Redux Wage Rigidity Proof

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1 Advantages of this Proof

- This proof is more general than what we have previously done in at least four ways:
  - Non separable and non log utility is allowed
  - Idiosyncratic shocks with arbitrary processes are allowed
  - The first order approach not required for the theorem showing that incentive wage flexibility does not mute unemployment dynamics.
    - This is a significant technical advance relative to the whole literature, which has made almost no progress without the first order approach.
    - We do not need to assume convexity anywhere
  - The proof does not rely on heuristic approaches anywhere (as far as I can tell)
- To do the proof we need a generalized envelope theorem for functionals which applies when there is a continuum of constraints; we use results from the mathematics literature on "sensitivity analysis".

2 Bargained Wage Cyclicality

- We are going to use a simpler formulation for bargained wage cyclicality, since the old theorem for bargained wage cyclicality was wrong, but this one is correct

3 Brief Notation Notes

- To try make this easy to read I am going to do a few things we might not do in the actual paper:
  - I will use the "innovation" notation for the aggregate shock (we might want this only in the Appendix and not in the main body)
  - In the setup I will only focus on the features that are different from what we have previously been studying, so it will be abbreviated
  - I have emboldened some key assumptions in the setup which will be technically important for the proofs
  - In the proofs I will use some functional analysis theorems. I will color code these theorems and the notation that they use so we can distinguish what is the textbook theorem versus our application

- I am also going to change to a new notation for the participation constraint and the budget constraint, which we will use elsewhere, which subsumes the value of unemployment into promised utility
  - This is going to make the expressions cleaner without any loss of meaning

4 Setup

4.1 Probabilities and Shock Distributions

Defining probabilities:

- z_t is a bounded and continuous process measuring shocks to aggregate productivity, with strictly positive support
  - In the main text we can simply define z_t as a continuous Markovian process as we have previously been doing, and then consider a shock to z_t at time 0 which is uncorrelated with subsequent innovations
  - Here I am going to adopt a more careful notation which formally defines an innovation
  - In the paper, we can put this more careful notation in the Appendix only
  - Define $z_t = E\left[z_t|z_0\right]+\varepsilon_t$, where by definition, ε_t is the cumulative innovation to the process for z between 0 and t and ε_0 is known to be 0.
  - An AR(1) example of this process is

$$z_t = \rho z_{t-1} + \nu_t \quad \nu_t \sim i.i.d.N(0,1)$$

$$\implies z_t = \underbrace{\rho^t\left(z_0-\bar{z}\right)+\left(1-\rho\right)\rho^{t-1}\bar{z}}_{E[z_t|z_0]} + \underbrace{\sum_{j=0}^{t}\rho^j\nu_{t-j}}_{\varepsilon_t}$$

  - Let's assume that the probability distribution of innovations is independent from z_0, with a distribution function $\pi_t\left(\varepsilon^t\right)$ that does not depend on z_0 .
  - The notion of an impulse response is a shock to z_0 that is uncorrelated with innovations ε_t after time 0.
  - This distribution function implicitly defines a marginal distribution of z_{t+1} given by $\hat{\pi}\left(z_{t+1}|z_0\right)$
- η_t is a discrete shock to idiosyncratic productivity, drawn from a set $\eta_t \in \Xi = \left\{\underline{\eta},...,\overline{\eta}\right\}$ with strictly positive support, we let Ξ^t be the set of idiosyncratic shock histories
  - Let's assume that the realization of idiosyncratic shocks in period t are characterized by

$$\pi_t\left(\eta_t|\eta^{t-1},a^t,\varepsilon^t\right)$$

    i.e. the probability of an idiosyncratic shock in period t depends on (i) past idiosyncratic shocks (ii) the sequence of effort (iii) innovations to the aggregate state
  - We assume that π_t is continously differentiable in a^t.
  - The probability of the initial shock is simply

$$\pi_0\left(\eta_0|a^0\right)$$

    which, since it is the first shock, is independent of any other shocks or past innovations.

- **The assumption that η_0 is a discrete shock will be technically important**

- Crucially, the distribution of η_t does not depend on z_0 other than via a^t, though η_t can depend on innovations to the aggregate state (there is mixed evidence that dispersion of idiosyncratic TFP rises during recessions but the best evidence from Berger/Vavra suggests not)

4.2 Firm Problem

- The optimal contract of the firm is

$$(\mathbf{w}, \mathbf{a}) = \left\{ w_t \left(\eta^t, \varepsilon^t; z_0 \right), a_t \left(\eta^{t-1}, \varepsilon^t; z_0 \right) \right\}_{t=0,\eta^t,\varepsilon^t}^{\infty}$$

where $w_t \left(\eta^t, \varepsilon^t; z_0 \right), a_t \left(\eta^{t-1}, \varepsilon^t; z_0 \right)$ are continuous functions mapping from the history of idiosyncratic and aggregate shocks, and the initial state, to wages and effort; and we will write $w_0 \left(\eta^0; z_0 \right), a_0 \left(z_0 \right)$ to refer to wages and effort in the initial period.

  - **We will assume that $w_t \left(\eta^t, \varepsilon^t; z_0 \right) \in [\underline{w}, \overline{w}]$ and $a_t \left(\eta^{t-1}, \varepsilon^t; z_0 \right) \in [\underline{a}, \overline{a}]$, with $\underline{a}, \underline{w} > 0$, so that the set of feasible contracts is compact**

- The firm solves

$$J(z_0) = \max_{\mathbf{w}(z_0),\mathbf{a}(z_0)} \sum_{t=0}^{\infty} \left(\beta \left(1 - s \right) \right)^t \int \sum_{\eta^t \in \Xi^t} \left(f(z_t, \eta_t) - w_t(\eta^t, \varepsilon^t; z_0) \right) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}(z_0) \right) d\varepsilon^t \tag{1}$$

where we have defined the probability distribution

$$\pi_t \left(\eta^t, \varepsilon^t | \mathbf{a}(z_0) \right) = \Pi_{\tau=0}^t \pi_\tau \left(\eta_\tau | \eta^{\tau-1}, a^\tau \left(\eta^{\tau-1}, \varepsilon^\tau; z_0 \right), \varepsilon^\tau \right) \pi_\tau \left(\varepsilon^\tau \right),$$

which defines an optimal contract

$$\left(\mathbf{w}^*, \mathbf{a}^* \right) = \left\{ w_t^* \left(\eta^t, \varepsilon^t; z_0 \right), a_t^* \left(\eta^{t-1}, \varepsilon^t; z_0 \right) \right\}_{t=0,\eta^t,\varepsilon^t}^{\infty}.$$

  - **We of course need that $f(z_t, \eta_t)$ is continuously differentiable in z_t and η_t**

- The maximization is subject to the participation constraint

$$\sum_{t=0}^{\infty} \left(\beta \left(1 - s \right) \right)^t \left[\int \sum_{\eta^t \in \Xi^t} u \left(w_t(\eta^t, \varepsilon^t; z_0), a_t(\eta^{t-1}, \varepsilon^t; z_0) \right) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}(z_0) \right) d\varepsilon^t = \mathcal{B}(z_0), \right. \tag{2}$$

which we assume binds with equality.

