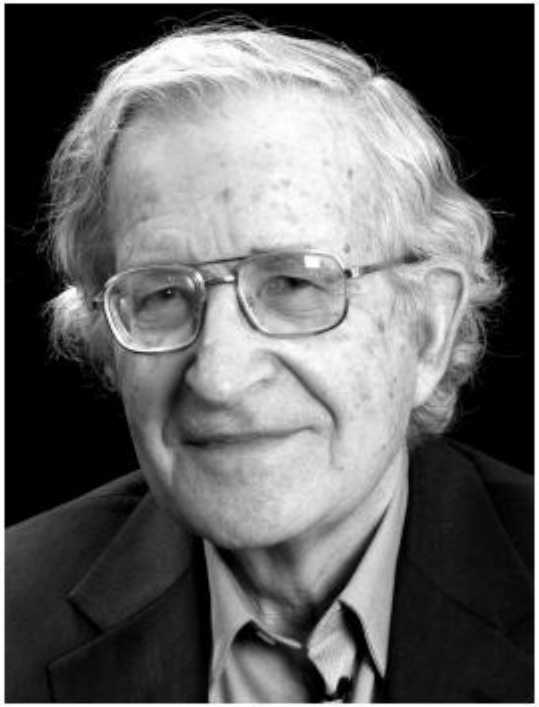
# Context Free Languages

Context Free Grammars

# Context-Free Grammars

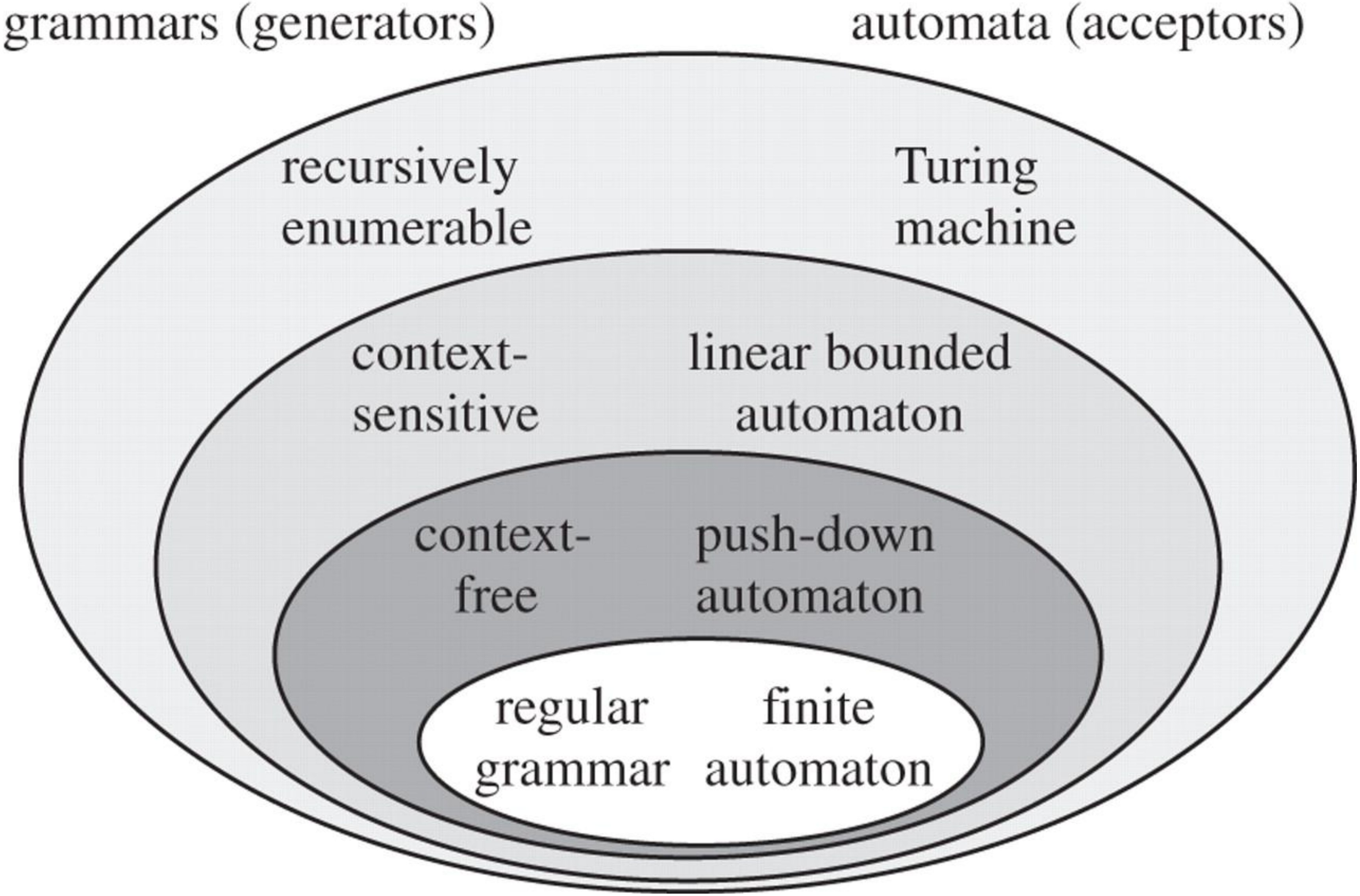


Noam Chomsky

(linguist, philosopher, logician, and activist)

In the formal languages of computer science and linguistics, the **Chomsky hierarchy** is a **hierarchy** of classes of formal grammars. This **hierarchy** of grammars was described by Noam **Chomsky** in 1956.

# Chomsky Hierarchy



grammars (generators)

automata (acceptors)

