Bargained Wage Flexibility

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1 Model

- We are going to introduce bargaining into our model
  - Take the model from the main text of the slides, but now parameterize the value of unemployment as $\bar{u}(z)$, where $\bar{u}(z)$ is weakly increasing in z.
  - This modification is barely a change! However we are swapping ω for $\bar{u}$ as notation.
  - Let me recap the model.
- Define a dynamic incentive contract as

$$\{\mathbf{a}, \mathbf{w}\} = \left\{a\left(\eta_i^{t-1}, z^t; z_0\right), w\left(\eta_i^t, z^t; z_0\right)\right\}_{t=0, \eta_i^t, z^t}^{\infty}$$

- The value of the firm's vacancy is

$$V(\mathbf{w}, \mathbf{a}; z_0) = \sum_{t=0}^{\infty} \int \int (\beta(1-s))^t \left(f(z_t, \eta_{it}) - w\left(\eta_i^t, z^t; z_0\right)\right) \pi\left(\eta_i^t, z^t | z_0, a_i^t(\cdot)\right) d\eta_i^t dz^t$$

- The participation constraint is

$$\sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u(w_{it}, a_{it}) + \beta s \bar{u}(z_{t+1}) | z_0, a_i^t\right] = \bar{u}(z_0) \tag{1}$$

  which we can use to define the participation constraint with the mapping

$$PC(\mathbf{w}, \mathbf{a}; z_0) \equiv \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u(w_{it}, a_{it}) + \beta s \bar{u}(z_{t+1}) | z_0, a_i^t\right] - \bar{u}(z_0)$$

  and $PC(\mathbf{w}, \mathbf{a}; z_0) = 0$.

- The incentive compatibility constraints are that for all $\left\{\tilde{a}\left(\eta_i^{t-1}, z^t; z_0\right)\right\}_{t=0, \eta_i^t, z^t}^{\infty}$

$$\sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u(w_{it}, \tilde{a}_{it}) + \beta s \bar{u}(z_{t+1}) | z_0, \tilde{a}_i^t\right] \leq \bar{u}(z_0) \tag{2}$$

  which defines a mapping $IC(\mathbf{w}, \mathbf{a}; z_0)$ and a constraint $IC(\mathbf{w}, \mathbf{a}; z_0) \leq 0$.

- Then the maximized value of a filled vacancy is given by the Lagrangian

$$J(z_0) \equiv \max_{\mathbf{w},\mathbf{a},\mu,\boldsymbol{\lambda}} \sum_{t=0}^{\infty} \int\int (\beta(1-s))^t \left(f(z_t,\eta_{it}) - w\left(\eta_i^t, z^t; z_0\right)\right) \pi\left(\eta_i^t, z^t | z_0, a_i^t(\cdot)\right) d\eta_i^t dz^t$$
$$+ \mu(z_0) \left[\sum_{t=0}^{\infty} (\beta(1-s))^t \int\int \left(u\left(w\left(\eta_i^t, z^t; z_0\right), a\left(\eta_i^{t-1}, z^t; z_0\right)\right) + \beta s \bar{u}(z_{t+1})\right) \pi\left(\eta_i^t, z^t | z_0, a_i^t(\cdot)\right) d\eta_i^t dz^t - \bar{u}(z_0)\right]$$
$$+ \langle \boldsymbol{\lambda}(z_0), IC(\mathbf{w},\mathbf{a}; z_0)\rangle \tag{3}$$

- This model of bargaining is general in the sense that $\bar{u}(z)$ includes many reasons for bargaining
  - There can be bargaining due to changes in the "outside option", because unemployment benefits vary (e.g. Mitman/Rabinovich, Hagedorn et al x 2, Jager et al) or because the value of not working varies (Chodorow-Reich & Karabarbounis)
  - There can be bargaining due to changes in the "inside option", due to Nash bargaining, Hall & Milgrom bargaining, or other bargaining protocols.[1]
  - This model of bargaining also evokes a notion of "efficiency wages" similar to Shapiro & Stiglitz (1982). Workers might accept lower wages or work harder during recessions, because being unemployed is more painful during recessions.

