On the Equivalence between Logic Programming and SETAF

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Abstract

A framework with sets of attacking arguments (SETAF) is an extension of the well-known Dung's Abstract Argumentation Frameworks (AAFs) that allows joint attacks on arguments. In this paper, we provide a translation from Normal Logic Programs (NLPs) to SETAFs and vice versa, from SETAFs to NLPs. We show that there is pairwise equivalence between their semantics, including the equivalence between L-stable and semi-stable semantics. Furthermore, for a class of NLPs called Redundancy-Free Atomic Logic Programs (RFALPs), there is also a structural equivalence as these back-and-forth translations are each other's inverse. Then, we show that RFALPs are as expressive as NLPs by transforming any NLP into an equivalent RFALP through a series of program transformations already known in the literature. We also show that these program transformations are confluent, meaning that every NLP will be transformed into a unique RFALP. The results presented in this paper enhance our understanding that NLPs and SETAFs are essentially the same formalism. Under consideration in Theory and Practice of Logic Programming (TPLP).

KEYWORDS: Abstract Argumentation, SETAF, Logic Programming Semantics, Program Transformations

1 Introduction

Argumentation and logic programming are two of the most successful paradigms in artificial intelligence and knowledge representation. Argumentation revolves around the idea of constructing and evaluating arguments to determine the acceptability of a claim. It models complex reasoning by considering various pieces of evidence and their interrelationships, making it a powerful tool for handling uncertainty and conflicting information. On the other hand, logic programming provides a formalism for expressing knowledge and defining computational processes through a set of logical rules.

In this scenario, the Abstract Argumentation Frameworks (AAFs) proposed by Dung in his seminal paper (Dung 1995b) have exerted a dominant influence over the devel-

opment of formal argumentation. We can depict such frameworks simply as a directed graph whose nodes represent arguments and edges represent the attack relation between them. Indeed, in AAFs, the content of these arguments is not considered, and the attack relation stands as the unique relation. The simplicity and elegance of AAFs have made them an appealing formalism for computational applications.

In Dung's proposal, the semantics for AAFs are given in terms of extensions, which are sets of arguments satisfying certain criteria of acceptability. Naturally, different criteria of acceptability will lead to different extension-based semantics, including Dung's original concepts of complete, stable, preferred and grounded semantics (Dung 1995b), and semi-stable semantics (Caminada 2006; Verheij 1996). A richer characterisation based on labellings was proposed by Caminada and Gabbay (Caminada and Gabbay 2009) to describe these semantics. Differently from extensions, which explicitly regard solely the accepted arguments, the labelling-based approach permits a more fine-grained setting, where each argument is assigned a label in, out, or undec. Intuitively, we accept an argument labelled as in, reject one labelled as out, and consider one labelled as undec as undecided, meaning it is neither accepted nor rejected.

Despite providing distinct perspectives on reasoning and decision-making, argumentation and logic programming have clear connections. Indeed, we can see in Dung's work (Dung 1995b) how to translate a Normal Logic Program (NLP) into an AAF. Besides, the author proved that stable models (resp. the well-founded model) of an NLP correspond to stable extensions (resp. the grounded extension) of the associated AAF. These results led to several studies concerning connections between argumentation and logic programming (Dung 1995a; Nieves et al. 2008; Wu et al. 2009; Toni and Sergot 2011; Dvořák et al. 2013; Caminada et al. 2015b; 2022). In particular, (Wu et al. 2009) established the equivalence between complete semantics and partial stable semantics. These semantics generalise a series of other relevant semantics for each system, as extensively documented in (Caminada et al. 2015b). However, one equivalence formerly expected to hold remained elusive: the correspondence between the semi-stable semantics (Caminada 2006) in AAF and the L-stable semantics in NLP (Eiter et al. 1997) could not be attained. They even showed in (Caminada et al. 2015b) that with their proposed translation from NLPs to AAFs, there cannot be a semantics for AAFs equivalent to L-stable semantics.

In (Caminada and Schulz 2017), the authors showed how to translate Assumption-Based Argumentation (ABA) (Bondarenko et al. 1997; Dung et al. 2009; Toni 2014) to NLPs and how this translation can be reapplied for a reverse translation from NLPs to ABA. Curiously, the problematic direction here is from ABA to NLP. In (Caminada and Schulz 2017), they have shown that with their translation, there cannot be a semantics for NLPs equivalent to the semi-stable semantics (Caminada et al. 2015a; Schulz and Toni 2015) for ABA.

Since then, a great effort has been made to identify paradigms where semi-stable and L-stable semantics are equivalent. In (Alcântara et al. 2019), the strategy was to look for more expressive argumentation frameworks than AAFs: Attacking Dialectical Frameworks, a support-free fragment of Abstract Dialectical Frameworks (Brewka and Woltran 2010; Brewka et al. 2013), a generalisation of AAFs designed to express arbitrary relationships among arguments. A translation from NLP to ADF^+ was proved in (Alcântara et al. 2019) to account for various equivalences between their semantics, in-

cluding the definition of a semantics for ADF^+ corresponding to the L-stable semantics for NLPs.

In a similar vein, other relevant proposals explored the equivalence between L-stable and semi-stable semantics for Claim-augmented Argumentation Frameworks (CAFs) (Dvořák et al. 2023; Rapberger 2020; Rocha and Cozman 2022b), which are a generalisation of AAFs where each argument is explicitly associated with a claim, and for Bipolar Argumentation Frameworks (BAFs) with conclusions (Rocha and Cozman 2022a), a generalisation of CAFs with the inclusion of an explicit notion of support between arguments. In both frameworks, the equivalence with NLPs does not just involve their semantics; it is also structural as there is a one-to-one mapping from them to NLPs.

In (Sá and Alcântara 2021a), instead of looking for more expressive argumentation frameworks, the idea was to introduce more fine-grained semantics to deal with AAFs. Then a five-valued setting was employed rather than the usual three-valued one. As in the previous cases, this approach also captures the correspondence between the semantics for AAFs and NLPs. Specifically, it captures the correspondence involving L-stable semantics.

The connections between ABA and logic programming were later revisited in (Sá and Alcântara 2019; 2021b), where they proposed a new translation from ABA frameworks to NLPs. The correspondence between their semantics (including L-stable) is obtained by selecting specific atoms in the characterisation of the NLP semantics.

In summary, in the connections between *NLP* and argumentation semantics, the Achilles' heel is the relation between *L*-stable and semi-stable semantics.

In this paper, we focus on the relationship between logic programming and SETAF (Nielsen and Parsons 2006), an extension of Dung's AAFs to allow joint attacks on arguments. Following the strategy adopted in (Caminada et al. 2015b; Alcântara et al. 2019), we resort to the characterisation of the SETAF semantics in terms of labellings (Flouris and Bikakis 2019). As a starting point, we provide a mapping from NLPs to SETAFs (and vice versa) and show that NLPs and SETAFs are pairwise equivalent under various semantics, including the equivalence between L-stable and semi-stable. These results were inspired directly by two of our previous works: the equivalence between NLPs and ADF^+ s (Alcântara et al. 2019), and the equivalence between ADF^+ s and SETAFs (Alcântara and Sá 2021).

Furthermore, we investigate a class of NLPs called Redundancy-Free Atomic Logic Programs (RFALPs) (König et al. 2022). In RFALPs, the translations from NLPs to SETAFs and vice versa preserve the structure of each other's theories. In essence, these translations become inverses of each other. Consequently, the equivalence results concerning NLPs and SETAFs have deeper implications than the correspondence results between NLPs and AAFs: they encompass equivalence in both semantics as well as structure.

Some of these results are not new as recently they have already been obtained independently by König et al. (König et al. 2022). In fact, their translation from NLPs to SETAFs and vice versa coincide with ours, and the structural equivalence between RFALPs and SETAFs has also been identified there. However, their focus differs from ours. While their work establishes the equivalence between stable models and stable extensions, it does not explore equivalences involving labelling-based semantics or address the controversy relating semi-stable semantics and L-stable semantics, which is a key motivation for this work. In comparison with König et al.'s work, the novelty of our proposal lies essentially in the aspects below:

- Our proofs of these results follow a significantly distinct path as they are based on properties of argument labellings and are deeply rooted in works such as (Caminada et al. 2015b; Alcântara et al. 2019; Alcântara and Sá 2021).
- We prove the equivalence between partial stable, well-founded, regular, stable, and semi-stable model semantics for NLPs and respectively complete, grounded, preferred, stable, and semi-stable labellings for SETAFs. In particular, for the first time, an equivalence between L-stable model semantics for NLPs and semi-stable labellings for SETAFs is established.
- We provide a more in-depth analysis of the relationship between *NLP*s and *SETAF*s. Going beyond just proving semantic equivalence, we define functions that map labellings to interpretations, and interpretations to labellings. These functions allow us to see interpretations and labellings as equivalent entities, further strengthening the connections between *NLP*s and *SETAF*s. In substance, we demonstrate that the equivalence also holds at the level of interpretations/labellings.

The strong connection we establish between interpretations and labellings opens doors for future exploration. This extends the applicability of our equivalence results to novel semantics beyond those investigated here, potentially even encompassing multivalued settings. This holds particular significance for the logic programming community. Wellestablished concepts from argumentation, such as argument strength (Beirlaen et al. 2018), can now be translated and investigated within the context of logic programming. This underscores the value of our decision to employ labellings instead of extensions as a more suitable approach to bridge the gap between *NLPs* and *SETAFs*.

Our research offers another key contribution, particularly relevant to the logic programming community: it explores the expressiveness of RFALPs. We demonstrate that a specific combination (denoted by \mapsto_{UTPM}) of program transformations can transform any NLP into an RFALP with exactly the same semantics. In simpler terms, RFALPs possess the same level of expressiveness as NLPs. Although each program transformation in \mapsto_{UTPM} was proposed by Brass and Dix (Brass and Dix 1994; 1997; 1999), the combination of these program transformations (to our knowledge) has not been investigated yet. Then we establish several properties of \mapsto_{UTPM} . Amongst other original contributions of our work related to \mapsto_{UTPM} , we highlight the following results:

- Given an NLP, if repeatedly applying \mapsto_{UTPM} leads to a program where no further transformations are applicable (irreducible program), then the resulting program is guaranteed to be an RFALP.
- We show that \mapsto_{UTPM} is confluent, i.e., given an NLP, it does not matter the order by which we apply repeatedly these program transformations, whenever we arrive at an irreducible program, they will always result in the same RFALP (and in the same corresponding SETAF). Hence, besides NLPs and RFALPs being equally expressive, each NLP is associated with a unique RFALP.
- The SETAF corresponding to an NLP is invariant with respect to \mapsto_{UTPM} , i.e., if P_2 is obtained from P_1 via \mapsto_{UTPM} (denoted by $P_1 \mapsto_{UTPM} P_2$), both P_1 and P_2 will be translated into the same SETAF.

• We show that \mapsto_{UTPM} preserves the semantics for NLPs studied in this paper: if $P_1 \mapsto_{UTPM} P_2$, then P_1 and P_2 have the same partial stable models, well-founded models, regular models, stable models, and L-stable models.

The structure of the paper unfolds as follows: in Section 2, we establish the fundamental definitions related to SETAFs and NLPs. In Section 3, we adapt the procedure from (Caminada et al. 2015b; Alcântara et al. 2019) to translate NLPs into SETAFs, and subsequently, in the following section, we perform the reverse translation from SETAFs to NLPs. In both directions, we demonstrate that our labelling-based approach effectively preserves semantic correspondences, including the challenging case involving the equivalence between semi-stable semantics (on the SETAFs side) and L-stable semantics (on the NLPs side). In Section 5, we focus on RFALPs and reveal that, when restricted to them, the translation processes between NLPs and SETAFs are each other's inverse. Then, in Section 6, we guarantee that RFALPs are as expressive as NLPs. We conclude the paper with a discussion of our findings and outline potential avenues for future research endeavours.

The proofs for all novel results are presented in Appendix A.

2 Preliminaries

2.1 SETAF

In (Nielsen and Parsons 2006), an extension of Dung's Abstract Argumentation Frameworks (AAFs) (Dung 1995b) to allow joint attacks on arguments was proposed. The resulting framework, called SETAF, is defined next:

Definition 1 (SETAF (Nielsen and Parsons 2006))

A framework with sets of attacking arguments (SETAF for short) is a pair $\mathfrak{A} = (\mathcal{A}, Att)$, in which \mathcal{A} is a finite set of arguments and $Att \subseteq (2^{\mathcal{A}} \setminus \{\emptyset\}) \times \mathcal{A}$ is an attack relation such that if $(\mathcal{B}, a) \in Att$, there is no $\mathcal{B}' \subset \mathcal{B}$ such that $(\mathcal{B}', a) \in Att$, i.e., \mathcal{B} is a minimal set (w.r.t. \subseteq) attacking a^1 . By $Att(a) = \{\mathcal{B} \subseteq \mathcal{A} \mid (\mathcal{B}, a) \in Att\}$, we mean the set of attackers of a.

In AAFs, only individual arguments can attack arguments. In SETAFs, the novelty is that sets of two or more arguments can also attack arguments. This means that SETAFs (\mathcal{A}, Att) with $|\mathcal{B}| = 1$ for each $(\mathcal{B}, a) \in Att$ amount to (standard Dung) AAFs.

The semantics for SETAFs are generalisations of the corresponding semantics for AAFs (Nielsen and Parsons 2006) and can be defined equivalently in terms of extensions or labellings (Flouris and Bikakis 2019). Our focus here will be on their labelling-based semantics.

Definition 2 (Labellings (Flouris and Bikakis 2019))

Let $\mathfrak{A} = (\mathfrak{A}, Att)$ be a *SETAF*. A labelling is a function $\mathcal{L} : \mathcal{A} \to \{\text{in}, \text{out}, \text{undec}\}$. It is *admissible* iff for each $a \in \mathcal{A}$,

• If $\mathcal{L}(a) = \text{in}$, then for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{L}(b) = \text{out}$ for some $b \in \mathcal{B}$.

 $^{^{1}}$ In the original definition of SETAFs in (Nielsen and Parsons 2006), attacks are not necessarily subset-

• If $\mathcal{L}(a) = \mathtt{out}$, then there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{L}(b) = \mathtt{in}$ for each $b \in \mathcal{B}$.

A labelling \mathcal{L} is called *complete* iff it is admissible and for each $a \in \mathcal{A}$,

• If $\mathcal{L}(a) = \text{undec}$, then there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{L}(b) \neq \text{out for each } b \in \mathcal{B}$, and for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{L}(b) \neq \text{in for some } b \in \mathcal{B}$.

We write $\operatorname{in}(\mathcal{L})$ for $\{a \in \mathcal{A} \mid \mathcal{L}(a) = \operatorname{in}\}$, $\operatorname{out}(\mathcal{L})$ for $\{a \in \mathcal{A} \mid \mathcal{L}(a) = \operatorname{out}\}$, and $\operatorname{undec}(\mathcal{L})$ for $\{a \in \mathcal{A} \mid \mathcal{L}(a) = \operatorname{undec}\}$. As a labelling essentially defines a partition among the arguments, we sometimes write \mathcal{L} as a triple $(\operatorname{in}(\mathcal{L}), \operatorname{out}(\mathcal{L}), \operatorname{undec}(\mathcal{L}))$. Intuitively, an argument labelled in represents explicit acceptance; an argument labelled out indicates rejection; and one labelled undec is undecided, i.e., it is neither accepted nor rejected. We can now describe the SETAF semantics studied in this paper:

Definition 3 (Semantics (Flouris and Bikakis 2019)) Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a SETAF. A complete labelling \mathcal{L} is called

- grounded iff $in(\mathcal{L})$ is minimal (w.r.t. \subseteq) among all complete labellings of \mathfrak{A} .
- preferred iff $in(\mathcal{L})$ is maximal (w.r.t. \subseteq) among all complete labellings of \mathfrak{A} .
- $stable \text{ iff undec}(\mathcal{L}) = \emptyset.$
- semi-stable iff $undec(\mathcal{L})$ is minimal (w.r.t. \subseteq) among all complete labellings of \mathfrak{A} .

Let us consider the following example:

Example 1

Consider the SETAF $\mathfrak{A} = (A, Att)$ below:

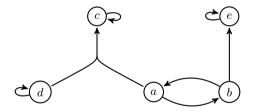


Fig. 1: A SETAF A

Concerning the semantics of \mathfrak{A} , we have

- Complete labellings: $\mathcal{L}_1 = (\emptyset, \emptyset, \{a, b, c, d, e\}), \mathcal{L}_2 = (\{a\}, \{b\}, \{c, d, e\})$ and $\mathcal{L}_3 = (\{b\}, \{a, e\}, \{c, d\});$
- Grounded labellings: $\mathcal{L}_1 = (\emptyset, \emptyset, \{a, b, c, d, e\});$
- Preferred labellings: $\mathcal{L}_2 = (\{a\}, \{b\}, \{c, d, e\})$ and $\mathcal{L}_3 = (\{b\}, \{a, e\}, \{c, d\})$;
- Stable labellings: none;
- Semi-stable labellings: $\mathcal{L}_3 = (\{b\}, \{a, e\}, \{c, d\}).$

2.2 Logic Programs and Semantics

Now, we take a look at propositional Normal Logic Programs. To delve into their definition and semantics, we will follow the presentation outlined in (Caminada et al. 2015b), which draws from the foundation laid out in (Przymusinski 1990).

