Simple Model

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1 Static Search Model

1.1 Overview and Workers

- Static model
- Key variable is labor productivity z
- There are two types of agents: firms and workers
- There is a large measure of risk neutral firms that post vacancies in order to match with workers.
- There is a unit mass of workers, searching for a job.
- The firms and workers that successfully match will produce output. The remaining vacancies posted by firms will go unfilled. The workers who do not find employment will remain unemployed.

1.2 Consumption and Savings of Workers

- Workers have risk averse preferences over consumption $c \geq 0$ and effort $a \geq 0$.
- Unemployed workers consume unemployment benefits b. Employed workers consume their wage w. There is a total mass of unemployed workers u and employed workers $n = 1 - u$.
- Workers have exponential preferences

$$u(c, a) = -e^{-r\left(c - \frac{\phi a^2}{2}\right)},$$

  r parameterizes the degree of absolute risk aversion, ϕ parameterizes the disutility from effort, unemployed workers exert zero effort.

1.3 Frictional Labor Market

- The labor market is a static version of DMP
- All workers search for jobs.
- Firms post v_t vacancies to recruit the workers.
- The key variable summarizing labor market tightness is v_t, the number of vacancies.
- The job finding rate of a worker is $f(v) = \Psi v^{1-\alpha}, \alpha \in (0, 1,)$ the probability that a vacancy is filled is $q(v) = \Psi v^{-\alpha}$. So the chance that a worker finds a job is increasing in the number of vacancies, whereas the chance that an individual vacancy is filled is decreasing in the total number of vacancies.

- **Analogy to dynamic DMP:**
  - Tightness in this economy is $\theta_t = v_t/1$ since all workers are initially unemployed and searching for work
  - The matching function is $h(1, v_t)$, again all workers are initially searching.
- Keeping vacancy open has period cost κ, a firm must post $1/q(v)$ vacancies to hire one worker.

1.4 Firms

- Consider a firm that successfully matches with a worker. The firm produces output.
- The firm's output y_i is

  $$y_i = za_i + \eta_i,$$

  where z is aggregate labor productivity, a_i is the worker's effort, η_i is an idiosyncratic shock to output, normally distributed with mean zero and variance σ^2.
- The firm observes aggregate labor productivity z and output y_i, but not effort or the idiosyncratic shock.
- The value to the firm of a vacancy is then

  $$\begin{aligned} \pi_i &= E\left[y_i - w_i - \frac{\kappa}{q(v)}\right] \\ &= E\left[za_i + \eta_i - w_i - \frac{\kappa}{q(v)}\right], \end{aligned} \tag{1}$$

  i.e. the value of a vacancy is output, minus the wage paid to the worker w_i, and also deducing the cost of posting the vacancy.

1.5 Timing

- The timing of the static model is important and goes as follows. The firm and worker match. They observe aggregate TFP z. They agree on a wage contract as a function of output y_i. Then the idiosyncratic shock η_i, and hence output y_i, are realized.
- Therefore the firm's expectation operator will always reflect uncertainty about the value of the idiosyncratic shock and only this uncertainty (everything else is deterministic in the static model)

1.6 Wages

- We entertain two possibilities for wages.

Rigid Wages

- This process is a static version of Hall (2005). Wages take a constant value $w_i = \hat{w}$.
- The worker's effort $a_i = \hat{a}$ is exogenous.

Bonus Wages

- This process for wages is a static version of Holmstrom & Milgrom (1987), similar to Doligalski, Ndiaye & Werquin (2020). I.e. we assume the firm sets an optimal contract for the worker given moral hazard **within the class of linear contracts.**
  - **Reminder:** there are nonlinear contracts that can provide better incentives, but the intuition here will be nice.
- Specifically the firm offers the worker a linear contract

$$w_i = \alpha + \beta y_i.$$

- The contract maximizes the value of a filled vacancy

$$\begin{aligned} V(z) &= \max_{a,\alpha,\beta} E\left[y_i - w_i\right] \\ &= \max_{a,\alpha,\beta} E\left[z a_i + \varepsilon_i - \left(\alpha + \beta\left(z a_i + \varepsilon_i\right)\right)\right] \\ &= \max_{a,\alpha,\beta} E\left[\left(1-\beta\right)\left(z a_i + \varepsilon_i\right) - \alpha\right] \end{aligned} \tag{2}$$

  subject to the worker's incentive compatibility and participation constraints.