recursively enumerable

Turing machine

context-sensitive

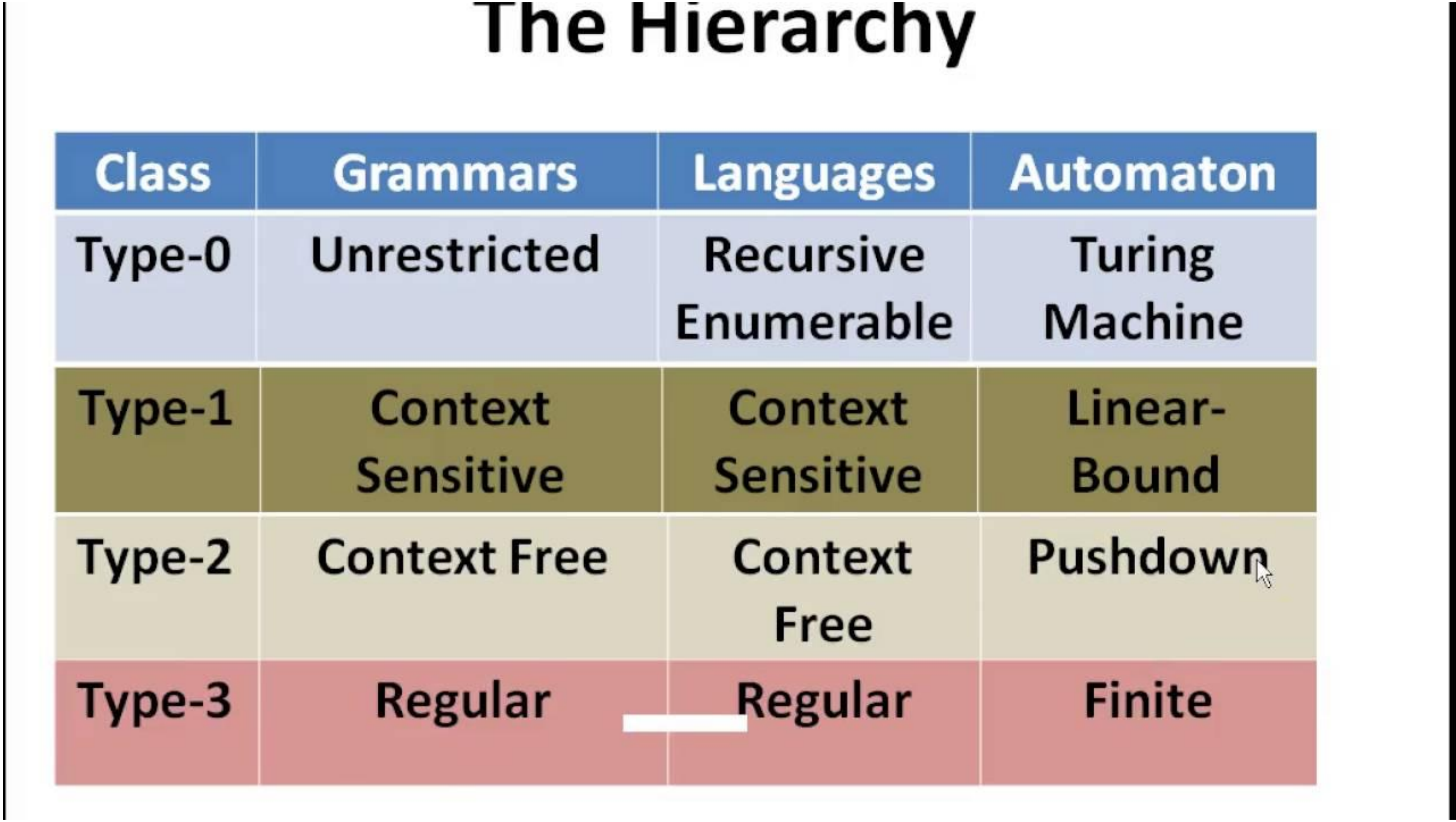
linear bounded automaton

context-free

push-down automaton

regular grammar

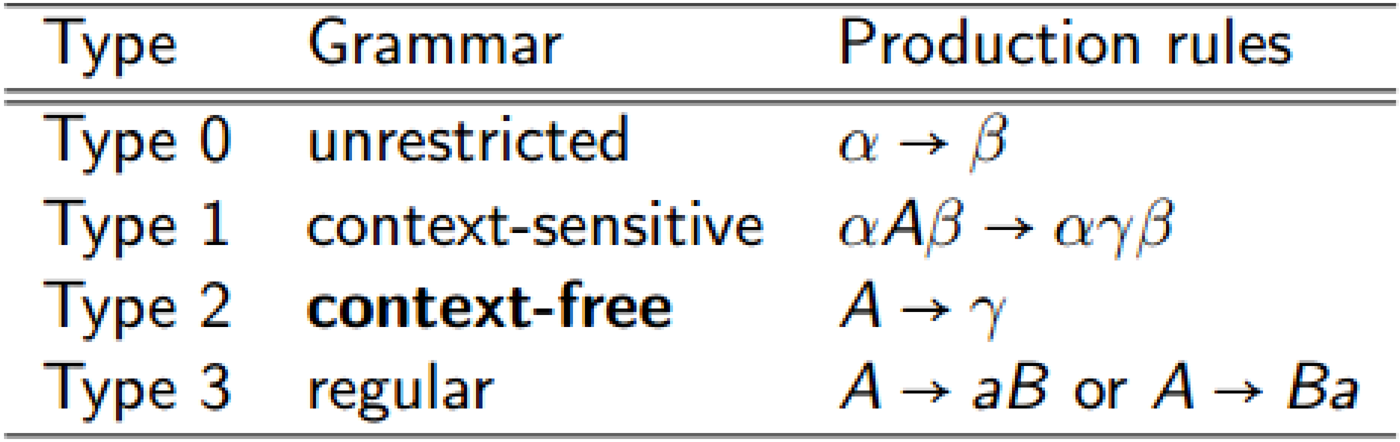
finite automaton



# The Hierarchy

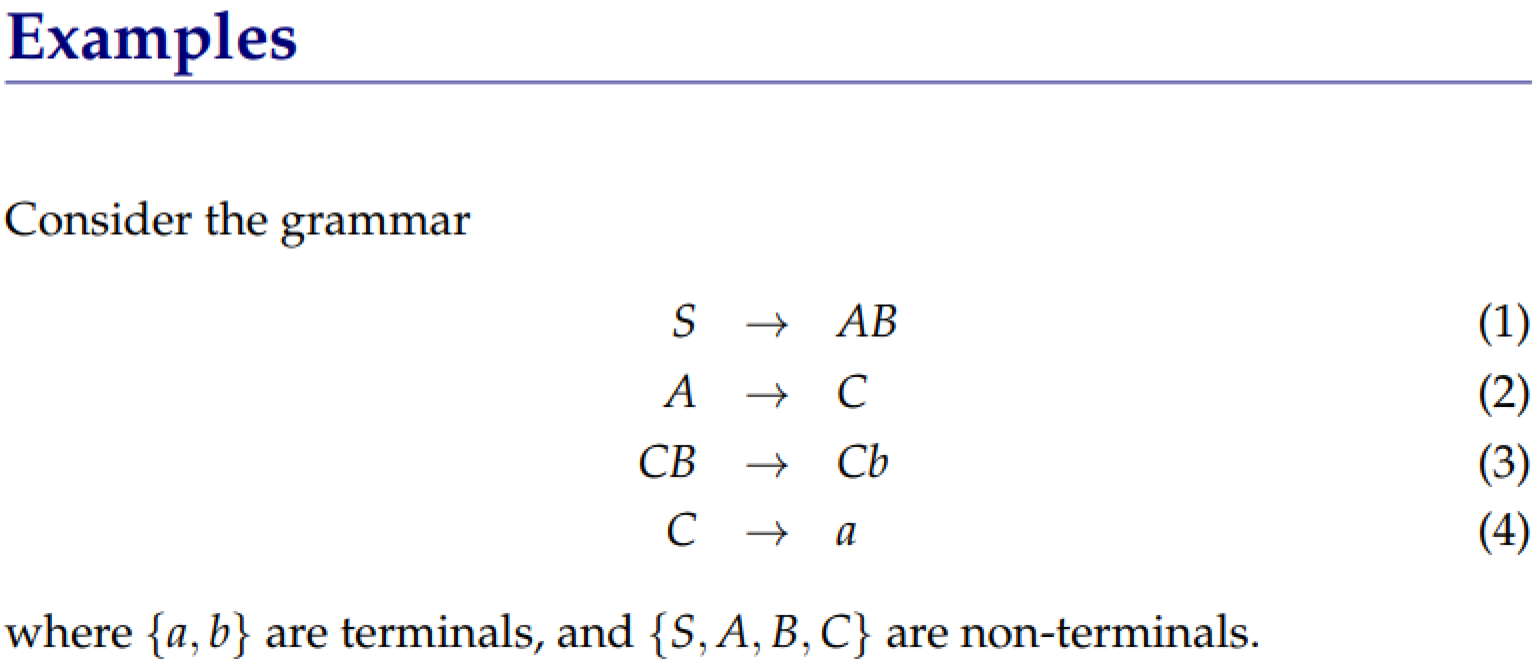
| Class | Grammars | Languages | Automaton |
|---|---|---|---|
| Type-0 | Unrestricted | Recursive Enumerable | Turing Machine |
| Type-1 | Context Sensitive | Context Sensitive | Linear-Bound |
| Type-2 | Context Free | Context Free | Pushdown |
| Type-3 | Regular | Regular | Finite |

# How production rules look like



| Type | Grammar | Production rules |
|---|---|---|
| Type 0 | unrestricted | $\alpha \rightarrow \beta$ |
| Type 1 | context-sensitive | $\alpha A \beta \rightarrow \alpha \gamma \beta$ |
| Type 2 | **context-free** | $A \rightarrow \gamma$ |
| Type 3 | regular | $A \rightarrow aB$ or $A \rightarrow Ba$ |

# A grammar generates sentences (strings) in a language



## Examples

Consider the grammar

$$
\begin{aligned}
S &\rightarrow AB \qquad (1)\\
A &\rightarrow C \qquad (2)\\
CB &\rightarrow Cb \qquad (3)\\
C &\rightarrow a \qquad (4)
\end{aligned}
$$

where $\{a, b\}$ are terminals, and $\{S, A, B, C\}$ are non-terminals.