  - **Assuming that the participation constraint binds with equality will be important, unfortunately it wasn't easy for me to prove that the participation constraint will always bind, because (i) we cannot write down a Lagrangian without assuming the participation constraint is an equality constraint; and (ii) I could not find a perturbation argument to rule out a slack PC constraint (but there probably is one)**
  - **We are also subsuming the value of unemployment into the $B(z_0)$ term. We can formally link this to Nash bargaining and take it or leave it wage offers in an appendix (take it or leave it wage offers with acyclical unemployment benefits is the only protocol that delivers acyclical $\mathcal{B}$)**

- There are also incentive compatibility constraints that for all $\tilde{\mathbf{a}} = \left\{\tilde{a}_t\left(\eta^{t-1},\varepsilon^t\right)\right\}_{t=0,\eta^t,\varepsilon^t}^{\infty}$ such that $\tilde{a}_t\left(\eta^{t-1},\varepsilon^t\right) \in [\underline{a},\overline{a}]$, we have

$$\sum_{t=0}^{\infty}(\beta(1-s))^t\left[\int\sum_{\eta^t\in\Xi^t}u\Big(w_t(\eta^t,\varepsilon^t;z_0),\tilde{a}_t(\eta^{t-1},\varepsilon^t)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\tilde{\mathbf{a}}\right)d\varepsilon^t\right]$$
$$\leq$$
$$\sum_{t=0}^{\infty}(\beta(1-s))^t\left[\int\sum_{\eta^t\in\Xi^t}u\Big(w_t(\eta^t,\varepsilon^t;z_0),a_t(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|,\mathbf{a}\left(z_0\right)\right)d\varepsilon^t\right] \tag{3}$$

  - **For symmetry with the participation constraint, we are getting rid of the continuation value of unemployment from this expression, but it can easily be added back in**
  - **We can link to an appendix section that includes the value of unemployment**

5 Theorems

5.1 General Decomposition of Profits

Assume that effort is interior in at least one state, i.e. on the optimal contract, there is some η^t,ε^t such that $a^*\left(\eta^t,\varepsilon^t;z_0\right)\in(\underline{a},\overline{a})$. Then generically, the response of firm profits to aggregate shocks is

$$\frac{dJ\left(z_0\right)}{dz_0} = \underbrace{\frac{\partial}{\partial z_0}V\left(\mathbf{w}^*,\mathbf{a}^*;z_0\right)}_{\text{(A) direct productivity effect on profits}} - \underbrace{\left\langle\frac{\partial}{\partial z_0}G\left(\mathbf{w}^*,\mathbf{a}^*\right),\lambda^*\left(z_0\right)\right\rangle}_{\text{(B) direct effect on participation and incentives}}$$
$$+\underbrace{\sum_{x\in\{\mathbf{w}^*,\mathbf{a}^*\}}\left[\partial_x V\left(\mathbf{w}^*,\mathbf{a}^*;z_0\right)-\left\langle\partial_x G\left(\mathbf{w}^*,\mathbf{a}^*\right),\lambda^*\left(z_0\right)\right\rangle\right]\cdot\frac{dx}{dz_0}-\left\langle G\left(\mathbf{w}^*,\mathbf{a}^*\right),\frac{d\lambda^*(z_0)}{dz_0}\right\rangle}$$

where ∂_x represents the vector of partial derivatives with respect to x.

- Quick discussion of assumptions/constraint qualification:
  - The main challenge to writing this proof is verifying the constraint qualification, which is difficult because there are a continuum of incentive constraints
  - Assuming that effort is interior in at least one state is required to verify the constraint qualification of the Lagrangian—a minimal assumption and much weaker than the first order approach. We are just ignoring the pathological contracts in which effort is always at its lower or upper bound in all times and states.
  - The constraint qualification is satisfied generically (there are knife edge values of the mapping between $\mathbf{a}\left(z_0\right)$ and $\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}\left(z_0\right)\right)$ such that the constraint qualification will fail).
  - We verify the functional analogue of the Mangasarian-Fromowitz constraint qualification, which implies that a Lagrangian characterizes the constrained optimum. We do not verify that the set of Lagrange multipliers is unique, which requires a stronger constraint qualification such as the linear independence constraint qualification, which is difficult to verify at our level of generality.

5.2 Incentive Wage Cyclicality Does Not Mute Unemployment Fluctuations

- *Suppose that there are take it or leave it wage offers and acyclical unemployment benefits, so $\mathcal{B}$ is independent of z_0. Assume that the set of feasible contracts $(\mathbf{w}, \mathbf{a}) \in \chi$ that satisfy the incentive constraints (3) and the participation constraint (2) is non-empty. Then the elasticity of market tightness with respect to aggregate shocks in a flexible incentive pay economy is*

$$\frac{d\log\theta_0}{d\log z_0} = \frac{1}{\nu_0}\frac{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \frac{\partial}{\partial\log z_0}\mathbb{E}\left[f(z_t,\eta_t)|z_0,\mathbf{a}^*\right]}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}\left[f(z_t,\eta_t)-w_t^*|z_0,\mathbf{a}^*\right]}, \tag{4}$$

where $\mathbf{a}^*$ and $\mathbf{w}^*$ are effort and wages under the firm's optimal incentive wage contract, and ν_0 is the elasticity of job filling with respect to tightness. The dynamics of market tightness in a rigid wage economy with $w = \bar{w}$ and $a = \bar{a}$ are

$$\frac{d\log\theta_0}{d\log z_0} = \frac{1}{\nu_0}\frac{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \frac{\partial}{\partial\log z_0}\mathbb{E}\left[f(z_t,\eta_t)|z_0,\bar{\mathbf{a}}\right]}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}\left[f(z_t,\eta_t)-\bar{w}|z_0,\bar{\mathbf{a}}\right]}. \tag{5}$$

Assume further that the production function $f(\cdot)$ is homogeneous of degree 1 in aggregate productivity z, and z is a driftless random walk. Then the response of market tightness to z in both economies, in the neighborhood of a steady state for z, is equal to

$$\frac{d\log\theta_0}{d\log z_0} = \frac{1}{\bar{\nu}}\left(\frac{1}{1-\Lambda}\right) \tag{6}$$

In both economies. Λ is the steady state labor share defined as

$$\Lambda \equiv \frac{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}_0 w_t}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}_0 f(\bar{z},\eta_t)}, \tag{7}$$

where expectations are evaluated in a steady state with constant aggregate productivity $z_t = \bar{z}$, and $\bar{\nu}$ is the steady state elasticity of job filling with respect to tightness.

5.3 Bargained Wage Cyclicality Does Mute Unemployment Fluctuations

- Let's define the present value of wages on the optimal contract

$$\mathcal{W}\left(z_0\right) = \sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}\left[w_t^*\left(z_0\right)|z_0,\mathbf{a}^*\left(z_0\right)\right],$$

and the the present value of output on the optimal contract as

$$\mathcal{Y}\left(\mathbf{a}^*\left(z_0\right);z_0\right) = \sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}\left[f(z_t,\eta_t)|z_0,\mathbf{a}^*\left(z_0\right)\right].$$

- Then "overall wage cyclicality", i.e. the response of the present value of wages on the optimal contract to z_0, is simply $d\mathcal{W}\left(z_0\right)/dz_0$.

- Now, we can define incentive and bargained wage cyclicality

- "Incentive wage cyclicality" is defined as

$$\frac{d\mathcal{W}^{\text{incentive}}\left(z_0\right)}{dz_0} = \partial_{\mathbf{a}}\mathcal{Y}\left(\mathbf{a}^*\left(z_0\right);z_0\right)\frac{d\mathbf{a}^*}{dz_0}.$$

therefore incentive wage cyclicality is the component of overall wage cyclicality that is associated with movements in effort $d\mathbf{a}^*/dz_0$. The $\partial_{\mathbf{a}}\mathcal{Y}$ term on the right hand side of the equation converts effort into the same units in wages, since $\partial_{\mathbf{a}}\mathcal{Y}$ is the marginal effect of effort on production.

- We can define "bargained wage cyclicality" as the residual component of overall wage cyclicality, excluding incentives, so that

$$\frac{d\mathcal{W}^{\text{incentive}}\left(z_0\right)}{dz_0} + \frac{d\mathcal{W}^{\text{bargained}}\left(z_0\right)}{dz_0} = \frac{d\mathcal{W}\left(z_0\right)}{dz_0}.$$

- This decomposition is useful:
  - In general, wage cyclicality is split into two terms: a term associated with cyclical movements in effort and incentives, and a term that reflects bargaining
  - In standard labor search models such as XX Hall and XX Shimer, overall wage cyclicality equals bargained wage cyclicality and incentive wage cyclicality is zero
  - By contrast, in our incentive pay model with acyclical promised utility, overall wage cyclicality equals incentive wage cyclicality and bargained wage cyclicality is zero
- Let's consider the case in which both bargained and incentive wage cyclicality is non zero. I will present two theorems. The first theorem is for the paper and the second theorem is for the slides.