Definition 4 ((Caminada et al. 2015b))

A rule r is an expression

$$r: c \leftarrow a_1, \dots, a_m, \text{not } b_1, \dots, \text{not } b_n$$
 (1)

where $(m, n \geq 0)$; c, each a_i $(1 \leq i \leq m)$ and each b_j $(1 \leq j \leq n)$ are atoms, and not represents negation as failure. A literal is either an atom a (positive literal) or a negated atom not a (negative literal). Given a rule r as above, c is called the head of r, which we denote as head(r), and $body(r) = \{a_1, \ldots, a_m, \text{not } b_1, \ldots, \text{not } b_n\}$ is called the body of r. Further, we divide body(r) into two sets $body^+(r) = \{a_1, \ldots, a_m\}$ and $body^-(r) = \{\text{not } b_1, \ldots, \text{not } b_n\}$. A fact is a rule where m = n = 0. A Normal Logic Program (NLP) or simply a program P is a finite set of rules. If every $r \in P$ has $body^-(r) = \emptyset$, P is a positive program. The Herbrand Base of P is the set HB_P of all atoms appearing in P.

A wide range of NLP semantics are based on the 3-valued interpretations of programs (Przymusinski 1990):

Definition 5 (3-Valued Herbrand Interpretation (Przymusinski 1990))

A 3-valued Herbrand Interpretation \mathcal{I} (or simply interpretation) of an NLP P is a pair $\langle T, F \rangle$ with $T, F \subseteq HB_P$ and $T \cap F = \emptyset$. The atoms in T are true in \mathcal{I} , the atoms in F are false in \mathcal{I} , and the atoms in $HB_P \setminus (T \cup F)$ are undefined in \mathcal{I} . For convenience, when the NLP P is clear from the context, we will refer to the set of undefined atoms in $HB_P \setminus (T \cup F)$ simply as $\overline{T \cup F}$.

Now we will consider the main semantics for NLPs. Let $\mathcal{I} = \langle T, F \rangle$ be a 3-valued Herbrand interpretation of an NLP P; the reduct of P with respect to \mathcal{I} (written as P/\mathcal{I}) is the NLP constructed using the following steps:

- 1. Remove any $a \leftarrow a_1, \ldots, a_m$, not b_1, \ldots , not $b_n \in P$ such that $b_j \in T$ for some j $(1 \le j \le n)$;
- 2. Afterwards, remove any occurrence of not b_i from P such that $b_i \in F$;
- 3. Then, replace any occurrence of not b_i left by a special atom \mathbf{u} ($\mathbf{u} \notin HB_P$).

In the above procedure, **u** is assumed to be an atom not in HB_P which is undefined in all interpretations of P (a constant). Note that P/\mathcal{I} is a positive program since all negative literals have been removed. As a consequence, P/\mathcal{I} has a unique least 3-valued model (Przymusinski 1990), obtained by the Ψ operator:

Definition 6 (Ψ Operator (Przymusinski 1990))

Let P be a positive program and $\mathcal{J} = \langle T, F \rangle$ be an interpretation. Define $\Psi_P(\mathcal{J}) = \langle T', F' \rangle$, where

- $c \in T'$ iff $c \in HB_P$ and there exists $c \leftarrow a_1, \ldots, a_m \in P$ such that for all $i, 1 \le i \le m$, $a_i \in T$;
- $c \in F'$ iff $c \in HB_P$ and for every $c \leftarrow a_1, \ldots, a_m \in P$, there exists $i, 1 \le i \le m$, such that $a_i \in F$.

The least 3-valued model of P is given by $\Psi_P^{\uparrow \omega}$ (Przymusinski 1990), the least fixed point of Ψ_P iteratively obtained as follows:

$$\begin{split} &\Psi_{P}^{\uparrow 0} = \langle \emptyset, HB_{P} \rangle \\ &\Psi_{P}^{\uparrow i+1} = \Psi_{P}(\Psi_{P}^{\uparrow i}) \\ &\Psi_{P}^{\uparrow \omega} = \left\langle \bigcup_{i < \omega} \left\{ T_{i} \mid \Psi_{P}^{\uparrow i} = \langle T_{i}, F_{i} \rangle \right\}, \bigcap_{i < \omega} \left\{ F_{i} \mid \Psi_{P}^{\uparrow i} = \langle T_{i}, F_{i} \rangle \right\} \right\rangle \end{split}$$

where ω denotes the first infinite ordinal.

We can now describe the logic programming semantics studied in this paper:

Definition 7

Let P be an NLP and $\mathcal{I} = \langle T, F \rangle$ be an interpretation; by $\Omega_P(\mathcal{I}) = \Psi_{\overline{\mathcal{I}}}^{\uparrow \omega}$, we mean the least 3-valued model of $\frac{P}{\mathcal{I}}$. We say that

- \mathcal{I} is a partial stable model of P iff $\Omega_P(\mathcal{I}) = \mathcal{I}$ (Przymusinski 1990).
- \mathcal{I} is a well-founded model of P iff \mathcal{I} is a partial stable model of P where there is no partial stable model $\mathcal{I}' = \langle T', F' \rangle$ of P such that $T' \subset T$, i.e., T is minimal (w.r.t. set inclusion) among all partial stable models of P (Przymusinski 1990).
- \mathcal{I} is a regular model of P iff \mathcal{I} is a partial stable model of P where there is no partial stable model $\mathcal{I}' = \langle T', F' \rangle$ of P such that $T \subset T'$, i.e., T is maximal (w.r.t. set inclusion) among all partial stable models of P (Eiter et al. 1997).
- \mathcal{I} is a (2-valued) stable model of P iff \mathcal{I} is a partial stable model of P where $T \cup F = HB_P$ (Przymusinski 1990).
- \mathcal{I} is an L-stable model of P iff \mathcal{I} is a partial stable model of P where there is no partial stable model $\mathcal{I}' = \langle T', F' \rangle$ of P such that $T \cup F \subset T' \cup F'$, i.e., $T \cup F$ is maximal (w.r.t. set inclusion) among all partial stable models of P (Eiter et al. 1997).

Although some of these definitions are not standard in logic programming literature, their equivalence is proved in (Caminada et al. 2015b). This format helps us to relate NLP and SETAF semantics due to the structural similarities between Definition 7 and Definitions 2 and 3. We illustrate these semantics in the following example:

Example 2

Consider the following logic program P:

This program has

- Partial Stable Models: $\mathcal{M}_1 = \langle \emptyset, \emptyset \rangle$, $\mathcal{M}_2 = \langle \{a\}, \{b\} \rangle$ and $\mathcal{M}_3 = \langle \{b\}, \{a, e\} \rangle$;
- Well-founded model: $\mathcal{M}_1 = \langle \emptyset, \emptyset \rangle$;
- Regular models: $\mathcal{M}_2 = \langle \{a\}, \{b\} \rangle$ and $\mathcal{M}_3 = \langle \{b\}, \{a, e\} \rangle$;
- Stable models: none;
- L-stable model: $\mathcal{M}_3 = \langle \{b\}, \{a, e\} \rangle$.

3 From NLP to SETAF

In this section, we revisit the three-step process of argumentation framework instantiation as employed in (Caminada et al. 2015b) for translating an NLP into an AAF. This method is based on the approach introduced by (Wu et al. 2009) and shares similarities with the procedures used in ASPIC (Caminada and Amgoud 2005; 2007) and logic-based argumentation (Gorogiannis and Hunter 2011). Its first step involves taking an NLP and constructing its associated AAF. Then, we apply AAF semantics in the second step, followed by an analysis of the implications of these semantics at the level of conclusions (step 3). In our case, starting with an NLP P, we derive the associated SETAF (A_P , Att_P). Unlike the construction described in (Caminada et al. 2015b), rules with identical conclusions in P will result in a single argument in A_P . This distinction is capital for establishing the equivalence results between NLPs and SETAFs. Additionally, it simplifies steps 2 and 3, making them more straightforward to follow. We now detail this process.

3.1 SETAF Construction

We will devise one translation from NLP to SETAF that is sufficiently robust to guarantee the equivalence between various kinds of NLPs models and SETAFs labellings. Specifically, our approach will establish the correspondence between partial stable models and complete labellings, well-founded models and grounded labellings, regular models and preferred labellings, stable models and stable labellings, L-stable models and semistable labellings. Our method is built upon a translation from NLP to AAF proposed in (Caminada et al. 2015b), where NLP rules are directly translated into arguments. We will adapt this approach for SETAF by employing the translation method outlined in (Caminada et al. 2015b) to construct statements, and then statements corresponding to rules with the same head will be grouped to form a single argument. Taking an NLPP, we can start to construct statements recursively as follows:

Definition 8 (Statements and Arguments) Let P be an NLP.

- If $c \leftarrow \text{not } b_1, \dots, \text{not } b_n$ is a rule in P, then it is also a statement (say s) with

 - $Vul(s) = \{b_1, \ldots, b_n\}$, and
 - $-- Sub(s) = \{s\}.$
- If $c \leftarrow a_1, \ldots, a_m$, not b_1, \ldots , not b_n is a rule in P and for each a_i $(1 \le i \le m)$ there exists a statement s_i with $\operatorname{Conc}(s_i) = a_i$ and $c \leftarrow a_1, \ldots, a_m, \operatorname{not} b_1, \ldots, \operatorname{not} b_n$ is not contained in $\operatorname{Rules}(s_i)$, then $c \leftarrow (s_1), \ldots, (s_m)$, not $b_1, \ldots, \operatorname{not} b_n$ is a statement (say s) with
 - -- Conc(s) = c,
 - $Rules(s) = Rules(s_1) \cup ... \cup Rules(s_m) \cup \{c \leftarrow a_1, ..., a_m, not b_1, ..., not b_n\}$
 - $\operatorname{Vul}(s) = \operatorname{Vul}(s_1) \cup \ldots \cup \operatorname{Vul}(s_m) \cup \{b_1, \ldots, b_n\}, \text{ and }$
 - $--- Sub(s) = \{s\} \cup Sub(s_1) \cup \ldots \cup Sub(s_m).$

By \mathfrak{S}_P we mean the set of all statements we can construct from P as above. Then we define $\mathcal{A}_P = \{ \mathtt{Conc}(s) \mid s \in \mathfrak{S}_P \}$ as the set of all arguments we can construct from P. For an argument c from P ($c \in \mathcal{A}_P$), we have that

- Conc(c) = c,
- $\operatorname{Vul}_P(c) = {\operatorname{Vul}(s) \mid s \in \mathfrak{S}_P \text{ and } \operatorname{Conc}(s) = c}, \text{ and }$

If c is an argument, then Conc(c) is referred to as the *conclusion* of c, and $Vul_P(c)$ is referred to as the *vulnerabilities* of c in P. When the context is clear, we will write simply Vul(c) instead of $Vul_P(c)$.

Now we will clarify the connection between the existence of statements and the existence of a derivation in a reduct.

Lemma 1

Let P be an NLP, $\mathcal{I} = \langle T, F \rangle$ an interpretation and $\Omega_P(\mathcal{I}) = \langle T', F' \rangle$ the least 3-valued model of $\frac{P}{\mathcal{I}}$. It holds

- (i) $c \in T'$ iff there exists a statement s constructed from P such that $\mathtt{Conc}(s) = c$ and $\mathtt{Vul}(s) \subseteq F$.
- (ii) $c \in F'$ iff for every statement s constructed from P such that $\mathtt{Conc}(s) = c$, we have $\mathtt{Vul}(s) \cap T \neq \emptyset$

We can prove both results in Lemma 1 by induction. Assuming that $\Psi_{\frac{P}{2}}^{\uparrow i} = \langle T_i, F_i \rangle$ for each $i \in \mathbb{N}$, we can prove the right-hand side of item (i) and the left-hand side of item (ii) by induction on the value of i after guaranteeing the following results:

- If $c \in T_i$, then there exists a statement s constructed from P such that Conc(s) = c and $Vul(s) \subseteq F$.
- If $c \notin F_i$, then there exists a statement s constructed from P such that Conc(s) = c and $Vul(s) \cap T = \emptyset$.

The remaining cases of Lemma 1 can be proved by structural induction on the construction of a statement s (see a detailed account of the proof of Lemma 1 in Section A.1 of Appendix A).

Lemma 1 ensures that statements are closely related to derivations in a reduct. An atom c is true in the least 3-valued model of $\frac{P}{\mathcal{I}}$ iff we can construct a statement with conclusion c and whose vulnerabilities are false according to \mathcal{I} ; otherwise, c is false in the least 3-valued model of $\frac{P}{\mathcal{I}}$ iff for every statement whose conclusion is c, at least one of its vulnerabilities is true in \mathcal{I} . The next result is a direct consequence of Lemma 1:

Corollary 2

Let P be an NLP.

- Assume $\mathcal{I} = \langle \emptyset, HB_P \rangle$ and $\Omega_P(\mathcal{I}) = \langle T', F' \rangle$. It holds that $c \in T'$ iff there exists a statement s constructed from P such that Conc(s) = c.
- There is no statement s constructed from P such that Conc(s) = c iff $c \in F'$ for every interpretation \mathcal{I} with $\Omega_P(\mathcal{I}) = \langle T', F' \rangle$.

The reduct of P with respect to $\langle \emptyset, HB_P \rangle$ gives us all the possible derivations of P, and from these derivations, we can construct all the statements associated with P. On the other hand, the atoms that are lost in the translation, i.e., the atoms not associated with

statements are simply those that are false in the least 3-valued model of every possible reduct of P. Besides establishing connections between statements and derivations in a reduct, Lemma 1 also plays a central role in the proof of Theorems 4 and 5.

Apart from that, intuitively, we can see a statement as a tree-like structure representing a possible derivation of an atom from the rules of a program. In contrast, an argument for c in P is associated with the (derivable) atom c itself and can be obtained by collecting all the statements with the same conclusion c (i.e., all the possible ways of deriving c in P).

Example 3

Consider the *NLP P* below with rules $\{r_1, \ldots, r_8\}$:

According to Definition 8, we can construct the following statements from P:

In the next table, we give the conclusions and vulnerabilities of each statement:

Alternatively, we can depict statements as possible derivations as in Fig. 2:

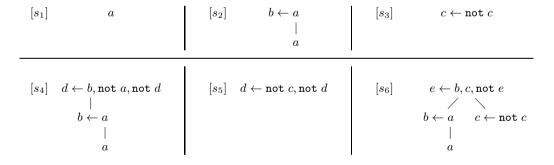


Fig. 2: Statements constructed from P

The vulnerabilities of a statement s are associated with the negative literals found in the derivation of s. If not a is one of them, we know that a is one of its vulnerabilities. This means that if a is derived, then Conc(s) cannot be obtained via this derivation represented by s. However, it can still be obtained via other derivations/statements. For instance, in the program P of Example 3, the derivation of a suffices to prevent the derivation of a via statement a (for that reason, $a \in Vul(s_4)$), but we still can derive a via a via a Notice also that there are no statements with conclusions a and a From Corollary

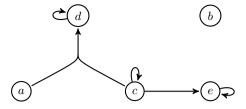


Fig. 3: A SETAF $\mathfrak{A}_P = (A_P, Att_P)$

2, we know that it is not possible to derive them in P as they are false in the least 3-valued model of each reduct of P. In addition, to determine the vulnerabilities of an atom (and not only of a specific derivation leading to this atom), we collect these data about the statements with the same conclusions to give the conclusions and vulnerabilities of each argument. In our example, we obtain the following results:

	a	b	c	d	e	
<pre>Conc(.) Vul(.)</pre>	a $\{\emptyset\}$	<i>b</i> {Ø}	c $\{c\}$	$d \\ \left\{ \left\{ a,d\right\} ,\left\{ c,d\right\} \right\}$	$e \\ \{\{c,e\}\}$	

As the vulnerabilities of an atom/argument a are a collection of the vulnerabilities of the statements whose conclusion is a, any set containing at least one atom in each of these statements suffices to prevent the derivation of a in P. In our example, there are two statements with the same conclusion d and $\operatorname{Vul}(d) = \{\{a,d\},\{c,d\}\}$. Thus any set of atoms containing $\{d\}$ or $\{a,c\}$ prevents the conclusion of d in P. We will resort to these minimal sets to determine the attack relation:

Definition 9

Let P be an NLP and let \mathcal{B} and a be respectively a set of arguments and an argument in the sense of Definition 8. We say that $(\mathcal{B}, a) \in Att_P$ iff \mathcal{B} is a minimal set (w.r.t. set inclusion) such that for each $V \in Vul_P(a)$, there exists $b \in \mathcal{B} \cap V$.

