On the Use of Multitone Techniques for Assessing RF Components' Intermodulation Distortion

José Carlos Pedro, *Member, IEEE,* and Nuno Borges de Carvalho, *Student Member, IEEE*

***Abstract*— A comprehensive analysis of various techniques currently used for assessing microwave components' nonlinear distortion behavior is presented in this paper. The output of a third-order system subject to a two- or three-tone input is given, and then used as the comparison reference for studying the response to a general multitone or random excitation. Theoretical results thus obtained allowed the generalization of standard two-tone intermodulation (IMD) figures of merit, to multitone IMD ratio, multitone or noise adjacent channel power ratio, and noise power ratio (NPR). This approach proved that normal NPR tests produce optimistic results that can be as large as 7 dB when evaluating in-band co-channel power interference.**

***Index Terms*—Intermodulation distortion, nonlinear systems.**

I. Introduction

Although two-tone measurements still represent the industry standard in intermodulation (IMD) distortion characterization, today, engineers seek alternative test procedures closer to the system's final operation regime [1], [2]. In fact, microwave circuits intended for telecommunications applications are expected to handle one or more carriers modulated with nonnull information signals, i.e., finite bandwidth excitations, which are generally modeled as multitone spectra.

The derivation of a set of analytic expressions capable of describing the system's response under any number of K tones is thus a very useful result, as it would allow the integration (and comparison) of the more usual ways of IMD characterization: under two-, three-tone, general multitone, and band-limited noise. Unfortunately, the number of possible frequency combinations among a general set of K tones is so large, which makes the problem intractable in analytical form even for the simplest case of IMD on third-order memoryless systems. Although some computer routines were already proposed to make these combinatory calculations [3], [4], they can not provide qualitative information useful for studying the IMD generation process or for comparison purposes between the various distortion figures of merit.

One possible way of simplifying this problem is to restrict the input excitation. The first thing that comes to mind is to consider all tones of equal amplitude. That is not too restrictive, as such a spectrum encounters many practical applications like distortion noise tests [5] (which generally use band-limited white noise) or modern CDMA mobile communications signals [2], [6]. The second simplifying assumption is to consider a frequency arrangement where all the tones are equally spaced. This resolves the output mixing products ordering problem faced whenever uncommensurated tones are dealt with. The drawback associated with the uniform frequency mapping is that now there will be various products falling on the same position, which may be added in voltage or power. As it will be shown in the following sections, this may be overcome if all input tones are supposed to have uncorrelated phases.

A work by Leffel [7] already proposed a first set of formulas to compute the number of mixing products appearing in a certain position inside (or close to) the input signal, for such an excitation spectrum. However, those results still suffer from several limitations. First, they assume that co-channel distortion is always measured within a notch and, thus, neglected all the products involving the nullified tone. Second, characterization of adjacent channel distortion is restricted to the amplitude of the tone closer to the input spectrum. This obviates important adjacent channel power ratio (ACPR) calculations, either of total integrated spectrum regrowth power or of integrated power in a prescribed bandwidth [6]. Finally, those formulas are expressed in terms of recursive summations (which makes them difficult to apply and justifies the tables of values also given) and some of them were derived from a trial-and-error empirical basis.

The first goal of this paper is to derive a more general and rigorous mathematical result valid for both two-tone, multitone, or noise excitations. That will be used to discuss the most important types of distortion specs in the microwave field, and to provide comparison relations of practical significance between these alternative IMD figures of merit.

To accomplish that, first the reference cases of two and three tones are analyzed in Section II. Section III generalizes those results to an input composed of K tones of equal amplitude and uniformly separated in frequency. Section IV analyses the system's response to random inputs, and Section V integrates the results obtained for continuous and discretized spectra. Finally, Section VI is dedicated to the comparison of the most important IMD figures of merit, and Section VII presents conclusions.

II. Two- and Three-Tone Intermodulation Response Calculations

The analytical development that follows uses the formalism of Volterra–Wiener techniques [8]. It is based on the statement

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The authors are with the Instituto de Telecomunicações, Universidade de Aveiro, 3810 Aveiro, Portugal (e-mail: jcpedro@ieee.org, nborges@ieee.org).

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that a nonlinear system $y(t) = S[x(t)]$ with or without memory, which is stable and its nonlinearities are differentiable in their domains, can be represented by a truncated Volterra series expansion around some quiescent point. Thus, assuming the input $x(t)$ is composed by K, equally separated tones of the form

$$x(t) = \frac{1}{2} \sum_{\substack{k=-K \\ k\neq 0}}^{K} X_k \cdot e^{j[\omega \kappa t + \phi k]} \tag{1}$$

where tone frequencies are given by $\omega_k = \omega_0 + (k-1)\Delta\omega$ $(1 \leq k \leq K)$ and $\omega_k = -\omega_0 + (k+1)\Delta\omega$ $(-K \leq k \leq -1)$, the output $y(t)$ can be written as

$$y(t) = \sum_{n=1}^{N} y_n(t) \tag{2a}$$

in which the $y_n(t)$ are the nth-order output responses given by

$$y_n(t) = \frac{1}{2^n} \sum_{k_1=-K}^{K}, \cdots, \sum_{k_n=-K}^{K} X_{k1} \cdots X_{kn} H_n(\omega_{k1}, \cdots, \omega_{kn}) \times \exp\Big(j\big[(\omega_{k1} + \cdots + \omega_{kn})t + \phi_1 + \cdots + \phi_n\big]\Big). \tag{2b}$$

This expression states that the output signal will be composed of a set of mixing products whose frequency is a linear combination of the input ω_k and whose amplitude is proportional to the number of different ways that each product can be generated as follows:

$$\omega_m = m_{-K}\omega_{-K} + \cdots + m_{-1}\omega_{-1} + m_1\omega_1 + \cdots + m_K\omega_K \tag{3}$$

where all m_k are positive or null integers and represent the number of times frequency ω_k enters in the desired mixing product.

Therefore, the output spectral lines will also be equally separated by $\Delta\omega$, and its amplitude will be proportional to the multinomial coefficient [9]

$$t_n = \frac{n!}{m_{-K}! \cdots m_{-1}! m_1! \cdots m_K!} \tag{4}$$

where n is the order.

As was said, in (1), the input tones ω_k were considered equally spaced in frequency and having a phase of ϕ_k relative to a common reference. Although defining relative phases of different frequency signals may sound strange, it allows the simultaneous treatment of two distinct spectrum generation arrangements. In the first one, it is assumed that all ω_k were created by frequency synthesis from a single reference. Thus, they are all harmonically related to that reference, and relative phases are clearly defined. On the other case, each ω_k has a different reference, their frequencies or phases are uncorrelated, and each of the ϕ_k have to be considered as a random variable. This is equivalent to a set of K uncommensurated tones.