- The incentive compatibility constraint is that the worker's effort choice is optimal, i.e.

$$\begin{aligned} a &\in \arg\max_{\tilde{a}_i} E\left[-e^{-r\left(w_i - \frac{\phi \tilde{a}_i^2}{2}\right)}\right] \\ &= \arg\max_{\tilde{a}_i} E\left[-e^{-r\left(\alpha + \beta x_i - \frac{\phi \tilde{a}_i^2}{2}\right)}\right] \\ &= \arg\max_{\tilde{a}_i} E\left[-e^{-r\left(\alpha + \beta(z a_i + \eta_i) - \frac{\phi \tilde{a}_i^2}{2}\right)}\right] \end{aligned} \tag{3}$$

- The participation constraint is that the worker is better off in employment than unemployment, i.e.

$$E\left[-e^{-r\left(\alpha + \beta(z a_i + \eta_i) - \frac{\phi a_i^2}{2}\right)}\right] \geq -e^{-rb} \tag{4}$$

  where the worker's value from unemployment benefits accounts for exerting zero effort and having no risk.

1.7 Equilibrium

- Workers and firms optimize
- There is **free entry in vacancy creation**. So $\pi_i = 0$, where $\pi_i$is defined in equation (1).
- Intuitively, when profits from production are high, firms create many vacancies. Then vacancies are hard to fill, so that even though profits ex post of the vacancy being filled are high, profits ex ante of the vacancy being filled are zero.

2 Model Results

2.1 Overview of Results

This section shows 3 results:

1. First, the main result of the paper. When calibrated to the same sufficient statistic—the ratio of wages to output—the bonus wage economy and the rigid wage economy have the same first order response of unemployment to labor demand shocks.

2. Second, the intuition for this result. Higher productivity lowers unemployment in this economy by raising profits from job creation. Firms create lots of vacancies to realize the profits. Then the vacancies lead to more employed workers. Intuitively, higher productivity raises profits through three effects in the bonus wage economy (I am not attached to the labels).

   - The **direct productivity effect.** When TFP is higher, then the marginal product of labor is higher.

   Then higher TFP causes the firm to pay higher wages which as two effects.

   - The **incentive effect**—workers' incentives are better, which leads them to produce more.
   - The **marginal cost effect**—higher wages reduce profits.

   I will show that on the optimal contract, the incentive effect and the marginal cost effect have identical and offsetting effects on profits. They cancel out (the envelope theorem) and so only the direct productivity effect matters. This effect is the only effect present in the rigid wage economy.

3. Third, I show the equivalence result holds even if wages are arbitrarily flexible, i.e. even if the pass through of output into wages is arbitrarily close to 1. This step is important because it shows that *even if wages are very flexible* the economy behaves *as if* wages are completely rigid.

- On the way, I solve for the optimal contract.

- Finally, note that the equivalence result holds in a completely static economy. Wages are paid once, the timing of wages is determinant. Thus the results are fully distinct from the issue that John was raising before about the timing of wage payments and so on (I think?).

2.2 Optimal Contract

- In the Appendix, I show that the optimal contract is

$$\beta = \frac{z^2}{z^2 + \phi r \sigma^2}$$

$$\alpha = b + \frac{\left(\frac{z^2}{z^2+\phi r\sigma^2}\right)^2 \left(\phi r \sigma^2 - z^2\right)}{2\phi}$$

  with optimal effort

$$a_i = \frac{\beta z}{\phi}$$

  and an optimal contract

$$w_i = \alpha + \beta y_i. \tag{5}$$

- Note that β is the "pass through" of output into wages. There is a pass through coefficient of 1 if $\phi = 0$ (effort is not costly to the agent), $\sigma^2 = 0$ (action is perfectly observable so no moral hazard) or $r = 0$, (no risk aversion).

- Also note that this contract nests the contract from Abdou's paper, it is the same when $z = 1$.

2.3 Result: Response of Employment to Labor Demand Shocks

- The key object of interest is the elasticity of employment with respect to aggregate labor productivity, $d\log n / d\log z$. This is the measure of the response of employment to labor demand shocks. I will show that this object is the same in the bonus wage and rigid wage economy, as long as they are calibrated to the same sufficient statistic (the labor share).