For the arguments of Example 3, it holds that both a and b are not attacked, c attacks itself, c attacks e, e attacks itself, d attacks itself, d and d (collectively) attack d. This strategy of extracting statements from NLPs rules and then gathering those with identical conclusions into arguments is not novel; in (Alcântara et al. 2019), the authors proposed a translation from NLPs into Abstract Dialectical Frameworks (Brewka et al. 2013; Brewka and Woltran 2010) by following a similar path. Using the thus-defined notions of arguments and attacks, we define the SETAF associated with an NLP.

Definition 10

Let P be an NLP. We define its associated SETAF as $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$, where \mathcal{A}_P is the set of arguments in the sense of Definition 8 and Att_P is the attack relation in the sense of Definition 9.

As an example, the SETAF $\mathfrak{A}_P = (A_P, Att_P)$ associated with the NLP of Example 3 is depicted in Fig. 3.

3.2 Equivalence Results

Once the SETAF has been constructed, we show the equivalence between the semantics for an NLP P and their counterpart for the associated $SETAF \mathfrak{A}_P$. One distinguishing characteristic of our approach in comparison with König et al.'s proposal (König et al. 2022) is that it is more organic. We prove the equivalence results by identifying connections between fundamental notions used in the definition of the semantics for NLPs and SETAFs. With this purpose, we introduce two functions: $\mathcal{L}2\mathcal{I}_P$ associates an interpretation to each labelling while $\mathcal{I}2\mathcal{L}_P$ associates a labelling to each interpretation. We then investigate the conditions under which they are each other's inverse and employ these results to prove the equivalence between the semantics. These functions essentially permit us to treat interpretations and labellings interchangeably.

Definition 11 ($\mathcal{L}2\mathcal{I}_P$ and $\mathcal{I}2\mathcal{L}_P$ Functions)

Let P be an NLP, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be its associated SETAF, $\mathcal{I}nt$ be the set of all the 3-valued interpretations of P and $\mathcal{L}ab$ be the set of all labellings of \mathfrak{A}_p . We introduce a function $\mathcal{L}2\mathcal{I}_P : \mathcal{L}ab \to \mathcal{I}nt$ such that $\mathcal{L}2\mathcal{I}_P(\mathcal{L}) = \langle T, F \rangle$, in which

- $T = \{c \in HB_P \mid c \in \mathcal{A}_P \text{ and } \mathcal{L}(c) = \mathtt{in}\};$
- $F = \{c \in HB_P \mid c \notin A_P \text{ or } c \in A_P \text{ and } \mathcal{L}(c) = \mathsf{out}\};$
- $\overline{T \cup F} = \{c \in HB_P \mid c \in \mathcal{A}_P \text{ and } \mathcal{L}(c) = \text{undec}\}.$

We introduce a function $\mathcal{I}2\mathcal{L}_P: \mathcal{I}nt \to \mathcal{L}ab$ such that for any $\mathcal{I} = \langle T, F \rangle \in \mathcal{I}nt$ and any $c \in \mathcal{A}_P$,

- $\mathcal{I}2\mathcal{L}_P(\mathcal{I})(c) = \text{in if } c \in T;$
- $\mathcal{I}2\mathcal{L}_P(\mathcal{I})(c) = \text{out if } c \in F$;
- $\mathcal{I}2\mathcal{L}_P(\mathcal{I})(c) = \text{undec if } c \notin T \cup F.$

 $\mathcal{I}2\mathcal{L}_P(\mathcal{I})(c)$ is not defined if $c \notin \mathcal{A}_P$.

The correspondence between labellings and interpretations is clear for those atoms $c \in HB_P$ in which $c \in \mathcal{A}_P$. In this case, we have that c is interpreted as true iff c is labelled as in; c is interpreted as false iff c is labelled as out. In contradistinction, those atoms $c \in HB_P$ not associated with arguments $(c \notin \mathcal{A}_P)$ are simply interpreted as false. This will suffice to guarantee our results; next theorem assures us that $\mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L})) = \mathcal{L}$:

Theorem 3

Let P be an NLP and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated SETAF. For any labelling \mathcal{L} of \mathfrak{A}_P , it holds $\mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L})) = \mathcal{L}$.

In general, $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{I}))$ is not equal to \mathcal{I} , because of those atoms c occurring in an NLPP, but not in \mathcal{A}_P . However, when \mathcal{M} is a partial stable model, $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M})) = \mathcal{M}$:

Theorem 4

Let P be an NLP, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated SETAF and $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of P. It holds that $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M})) = \mathcal{M}$.

This means that when restricted to partial stable models and complete labellings, $\mathcal{L}2\mathcal{I}_P$ and $\mathcal{I}2\mathcal{L}_P$ are each other's inverse. From Lemma 1, and Theorems 3 and 4, we can obtain the following result:

Theorem 5

Let P be an NLP and $\mathfrak{A}_P = (A_P, Att_P)$ be the associated SETAF. It holds

- \mathcal{L} is a complete labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P.
- \mathcal{M} is a partial stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of \mathfrak{A}_P .

Theorem 5 is one of the main results of this paper. It plays a central role in ensuring the equivalence between the semantics for NLP and their counterpart for SETAF:

Theorem 6

Let P be an NLP and $\mathfrak{A}_P = (A_P, Att_P)$ be the associated SETAF. It holds

- 1. \mathcal{L} is a grounded labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a well-founded model of P.
- 2. \mathcal{L} is a preferred labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a regular model of P.
- 3. \mathcal{L} is a stable labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a stable model of P.
- 4. \mathcal{L} is a semi-stable labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is an L-stable model of P.

The following result is a direct consequence of Theorems 4 and 6:

Corollary 7

Let P be an NLP and $\mathfrak{A}_P = (A_P, Att_P)$ be the associated SETAF. It holds

- 1. \mathcal{M} is a well-founded model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a grounded labelling of \mathfrak{A}_P .
- 2. \mathcal{M} is a regular model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a preferred labelling of \mathfrak{A}_P .
- 3. \mathcal{M} is a stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a stable labelling of \mathfrak{A}_P .
- 4. \mathcal{M} is an L-stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a semi-stable labelling of \mathfrak{A}_P .

Next, we consider the NLP exploited by Caminada et al. (Caminada et al. 2015b) as a counterexample to show that in general, L-stable models and semi-stable labellings do not coincide with each other in their translation from NLPs to AAFs:

Example 4

Let P be the NLP and \mathfrak{A}_P be the associated SETAF depicted in Fig. 4:

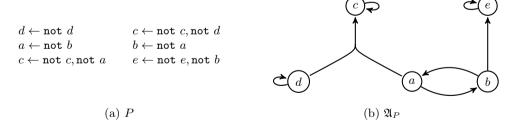


Fig. 4: NLP P and its associated SETAF \mathfrak{A}_P

As expected from Theorems 5 and 6, we obtain in Table 1 the equivalence between partial stable models and complete labellings, well-founded models and grounded labellings, regular models and preferred labellings, stable models and stable labellings, L-stable models and semi-stable labellings. We emphasise the coincidence between L-stable models and semi-stable labellings in Table 1 as it does not occur in (Caminada et al. 2015b). In that reference, the associated AAF possesses two semi-stable labellings in contrast

with the unique L-stable model \mathcal{M}_3 of P. In the next two sections, we will show that this relation between NLPs and SETAFs has even deeper implications.

Table 1: Semantics for P and \mathfrak{A}_P

Partial Stable Models	$\mathcal{M}_1 = \langle \emptyset, \emptyset \rangle$ $\mathcal{M}_2 = \langle \{a\}, \{b\} \rangle$ $\mathcal{M}_3 = \langle \{b\}, \{a, e\} \rangle$	Complete Labellings	$\mathcal{L}_{1} = (\emptyset, \emptyset, \{a, b, c, d, e\})$ $\mathcal{L}_{2} = (\{a\}, \{b\}, \{c, d, e\})$ $\mathcal{L}_{3} = (\{b\}, \{a, e\}, \{c, d\})$
Well-Founded Models	$\mathcal{M}_1 = \langle \emptyset, \emptyset angle$	Grounded Labellings	$\mathcal{L}_1 = (\emptyset, \emptyset, \{a, b, c, d, e\})$
Regular Models	$\mathcal{M}_{2} = \langle \{a\}, \{b\} \rangle$ $\mathcal{M}_{3} = \langle \{b\}, \{a, e\} \rangle$	Preferred Labellings	$\mathcal{L}_{2} = (\{a\}, \{b\}, \{c, d, e\})$ $\mathcal{L}_{3} = (\{b\}, \{a, e\}, \{c, d\})$
Stable Models	None	Stable Labellings	None
L-stable Models	$\mathcal{M}_3 = \langle \{b\}, \{a, e\} \rangle$	Semi-stable Labellings	$\mathcal{L}_{3} = (\{b\}, \{a, e\}, \{c, d\})$

4 From SETAF to NLP

Now we will provide a translation in the other direction, i.e., from SETAFs to NLPs. As in the previous section, this translation guarantees the equivalence between the semantics for NLPs and their counterpart for SETAFs.

Definition 12

Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a *SETAF*. For any argument $a \in \mathcal{A}$, we will assume $\mathcal{V}_a = \{V \mid V \text{ is a minimal set (w.r.t. set inclusion) such that for each <math>\mathcal{B} \in Att(a)$, there exists $b \in \mathcal{B} \cap V\}$. We define the associated *NLP P*_{\mathfrak{A}} as follows:

$$P_{\mathfrak{A}} = \{ a \leftarrow \mathtt{not} \ b_1, \ldots \mathtt{not} \ b_n \mid a \in \mathcal{A}, V \in \mathcal{V}_a \ and \ V = \{b_1, \ldots, b_n\} \}.$$

Example 5

Recall the SETAF \mathfrak{A} of Example 1 (it is the same as that in Fig 4b). The associated NLP $P_{\mathfrak{A}}$ is

$$\begin{array}{ll} d \leftarrow \mathtt{not} \ d & c \leftarrow \mathtt{not} \ c, \mathtt{not} \ d \\ a \leftarrow \mathtt{not} \ b & b \leftarrow \mathtt{not} \ a \\ c \leftarrow \mathtt{not} \ c, \mathtt{not} \ a & e \leftarrow \mathtt{not} \ e, \mathtt{not} \ b \end{array}$$

Notice that $P_{\mathfrak{A}}$ and the NLP P of Example 4 are the same. As it will be clear in the next section, this is not merely a coincidence. Besides, from Definition 12, it is clear that $HB_{P_{\mathfrak{A}}} = \mathcal{A}$. Consequently, when considering a SETAF \mathfrak{A} and its associated NLP $P_{\mathfrak{A}}$, the definition of the function $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}$ (resp. $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}$), which associates labellings with interpretations (resp. interpretations with labellings), will be simpler than the definition of $\mathcal{L}2\mathcal{I}_{P}$ (resp. $\mathcal{I}2\mathcal{L}_{P}$) presented in the previous section.

Definition 13 ($\mathcal{L}2\mathcal{I}_{\mathfrak{A}}$ and $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}$ Functions)

Let \mathfrak{A} be a *SETAF* and *P* be its associated *NLP*, $\mathcal{L}ab$ be the set of all labellings of \mathfrak{A} and $\mathcal{I}nt$ be the set of all the 3-valued interpretations of $P_{\mathfrak{A}}$. We introduce the functions

• $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}:\mathcal{L}ab\to\mathcal{I}nt$, in which

$$\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}) = \langle \mathtt{in}(\mathcal{L}), \mathtt{out}(\mathcal{L}) \rangle$$
.

Obviously $\overline{\operatorname{in}(\mathcal{L}) \cup \operatorname{out}(\mathcal{L})} = \operatorname{undec}(\mathcal{L})$.

• $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}: \mathcal{I}nt \to \mathcal{L}ab$, in which for $\mathcal{M} = \langle T, F \rangle \in \mathcal{I}nt$,

$$\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M}) = (T, F, \overline{T \cup F}).$$

In contrast with $\mathcal{L}2\mathcal{I}_P$ and $\mathcal{I}2\mathcal{L}_P$, the functions $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}$ and $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}$ are each other's inverse in the general case:

Theorem 8

Let $\mathfrak{A} = (A, Att)$ be a SETAF and $P_{\mathfrak{A}}$ its associated NLP.

- For any labelling \mathcal{L} of \mathfrak{A} , it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})) = \mathcal{L}$.
- For any interpretation \mathcal{I} of $P_{\mathfrak{A}}$, it holds $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})) = \mathcal{I}$.

A similar result to Theorem 5 also holds here:

Theorem 9

Let \mathfrak{A} be a SETAF and $P_{\mathfrak{A}}$ be its associated NLP. It holds

- \mathcal{L} is a complete labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$.
- \mathcal{M} is a partial stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of \mathfrak{A} .

From Theorem 9, we can ensure the equivalence between the semantics for NLP and their counterpart for SETAF:

Theorem 10

Let \mathfrak{A} be a *SETAF* and $P_{\mathfrak{A}}$ its associated *NLP*. It holds

- 1. \mathcal{L} is a grounded labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a well-founded model of $P_{\mathfrak{A}}$.
- 2. \mathcal{L} is a preferred labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a regular model of $P_{\mathfrak{A}}$.
- 3. \mathcal{L} is a stable labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a stable model of $P_{\mathfrak{A}}$.
- 4. \mathcal{L} is a semi-stable labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is an L-stable model of $P_{\mathfrak{A}}$.

The following result is a direct consequence of Theorems 8 and 10:

Corollary 11

Let \mathfrak{A} be a SETAF and $P_{\mathfrak{A}}$ its associated NLP. It holds

- 1. \mathcal{M} is a well-founded model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a grounded labelling of \mathfrak{A} .
- 2. \mathcal{M} is a regular model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a preferred labelling of \mathfrak{A} .
- 3. \mathcal{M} is a stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a stable labelling of \mathfrak{A} .
- 4. \mathcal{M} is an L-stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a semi-stable labelling of \mathfrak{A} .

Recalling the $SETAF \mathfrak{A}$ and its associated $P_{\mathfrak{A}}$ of Example 5, we obtain the expected equivalence results related to their semantics (see Table 1). In the next section, we will identify a class of NLPs in which the translation from a SETAF to an NLP (Definition 12) behaves as the inverse of the translation from an NLP to a SETAF (Definition 10).

5 On the relation between RFALPs and SETAFs

We will recall a particular kind of NLPs, called Redundancy-Free Atomic Logic Programs (RFALPs). From an RFALP P, we obtain its associated SETAF \mathfrak{A}_P via Definition 10; from \mathfrak{A}_P , we obtain its associated NLP $P_{\mathfrak{A}_P}$ via Definition 12. By following the other direction, from a SETAF \mathfrak{A} , we obtain its associated NLP $P_{\mathfrak{A}}$, and from $P_{\mathfrak{A}}$, its associated SETAF $\mathfrak{A}_{P_{\mathfrak{A}}}$. An important result mentioned in this section is that $P = P_{\mathfrak{A}_P}$ and $\mathfrak{A} = \mathfrak{A}_{P_{\mathfrak{A}}}$, i.e., the translation from an NLP to a SETAF and the translation from a SETAF to an NLP are each other's inverse. Next, we define RFALPs:

Definition 14 (RFALP (König et al. 2022))

We define a Redundancy-Free Atomic Logic Program (RFALP) P as an NLP such that

- 1. P is redundancy-free, i.e., $HB_P = \{head(r) \mid r \in P\}$ and if $c \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P$, there is no rule $c \leftarrow \text{not } c_1, \ldots, \text{not } c_{n'} \in P$ such that $\{c_1, \ldots, c_{n'}\} \subset \{b_1, \ldots, b_n\}$.
- 2. P is atomic, i.e., each rule has the form $c \leftarrow \text{not } b_1, \ldots, \text{not } b_n \ (n \geq 0)$.

Firstly, Proposition 12 sustains that for any SETAF \mathfrak{A} , its associated NLP $P_{\mathfrak{A}}$ will always be an RFALP:

Proposition 12

Let $\mathfrak{A} = (A, Att)$ be a SETAF and $P_{\mathfrak{A}}$ its associated NLP. It holds $P_{\mathfrak{A}}$ is an RFALP.