2 Definitions—Overall, Bargained and Incentive Wage Flexibility

- We are going to argue that, once our model allows for cyclical bargaining, only "bargained wage flexibility" is allocative for unemployment fluctuations. However "incentive wage flexibility" is irrelevant.
- As such, we define these notions of wage flexibility. Our notion of overall wage flexibility is the cyclicality of the present value of wages, following Kudlyak. We can decompose overall wage flexibility into components associated with bargaining and incentives, as

$$\overbrace{\frac{d}{dz_0}\mathbb{E}_{0,a^*}\left[\sum_{t=0}^{\infty}(\beta(1-s))^t w_{it}^*\right]}^{\text{overall wage flexibility}} = \underbrace{\partial_{\mathbf{a}}\mathbb{E}_{0,a^*}\left[\sum_{t=0}^{\infty}(\beta(1-s))^t w_{it}\right]\cdot\frac{d\mathbf{a}^*}{dz_0}}_{\text{incentive wage flexibility}} + \overbrace{\mathbb{E}_{0,a^*}\left[\sum_{t=0}^{\infty}(\beta(1-s))^t \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}}\right]}^{\text{bargained wage flexibility}}. \tag{4}$$

The first term is overall wage flexibility—how the expected discounted present value of wages varies with the aggregate state, evaluated on the optimal contract.

  - On the right hand side of equation (4), first, there is incentive wage flexibility. This component is the cyclical changes in wages that are associated with changes in effort. As such, this component is compensation for effort, which we associate with incentives. Inside this component, the $\partial_{\mathbf{a}}$ term captures how wages vary with effort, whereas the $d\mathbf{a}^*/dz_0$ term captures how effort varies with the aggregate state.
  - The second term is bargained wage flexibility. Bargained wage flexibility is how much wages respond to the aggregate state, holding fixed effort (which we denote by the partial derivative $\partial/\partial z_0|_{\mathbf{a}}$). In our model, holding fixed effort, all remaining wage variation is due to changes in $\bar{u}(z)$, which represents bargaining.

[1] With few modifications, this formulation will also encompass models of directed search such as Moen.

- The decomposition into bargained and incentive wage flexibility is natural, and encompasses several models.
  - For instance, in standard labor search models with exogenous effort (for instance Shimer, Hall, Hall & Milgrom or Christiano, Eichenbaum & Trabandt), incentive wage flexibility is zero, so that all wage flexibility is due to bargaining.
  - On the other hand, in the version of our model with constant bargaining power, with $\bar{u}$ independent of z, all wage flexibility is due to incentives.

3 Result: Bargained Wage Flexibility is Allocative for Unemployment Fluctuations

- We now ask what kind of wage flexibility matters for unemployment dynamics, when wages can vary due to both bargaining and incentives.

Proposition 1. *Let the worker's value of unemployment $\bar{u}(z)$ vary with z, but otherwise make the same assumptions as Theorem XX. Additionally, assume that the Lagrange multipliers on the incentive compability constraints defined by equation XX are non-negative. Then the elasticity of market tightness with respect to aggregate shocks is to a first order*

$$\frac{d\log\theta_0}{d\log z_0} \geq \frac{1}{\nu}\frac{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \mathbb{E}_{0,a^*}[\overbrace{f_z\left(z_t,\eta_{it}\right)\frac{\partial z_t}{\partial z_0}}^{\substack{\textit{direct}\\ \textit{productivity}\\ \textit{effect}}} - \overbrace{\left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}}}^{\substack{\textit{bargained}\\ \textit{wage}\\ \textit{flexibility}}}]}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\left(\mathbb{E}_{0,a^*}f\left(z_t,\eta_{it}\right)-\mathbb{E}_{0,a^*}w_{it}^*\right)} \tag{5}$$

