + \cdots \lambda}{n}$$

$$= \frac{n\lambda}{n}$$

$$= \lambda$$

$\therefore$ MLE is unbiased.

$$Var(\lambda) = var\left(\frac{\Sigma x_i}{n}\right)$$

$$= \frac{E[x^2] - (E[x])^2}{n} \quad \text{———— (A)}$$

$$E[x^2] = \int_0^\infty \frac{x^2}{\lambda} \cdot e^{-x/\lambda} \, dx.$$

$$= -\frac{x^2}{\lambda} \cdot e^{-x/\lambda} \cdot \lambda + \int \frac{2x}{\lambda} \cdot e^{-x/\lambda} \cdot \lambda \, dx$$

$$= -x^2 \cdot e^{-x/\lambda} + 2\int x \cdot e^{-x/\lambda} \, dx$$

$$= -x^2 e^{-x/\lambda} + 2\left[-x \, e^{-x/\lambda} \cdot \lambda + \int \lambda \cdot e^{-x/\lambda} \, dx\right]$$

$$= -x^2 e^{-x/\lambda}\Big]_0^\infty + 2\left[-x \cdot e^{-x/\lambda} \cdot \lambda - \lambda^2 \cdot e^{-x/\lambda}\right]_0^\infty$$

$$= 0 + 2\left[-\lambda^2 e^{-\infty/\lambda} + \lambda^2 e^{-0/\lambda}\right]$$

$$= 2\lambda^2$$

Put the above in (A)

$$var(\lambda) = \frac{2\lambda^2 - \lambda^2}{n} = \frac{\lambda^2}{n}$$

(d) unbiased estimate for smaller variance.

We can use Cramer Rao lower bound for this Problem.

CR. lower bound = $\frac{1}{n I(\lambda)}$

$$f(x) = \frac{1}{\lambda} \cdot e^{-x/\lambda}$$

$$L = \log(f(x)) = \frac{-x}{\lambda} + \log\left(\frac{1}{\lambda}\right)$$

$$L' = \frac{x}{\lambda^2} + \lambda$$

$$L'' = \frac{-2x}{\lambda^3} + 1$$

$$I(\lambda) = -E[L''] = -E\left[\frac{-2x}{\lambda^3} + 1\right] = \frac{E[2x]}{\lambda^3}$$

$$= \frac{\lambda}{\lambda^3}$$

$$I[\lambda] = \frac{1}{\lambda^2}$$

CR lower bound $= \frac{\lambda^2}{n}$ = variance of $\bar{x}$

Thus there is no other unbiased estimate of λ with smaller variance than $\bar{x}$.

Q3

Binomial distribution :-

$$f(u,x) = \binom{n}{x} u^x (1-u)^{n-x}.$$

$$l = \log f(u,x) = x\log u + (n-x)\log(1-u) + \log\binom{n}{x}$$

differentiate wrt u :-

$$l' = \cancel{\#} \left[\frac{x}{u} - \frac{(n-x)}{1-u}\right]$$

$$l'(u,x)^2 = \left[\frac{x}{u} - \frac{(n-x)}{1-u}\right]^2 = \frac{x^2}{\cancel{\mu}^2} + \frac{(n-x)^2}{(1-u)^2} - \frac{2x}{u}\frac{(n-x)}{(1-u)}$$

$$= \frac{x^2(1-u)^2 + (n-x)^2u^2 - 2x(n-x)u(1-u)}{u^2(1-u)^2}$$

$$= \frac{x^2(1+u^2-2u) + (n^2+x^2 \cancel{*} - 2nx)u^2 - 2(nx-x^2)(u-u^2)}{u^2(1-u)^2}$$

$$= \frac{x^2 - 2nx\cancel{\mu}u + n^2u^2}{u^2(1-u)^2} \qquad \text{———} (1)$$

$$E[l'(u,x)^2] = \Sigma\left(\frac{x}{u} - \frac{n-x}{1-u}\right)^2 \binom{n}{x} u^x (1-u)^{n-x}$$