- I will derive the results separately in each economy

- In both cases, we will use the following relationship

$$\begin{aligned} \pi_i &= 0 \\ \implies E\left[za_i + \eta_i - w_i - \frac{\kappa}{q(v)}\right] &= 0 \\ \implies E\left[za_i - w_i\right] &= \frac{\kappa}{\Psi v^{-\alpha}} \end{aligned}$$

where the first line is the free entry condition in vacancy creation, and the second line substitutes in the value of a vacancy (1).

- Next, note that employment is

$$\begin{aligned} n &= f \\ &= \Psi v^{1-\alpha} \\ \implies \left(\frac{n}{\Psi}\right)^{\frac{1}{1-\alpha}} &= v. \end{aligned}$$

The first line uses the fact that employment is the probability of job finding times the number of job seekers (equal to 1), the second line substitutes in the definition of the job finding rate.

- Substituting the previous two equations and log differentiating leads to

$$\begin{aligned} E\left[za_i - w_i\right] &= \frac{\kappa}{\Psi\left(\left(\frac{n}{\Psi}\right)^{\frac{1}{1-\alpha}}\right)^{-\alpha}} \\ &= \frac{\kappa}{\Psi\left(\frac{n}{\Psi}\right)^{\frac{-\alpha}{1-\alpha}}} \\ \implies \frac{d\log\left(E\left[za_i - w_i\right]\right)}{d\log z} &= \frac{\alpha}{1-\alpha}\frac{d\log n}{d\log z} \\ \implies \frac{d\log n}{d\log z} &= \frac{1-\alpha}{\alpha}\frac{d\log\left(E\left[za_i - w_i\right]\right)}{d\log z}. \end{aligned}$$

- This expression is helpful! It shows that the elasticity of employment with respect to the labor demand shock is the elasticity of profits $E\left[za_i - w_i\right]$ scaled by a term for search frictions $(1-\alpha)/\alpha$. Intuitively, in both models, an increase in labor demand raises profits, which causes firms to create more vacancies in order to realize the profits—as a result, employment rises.

- It remains to solve out for the endogenous object

$$\frac{d\log\left(E\left[za_i - w_i\right]\right)}{d\log z} = \frac{dE\left[za_i - w_i\right]}{dz}\frac{z}{E\left[za_i - w_i\right]}$$

in the bonus and rigid wage economy.

2.3.1 Bonus Wage Economy

- In the bonus wage economy, we have

$$\begin{aligned}\frac{d\max_{\alpha,\beta} E\left[za_i\left(\alpha,\beta\right)-w_i\left(\alpha,\beta\right)\right]}{dz} &= \frac{\partial E\left[za_i-w_i\right]}{\partial z}\\ &= a_i.\end{aligned}$$

The first expression makes explicit that the firm is optimally choosing the contract given values of α and β. The crucial second step is to use partial instead of total derivatives by the envelope theorem.

- Immediately we have

$$\begin{aligned}\frac{d\log\left(E\left[za_i-w_i\right]\right)}{d\log z} &= \frac{dE\left[za_i-w_i\right]}{dz}\frac{z}{E\left[za_i-w_i\right]}\\ &= a_i\frac{z}{E\left[za_i-w_i\right]}\\ &= \frac{E\left[za_i\right]}{E\left[za_i-w_i\right]}\\ &= \frac{E\left[y_i\right]}{E\left[y_i-w_i\right]}\\ &= \frac{E\left[y_i\right]}{E\left[y_i\right]-E\left[w_i\right]}\\ &= \frac{1}{1-\frac{E[w_i]}{E[y_i]}},\end{aligned}$$

which means we can characterize the elasticity of employment with respect to tightness as

$$\frac{d\log n}{d\log z} = \frac{1-\alpha}{\alpha}\frac{1}{1-\frac{E[w_i]}{E[y_i]}}. \tag{6}$$