1. *Assume that the set of feasible contracts* $(\mathbf{w},\mathbf{a}) \in \chi$ *that satisfy the incentive constraints (3) and the participation constraint (2) is non-empty. Then the elasticity of market tightness with respect to aggregate shocks in the flexible incentive pay economy is*

$$\frac{d\log\theta_0}{d\log z_0} = \frac{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}\left[f_z(z_t,\eta_t)|z_0,\mathbf{a}^*\left(z_0\right)\right]\frac{\partial\mathbb{E}[z_t|z_0]}{\partial\log z_0} - \frac{\partial\mathcal{W}^{bargained}(z_0)}{\partial\log z_0}}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\mathbb{E}\left[f(z_t,\eta_t) - w_t^*(z_0)|z_0,\mathbf{a}^*\left(z_0\right)\right]}$$

where

$$\frac{\partial\mathcal{W}^{bargained}\left(z_0\right)}{\partial\log z_0} > 0 \quad\Longleftrightarrow\quad \mathcal{B}'\left(z_0\right) > 0,$$

that is, bargained wage cyclicality is positive if and only if bargained utility is cyclical.

2. *Assume that the set of feasible contracts* $(\mathbf{w},\mathbf{a}) \in \chi$ *that satisfy the incentive constraints (3) and the participation constraint (2) is non-empty; that the production function* $f(\cdot)$ *is homogeneous of degree 1 in aggregate productivity* z*; and that* z *is a driftless random walk. Then the response of market tightness to* z*, in the neighborhood of a steady state for* z*, is equal to*

$$\frac{d\log\theta_0}{d\log z_0} = \frac{1}{\bar{\nu}}\left(\frac{1-\xi}{1-\Lambda}\right)$$

where expectations are evaluated in a steady state with constant aggregate productivity $z_t = \bar{z}$*, and*

$$\xi > 0 \quad\Longleftrightarrow\quad \mathcal{B}'\left(z_0\right) > 0$$

where ξ is the share of bargained wage cyclicality, defined as

$$\xi \equiv \frac{\frac{\partial \mathcal{W}^{bargained}(z_0)}{\partial \log z_0}}{\sum_{t=0}^{\infty} (\beta (1-s))^t \, \mathbb{E}\left[f(\bar{z}, \eta_t) | \bar{z}, \mathbf{a}^*(\bar{z})\right]}.$$

- I think we should use only theorem (1) in the paper and only theorem (2) in the slides, but not bother defining ξ on the slides.

6 Proofs

6.1 Outline of the Proof Strategy

- First, I will derive a general expression for dJ/dz_0 and $d\log\theta_0/d\log z_0$ when $\mathcal{B}(z_0)$ can vary with z_0
  - En route I will develop a lemma to apply a general envelope theorem
- Then I will use the general expression to solve the "incentive wage cyclicality does not mute unemployment fluctuations" theorem
- Then I will use this general expression, and formally define a Lagrangian, to do the "general decomposition of profits" proposition
- Then I will use the general expression to solve the "bargained wage cyclicality does mute unemployment fluctuations" theorem

6.2 Deriving a General Expression for dJ/dz_0 and $d\log\theta_0/d\log z_0$

- To derive a general expression for dJ/dz_0, I will make the following argument
  - **Step 1:** I show that because participation constraint is an equality constraint, it can be substituted into the firm's objective function.
  - **Step 2:** In a lemma, I apply an envelope theorem:
    * I observe that the firm problem is a Lagrangian with a continuum of constraints.
    * Since the participation constraint has been substituted into the envelope theorem and the incentive constraints do not depend on z_0, the constraint set does not depend on z_0
    * Then a generalized envelope theorem for functionals can be applied, from XX Bonnans and Shapiro (2000), as I verify.
  - **Step 3:** With the envelope theorem done, the remaining steps to derive the general expression are straightforward
- **Step 1:** we can rewrite the participation constraint as an equality relationship between $w_0\left(\underline{\eta}; z_0\right)$ and the other wages. $w_0\left(\underline{\eta}; z_0\right)$ is the wage in the initial period for the lowest idiosyncratic shock. This is a normalization—any other wage in an initial period will work as well. This will allow us to substitute the participation constraint into the objective function. We have from equation (2) that

$$\sum_{t=0}^{\infty} (\beta(1-s))^t \left[\int \sum_{\eta^t \in \Xi^t} u\bigg(w_t(\eta^t, \varepsilon^t; z_0), a_t(\eta^{t-1}, \varepsilon^t; z_0) \bigg) \tilde{\pi}_t\left(\eta^t, \varepsilon^t | \mathbf{a}(z_0)\right) d\varepsilon^t = \mathcal{B}(z_0) \right.$$

$$
\begin{aligned}
&\implies \sum_{\eta^0\in\Xi^0} u\Big(w_0\left(\eta^0;z_0\right),a_0\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}\left(z_0\right)\right)\\
&+\sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t}u\Big(w_t(\eta^t,\varepsilon^t;z_0),a_t(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}\left(z_0\right)\right)d\varepsilon^t\right]=\mathcal{B}\left(z_0\right)\\
&\implies \sum_{\eta^0\in\Xi^0} u\Big(w_0\left(\eta;z_0\right),a_0\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta|\mathbf{a}\left(z_0\right)\right)=\\
&\mathcal{B}\left(z_0\right)-\sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t}u\Big(w_t(\eta^t,\varepsilon^t;z_0),a_t(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}\left(z_0\right)\right)d\varepsilon^t\right.\\
&\implies u\Big(w_0\left(\underline{\eta};z_0\right),a_0\left(z_0\right)\Big)\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}\left(z_0\right)\right)+\sum_{\eta^0\in\Xi^0\backslash\underline{\eta}}u\Big(w_0\left(\eta^0;z_0\right),a_0\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}\left(z_0\right)\right)=\\
&\mathcal{B}\left(z_0\right)-\sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t}u\Big(w_t(\eta^t,\varepsilon^t;z_0),a_t(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}\left(z_0\right)\right)d\varepsilon^t\right]\\
&\implies u\Big(w_0\left(\underline{\eta};z_0\right),a_0\left(z_0\right)\Big)=\\
&\frac{\mathcal{B}\left(z_0\right)-\sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t}u\Big(w_t(\eta^t,\varepsilon^t;z_0),a_t(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}\left(z_0\right)\right)d\varepsilon^t\right]}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}\left(z_0\right)\right)}\\
&-\frac{\sum_{\eta^0\in\Xi^0\backslash\underline{\eta}}u\Big(w_0\left(\eta^0;z_0\right),a_0\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}\left(z_0\right)\right)}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}\left(z_0\right)\right)}\\
&\implies w_0\left(\underline{\eta};z_0\right)=\\
&u^{-1}\left(\frac{\mathcal{B}\left(z_0\right)-\sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t}u\Big(w_t(\eta^t,\varepsilon^t;z_0),a_t(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}\left(z_0\right)\right)d\varepsilon^t\right]}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}\left(z_0\right)\right)}\right.\\
&\left.-\frac{\sum_{\eta^0\in\Xi^0\backslash\underline{\eta}}u\Big(w_0\left(\eta^0;z_0\right),a_0\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}\left(z_0\right)\right)}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}\left(z_0\right)\right)},a_0\left(z_0\right)\right)
\end{aligned}
\tag{8}
$$

where the second implication separates period 0 utility from the rest, and exploits that z_0 is known at time 0; and the other implications are straightforward algebra.

- Therefore we can write the firm's problem as

$$
J\left(z_0\right)=\max_{\mathbf{w},\mathbf{a}}\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t\int\sum_{\eta^t\in\Xi^t}\left(f(z_t,\eta_t)-w_t(\eta^t,\varepsilon^t;z_0)\right)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}\left(z_0\right)\right)d\varepsilon^t \tag{9}
$$

where $w_0(\eta_1; z_0)$ satisfies equation (8) and $(\mathbf{w}, \mathbf{a}) \in \Phi$, where Φ is a constraint set implicitly defined by the incentive constraints (3) and the requirement that $(\mathbf{w}, \mathbf{a}) \in \chi$, where χ is the set of feasible contracts (given by the bounds on wages and effort).