If one is interested in studying system's response up to order n, only the first n nonlinear transfer functions (NLTF's) $H_1(\omega), \cdots, H_n(\omega_1, \cdots, \omega_n)$ have to be known. In this context, nonlinear distortion is often classified as small- or large-signal distortion, whether an expansion up to third order is or is not sufficient to describe the system with enough accuracy. In telecommunications systems of practical interest, where linearity is a prime concern, an expansion up to third order generally gives all the required information [2], [9]. This assumes signal excitation reasonably below output power saturation or, in more practical terms, up to the 1-dB compression point. Thus, it becomes clear from (2) that a signal composed of three different tones is enough to completely characterize all of the first three NLTF's. Obviously, if this characterization is to be done by laboratory measurements, $(\omega_1, \omega_2, \omega_2)$ must be selected among all possible combinations inside the band of interest. Thus, if it is clear that a two-tone test is not sufficient to uniquely identify $H_3(\omega_1, \omega_2, \omega_3)$, it should also be evident that, provided the excitation is always kept in the small-signal range (high carrier-to-distortion-ratios required), there is no theoretical benefit in increasing the number of excitation tones beyond three. Furthermore, in systems where $S[x(t)]$ is a memoryless mapping or in which operation bandwidth is very small compared to the system's available bandwidth, and low-frequency behavior is constant for all possible baseband products, $H_n(\omega_1, \cdots, \omega_n)$ do not depend on frequency. Even a continuous wave (CW) test should then be enough for system identification. The problem with this procedure is that third-order products would appear at $\omega_0 + \omega_0 - \omega_0 = \omega_0$ as co-channel power interference (CCP), which is indistinguishable from the much larger linear response $H_1(\omega_0)$, and at $\omega_0 + \omega_0 + \omega_0 = 3\omega_0$, which usually falls outside the useful bandwidth and, thus, is strongly affected by $H_1(\omega)$ bandpass behavior.

In fact, the simplest way to generate in-band third-order products at frequencies distinct from the linear response is to excite the system with two equal amplitude tones at ω_1 and ω_2 $(X_1 = X_2 = X/\sqrt{2}$ such that input power is $|X|^2)$. The in-band mixing products would then appear at $\omega_1 + \omega_1 - \omega_1 = \omega_1$, $\omega_1 + \omega_2 - \omega_2 = \omega_1$, $\omega_2 + \omega_1 - \omega_1 = \omega_2$, $\omega_2 + \omega_2 - \omega_2 = \omega_2$, but also at $\omega_1 + \omega_1 - \omega_2 = 2\omega_1 - \omega_2$ and $\omega_2 + \omega_2 - \omega_1 = 2\omega_2 - \omega_1$. According to (2) and (4)

$$\begin{aligned} Y_3(\omega_1)\Big[=Y_3(\omega_2)\Big] &= Y_3(\omega_1 + \omega_1 - \omega_1) + Y_3(\omega_1 + \omega_2 - \omega_2) \\ &= \frac{3}{8} H_3(\omega_1, \omega_1, -\omega_1)|X_1|^2 X_1 \\ &\quad + \frac{6}{8} H_3(\omega_1, \omega_2, -\omega_2)|X_2|^2 X_1 \\ &= \frac{9}{8} H_3(\omega, \omega, -\omega) \frac{|X|^2 X}{2\sqrt{2}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} Y_3(2\omega_1 - \omega_2)\Big[=Y_3(2\omega_2 - \omega_1)\Big] &= Y_3(\omega_1 + \omega_1 - \omega_2) \\ &= \frac{3}{8} H_3(\omega_1, \omega_1, -\omega_2)|X_1|^2 X_2^* \\ &= \frac{3}{8} H_3(\omega, \omega, -\omega) \frac{|X|^2 X}{2\sqrt{2}}. \end{aligned} \tag{6}$$

Thus, from the measurement of ACP_2, $|Y_3(2\omega_1 - \omega_2)|^2$, one can evaluate $H_3(\omega, \omega, -\omega)$ and, therefore, assess system's

CCP_2 $|Y_3(\omega_1)|^2$. Also, assuming that $Y_3(\omega_1)$ is negligible compared to $Y_1(\omega_1)$ ($P_{\rm out}/P_{\rm in}$ follows a 1-dB/dB straight line), output third-order intercept point for ACP_2 can be calculated as

$$\mathrm{IP}_{3\mathrm{ACP}_2} = \frac{4}{3}\frac{|H_1(\omega_1)|^3}{|H_3(\omega_1,\omega_1,-\omega_2)|} \tag{7}$$

which stands 4.77 dB above the third-order intercept point for CCP_2.

Another more demanding way of doing the characterization is to use an excitation of three equal amplitude tones ($X_1 = X_2 = X_3 = X/\sqrt{3}$) of equally spaced frequencies ω_1, ω_2, and ω_3, and then rely on composite triple-beat (CTB) products. Some extra care must now be taken because, contrary to the previous two-tone test, the phase (or frequency) correlation between the three tones ϕ_1, ϕ_2, and ϕ_3 does matter. Indeed, for example, for CCP_3 at ω_2, $H_3(\omega_1,\omega_3,-\omega_2)$ contributes with a phase of $\phi_1-\phi_2+\phi_3$, and $H_3(\omega_1,\omega_2,-\omega_1)$, $H_3(\omega_2,\omega_2,-\omega_2)$, and $H_3(\omega_2,\omega_3,-\omega_3)$ all contribute with a phase of ϕ_2. If ϕ_1, ϕ_2, and ϕ_3 are uncorrelated, these two groups add in power, while if they were correlated, they would add up linearly. The same happens with ACP_3 measured at $\omega_4 = \omega_3 + \Delta\omega$, for which $H_3(\omega_2,\omega_3,-\omega_1)$ has a phase of $\phi_2+\phi_3-\phi_1$, while $H_3(\omega_3,\omega_3,-\omega_2)$ contributes with $2\phi_3-\phi_2$.

Nevertheless, it is still possible to relate two- and three-tone excitation results. Assuming uncorrelated phases and a memoryless system, ACP_3 can be given by

$$\begin{aligned}|Y_3(\omega_4)|^2 &= \left|\frac{6}{8}H_3(\omega_2,\omega_3,-\omega_1)\frac{X_1^*X_2X_3}{3\sqrt{3}}\right|^2 \\ &\quad + \left|\frac{3}{8}H_3(\omega_3,\omega_3,-\omega_2)\frac{X_2^*X_3X_3}{3\sqrt{3}}\right|^2 \\ &= \frac{45}{64}|H_3(\omega,\omega,-\omega)|^2\frac{|X|^6}{3^3}\end{aligned} \tag{8}$$

and

$$\begin{aligned}|Y_3(\omega_5)|^2 &= \left|\frac{3}{8}H_3(\omega_3,\omega_3,-\omega_1)\frac{X_1^*X_3^2}{3\sqrt{3}}\right|^2 \\ &= \frac{9}{64}|H_3(\omega,\omega,-\omega)|^2\frac{|X|^6}{3^3}\end{aligned} \tag{9}$$

while CCP_3 is

$$\begin{aligned}|Y_3(\omega_1)|^2 &= \left[|Y_3(\omega_3)|^2\right] \\ &= \left|\frac{3}{8}H_3(\omega_1,\omega_1,-\omega_1)\frac{|X_1|^2X_1}{3\sqrt{3}}\right. \\ &\quad + \frac{6}{8}H_3(\omega_1,\omega_2,-\omega_2)\frac{|X_2|^2X_1}{3\sqrt{3}} \\ &\quad \left. + \frac{6}{8}H_3(\omega_1,\omega_3,-\omega_3)\frac{|X_3|^2X_1}{3\sqrt{3}}\right|^2 \\ &\quad + \left|\frac{3}{8}H_3(\omega_2,\omega_2,-\omega_3)\frac{X_2^2X_3^*}{3\sqrt{3}}\right|^2 \\ &= \frac{234}{64}|H_3(\omega,\omega,-\omega)|^2\frac{|X|^6}{3^3}\end{aligned} \tag{10}$$