2.3.2 Rigid Wage Economy

- In the rigid wage economy we have with exogenous effort and wage $\hat{a}$ and $\hat{w}$

$$\frac{dE\left[z\hat{a}-\hat{w}\right]}{dz} = \hat{a}.$$

- Immediately we have

$$\begin{aligned}\frac{d\log\left(E\left[za_i-w_i\right]\right)}{d\log z} &= \frac{dE\left[za_i-w_i\right]}{dz}\frac{z}{E\left[za_i-w_i\right]}\\ &= \hat{a}\frac{z}{E\left[za_i-w_i\right]}\\ &= \frac{E\left[\hat{a}z\right]}{E\left[z\hat{a}-\hat{w}\right]}\\ &= \frac{1}{1-\frac{E[w_i]}{E[y_i]}}\end{aligned}$$

which means

$$\frac{d\log n}{d\log z} = \frac{1-\alpha}{\alpha}\frac{1}{1-\frac{E[w_i]}{E[y_i]}} \tag{7}$$

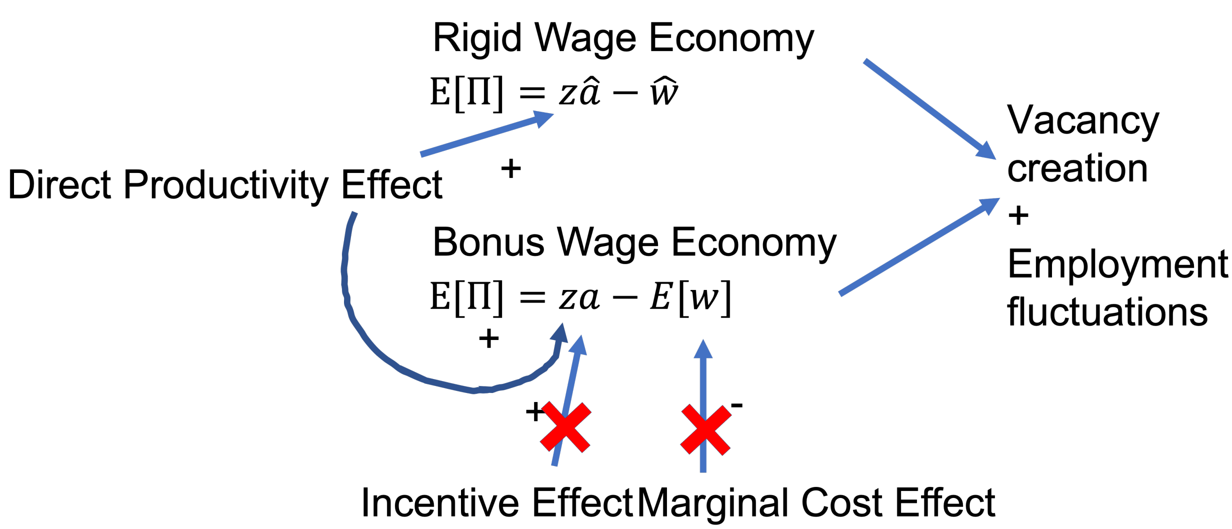
in the rigid wage economy.

2.3.3 Statement of the Main Result

- Equations (6) and (7) show the main result. If one calibrates the bonus and the rigid wage economy to the same statistics on search frictions α and average labor shares $E[w_i]/E[y_i]$, the response of employment to labor demand shocks is the same in both economies.
- Later on I believe we will calibrate $E[w_i]/E[y_i]$, perhaps using a fully dynamic model.

2.4 Intuition for the Main Result

- This result relies heavily on the envelope theorem. I am now going to explain this point further with equations and diagrams.
- In both economies, an increase in TFP raises profits. Then higher profits lead to more job creation. In both economies, there is a common effect:
  - The **direct productivity effect.** When TFP is higher, then the marginal product of labor is higher. So, profits are higher. Firms create more vacancies and employment increases.
- The rigid wage economy has no other effects at work.
- The bonus wage economy has two more effects at work after a TFP shock that raises wages:
  - The **incentive effect**—workers' incentives are better, which leads them to produce more and raises profits
  - The **marginal cost effect**—higher wages increase marginal costs and lower profits.
- In the bonus wage economy, the incentive effect and the marginal cost effect cancel out exactly. So, only the direct productivity effect remains. Therefore the response of employment is the same in both economies.
- Why is this interesting? Most people have lazily hypothesized that flexible bonus wages will make employment less response to labor demand shocks. We are showing this logic is false—employment responds very strongly to labor demand shocks in the bonus wage economy, just as strongly as in an economy with fully rigid wages.
- Here is the point in a simple diagram. Hopefully this is something we can use:



- I will now make this point more formally. We are interested in the response of profits in the bonus wage economy to labor productivity shocks. That is, we are interested in

$$\frac{d}{dz}\pi_i^* = \frac{d}{dz}E\left[za_i^* - w_i^*\right]$$

where π_i^*, a_i^* and w_i^* are optimal effort, wages and profits on the optimal contract.

