CS 5114
Solutions to Homework Assignment 1

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January 27, 2017

[20] 1. CLRS Problem 3-2. Relative asymptotic growths
Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1, \epsilon > 0$, and $c > 1$ are constants. Your answer should be in the form of the table with "yes" or "no" written in each box. For those of you using LaTeX, Figure 1 gives the table to fill in.

	A	B	O	o	Ω	ω	Θ
i.	$\lg^k n$	n^ϵ	*yes*	*yes*	*no*	*no*	*no*
ii.	n^k	c^n	*yes*	*yes*	*no*	*no*	*no*
iii.	$\sqrt{n}$	$n^{\sin n}$	*no*	*no*	*no*	*no*	*no*
iv.	2^n	$2^{n/2}$	*no*	*no*	*yes*	*yes*	*no*
v.	$n^{\lg c}$	$c^{\lg n}$	*yes*	*no*	*yes*	*no*	*yes*
vi.	$\lg(n!)$	$\lg(n^n)$	*yes*	*no*	*yes*	*no*	*yes*

Figure 1: Table for Problem 3-2.

i. Here A = $\lg^k n$ (a ***polylogarithmic function***) and B = n^ϵ (a ***polynomial function***), for $k \geq 1$ and $\epsilon > 0$. Since, any positive polynomial function grows faster than any polylogarithmic function, we have:

$$\lim_{n\to\infty} \frac{\lg^b n}{n^a} = 0$$

for $a, b > 0$. Replacing a and b by k and ϵ respectively, we get:

$$\lim_{n\to\infty} \frac{\lg^k n}{n^\epsilon} = 0 \tag{1}$$

Using (1) we can conclude that $\lg^k n = \mathrm{o}(n^\epsilon)$. Since, o is a more restrictive form of O, we can also conclude that: $\lg^k n = \mathrm{O}(n^\epsilon)$. While, o notation denotes an upper bound that is not asymptotically tight, O notation may or may not be asymptotically tight. Therefore, if A is o(B), then A should also be O(B), but the inverse may or may not be true.

ii. Here A = n^k (a ***polynomial function***) and B = c^n (an ***exponential function***), for $k \geq 1$ and $c > 1$.Since, any exponential function with a base strictly greater than 1 grows faster than any polynomial function, we have:

$$\lim_{n\to\infty} \frac{n^k}{c^n} = 0 \tag{2}$$

Using (2) we can conclude that $n^k = \text{o}(c^n)$. Once again since, o is a more restrictive form of O, we can conclude that: $n^k = \text{O}(c^n)$.

iii. Here A $= \sqrt{n}$ and B $= n^{\sin n}$ (a ***periodic function***). Since, the range of $\sin n$ is [-1,1], we can compute the following limit to estimate the asymptotic notation:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^{\sin n}}$$

For $\sin n = -1$, we have:

$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{n}}{n^{-1}} &= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{n^{-1}} \\ &= \lim_{n\to\infty} n^{\frac{3}{2}} \\ &= \infty \end{aligned}$$

Similarly, for $\sin n = 1$, we have:

$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{n}}{n^{1}} &= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{n^{1}} \\ &= \lim_{n\to\infty} n^{-\frac{1}{2}} \\ &= 0 \end{aligned}$$

Here, as the two boundary values of $\sin n$ result in different limits (∞ and 0), we can't determine the asymptotic notation between A and B.

iv. Here A $= 2^n$ and B $= 2^{\frac{n}{2}}$. Since,

$$\lim_{n\to\infty} \frac{2^n}{2^{\frac{n}{2}}} = \lim_{n\to\infty} 2^{\frac{n}{2}} = \infty \tag{3}$$

Using (3), we can conclude that $2^n = \omega(2^{\frac{n}{2}})$. Since, ω is a more restrictive form of Ω, we can also conclude that: $2^n = \Omega(2^{\frac{n}{2}})$. While, ω notation denotes a lower bound that is not asymptotically tight, Ω notation may or may not be asymptotically tight. Therefore, if A is ω(B), then A should also be Ω(B), but the inverse may or may not be true.