The following results guarantee that $\mathfrak{A} = \mathfrak{A}_{P_{\mathfrak{A}}}$ (Theorem 13) and $P = P_{\mathfrak{A}_P}$ (Theorem 14):

Theorem 13

Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a SETAF, $P_{\mathfrak{A}}$ its associated NLP and $\mathfrak{A}_{P_{\mathfrak{A}}}$ the associated SETAF of $P_{\mathfrak{A}}$. It holds that $\mathfrak{A} = \mathfrak{A}_{P_{\mathfrak{A}}}$.

Theorem 14

Let P be an RFALP, \mathfrak{A}_P its associated SETAF and $P_{\mathfrak{A}_P}$ the associated NLP of \mathfrak{A}_P . It holds that $P = P_{\mathfrak{A}_P}$.

Remark 1

Minimality is crucial to ensure that the translation from an NLP to a SETAF and the translation from a SETAF to an NLP are each other's inverse. If the minimality requirement in Definition 1 (and consequently in Definition 9) were dropped, any SETAF (among other combinations) in Fig. 5 could be a possible candidate to be the associated $SETAF \mathfrak{A}_P$ of the RFALP P

$$\begin{array}{ccc} c \leftarrow \mathtt{not} \ a, \mathtt{not} \ c & c \leftarrow \mathtt{not} \ b, \mathtt{not} \ c \\ a & b \end{array}$$

As a result, Theorem 13 would no longer hold, and these translations would not be each other's inverse. Notice also that the *SETAFs* in Fig. 5 have the same complete labellings as non-minimal attacks are irrelevant and can be ignored when determining semantics based on complete labellings.

Theorems 13 and 14 reveal that SETAFs and RFALPs are essentially the same formalism. The equivalence between them involves their semantics and is also structural:

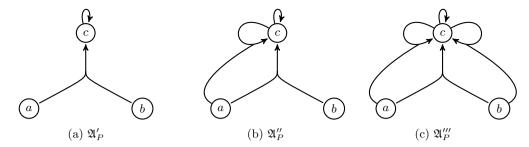


Fig. 5: Possible SETAFs associated with P

two distinct SETAFs will always be translated into two distinct RFALPs and vice versa. In contradistinction, Theorem 14 would not hold if we had replaced our translation from NLP to SETAF (Definition 10) with that from NLP to AAF presented in (Caminada et al. 2015b). Thus, the connection between NLPs and SETAFs is more robust than that between NLPs and AAFs. In the forthcoming section, we will explore how expressive RFALPs can be; we will ensure they are as expressive as NLPs.

6 On the Expressiveness of RFALPs

Dvorák et al. comprehensively characterised the expressiveness of SETAFs (Dvořák et al. 2019). Now we compare the expressiveness of NLPs with that of RFALPs. In the previous section, we established that SETAFs and RFALPs are essentially the same formalism. We demonstrated that from the SETAF \mathfrak{A}_P associated with an NLP P, we can obtain P; and conversely, from the NLP $P_{\mathfrak{A}}$ associated with a SETAF \mathfrak{A} , we can obtain \mathfrak{A} . Here, we reveal that this connection between SETAFs and RFALPs is even more substantial: RFALPs are as expressive as NLPs when considering the semantics for NLPs we have exploited in this paper. With this aim in mind, we transform any NLPP into an RFALP P^* by resorting to a specific combination (denoted by \mapsto_{UTPM}) of some program transformations proposed by Brass and Dix (Brass and Dix 1994; 1997; 1999). Although each program transformation in \mapsto_{UTPM} was proposed in (Brass and Dix 1994; 1997; 1999), the combination of these program transformations (as far as we know) has not been investigated yet. Then, we show that P and P^* share the same partial stable models. Since well-founded models, regular models, stable models, and L-stable models are all settled on partial stable models, it follows that both P and P^* also coincide under these semantics. Based on Dunne et al.'s work (Dunne et al. 2015), where they define the notion of expressiveness of the semantics for AAFs, we define formally expressiveness in terms of the signatures of the semantics for NLPs:

Definition 15 (Expressiveness)

Let \mathcal{P} be a class of NLPs. The signature $\Sigma_{PSM}^{\mathcal{P}}$ of the partial stable models associated with \mathcal{P} is defined as

$$\Sigma_{PSM}^{\mathcal{P}} = \left\{ \sigma(P) \mid P \in \mathcal{P} \right\},\,$$

where $\sigma(P) = \{ \mathcal{I} \mid \mathcal{I} \text{ is a partial stable model of } P \}$ is the set of all partial stable models of P.

Given two classes \mathcal{P}_1 and \mathcal{P}_2 of NLPs, we say that \mathcal{P}_1 and \mathcal{P}_2 have the same expressiveness for the partial stable models semantics if $\Sigma_{PSM}^{\mathcal{P}_1} = \Sigma_{PSM}^{\mathcal{P}_2}$

In other words, \mathcal{P}_1 and \mathcal{P}_2 have the same expressiveness if

- For every $P_1 \in \mathcal{P}_1$, there exists $P_2 \in \mathcal{P}_2$ such that P_1 and P_2 have the same set of partial stable models.
- For every $P_2 \in \mathcal{P}_2$, there exists $P_1 \in \mathcal{P}_1$ such that P_1 and P_2 have the same set of partial stable models.

Similarly, we can define when \mathcal{P}_1 and \mathcal{P}_2 have the same expressiveness for the well-founded, regular, stable, and L-stable semantics.

As the class of RFALPs is contained in the class of all NLPs, to show that these classes have the same expressiveness for these semantics, it suffices to prove that for every NLP, there exists an RFALP with the same set of partial stable models. We will obtain this result by resorting to a combination of program transformations:

Definition 16 (Program Transformation (Brass and Dix 1994; 1997; 1999))

A program transformation is any binary relation \mapsto between NLPs. By \mapsto^* we mean the reflexive and transitive closure of \mapsto .

Thus, $P \mapsto^* P'$ means that there is a finite sequence $P = P_1 \mapsto \cdots \mapsto P_n = P'$. We are particularly interested in program transformations preserving partial stable models:

Definition 17 (Equivalence Transformation (Brass and Dix 1994; 1997; 1999))

We say a program transformation \mapsto is a partial stable model equivalence transformation if for any NLPs P_1 and P_2 with $P_1 \mapsto P_2$, it holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

From Definitions 18 to 21, we focus on the following program transformations introduced in (Brass and Dix 1994; 1997; 1999): Unfolding (it is also known as Generalised Principle of Partial Evaluation (GPPE)), Elimination of Tautologies, Positive Reduction, and Elimination of Non-Minimal Rules. They are sufficient for our purposes.

Definition 18 (Unfolding (Brass and Dix 1994; 1997; 1999))

An NLP P_2 results from an NLP P_1 by unfolding $(P_1 \mapsto_U P_2)$ iff there exists a rule $c \leftarrow a, a_1, \ldots, a_m, \text{not } b_1, \ldots, \text{not } b_n \in P_1$ such that

$$\begin{split} P_2 &= (P_1 - \{c \leftarrow a, a_1, \dots, a_m, \mathtt{not}\ b_1, \dots, \mathtt{not}\ b_n\}) \\ &\qquad \qquad \cup \{c \leftarrow a_1', \dots, a_p', a_1, \dots, a_m, \mathtt{not}\ b_1', \dots, \mathtt{not}\ b_q', \mathtt{not}\ b_1, \dots, \mathtt{not}\ b_n\mid \\ &\qquad \qquad \qquad a \leftarrow a_1', \dots, a_p', \mathtt{not}\ b_1', \dots, \mathtt{not}\ b_q' \in P_1\}. \end{split}$$

Definition 19 (Elimination of Tautologies (Brass and Dix 1994; 1997; 1999)) An NLP P_2 results from an NLP P_1 by elimination of tautologies $(P_1 \mapsto_T P_2)$ iff there exists a rule $r \in P_1$ such that $head(r) \in body^+(r)$ and $P_2 = P_1 - \{r\}$.

Definition 20 (Positive Reduction (Brass and Dix 1994; 1999))

An NLP P_2 results from an NLP P_1 by positive reduction $(P_1 \mapsto_P P_2)$ iff there exists a rule $c \leftarrow a_1, \ldots, a_m, \text{not } b, \text{not } b_1, \ldots, \text{not } b_n \in P_1$ such that $b \notin \{head(r) \mid r \in P_1\}$ and

$$P_2 = (P_1 - \{c \leftarrow a_1, \dots, a_m, \text{not } b, \text{not } b_1, \dots, \text{not } b_n\})$$
$$\cup \{c \leftarrow a_1, \dots, a_m, \text{not } b_1, \dots, \text{not } b_n\}.$$

Definition 21 (Elimination of Non-Minimal Rules (Brass and Dix 1994; 1999)) An NLP P_2 results from an NLP P_1 by elimination of non-minimal rules $(P_1 \mapsto_M P_2)$ iff there are two distinct rules r and r' in P_1 such that head(r) = head(r'), $body^+(r') \subseteq body^+(r)$, $body^-(r') \subseteq body^-(r)$ and $P_2 = P_1 - \{r\}$.

Now we combine these program transformations and define \mapsto_{UTPM} as follows:

Definition 22 (Combined Transformation) Let $\mapsto_{UTPM} = \mapsto_U \cup \mapsto_T \cup \mapsto_P \cup \mapsto_M$.

We call an $NLP\ P$ irreducible concerning \mapsto if there is no $NLP\ P' \neq P$ with $P \mapsto^* P'$. Besides, we say \mapsto is strongly terminating iff every sequence of successive applications of \mapsto eventually leads to an irreducible NLP. As displayed in (Brass and Dix 1998), not every program transformation is strongly terminating. For instance, in the NLP

$$\begin{array}{lll} a & \leftarrow & b \\ b & \leftarrow & a \\ c & \leftarrow & a, \mathtt{not} \ c \end{array}$$

if we apply unfolding (\mapsto_U) to the third rule, this rule is replaced by $c \leftarrow b$, not c. We can now apply unfolding again to this rule and get the original program; such an oscillation can repeat indefinitely. Thus we have a sequence of program transformations that do not terminate. However, if we restrict ourselves to fair sequences of program transformations, the termination is guaranteed:

Definition 23 (Fair Sequences (Brass and Dix 1998)) A sequence of program transformations $P_1 \mapsto \cdots \mapsto P_n$ is fair with respect to \mapsto if

- Every positive body atom occurring in P_1 is eventually removed in some P_i with $1 < i \le n$ (either by removing the whole rule using a suitable program transformation or by an application of \mapsto_U);
- Every rule $r \in P_i$ such that $head(r) \in body^+(r)$ is eventually removed in some P_j with $i < j \le n$ (either by applying \mapsto_T or another suitable program transformation).

The sequence above of program transformations is not fair, because it does not remove the positive body atoms occurring in the program. In contrast, the sequence of program transformations given by

is not only fair but also terminates. The next result guarantees that it is not simply a coincidence:

Theorem 15

The relation \mapsto_{UTPM} is strongly terminating for fair sequences of program transformations, i.e., such fair sequences always lead to irreducible programs.

Theorem 15 is crucial to obtain the following result:

Theorem 16

For any NLP P, there exists an irreducible NLP P^* such that $P \mapsto_{UTPM}^* P^*$.

This means that from an NLP P, it is always possible to obtain an irreducible NLP P^* after successive applications of \mapsto_{UTPM} . Indeed, P^* is an RFALP:

Theorem 17

Let P be an NLP and P^* be an NLP obtained after applying repeatedly the program transformation \mapsto_{UTPM} until no further transformation is possible, i.e., $P \mapsto_{UTPM}^* P^*$ and P^* is irreducible. Then P^* is an RFALP.

From Theorems 15 and 17, we can infer that for fair sequences, after applying repeatedly \mapsto_{UTPM} , we will eventually produce an *RFALP*. In fact, every *RFALP* is irreducible:

Theorem 18

Let P be an RFALP. Then P is irreducible with respect to \mapsto_{UTPM} .

Theorems 16 and 17 guarantee that every $NLP\ P$ can be transformed into an $RFALP\ P^*$ by applying \mapsto_{UTPM} a finite number of times. It remains to show that P and P^* share the same partial stable models (and consequently, the same well-founded, regular, stable, and L-stable models). Before, however, note that \mapsto_{UTPM} does not introduce new atoms; instead, it can eliminate the occurrence of existing atoms in an NLP. For simplicity in notation, we assume throughout the rest of this section that $HB_P = HB_{P'}$ whenever $P \mapsto_{UTPM}^* P'$. Next, we recall that these program transformations preserve the least models of positive programs:

Lemma 19 ((Brass and Dix 1995; 1997))

Let P_1 and P_2 be positive programs such that $P_1 \mapsto_x P_2$, in which $x \in \{U, T, P, M\}$. It holds \mathcal{M} is the least model of P_1 iff \mathcal{M} is the least model of P_2 .

In the sequel, we aim to extend Lemma 19 to NLPs. Notice, however, that we already have the result for the program transformation \mapsto_U :

Theorem 20 ((Aravindan and Minh 1995))

Let P_1 and P_2 be NLPs such that $P_1 \mapsto_U P_2$. It holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

It remains to guarantee the result for the program transformation \mapsto_T , \mapsto_P and \mapsto_M :

Theorem 21

Let P_1 and P_2 be NLP_3 such that $P_1 \mapsto_T P_2$. It holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

Theorem 22

Let P_1 and P_2 be NLPs such that $P_1 \mapsto_P P_2$. It holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

Theorem 23

Let P_1 and P_2 be NLPs such that $P_1 \mapsto_M P_2$. It holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

Consequently, if $P_1 \mapsto_{UTPM} P_2$, then P_1 and P_2 share the same partial stable models. By repeatedly resorting to this result, we can even show that for any NLP, there exists an irreducible NLP with the same set of partial stable models, well-founded models, regular models, stable models, and L-stable models:

Theorem 24

Let P be an NLP and P^* be an irreducible NLP such that $P \mapsto_{UTPM}^* P^*$. It holds \mathcal{M} is a partial stable model of P iff \mathcal{M} is a partial stable model of P^* .

Corollary 25

Let P be an NLP and P^* be an irreducible NLP such that $P \mapsto_{UTPM}^* P^*$. It holds \mathcal{M} is a well-founded, regular, stable, L-stable model of P iff \mathcal{M} is respectively a well-founded, regular, stable, L-stable model of P^* .

As any irreducible *NLP* is an *RFALP* (Theorem 17), the following result is immediate:

Corollary 26

For any NLP P, there exists an $RFALP P^*$ such that \mathcal{M} is a partial stable, well-founded, regular, stable, L-stable model of P iff \mathcal{M} is respectively a partial stable, well-founded, regular, stable, L-stable model of P^* .

Given that each NLP can be associated with an RFALP preserving the semantics above, it follows that NLP and RFALPs have the same expressiveness for those semantics:

Theorem 27

NLPs and RFALPs have the same expressiveness for partial stable, well-founded, regular, stable, and L-stable semantics.

Another important result is that the *SETAF* corresponding to an *NLP* is invariant with respect to \mapsto_{UTPM} :

Theorem 28

For any NLPs P_1 and P_2 , if $P_1 \mapsto_{UTPM} P_2$, then $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$

This means that any NLP in a sequence of program transformations from \mapsto_{UTPM} has the same corresponding SETAF. For instance, every NLP in this sequence

leads to the same corresponding SETAF, constituted by a unique (unattacked) argument:



Theorem 28 also suggests an alternative way to find the SETAF corresponding to an NLP P: instead of resorting directly to Definition 8 to construct the arguments, we can apply (starting from P) \mapsto_{UTPM} successively by following a fair sequence of

program transformations. By Theorems 15 and 17, we know that eventually, we will reach an RFALP whose corresponding SETAF is identical to that of the original program P (Theorem 28). Then, we apply Definition 8 to this RFALP to obtain the arguments and Definition 9 for the attack relation. Notably, when P is an RFALP, Definition 8 becomes considerably simpler, requiring only its first item to characterise the statements.

In addition, from the same NLP, various fair sequences of program transformations can be conceived. Recalling the NLP

$$\begin{array}{lll} a & \leftarrow & b \\ b & \leftarrow & a \\ c & \leftarrow & a, \mathtt{not} \ c \end{array}$$

exploited above, we can design the following alternative fair sequence

This sequence produced the same RFALP as before; it is not a coincidence. Apart from being strongly terminating for fair sequences of program transformations, the relation \mapsto_{UTPM} has an appealing property; it is also confluent:

Theorem 29

The relation \mapsto_{UTPM} is confluent, i.e., for any $NLPs\ P$, P' and P'', if $P\mapsto_{UTPM}^* P'$ and $P\mapsto_{UTPM}^* P''$ and both P' and P'' are irreducible, then P'=P''.