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where $\{a, b\}$ are terminals, and $\{S, A, B, C\}$ are non-terminals.
We can derive the phrase “ab” from this grammar in the following way:

$$\begin{aligned} S &\rightarrow AB, \text{ from (1)}\\ &\rightarrow CB, \text{ from (2)}\\ &\rightarrow Cb, \text{ from (3)}\\ &\rightarrow ab, \text{ from (4)} \end{aligned}$$

## Examples

Consider the grammar

$$S \rightarrow \mathit{NounPhrase} \quad \mathit{VerbPhrase} \tag{5}$$

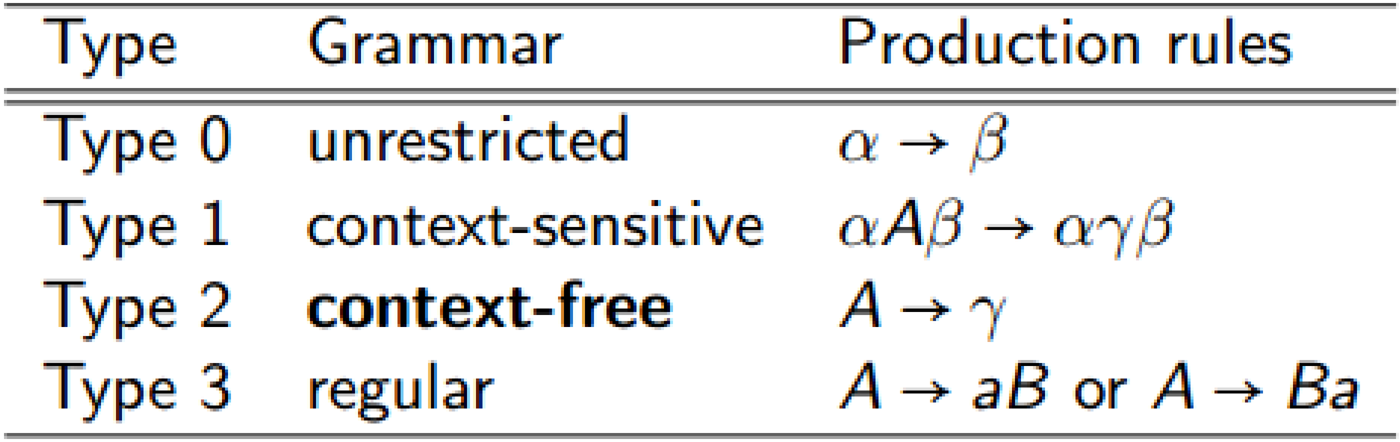
$$\mathit{NounPhrase} \rightarrow \mathit{SingularNoun} \tag{6}$$

$$\mathit{SingularNoun} \quad \mathit{VerbPhrase} \rightarrow \mathit{SingularNoun} \quad \mathit{comes} \tag{7}$$

$$\mathit{SingularNoun} \rightarrow \mathit{John} \tag{8}$$

We can derive the phrase “John comes” from this grammar in the following way:

$$\begin{aligned} S &\rightarrow \mathit{NounPhrase} \quad \mathit{VerbPhrase}, \text{ from (1)} \\ &\rightarrow \mathit{SingularNoun} \quad \mathit{VerbPhrase}, \text{ from (2)} \\ &\rightarrow \mathit{SingularNoun} \quad \mathit{comes}, \text{ from (3)} \\ &\rightarrow \mathit{John} \quad \mathit{comes}, \text{ from (4)} \end{aligned}$$



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### Definition (Context-Free Grammar)

A context-free grammar is a tuple $G = (V, T, P, S)$ where

- $V$ is a finite set of variables (nonterminals, nonterminals vocabulary);
- $T$ is a finite set of terminals (letters);
- $P \subseteq V \times (V \cup T)^*$ is a finite set of rewriting rules called productions,
  - We write $A \to \beta$ if $(A, \beta) \in P$;
- $S \in V$ is a distinguished start or "sentence" symbol.

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Example: $G_{0^n1^n} = (V, T, P, S)$ where

- $V = \{S\}$;
- $T = \{0, 1\}$;
- $P$ is defined as

$$\begin{aligned} S &\to \varepsilon \\ S &\to 0S1 \end{aligned}$$

- $S = S$.

# Palindromes

$G_{pal} = (\{P\}, \{0, 1\}, A, P)$

$$\begin{array}{llll}
1. & P & \rightarrow & \epsilon \\
2. & P & \rightarrow & 0 \\
3. & P & \rightarrow & 1 \\
4. & P & \rightarrow & 0P0 \\
5. & P & \rightarrow & 1P1
\end{array}$$

A context-free grammar for palindromes

## Derivation:

- Let $G = (V, T, P, S)$ be a context-free grammar.
- Let $\alpha A \beta$ be a string in $(V \cup T)^* V (V \cup T)^*$
- We say that $\alpha A \beta$ yields the string $\alpha \gamma \beta$, and we write $\alpha A \beta \Rightarrow \alpha \gamma \beta$ if

$$A \rightarrow \gamma \text{ is a production rule in } G.$$

- For strings $\alpha, \beta \in (V \cup T)^*$, we say that $\alpha$ derives $\beta$ and we write $\alpha \overset{*}{\Rightarrow} \beta$ if there is a sequence $\alpha_1, \alpha_2, \ldots, \alpha_n \in (V \cup T)^*$ s.t.

$$\alpha \Rightarrow \alpha_1 \Rightarrow \alpha_2 \cdots \alpha_n \Rightarrow \beta.$$

$\Rightarrow$ is also called direct derivation.

$\overset{i}{\Rightarrow}$ is to mean that the i th production is used in the direct derivation.