v. Here A $= n^{\lg c}$ and B$= c^{\lg n}$. These two functions are the same function, we can derive the relationship between them as follows:

$$n^{\lg c} = (\lg c)(\lg n) \tag{4}$$

And,

$$c^{\lg n} = (\lg n)(\lg c) \tag{5}$$

From (4) and (5), we can conclude that $n^{\lg c} = \Theta(c^{\lg n})$. This also implies that, $n^{\lg c} = O(c^{\lg n})$ and $n^{\lg c} = \Omega(c^{\lg n})$.

vi. Here A = $\lg(n!)$ and B = $\lg(n^n)$. By ***Sterling's approximation***, we have:

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n\left(1 + \Theta\Big(\frac{1}{n}\Big)\right)$$

applying lg on both sides, we get:

$$\begin{aligned}
\lg(n!) &\approx \lg\left(\sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n\right)\\
&= \lg(\sqrt{2\pi n}) + \lg\big(\frac{n}{e}\big)^n\\
&= \lg(\sqrt{2\pi}) + \lg(\sqrt{n}) + n\lg\big(\frac{n}{e}\big)\\
&= \Theta(1) + \Theta(\lg n) + \Theta(n\lg n) - \Theta(n)\\
&= \Theta(n\lg n)
\end{aligned}$$

Therefore,

$$\lg(n!) = \Theta(n\lg n) = \Theta(\lg(n^n)) \tag{6}$$

Using (6), we can also conclude that $\lg(n!) = O(\lg(n)^n)$ and $\lg(n!) = \Omega(\lg(n)^n)$.

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[20] 2. CLRS Exercise 4.4-6.
Argue that the solution to the recurrence $T(n) = T(n/3) + T(2n/3) + cn$, where c is a constant, is $\Omega(n\lg n)$ by appealing to a recursion tree.

We can draw the recursion tree for the recurrence as shown in Figure 2. To compute the asymptotic lower bound (Ω), we need to find the shortest path from the root to a leaf of the recursion tree. Assuming that the first leaf is at level k, hence:

$$\begin{aligned}
\frac{n}{3^k} &= 1\\
n &= 3^k\\
\log n &= k\log 3\\
k &= \frac{\log n}{\log 3}\\
k &= \log_3 n
\end{aligned}$$

Upon adding up the costs till level $\log_3 n$ of the recursion tree, we get the following