and

$$\begin{aligned}|Y_3(\omega_2)|^2 &= \left|\frac{6}{8}H_3(\omega_1,\omega_2,-\omega_1)\frac{|X_1|^2X_2}{3\sqrt{3}}\right. \\ &\quad + \frac{3}{8}H_3(\omega_2,\omega_2,-\omega_2)\frac{|X_2|^2X_2}{3\sqrt{3}} \\ &\quad \left. + \frac{6}{8}H_3(\omega_3,\omega_2,-\omega_3)\frac{X_2|X_3|^2}{3\sqrt{3}}\right|^2 \\ &\quad + \left|\frac{6}{8}H_3(\omega_1,\omega_3,-\omega_2)\frac{X_1X_2^*X_3}{3\sqrt{3}}\right|^2 \\ &= \frac{261}{64}|H_3(\omega,\omega,-\omega)|^2\frac{|X|^6}{3^3}.\end{aligned} \tag{11}$$

Thus, knowing $\mathrm{IP}_{3\mathrm{ACP}_2}$ and the system's linear response, it is possible to predict CTB behavior or, conversely, one can deduce two-tone test responses from CTB measurements. However, again keep in mind that if the system has memory, these conclusions may not be necessarily true.

Another interesting conclusion that may be drawn from these three-tone test results is that CCP can never be fully observed, but simply inferred. Note, for instance, the results of (10), where CCP_3 at ω_1 was calculated. $Y_3(\omega_1)$ depends not only on the adjacent tones ω_2, ω_3, but also on ω_1 itself. This implies that the measurement of full $Y_3(\omega_1)$ power will be masked by $Y_1(\omega_1)$, while the common action of making X_1 null will dramatically perturb $Y_3(\omega_1)$, reducing its value by 8.87 dB for the same amount of output power. If CCP at ω_2 was to be studied, the situation would even be more severe, as it is shown that all terms of $Y_3(\omega_2)$ depend on X_2. (In fact, if tone X_2 was disabled, the experience would be converted into a conventional two-tone test of ω_1 and ω_3, which cannot produce any mixing product between the two).

A generalization of these results to a multitone input spectrum composed of K equally spaced, but uncorrelated, tones is not too difficult, although requiring some laborious combinatory calculations.

III. Multitone IMD Response Calculations

Consider an input spectrum composed of K equally space tones like (1), whose positive frequencies can be given by $\omega_k = \omega_0 + (k-1)\Delta\omega$; $1 \le k \le K$, which excites a third-order memoryless system. In this case, $H_n(\omega_1,\cdots,\omega_n)$ do not depend on frequency and, thus, $H_n(\omega_1,\cdots,\omega_n) = C_n$. According to (2b), the system's output frequencies will be given as all possible combinations ($\omega_m = \omega_{kn} + \omega_{kn} + \cdots + \omega_{kn}$) of the input frequencies ω_{ki}, where n is the order of the system's component in the Volterra series sense. Therefore, the system's third-degree term $C_3[x(t)]^3$ generates two distinct frequency clusters, one around the excitation spectrum and another at the third harmonic. The former, herein represented by ω_r, includes $K-1$ new adjacent tones below the input spectrum ($\omega_0 - (K-1)\Delta\omega \le \omega_r \le \omega_0 - \Delta\omega$), $K-1$ new adjacent tones above the input spectrum ($\omega_0 + K\Delta\omega \le \omega_r \le \omega_0 + 2(K-1)\Delta\omega$), and K tones exactly at the input frequencies ($\omega_0 \le \omega_r \le \omega_0 + (K-1)\Delta\omega$). The cluster near

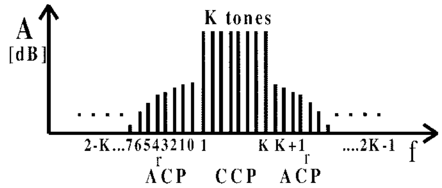

Fig. 1. Output signal spectrum of a third-order nonlinear circuit excited by K equally spaced tones of constant amplitude.

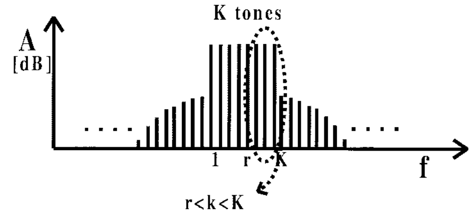

Fig. 2. Division of the input spectrum into two blocks of tones, for computing the number of co-channel mixing products.

third-harmonic, like the ones produced by the second-degree terms are out-of-band products and, thus, are usually discarded in narrow-band microwave applications.

To calculate the amplitude of each of these generated tones, it is necessary to first evaluate the number of different mixing products appearing on each position, and then adding up them linearly or quadraticaly (in voltage or power) according to their phase relation. For that, we will divide the generated spectrum according to the above classification and then present formulas for the corresponding mixing products. These formulas were derived using straightforward, but lengthy, combinatory calculus, which is the reason why their development will not be discussed here. Their proofs can be found in the Appendix.

In general, there will be nth-order mixing products involving only one input frequency $(\omega_{k1}+\cdots+\omega_{k1})$ or involving $2,\cdots,n$ such of tones $(\omega_{k1}+\omega_{k1}+\cdots+\omega_{k2}),\cdots,(\omega_{k1}+\cdots+\omega_{kn})$, which may fall into the same output frequency position. However, each of these sets has a distinct multinomial coefficient and is affected by a different phase arrangement. For example, although $(\omega_2+\omega_3-\omega_4)$ and $(\omega_1+\omega_2-\omega_2)$ correspond to the same frequency (ω_1), there are six different ways in which the first may be constructed, while there are only three ways for the second. Furthermore, the phase of the first is $\phi_2+\phi_3-\phi_4$, while it values ϕ_1 for the second, no matter each input tone considered for ω_2. If the input tones are not correlated in phase, these two mixing products can not be added linearly and, thus, it results are convenient to derive distinct expressions for each of these frequency arrangement types.

A. Number of Third-Order In-Band Mixing Products $(2-K\leq r\leq 2K-1)$

A rapid glance on the output spectrum generation process may allow the conclusion that third-order clusters will be symmetric. The symmetry axis for the in-band mixing products corresponds exactly to the middle of the excitation spectrum $(K+1)/2$. Therefore, it is only necessary to study cases where $K+1\leq r\leq 2K-1$ for adjacent channel products and $\mathrm{round}[(K+1)/2]\leq r\leq K$ for co-channel mixing products, as shown in Fig. 1.