- By the chain rule we have

$$\frac{d}{dz}\pi_i^* = \overbrace{\frac{\partial}{\partial z}E\left[za_i^* - w_i^*\right]}^{\text{Direct productivity effect}} + \overbrace{\frac{\partial}{\partial a_i}E\left[za_i^* - w_i^*\right]\frac{da_i^*}{dz}}^{\text{Incentive Effect}} + \overbrace{\frac{\partial}{\partial w_i}E\left[za_i^* - w_i^*\right]\frac{dw_i^*}{dz}}^{\text{Marginal Cost Effect}}.$$

- The first effect is also present in the rigid wage economy. The second effect is absent (since effort is fixed) and the third effect is also absent (since wages are constant).
- By the envelope theorem, the incentive effect and the marginal cost effect must sum to zero.
- In particular, we can show that the incentive effect is

$$\begin{aligned}\frac{\partial}{\partial a_i}E\left[za_i^* - w_i^*\right]\frac{da_i^*}{dz} &= z\frac{da_i^*}{dz}\\ &= z\frac{d}{dz}\left[\frac{z^2}{z^2+\phi r\sigma^2}\frac{z}{\phi}\right]\\ &= \frac{1}{\phi}\frac{z^2\left(z^2+3\phi r\sigma^2\right)}{\left(z^2+\phi r\sigma^2\right)^2} > 0\end{aligned}$$

where the second line uses tedious calculations from the appendix and the third line also solves the tedious calculations.

- What is driving the incentive effect? Intuitively, effort and labor productivity are complements. So the firm wants to pay higher wages when TFP is high. The higher wage raises the marginal value of effort, causing the worker to worker more—therefore aligning the worker's incentives better with the firm.
- The marginal cost effect is the negative of the incentive effect. So the effect of TFP on profits is simply

$$\begin{aligned}\frac{d}{dz}\pi_i^* &= \overbrace{\frac{\partial}{\partial z}E\left[za_i^* - w_i^*\right]}^{\text{Direct productivity effect}} + \overbrace{\frac{\partial}{\partial a_i}E\left[za_i^* - w_i^*\right]\frac{da_i^*}{dz}}^{\text{Incentive Effect}} + \overbrace{\frac{\partial}{\partial w_i}E\left[za_i^* - w_i^*\right]\frac{dw_i^*}{dz}}^{\text{Marginal Cost Effect}}\\ &= \overbrace{a_i^*}^{\text{Direct productivity effect}} + \overbrace{\frac{1}{\phi}\frac{z^2\left(z^2+3\phi r\sigma^2\right)}{\left(z^2+\phi r\sigma^2\right)^2}}^{\text{Incentive Effect}} - \overbrace{\frac{1}{\phi}\frac{z^2\left(z^2+3\phi r\sigma^2\right)}{\left(z^2+\phi r\sigma^2\right)^2}}^{\text{Marginal Cost Effect}}\\ &= \overbrace{a_i^*}^{\text{Direct productivity effect}}.\end{aligned}$$

- So we have shown how the incentive effect and the marginal cost effect "cancel out".

2.5 Equivalence Result Even if Wage are Flexible

- Finally I show that the same result holds even if wages are arbitrarily flexible.
- This step is important because it shows that dynamics are the same in the bonus wage and the rigid wage economy even when wages are very flexible in the bonus wage economy—contra the intuition that flexible bonus wages should stabilize the economy.
- Using equation (5), the optimal contract is

$$w_i = \alpha + \beta y_i,$$

  β is the "pass through" from output into wages.
- Clearly $\beta \to 1$ is perfect pass through of output into wages, i.e. perfectly flexible wages.
- We have

$$\beta = \frac{z^2}{z^2 + \phi r \sigma^2}.$$

- Therefore if $\phi r \sigma^2$ is positive but small, β can be arbitrarily close to 1, and yet the same equivalence results will apply.