$\overset{*}{\Rightarrow}$ is reflexive and transitive closure of $\Rightarrow$

$$
\begin{array}{llll}
1. & E & \rightarrow & I \\
2. & E & \rightarrow & E + E \\
3. & E & \rightarrow & E * E \\
4. & E & \rightarrow & (E) \\
 & & & \\
5. & I & \rightarrow & a \\
6. & I & \rightarrow & b \\
7. & I & \rightarrow & Ia \\
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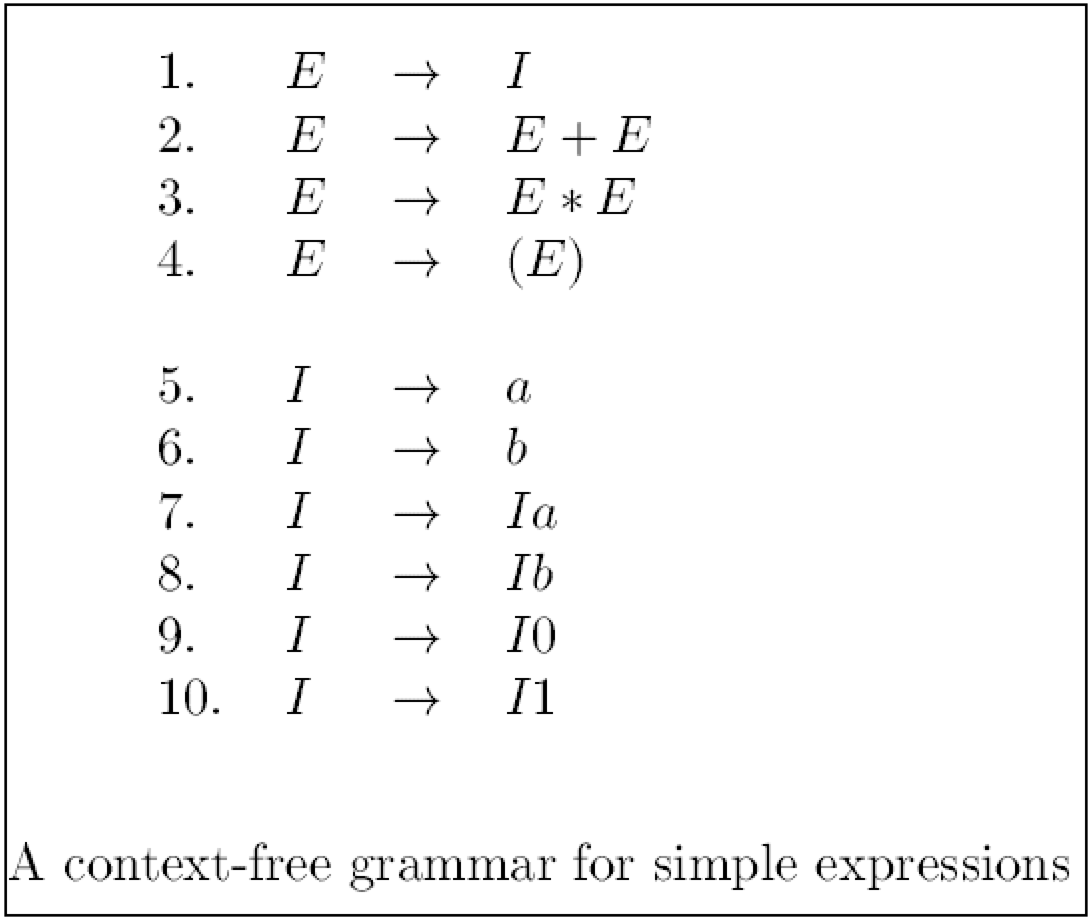
9. & I & \rightarrow & I0 \\
10. & I & \rightarrow & I1
\end{array}
$$

A context-free grammar for simple expressions



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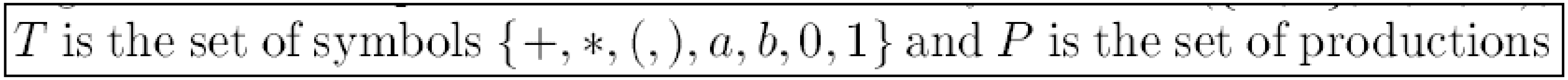
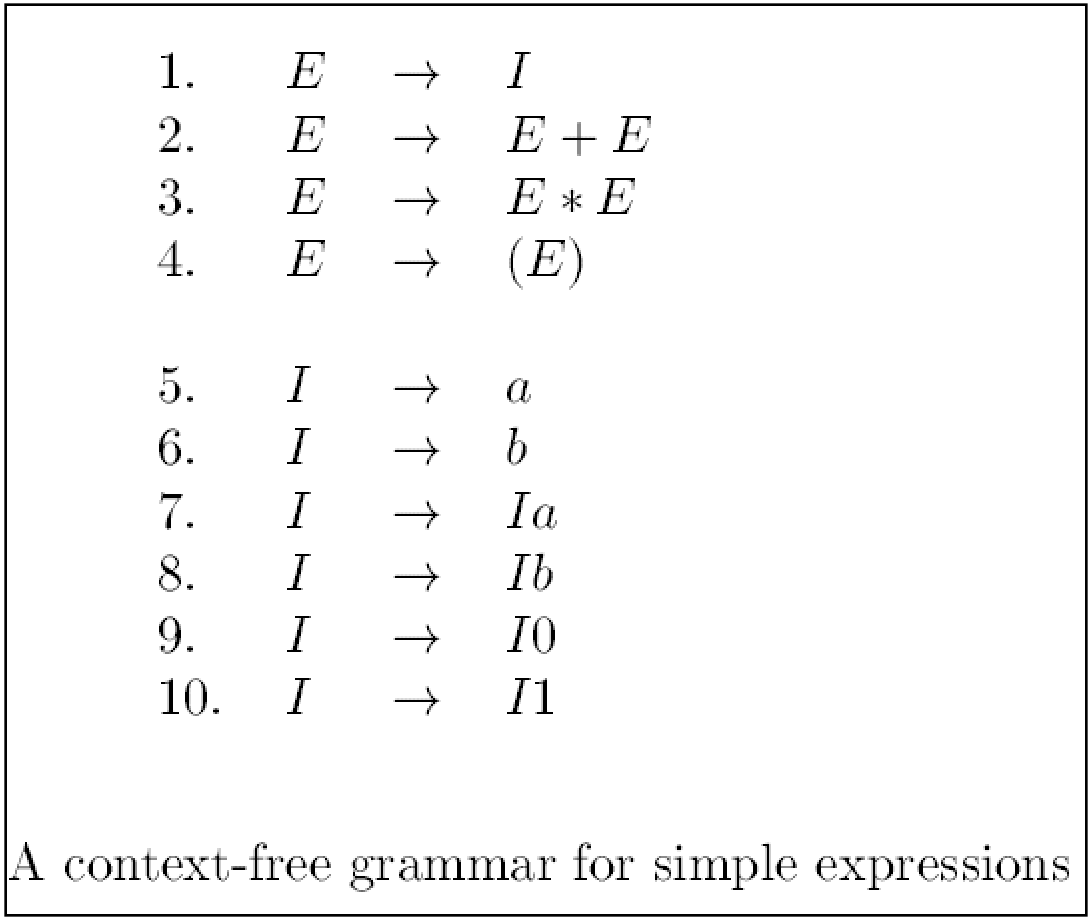
A context-free grammar for simple expressions



$T$ is the set of symbols $\{+, *, (, ), a, b, 0, 1\}$ and $P$ is the set of productions



1. $E \rightarrow I$
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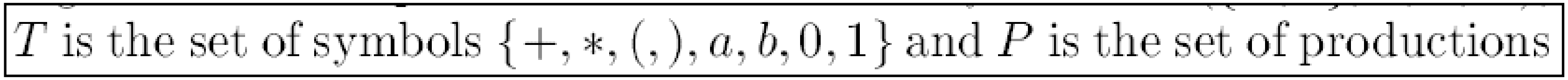
A context-free grammar for simple expressions



$T$ is the set of symbols $\{+, *, (, ), a, b, 0, 1\}$ and $P$ is the set of productions

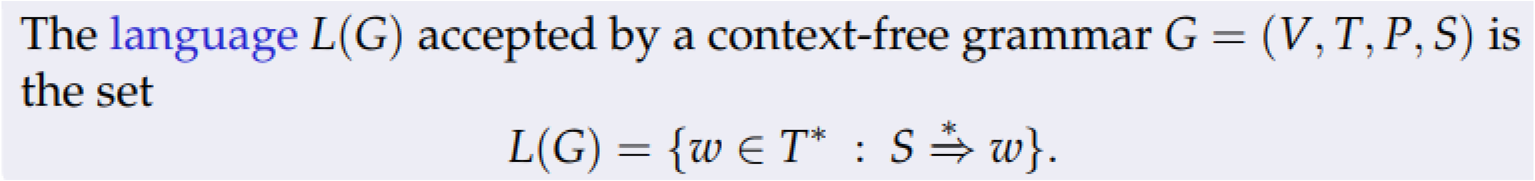
- Can you find how the following is true.