1) Number of Adjacent Channel Mixing Products $(K+1\leq k\leq 2K-1)$*:* At adjacent channel position $\omega_r=\omega_{k1}+\omega_{k2}-\omega_{k3}$, there may appear mixing products of type A, in which $k_1\neq k_2\neq k_3$, or B, in which $k_1=k_2\neq k_3$. The multinomial coefficient for type A is $t_3=6$ and B is $t_3=3$, thus ACP at $\omega_r=\omega_{k1}+\omega_{k2}-\omega_{k3},\ 1\leq k_1,k_2,k_3\leq K$, and $K+1\leq r\leq 2K-1$ where:

Type A: $k_1\neq k_2\neq k_3$:

$$N_A(K,r)=6\left[\left(\frac{2K-r}{2}\right)^2-\frac{\varepsilon}{4}\right] \tag{12}$$

Type B: $k_1=k_2\neq k_3$:

$$N_B(K,r)=3\left[\left(\frac{2K-r}{2}\right)+\frac{\varepsilon}{2}\right] \tag{13}$$

where $\varepsilon=\mathrm{mod}[r/2]$, and $\mathrm{mod}(p/q)$ is the remainder of p/q.

It can be shown that for $r=K+1$, these expressions coincide with the sum of the recursive equation presented by Leffel in [7].

2) Number of Co-Channel Mixing Products $(\mathrm{round}[(K+1)/2]<k\leq K)$*:* At co-channel position $\omega_r=\omega_{k1}+\omega_{k2}-\omega_{k3}$, there may appear mixing products of type A, in which $k_1\neq k_2\neq k_3$, B, in which $k_1=k_2\neq k_3$, C, in which $k_1\neq k_2=k_3$, or even D, in which $k_1=k_2=k_3$. The multinomial coefficient for types A and C is $t_3=6$, and for B and D $t_3=3$. Since we are interested in calculating the number of co-channel products that fall on position r, it will be useful to divide the input spectrum in two different blocks: $1\leq k<r$ of $(r-1)$ tones and $r<k\leq K$ of $K-r$ positions, as is shown in Fig. 2.

The mixing products that fall on ω_r may then be seen as including an upper adjacent power channel (ACP) from the first block, lower ACP from the second, and CCP of mixing frequencies from both and ω_r. This approach enables the use of the formulas already presented for the first two cases

Type A$_1$: $k_1\neq k_2\neq k_3$ and $1\leq k_1,k_2,k_3<r$:

$$N_{A_1}(K,r)=6\left[\left(\frac{r-2}{2}\right)^2-\frac{\varepsilon_1}{4}\right] \tag{14}$$

Type A$_2$: $k_1\neq k_2\neq k_3$ and $r<k_1,k_2,k_3\leq K$:

$$N_{A_2}(K,r)=6\left[\left(\frac{K-r-1}{2}\right)^2-\frac{\varepsilon_2}{4}\right] \tag{15}$$

Type A$_3$: $k_1\neq k_2\neq k_3$ and $1\leq k_1<r$, $r<k_2\leq K$, $k_2\leq k_3<k_1$:

$$N_{A_3}(K,r)=6(K-r)(r-1). \tag{16}$$

Then

$$N_A(K,r)=6\left[\left(\frac{r-2}{2}\right)^2-\frac{\varepsilon_1}{4}+\left(\frac{K-r-1}{2}\right)^2-\frac{\varepsilon_2}{4}+(K-r)(r-1)\right] \tag{17}$$

where $\varepsilon_1 = \mathrm{mod}[(2K-r)/2]$, and $\varepsilon_2 = \mathrm{mod}[(K-r+1)/2]$.

Type B_1: $k_1 \neq k_2$ and $1 \leq k_1, k_2 < r$:

$$N_{B_1}(K,r) = 3\left[\left(\frac{r-2}{2}\right) + \frac{\varepsilon_1}{2}\right] \tag{18}$$

Type B_2: $k_1 \neq k_2$ and $r < k_1, k_2 < K$:

$$N_{B_2}(K,r) = 3\left[\left(\frac{K-r-1}{2}\right) + \frac{\varepsilon_2}{2}\right]. \tag{19}$$

Then

$$N_B(K,r) = 3\left[\left(\frac{r-2}{2}\right) + \frac{\varepsilon_1}{2} + \left(\frac{K-r-1}{2}\right) + \frac{\varepsilon_2}{2}\right] \tag{20}$$

where $\varepsilon_1 = \mathrm{mod}[(2K-r)/2]$, and $\varepsilon_2 = \mathrm{mod}[(K-r+1)/2]$.

Type C: $k_1 \neq k_2$ and $k_1 = r$, $k_2 \leq K$:

$$N_C(K) = 6(K-1) \tag{21}$$

and

$$\text{Type D:} \quad k_1 = k_2 = r: \quad N_D = 3. \tag{22}$$

IV. Nonlinear Distortion Response to Random Inputs

Another common way of evaluating the component's nonlinear distortion performance is by performing a noise test [2].

According to the previous scenario, in this section, we will restrict our analysis to a third-order memoryless system, subject to a narrow bandwidth signal. The random signal $x(t)$ is assumed to have zero mean with Gaussian probability distribution and $R_{xx}(\tau)$, $S_{xx}(\omega)$ as its auto-correlation and power spectrum density functions, respectively [10]. The system's output $y(t)$ will also be random and, thus, its power spectrum density function can be deduced from the auto-correlation function, $R_{yy}(\tau)$ [2], [8]:

$$R_{yy}(\tau) = C_2^2 R_{xx}(0)^2 + \left[C_1^2 + 6C_1C_3R_{xx}(0) + 9C_3^2R_{xx}(0)^2\right] \cdot R_{xx}(\tau) + 2C_2^2R_{xx}(\tau)^2 + 6C_3^2R_{xx}(\tau)^3. \tag{23}$$

By calculating the Fourier transform of $R_{yy}(\tau)$, the power spectrum density function of $y(t)$ becomes

$$\begin{aligned} S_{yy}(\omega) &= C_2^2R_{xx}(0)^2\delta(\omega) + \left[C_1^2 + 6C_1C_3R_{xx}(0) + 9C_3^2R_{xx}(0)^2\right] \\ &\quad \cdot S_{xx}(\omega) + 2C_2^2S_{xx}(\omega) * S_{xx}(\omega) + 6C_3^2S_{xx}(\omega) * S_{xx}(\omega) \\ &\quad * S_{xx}(\omega) \end{aligned} \tag{24}$$

in which $\delta(\omega)$ is a Dirac delta function at dc, and the operator $*$ represents spectral convolution.

For common practical tests, such as noise power ratio (NPR), excitation signals are usually band-limited white Gaussian noise with, or without a notch. Thus, in the following, we will calculate $S_{yy}(\omega)$ of such a random input where $S_{xx}(\omega)$ is

$$S_{xx}(w) = \begin{cases} \dfrac{N_0}{2}, & -\omega_h < \omega < -\omega_l; \omega_l < \omega < \omega_h \\ 0, & \text{elsewhere.} \end{cases} \tag{25}$$

Using (24), we found that $S_{yy}(\omega)$ components falling inside, or close to, the input bandwidth are

$$\begin{aligned} S_{yy}(\omega) &= 18C_3^2\left(\frac{N_0}{2}\right)^3\left(\frac{\omega^2}{2} + (B_w - \omega_l)\omega + \frac{(B_w-\omega_l)^2}{2}\right), \\ &\qquad \omega_l - B_w < \omega < \omega_l \\ &= C_1^2\frac{N_0}{2} + 6C_1C_3\frac{N_0^2}{2}B_w + 9C_3^2\frac{N_0^3}{2}B_w^2 + 18C_3^2\left(\frac{N_0}{2}\right)^3 \\ &\quad \cdot\left(-\omega^2 + (\omega_h+\omega_l)\omega + \frac{B_w^2}{2} - \omega_h\omega_l\right), \qquad \omega_l < \omega < \omega_h \\ &= 18C_3^2\left(\frac{N_0}{2}\right)^3\left(\frac{\omega^2}{2} - (B_w + \omega_h)\omega + \frac{(B_w+\omega_h)^2}{2}\right), \\ &\qquad \omega_h < \omega < \omega_h + B_w \\ &= 0, \qquad \text{elsewhere} \end{aligned} \tag{26}$$

where $B_w = \omega_h - \omega_l$, and $S_{yy}(-\omega) = S_{yy}(\omega)$.