- Proposition 1 shows that bargained wage flexibility is allocative for unemployment fluctuations, but incentive wage flexibility is irrelevant.
  - According to its assumptions, Proposition 1 asks how the dynamics of tightness change when there is bargaining—so that workers' value of unemployment can vary with the aggregate state—but otherwise considers the same environment as Theorem XX. Proposition 1 makes an additional technical assumption, that the Lagrange multipliers on the incentive compability constraints are non-negative. This regularity condition is similar to the Monotone Likelihood Ratio Principle of XX Milgrom, but extended to our general dynamic setting.
  - Proposition 1 shows that with wage bargaining, a new term affects tightness dynamics, alongside the direct productivity effect emphasized by Theorem XX.
  - When bargained wage flexibility $\left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}}$ is high, then $d\log\theta_0/d\log z_0$ is smaller and unemployment is less sensitive to shocks.
  - However, *only* bargained wage flexibility matters for tightness dynamics. Incentive wage flexibility does not affect $d\log\theta_0/d\log z_0$.
  - Proposition XX identifies a lower bound for $d\log\theta_0/d\log z_0$ and not an equality.
    - Therefore, in principle, market tightness can be even more sensitive to aggregate shocks than equation (5) suggests.
    - We derive a bound for technical reasons, to be discussed shortly.

- As in Theorem XX, the denominator of the expression for $d\log\theta_0/d\log z_0$ is simply the present value of profits, which converts into elasticity units.
- The proposition studies market tightness, but as we pointed out when discussing Theorem XX, the dynamics of tightness are closely related to the dynamics of unemployment.

- Let us explain the intuition for this result.
  - Incentive wage flexibility is irrelevant for unemployment dynamics for precisely the reasons laid out in Theorem XX. If wages rise due to better incentives, effort must rise at the same time—meaning profits and job creation does not change.
  - However greater bargained wage flexibility dampens unemployment dynamics for standard reasons. If wages rise, *holding fixed effort,* then all else equal firms' profits must fall. As such, with flexible bargained wages, firms' profits increase by less as the aggregate state z increases, which dampens cyclical changes in job creation.
- Proposition 1 suggests that empirical work on unemployment fluctuations should measure bargained wage flexibility.
  - Some empirical measures of wages are flexible. As such, some researchers conclude that wage rigidity cannot account for unemployment fluctuations.
  - However, wages might be flexible due to incentives, which according to our model is irrelevant for unemployment fluctuations.
  - The bargained wage, which is allocative for unemployment fluctuations, could still be rigid.
  - As such, we advocate that researchers measure bargained wage flexibility, and devise strategies to separate bargaining from incentives.
  - **Can we be stronger? Would studying overall wage flexibility be "misleading"?**
  - In the section to come, we will propose one such strategy.
- Last, though equation (5) is intuitive, the derivation of this equation is not straightforward.
  - There are two technical challenges.
  - First, given moral hazard and risk aversion, the Lagrange multipliers on the incentive compatibility and participation constraints cannot be characterized in closed form. However, we derive informative bounds on the multipliers, leading to the bounds of equation (5).
  - Second, the incentive compatibility constraint is complex, which makes effort and wages hard to characterize. To make progress, we rewrite the incentive compatibility constraint as

$$\max_{\tilde{\mathbf{a}}} \sum_{t=0}^{\infty} \left(\beta\left(1-s\right)\right)^t \mathbb{E}\left[u\left(w_{it},\tilde{a}_{it}\right)+\beta s\bar{u}\left(z_{t+1}\right)|z_0,\tilde{a}_i^t\right]=\bar{u}\left(z_0\right). \tag{6}$$

  - This form of the incentive constraint acknowledges that the worker chooses effort $\tilde{\mathbf{a}}$ optimally, given the wage $\mathbf{w}$ proposed by the firm. As such, one can apply an envelope theorem to the left hand side of equation (6), in order to derive the dynamics of $\bar{u}$ without solving for optimal effort.

4 Calibrated Model: Separately Measuring Bargained vs. Incentive Wage Flexibility

- Outline of this part:
  - Equipped with the observation that only bargained wage flexibility is allocative for unemployment dynamics, we set out to measure how much of the overall wage flexibility in the data is due to bargaining versus incentives.
  - Why not just measure bonuses? Two reasons:
    * Bonus is not the same as incentives
    * Incentives beyond bonuses e.g. promotions
  - We pursue a different strategy to measure, though we admit it is a first step
  - Identification argument—ex ante wage fluctuations are bargaining, ex post moments are incentives
  - Parameterize model
    * Tractable form of the incentive contract that can be simulated
    * Parameterize $\bar{u}(z_0)$. Agnostic about the form of bargaining since there are many possibilities. Instead we take an exogenous process to the data.
  - Show that unemployment fluctuations are large and close to Hall
    * Calculate incentive vs. bargained wage flexibility and show that a large share of overall wage flexibility is due to incentives
  - User guide:
    * There exists an economy with a wage, which is an exogenous function of z, which can perfectly match the incentive pay economy dynamics
    * This function---and how sensitive it is to z---is the flexibility of the bargained wage.
    * People should just calibrate that exogenous function

