$$\frac{d}{dz}\pi_i = \frac{d}{dz} E[za - w_i]$$

$$= a + \frac{z\,da}{dw} - \frac{dw}{dz}$$

$$= a_i + z\frac{da_i}{dz} - \frac{dw}{dz}$$ ← includes $\frac{\partial\alpha}{\partial z}, \frac{\partial a}{\partial z}, \frac{d\beta}{\partial z}$

$$= a_i + z\frac{d}{dz}\left(\beta z / \phi\right) - \frac{dw}{dz}$$

$$= a_i + \frac{z^3(z^2 + 3\phi r a^2)}{\phi(z^2 + \phi r a^2)^2} - \frac{dw}{dz}$$, where

$$\frac{z^5 + 3\phi r a^2 z^3}{\phi(z^2 + \phi r a^2)^2}$$

$$= \frac{z(z^4 + 3\phi r a^2 z^2)}{\phi(z^2 + \phi r a^2)^2}$$

$$\frac{dw}{dz} = \frac{2\phi a^2 r z^3 + z^5}{\phi(\phi a^2 r + z^2)^2}$$

$\Rightarrow$

$$\boxed{\frac{d\pi}{dz} = a_i + \underbrace{\frac{\phi r a^2 z^3}{\phi(z^2 + \phi r a^2)^2}}_{\text{extra term}}}$$

$$= \beta^2 \cdot z r a^2$$

* Double-Check *

$$\alpha = b + \beta^2/2\,(ra^2 - z^2/\phi)$$

$$\beta = \frac{z^2}{z^2 + \phi r a^2}\,, \quad w = \alpha + \beta\,(za)$$

$$a = \beta z/\phi$$

$$\Rightarrow \quad a = \frac{z^2}{z^2 + \phi r a^2} \cdot \frac{z}{\phi} \Rightarrow \frac{da}{dz} = \frac{3\phi r a^2 z^2 + z^4}{\phi(\phi r a^2 + z^2)^2}$$

$$\frac{dw}{dz} = \frac{d}{dz}\left[\beta^2/2\,(ra^2 - z^2/\phi)\right]$$

$$+ \frac{d}{dz}\left[\underbrace{\left(\frac{z^2}{z^2 + \phi r a^2}\right)^2 \cdot z^2 \cdot /\phi}_{\beta a z}\right]$$

$$= \frac{d}{dz}\left[\left(\frac{z^2}{z^2 + \phi r a^2}\right)^2 \cdot \frac{1}{2}\left(ra^2 - z^2/\phi\right)\right.$$

$$\left. + \left(\frac{z^2}{z^2 + \phi r a^2}\right)^2 z^2/\phi\right]$$

$$\max_{\alpha, \beta, a} \quad \pi + \mu [IR] + \eta (IC)$$

$$\to \frac{d}{dz}(\mathcal{L}) = \frac{\partial \pi}{\partial z} + \frac{\partial}{\partial z}(\mu \cdot IR) + \frac{\partial}{\partial z}\eta(IC)$$

$$+ \frac{\partial \alpha}{\partial z}\frac{d}{d\alpha}\left[\pi + \mu[IR] + \eta(IC)\right]$$

$$+ \frac{\partial \beta}{dz}\frac{d}{d\beta}\left[\pi + \mu[IR] + \eta(IC)\right]$$

$$+ \frac{da}{dz}\frac{d}{da}\left[\pi + \mu[IR] + \eta IC\right]$$

$$\frac{d\pi}{\partial z} = \frac{\partial}{\partial z}(az - (\alpha + \beta z a))$$

$$= a - \beta a + \frac{d}{dz}\underbrace{\mu IR}_{\frac{r}{\phi}} + \frac{d}{dz}\underbrace{\eta IC}_{\frac{-2zr\beta^2}{2\phi}}$$

Determine Analytical Derivatives

Fixed w, a

$$\frac{dJ}{dz} = a \quad \checkmark$$

Nash Bargaining

$$\frac{dJ}{dz} = \frac{\partial J}{\partial z} + \frac{\partial J}{\partial a}\underset{\sim}{\frac{\partial a}{\partial z}} + \frac{\partial J}{\partial w}\frac{\partial w}{\partial z}$$

$$= a + 0 - \beta a$$

Fixed a

$$\frac{dJ}{dz} = a + \frac{dJ}{da}\frac{\partial a}{\partial z} + \frac{\partial J}{\partial w}\frac{\partial w}{\partial z}$$

$$= a + 0 - \frac{2\phi r a^2 z^3 + z^5}{\phi(\phi a^2 r + z^2)^2}$$

Fixed w

$$\frac{dJ}{\partial z} = a + \frac{dJ}{da}\frac{da}{dz}$$

$$= a + \frac{dJ}{da}\left[az - w\right]\frac{da}{dz}$$

$$= a + \underbrace{\frac{z^5 + 3\phi r a^2 z^3}{\phi(ra^2\phi + z^2)^2}}_{z \cdot \frac{da}{dz}}$$

Contract

Envelope

$$\frac{dJ}{dz} = \frac{d}{dz}\left[az - w\right]$$

$$= \frac{d}{dz}\Big[\overbrace{a(\beta(z),z)\,z}^{y}\Big]$$

$$- \frac{d}{dz}\Big[\overbrace{\alpha(\beta(z),z) - \beta(z)\big(a(\beta(z),z)\,z\big)}^{w}\Big]$$

$$= \underbrace{\frac{\partial J}{\partial \beta}}_{0 \text{ by FOC}}\frac{d\beta}{dz} + \frac{\partial J}{\partial z}, \text{ where}$$

$$\frac{\partial J}{\partial z} = \overset{①}{a(\beta(z),z)} + \overset{② = \beta z/\phi}{a_z(\beta(z),z)\,z}$$

$$- \Big[\underbrace{\alpha_z(\beta(z),z)}_{③}\Big] - \underbrace{\beta(z)\frac{\partial\big(a(\beta(z),z)\cdot z\big)}{\partial z}}_{④}.$$

Rewrite ④ as

$$\beta(z)\,a_z(\beta(z),z)\,z + \beta(z)\,a(\beta(z),z)$$

$$= \beta^2 z/\phi + \beta^2 z/\phi = 2\beta^2 z/\phi$$

Rewrite ③ as

$$\frac{\partial}{\partial z}\left(b + \frac{\beta^2}{2}\left(ra^2 - \frac{z^2}{\phi}\right)\right)$$

$$= -\frac{\beta^2}{2} \cdot 2z/\phi = -\beta^2 \frac{z}{\phi}$$

$$\Rightarrow \frac{dJ}{dz} = \overset{①}{a} + \overset{②}{\beta z/\phi} + \overset{③}{\beta^2 z/\phi} - \overset{④}{2\beta^2 z/\phi}$$

$$= a + \beta z/\phi - \beta^2 z/\phi$$

$$// \quad \text{(checked on Wolfram)}$$

$$= a + \frac{\sigma^2 r z^3}{(\phi r a^2 + z^2)^2}$$