By confluent \mapsto_{UTPM} , we mean that it does not matter the path we take by repeatedly applying \mapsto_{UTPM} , if it ends, it will always lead to the same irreducible NLP. In addition, as any irreducible NLP is an RFALP (Theorem 17), and the translations from SETAF to RFALPs and conversely, from RFALPs to SETAF are each other's inverse (Theorems 13 and 14), we obtain that two distinct SETAFs will always be associated with two distinct NLPs. The confluence of \mapsto_{UTPM} is of particular significance from the logic programming perspective as it guarantees that the ordering of the transformations in UTPM does not matter: we are free to choose always the "best" transformation, which maximally reduces the program. Consequently, Theorem 29 also sheds light on the search for efficient implementations in NLPs.

From the previous section, we know that the equivalence between SETAFs and RFALPs is not only of a semantic nature but also structural: two distinct SETAFs will always be translated into two distinct RFALPs and vice versa. Now we enhance our understanding of this result still more by establishing that

- RFALPs are as expressive as NLPs.
- The SETAF corresponding to an NLP is invariant with respect to \mapsto_{UTPM} , i.e., if $P_1 \mapsto_{UTPM} P_2$, then $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.
- Each $NLP\ P$ leads to a unique $RFALP\ P^*$ via the relation \mapsto_{UTPM} . Besides, P and P^* have the same partial stable, grounded, regular, stable, and L-stable models.

Beyond revealing the connections between SETAFs and NLPs, the results in this paper also enhance our understanding of NLPs themselves. To give a concrete example, let us

consider the following issue: in the sequence of program transformations in \mapsto_{UTPM} , atoms can be removed. Are these atoms underivable and set to false in the partial stable models of the program or true/undecided atoms can be removed in this sequence? Such questions can be answered by considering some results from Section 3 and the current section. In more formal terms, let P and P^* be NLPs such that $P\mapsto_{UTPM}^* P^*$ and P^* is irreducible. We have

- P^* is an RFALP (Theorem 17), and the set of atoms occurring in P^* is $\{head(r) \mid r \in P^*\}$ (Definition 14);
- $\mathcal{A}_{P^*} = \{head(r) \mid r \in P^*\}$ is the set of all arguments we can construct from P^* (Definition 8), and $\mathfrak{A}_P = \mathfrak{A}_{P^*}$ (Theorem 28), i.e., $\mathcal{A}_P = \mathcal{A}_{P^*}$;
- Thus c occurs in P, but does not occur in P^* iff there is no statement s constructed from P such that $\operatorname{Conc}(s) = c$. According to Corollary 2, $c \in F'$ for every interpretation \mathcal{I} with $\Omega_P(\mathcal{I}) = \langle T', F' \rangle$.

Consequently, every atom occurring in P, but not occurring in P^* is set to false in the least 3-valued model of each disjunct of P. In particular, they will be false in its partial stable models.

Supported by the findings presented in the current section, we can argue that SETAFs and RFALPs are essentially the same paradigm, and both are deeply connected with NLPs.

7 Conclusion and Future Works

This paper investigates the connections between frameworks with sets of attacking arguments (SETAFs) and Normal Logic Programs (NLPs). Building on the research in (Alcântara et al. 2019; Alcântara and Sá 2021), we employ the characterisation of the SETAF semantics in terms of labellings (Flouris and Bikakis 2019) to establish a mapping from NLPs to SETAFs (and vice versa). We further demonstrate the equivalence between partial stable, well-founded, regular, stable, and L-stable models semantics for NLPs and respectively complete, grounded, preferred, stable, and semi-stable labellings for SETAFs.

Our translation from NLPs to SETAFs offers a key advantage over the translation from NLPs to AAFs presented in (Caminada et al. 2015b). Our approach captures the equivalence between semi-stable labellings for SETAFs and L-stable models for NLPs. In addition, their translation is unable to preserve the structure of the NLPs. While an NLP can be translated to an AAF, recovering the original NLP from the corresponding AAF is generally not possible. In contradistinction, we have revisited a class of NLPs called Redundancy-Free Atomic Logic Programs (RFALPs). For RFALPs, the translations from NLPs to SETAFs, and from SETAFs to NLPs also preserve their structures as they are each other's inverse. Hence, when compared to the relationship between NLPs and AAFs, the relationship between NLPs and SETAFs is demonstrably more robust. It extends beyond semantics to encompass structural aspects.

Some of these results are not new as they have already been obtained independently in (König et al. 2022). In fact, their translation from *NLP*s to *SETAF*s and vice versa coincide with ours, and the structural equivalence between *RFALP*s and *SETAF*s has also been identified there. Notwithstanding, our proofs of these results stem from a signifi-

cantly distinct path as they are based on properties of argument labellings and are deeply rooted in works such as (Caminada et al. 2015b; Alcântara et al. 2019; Alcântara and Sá 2021). For instance, our equivalence results are settled on two important aspects:

- Properties involving the maximisation/minimisation of labellings adapted from (Caminada et al. 2015b) to deal with labellings for SETAFs.
- Again inspired in (Caminada et al. 2015b), we proposed a mapping from interpretations to labellings and a mapping from labellings to interpretations. We also showed that they are each other's inverse.

In contrast, in (König et al. 2022) the equivalence between the semantics is demonstrated in terms of extensions. They also have not tackled the controversy between semistable and L-stable, one of our leading motivations for developing this work.

In addition to showing this structural equivalence between RFALPs and SETAFs, we have also investigated the expressiveness of RFALPs. To demonstrate that they are as expressive as NLPs, we proved that any NLP can be transformed into an RFALP with the same partial stable models through repeated applications of the program transformation \mapsto_{UTPM} . It is worth noticing that \mapsto_{UTPM} results from the combination of the following program transformations presented in (Brass and Dix 1994; 1997; 1999): unfolding, elimination of tautologies, positive reduction, and elimination of non-minimal rules. In the course of our investigations, we also have obtained relevant findings as follows:

- RFALPs are irreducible with respect to \mapsto_{UTPM} : the application of \mapsto_{UTPM} to an RFALP will result in the same program.
- The mapping from NLPs to SETAFs is invariant with respect to the program transformation \mapsto_{UTPM} , i.e., if an $NLP\ P_2$ is obtained from an $NLP\ P_1$ via \mapsto_{UTPM} , then the SETAF corresponding to P_1 is the same corresponding to P_2 .
- The program transformation \mapsto_{UTPM} is confluent: any NLP will lead to a unique RFALP after repeatedly applying \mapsto_{UTPM} . Consequently, two distinct RFALPs will always be associated with two distinct NLPs.

In summary, RFALPs (which are as expressive as NLPs) and SETAFs are essentially the same formalism. Roughly speaking, we can consider a SETAF as a graphical representation of an RFALP, and an RFALP as a rule-based representation of a SETAF. Any change in one formalism is mirrored by a corresponding change in the other. Thus, SETAFs emerge as a natural candidate for representing argumentation frameworks corresponding to NLPs.

Regarding the significance and potential impact of our results, we highlight that by pursuing this line of research, one gains insight into what forms of non-monotonic reasoning can and cannot be represented by formal argumentation. In particular, by enlightening these connections between SETAFs and NLPs, many approaches, semantics and techniques naturally developed for the former may be applied to the latter, and vice versa. While SETAFs serve as an inspiration for defining RFALPs, the representation of NLPs as SETAFs is an alternative for intuitively visualising logic programs.

In addition, our results associated with the confluence of \mapsto_{UTPM} are of particular significance from the logic programming perspective as they guarantee that the ordering of the transformations in \mapsto_{UTPM} does not matter: we are free to choose always the

"best" transformation, which maximally reduces the program. Consequently, our paper also sheds light on the search for efficient implementations in NLPs.

Natural ramifications of this work include an in-depth analysis of other program transformations beyond those studied here and their impact on SETAF and argumentation in general. Given the close relationship between Argumentation and Logic Programming, a possible line of research is to investigate how Argumentation can benefit from these program transformations in the development of more efficient algorithms. The structural connection involving RFALPs and SETAFs gives rise to exploiting other extensions of Dung AAFs; in particular, we are interested in identifying which of them are robust enough to preserve the structure of logic programs. Along this same line of research, it is also our aim to study connections between extensions of NLPs (including their paraconsistent semantics) and Argumentation.

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Appendix A Proofs of Theorems

A.1 Theorems and Proofs from Section 3

Lemma 1

Let P be an NLP, $\mathcal{I} = \langle T, F \rangle$ an interpretation and $\Omega_P(\mathcal{I}) = \langle T', F' \rangle$ the least 3-valued model of $\frac{P}{\mathcal{I}}$. It holds

- (i) $c \in T'$ iff there exists a statement s constructed from P such that $\mathtt{Conc}(s) = c$ and $\mathtt{Vul}(s) \subseteq F$.
- (ii) $c \in F'$ iff for every statement s constructed from P such that $\mathtt{Conc}(s) = c$, we have $\mathtt{Vul}(s) \cap T \neq \emptyset$

Proof

- Proving that $c \in T'$ iff there exists a statement s constructed from P such that $\mathtt{Conc}(s) = c$ and $\mathtt{Vul}(s) \subseteq F$:
 - \Rightarrow Consider $\Psi_{\frac{P}{2}}^{\uparrow i} = \langle T_i, F_i \rangle$ for each $i \in \mathbb{N}$. It suffices to prove by induction on the value of i that if $c \in T_i$, then there exists a statement s constructed from P such that $\operatorname{Conc}(s) = c$ and $\operatorname{Vul}(s) \subseteq F$:
 - Basis. For i = 0, the result is trivial as $T_0 = \emptyset$.

- Step. Assume that for every $c' \in T_n$, there exists a statement s' constructed from P such that $\operatorname{Conc}(s') = c'$ and $\operatorname{Vul}(s') \subseteq F$. We will prove that if $c \in T_{n+1}$, there exists a statement s constructed from P such that $\operatorname{Conc}(s) = c$ and $\operatorname{Vul}(s) \subseteq F$: If $c \in T_{n+1}$, there exists a rule $c \leftarrow a_1, \ldots, a_m, \operatorname{not} b_1, \ldots, \operatorname{not} b_n (m \ge 0, n \ge 0) \in P$ such that $\{a_1, \ldots, a_m\} \subseteq T_n$ and $\{b_1, \ldots, b_n\} \subseteq F$. It follows via inductive step that for every $j \in \{1, \ldots, m\}$, there exists a statement s_j constructed from P such that $\operatorname{Conc}(s_j) = a_j$ and $\operatorname{Vul}(s_j) \subseteq F$. But then, we can construct from P a statement s with $\operatorname{Conc}(s) = c$ where $\operatorname{Vul}(s) = \operatorname{Vul}(s_1) \cup \cdots \cup \operatorname{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. This implies that $\operatorname{Vul}(s) \subseteq F$.
- \Leftarrow We will prove by structural induction on the construction of statements that for each statement s constructed from P such that $\mathtt{Vul}(s) \subseteq F$, it holds $\mathtt{Conc}(s) \in T'$:
 - Basis. Let s be a statement $c \leftarrow \text{not } b_1, \ldots, \text{not } b_n \ (n \geq 0)$ such that $\{b_1, \ldots, b_n\} = \text{Vul}(s) \subseteq F$. It follows the fact $c \in \frac{P}{\mathcal{I}}$. Then $c \in T'$.
 - Step. Assume $s_1, \ldots, s_m \ (m \geq 1)$ are arbitrary statements constructed from P such that for each $i \in \{1, \ldots, m\}$, if $\operatorname{Vul}(s_i) \subseteq F$, then $\operatorname{Conc}(s_i) \in T'$. We will prove that if s is a statement $c \leftarrow (s_1), \ldots, (s_m), \operatorname{not} b_1, \ldots, \operatorname{not} b_n \ (n \geq 0)$ constructed from P such that $\operatorname{Vul}(s) \subseteq F$, then $c \in T'$:

 Let s be such a statement. By Definition 8, there exists a rule $c \leftarrow a_1, \ldots, a_m, \operatorname{not} b_1, \ldots, \operatorname{not} b_n \in P$ such that $\operatorname{Conc}(s_i) = a_i$ for each $i \in \{1, \ldots, m\}$ and $\operatorname{Vul}(s) = \operatorname{Vul}(s_1) \cup \cdots \cup \operatorname{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. As $\operatorname{Vul}(s) \subseteq F$, we obtain $\{b_1, \ldots, b_n\} \subseteq F$ and $\operatorname{Vul}(s_i) \subseteq F$ for each $i \in \{1, \ldots, m\}$. By inductive hypothesis, it follows $\{a_1, \ldots, a_m\} \subseteq T'$. Then $c \in T'$.
- Proving that $c \in F'$ iff for every statement s constructed from P such that Conc(s) = c, we have $Vul(s) \cap T \neq \emptyset$:
 - \Rightarrow Firstly, we will prove by structural induction on the construction of statements that for each statement s constructed from P such that $\mathtt{Vul}(s) \cap T = \emptyset$, it holds $\mathtt{Conc}(s) \not\in F'$:
 - Basis. Let s be a statement $c \leftarrow \text{not } b_1, \ldots, \text{not } b_n \ (n \geq 0)$ such that $\{b_1, \ldots, b_n\} \cap T = \emptyset$. It follows the fact $c \in \frac{P}{\mathcal{I}}$ or $c \leftarrow \mathbf{u} \in \frac{P}{\mathcal{I}}$. Then $c \notin F'$.
 - Step. Assume $s_1, \ldots, s_m \ (m \geq 1)$ are arbitrary statements constructed from P such that for each $i \in \{1, \ldots, m\}$, if $\operatorname{Vul}(s_i) \cap T = \emptyset$, then $\operatorname{Conc}(s_i) \not\in F'$. We will prove that if s is a statement $c \leftarrow (s_1), \ldots, (s_m), \operatorname{not} b_1, \ldots, \operatorname{not} b_n \ (n \geq 0)$ constructed from P such that $\operatorname{Vul}(s) \cap T = \emptyset$, then $c \notin F'$:
 - Let s be such a statement. By Definition 8, there exists a rule $c \leftarrow a_1, \ldots, a_m, \text{not } b_1, \ldots, \text{not } b_n \in P$ such that $\text{Conc}(s_i) = a_i$ for each $i \in \{1, \ldots, m\}$ and $\text{Vul}(s) = \text{Vul}(s_1) \cup \cdots \cup \text{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. As $\text{Vul}(s) \cap T = \emptyset$, we obtain $\{b_1, \ldots, b_n\} \cap T = \emptyset$ and $\text{Vul}(s_i) \cap T = \emptyset$ for each $i \in \{1, \ldots, m\}$. By inductive hypothesis, it follows $\{a_1, \ldots, a_m\} \cap F' = \emptyset$. Then, $c \notin F'$.

Hence, if $c \in F'$, for every statement s constructed from P such that Conc(s) = c, we have $Vul(s) \cap T \neq \emptyset$.

 \Leftarrow Assume that for every statement s constructed from P such that $\mathtt{Conc}(s) = c$, we have $\mathtt{Vul}(s) \cap T \neq \emptyset$. The proof is by contradiction: suppose that $c \notin F'$.

Consider $\Psi_{\frac{P}{2}}^{\uparrow i} = \langle T_i, F_i \rangle$ for each $i \in \mathbb{N}$. It suffices to prove by induction on the value of i that if $c \notin F_i$, then there exists a statement s constructed from P such that $\mathtt{Conc}(s) = c$ and $\mathtt{Vul}(s) \cap T = \emptyset$:

- Basis. For i = 0, the result is trivial as $F_0 = HB_P$.
- Step. Assume that for every $c' \notin F_n$, there exists a statement s' constructed from P such that $\operatorname{Conc}(s') = c'$ and $\operatorname{Vul}(s') \cap T = \emptyset$. We will prove that if $c \notin F_{n+1}$, there exists a statement s constructed from P such that $\operatorname{Conc}(s) = c$ and $\operatorname{Vul}(s) \cap T = \emptyset$:

If $c \notin F_{n+1}$, there exists a rule $c \leftarrow a_1, \ldots, a_m, \text{not } b_1, \ldots, \text{not } b_n (m \geq 0, n \geq 0) \in P$ such that $\{a_1, \ldots, a_m\} \cap F_n = \emptyset$ and $\{b_1, \ldots, b_n\} \cap T = \emptyset$. It follows via inductive step that for every $j \in \{1, \ldots, m\}$, there exists a statement s_j constructed from P such that $\operatorname{Conc}(s_j) = a_j$ and $\operatorname{Vul}(s_j) \cap T = \emptyset$. But then, we can construct from P a statement s with $\operatorname{Conc}(s) = c$ where $\operatorname{Vul}(s) = \operatorname{Vul}(s_1) \cup \cdots \cup \operatorname{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. This implies that $\operatorname{Vul}(s) \cap T = \emptyset$.

Theorem 3

Let P be an NLP and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated SETAF. For any labelling \mathcal{L} of \mathfrak{A}_P , it holds $\mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L})) = \mathcal{L}$.