5 Proof of Proposition 1

- The first order condition of the Lagrangian, with respect to wages, is

$$\frac{\partial}{\partial w\left(\eta_i^t, z^t; z_0\right)}[\sum_{t=0}^{\infty} \int\int \left(\beta\left(1-s\right)\right)^t \left(f\left(z_t, \eta_{it}\right) - w\left(\eta_i^t, z^t; z_0\right)\right) \pi\left(\eta_i^t, z^t | z_0, a_i^t\left(\cdot\right)\right) d\eta_i^t dz^t$$

$$+\mu\left(z_0\right)\left[\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \int\int \left(u\left(w\left(\eta_i^t, z^t; z_0\right), a\left(\eta_i^{t-1}, z^t; z_0\right)\right) + \beta s \bar{u}\left(z_{t+1}\right)\right) \pi\left(\eta_i^t, z^t | z_0, a_i^t\left(\cdot\right)\right) d\eta_i^t dz^t - \bar{u}\left(z_0\right)\right]$$

$$+\left\langle \boldsymbol{\lambda}\left(z_0\right), IC\left(\mathbf{w}, \mathbf{a}; z_0\right)\right\rangle] = 0$$

$$\implies \left(\beta\left(1-s\right)\right)^t \left(-1\right) \pi\left(\eta_i^t, z^t | z_0, a_i^t\left(\cdot\right)\right)$$

$$+\mu\left(z_0\right)\left[\left(\beta\left(1-s\right)\right)^t u_w\left(w\left(\eta_i^t, z^t; z_0\right), a\left(\eta_i^{t-1}, z^t; z_0\right)\right) \pi\left(\eta_i^t, z^t | z_0, a_i^t\left(\cdot\right)\right)\right]$$

$$+\frac{\partial}{\partial w\left(\eta_i^t, z^t; z_0\right)} \left\langle \boldsymbol{\lambda}\left(z_0\right), IC\left(\mathbf{w}, \mathbf{a}; z_0\right)\right\rangle = 0$$

$$\implies \mu(z_0)\left[(\beta(1-s))^t u_w\left(w\left(\eta_i^t, z^t; z_0\right), a\left(\eta_i^{t-1}, z^t; z_0\right)\right)\pi\left(\eta_i^t, z^t | z_0, a_i^t(\cdot)\right)\right] + \left\langle \boldsymbol{\lambda}(z_0), \frac{\partial IC(\mathbf{w}, \mathbf{a}; z_0)}{\partial w\left(\eta_i^t, z^t; z_0\right)}\right\rangle$$

$$= (\beta(1-s))^t \pi\left(\eta_i^t, z^t | z_0, a_i^t(\cdot)\right)$$

$$\implies \mu(z_0) u_w\left(w\left(\eta_i^t, z^t; z_0\right), a\left(\eta_i^{t-1}, z^t; z_0\right)\right)$$

$$= 1 - \left\langle \frac{\boldsymbol{\lambda}(z_0)}{(\beta(1-s))^t \pi\left(\eta_i^t, z^t | z_0, a_i^t(\cdot)\right)}, \frac{\partial IC(\mathbf{w}, \mathbf{a}; z_0)}{\partial w\left(\eta_i^t, z^t; z_0\right)}\right\rangle.$$