A rapid glance over this expression shows that, due to constant input spectrum density and the successive convolution process, third-order components present a parabolic pattern, except the term $9C_3^2R_{xx}(0)^2S_{xx}(\omega) = 9C_3^2(N_0^3/2)B_w^2$, which is constant in frequency. In fact, it is indistinguishable from the flat linear response. Note, however, that it is indeed part of third-order response, as its amplitude exhibits a cubic dependence on total normalized driving power N_0B_w. The other $S_{yy}(\omega)$ term, which appears at the input band and is proportional to $6C_1C_3N_0^2$, may origin some confusion as it seems a second-order in-band component, dependent on the first- and third-degree coefficients of the nonlinearity, but not on the more natural second-degree C_2. The reason for that resides on the quadratic (average power) nature of the spectrum density function. That term is really the cross product of the output fundamental power $[C_1 + 3C_3R_{xx}(0)]^2$ and describes the existing correlation between the first-order response and part of the third-order response.

In macroscopic terms, the role played by $9C_3^2R_{xx}(0)^2S_{xx}(\omega) = 9C_3^2(N_0^3/2)B_w^2$ is to describe gain compression or expansion—AM–AM—(if a nonlinear system with memory were considered, that component would also lead to AM–PM conversion) and system's loss of sensitivity to one weak signal due to the presence of a stronger one. The remaining parabolic term constitutes the origin of what is normally called IMD, spectral regrowth, or adjacent channel power, observed in real systems subject to random inputs.

Another more revealing conclusion that may be gathered from those results refers to common NPR tests. For that purpose, consider now that a slice of infinitesimal bandwidth $\delta\omega$ was cut off from the driving spectrum at a certain ω_T. Since a finite power density spectrum presumes null power in a vanishing bandwidth, total input power is unaltered. Also, it can be easily understood that all $S_{yy}(\omega)$ components will tend to the previous $S_{yy}(\omega)$ as $\delta\omega$ tends to zero. Thus, if one was measuring noise density power exactly within the notch, the results would remain essentially unaltered, except since $S_{xx}(\omega_T)$ is null, both measured first-order response and third-order term $9C_3^2R_{xx}(0)^2S_{xx}(\omega_T) = 9C_3^2(N_0^3/2)B_w^2$

are zero. This means the impact of measured third-order nonlinear distortion when the notch is present, compared to the untouched excitation, tends to (27), shown at the bottom of this page, which can vary from 5.64 dB in the middle of input bandwidth $(\omega_T = (\omega_l + \omega_h)/2)$ to 6.99 dB in its extremes $(\omega_T = \omega_l$ or $\omega_T = \omega_h)$. Therefore, despite the observed unaltered spectral regrowth, which may support the intuitive idea that a narrow bandwidth notch has no impact on the in-band distortion, this method induces a misleading evaluation that can be as optimistic as 7 dB.

V. Generalization of Multitone Excitation

In this section, we discuss the use of multitone signal excitations to simulate NPR tests. Beyond the direct interest that this analysis has, it also provides a very useful insight to distortion generation under noise or multitone excitations. To do that, we now consider a driving signal composed of K tones, which can be understood as the discrete representation of the band limited power spectrum density function of the previous section. Thus, each tone will have a constant amplitude of

$$|X_k| = \sqrt{\frac{N_o}{2}\Delta\omega} \tag{28}$$

which corresponds to total integrated power of $(N_0/2)\Delta\omega$ in one of the K divisions of B_w. When K tends to infinity, $\Delta\omega = B_w/K$ goes to zero, this input tends to the power density spectrum of (25), and the results of Section III must coincide with the ones of Section IV. This is equal to saying that the following limit must equal $S_{yy}(\omega)$ previously calculated

$$\lim_{\substack{\Delta\omega\to 0\\ K\to\infty}} \frac{1}{\Delta\omega} N(K,k) C_n \left(\frac{N_o}{2}\Delta\omega\right)^n = S_{yy}(\omega) \tag{29}$$

where $N(K,k)$ is the number of different mixing products appearing on k and n is the nonlinearity degree.

Before proceeding with the application of Section III's results, it is necessary to discuss the tones' phase correlation. In order to consider the statistics central limit theorem as a justification to use a K tones signal as a reasonable approximation of a Gaussian distributed random process [10], it is necessary to guarantee that each of the tones is a random variable independent from all the others. Thus, each tone should have a random phase that is completely uncorrelated to all the others, which implies that the K tones must be generated from K distinct source references. (Such an uniformly located multitone spectrum is equivalent to another one composed of K uncommensurated tones). If a pseudorandom (periodic and deterministic) signal is used to generate the input tones, uncorrelated phases are not granted, and the test should be performed by averaging partial results obtained with various phase arrangements. (That also applies for computer simulations of NPR.)

Considering then K uncorrelated tones, normalized power of co-channel tones at $\omega_T = \omega_l + r\Delta\omega$ becomes

$$\begin{aligned} |X(K,r)|^2 &= C_1^2 \frac{N_0}{2}\Delta\omega + C_3^2\left(\frac{N_0}{2}\right)^3 \Delta\omega \Big\{6N_A(K,r) + 3N_B(K,r) \\ &\quad + \big[N_C(K,r) + N_D(K,r)\big]^2\Big\} \end{aligned} \tag{30}$$

which, when $\Delta\omega$ tends to zero, is approximately equal to

$$\begin{aligned} &\lim_{\substack{\Delta\omega\to 0\\ K\to\infty}} \frac{1}{\Delta\omega}|X(K,r)|^2 \\ &\quad = C_1^2\frac{N_0}{2} + C_3^2\left(\frac{N_0}{2}\right)^3 \\ &\qquad \times \Delta\omega^2\left[9K^2 - 18r^2 + 18Kr + 36K^2\right] \\ &\quad = S_{yy}(\omega_l + r\Delta\omega). \end{aligned} \tag{31}$$

For the adjacent channel tones, we have

$$|X(K,r)|^2 = C_3^2\left(\frac{N_0}{2}\right)^3 \Delta\omega^3\left[6N_A(K,r) + 3N_B(K,r)\right] \tag{32}$$

and

$$\begin{aligned} &\lim_{\substack{\Delta\omega\to 0\\ K\to\infty}} \frac{1}{\Delta\omega}|X(K,r)|^2 \\ &\quad = C_3^2\left(\frac{N_0}{2}\right)^3 \Delta\omega^2\left[36K^2 - 36Kr + 9r^2\right] \\ &\quad = S_{yy}(\omega_l + r\Delta\omega). \end{aligned} \tag{33}$$

Beyond the proof of consistency between the results of the previous sections, these expressions provide two other interesting conclusions.