$$E \overset{*}{\Rightarrow} (a1 + b0 * a1)$$

## Compact Notation for Productions

It is convenient to think of a production as "belonging" to the variable of its head. We shall often use remarks like "the productions for $A$" or "$A$-productions" to refer to the productions whose head is variable $A$. We may write the productions for a grammar by listing each variable once, and then listing all the bodies of the productions for that variable, separated by vertical bars. That is, the productions $A \to \alpha_1,\ A \to \alpha_2, \ldots, A \to \alpha_n$ can be replaced by the notation $A \to \alpha_1 | \alpha_2 | \cdots | \alpha_n$. For instance, the grammar for palindromes from Fig. 5.1 can be written as $P \to \epsilon \mid 0 \mid 1 \mid 0P0 \mid 1P1$.

# CFL Definition

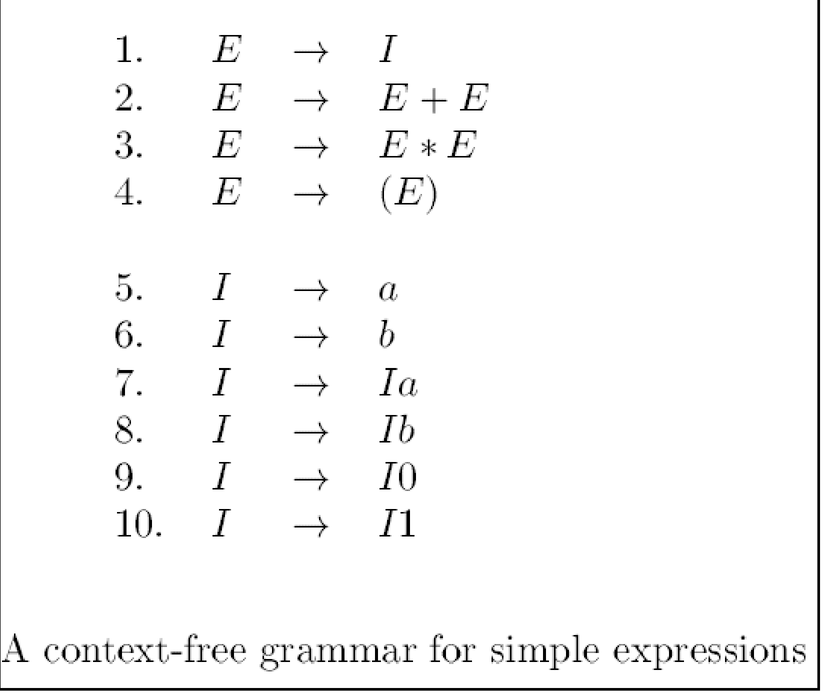


The language $L(G)$ accepted by a context-free grammar $G = (V, T, P, S)$ is the set

$$L(G) = \{w \in T^* \ : \ S \overset{*}{\Rightarrow} w\}.$$

# Leftmost and Rightmost Derivations

- Derivations are not unique.



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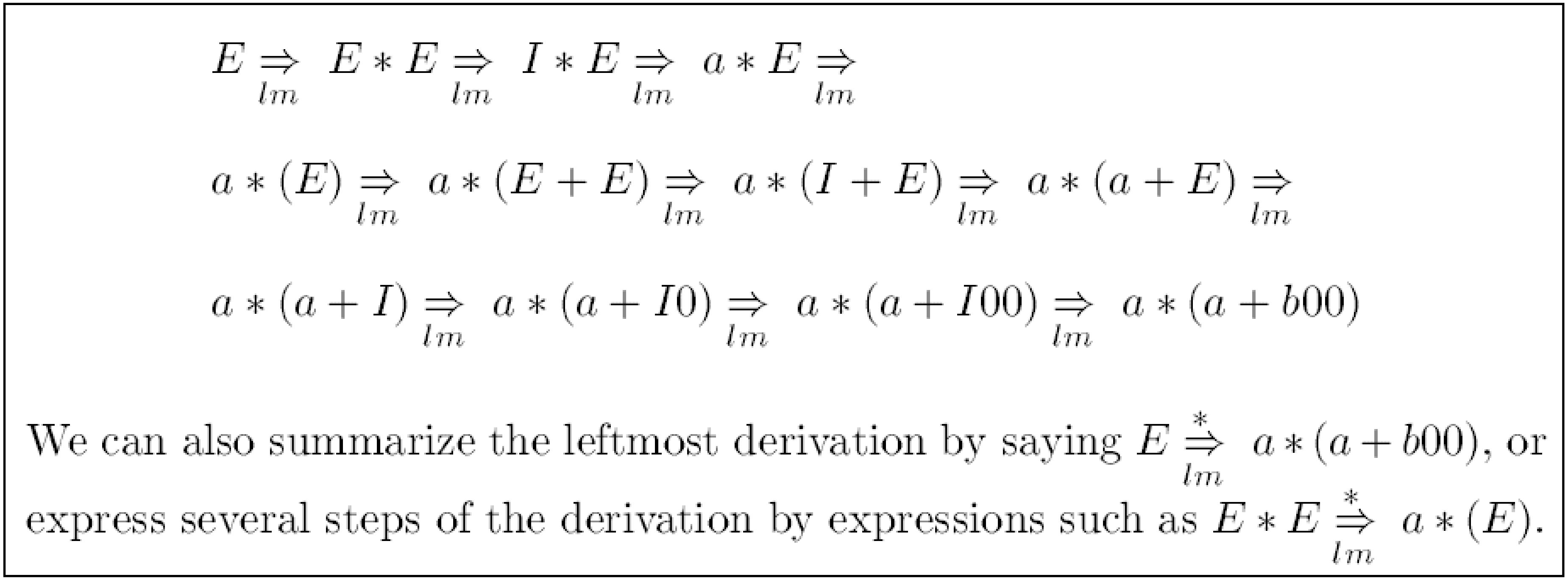
A context-free grammar for simple expressions

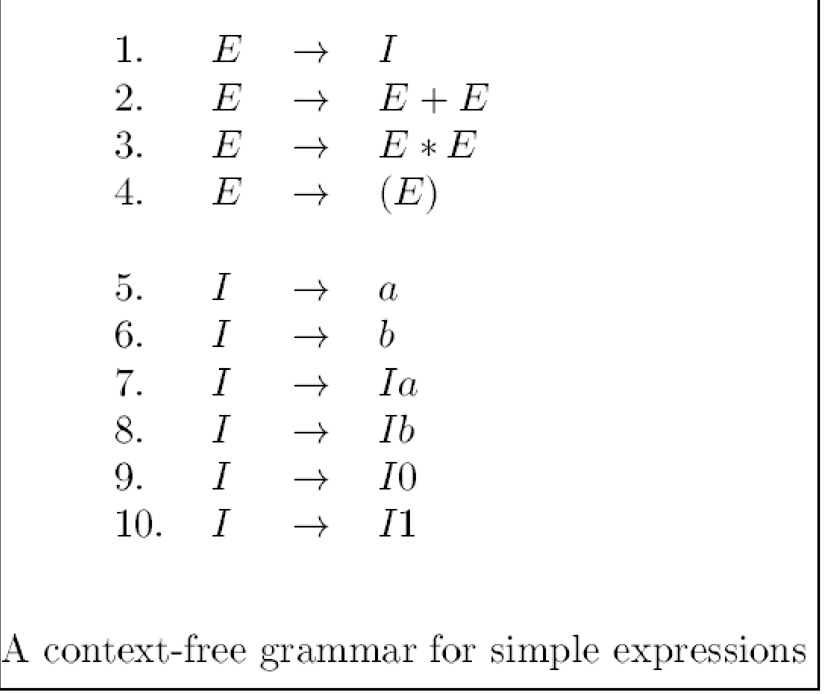


$$E \underset{lm}{\Rightarrow} E * E \underset{lm}{\Rightarrow} I * E \underset{lm}{\Rightarrow} a * E \underset{lm}{\Rightarrow}$$