- Let us show that the term in chevrons is non-negative.
  - We know from equation (2) that $\frac{\partial IC(\mathbf{w},\mathbf{a};z_0)}{\partial w\left(\eta_i^t,z^t;z_0\right)}$ is positive, i.e. every element in $IC(\mathbf{w}, \mathbf{a}; z_0)$ is non-decreasing in $w(\eta_i^t, z^t; z_0)$.
  - We know that $(\beta(1-s))^t \pi(\eta_i^t, z^t | z_0, a_i^t(\cdot))$ is positive.
  - By the assumptions of the proposition we know that all elements in $\boldsymbol{\lambda}(z_0)$ are non-negative.
- Therefore we must have
$$\mu(z_0) u_w\left(w\left(\eta_i^t, z^t; z_0\right), a\left(\eta_i^{t-1}, z^t; z_0\right)\right) \leq 1. \tag{7}$$
- Next, note that by Theorem XX and equation (3), we have

$$\begin{aligned}
\frac{dJ(z_0)}{dz_0} &= \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} f_z(z_t, \eta_{it}) \frac{\partial z_t}{\partial z_0} \\
&+ \mu(z_0) \frac{\partial}{\partial z_0}\left[\sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u(w_{it}, a_{it}) + \beta s \bar{u}(z_{t+1}) | z_0, a_i^t\right] - \bar{u}(z_0)\right] \\
&= \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} f_z(z_t, \eta_{it}) \frac{\partial z_t}{\partial z_0} \\
&+ \mu(z_0)\left[\sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[\frac{\partial}{\partial z_0}\beta s \bar{u}(z_{t+1}) | z_0, a_i^t\right] - \frac{\partial}{\partial z_0}\bar{u}(z_0)\right] \\
&= \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} f_z(z_t, \eta_{it}) \frac{\partial z_t}{\partial z_0} \\
&+ \mu(z_0)\left[\sum_{t=0}^{\infty} (\beta(1-s))^t \beta s \mathbb{E}\left[\frac{d\bar{u}(z_{t+1})}{dz_0} | z_0\right] - \frac{d}{dz_0}\bar{u}(z_0)\right].
\end{aligned} \tag{8}$$

  Note that, as in Theorem XX, we are using the argument that to a first order shocks to z_0 do not affect the probability measure $\mathbb{E}[\cdot | z_0]$. Also, we have swapped partial for total derivatives for the $\bar{u}(z)$ terms in the final line, which is without loss because there is a single argument in $\bar{u}$.

- Next, note that at the optimal contract, the IC and PC constraints (1) and (2) imply

$$\sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u(w_{it}, a_{it}) + \beta s \bar{u}(z_{t+1}) | z_0, a_i^t\right] = \bar{u}(z_0)$$

$$\max_{\tilde{\mathbf{a}}} \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u(w_{it}, \tilde{a}_{it}) + \beta s \bar{u}(z_{t+1}) | z_0, \tilde{a}_i^t\right] = \bar{u}(z_0),$$

the first equation is the PC constraint, the second equation is a rewritten version of the IC constraint.