$$= \Sigma\left(\frac{x^2 - 2nxu + n^2u^2}{u^2(1-u)^2}\right)\binom{n}{x} u^x (1-u)^{n-x}$$

$$= \frac{n^2u^2 + nu(1-u) - 2n^2u^2 + n^2u^2}{u^2(1-u)^2}$$

$$\boxed{I(u) = \frac{n}{u(1-u)}}$$

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Q4

$$f(x|\mu,\sigma) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$\mu = 0$ (given)

$$f(x|\sigma) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$L = \log f(x|\sigma) = \frac{-x^2}{2\sigma^2} - \frac{1}{2}\log(2\pi) - \log\sigma$$

differentiate w.r.t σ :-

$$L' = \frac{x^2}{\sigma^3} - \frac{1}{\sigma}$$

$$L'' = \frac{d}{d\sigma}\left[x^2\sigma^{-3} - \frac{1}{\sigma}\right]$$

$$= \frac{-3x^2}{\sigma^4} + \frac{1}{\sigma^2}$$

$$I(\sigma) = -E(L'')$$

$$= -E\left[\frac{-3x^2}{\sigma^4} + \frac{1}{\sigma^2}\right] = \frac{3\,E[x^2]}{\sigma^4} - \frac{1}{\sigma^2} \quad \text{——(1)}$$

$$\text{Var}(X) = E(X^2) - (E[X])^2$$

$$E[X^2] = \text{Var}(X) + (E[X])^2.$$

put the above in (1) we get :-

$$I(\sigma) = \frac{3\left(\text{Var}(X) + (E[X])^2\right)}{\sigma^4} - \frac{1}{\sigma^2} = \frac{3[\sigma^2 + 0]}{\sigma^4} - \frac{1}{\sigma^2} \quad (\text{given } \mu = E[X] = 0)$$

$$= \frac{3}{\sigma^2} - \frac{1}{\sigma^2}$$

$$\boxed{I(\sigma) = \frac{2}{\sigma^2}}$$

Q5 In MAP, we consider prior distribution also,

(a) $\therefore P(x \mid \mu, \sigma, v, \sqrt{\beta}) = P(x \mid \mu, \sigma) \times P(\mu \mid v, \sqrt{\beta})$

where $P(\mu \mid v, \sqrt{\beta})$ is prior.

$$\Rightarrow P = \frac{e^{-\frac{\Sigma (x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \times \frac{e^{\frac{(\mu-v)^2}{2\beta}}}{\sqrt{2\pi\beta}}$$

$$L = \log P = -\frac{\Sigma (x-\mu)^2}{2\sigma^2} - \frac{(\mu-v)^2}{2\beta} - \log \sigma\sqrt{2\pi} - \log \sqrt{2\pi\beta}$$

differentiate w.r.t μ:-

$$L' = \frac{\Sigma (x-\mu)}{\sigma^2} - \frac{(\mu - v)}{\beta} = 0$$

$$= \frac{\Sigma x_i - n\mu}{\sigma^2} - \frac{\mu}{\beta} + \frac{v}{\beta} = 0$$

$$\frac{\Sigma x_i}{\sigma^2} - \frac{n\mu}{\sigma^2} - \frac{\mu}{\beta} + \frac{v}{\beta} = 0$$

$$\frac{\Sigma x_i}{\sigma^2} + \frac{v}{\beta} = \left(\frac{n}{\sigma^2} + \frac{1}{\beta}\right)\mu$$

$$\mu = \frac{\left(\frac{\Sigma x_i}{\sigma^2} + \frac{v}{\beta}\right)}{\left(\frac{n}{\sigma^2} + \frac{1}{\beta}\right)}$$

(b) MLE:-

$$f(x \mid \mu, \sigma) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$\text{likelihood} = \frac{e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \times \frac{e^{-\frac{(x_2-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \times \cdots$$ n^{th} term.