Proof

Let $c \in \mathcal{A}_P$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}) = \langle T, F \rangle$; there are three possibilities:

- $\mathcal{L}(c) = \text{in} \Rightarrow c \in T \Rightarrow \mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))(c) = \text{in}.$
- $\mathcal{L}(c) = \text{out} \Rightarrow c \in F \Rightarrow \mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))(c) = \text{out}.$
- $\mathcal{L}(c) = \mathtt{undec} \Rightarrow c \in \overline{T \cup F} \Rightarrow \mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))(c) = \mathtt{undec}.$

Theorem 4

Let P be an NLP, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated SETAF and $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of P. It holds that $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M})) = \mathcal{M}$.

Proof

Let $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of P, $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M})) = \langle T', F' \rangle$ and $c \in HB_P$. It suffices to prove the following results:

- $c \in T$ iff $c \in T'$.
 - Assume $c \in T$. As $\Omega_P(\mathcal{M}) = \mathcal{M}$, by Lemma 1, there exists a statement s with $\mathtt{Conc}(s) = c$ such that $\mathtt{Vul}(s) \subseteq F$. In particular, it follows that $c \in \mathcal{A}_P$. This implies $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) = \mathtt{in}$ and $c \in T'$.
 - Assume $c \in T'$. Then $c \in \mathcal{A}_P$ and $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) = \text{in.}$ From Definition 11, we obtain $c \in T$.
- $c \in F$ iff $c \in F'$.

- Assume $c \notin F'$. Then $c \in \mathcal{A}_P$ and $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) \neq \text{out.}$ From Definition 11, we obtain $c \notin F$.
- Assume $c \notin F$. As $\Omega_P(\mathcal{M}) = \mathcal{M}$, by Lemma 1, there exists a statement s with $\operatorname{Conc}(s) = c$ such that $\operatorname{Vul}(s) \cap T = \emptyset$. In particular, it follows that $c \in \mathcal{A}_P$. This implies $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) \neq \operatorname{out}$ and $c \notin F'$.

Lemma~30

Let P be an NLP, $\mathfrak{A}_P = (A_P, Att_P)$ be the associated SETAF and $v \in \{in, out, undec\}$. It holds that

- For each $\mathcal{B} \in Att(c)$, $\mathcal{L}(b) = v$ for some $b \in \mathcal{B}$ iff there exists $V \in Vul(c)$ such that $\mathcal{L}(b) = v$ for every $b \in \mathcal{A}_P \cap V$.
- For each $\mathcal{B} \in Att(c)$, $\mathcal{L}(b) \neq v$ for some $b \in \mathcal{B}$ iff there exists $V \in Vul(c)$ such that $\mathcal{L}(b) \neq v$ for every $b \in \mathcal{A}_P \cap V$.

Proof

We will prove the result in the first item; the proof of the other result follows a similar path:

 \Rightarrow Assume that for each $\mathcal{B} \in Att(c)$, $\mathcal{L}(b) = v$ for some $b \in \mathcal{B}$.

By absurd, suppose that for each $V \in Vul(c)$, it holds that $\mathcal{L}(b) \neq v$ for some $b \in \mathcal{A}_P \cap V$. Then we can construct a set $\mathcal{B}' \subseteq \mathcal{A}_P$ by selecting for each $V \in Vul(c)$, an element $b \in \mathcal{V}$ such that $\mathcal{L}(b) \neq v$. From Definition 9, we know that there exists $\mathcal{B} \subseteq \mathcal{B}'$ such that $(\mathcal{B}, c) \in Att_P$. But then, there exists $\mathcal{B} \in Att(c)$ such that $\mathcal{L}(b) \neq v$ for each $b \in \mathcal{B}$. It is absurd as it contradicts our hypothesis.

 \Leftarrow Assume that there exists $V \in Vul(c)$ such that $\mathcal{L}(b) = v$ for every $b \in \mathcal{A}_P \cap V$. The result is immediate as according to Definition 9, every set \mathcal{B} of arguments attacking c contains an element $b \in \mathcal{A}_P \cap V$.

Theorem 5

Let P be an NLP and $\mathfrak{A}_P = (A_P, Att_P)$ be the associated SETAF. It holds

- \mathcal{L} is a complete labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P.
- \mathcal{M} is a partial stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of \mathfrak{A}_P .

Proof

1. If \mathcal{L} is a complete labelling of \mathfrak{A}_P , then $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P:

Let $\mathcal{M} = \mathcal{L}2\mathcal{I}_P(\mathcal{L}) = \langle T, F \rangle$. We will show \mathcal{M} is a partial stable model of P, i.e., $\Omega_P(\mathcal{M}) = \langle T', F' \rangle = \langle T, F \rangle$:

• $c \in T$ iff $c \in \mathcal{A}_P$ and $\mathcal{L}(c) = \text{in}$ iff for each $\mathcal{B} \in Att(c)$, it holds $\mathcal{L}(b) = \text{out}$ for some $b \in \mathcal{B}$ iff (Lemma 30) there exists $V \in \text{Vul}(c)$ such that $\mathcal{L}(b) = \text{out}$ for every $b \in \mathcal{A}_P \cap V$ iff there exists a statement s with Conc(s) = c and $\text{Vul}(s) \subseteq F$ iff (Lemma 1) $c \in T'$.

- $c \notin F$ iff $c \in \mathcal{A}_P$ and $\mathcal{L}(c) \neq \text{out}$ iff for each $\mathcal{B} \in Att(c)$, it holds $\mathcal{L}(b) \neq \text{in}$ for some $b \in \mathcal{B}$ iff (Lemma 30) there exists $V \in \text{Vul}(c)$ such that $\mathcal{L}(b) \neq \text{in}$ for every $b \in \mathcal{A}_P \cap V$ iff there exists a statement s with Conc(s) = c and $\text{Vul}(s) \cap T = \emptyset$ iff (Lemma 1) $c \notin F'$.
- 2. If \mathcal{M} is a partial stable model of P, then $\mathcal{I}2\mathcal{L}_{P}(\mathcal{M})$ is a complete labelling of \mathfrak{A}_{P} :

Let $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of P. Then $\Omega_P(\mathcal{M}) = \langle T, F \rangle$. Let c be an argument in \mathcal{A}_P . We will prove $\mathcal{L} = \mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of \mathfrak{A}_P :

- $\mathcal{L}(c) = \text{in iff } c \in T \text{ iff (Lemma 1) there exists a statement } s \text{ with } \text{Conc}(s) = c \text{ and } \text{Vul}(s) \subseteq F \text{ iff there exists } V \in \text{Vul}(c) \text{ such that } \mathcal{L}(b) = \text{out for every } b \in \mathcal{A}_P \cap V \text{ iff (Lemma 30) for each } \mathcal{B} \in Att(c), \text{ it holds } \mathcal{L}(b) = \text{out for some } b \in \mathcal{B}.$
- $\mathcal{L}(c) \neq \text{out iff } c \neq F \text{ iff (Lemma 1) there exists a statement } s \text{ with } \text{Conc}(s) = c \text{ and } \text{Vul}(s) \cap T = \emptyset \text{ iff there exists } V \in \text{Vul}(c) \text{ such that } \mathcal{L}(b) \neq \text{in for every } b \in \mathcal{A}_P \cap V \text{ iff (Lemma 30) for each } \mathcal{B} \in Att(c), \text{ it holds } \mathcal{L}(b) \neq \text{in for some } b \in \mathcal{B}.$
- 3. If $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, then \mathcal{L} is a complete labelling of \mathfrak{A}_P :

It holds that $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of $P \Rightarrow$ according to item 2 above, $\mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))$ is a complete labelling of $\mathfrak{A}_P \Rightarrow$ (via Theorem 3) \mathcal{L} is a complete labelling of \mathfrak{A}_P .

4. If $\mathcal{I}2\mathcal{L}_{P}(\mathcal{M})$ is a complete labelling of \mathfrak{A}_{P} , then \mathcal{M} is a partial stable model of P:

It holds that $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of $\mathfrak{A}_P \Rightarrow$ according to item 1 above, $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M}))$ is a partial stable model of $P \Rightarrow$ (via Theorem 4) \mathcal{M} is a partial stable model of P.

Lemma 31

Let P be an NLP, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be its associated SETAF. Let \mathcal{L}_1 and \mathcal{L}_2 be β -complete labellings of \mathfrak{A}_P , and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_1) = \langle T_1, F_1 \rangle$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2 \rangle$. It holds

- 1. $\operatorname{in}(\mathcal{L}_1) \subseteq \operatorname{in}(\mathcal{L}_2)$ iff $T_1 \subseteq T_2$;
- 2. $\operatorname{in}(\mathcal{L}_1) = \operatorname{in}(\mathcal{L}_2)$ iff $T_1 = T_2$;
- 3. $in(\mathcal{L}_1) \subset in(\mathcal{L}_2)$ iff $T_1 \subset T_2$.

Proof

- 1.(\Rightarrow): Suppose $\operatorname{in}(\mathcal{L}_1) \subseteq \operatorname{in}(\mathcal{L}_2)$. If $c \in T_1$, by Definition 11, $c \in \mathcal{A}_P$ and $\mathcal{L}_1(A) = \operatorname{in}$. From our initial assumption, it follows $\mathcal{L}_2(c) = \operatorname{in}$. So, by Definition 11, $c \in T_2$.
 - (\Leftarrow): Suppose $T_1 \subseteq T_2$. If $\mathcal{L}_1(c) = \mathtt{in}$, by Definition 11, $c \in T_1$. From our initial assumption, it follows $c \in T_2$. So, by Definition 11, $\mathcal{L}_2(c) = \mathtt{in}$.
- 2. It follows directly from point 1.
- 3. It follows directly from points 1 and 2.

Let P be an NLP, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be its associated SETAF. Let \mathcal{L}_1 and \mathcal{L}_2 be complete labellings of \mathfrak{A}_P , and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_1) = \langle T_1, F_1 \rangle$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2 \rangle$. It holds

- 1. $\operatorname{out}(\mathcal{L}_1) \subseteq \operatorname{out}(\mathcal{L}_2)$ iff $F_1 \subseteq F_2$;
- 2. $\operatorname{out}(\mathcal{L}_1) = \operatorname{out}(\mathcal{L}_2)$ iff $F_1 = F_2$;
- 3. $\operatorname{out}(\mathcal{L}_1) \subset \operatorname{out}(\mathcal{L}_2)$ iff $F_1 \subset F_2$.

Proof

- 1. (\Rightarrow) : Suppose $\operatorname{out}(\mathcal{L}_1) \subseteq \operatorname{out}(\mathcal{L}_2)$. If $c \in F_1$, by Definition 11, there are two possibilities:
 - $c \notin \mathcal{A}_P$. As $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2 \rangle$, we obtain that $c \in F_2$.
 - $c \in \mathcal{A}_P$ and $\mathcal{L}_1(c) = \text{out}$. From our initial assumption, it follows $\mathcal{L}_2(c) = \text{out}$. So, by Definition 11, $c \in F_2$.
 - (\Leftarrow): Suppose $F_1 \subseteq F_2$. If $\mathcal{L}_1(c) = \text{out}$, by Definition 11, $c \in F_1$. From our initial assumption, it follows $c \in F_2$. So, by Definition 11, $\mathcal{L}_2(c) = \text{out}$.
- 2. It follows directly from point 1.
- 3. It follows directly from points 1 and 2.

Lemma~33

Let P be an NLP, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be its associated SETAF. Let \mathcal{L}_1 and \mathcal{L}_2 be complete labellings of \mathfrak{A}_P , and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_1) = \langle T_1, F_1 \rangle$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2 \rangle$. It holds

- 1. $\operatorname{undec}(\mathcal{L}_1) \subseteq \operatorname{undec}(\mathcal{L}_2)$ iff $\overline{T_1 \cup F_1} \subseteq \overline{T_2 \cup F_2}$;
- 2. $\operatorname{undec}(\mathcal{L}_1) = \operatorname{undec}(\mathcal{L}_2)$ iff $\overline{T_1 \cup F_1} = \overline{T_2 \cup F_2}$;
- 3. $\operatorname{undec}(\mathcal{L}_1) \subset \operatorname{undec}(\mathcal{L}_2)$ iff $\overline{T_1 \cup F_1} \subset \overline{T_2 \cup F_2}$.

Proof

- 1.(\Rightarrow): Suppose $\operatorname{undec}(\mathcal{L}_1) \subseteq \operatorname{undec}(\mathcal{L}_2)$. If $c \in \overline{T_1 \cup F_1}$, by Definition 11, $c \in \mathcal{A}_P$ and $\mathcal{L}_1(c) = \operatorname{undec}$. From our initial assumption, it follows $\mathcal{L}_2(c) = \operatorname{undec}$. So, by Definition 11, $c \in \overline{T_2 \cup F_2}$.
 - (\Leftarrow): Suppose $\overline{T_1 \cup F_1} \subseteq \overline{T_2 \cup F_2}$. If $\mathcal{L}_1(c) = \text{undec}$, by Definition 11, $c \in \overline{T_1 \cup F_1}$. From our initial assumption, it follows $c \in \overline{T_2 \cup F_2}$. So, by Definition 11, $\mathcal{L}_2(c) = \text{undec}$.
- 2. It follows directly from point 1.
- 3. It follows directly from points 1 and 2.

Theorem 6

Let P be an NLP and $\mathfrak{A}_P = (A_P, Att_P)$ be the associated SETAF. It holds

- 1. \mathcal{L} is a grounded labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a well-founded model of P.
- 2. \mathcal{L} is a preferred labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a regular model of P.
- 3. \mathcal{L} is a stable labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a stable model of P.
- 4. \mathcal{L} is a semi-stable labelling of \mathfrak{A}_P iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is an L-stable model of P.

Proof

Let \mathcal{L} be an argument labelling of \mathfrak{A}_P and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}) = \langle T, F \rangle$. The proof is straightforward:

- 1. \mathcal{L} is a grounded labelling of \mathfrak{A}_P iff \mathcal{L} is a complete labelling of \mathfrak{A}_P , and $in(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of \mathfrak{A}_P iff (Theorem 5 and Lemma 31) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of P such that $T' \subset T$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a well-founded model of P;
- 2. \mathcal{L} is a preferred labelling of \mathfrak{A}_P iff \mathcal{L} is a complete labelling of \mathfrak{A}_P , and $in(\mathcal{L})$ is maximal (w.r.t. set inclusion) among all complete labellings of \mathfrak{A}_P iff (Theorem 5 and Lemma 31) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of P such that $T \subset T'$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a regular model of P;
- 3. \mathcal{L} is a stable labelling of \mathfrak{A}_P iff \mathcal{L} is a complete labelling of \mathfrak{A}_P such that $\operatorname{undec}(\mathcal{L}) = \emptyset$ iff (Theorem 5) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model such that $\overline{T \cup F} = \emptyset$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a stable model of P;
- 4. \mathcal{L} is a semi-stable labelling of \mathfrak{A}_P iff \mathcal{L} is a complete labelling of \mathfrak{A}_P , and $undec(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of \mathfrak{A}_P iff (Theorem 5 and Lemma 33) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of P such that $\overline{T'} \cap \overline{F'} \subset \overline{T \cup F}$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is an L-stable model of P.

Corollary 7

Let P be an NLP and $\mathfrak{A}_P = (A_P, Att_P)$ be the associated SETAF. It holds

- 1. \mathcal{M} is a well-founded model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a grounded labelling of \mathfrak{A}_P .
- 2. \mathcal{M} is a regular model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a preferred labelling of \mathfrak{A}_P .
- 3. \mathcal{M} is a stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a stable labelling of \mathfrak{A}_P .
- 4. \mathcal{M} is an L-stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a semi-stable labelling of \mathfrak{A}_P .

Proof

These results come from Theorems 4 and 6.

A.2 Theorems and Proofs from Section 4

Theorem 8

Let $\mathfrak{A} = (A, Att)$ be a SETAF and $P_{\mathfrak{A}}$ its associated NLP.

- For any labelling \mathcal{L} of \mathfrak{A} , it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})) = \mathcal{L}$.
- For any interpretation \mathcal{I} of $P_{\mathfrak{A}}$, it holds $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})) = \mathcal{I}$.

Proof

Both results are immediate:

• Proving that for any labelling \mathcal{L} of \mathfrak{A} , it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})) = \mathcal{L}$: Let $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}) = \langle T, F \rangle$.

$$\mathcal{L}(a) = \mathtt{out} \Rightarrow a \in F \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}))(a) = \mathtt{out};$$

$$\mathcal{L}(a) = \mathtt{undec} \Rightarrow a \in \overline{T \cup F} \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}))(a) = \mathtt{undec}.$$

- Proving that for any interpretation \mathcal{I} of $P_{\mathfrak{A}}$, it holds $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})) = \mathcal{I}$. Let $\mathcal{I} = \langle T, F \rangle$ be an interpretation of $P_{\mathfrak{A}}$, and $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})) = \langle T', F' \rangle$. We will show T = T' and F = F':
 - $-a \in T \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})(a) = \text{in} \Rightarrow a \in T';$
 - $-a \in F \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})(a) = \mathsf{out} \Rightarrow a \in F';$
 - $--a \in \overline{T \cup F} \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})(a) = \mathtt{undec} \Rightarrow a \in \overline{T' \cup F'};$

Theorem 9

Let \mathfrak{A} be a SETAF and $P_{\mathfrak{A}}$ be its associated NLP. It holds

- \mathcal{L} is a complete labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$.
- \mathcal{M} is a partial stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of \mathfrak{A} .