2.6 Appendix Derivations

- To get to the main results, I will quickly solve the optimal contract.
- To start with, we can simplify the IC constraint (3). We have

$$\begin{aligned} a &= \arg\max_{\tilde{a}_i} E\left[-e^{-r\left(\alpha+\beta(z\tilde{a}_i+\eta_i)-\frac{\phi\tilde{a}_i^2}{2}\right)}\right] \\ &= \arg\max_{\tilde{a}_i} -e^{-rE[\alpha+\beta(z\tilde{a}_i+\eta_i)]+\frac{r^2}{2}Var[\alpha+\beta(z\tilde{a}_i+\eta_i)]+\frac{r\phi\tilde{a}_i^2}{2}} \\ &= \arg\max_{\tilde{a}_i} -e^{-r[\alpha+\beta z\tilde{a}_i]+\frac{r^2\beta^2}{2}\sigma^2+\frac{r\phi\tilde{a}_i^2}{2}}. \end{aligned}$$

- The first order necessary and sufficient condition is

$$\begin{aligned} \frac{\partial}{\partial a}\left[-e^{-r[\alpha+\beta z\tilde{a}_i]+\frac{r^2\beta^2}{2}\sigma^2+\frac{r\phi\tilde{a}_i^2}{2}}\right] &= 0 \\ \implies \left[-e^{-r[\alpha+\beta z a_i]+\frac{r^2\beta^2}{2}\sigma^2+\frac{r\phi a_i^2}{2}}\right] \times (-r\beta z + r\phi a_i) &= 0 \\ \implies -r\beta z + r\phi a_i &= 0 \\ \implies a_i &= \frac{\beta z}{\phi}. \end{aligned}$$

- The agent's utility is then

$$\begin{aligned} -e^{-r[\alpha+\beta z a_i]+\frac{r^2\beta^2}{2}\sigma^2+\frac{r\phi a_i^2}{2}} &= -e^{-r\left[\alpha+\beta z\frac{\beta z}{\phi}\right]+\frac{r^2\beta^2}{2}\sigma^2+\frac{r\phi\left(\frac{\beta z}{\phi}\right)^2}{2}} \\ &= -e^{-r\left[\alpha+\frac{\beta^2 z^2}{\phi}\right]+\frac{r^2\beta^2}{2}\sigma^2+r\frac{\beta^2 z^2}{2\phi}} \\ &= -e^{-r\alpha-r\frac{\beta^2 z^2}{\phi}+\frac{r^2\beta^2}{2}\sigma^2+r\frac{\beta^2 z^2}{2\phi}} \\ &= -e^{-r\alpha-r\frac{\beta^2 z^2}{2\phi}+\frac{r^2\beta^2}{2}\sigma^2} \\ &= -e^{-r\alpha+r\frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right)}. \end{aligned}$$

- We can substitute into the IR constraint (4) to get

$$
\begin{aligned}
-e^{-r\alpha+r\frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right)} &\geq -e^{-rb} \\
\implies -r\alpha + r\frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right) &= -rb \\
\implies \alpha - \frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right) &= b \\
\implies \alpha = b + \frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right)&.
\end{aligned}
$$

- So, we can write the wage as

$$
\begin{aligned}
w_i &= b + \frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right) + \beta\left(a_i z + \eta_i\right) \\
&= b + \frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right) + \beta\left(\frac{\beta z}{\phi}z + \eta_i\right) \\
&= b + \frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right) + \frac{\beta^2 z^2}{\phi} + \beta\eta_i
\end{aligned}
$$

- We can substitute all the constraints into the value of a filled vacancy (2) to get

$$
\begin{aligned}
V(z) &= \max_\beta E\left[(1-\beta)(za_i+\varepsilon_i)-\alpha\right] \\
&= \max_\beta E\left[(1-\beta)\left(z\frac{\beta z}{\phi}+\varepsilon_i\right)-\left(b+\frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right)\right)\right] \\
&= \max_\beta\left[(1-\beta)\left(\frac{\beta z^2}{\phi}\right)-b-\frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right)\right].
\end{aligned}
$$