- Taking the total derivative of the Lagrangian implicitly defined by problem (9) with respect to z_0, we have

$$\frac{dJ(z_0)}{dz_0} = \frac{\partial J(z_0)}{\partial z_0} \tag{10}$$

$$= \frac{\partial}{\partial z_0}\left[\sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t}\left(f(z_t,\eta_t) - w_t^*(\eta^t,\varepsilon^t;z_0)\right)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t\right] \tag{11}$$

$$-\frac{\partial}{\partial z_0}\left[w_0^*\left(\underline{\eta}; z_0\right)\right]. \tag{12}$$

The first equality, which is **Step 3**, invokes a generalized envelope theorem for functionals in order to swap the total for partial derivative and in order to ignore perturbations in the constraint set Φ. We apply this step in Lemma 6.3 below. We have also substituted in the optimal wage and effort, $w_t^*(\eta^t, \varepsilon^t; z_0)$ and $\mathbf{a}^*(z_0)$.

- **Step 4:** we now finish deriving the general expression with algebra. Taking each of these terms separately we have for the first part (11) that

$$\frac{\partial}{\partial z_0}\left[\sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t}\left(f(z_t,\eta_t) - w_t^*(\eta^t,\varepsilon^t;z_0)\right)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t\right]$$

$$= \sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t}\frac{\partial}{\partial z_0}\left(f(z_t,\eta_t) - w_t^*(\eta^t,\varepsilon^t;z_0)\right)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t$$

$$= \sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t}\frac{\partial}{\partial z_0}\left(f(\mathbb{E}[z_t|z_0]+\varepsilon_t,\eta_t) - w_t^*(\eta^t,\varepsilon^t;z_0)\right)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t$$

$$= \sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t}\frac{\partial}{\partial z_0}\left(f(\mathbb{E}[z_t|z_0]+\varepsilon_t,\eta_t)\right)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t$$

$$= \sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t} f_z(z_t,\eta_t)\frac{\partial\left(\mathbb{E}[z_t|z_0]+\varepsilon_t\right)}{\partial z_0}\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t$$

$$= \sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t} f_z(z_t,\eta_t)\frac{\partial\mathbb{E}[z_t|z_0]}{\partial z_0}\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t$$

$$= \sum_{t=0}^{\infty}(\beta(1-s))^t \int \sum_{\eta^t\in\Xi^t} f_z(z_t,\eta_t)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*(z_0)\right)d\varepsilon^t\frac{\partial\mathbb{E}[z_t|z_0]}{\partial z_0}$$

$$= \sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f_z(z_t,\eta_t)|z_0,\mathbf{a}^*(z_0)\right]\frac{\partial\mathbb{E}[z_t|z_0]}{\partial z_0}, \tag{13}$$

which is the direct productivity effect or "A term".

- For the second part (12), we have, using equation (8), that

$$\frac{\partial}{\partial z_0}\left[w_0^*\left(\eta_1; z_0\right)\right]$$

$$= \frac{\partial}{\partial z_0}\left[u^{-1}\left(\frac{\mathcal{B}\left(z_0\right) - \sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t} u\Big(w_t^*(\eta^t,\varepsilon^t;z_0), a_t^*(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*\left(z_0\right)\right)d\varepsilon^t\right]}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)}\right.\right.$$

$$\left.\left. - \frac{\sum_{\eta^0\in\Xi^0\backslash\underline{\eta}} u\Big(w_0^*\left(\eta^0;z_0\right), a_0^*\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}^*\left(z_0\right)\right)}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)}, a_0^*\left(z_0\right)\right)\right]$$

$$= u_1^{-1}\left(\frac{\mathcal{B}\left(z_0\right) - \sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t} u\Big(w_t^*(\eta^t,\varepsilon^t;z_0), a_t^*(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*\left(z_0\right)\right)d\varepsilon^t\right]}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)}\right.$$

$$\left. - \frac{\sum_{\eta^0\in\Xi^0\backslash\underline{\eta}} u\Big(w_0^*\left(\eta^0;z_0\right), a_0^*\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}^*\left(z_0\right)\right)}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)}, a_0^*\left(z_0\right)\right)$$

$$\times\frac{\partial}{\partial z_0}\left[\frac{\mathcal{B}\left(z_0\right) - \sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t} u\Big(w_t^*(\eta^t,\varepsilon^t;z_0), a_t^*(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*\left(z_0\right)\right)d\varepsilon^t\right]}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)}\right.$$

$$\left. - \frac{\sum_{\eta^0\in\Xi^0\backslash\underline{\eta}} u\Big(w_0^*\left(\eta^0;z_0\right), a_0^*\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}^*\left(z_0\right)\right)}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)}\right]$$

$$= \mu\left(z_0\right)\frac{\partial}{\partial z_0}\left[\mathcal{B}\left(z_0\right) - \sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^t\left[\int\sum_{\eta^t\in\Xi^t} u\Big(w_t^*(\eta^t,\varepsilon^t;z_0), a_t^*(\eta^{t-1},\varepsilon^t;z_0)\Big)\tilde{\pi}_t\left(\eta^t,\varepsilon^t|\mathbf{a}^*\left(z_0\right)\right)d\varepsilon^t\right]\right.$$

$$\left. - \sum_{\eta^0\in\Xi^0\backslash\underline{\eta}} u\Big(w_0^*\left(\eta^0;z_0\right), a_0^*\left(z_0\right)\Big)\tilde{\pi}_0\left(\eta^0|\mathbf{a}^*\left(z_0\right)\right)\right]$$

$$= \mu\left(z_0\right)\left[\frac{\partial}{\partial z_0}\mathcal{B}\left(z_0\right)\right], \tag{14}$$

where we have defined

$$\mu\left(z_0\right) \equiv \frac{1}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)} \times u_1^{-1}\left(\frac{\mathcal{B}\left(z_0\right)}{\tilde{\pi}_0\left(\underline{\eta}|\mathbf{a}^*\left(z_0\right)\right)}\right.$$

$$-\frac{\sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^{t}\left[\int\sum_{\eta^{t}\in\Xi^{t}}u\Bigg(w_{t}^{*}(\eta^{t},\varepsilon^{t};z_{0}),a_{t}^{*}(\eta^{t-1},\varepsilon^{t};z_{0})\Bigg)\tilde{\pi}_{t}\left(\eta^{t},\varepsilon^{t}|\mathbf{a}^{*}\left(z_{0}\right)\right)d\varepsilon^{t}\right]}{\tilde{\pi}_{0}\left(\underline{\eta}|\mathbf{a}^{*}\left(z_{0}\right)\right)}$$

$$\left.-\frac{\sum_{\eta^{0}\in\Xi^{0}\setminus\underline{\eta}}u\Bigg(w_{0}^{*}\left(\eta^{0};z_{0}\right),a_{0}^{*}\left(z_{0}\right)\Bigg)\tilde{\pi}_{0}\left(\eta^{0}|\mathbf{a}^{*}\left(z_{0}\right)\right)}{\tilde{\pi}_{0}\left(\underline{\eta}|\mathbf{a}^{*}\left(z_{0}\right)\right)},a_{0}\left(z_{0}\right)\right), \tag{15}$$

and where $u_{1}^{-1}\left(\cdot,\cdot\right)$ denotes the partial derivative of u^{-1} with respect to its first argument. Equation (14) is the "B term".

- Also note that $\mu\left(z_{0}\right)>0$ since on the optimal contract we have

$$\frac{\mathcal{B}\left(z_{0}\right)}{\tilde{\pi}_{0}\left(\underline{\eta}|\mathbf{a}^{*}\left(z_{0}\right)\right)}-\frac{\sum_{t=1}^{\infty}\left(\beta\left(1-s\right)\right)^{t}\left[\int\sum_{\eta^{t}\in\Xi^{t}}u\Bigg(w_{t}^{*}(\eta^{t},\varepsilon^{t};z_{0}),a_{t}^{*}(\eta^{t-1},\varepsilon^{t};z_{0})\Bigg)\tilde{\pi}_{t}\left(\eta^{t},\varepsilon^{t}|\mathbf{a}^{*}\left(z_{0}\right)\right)d\varepsilon^{t}\right]}{\tilde{\pi}_{0}\left(\underline{\eta}|\mathbf{a}^{*}\left(z_{0}\right)\right)}$$

$$-\frac{\sum_{\eta^{0}\in\Xi^{0}\setminus\underline{\eta}}u\Bigg(w_{0}^{*}\left(\eta^{0};z_{0}\right),a_{0}^{*}\left(z_{0}\right)\Bigg)\tilde{\pi}_{0}\left(\eta^{0}|\mathbf{a}^{*}\left(z_{0}\right)\right)}{\tilde{\pi}_{0}\left(\underline{\eta}|\mathbf{a}^{*}\left(z_{0}\right)\right)}=w_{0}^{*}\left(\eta_{1};z_{0}\right)$$

  and $w_{0}^{*}\left(\eta_{1};z_{0}\right)$ is strictly positive on the feasible contract.