$$a * (E) \underset{lm}{\Rightarrow} a * (E + E) \underset{lm}{\Rightarrow} a * (I + E) \underset{lm}{\Rightarrow} a * (a + E) \underset{lm}{\Rightarrow}$$

$$a * (a + I) \underset{lm}{\Rightarrow} a * (a + I0) \underset{lm}{\Rightarrow} a * (a + I00) \underset{lm}{\Rightarrow} a * (a + b00)$$

We can also summarize the leftmost derivation by saying $E \overset{*}{\underset{lm}{\Rightarrow}} a * (a + b00)$, or express several steps of the derivation by expressions such as $E * E \overset{*}{\underset{lm}{\Rightarrow}} a * (E)$.



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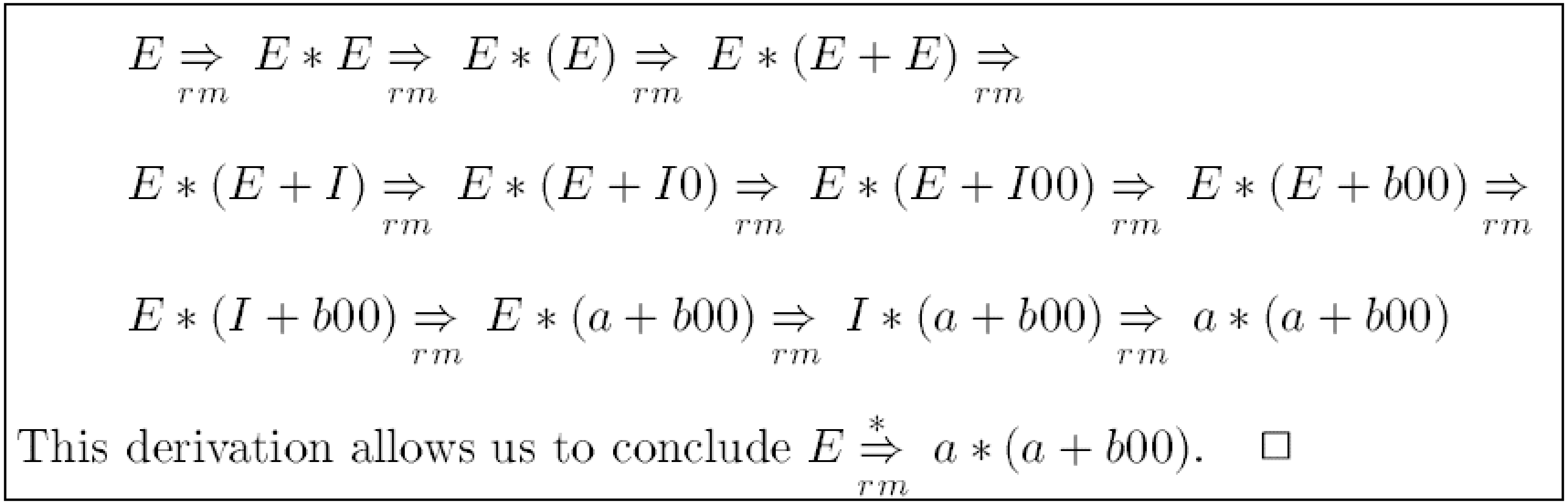


$$E \underset{rm}{\Rightarrow} E * E \underset{rm}{\Rightarrow} E * (E) \underset{rm}{\Rightarrow} E * (E + E) \underset{rm}{\Rightarrow}$$

$$E * (E + I) \underset{rm}{\Rightarrow} E * (E + I0) \underset{rm}{\Rightarrow} E * (E + I00) \underset{rm}{\Rightarrow} E * (E + b00) \underset{rm}{\Rightarrow}$$

$$E * (I + b00) \underset{rm}{\Rightarrow} E * (a + b00) \underset{rm}{\Rightarrow} I * (a + b00) \underset{rm}{\Rightarrow} a * (a + b00)$$

This derivation allows us to conclude $E \overset{*}{\underset{rm}{\Rightarrow}} a * (a + b00)$. □

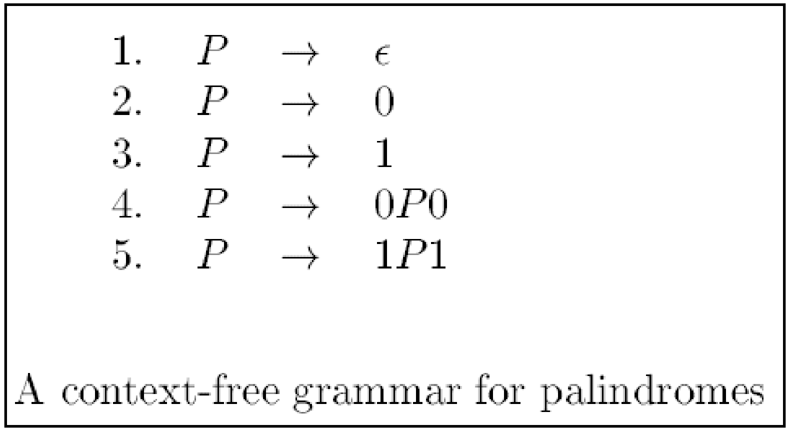
## Exercise

Consider the following grammar:

$$\begin{aligned} S &\rightarrow AS \mid \varepsilon. \\ S &\rightarrow aa \mid ab \mid ba \mid bb \end{aligned}$$

Give leftmost and rightmost derivations of the string *aabbba*.

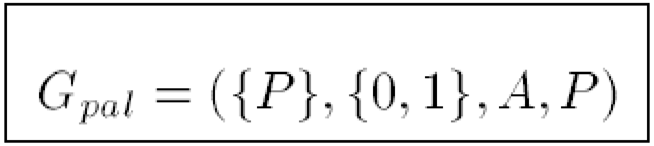
$$G_{pal} = (\{P\}, \{0,1\}, A, P)$$



$$\begin{array}{llll} 1. & P & \rightarrow & \epsilon \\ 2. & P & \rightarrow & 0 \\ 3. & P & \rightarrow & 1 \\ 4. & P & \rightarrow & 0P0 \\ 5. & P & \rightarrow & 1P1 \end{array}$$

A context-free grammar for palindromes

Prove that $L(G_{pal})$ is the set of palindromes over the given alphabet.