First, the comparison between pairs $(12, 13)$, (26), and $(17, 20, 21, 22)$, (26) allows an error analysis that support the statement, many times gathered from laboratory observation, that a reduced number of uncorrelated tones (usually in the order of ten [11]) is sufficient to simulate a continuous noise spectrum. In fact, when the limit was taken, first-order terms in K and r were neglected in comparison to second-order ones in K^2, Kr, and r^2. Thus, and despite the fact that the approximation error changes with position ω_T, roughly speaking, we can expect an error of less then 1 dB in power when K is greater then ten.

$$\Delta_{\mathrm{NPR}} = \frac{9C_3^2\dfrac{N_0^3}{2}B_w^2 + 18C_3^2\left(\dfrac{N_0}{2}\right)^3\left(-\omega_T^2 + (\omega_h+\omega_l)\omega_T + \dfrac{B_w^2}{2} - \omega_h\omega_l\right)}{18C_3^2\left(\dfrac{N_0}{2}\right)^3\left(-\omega_T^2 + (\omega_h+\omega_l)\omega_T + \dfrac{B_w^2}{2} - \omega_h\omega_l\right)} \tag{27}$$

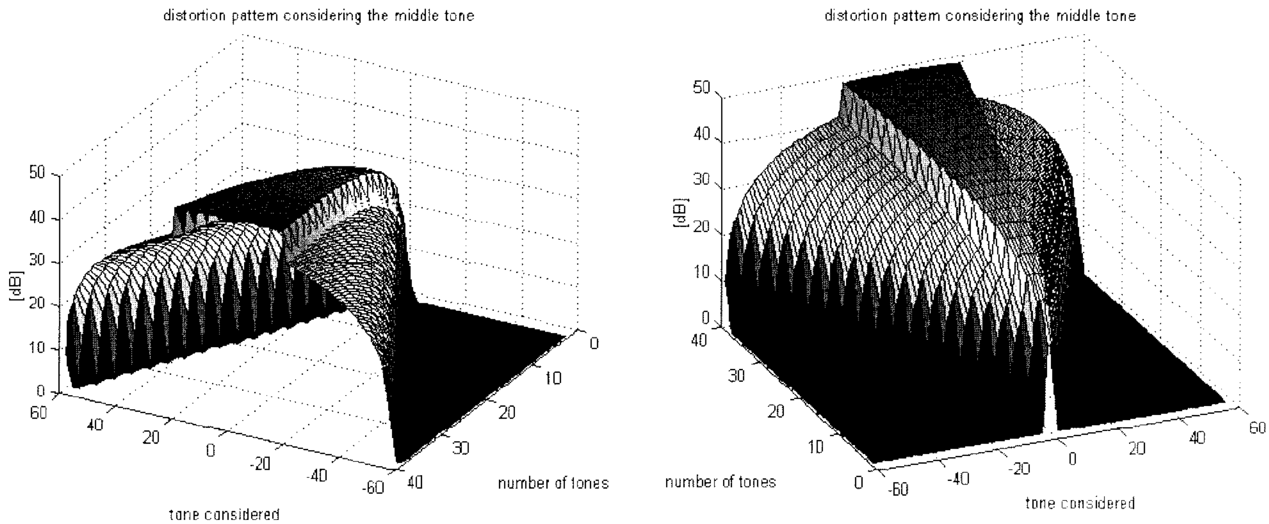

Fig. 3. Three-dimensional view of ACP and CCP distortion components when no in-band tone is shut down (corrected NPR).

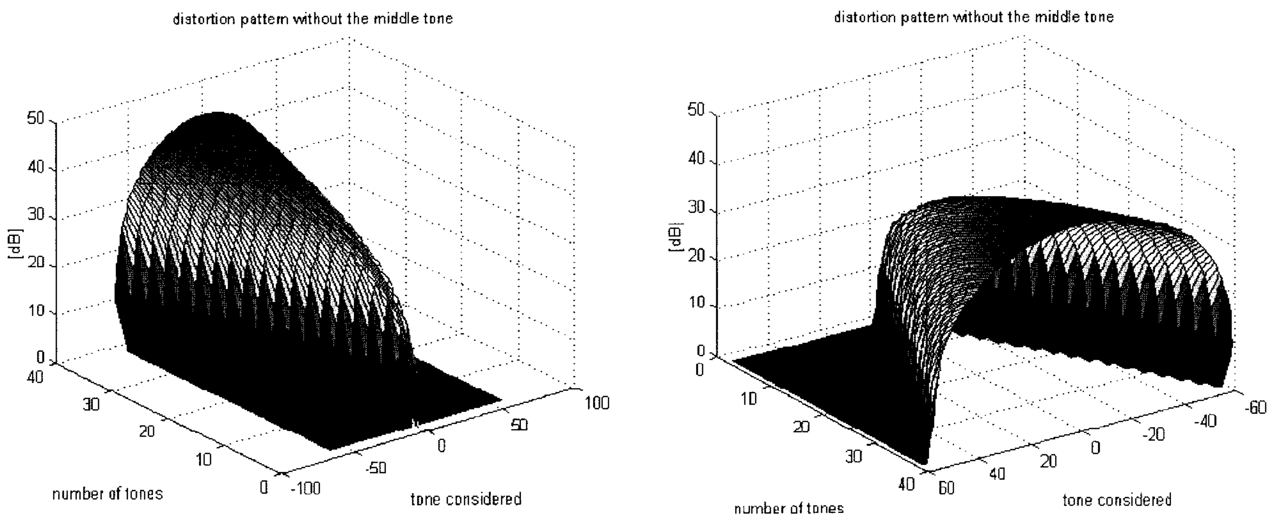

Fig. 4. Three-dimensional view of ACP and CCP distortion components as measured within a notch (usual NPR).

The second conclusion is that co-channel interference, which was modeled by the term $9C_3^2R_{xx}(0)^2S_{xx}(\omega_T) = 9C_3^2(N_0^3/2)B_w^2$ in the noise analysis, is represented in the multitone excitation by mixing products of Types C and D: $\omega_T = \omega_x + \omega_y - \omega_y$ or $\omega_T = \omega_x + \omega_x - \omega_x$. Therefore, eliminating tone $\omega_T = \omega_x$ to read there co-channel distortion corresponds to eliminate about $6K$ mixing products in a total of about $\sqrt{[9K^2 - 18r^2 + 18Kr + 36K^2]}$, which, again, corresponds to an error from 5.64 dB for $r = K/2$ to 7 dB when $r = K$.

Figs. 3 and 4 illustrate these two situations of NPR evaluation as a function of the number of input tones K. The first one represents ACP and CCP as would be observed if no tone were shut down, while the second shows the observed result measured within the notch. The referred difference in measured CCP is obvious.

VI. Comparison Between Various IMD Figures of Merit

In this section we will derive some relational formulas for various IMD figures of merit, e.g., IMD ratio (IMR), ACPR, co-channel power ratio (CCPR), and NPR, obtained under two-tone multitone and band-limited white Gaussian noise excitations. They will be expressed in terms of two-tone IMR, although they can be easily referred to IP_3 using (7), derived in Section II.