- Equations (13) and (14) imply

$$\frac{dJ\left(z_{0}\right)}{dz_{0}}=\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^{t}\mathbb{E}\left[f_{z}(z_{t},\eta_{t})|\mathbf{a}^{*}\left(z_{0}\right)\right]\frac{\partial\mathbb{E}\left[z_{t}|z_{0}\right]}{\partial z_{0}}-\mu\left(z_{0}\right)\frac{\partial}{\partial z_{0}}\mathcal{B}\left(z_{0}\right). \tag{16}$$

  This is the general expression for $\frac{dJ(z_{0})}{dz_{0}}$ that we will use elsewhere.

- We can also convert into a general expression for $d\log\theta_{0}/d\log z_{0}$. The free entry condition (**??**) implies

$$J(z_{0})=\frac{\kappa}{q(\theta_{0})}.$$

  Taking logs and totally differentiating both sides with respect to z_{0} implies

$$\frac{d\log J\left(z_{0}\right)}{d\log z_{0}}=\nu_{0}\frac{d\log\theta_{0}}{d\log z_{0}} \tag{17}$$

$$\implies\frac{d\log\theta_{0}}{d\log z_{0}}=\frac{1}{\nu_{0}}\frac{dJ\left(z_{0}\right)}{dz_{0}}\frac{z_{0}}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^{t}\mathbb{E}\left[f(z_{t},\eta_{t})-w_{t}^{*}|z_{0},\mathbf{a}^{*}\right]}$$

$$\implies\frac{d\log\theta_{0}}{d\log z_{0}}=\frac{1}{\nu_{0}}\frac{z_{0}\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^{t}\mathbb{E}\left[f_{z}(z_{t},\eta_{t})|\mathbf{a}^{*}\left(z_{0}\right)\right]\frac{\partial\mathbb{E}\left[z_{t}|z_{0}\right]}{\partial z_{0}}-\mu\left(z_{0}\right)\frac{\partial}{\partial z_{0}}\mathcal{B}\left(z_{0}\right)}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^{t}\mathbb{E}\left[f(z_{t},\eta_{t})-w_{t}^{*}|z_{0},\mathbf{a}^{*}\right]} \tag{18}$$

  where ν_{0} is the elasticity of job filling with respect to tightness, where we have substituted in the value for $J\left(z_{0}\right)$ on the optimal contract in the first implication, and where we have substituted in equation (16) in the second implication.

6.3 Envelope Theorem Lemma

- This Lemma proves that we can apply an envelope theorem to problem (9), as we have done in equality (10).
- Recall that the problem is

$$J(z_0) = \max_{\mathbf{w},\mathbf{a}} V(\mathbf{w},\mathbf{a};z_0) = \max_{\mathbf{w},\mathbf{a}} \sum_{t=0}^{\infty} (\beta(1-s))^t \int \sum_{\eta^t \in \Xi^t} \left(f(z_t,\eta_t) - w_t(\eta^t,\varepsilon^t;z_0)\right) \tilde{\pi}_t\left(\eta^t,\varepsilon^t | \mathbf{a}(z_0)\right) d\varepsilon^t \quad (19)$$

  where $w_0(\eta_1;z_0)$ satisfies equation (8) and $(\mathbf{w},\mathbf{a}) \in \Phi$, where Φ is a constraint set implicitly defined by the incentive constraints (3) and the bounds on the set of feasible contracts $(\mathbf{w},\mathbf{a}) \in \chi$.
- We wish to show that

$$\frac{dJ(z_0)}{dz_0} = \frac{\partial J(z_0)}{\partial z_0}.$$

- The key observation is that the constraint set Φ does not depend on z_0. Therefore we can apply a theorem from Bonnans & Shapiro (2000).
- In this Lemma, I will first write the theorem from Bonnans & Shapiro and map between our notation and the notation of Bonnans & Shapiro. Then I will show that our problem satisfies the assumptions of their theorem. I will use a blue color to denote the Bonnans & Shapiro notation and the associated text that uses this notation. There is no ambiguity between our notation and theirs, but it will hopefully make the proof easier to follow.
- Bonnans and Shapiro consider in their equation (4.26) the problem

$$\min_{x \in X} h(x,u) \quad \text{subject to } x \in \Phi$$

  where U is a Banach space, X is a Hausdorff topological space, $\Phi \subset X$ is nonempty and closed, and $h : X \times U \to \mathbb{R}$ is continuous. They also define the value function

$$v(u) \equiv \inf_{x \in \Phi(u)} h(x,u)$$

  and the associated set

$$\mathcal{S}(u) \equiv \arg\min_{x \in \Phi(u)} f(x,u).$$

- Let us map between our notation and the notation of Bonnans & Shapiro and in doing so confirm that our problem is described by the problem of XXBonnans & Shapiro.
  - Here, u in the Bonnans & Shapiro notation is z_0 in our notation. Then U is the set of values that z_0 can take on. Since z_0 can take on values on a closed and bounded interval of the real numbers, U (in the Bonnans & Shapiro notation) is a Banach space as required.
  - An element x, in the Bonnans & Shapiro notation, is equivalent to an element $(\mathbf{w},\mathbf{a})$ in our notation. All metric spaces are Hausdorff spaces and clearly the set of feasible contracts belongs to a metric space, therefore X (in the Bonnans & Shapiro notation) is a Hausdorff space as required.
  - In both our notation and in Bonnans & Shapiro, Φ is the constraint set defined by the incentive constraints (3) and bounds on the feasible contract. Since these are non-strict inequality constraints, Φ is closed (i.e. it contains its endpoints). Φ is non-empty by the assumptions of the theorem.

- The function $h(x,u)$ in the Bonnans & Shapiro notation is equivalent to $V(\mathbf{w},\mathbf{a};z_0)$ defined in problem (19), which is continuous by assumption.
- Then $v(u)$, in the Bonnans & Shapiro notation, is the same as $J(z_0)$ in our notation.

- Some useful notation from Bonnans & Shapiro:
  - $D_u h(x,u)$ is the partial Frechet derivative of h with respect to u

- The relevant envelope theorem of Bonnans & Shapiro (Theorem 4.13) states:
  - *Suppose that*
    1. *For all $x \in X$ the function $h(x,\cdot)$ is Gateaux differentiable*
    2. *$h(x,u)$ and $D_u h(x,u)$ are continuous on $X \times U$*
    3. *There exists $\alpha \in \mathbb{R}$ and a compact set $C \subset X$ such that for every u near u_0 the set $\{x \in \Phi : h(x,u) \le \alpha\}$ is non-empty and contained in C.*
  - *Then the optimal value function $v(\cdot)$ is Frechet directionally differentiable at u_0 and*

$$v'(u_0,d) = \inf_{x \in \mathcal{S}(u_0)} D_u h(x,u_0)\, d,$$

    where d is the direction of the Frechet derivative.

- We have verified that problem (9) corresponds to the problem of Bonnans & Shapiro. All that remains is to verify that the conditions of the envelope theorem of Bonnans & Shapiro apply to our setting. Conditions (1) and (2) of the envelope theorem are clearly true.

- Condition (3) is also straightforward to verify.
  - Define $\bar{h}$ as the maximum value that $h(x,u)$ (in the Bonnans & Shapiro notation) can take, which exists because u and x in the Bonnans and Shapiro notation, or z_0 and $(\mathbf{w},\mathbf{a})$ in our notation, are bounded.
  - Now, to verify condition (3) of Bonnans & Shapiro, set $\alpha = \bar{h}$ and also set $C = \Phi$. Clearly for any u, the set $\left\{x \in \Phi : h(x,u) \le \bar{h}\right\} = \Phi$ because all x in the constraint set imply a value of h less than the upper bound. We have assumed that Φ is non-empty in the conditions of the theorem. Φ is also compact, since the bounds on the contract and the IC constraints are non-strict. We have verified condition (3).

- Now, we have verified that we can apply the envelope theorem, Theorem 4.13 of Bonnans & Shapiro, as required for the lemma.