Proof

1. Proving that if \mathcal{L} is a complete labelling of \mathfrak{A} , then $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$:

Let $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}) = \langle T, F \rangle$ and $\Omega_{P_{\mathfrak{A}}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})) = \langle T', F' \rangle$. It suffices to show $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a fixpoint of $\Omega_{P_{\mathfrak{A}}}$: T = T' and F = F'. For any argument $a \in \mathcal{A} = HB_{P_{\mathfrak{A}}}$, there are three possibilities:

- $a \in T$. Then $\mathcal{L}(a) = \text{in}$. From Definition 2, we know that for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{L}(b) = \text{out}$ for some $b \in \mathcal{B}$. It follows from Definition 12 that there exists $V \in \mathcal{V}_a$ such that $\mathcal{L}(b) = \text{out}$ for every $b \in V$. This means the fact $a \in \frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$, i.e., $a \in T'$.
- $a \in F$. Then $\mathcal{L}(a) = \text{out}$. From Definition 2, we know that there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{L}(b) = \text{in}$ for each $b \in \mathcal{B}$. It follows from Definition 12 that for each $V \in \mathcal{V}_a$, there exists $b \in V$ such that $\mathcal{L}(b) = \text{in}$. This means that there exists no rule for a in $\frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$, i.e., $a \in F'$.
- $a \in \overline{T \cup F}$. Then $\mathcal{L}(a) = \text{undec}$. From Definition 2, we know that (i) there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{L}(b) \neq \text{out}$ for each $b \in \mathcal{B}$, and (ii) for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{L}(b) \neq \text{in}$ for some $b \in \mathcal{B}$. It follows from Definition 12 that (i) there does not exist $V \in \mathcal{V}_a$ such that $\mathcal{L}(b) = \text{out}$ for every $b \in V$, and (ii) there exists $V \in \mathcal{V}_a$ such that $\mathcal{L}(b) \neq \text{in}$ for each $b \in V$. This means (i) the fact $a \notin \frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$, and (ii) there exists rule for a in $\frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$. Thus $body(r) = \mathbf{u}$ for any $r \in \frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$ such that head(r) = a, i.e., $a \in \overline{T' \cup F'}$.
- 2. Proving that if \mathcal{M} is a partial stable model of $P_{\mathfrak{A}}$, then $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of \mathfrak{A} :

Let $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of $P_{\mathfrak{A}}$. Thus \mathcal{M} is a fixpoint of $\Omega_{P_{\mathfrak{A}}}$, i.e., $\Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}$. We now prove $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of \mathfrak{A} . For any $a \in HB_{P_{\mathfrak{A}}} = \mathcal{A}$, there are three possibilities:

- $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(a) = \text{in. Then } a \in T. \text{ As } \Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}, \text{ the fact } a \in \frac{P_{\mathfrak{A}}}{\mathcal{M}}.$ This means that there exists a rule $a \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P_{\mathfrak{A}} \ (n \geq 0)$ such that $\{b_1, \ldots, b_n\} \subseteq F.$ It follows from Definition 12 that for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) = \text{out for some } b \in \mathcal{B};$
- $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(a) = \text{out.}$ Then $a \in F$. As $\Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}$, there exists no rule for a in $\frac{P_{\mathfrak{A}}}{\mathcal{M}}$. This means that for every rule $a \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P_{\mathfrak{A}} \ (n \geq 0)$, there exists $b_i \in T \ (1 \leq i \leq n)$. It follows from Definition 12 that there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) = \text{in for each } b \in \mathcal{B}$;
- $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(a) = \text{undec.}$ Then $a \in \overline{T \cup F}$. As $\Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}$, the fact $a \notin \frac{P_{\mathfrak{A}}}{\mathcal{M}}$, but there exists a rule r in $\frac{P_{\mathfrak{A}}}{\mathcal{M}}$ such that head(r) = a and $body(r) = \mathbf{u}$. This means that (i) for each rule $a \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P_{\mathfrak{A}} \ (n \geq 0)$, it holds $\{b_1, \ldots, b_n\} \not\subseteq F$, and (ii) there exists a rule $a \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P_{\mathfrak{A}} \ (n \geq 0)$ such that $\{b_1, \ldots, b_n\} \cap T = \emptyset$. It follows from Definition 12 that (i) there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) \neq \text{out for each } b \in \mathcal{B}$, and (ii) for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) \neq \text{in for some } b \in \mathcal{B}$.

Hence, $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of \mathfrak{A} .

- 3. Proving that if $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$, then \mathcal{L} is a complete labelling of \mathfrak{A} :
 - $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}} \Rightarrow$ according to item 2 above, $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}))$ is a complete labelling of $\mathfrak{A} \Rightarrow$ (Theorem 8) \mathcal{L} is a complete labelling of \mathfrak{A} .
- 4. Proving that if $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of \mathfrak{A} , then \mathcal{M} is a partial stable model of $P_{\mathfrak{A}}$:
 - $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is complete labelling of $\mathfrak{A} \Rightarrow$ according to item 1 above, $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M}))$ is a partial stable model of $P_{\mathfrak{A}} \Rightarrow$ (Theorem 8) \mathcal{M} is a partial stable model of $P_{\mathfrak{A}}$.

Theorem 10

Let \mathfrak{A} be a *SETAF* and $P_{\mathfrak{A}}$ its associated *NLP*. It holds

- 1. \mathcal{L} is a grounded labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a well-founded model of $P_{\mathfrak{A}}$.
- 2. \mathcal{L} is a preferred labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a regular model of $P_{\mathfrak{A}}$.
- 3. \mathcal{L} is a stable labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a stable model of $P_{\mathfrak{A}}$.
- 4. \mathcal{L} is a semi-stable labelling of \mathfrak{A} iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is an L-stable model of $P_{\mathfrak{A}}$.

Proof

Let \mathcal{L} be an argument labelling of \mathfrak{A} . Recall that $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}) = \langle \mathtt{in}(\mathcal{L}), \mathtt{out}(\mathcal{L}) \rangle$. The proof is straightforward:

- 1. \mathcal{L} is a grounded labelling of \mathfrak{A} iff \mathcal{L} is a complete labelling of \mathfrak{A} and $in(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of \mathfrak{A} iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of $P_{\mathfrak{A}}$ such that $T' \subset in(\mathcal{L})$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a well-founded model of $P_{\mathfrak{A}}$;
- 2. \mathcal{L} is a preferred labelling of \mathfrak{A} iff \mathcal{L} is a complete labelling of \mathfrak{A} and $in(\mathcal{L})$ is maximal (w.r.t. set inclusion) among all complete labellings of \mathfrak{A} iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a

partial stable model of $P_{\mathfrak{A}}$ and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of P such that $in(\mathcal{L}) \subset T'$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a regular model of $P_{\mathfrak{A}}$;

- 3. \mathcal{L} is a stable labelling of \mathfrak{A} iff \mathcal{L} is a complete labelling of \mathfrak{A} such that $\underline{\mathsf{undec}(\mathcal{L})} = \emptyset$ iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ such that $\underline{\mathsf{in}(\mathcal{L})} \cup \mathsf{out}(\mathcal{L}) = \emptyset$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a stable model of $P_{\mathfrak{A}}$;
- 4. \mathcal{L} is a semi-stable labelling of \mathfrak{A} iff \mathcal{L} is a complete labelling of \mathfrak{A} and $\operatorname{undec}(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of \mathfrak{A} iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of $P_{\mathfrak{A}}$ such that $\overline{T' \cup F'} \subset \operatorname{in}(\mathcal{L}) \cup \operatorname{out}(\mathcal{L})$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is an L-stable model of $P_{\mathfrak{A}}$.

Corollary 11

Let \mathfrak{A} be a SETAF and $P_{\mathfrak{A}}$ its associated NLP. It holds

- 1. \mathcal{M} is a well-founded model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a grounded labelling of \mathfrak{A} .
- 2. \mathcal{M} is a regular model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a preferred labelling of \mathfrak{A} .
- 3. \mathcal{M} is a stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a stable labelling of \mathfrak{A} .
- 4. \mathcal{M} is an L-stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a semi-stable labelling of \mathfrak{A} .

Proof

These results come from Theorems 8 and 10.

A.3 Theorems and Proofs from Section 5

Proposition 12

Let $\mathfrak{A} = (A, Att)$ be a SETAF and $P_{\mathfrak{A}}$ its associated NLP. It holds $P_{\mathfrak{A}}$ is an RFALP.

Proof

It follows that

- 1. Each rule in $P_{\mathfrak{A}}$ has the form $a \leftarrow \text{not } b_1, \ldots, \text{not } b_n$;
- 2. for each rule $a \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P_{\mathfrak{A}}$, if $b \in \{b_1, \ldots, b_n\}$, there exists $(\mathcal{B}, a) \in Att$ such that $b \in \mathcal{B}$, i.e., $b \in \mathcal{A}_P$. Then there exists a rule $r \in P_{\mathfrak{A}}$ such that b = head(r). This suffices to guarantee $HB_{P_{\mathfrak{A}}} = \{head(r) \mid r \in P_{\mathfrak{A}}\}$;
- 3. A rule $a \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P_{\mathfrak{A}}$ iff there exists a minimal set (w.r.t. set inclusion) $V = \{b_1, \ldots, b_n\}$ such that for each $\mathcal{B} \in Att(a)$, there exists $b \in \mathcal{B} \cap V$. This means there exists no rule $a \leftarrow \text{not } c_1, \ldots, \text{not } c_{n'} \in P_{\mathfrak{A}}$ such that $\{c_1, \ldots, c_{n'}\} \subset \{b_1, \ldots, b_n\}$.

ŀ.	lence,	$P_{\mathfrak{N}}$	is	an	RFA	LP	

Lemma 34

Let P be an RFALP, $Head_P = \{head(r) \mid r \in P\}$ and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ its corresponding SETAF. It holds $Head_P = \mathcal{A}_P$.

Proof

The result is straightforward: $c \in Head_P$ iff there exists a rule $c \leftarrow \text{not } b_1, \dots, \text{not } b_n \in P$ $(n \ge 0)$ iff $c \in A_P$ (Definition 8).

Theorem 13

Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a SETAF, $P_{\mathfrak{A}}$ its associated NLP and $\mathfrak{A}_{P_{\mathfrak{A}}}$ the associated SETAF of $P_{\mathfrak{A}}$. It holds that $\mathfrak{A} = \mathfrak{A}_{P_{\mathfrak{A}}}$.

Proof

Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a SETAF with $\mathcal{A} = \{a_1, \ldots, a_n\}$ and for each $a_i \in \mathcal{A}$, we define $R_i = \{r \in P_{\mathfrak{A}} \mid head(r) = a_i\}$, i.e., $P_{\mathfrak{A}} = R_1 \cup R_2 \cup \cdots \cup R_n$. It follows from Proposition 12 and Lemma 34 that $\mathfrak{A}_{P_{\mathfrak{A}}} = (\mathcal{A}_{P_{\mathfrak{A}}}, Att_{P_{\mathfrak{A}}})$ with $\mathcal{A}_{P_{\mathfrak{A}}} = \{a_1, \ldots, a_n\} = \mathcal{A}$. It remains to prove that $Att = Att_{P_{\mathfrak{A}}}$:

 $(\mathcal{B}, a_j) \in Att$ iff $(\mathcal{B}, a_j) \in Att$ and there exists no $\mathcal{B}' \subset \mathcal{B}$ such that $(\mathcal{B}', a_j) \in Att$ iff \mathcal{B} is a minimal set (w. r. t. set inclusion) in which for each rule $r \in R_j$, there exists $b \in \mathcal{B}$ such that **not** $b \in body^-(r)$ iff \mathcal{B} is a minimal set (w. r. t. set inclusion) in which for each $V \in Vul(a_j)$, there exists $b \in \mathcal{B} \cap V$ iff $(\mathcal{B}, a_j) \in Att_{P_{\mathfrak{A}}}$.

Lemma 35

Let P be an RFALP, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ the corresponding SETAF and $c \in \mathcal{A}_P$. If $\{a_1, \ldots, a_n\}$ is a minimal set such that for each $\mathcal{B} \in Att_P(c)$, there exists $a_i \in \mathcal{B}$ $(1 \leq i \leq n)$, then $c \leftarrow \text{not } a_1, \ldots, \text{not } a_n \in P$.

Proof

As for each $\mathcal{B} \in Att_P(c)$, there exists $a_i \in \mathcal{B}$ $(1 \leq i \leq n)$, it follows from Definition 9 that there exists $V \in Vul(c)$ such that $V \subseteq \{a_1, \ldots, a_n\}$. Note that for each $\mathcal{B} \in Att_P(c)$, there exists $b \in V \cap \mathcal{B}$. As $\{a_1, \ldots, a_n\}$ is a minimal set with this property, it holds $V = \{a_1, \ldots, a_n\}$. Then (Definition 8) $c \leftarrow \text{not } a_1, \ldots, \text{not } a_n \in P$.

Theorem 14

Let P be an RFALP, \mathfrak{A}_P its associated SETAF and $P_{\mathfrak{A}_P}$ the associated NLP of \mathfrak{A}_P . It holds that $P = P_{\mathfrak{A}_P}$.

Proof

Let P be an RFALP with $HB_P = \{a_1, \ldots, a_n\}$, and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ the corresponding SETAF. For each $a_i \in HB_P$ $(1 \leq i \leq n)$, we define $R_i = \{r \in P_{\mathfrak{A}} \mid head(r) = a_i\}$. It follows that $\mathcal{A}_P = \{a_1, \ldots, a_n\}$. Hence, $HB_{P_{\mathfrak{A}_P}} = \{a_1, \ldots, a_n\}$. We will prove $P = P_{\mathfrak{A}_P}$:

- If $a_i \leftarrow \text{not } a_{i_1}, \ldots, \text{not } a_{i_m} \in P$. then $a_i \in \mathcal{A}_P$ and $\{a_{i_1}, \ldots, a_{i_m}\}$ is a minimal set (w.r.t. set inclusion) such that for each $\mathcal{B} \in Att_P(a_i)$, there exists $a_{i_k} \in \mathcal{B}$ $(k \in \{1, \ldots, m\})$. This implies (Definition 12) $a_i \leftarrow \text{not } a_{i_1}, \ldots, \text{not } a_{i_m} \in P_{\mathfrak{A}_P}$.
- If $a_i \leftarrow \text{not } a_{i_1}, \ldots, \text{not } a_{i_m} \in P_{\mathfrak{A}_P}$, then (Definition 12) $\{a_{i_1}, \ldots, a_{i_m}\}$ is a minimal set (w.r.t. set inclusion) such that for each $\mathcal{B} \in Att_P(a_i)$, there exists $a_{i_k} \in \mathcal{B}$ $(k \in \{1, \ldots, m\})$. Thus (Lemma 35) $a_i \leftarrow \text{not } a_{i_1}, \ldots, \text{not } a_{i_m} \in P$.

A.4 Theorems and Proofs from Section 6

Theorem 15

The relation \mapsto_{UTPM} is strongly terminating for fair sequences of program transformations, i.e., such fair sequences always lead to irreducible programs.

Proof

Let $P_1 \mapsto_{UTPM} P_2 \mapsto_{UTPM} \cdots \mapsto_{UTPM} P_k \mapsto_{UTPM} \cdots \mapsto_{UTPM} P_{k'} \mapsto_{UTPM} \cdots$ be a fair sequence of \mapsto_{UTPM} . This fairness condition implies that for every atom a, there exists a natural number k such that for each NLP P_i with i > k in the sequence of \mapsto_{UTPM} above, it holds $a \notin body^+(r)$ for each $r \in P_i$. As each NLP is a finite set of rules, from some natural number k' on, $body^+(r) = \emptyset$ for any $r \in P_{k'}$. Then for each $k'' \geq k'$, \mapsto_{U} and \mapsto_{T} cannot be applied in $P_{k''}$. It remains the program transformations \mapsto_{P} and \mapsto_{M} . For each of these $P_{k''}$, there are two possibilities:

- \mapsto_M strictly decreases the number of rules of $P_{k''}$ or
- \mapsto_P strictly decreases the number of negative literals in $body^-(r)$ for some $r \in P_{k''}$.

