This document shows a first-order equivalence between a complicated model with bargaining over promised utilities, and a simpler model in which the firm makes take it or leave it offers to a worker whose flow value of unemployment exogenously moves with the cycle. For now, we abstract from savings and assume workers consume their wage every period.

We have

$$J(z,\omega) = \max_{\{w_{it}(\eta_i^t,z^t),a_{it}(\eta_i^t,z^t)\}_t} \sum_{t=0}^{\infty}(\beta(1-s))^t \int\int (f(z_t,\eta_{it}) - w_{it})\pi(\eta_i^t, z^t|z_0, a_i^t(\cdot))d\eta_i^t dz^t \tag{1}$$

$s.t.$

IC

$$\sum_{t=0}^{\infty}(\beta(1-s))^t \int\int (u(w_{it},a_{it}) + \beta s U(z_t)\pi(\eta_i^t, z^t|z_0, a_i^t)d\eta_i^t dz^t \geq \omega(z_0)$$

Letting λ denote the Lagrange multiplier on the IR constraint, we have that the first order change in firm value from a small change in z_0 is, through Abdou's decomposition:

$$\frac{dJ}{dz_0} = Direct + \lambda\left(\frac{d\omega}{dz_0} - \sum_{t=0}^{\infty}(\beta(1-s))^t\beta s\frac{d\mathbb{E}[U(z_t)|z_0]}{dz_0}\right) \tag{2}$$

The key thing to note is that it does not matter whether the IR constraint is loosened because the promised utility falls or because the expected PDV of the continuation value from separating to unemployment rises.[1]

Now suppose that the firm has all of the bargaining power and makes take it of leave it (TIOLI) offers to the worker when they match, but the worker's flow value of unemployment is procyclical. That is, the value of unemployment is

$$U(z) = b(z) + \beta\mathbb{E}[U(z')|z]$$

If $b(z)$ follows a random walk (e.g. z follows a random walk and $b(z) = b_0 + \gamma z$ for some $(b_0,\gamma) \in \mathbb{R}^2$), we can simplify this further to

$$U(z) = \frac{b(z)}{1-\beta}$$

[1] I conjecture that this will be helpful with solving bargaining games if we went that route, but, as we shall see below, I don't think we need to.

Proof is by guess and verify. We have

$$U(z) = b_0 + \gamma z + \beta \mathbb{E}\left[\frac{b_0 + \gamma z'}{1-\beta}|z\right]$$

$$= b_0 + \gamma z + \beta \frac{b_0 + \gamma z}{1-\beta}$$

$$= \frac{(1-\beta)(b_0 + \gamma z) + \beta(b_0 + \gamma z)}{1-\beta}$$

$$= \frac{b(z)}{1-\beta}$$

In this case, we have $\omega(z) = U(z) = b(z)/(1-\beta)$, so that:

$$\begin{aligned}
\frac{dJ}{dz_0} &= Direct + \frac{\lambda}{1-\beta}\left(\frac{db}{dz} - \sum_{t=0}^{\infty}(\beta(1-s))^t \beta s \frac{db}{dz}\right) \\
&= Direct + \frac{\lambda}{1-\beta}\left(1 - \frac{\beta s}{1-\beta(1-s)}\right)\frac{db}{dz} \\
&= Direct + \frac{\lambda}{1-\beta(1-s)}\frac{db}{dz} \qquad (3) \\
&= Direct + \frac{\lambda\gamma}{1-\beta(1-s)} \qquad (4)
\end{aligned}$$

More generally, suppose we had a model where z followed an AR(1) process and $b(z) = b_0 + \gamma z$ or, equivalently after a relabeling, $\ln z$ followed AR(1) and $b(z) = b_0 + \gamma \ln z$. Note this is also equivalent to $b(z) = \tilde{b}_0 + \gamma(z-1)$ if $\tilde{b}_0 = b_0 + \gamma$. Then guess and verify that $U(z) = A + Bz$ for some scalars A and B:

$$A + Bz = U(z) = b_0 + \gamma z + \beta \mathbb{E}\left[A + Bz'|z\right]$$

$$= b_0 + \gamma z + \beta(A + B\rho z)$$

$$= b_0 + \beta A + (\gamma + \beta B \rho)z$$

$$\implies$$

$$A = \frac{b_0}{1-\beta}$$

$$B = \frac{\gamma}{1-\beta\rho}$$

so $U(z) = \dfrac{b_0}{1-\beta} + \dfrac{\gamma z}{1-\beta\rho}$. In this case, we have:

$$
\begin{aligned}
\frac{dJ}{dz_0} &= Direct + \lambda\left(\frac{\gamma}{1-\beta\rho} - \sum_{t=0}^{\infty}(\beta(1-s))^t \beta s \frac{d\mathbb{E}[b_0/(1-\beta)+\gamma z_t/(1-\beta\rho)|z_0]}{dz_0}\right) \\
&= Direct + \frac{\lambda\gamma}{1-\beta\rho}\left(1 - \sum_{t=0}^{\infty}(\beta(1-s))^t \beta s \frac{d\mathbb{E}[z_t|z_0]}{dz_0}\right) \qquad (5) \\
\xrightarrow{?} &= Direct + \frac{\lambda\gamma}{1-\beta\rho}\left(1 - \sum_{t=0}^{\infty}(\beta(1-s)\rho)^t \beta s\right) \\
&= Direct + \frac{\lambda\gamma}{1-\beta\rho}\left(\frac{1-\beta\rho(1-s)-\beta s}{1-\beta\rho(1-s)}\right)
\end{aligned}
$$

Now consider a more general model in which ω (and thus U) may be influenced by some more complicated bargaining or rent sharing between worker and firm. Such a model will generate the same first-order fluctuations in profits per worker (and thus market tightness) as a simpler model with TIOLI offers and exogenously fluctuating unemployment benefits so long as equation (2) equals equation (3), or, with AR(1), equal to 5. The Direct effects will be the same in both, so this boils down to

$$
\lambda^{FULL}\left(\frac{d\omega}{dz_0} - \sum_{t=0}^{\infty}(\beta(1-s))^t \beta s \frac{d\mathbb{E}[U(z_t)|z_0]}{dz_0}\right) = \lambda^{SIMPLE}\left(\frac{1}{1-\beta(1-s)}\right)\cdot\frac{db}{dz}
$$

That is, so long as we choose γ (i.e. db/dz) to be equal to the change in the outside option minus the EPDV of unemployment movements, and so long as we choose b_0 such that $\lambda^{SIMPLE} = \lambda^{FULL}$, these two models generate identical first order movements in profits per worker and, therefore, identical movements in both vacancies and employment. THIS IS TRUE IN AFFINE CASE WITH AR(1). It also suggests we should stick with affine (probably in $\ln z$) for our specification of $b(z)$.

How does this help us? Most importantly, we can simulate the model completely abstracting from bargaining. It also shows that bargaining and flow unemployment benefits have identical first order effects on the contract. Thus so long as we calibrate models to the same cyclical fluctuation in the flow value of unemployment/promised utility value (i.e. we target the cyclicality new hire wages), we can remain agnostic about the bargaining model that led to those fluctuations in promised utility values, while still separating out the extent to which wage fluctuations come from fluctuations in outside options versus from incentives. That is, we can work with the simple model, calibrate the parameters b_0 (maybe = 0.4 as per Shimer?) and γ, and use those to infer how much is coming from bargaining (IR constraint) versus incentives (IC constraint).