6.4 Proof: General Decomposition of Profits (Proposition 5.1)

- The strategy of the proof is as follows:
  - **Step 1:** we will set up the functional Lagrangian that characterizes the optimal contract (9).
  - **Step 2:** we will verify the constraint qualification of the Lagrangian
  - **Step 3:** we will take the total derivative of the Lagrangian with respect to z_0 in order to arrive at profits

- **Step 1:** Let's start by writing the Kuhn-Tucker Lagrangian.

- To write problem (9) as a Kuhn-Tucker Lagrangian, we follow Theorem 1, Section 9.4 of XXLuenberger (1967) (hereafter the generalized Kuhn-Tucker Lagrangian Theorem) and write

$$J(z_0) = \sum_{t=0}^{\infty} (\beta(1-s))^t \int \sum_{\eta^t \in \Xi^t} \left(f(z_t, \eta_t) - w_t^*(\eta^t, \varepsilon^t; z_0) \right) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}^*(z_0) \right) d\varepsilon^t + \langle \lambda^*, G(\mathbf{w}, \mathbf{a}) \rangle \tag{20}$$

where $w_0(\eta_1; z_0)$ satisfies equation (8), where $G(\mathbf{w}, \mathbf{a}) \leq 0$ is the constraint set defined by the incentive constraints (3) and the bounds on the feasible contract $(\mathbf{w}, \mathbf{a}) \in \chi$. The maximization is with respect to $(\mathbf{w}, \mathbf{a})$. The notation $< q, q' >$ denotes the value of the linear functional q at the point q'.

- One useful piece of notation from Luenberger is that $\delta G(\mathbf{w}, \mathbf{a}; h)$ denotes the Gateaux derivative of $G(\mathbf{w}, \mathbf{a})$ with increment h, i.e.

$$\delta G(\mathbf{w}, \mathbf{a}; h) = \lim_{\alpha \to 0} \frac{1}{\alpha} \left[G((\mathbf{w}, \mathbf{a}) + \alpha h) - G(\mathbf{w}, \mathbf{a}) \right]$$

where h is an element in the feasible contract space $h \in \chi$.

- To define the constraint qualification of the Kuhn-Tucker Lagrangian, define a "regular point" $(\mathbf{w}_0, \mathbf{a}_0)$, of the inequality $G(\mathbf{w}, \mathbf{a}) \leq 0$, if:

  1. $G(\mathbf{w}_0, \mathbf{a}_0) \leq 0$; and
  2. There is a direction h such that $G(\mathbf{w}_0, \mathbf{a}_0) + \delta G(\mathbf{w}_0, \mathbf{a}_0; h) < 0$, where the strict inequality denotes an interior point.

- *The generalized Kuhn-Tucker Lagrangian Theorem of Luenberger (1967) states that if*

  1. $(\mathbf{w}^*, \mathbf{a}^*)$ *maximizes*

  $$\sum_{t=0}^{\infty} (\beta(1-s))^t \int \sum_{\eta^t \in \Xi^t} \left(f(z_t, \eta_t) - w_t^*(\eta^t, \varepsilon^t; z_0) \right) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}^*(z_0) \right) d\varepsilon^t$$

  subject to the constraint set G; and

  2. $(\mathbf{w}^*, \mathbf{a}^*)$ *is a "regular point" of the constrained inequality* $G(\mathbf{w}, \mathbf{a}) \leq 0$

  – *Then there exists a linear functional* $\lambda^* \geq 0$ *such that the Lagrangian (20) is stationary at* $(\mathbf{w}^*, \mathbf{a}^*)$ *and* $\langle \lambda^*, G(\mathbf{w}^*, \mathbf{a}^*) \rangle = 0$.

- **Step 2:** for the Kuhn-Tucker Lagrangian Theorem to apply we simply need to verify that $(\mathbf{w}^*, \mathbf{a}^*)$ must be a "regular point", i.e. there is a feasible perturbation to the contract, starting from $(\mathbf{w}^*, \mathbf{a}^*)$, such that all constraints are slack.

  – Let us rewrite the constraint set $G(\mathbf{w}, \mathbf{a}) \leq 0$ for simplicity. The constraint set is defined by the boundary conditions $w_t(\eta^t, \varepsilon^t; z_0) \in [\underline{w}, \overline{w}]$, $a_t(\eta^{t-1}, \varepsilon^t; z_0) \in [\overline{a}, \overline{a}]$, and the incentive constraints that for all feasible $\tilde{\mathbf{a}}$ that $G(\mathbf{w}, \mathbf{a}; \tilde{\mathbf{a}}) \leq 0$ where we define

$$G(\mathbf{w}, \mathbf{a}; \tilde{\mathbf{a}}) \equiv \sum_{t=0}^{\infty} (\beta(1-s))^t \left[\int \sum_{\eta^t \in \Xi^t} \left\{ u\left(w_t(\eta^t, \varepsilon^t; z_0), \tilde{a}_t(\eta^{t-1}, \varepsilon^t) \right) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \tilde{\mathbf{a}} \right) \right. \right. \tag{21}$$

$$\left. -u\left(w_t(\eta^t, \varepsilon^t; z_0), a_t(\eta^{t-1}, \varepsilon^t; z_0) \right) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}(z_0) \right) \right\} d\varepsilon^t.$$

– Therefore $G(\mathbf{w}, \mathbf{a}; \tilde{\mathbf{a}})$ is the incentive constraint associated with a given feasible effort choice $\tilde{\mathbf{a}}$. The incentive constraint associated with an individual effect vector $\tilde{\mathbf{a}}$ implies the following. Consider the expected present value of the worker, given the wage contract $\mathbf{w}$ offered by the firm, and the effort choice $\tilde{\mathbf{a}}$ which is implemented across all dates and states. This expected present value must be less than what the worker receives from implementing the recommended effort $\mathbf{a}^*$ across dates and states, given the firm's wage contract.

– For any $\tilde{\mathbf{a}}$, the partial derivative of $G(\mathbf{w}, \mathbf{a}; \tilde{\mathbf{a}})$ with respect to a given effort choice $a_\tau(\eta^{\tau-1}, \varepsilon^\tau; z_0)$ is

$$\frac{\partial G(\mathbf{w}, \mathbf{a}; \tilde{\mathbf{a}})}{\partial a_\tau(\eta^{\tau-1}, \varepsilon^\tau; z_0)}$$

$$= -\frac{\partial}{\partial a_\tau(\eta^{\tau-1}, \varepsilon^\tau; z_0)} \left[\sum_{t=0}^{\infty} (\beta(1-s))^t \left[\int \sum_{\eta^t \in \Xi^t} u\Big(w_t(\eta^t, \varepsilon^t; z_0), a_t(\eta^{t-1}, \varepsilon^t; z_0) \Big) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}(z_0)\right) d\varepsilon^t \right] \right.$$

$$= -\left[\sum_{t=0}^{\infty} (\beta(1-s))^t \left[\int \sum_{\eta^t \in \Xi^t} \frac{\partial}{\partial a_\tau(\eta^{\tau-1}, \varepsilon^\tau; z_0)} \left[u\Big(w_t(\eta^t, \varepsilon^t; z_0), a_t(\eta^{t-1}, \varepsilon^t; z_0) \Big) \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}(z_0)\right) \right] d\varepsilon^t \right] \right.$$

$$= -(\beta(1-s))^\tau u_a \Big(w_\tau(\eta^\tau, \varepsilon^\tau; z_0), a_\tau(\eta^{\tau-1}, \varepsilon^\tau; z_0) \Big) \tilde{\pi}_\tau \left(\eta^\tau, \varepsilon^\tau | \mathbf{a}(z_0)\right)$$

$$- \sum_{t=0}^{\infty} (\beta(1-s))^t \left[\int \sum_{\eta^t \in \Xi^t} \left[u\Big(w_t(\eta^t, \varepsilon^t; z_0), a_t(\eta^{t-1}, \varepsilon^t; z_0) \Big) \frac{\partial}{\partial a_\tau(\eta^{\tau-1}, \varepsilon^\tau; z_0)} \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}(z_0)\right) \right] d\varepsilon^t. \right. \quad (22)$$

– Now, consider the set of "active" incentive constraints that are binding at the optimum, i.e. the a set of effort deviations $\tilde{\mathbf{a}} \in \mathcal{A}$ such that for $\tilde{\mathbf{a}} \in \mathcal{A}$ we have $G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}}) = 0$. (The inactive incentive constraints at the optimum are $\tilde{\mathbf{a}} \notin \mathcal{A}$, for which we have $G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}}) < 0$).