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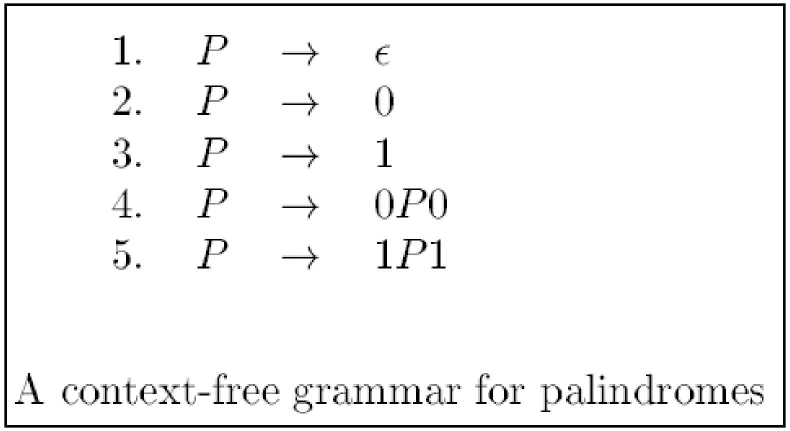

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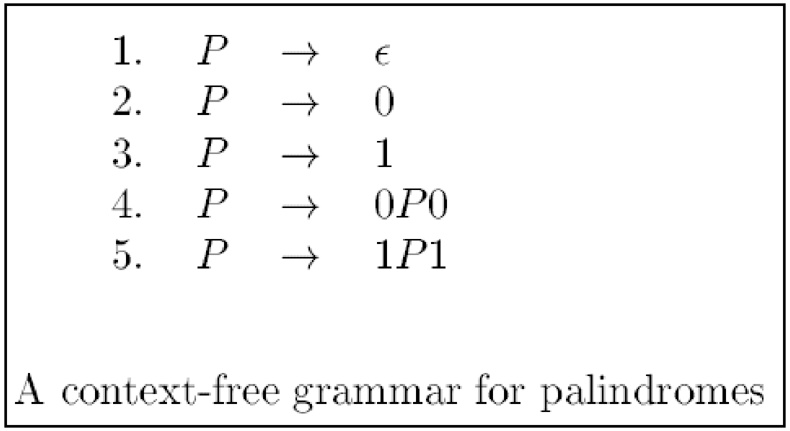

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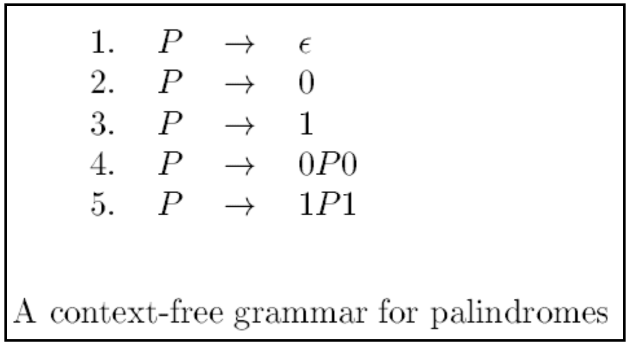
Prove that $L(G_{pal})$ is the set of palindromes over the given alphabet.

- This proof has two parts (⇒ and ⇐ )

1) $(w = w^R) \Rightarrow w \in L(G_{pal})$

2) $w \in L(G_{pal}) \Rightarrow (w = w^R)$

$$\boxed{(w = w^R) \Rightarrow w \in L(G_{pal})}$$



| | | | |
|---|---|---|---|
| 1. | $P$ | $\rightarrow$ | $\epsilon$ |
| 2. | $P$ | $\rightarrow$ | $0$ |
| 3. | $P$ | $\rightarrow$ | $1$ |
| 4. | $P$ | $\rightarrow$ | $0P0$ |
| 5. | $P$ | $\rightarrow$ | $1P1$ |

A context-free grammar for palindromes

- Proof [by induction on $|w|$:

**BASIS**: We use lengths 0 and 1 as the basis.

If $|w| = 0$ or $|w| = 1$, then $w$ is $\epsilon$, 0, or 1.

Since there are productions $P \rightarrow \epsilon$, $P \rightarrow 0$, and $P \rightarrow 1$, we conclude that $P \overset{*}{\Rightarrow} w$ in any of these basis cases.

**INDUCTION**: Suppose $|w| \geq 2$. Since $w = w^R$, $w$ must begin and end with the same symbol

- Note, $w \in L(G_{pal})$ is same $P \overset{*}{\Rightarrow}$ w

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**INDUCTION**: Suppose $|w| \geq 2$. Since $w = w^R$, $w$ must begin and end with the same symbol

**Inductive Hypothesis:** Let for $|w| \leq k$ *where* $(w = w^R)$, $P \overset{*}{\Rightarrow}$ w is true.

**Inductive Step:** We need to show for $|w| = k + 1$, $P \overset{*}{\Rightarrow}$ w is true.

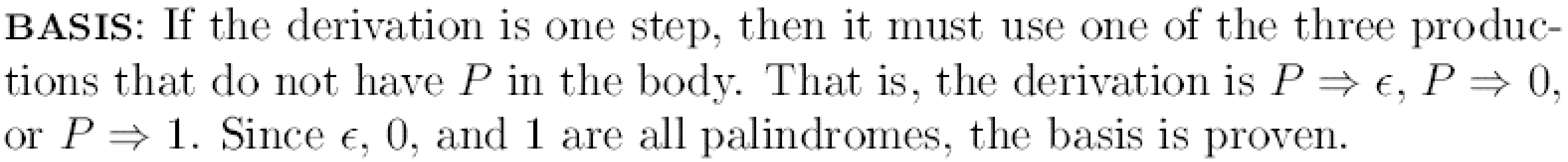
Note, $w = 0x0$ or w $= 1x1$, where $|x| = k - 1$.

Then, $P \Rightarrow 0P0 \overset{*}{\Rightarrow} 0x0$ (Since $|x| \leq k$, so $P \overset{*}{\Rightarrow} x$ is true).

So, $P \overset{*}{\Rightarrow}$ w is true. With a similar argument, $P \Rightarrow 1P1 \overset{*}{\Rightarrow} 1x1$

$$w \in L(G_{pal}) \Rightarrow (w = w^R)$$

- Proof [by induction on number of steps in the derivation]:



**BASIS:** If the derivation is one step, then it must use one of the three productions that do not have $P$ in the body. That is, the derivation is $P \Rightarrow \epsilon$, $P \Rightarrow 0$, or $P \Rightarrow 1$. Since $\epsilon$, 0, and 1 are all palindromes, the basis is proven.

**INDUCTION:**

- Assume for n steps it is true.
- Then, show for (n+1) steps it must be true.

**Left as an exercise.**

## Sentential Forms

$G = (V, T, P, S)$ is a CFG, then any string $\alpha$ in $(V \cup T)^*$ such that $S \overset{*}{\Rightarrow} \alpha$ is a *sentential form.*

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 & & & \\
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\end{array}
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A context-free grammar for simple expressions

$$\boxed{E \Rightarrow E * E \Rightarrow E * (E) \Rightarrow E * (E + E) \Rightarrow E * (I + E)}$$

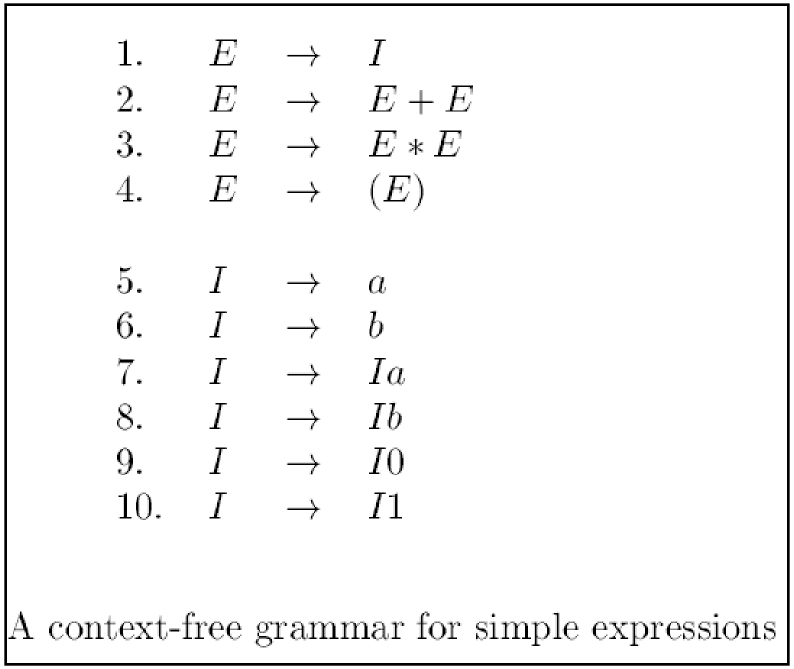
# Sentential Forms

$G = (V, T, P, S)$ is a CFG, then any string $\alpha$ in $(V \cup T)^*$ such that $S \overset{*}{\Rightarrow} \alpha$ is a *sentential form.*