For comparison purposes, we will consider all driving signals as having a constant input power of N_0B_w, which corresponds to an amplitude of $\sqrt{(N_0/2)(B_w/K)}$ per tone in a K-tone excitation or to a spectrum density function of $(N_0/2)$ in a B_w bandwidth.

A. IMR

IMR is herein defined as the ratio between linear output power per tone and output power of adjacent channel tones. Thus, IMR for a two-tone test is

$$\mathrm{IMR}_2 = \frac{C_1^2|X|^2}{3^2C_3^2|X|^6} = \left(\frac{4}{3}\frac{C_1}{C_3}\right)^2\frac{1}{N_0^2B_w^2} \tag{34}$$

while for a K-tone excitation

$$\mathrm{IMR}_M(K,r) = \frac{3}{4}\frac{K^2}{2N_A(K,r) + N_B(K,r)}\mathrm{IMR}_2 \tag{35}$$

and for a noise input

$$\mathrm{IMR}_N(\omega) = \frac{1}{8}\frac{B_w^2}{\dfrac{\omega^2}{2} - (B_w+\omega_h)\omega + \dfrac{(B_w+\omega_h)^2}{2}}\mathrm{IMR}_2. \tag{36}$$

The minimum value of $\mathrm{IMR}_M(K,r)$ for K large, which tends to $\mathrm{IMR}_N(\omega=\omega_h)$, is obtained for $r=K+1$ and values $(1/4)$ IMR_2.

B. ACPR

ACPR is considered to be the ratio between total linear output power and total output power collected in the upper or lower adjacent channel. Thus, ACPR for the two-tone test must double IMR_2, while for the K-tone input, it is

$$\begin{aligned}\mathrm{ACPR}_M(K) &= \frac{9}{4}\frac{K^3}{\sum_{r=K+1}^{2K-1}\left[6N_A(K,r)+3N_B(K,r)\right]}\mathrm{IMR}_2\\ &= \frac{3K^3}{4K^3-3K^2-4K+3\left(1-\mathrm{mult}_2(K)\right)}\mathrm{IMR}_2\end{aligned} \tag{37}$$

and for a random input, it is (38), shown at the bottom of this page.

C. CCPR

CCPR is defined here as the ratio between total linear output power and total distortion power collected in the input bandwidth. Two-tone CCPR is, consequently,

$$\mathrm{CCPR}_2 = \frac{1}{9}\mathrm{IMR}_2 \tag{39}$$

and for the K-tone input, it is (40), shown at the bottom of this page.

When the input is band-limited white Gaussian noise, CCPR_N is then (41), shown at the bottom of this page.

D. NPR

According to the usual definition, NPR is the ratio between the linear output power spectrum density to the power spectrum density measured within a prelocated notch. If a multitone signal is used to estimate NPR, power spectrum densities should be substituted by powers per tone. Since a two-tone test does not produce any mixing product between the exciting tones, NPR_2 can not be defined. For a K-tone test, NPR is

$$\begin{aligned}&\mathrm{NPR}_M(K,r)\\ &\quad= \frac{9}{4}\frac{K^2}{6\left[N_A(K,r)-6(K-r)\right]+3N_B(K,r)}\mathrm{IMR}_2\\ &\quad= \frac{K^2}{4K^2-8r^2+8Kr-38K+24r+14-2(\varepsilon_1+\varepsilon_2)}\\ &\qquad\times \mathrm{IMR}_2\end{aligned} \tag{42}$$

where ε_1 and ε_2 are defined as in Section III.

For a noise test, NPR is

$$\begin{aligned}&\mathrm{NPR}_N(\omega)\\ &\quad= \frac{C_1^2\dfrac{N_0}{2}}{18C_3^2\left(\frac{N_0}{2}\right)^3\left(-\omega^2+(\omega_l+\omega_h)\omega+\dfrac{B_w^2}{2}-\omega_h\omega_l\right)}\\ &\quad= \frac{B_w^2}{\left(-8\omega^2+8(\omega_l+\omega_h)\omega+4B_w^2-8\omega_h\omega_l\right)}\mathrm{IMR}_2\end{aligned} \tag{43}$$

$$\mathrm{ACPR}_N(\omega) = \frac{\int_{\omega_l}^{\omega_h} C_1^2\dfrac{N_0}{2}\,d\omega}{\int_{\omega_h}^{\omega_h+B_w} 18C_3^2\left(\dfrac{N_0}{2}\right)^3\left(\dfrac{\omega^2}{2}-(B_w+\omega_h)\omega+\dfrac{(B_w+\omega_h)^2}{2}\right)d\omega} = \frac{3}{4}\mathrm{IMR}_2 \tag{38}$$

$$\begin{aligned}\mathrm{CCPR}_M(K) &= \frac{9}{4}\frac{K^3}{\sum_{r=1}^{K}\left[6N_A(K,r)+3N_B(K,r)\right]+\sum_{r=1}^{K}\left[N_C(K,r)+N_D(K,r)\right]^2}\mathrm{IMR}_2\\ &= \frac{3K^3}{64K^3-102K^2+56K+6\,\mathrm{mod}\left(\dfrac{K}{2}\right)}\mathrm{IMR}_2\end{aligned} \tag{40}$$

$$\mathrm{CCPR}_N(\omega) = \frac{\int_{\omega_l}^{\omega_h} C_1^2\dfrac{N_0}{2}\,d\omega}{\int_{\omega_l}^{\omega_h} 9C_3^2\dfrac{N_0^3}{2}B_w^2\,d\omega + \int_{\omega_l}^{\omega_h} 18C_3^2\left(\dfrac{N_0}{2}\right)^3\left(-w^2+(\omega_l+\omega_h)\omega+\dfrac{B_w^2}{2}-\omega_h\omega_l\right)d\omega} = \frac{3}{4^3}\mathrm{IMR}_2 \tag{41}$$

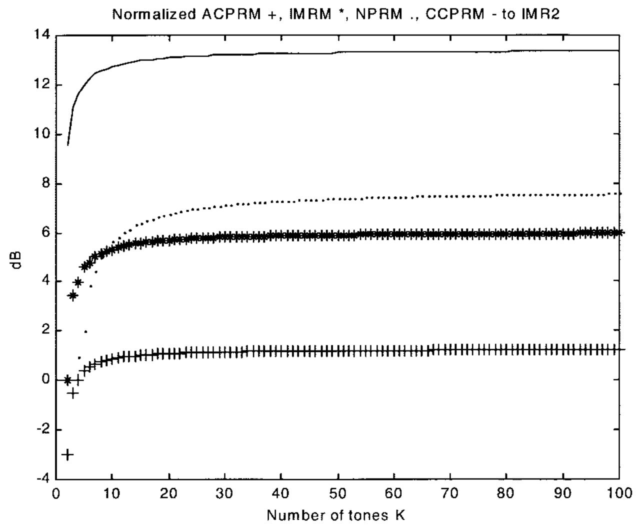

Fig. 5. Relation between multitone ACPR_M, IMR_M, CCPR_M, $\mathrm{NPR}_M(K/2)$, and IMR_2 versus the number of input tones.