$$L = \frac{e^{-\Sigma(x_i-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

$$\log L = -\frac{\Sigma(x_i-\mu)^2}{2\sigma^2} - \log\sigma\sqrt{2\pi}$$

differentiate wrt μ :-

$$\frac{1}{L}\frac{dL}{d\mu} = \frac{\Sigma(x_i-\mu)}{\sigma^2}$$

$$= \frac{\Sigma x_i - n\mu}{\sigma^2} = 0$$

$$\mu = \frac{\Sigma x_i}{n} = \overline{X}$$

MAP :- $$\frac{\left(\frac{\Sigma x_i}{\sigma^2} + \frac{v}{\beta}\right)}{\left(\frac{n}{\sigma^2} + \frac{1}{\beta}\right)} = \frac{\Sigma x_i\beta + v\sigma^2}{n\beta + \sigma^2}$$

$$= \frac{\sigma^2 v}{n\beta+\sigma^2} + \frac{\beta\Sigma x_i}{n\beta+\sigma^2}$$

$$= \frac{\sigma^2 v}{n\beta+\sigma^2} + \frac{\Sigma x_i}{n+\sigma^2/\beta} \quad \text{(dividing II by } \beta)$$

$$= \frac{\sigma^2 v}{n\beta+\sigma^2} + \frac{\Sigma x_i/n}{1+\sigma^2/\beta n} \quad \text{(dividing II by } n)$$

when $n \to \infty$ $\quad \frac{\sigma^2 v}{n\beta+\sigma^2} \to 0, \quad \frac{\sigma^2}{\beta n} \to 0$

$$\therefore \text{ MAP} = \frac{\Sigma x_i}{n} = \text{MLE}.$$

Therefore we can say :- when $n \to \infty$ $\quad$ MLE = MAP

Q6 (a) $y_i = \alpha + \beta x_i + \varepsilon_i$

$\varepsilon_i = y_i - \alpha - \beta x_i$

given ε_i has mean $= 0$ & st. deviation $= \sigma$ with normal distribution.

then likelihood f^n is as follows :-

$$L(\varepsilon_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\varepsilon_i^2}{2\sigma^2}} \quad (\because \mu = 0)$$

$$L(\varepsilon_1, \varepsilon_2 \ldots \varepsilon_n) = \prod_{i=1}^{n} \frac{e^{-\Sigma\varepsilon_i^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

$$\log L = \frac{-\Sigma\varepsilon_i^2}{2\sigma^2} - n\log(\sigma\sqrt{2\pi})$$

for MLE of β, we need to differentiate wrt β, but that requires value of α.

$$\therefore \frac{dLL}{d\alpha} = \frac{d}{d\alpha}\left[-\frac{\Sigma(y_i - \alpha - \beta x_i)^2}{2\sigma^2} - n\log(\sigma\sqrt{2\pi})\right]$$

$$\Rightarrow -\frac{\Sigma(y_i - \alpha - \beta x_i)}{\sigma^2} = 0$$

$$\Rightarrow -\Sigma y_i + n\alpha + \beta\Sigma x_i = 0$$

$$\Rightarrow n\alpha = \Sigma y_i - \beta\Sigma x_i$$

$$\Rightarrow \alpha = \frac{\Sigma y_i}{n} - \beta\frac{\Sigma x_i}{n}$$

$$\Rightarrow \boxed{\alpha = \bar{y} - \beta\bar{x}} \quad \text{———— (1)}$$

$$\frac{dLL}{d\beta} = \frac{d}{d\beta}\left[-\frac{\Sigma(y_i - \bar{y} + \beta\bar{x} - \beta x_i)^2}{2\sigma^2} - n\log(\sigma\sqrt{2\pi})\right] \quad \text{from eq}^n \text{ (1)}$$

$$= -\frac{\Sigma(y_i - \bar{y} + \beta\bar{x} - \beta x_i)(\bar{x} - x_i)}{\sigma^2} = 0 \quad \Rightarrow \boxed{\beta = \frac{\Sigma(y_i - \bar{y})(\bar{x} - x_i)}{\Sigma(\bar{x} - x_i)^2}} \text{ —— (2)}$$