If $S \underset{lm}{\overset{*}{\Rightarrow}} \alpha$, then $\alpha$ is a *left-sentential form,*

and if $S \underset{rm}{\overset{*}{\Rightarrow}} \alpha$, then $\alpha$ is a *right-sentential form.*

Note that the language $L(G)$ is those sentential forms that are in $T^*$; i.e., they consist solely of terminals.



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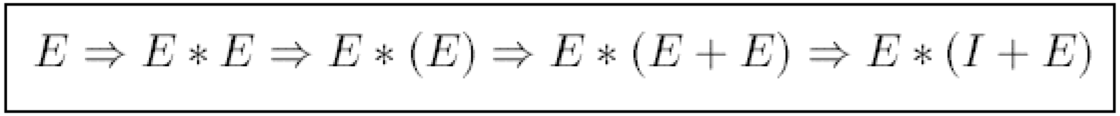
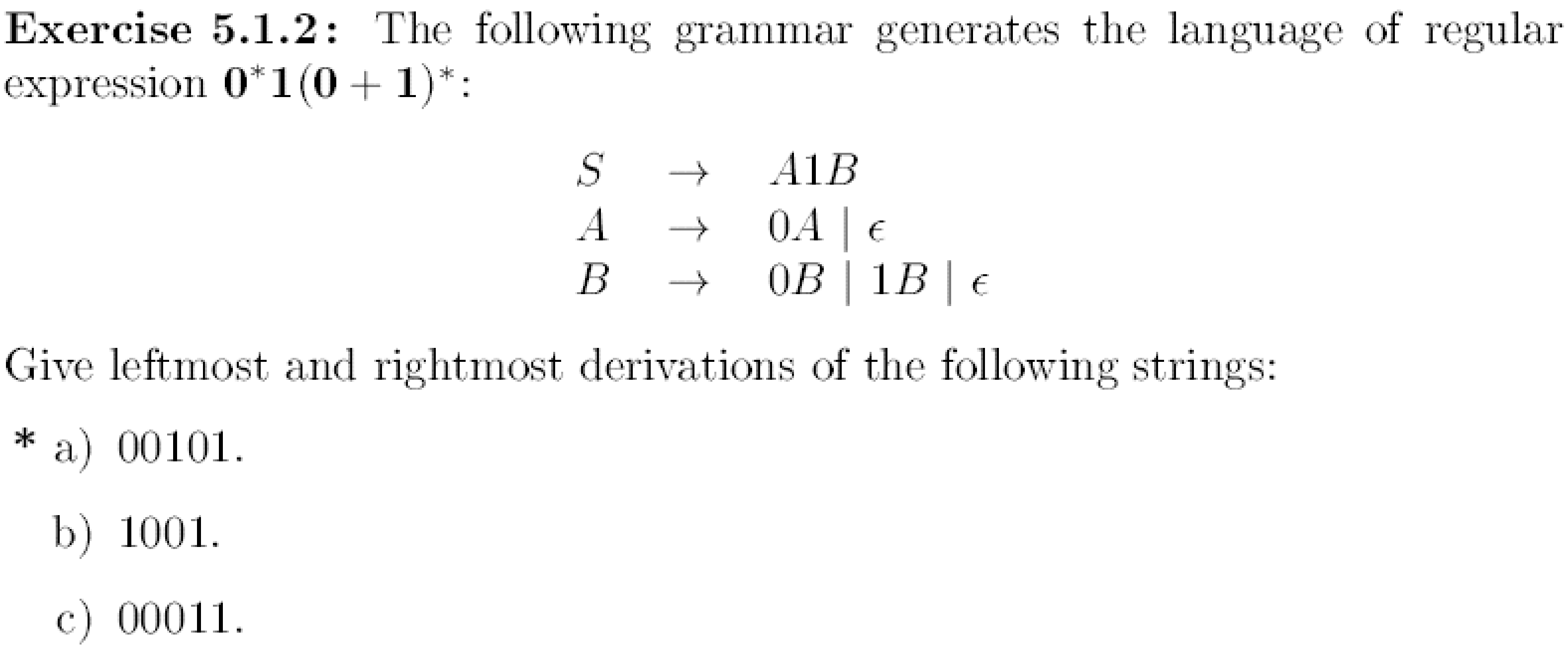
A context-free grammar for simple expressions



$$E \Rightarrow E * E \Rightarrow E * (E) \Rightarrow E * (E + E) \Rightarrow E * (I + E)$$

- Is this sentential form left-sentential? Or right-sentential?



**Exercise 5.1.2:** The following grammar generates the language of regular expression $\mathbf{0}^*\mathbf{1}(\mathbf{0}+\mathbf{1})^*$:

$$\begin{array}{lcl} S & \rightarrow & A1B \\ A & \rightarrow & 0A \mid \epsilon \\ B & \rightarrow & 0B \mid 1B \mid \epsilon \end{array}$$

Give leftmost and rightmost derivations of the following strings:

* a) 00101.

b) 1001.

c) 00011.

Note, the given grammar is not a regular grammar (even-though it generates a regular language).

# Can you find $L(G)$?

- $S \rightarrow aS|bS|a|b|\epsilon$

# Can you find $L(G)$?

- $S \to aS|bS|a|b|\epsilon$

- Answer: All strings. $\Sigma^*$

# Can you find $L(G)$?

1. $S \rightarrow S_1 S | \epsilon$,
2. $S_1 \rightarrow a S_1 b | ab$

# Can you find $L(G)$?

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2. $S_1 \rightarrow aS_1 b | ab$

Recall that the $S \rightarrow aSb|\epsilon$ generates $\{a^n b^n | n \geq 0\}$.

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The answer:

$a^{n_1}\mathrm{b}^{\mathrm{n}_1}a^{n_2}\mathrm{b}^{\mathrm{n}_2} \ldots a^{n_k}\,\mathrm{b}^{\mathrm{n}_\mathrm{k}} \in L(G)$

$L(G) = \ (\{a^nb^n|n \geq 1\})^*$

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$$S \rightarrow SS \mid S \mid [S] \mid (\,) \mid [\,]$$

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$$S \rightarrow SS \mid [S] \mid (S) \mid [\,] \mid (\,)$$

Set of all balanced parentheses with alphabet $\{\, (, ), [, ] \,\}$

# Can you find $L(G)$?

1. $S \rightarrow aB|bA$
2. $B \rightarrow b|bS|aBB$
3. $A \rightarrow a|aS|bAA$

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1. $S \rightarrow aB|bA$
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Produces strings with equal number of a’s and b’s.

# Can you find $L(G)$?

1. $S \rightarrow SaSbS|SbSaS|\epsilon$

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With one difference than the previous CFG. What is it?