– By the assumptions of the theorem, there exists at least one interior effort choice, i.e. some $\eta^{\sigma-1}, \varepsilon^\sigma$ such that $a^*_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0) \in (0, \overline{a})$.

– Equation (22) shows two useful properties of $\frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)}$.

1. $\frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)}$ is generically not equal to zero, i.e. the slope of the incentive constraint corresponding to $\tilde{\mathbf{a}}$, with respect to the single effort perturbation, at the optimal contract, is not equal to zero, except for knife edge values of the probability distribution of idiosyncratic shocks $\frac{\partial}{\partial a_\tau(\eta^{\tau-1}, \varepsilon^\tau; z_0)} \tilde{\pi}_t \left(\eta^t, \varepsilon^t | \mathbf{a}^*(z_0)\right)$.
Therefore we will assume that $\frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)} \neq 0$, which is generically true.

2. $\frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)}$ is independent of $\tilde{\mathbf{a}}$, i.e. the slope of the incentive constraint corresponding to $\tilde{\mathbf{a}}$, with respect to the single effort perturbation, does not depend on $\tilde{\mathbf{a}}$.

– Therefore there exists a scalar Δ_1 such that for all active incentive constraints associated with $\tilde{\mathbf{a}} \in \mathcal{A}$, we have

$$G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}}) + \frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)} \Delta_1 < 0,$$

this follows because for the active constraints $G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}}) = 0$ on the optimal contract; and because $\frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)} \neq 0$, and indeed is the same for all values of $\tilde{\mathbf{a}}$. Therefore there exists scalar perturbation Δ_1 to $a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)$, which slackens all active incentive constraints (though Δ_1 may be positive or negative.)

– By the same logic, all inactive constraint sets at the optimum will continue to be inactive after the perturbation Δ_1. Inactive constraints with $\tilde{\mathbf{a}} \notin \mathcal{A}$ have $G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}}) < 0$ on the optimal contract, which implies that for the perturbation Δ_1 we must have

$$G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}}) + \frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)} \Delta_1 < 0,$$

where again we exploit that $\frac{\partial G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})}{\partial a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)}$ is the same for all $\tilde{\mathbf{a}}$.

– We can easily show that the scalar perturbation Δ_1 to $a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)$, is feasible, in the sense that the values of $\mathbf{w}$ and $\mathbf{a}$ after perturbing the constract, remain in the feasible set.

– Moreover, since $a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)$ is interior by assumption, a small perturbation Δ_1 to $a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)$ will remain interior and hence is feasible.

– Therefore there is a feasible scalar perturbation Δ_1 to $a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)$, starting from the optimal contract $(\mathbf{w}^*, \mathbf{a}^*)$, such that all incentive constraints are slack, i.e. $G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}}) < 0$ for all $\tilde{\mathbf{a}}$.

– To finish verifying the constraint qualification, we need to find a perturbation that means all wage and effort choices are interior. Now, consider all effort and wage choices that are at their boundaries, i.e. all of the elements of $(\mathbf{w}^*, \mathbf{a}^*)$ that satisfy either $w_t^*(\eta^t, \varepsilon^t; z_0) = 0$, or $w_t^*(\eta^t, \varepsilon^t; z_0) = \overline{w}$, or $a_t^*(\eta^{t-1}, \varepsilon^t) = 0$ or $a_t^*(\eta^{t-1}, \varepsilon^t) = \overline{a}$. Consider a vector perturbation $\boldsymbol{\Delta_2}$ which brings all of these elements inside the boundary.

– Finally, consider a perturbation $\boldsymbol{\Delta_1} + \boldsymbol{\Delta_2}$, starting from $(\mathbf{w}^*, \mathbf{a}^*)$, where $\boldsymbol{\Delta_1}$ is the vector that perturbs only $a_\sigma(\eta^{\sigma-1}, \varepsilon^\sigma; z_0)$, but not any other elements of $(\mathbf{w}^*, \mathbf{a}^*)$. If $\boldsymbol{\Delta_2}$ is sufficiently small relative to $\boldsymbol{\Delta_1}$, this perturbation ensures that all incentive constraints $G(\mathbf{w}^*, \mathbf{a}^*; \tilde{\mathbf{a}})$ are slack and also ensures that all elements of $(\mathbf{w}, \mathbf{a})$ are interior.

– Therefore we have found a perturbation from the optimum $(\mathbf{w}^*, \mathbf{a}^*)$ such that all constraints are slack. Hence $(\mathbf{w}^*, \mathbf{a}^*)$ is a regular point and the constraint qualification is verified.

- **Step 3:** therefore we have verified that we can characterize the constrained optimum by the Kuhn-Tucker Lagrangian (20). Now, we can take the total derivative of the Lagrangian to arrive at the general decomposition for profits.

– We can write the Lagrangian (20) as

$$\begin{aligned} J(z_0) &= J^*(z_0, \mathbf{w}^*(z_0), \mathbf{a}^*(z_0)) \\ &= \sum_{t=0}^{\infty} (\beta(1-s))^t \int \sum_{\eta^t \in \Xi^t} \left(f(z_t, \eta_t) - w_t^*(\eta^t, \varepsilon^t; z_0)\right) \tilde{\pi}_t\left(\eta^t, \varepsilon^t | \mathbf{a}^*(z_0)\right) d\varepsilon^t + \langle \lambda^*, G(\mathbf{w}, \mathbf{a}) \rangle \\ &= V(\mathbf{w}^*, \mathbf{a}^*; z_0) + \langle \lambda^*, G(\mathbf{w}, \mathbf{a}) \rangle . \end{aligned}$$

– Taking total derivatives with respect to z_0 implies

$$\begin{aligned} \frac{dJ(z_0)}{dz_0} &= \frac{\partial}{\partial z_0} V(\mathbf{w}^*, \mathbf{a}^*; z_0) - \left\langle \frac{\partial}{\partial z_0} G(\mathbf{w}^*, \mathbf{a}^*), \lambda^*(z_0) \right\rangle \\ &+ \sum_{x \in \{\mathbf{w}^*, \mathbf{a}^*\}} \left[\partial_x V(\mathbf{w}^*, \mathbf{a}^*; z_0) - \langle \partial_x G(\mathbf{w}^*, \mathbf{a}^*), \lambda^*(z_0) \rangle\right] \cdot \frac{dx}{dz_0} - \left\langle G(\mathbf{w}^*, \mathbf{a}^*), \frac{d\lambda^*(z_0)}{dz_0} \right\rangle \end{aligned}$$

which is the result.

6.5 Proof: Incentive Wage Cyclicality Does Not Mute Unemployment Fluctuations (Proposition 5.2)

- Given the general expression for $\frac{d\log\theta_0}{d\log z_0}$ from equation (18), the proof is straightforward to finish.
- Simply note that with take it or leave it wage offers and acyclical unemployment benefits, $\frac{\partial}{\partial z_0}\mathcal{B}(z_0) =$ so that

$$\begin{aligned}\frac{d\log\theta_0}{d\log z_0} &= \frac{1}{\nu_0}\frac{z_0\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f_z(z_t,\eta_t)|\mathbf{a}^*(z_0)\right]\frac{\partial\mathbb{E}[z_t|z_0]}{\partial z_0}}{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(z_t,\eta_t)-w_t^*|z_0,\mathbf{a}^*\right]}\\ &= \frac{1}{\nu_0}\frac{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f_z(z_t,\eta_t)|\mathbf{a}^*(z_0)\right]\frac{\partial\mathbb{E}[z_t|z_0]}{\partial\log z_0}}{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(z_t,\eta_t)-w_t^*|z_0,\mathbf{a}^*\right]}\end{aligned}$$

  as required in equation (4) of the proposition.
- Tightness dynamics in the rigid wage economy, with $w_t=\bar{w}$ and $a_t=\bar{a}$ are trivially given by rearranging the free entry condition (**??**) to arrive at