Q 6 (b) $L(\varepsilon_i) = \frac{1}{B(n, m)} \varepsilon_i^{n-1} (1-\varepsilon_i)^{m-1}$

$$L(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_K | n, m) = \prod_{i=1}^{K} \frac{1}{B(n,m)} \varepsilon_i^{n-1} (1-\varepsilon_i)^{m-1}$$

$$LL = \log L = (n-1) \sum \log \varepsilon_i + (m-1) \sum \log(1-\varepsilon_i) - \log(B(n,m)) \quad \text{(A)}$$

$$\frac{dLL}{d\alpha} = \frac{d}{d\alpha}\left[n-1 \sum \log(y_i - \alpha - \beta x_i) + (m-1) \sum \log(1 - y_i + \alpha + \beta x_i) - \log B(n,m) \right]$$

$$= (n-1) \sum \frac{-1}{y_i - \alpha - \beta x_i} + (m-1) \sum \frac{1}{1 - y_i + \alpha + \beta x_i} = 0$$

$$(n-1) \sum \frac{1}{y_i - \alpha - \beta x_i} = (m-1) \sum \frac{1}{1 - y_i + \alpha + \beta x_i}$$

$$(n-1) \frac{1}{\sum y_i - K\alpha - \beta \sum x_i} = (m-1) \frac{1}{K - \sum y_i + K\alpha + \beta \sum x_i}$$

$$(n-1)(K - \sum y_i + K\alpha + \beta \sum x_i) = (m-1)[\sum y_i - K\alpha - \beta \sum x_i]$$

$$nK - n\sum y_i + nK\alpha + n\beta \sum x_i - K + \sum y_i - K\alpha - \beta \sum x_i - m\sum y_i + mK\alpha + m\beta\sum x_i + \sum y_i - K\alpha - \beta \sum x_i = 0$$

$$K\alpha[n + m - 2] = K(1-n) + 2\beta \sum x_i + (n+m)\sum y_i - \beta(n+m)\sum x_i - 2\sum y_i$$

$$\alpha = \frac{(1-n) + 2\beta \bar{x} + (n+m)\bar{y} - \beta(n+m)\bar{x} - 2\bar{y}}{n+m-2}$$

$$= \frac{\bar{y}(n+m-2) - \beta\bar{x}(n+m-2) + (1-n)}{n+m-2}$$

$$\boxed{\alpha = \bar{y} - \beta\bar{x} + \frac{(1-n)}{n+m-2}}$$

$$\left.\begin{aligned}\frac{dll}{du} &= 0 \\ \frac{dll}{dm} &= 0\end{aligned}\right\}$$ since all terms are constant.

$$\frac{dll}{d\beta} = \frac{d}{d\beta}\left[(n-1)\sum \log(y_i - \alpha - \beta x_i) + (m-1)\sum \log(1 - y_i + \alpha + \beta x_i) - \log B(m,n)\right]$$

put value of α (in terms of β)

$$\Rightarrow \frac{d}{d\beta}\left[(n-1)\sum \log\left[y_i - \left(\frac{(1-n)}{n+m-2} + \bar{y} - \beta\bar{x}\right) - \beta x_i\right] + (m-1)\sum \log\left(1 - y_i + \left(\frac{1-n}{n+m-2} + \bar{y} - \beta\bar{x}\right) + \beta x_i\right) - \log(B(n,m))\right]$$

$$= (n-1)\frac{\sum \bar{x} - x_i}{y_i - \alpha - \beta x_i} + (m-1)\frac{\sum x_i - \bar{x}}{1 - y_i + \alpha + \beta x_i} = 0$$

Solving the above we can find β.