It follows that the successive application of \mapsto_M or \mapsto_P in these $P_{k''}$ s will eventually lead to an irreducible NLP.

Theorem 16

For any NLP P, there exists an irreducible NLP P^* such that $P \mapsto_{UTPM}^* P^*$.

Proof

A simple method to obtain a fair sequence of program transformations with respect to \mapsto_{UTPM} is to apply \mapsto_{U} to a rule r only if \mapsto_{T} is not applicable to r and to ensure that whenever \mapsto_{U} has been applied to get rid of an occurrence of an atom a, then all such occurrences of a (in other rules of the same program) have also been removed before applying \mapsto_{U} to another occurrence of an atom $b \neq a$.

As for any $NLP\ P$, it is always possible to build such a fair sequence of program transformations with respect to \mapsto_{UTPM} , we obtain from Theorem 15 that there exists an irreducible $NLP\ P^*$ such that $P\mapsto_{UTPM}^* P^*$.

Theorem 17

Let P be an NLP and P^* be an NLP obtained after applying repeatedly the program transformation \mapsto_{UTPM} until no further transformation is possible, i.e., $P \mapsto_{UTPM}^* P^*$ and P^* is irreducible. Then P^* is an RFALP.

Proof

To prove it by contradiction, suppose P^* is not an *RFALP*. There are three possibilities:

- A rule $c \leftarrow a_1, \ldots, a_m, \text{not } b_1, \ldots, \text{not } b_n \in P^* \text{ with } m \geq 1 \text{ and } n \geq 0.$ Then
 - The program transformation \mapsto_U (unfolding) can be applied.
 - If $c \in \{a_1, \ldots, a_m\}$, the program transformation \mapsto_T (elimination of tautologies) can be applied.
- A rule $c \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P^*$, but there exists $b \in \{b_1, \ldots, b_n\}$ such that $b \not\in \{head(r) \mid r \in P^*\}$. Then the program transformation \mapsto_P (positive reduction) can be applied.

• A rule $c \leftarrow \text{not } b_1, \ldots, \text{not } b_n \in P^*$ and there is a rule $c \leftarrow \text{not } c_1, \ldots, \text{not } c_p \in P^*$ such that $\{c_1, \ldots, c_p\} \subset \{b_1, \ldots, b_n\}$. Then the program transformation \mapsto_M (elimination of non-minimal rules) can be applied.

It is absurd as in each case, there is still a program transformation to be applied. \Box

Theorem 18

Let P be an RFALP. Then P is irreducible with respect to \mapsto_{UTPM} .

Proof

Let P be an RFALP. It holds

- The program transformations \mapsto_U and \mapsto_T cannot be applied as they require a rule $c \leftarrow a_1, \ldots, a_m, \text{not } b_1, \ldots, \text{not } b_n \text{ in } P \text{ with } m \geq 1.$
- The program transformation \mapsto_P cannot be applied as it requires a rule $c \leftarrow a_1, \ldots, a_m, \text{not } b, \text{not } b_1, \ldots, \text{not } b_n$ in P such that $b \notin \{head(r) \mid r \in P\}$, but $\{head(r) \mid r \in P\} = HB_P$.
- The program transformation \mapsto_M cannot be applied as it requires two distinct rules r and r' in P such that head(r) = head(r') and $body^-(r') \subset body^-(r)$.

Theorem 21

Let P_1 and P_2 be NLP_3 such that $P_1 \mapsto_T P_2$. It holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

Proof

Let $P_2 = P_1 - \{r\}$ and $head(r) \in body^+(r)$. We have to show for any interpretation $\mathcal{M} = \langle T, F \rangle$, it holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 ; we distinguish two cases:

- $\{a \mid \text{not } a \in body^-(r)\} \cap T \neq \emptyset$: Then $\frac{P_1}{\mathcal{M}} = \frac{P_2}{\mathcal{M}}$. This trivially implies that \mathcal{M} is a partial stable model of P_1 iff it is a partial stable model of P_2 .
- $\{a \mid \text{not } a \in body^-(r)\} \cap T = \emptyset$: Then it is clear $\frac{P_1}{\mathcal{M}} \mapsto_T \frac{P_2}{\mathcal{M}}$. As both $\frac{P_1}{\mathcal{M}}$ and $\frac{P_2}{\mathcal{M}}$ are positive programs, according to Lemma 19, it holds \mathcal{M} is the least model of $\frac{P_1}{\mathcal{M}}$ iff \mathcal{M} is the least model of $\frac{P_2}{\mathcal{M}}$. Hence, \mathcal{M} is a partial stable model of P_1 iff it is a partial stable model of P_2 .

Theorem 22

Let P_1 and P_2 be NLPs such that $P_1 \mapsto_P P_2$. It holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

Proof Let

$$P_2 = P_1 - \{c \leftarrow a_1, \dots, a_m, \text{not } b, \text{not } b_1, \dots, \text{not } b_n\}$$

$$\cup \{c \leftarrow a_1, \dots, a_m, \text{not } b_1, \dots, \text{not } b_n\}$$

such that r is the rule $c \leftarrow a_1, \ldots, a_m, \text{not } b, \text{not } b_1, \ldots, \text{not } b_n \in P_1 \text{ and } b \notin \{head(r') \mid r' \in P_1\}$. We have to show that for any interpretation $\mathcal{M} = \langle T, F \rangle$, it holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 ; we distinguish two cases:

- $(\{a \mid \text{not } a \in body^-(r)\} \{b\}) \cap T \neq \emptyset \text{ or } b \in F$: Then $\frac{P_1}{\mathcal{M}} = \frac{P_2}{\mathcal{M}}$. This trivially implies that \mathcal{M} is a partial stable model of P_1 iff it is a partial stable model of P_2 .
- $(\{a \mid \text{not } a \in body^-(r)\} \{b\}) \cap T = \emptyset \text{ and } b \notin F. \text{ Let } \langle T_1, F_1 \rangle \text{ and } \langle T_2, F_2 \rangle \text{ be respectively the least models of } \frac{P_1}{\mathcal{M}} \text{ and } \frac{P_2}{\mathcal{M}}. \text{ As } b \notin \{head(r') \mid r' \in P_1\}, \text{ it is clear that } b \in F_1 \text{ and } b \in F_2. \text{ Given that } b \notin F, \text{ we obtain } \mathcal{M} = \langle T, F \rangle \text{ is different from both } \langle T_1, F_1 \rangle \text{ and } \langle T_2, F_2 \rangle. \text{ Hence, } \mathcal{M} \text{ is neither a partial stable model of } P_1 \text{ nor of } P_2. \text{ This implies that } \mathcal{M} \text{ is a partial stable model of } P_1 \text{ iff it is a partial stable model of } P_2.$

Theorem 23

Let P_1 and P_2 be NLP_3 such that $P_1 \mapsto_M P_2$. It holds \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 .

Proof

Suppose that there are two distinct rules r and r' in P_1 such that head(r) = head(r'), $body^+(r') \subseteq body^+(r)$, $body^-(r') \subseteq body^-(r)$ and $P_2 = P_1 - \{r\}$. We have to show that for any interpretation $\mathcal{M} = \langle T, F \rangle$, it holds that \mathcal{M} is a partial stable model of P_1 iff \mathcal{M} is a partial stable model of P_2 ; we distinguish two cases:

- $\{a \mid \text{not } a \in body^-(r)\} \cap T \neq \emptyset \text{ or } (\{a \mid \text{not } a \in body^-(r)\} \cap T = \emptyset \text{ and } body^+(r) = body^+(r'))$: Then $\frac{P_1}{\mathcal{M}} = \frac{P_2}{\mathcal{M}}$. This trivially implies that \mathcal{M} is a partial stable model of P_1 iff it is a partial stable model of P_2 .
- $\{a \mid \text{not } a \in body^-(r)\} \cap T = \emptyset \text{ and } body^+(r') \subset body^+(r)$: Then it is clear that $\frac{P_1}{\mathcal{M}} \mapsto_M \frac{P_2}{\mathcal{M}}$. As both $\frac{P_1}{\mathcal{M}}$ and $\frac{P_2}{\mathcal{M}}$ are positive programs, according to Lemma 19, it holds that \mathcal{M} is the least model of $\frac{P_1}{\mathcal{M}}$ iff \mathcal{M} is least model $\frac{P_2}{\mathcal{M}}$. Hence, \mathcal{M} is a partial stable model of P_1 iff it is a partial stable model of P_2 .

Theorem 24

Let P be an NLP and P^* be an irreducible NLP such that $P \mapsto_{UTPM}^* P^*$. It holds \mathcal{M} is a partial stable model of P iff \mathcal{M} is a partial stable model of P^* .

Proof

If $P \mapsto_{UTPM}^* P^*$, then there exists a finite sequence of program transformations $P = P_1 \mapsto_{UTPM} \cdots \mapsto_{UTPM} P_n = P^*$. According to Theorems 20, 21, 22 and 23, \mathcal{M} is a partial stable model of P_i iff \mathcal{M} is a partial stable model of P_{i+1} with $1 \leq i < n$. Thus by transitivity, \mathcal{M} is a partial stable model of P iff \mathcal{M} is a partial stable model of P^* .

Corollary 25

Let P be an NLP and P^* be an irreducible NLP such that $P \mapsto_{UTPM}^* P^*$. It holds \mathcal{M} is a well-founded, regular, stable, L-stable model of P iff \mathcal{M} is respectively a well-founded, regular, stable, L-stable model of P^* .

Proof

As P and P^* share the same set of partial stable models (Theorem 24), the result is straightforward.

Corollary 26

For any NLPP, there exists an $RFALPP^*$ such that \mathcal{M} is a partial stable, well-founded, regular, stable, L-stable model of P iff \mathcal{M} is respectively a partial stable, well-founded, regular, stable, L-stable model of P^* .

Proof

From Theorem 16, we know that for any $NLP\ P$, there exists an irreducible $NLP\ P^*$ such that $P\mapsto_{UTPM}^* P^*$. From Theorem 17, we obtain P^* is an RFALP. Besides, from Theorem 24 and Corollary 25, we infer \mathcal{M} is a partial stable, well-founded, regular, stable, L-stable model of P iff \mathcal{M} is respectively a partial stable, well-founded, regular, stable, L-stable model of P^* .

Theorem 27

NLPs and RFALPs have the same expressiveness for partial stable, well-founded, regular, stable, and L-stable semantics.

Proof

We have

- For any $NLP\ P$, there exists an $RFALP\ P^*$ such that $\mathcal M$ is a partial stable, well-founded, regular, stable, L-stable model of P iff $\mathcal M$ is respectively a partial stable, well-founded, regular, stable, L-stable model of P^* (Corollary 26).
- Obviously, any RFALP is an NLP.

Hence, NLPs and RFALPs have the same expressiveness for partial stable, well-founded, regular, stable and L-stable semantics.

Lemma 36

Let P_1 and P_2 be NLP_3 such that $P_1 \mapsto_U P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof

Let P_1 and P_2 be NLP_3 such that

$$\begin{split} P_2 = & P_1 - \{c \leftarrow a, a_1, \dots, a_m, \text{not } b_1, \dots, \text{not } b_n\} \\ & \cup \{c \leftarrow a'_1, \dots, a'_p, a_1, \dots, a_m, \text{not } b'_1, \dots, \text{not } b'_q, \text{not } b_1, \dots, \text{not } b_n \mid \\ & a \leftarrow a'_1, \dots, a'_p, \text{not } b'_1, \dots, \text{not } b'_q \in P_1\}, \end{split}$$

$$\mathfrak{A}_{P_1} = (\mathcal{A}_{P_1}, Att_{P_1})$$
 and $\mathfrak{A}_{P_2} = (\mathcal{A}_{P_2}, Att_{P_2})$. Note that

• For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that $\mathtt{Conc}(s) = \mathtt{Conc}(s')$, and $\mathtt{Vul}(s) = \mathtt{Vul}(s')$.

• For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that Conc(s') = Conc(s), and Vul(s') = Vul(s).

Hence,
$$A_{P_1} = A_{P_2}$$
, and $Att_{P_1} = Att_{P_2}$.

Lemma~37

Let P_1 and P_2 be NLP_3 such that $P_1 \mapsto_T P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof

Let $P_2 = P_1 - \{r\}$, where there exists a rule $r \in P_1$ such that $head(r) \in body^+(r)$. In addition, let $\mathfrak{A}_{P_1} = (\mathcal{A}_{P_1}, Att_{P_1})$ and $\mathfrak{A}_{P_2} = (\mathcal{A}_{P_2}, Att_{P_2})$. Note that

- For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that $\mathtt{Conc}(s) = \mathtt{Conc}(s')$, and for each $V \in \mathtt{Vul}(s)$, there exists $V' \in \mathtt{Vul}(s) \cap \mathtt{Vul}(s')$ such that $V' \subseteq V$.
- For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that Conc(s') = Conc(s), and Vul(s') = Vul(s).

Hence, $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$, and for each $c \in \mathcal{A}_{P_1}$, V is a minimal set (w.r.t. set inclusion) in $\operatorname{Vul}_{P_2}(c)$; if V is a minimal set (w.r.t. set inclusion) in $\operatorname{Vul}_{P_2}(c)$; it holds that $\operatorname{Att}_{P_1} = \operatorname{Att}_{P_2}$.

Lemma~38

Let P_1 and P_2 be NLP_3 such that $P_1 \mapsto_P P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof

Let $c \leftarrow a_1, \ldots, a_m, \text{not } b, \text{not } b_1, \ldots, \text{not } b_n \in P_1$ be a rule such that $b \notin \{head(r) \mid r \in P_1\},$

$$\begin{split} P_2 = & (P_1 - \{c \leftarrow a_1, \dots, a_m, \mathtt{not}\ b, \mathtt{not}\ b_1, \dots, \mathtt{not}\ b_n\}) \\ & \cup \{c \leftarrow a_1, \dots, a_m, \mathtt{not}\ b_1, \dots, \mathtt{not}\ b_n\}\,, \end{split}$$

 $\mathfrak{A}_{P_1}=(\mathcal{A}_{P_1},Att_{P_1})$ and $\mathfrak{A}_{P_2}=(\mathcal{A}_{P_2},Att_{P_2}).$ Note that

- For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that Conc(s) = Conc(s'), and $Vul(s) = \{V \mid \exists V' \in Vul(s') \text{ such that } V = V' \text{ or } V = V' \cup \{b\}\}.$
- For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that Conc(s') = Conc(s), and $Vul(s') = \{V' \mid \exists V \in Vul(s) \text{ such that } V' = V \text{ or } V' = V \{b\}\}.$

Hence,
$$\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$$
, and as $b \notin \mathcal{A}_{P_1} \cup \mathcal{A}_{P_2}$, it holds that $Att_{P_1} = Att_{P_2}$.

Lemma 39

Let P_1 and P_2 be NLP_3 such that $P_1 \mapsto_M P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof

Let $P_2 = P_1 - \{r\}$, where there are two distinct rules r and r' in P_1 such that head(r) = head(r'), $body^+(r') \subseteq body^+(r)$, $body^-(r') \subseteq body^-(r)$. In addition, let $\mathfrak{A}_{P_1} = (\mathcal{A}_{P_1}, Att_{P_1})$ and $\mathfrak{A}_{P_2} = (\mathcal{A}_{P_2}, Att_{P_2})$. Note that

• For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that Conc(s) = Conc(s'), and for each $V \in Vul(s)$, there exists $V' \in Vul(s) \cap Vul(s')$ such that $V' \subseteq V$.

• For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that Conc(s') = Conc(s), and Vul(s') = Vul(s).

Hence, $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$, and for each $c \in \mathcal{A}_{P_1}$, V is a minimal set (w.r.t. set inclusion) in $\operatorname{Vul}_{P_1}(c)$ iff V is a minimal set (w.r.t. set inclusion) in $\operatorname{Vul}_{P_2}(c)$; it holds that $Att_{P_1} = Att_{P_2}$.

Theorem 28

For any NLPs P_1 and P_2 , if $P_1 \mapsto_{UTPM} P_2$, then $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$

Proof

It follows straightforwardly from Lemmas 36, 37, 38 and 39.

Theorem 29

The relation \mapsto_{UTPM} is confluent, i.e., for any $NLPs\ P,\ P'$ and P'', if $P\mapsto_{UTPM}^* P'$ and $P\mapsto_{UTPM}^* P''$ and both P' and P'' are irreducible, then P'=P''.

Proof

From Theorem 28, we know that $\mathfrak{A}_P = \mathfrak{A}_{P'} = \mathfrak{A}_{P''}$. Thus $P_{\mathfrak{A}_{P'}} = P_{\mathfrak{A}_{P''}}$. As P' and P'' are RFALPs (Theorem 17), it holds (Theorem 14) that $P' = P_{\mathfrak{A}_{P'}} = P_{\mathfrak{A}_{P''}} = P''$.