$$J^{\text{rigid}}(z_0) = \frac{\kappa}{q(\theta_0)}$$

$$\frac{d\log J^{\text{rigid}}(z_0)}{d\log z_0} = \nu_0\frac{d\log\theta_0}{d\log z_0}$$

$$\implies \frac{d\log\theta_0}{d\log z_0} = \frac{1}{\nu_0}\frac{dJ^{\text{rigid}}(z_0)}{dz_0}\frac{z_0}{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(z_t,\eta_t)-\bar{w}|z_0,\bar{\mathbf{a}}\right]}$$

$$\implies \frac{d\log\theta_0}{d\log z_0} = \frac{1}{\nu_0}\frac{z_0\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f_z(z_t,\eta_t)|\bar{\mathbf{a}}(z_0)\right]\frac{\partial\mathbb{E}[z_t|z_0]}{\partial z_0}}{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(z_t,\eta_t)-\bar{w}|z_0,\bar{\mathbf{a}}\right]}$$

  as required.
- To arrive at equation (6) observe that in a neighborhood of a steady state $\bar{z}$, and if z_t is a random walk, then approximately

$$\begin{aligned}\frac{d\log\theta_0}{d\log z_0} &= \frac{1}{\bar{\nu}}\frac{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[\bar{z}f_z(\bar{z},\eta_t)|\bar{z},\mathbf{a}^*(\bar{z})\right]\frac{\partial\mathbb{E}[z_t|z_0]}{\partial z_0}}{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(\bar{z},\eta_t)-w_t^*(\bar{z})|\bar{z},\mathbf{a}^*(\bar{z})\right]}\\ &= \frac{1}{\bar{\nu}}\frac{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(\bar{z},\eta_t)|\bar{z},\mathbf{a}^*(\bar{z})\right]}{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(\bar{z},\eta_t)-w_t^*(\bar{z})|\bar{z},\mathbf{a}^*(\bar{z})\right]}\\ &= \frac{1}{\bar{\nu}}\frac{1}{1-\Lambda}\qquad \Lambda = \frac{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[w_t^*(\bar{z})|\bar{z},\mathbf{a}^*(\bar{z})\right]}{\sum_{t=0}^{\infty}(\beta(1-s))^t\,\mathbb{E}\left[f(\bar{z},\eta_t)|\bar{z},\mathbf{a}^*(\bar{z})\right]}.\end{aligned}$$

6.6 Proof: Bargained Wage Cyclicality Does Mute Unemployment Fluctuations (Proposition 5.3)

- We can write the optimized value of a vacancy as

$$\begin{aligned} J(z_0) &= V(\mathbf{w}^*(z_0), \mathbf{a}^*(z_0), z_0) \\ &= \sum_{t=0}^{\infty} (\beta(1-s))^t \, \mathbb{E}\left[f(z_t, \eta_t) - w_t^*(z_0) | z_0, \mathbf{a}^*(z_0)\right] \\ &= \sum_{t=0}^{\infty} (\beta(1-s))^t \, \mathbb{E}\left[f(z_t, \eta_t) | z_0, \mathbf{a}^*(z_0)\right] - \sum_{t=0}^{\infty} (\beta(1-s))^t \, \mathbb{E}\left[w_t^*(z_0) | z_0, \mathbf{a}^*(z_0)\right] \\ &= \mathcal{Y}(\mathbf{a}^*(z_0); z_0) - \mathcal{W}(z_0), \end{aligned}$$

  which implies

$$\begin{aligned} \frac{dJ(z_0)}{dz_0} &= \partial_{\mathbf{a}} \mathcal{Y}(\mathbf{a}^*(z_0); z_0) \frac{d\mathbf{a}^*}{dz_0} + \frac{\partial \mathcal{Y}(\mathbf{a}^*(z_0); z_0)}{\partial z_0} - \frac{d\mathcal{W}(z_0)}{dz_0} \\ &= \sum_{t=0}^{\infty} (\beta(1-s))^t \, \mathbb{E}\left[f_z(z_t, \eta_t) | z_0, \mathbf{a}^*(z_0)\right] \frac{\partial \mathbb{E}[z_t | z_0]}{\partial z_0} + \partial_{\mathbf{a}} \mathcal{Y}(\mathbf{a}^*(z_0); z_0) \frac{d\mathbf{a}^*}{dz_0} - \frac{d\mathcal{W}(z_0)}{dz_0}, \end{aligned} \tag{23}$$

  where the second equality solves for the "A term" using equation (13).

- Using the definition of incentive wage cyclicality as

$$\frac{d\mathcal{W}^{\text{incentive}}(z_0)}{dz_0} = \partial_{\mathbf{a}} \mathcal{Y}(\mathbf{a}^*(z_0); z_0) \frac{d\mathbf{a}^*}{dz_0}$$

  and the definition of bargained wage cyclicality as

$$\frac{d\mathcal{W}^{\text{incentive}}(z_0)}{dz_0} + \frac{d\mathcal{W}^{\text{bargained}}(z_0)}{dz_0} = \frac{d\mathcal{W}(z_0)}{dz_0}$$

  we have from equation (13) that

$$\frac{dJ(z_0)}{dz_0} = \sum_{t=0}^{\infty} (\beta(1-s))^t \, \mathbb{E}\left[f_z(z_t, \eta_t) | z_0, \mathbf{a}^*(z_0)\right] \frac{\partial \mathbb{E}[z_t | z_0]}{\partial z_0} - \frac{d\mathcal{W}^{\text{bargained}}(z_0)}{dz_0} \tag{24}$$

- Finally, comparing equations (16) and (24), we have

$$\frac{d\mathcal{W}^{\text{bargained}}(z_0)}{dz_0} = \mu(z_0) \mathcal{B}'(z_0).$$

  Since $\mu(z_0) > 0$ we have

$$\frac{d\mathcal{W}^{\text{bargained}}(z_0)}{dz_0} > 0 \quad \iff \quad \mathcal{B}'(z_0) > 0.$$

- In a neighborhood of a steady state $\bar{z}$, and if z_t is a random walk, then

$$\frac{d\log J\left(z_0\right)}{d\log z_0} \approx \frac{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \mathbb{E}\left[\bar{z}f_z(\bar{z},\eta_t)|\bar{z},\mathbf{a}^*\left(\bar{z}\right)\right]\frac{\partial\mathbb{E}[z_t|z_0]}{\partial z_0} - \left.\frac{d\mathcal{W}^{\text{bargained}}(z_0)}{dz_0}\right|_{z'=\bar{z}}}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \mathbb{E}\left[f(\bar{z},\eta_t) - w_t^*(\bar{z})|\bar{z},\mathbf{a}^*\left(\bar{z}\right)\right]}$$
$$= \frac{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \mathbb{E}\left[f(\bar{z},\eta_t)|\bar{z},\mathbf{a}^*\left(\bar{z}\right)\right] - \left.\frac{d\mathcal{W}^{\text{bargained}}(z_0)}{dz_0}\right|_{z'=\bar{z}}}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \mathbb{E}\left[f(\bar{z},\eta_t) - w_t^*(\bar{z})|\bar{z},\mathbf{a}^*\left(\bar{z}\right)\right]}$$
$$= \frac{1-\frac{\left.\frac{d\mathcal{W}^{\text{bargained}}(z_0)}{d\log z_0}\right|_{z'=\bar{z}}}{\sum_{t=0}^{\infty}(\beta(1-s))^t\mathbb{E}[f(\bar{z},\eta_t)|\bar{z},\mathbf{a}^*(\bar{z})]}}{1-\Lambda}$$
$$= \frac{1-\xi}{1-\Lambda} \quad \xi \equiv \frac{\left.\frac{d\mathcal{W}^{\text{bargained}}(z_0)}{d\log z_0}\right|_{z'=\bar{z}}}{\sum_{t=0}^{\infty}\left(\beta\left(1-s\right)\right)^t \mathbb{E}\left[f(\bar{z},\eta_t)|\bar{z},\mathbf{a}^*\left(\bar{z}\right)\right]}$$

  where ξ is the "bargained wage cyclicality share".

- Setting

$$\frac{d\log\theta_0}{d\log z_0} = \frac{1}{\bar{\nu}}\frac{d\log J\left(z_0\right)}{d\log z_0}$$

  (from equation 17) completes the proof.