- Equating and differentiating the two constraints implies

$$\begin{aligned}
\frac{d\bar{u}(z_0)}{dz_0} &= \frac{d}{dz_0} \max_{\tilde{\mathbf{a}}} \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u(w_{it}, \tilde{a}_{it}) + \beta s \bar{u}(z_{t+1}) | z_0, \tilde{a}_i^t\right] \\
&= \frac{d}{dz_0} \max_{\tilde{\mathbf{a}}} \sum_{t=0}^{\infty} (\beta(1-s))^t \int\int \left(u\left(w\left(\eta_i^t, z^t; z_0\right), \tilde{a}\left(\eta_i^{t-1}, z^t; z_0\right)\right) + \beta s \bar{u}(z_{t+1})\right) \\
&\quad \pi\left(\eta_i^t, z^t | z_0, \tilde{a}_i^t(\cdot)\right) d\eta_i^t dz^t \\
&= \left.\frac{\partial}{\partial z_0}\right|_{\mathbf{a}} \sum_{t=0}^{\infty} (\beta(1-s))^t \int\int \left(u\left(w\left(\eta_i^t, z^t; z_0\right), \tilde{a}\left(\eta_i^{t-1}, z^t; z_0\right)\right) + \beta s \bar{u}(z_{t+1})\right) \\
&\quad \pi\left(\eta_i^t, z^t | z_0, \tilde{a}_i^t(\cdot)\right) d\eta_i^t dz^t \\
&= \sum_{t=0}^{\infty} (\beta(1-s))^t \int\int \left(u_w\left(w\left(\eta_i^t, z^t; z_0\right), \tilde{a}\left(\eta_i^{t-1}, z^t; z_0\right)\right) \left.\frac{\partial w\left(\eta_i^t, z^t; z_0\right)}{\partial z_0}\right|_{\mathbf{a}} + \beta s \left.\frac{\partial \bar{u}(z_{t+1})}{\partial z_0}\right|_{\mathbf{a}}\right) \\
&\quad \pi\left(\eta_i^t, z^t | z_0, \tilde{a}_i^t(\cdot)\right) d\eta_i^t dz^t \\
&= \sum_{t=0}^{\infty} (\beta(1-s))^t \int\int \left(u_w\left(w\left(\eta_i^t, z^t; z_0\right), \tilde{a}\left(\eta_i^{t-1}, z^t; z_0\right)\right) \left.\frac{\partial w\left(\eta_i^t, z^t; z_0\right)}{\partial z_0}\right|_{\mathbf{a}} + \beta s \frac{\partial \bar{u}(z_{t+1})}{\partial z_0}\right) \\
&\quad \pi\left(\eta_i^t, z^t | z_0, \tilde{a}_i^t(\cdot)\right) d\eta_i^t dz^t \\
&= \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u_w(w_{it}, \tilde{a}_{it}) \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}} + \beta s \frac{\partial \bar{u}(z_{t+1})}{\partial z_0} | z_0, \tilde{a}_i^t\right] \\
&= \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u_w(w_{it}, \tilde{a}_{it}) \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}} | z_0, \tilde{a}_i^t\right] + \beta s \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[\frac{\partial \bar{u}(z_{t+1})}{\partial z_0} | z_0, \tilde{a}_i^t\right]
\end{aligned}$$

$$\implies \sum_{t=0}^{\infty} (\beta(1-s))^t \beta s \mathbb{E}\left[\frac{d\bar{u}(z_{t+1})}{dz_0} | z_0\right] - \frac{d\bar{u}(z_0)}{dz_0} = -\sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[u_w(w_{it}, \tilde{a}_{it}) \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}} | z_0, \tilde{a}_i^t\right]$$

where in the second equality we have expanded the compact notation for precision; in the third line we have used the envelope theorem to exchange total derivatives for partial derivatives, using the notation $\left.\frac{\partial}{\partial z_0}\right|_{\mathbf{a}}$ to indicate a partial derivative with $\mathbf{a}$ held fixed; in the fifth line we are using that $\bar{u}(z)$ is independent of effort deviations, so $\left.\frac{\partial \bar{u}(z_{t+1})}{\partial z_0}\right|_{\mathbf{a}} = \frac{\partial \bar{u}(z_{t+1})}{\partial z_0}$; and again throughout we are using the argument that to a first order shocks to z_0 do not affect the probability measure $\mathbb{E}[\cdot | z_0]$.

- Finally, since the preceding equation must hold along the optimal contract with $\tilde{\mathbf{a}} = \mathbf{a}^*$, we have

$$\beta s \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}\left[\frac{d\bar{u}(z_{t+1})}{dz_0} | z_0\right] - \frac{d\bar{u}(z_0)}{dz_0} = -\sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*}\left[u_w(w_{it}, a_{it}^*) \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}}\right]. \tag{9}$$

- Substituting equation (9) into equation (8) implies

$$\begin{aligned}\frac{dJ(z_0)}{dz_0} &= \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} f_z(z_t, \eta_{it}) \frac{\partial z_t}{\partial z_0} - \mu(z_0) \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} \left[u_w(w_{it}, a_{it}^*) \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}} \right] \\ &= \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} f_z(z_t, \eta_{it}) \frac{\partial z_t}{\partial z_0} - \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} \left[\mu(z_0) u_w(w_{it}, a_{it}^*) \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}} \right] \\ &\geq \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} f_z(z_t, \eta_{it}) \frac{\partial z_t}{\partial z_0} - \sum_{t=0}^{\infty} (\beta(1-s))^t \mathbb{E}_{0,a^*} \left.\frac{\partial w_{it}}{\partial z_0}\right|_{\mathbf{a}},\end{aligned}$$

  where in the third line we have used equation (7), which proves the result.