- So, we have written the value of a filled vacancy in terms of a single choice variable β.
- The first order necessary and sufficient condition is

$$
\begin{aligned}
\frac{\partial}{\partial\beta}\left[(1-\beta)\left(\frac{\beta z^2}{\phi}\right)-b-\frac{\beta^2}{2}\left(r\sigma^2-\frac{z^2}{\phi}\right)\right] &= 0 \\
\implies (1-\beta)\frac{z^2}{\phi}-\frac{\beta z^2}{\phi}-\beta\left(r\sigma^2-\frac{z^2}{\phi}\right) &= 0 \\
\implies \frac{z^2}{\phi}-\beta\frac{z^2}{\phi}-\frac{\beta z^2}{\phi}-\beta\left(r\sigma^2-\frac{z^2}{\phi}\right) &= 0 \\
\implies \frac{z^2}{\phi} = 2\beta\frac{z^2}{\phi}+\beta\left(r\sigma^2-\frac{z^2}{\phi}\right)& \\
\implies \frac{z^2}{\phi} = \beta\left[\frac{2z^2}{\phi}+\left(r\sigma^2-\frac{z^2}{\phi}\right)\right]& \\
\implies \frac{\frac{z^2}{\phi}}{\left[\frac{2z^2}{\phi}+\left(r\sigma^2-\frac{z^2}{\phi}\right)\right]} &= \beta \\
\implies \frac{z^2}{\left[2z^2+\left(\phi r\sigma^2-z^2\right)\right]} &= \beta \\
\implies \beta = \frac{z^2}{z^2+\phi r\sigma^2}&.
\end{aligned}
$$

- Therefore we have

$$\begin{aligned}
\alpha &= b + \frac{\beta^2}{2}\left(r\sigma^2 - \frac{z^2}{\phi}\right) \\
&= b + \frac{\left(\frac{z^2}{z^2+\phi r\sigma^2}\right)^2}{2}\left(r\sigma^2 - \frac{z^2}{\phi}\right) \\
&= b + \frac{\left(\frac{z^2}{z^2+\phi r\sigma^2}\right)^2\left(\phi r\sigma^2 - z^2\right)}{2\phi}
\end{aligned}$$

- Optimal effort is

$$a_i = \frac{z^2}{z^2 + \phi r\sigma^2}\frac{z}{\phi}$$

- Therefore the wage is

$$\begin{aligned}
w_i &= \alpha + \beta x_i \\
&= \frac{z^2}{z^2+\phi r\sigma^2}\frac{z}{\phi} + \frac{z^2}{z^2+\phi r\sigma^2}\left(z a_i + \eta_i\right) \\
&= \frac{z^2}{z^2+\phi r\sigma^2}\frac{z}{\phi} + \frac{z^2}{z^2+\phi r\sigma^2}\left(z\frac{z^2}{z^2+\phi r\sigma^2}\frac{z}{\phi} + \eta_i\right) \\
&= \frac{z^2}{z^2+\phi r\sigma^2}\frac{z}{\phi} + \frac{z^2}{z^2+\phi r\sigma^2}\left(\frac{z^4}{\phi\left(z^2+\phi r\sigma^2\right)} + \eta_i\right)
\end{aligned}$$

- It will be useful later on to solve for

$$\begin{aligned}
\frac{da_i^*}{dz} &= \frac{d}{dz}\left[\frac{z^2}{z^2+\phi r\sigma^2}\frac{z}{\phi}\right] \\
&= \frac{1}{\phi}\frac{d}{dz}\left[\frac{z^3}{z^2+\phi r\sigma^2}\right] \\
&= \frac{1}{\phi}\frac{3z^2\left(z^2+\phi r\sigma^2\right) - z^3\left(2z\right)}{\left(z^2+\phi r\sigma^2\right)^2} \\
&= \frac{1}{\phi}\frac{3z^4 + 3z^2\phi r\sigma^2 - 2z^4}{\left(z^2+\phi r\sigma^2\right)^2} \\
&= \frac{1}{\phi}\frac{z^4 + 3z^2\phi r\sigma^2}{\left(z^2+\phi r\sigma^2\right)^2} \\
&= \frac{1}{\phi}\frac{z^2\left(z^2 + 3\phi r\sigma^2\right)}{\left(z^2+\phi r\sigma^2\right)^2}
\end{aligned}$$