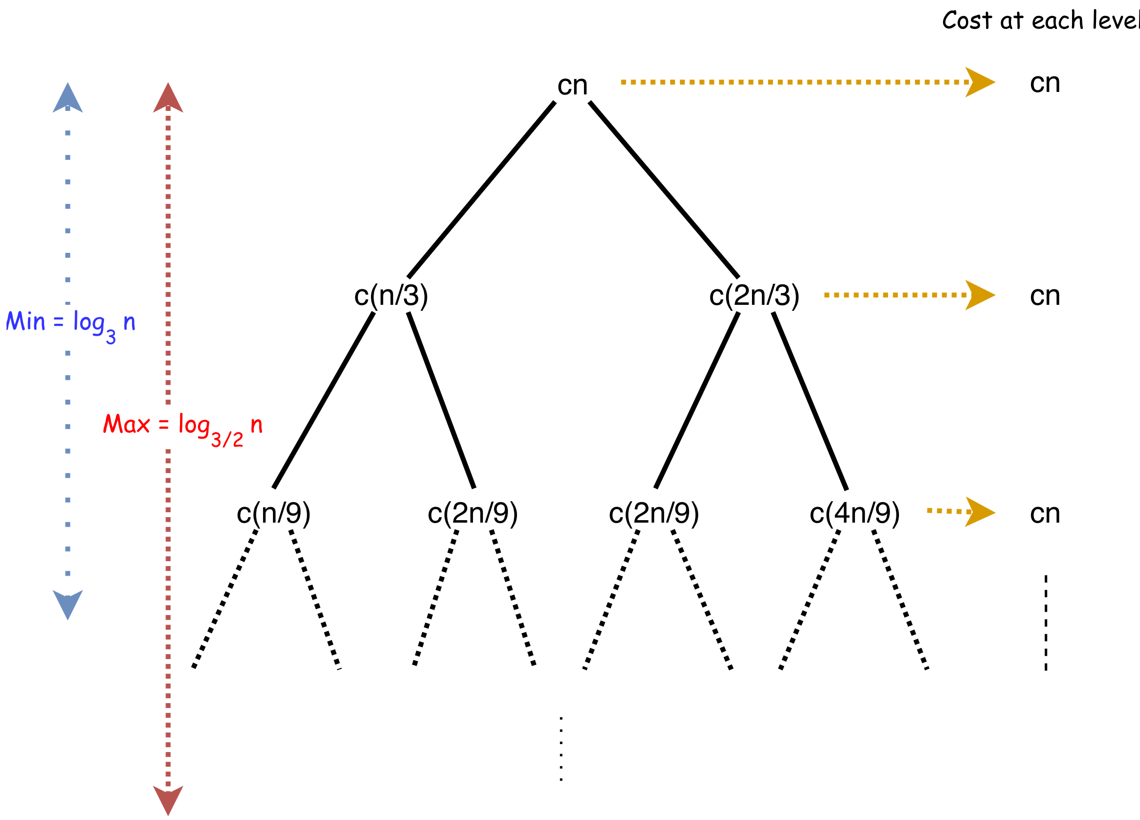

Figure 2: Recursion tree for $T(n) = T(n/3) + T(2n/3) + cn$

value for lower bound of T(n):

$$\begin{aligned}
T(n) &> \sum_{i=1}^{\log_3 n} (cn) \\
&> \log_3 n(cn) \\
&> \frac{\log n}{\log 3}(cn) \\
&> \frac{\log n}{\log 3}\frac{\log 2}{\log 2}(cn) \\
&> \frac{\log n}{\log 2}\frac{\log 2}{\log 3}(cn) \\
&> n \lg n(\frac{c \log 2}{\log 3}) \\
\implies T(n) &= \Omega(n \lg n)
\end{aligned}$$

[20] 3. CLRS Exercise 4.5-1.
Use the master method to give tight asymptotic bounds for the following recurrences.

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a. $T(n) = 2T(n/4) + 1$.
Here, a = 2, b = 4 and $f(n) = 1$. Hence,

$$\begin{aligned} n^{\log_b a} &= n^{\log_4 2} \\ &= n^{\frac{1}{2}} \\ &= n^{0.5} \end{aligned}$$

Since,

$$\begin{aligned} &f(n) = n^0 \\ &f(n) \in O(n^{0.5-0.5}) \end{aligned}$$

Therefore, by Case 1 of Master Theorem, we obtain:

$$\implies T(n) \in \Theta(n^{0.5})$$

b. $T(n) = 2T(n/4) + \sqrt{n}$.
Here, a = 2, b = 4 and $f(n) = \sqrt{n}$. Hence,

$$\begin{aligned} n^{\log_b a} &= n^{\log_4 2} \\ &= n^{\frac{1}{2}} \\ &= n^{0.5} \end{aligned}$$

Since,

$$\begin{aligned} &f(n) = n^{0.5} \\ &f(n) \in \Theta(n^{0.5}) \end{aligned}$$

Therefore, by Case 2 of Master Theorem, we obtain:

$$\implies T(n) \in \Theta(n^{0.5} \lg n)$$

c. $T(n) = 2T(n/4) + n$.
Here, a = 2, b = 4 and $f(n) = n$. Hence,

$$\begin{aligned} n^{\log_b a} &= n^{\log_4 2} \\ &= n^{\frac{1}{2}} \\ &= n^{0.5} \end{aligned}$$

Since,

$$\begin{aligned} &f(n) = n^1 \\ &f(n) \in \Omega(n^{0.5+0.5}) \end{aligned}$$

Therefore, by Case 3 of Master Theorem, we obtain:

$$\implies T(n) \in \Theta(n)$$

d. $T(n) = 2T(n/4) + n^2$
Here, a = 2, b = 4 and $f(n) = n^2$. Hence,

$$\begin{aligned} n^{\log_b a} &= n^{\log_4 2} \\ &= n^{\frac{1}{2}} \\ &= n^{0.5} \end{aligned}$$

Since,

$$\begin{aligned} &f(n) = n^2 \\ &f(n) \in O(n^{0.5+1.5}) \end{aligned}$$

Therefore, by Case 3 of Master Theorem, we obtain:

$$\implies T(n) \in \Theta(n^2)$$