$\mathrm{NPR}_N(\omega)$ is minimum in the middle of the input bandwidth where it values

$$\mathrm{NPR}_N\left(\frac{\omega_l+\omega_h}{2}\right)=\frac{1}{6}\mathrm{IMR}_2$$

and is maximum in the extremes

$$\mathrm{NPR}_N(\omega_l,\omega_h)=\frac{1}{4}\mathrm{IMR}_2.$$

Fig. 5 summarizes these results by showing ACPR_M, IMR_M, CCPR_M, and usual NPR_M (measured in the middle of operation bandwidth) normalized to IMR_2 as a function of the number of input tones K. The asymptotic behavior shown for large K corresponds to the random input. As expected for theory, the random approximation is completely fulfilled (the error is within 0.5 dB), except perhaps for usual NPR_M (where it is still 2 dB below its asymptotic value), if a number of discrete tones larger than ten is considered.

VII. Conclusion

In conclusion, the proposed formulas for the analysis of in-band mixing products that arise in a memoryless third-order system, subject to a multitone or band-limited white noise, allowed us the comparison between standard two-tone IMD and, more general specs, like ACPR or NPR. Also, the discussion on co-channel distortion power under either noise or multitone stimuli induced the counter-intuitive conclusion that no matter the notch width, it has a nonnegligible impact on measured distortion. Indeed, it was shown that usual NPR tests can produce misleading co-channel distortion evaluations that can be as optimistic as 5.6 or 7 dB in the middle or the extremes of the input bandwidth, respectively.

Appendix

The objective here is to give a proof of the expressions presented for the number of in-band mixing products generated in a memoryless third-order system driven by K equally spaced tones. Beyond the natural objective of those proofs, they were herein presented as they also provide an insight onto how the expressions were derived.

The method of proof follows mathematical induction. First, the expression under proof is shown to apply for a simple excitation of three tones, and then it will be shown to apply for $K+1$ tones, provided that it also applies for a general number of K input tones. Since the expressions' validity for a three-tone input is immediately proven by comparing their results with the coefficients of (8) to (11), it will be omitted. The latter part of the proof will be achieved by calculating the increment in the number of mixing products that fall on test position r, imposed by the addition of a new input tone $K+1$, $\Delta N(r)$, and comparing the results of the expression under proof for K and $K+1$, i.e., showing that $N(K+1,r)=N(K,r)+\Delta N(r)$.

A. Number of Adjacent Channel Mixing Products

1) Adjacent Channel Mixing Products of Type A: For ACP products $\omega_r=\omega_{k1}+\omega_{k2}-\omega_{k3}$, $1\le k_1,k_2,k_3\le K$, and $K+2\le r\le 2K-1$, where $k_1\ne k_2\ne k_3$, $N_A(K,r)$ is given by (12). $N_A(K+1,r)$ must equal $N_A(K,r)+\Delta N_A(r)$, where $\Delta N_A(r)$ is the increment in newly generated mixing products that fall on position r, due to the addition of a new input tone at $K+1$. Since we are studying mixing products of the form $\omega_r=\omega_{k1}+\omega_{k2}-\omega_{k3}$, only ω_{k1} can be made equal to ω_{K+1}, as $\omega_{k3}=\omega_{K+1}$ would not produce any product on ω_r, and products where $\omega_{k2}=\omega_{K+1}$ are already included in $\omega_{k1}=\omega_{K+1}$. Thus, the number of new combinations $\Delta N_A(r)$, which verify $\omega_r=\omega_{K+1}+\omega_{k2}-\omega_{k3}=\omega_0+(r-1)\Delta\omega$ is such that

$$\begin{aligned}\omega_0+K\Delta\omega+\omega_0+(k_2-1)\Delta\omega-\omega_0-(k_3-1)\Delta\omega\\ =\omega_0+(r-1)\Delta\omega,\qquad \text{while } k_2<K+1;\quad k_3\ge 1.\end{aligned}$$

Therefore, $K+k_2-k_3=r-1\wedge k_2<K+1\wedge k_3\ge 1$ or $k_2=k_3-K+r-1\wedge k_2<K+1\wedge k_3\ge 1$ or even $k_3<2K-r+2\wedge k_3\ge 1$. Thus, $k_3=1,2,\cdots,2K-r+1$ and $\Delta N_A(r)=6(2K-r+1)$. It is now straightforward to show that $N_A(K+1,r)$ is indeed equal to $N_A(K,r)+6(2K-r+1)$.

2) Adjacent Channel Mixing Products of Type B: For ACP products $\omega_r=\omega_{k1}+\omega_{k2}-\omega_{k3}$, $1\le k_1,k_2,k_3\le K$, and $K+2\le r\le 2K-1$, where $k_1=k_2\ne k_3$, $N_B(K,r)$ is given by (13). Once again, only ω_{k1} can be made equal to ω_{K+1} because $\omega_{k3}=\omega_{K+1}$ would not produce any product on ω_r. Thus, the number of new combinations $\Delta N_B(r)$, which verify $\omega_r=2\omega_{K+1}-\omega_{k3}=\omega_0+(r-1)\Delta\omega$ is such that

$$\begin{aligned}2\omega_0+2K\Delta\omega-\omega_0-(k_3-1)\Delta\omega\\ =\omega_0+(r-1)\Delta\omega;\ k_3\ge 1.\end{aligned}$$

Therefore, $k_3=2(K+1)-r\wedge k_3\ge 1$. This means only one new mixing product involving tone $K+1$ will be generated and $\Delta N_B(r)=3$. In can be easily verified that $N_B(K+1,r)$ indeed equals $N_B(K,r)+3$.

B. Co-Channel Mixing Products

1) Co-Channel Mixing Products of Type A: Since mixing products of types A_1 and A_2 are, in fact, ACP products from each of the blocks in which the K input tones were grouped, their expressions were already proven.

The number of Type A_3 co-channel mixing products, i.e., combinations that arise from beats involving tones of the two blocks, is given by (16). The products now under study have the form $\omega_r = \omega_{k1} + \omega_{k2} - \omega_{k3}$, $1 \leq k_1 < r$, $r < k_2 \leq K$, and $k_2 \leq k_3 < k_3$, where $k_1 \neq k_2 \neq k_3$. If a new input tone is inserted in position $K+1$, the left-hand-side block will be unchanged, maintaining its $r-1$ tones, but the right-hand-side one will have one more tone or a dimension of $K+1-r$. Therefore, to determine the number of new mixing components involving ω_{K+1}, only ω_{k2} can be made equal to that frequency, and it may be concluded that

$$\omega_0+(k_1-1)\Delta\omega+\omega_0+(K+1-1)\Delta\omega-\omega_0-(k_3-1)\Delta\omega = \omega_0+(r-1)\Delta\omega \text{ or } k_1-k_3=-(K-r+1).$$

Since $(K-r+1)$ is a constant, this means there will be an unique new pair (k_1, k_3) for each k_1. Thus, the total number of newly generated products of Type A_3 involving ω_{K+1} will be exactly equal to six times the number of possible different k_1 or $\Delta N_{A_3}(r) = 6(r-1)$. This, in turn, really proves that $N_{A_3}(K+1, r) = N_{A_3}(K, r) + \Delta N_{A_3}(r)$.

If, on the other hand, a new tone is inserted on the left-hand-side block, this block will now extend from $k=1$ to $k=r$ since the test point will be shifted one position to the right, becoming $r+1$. The right-hand-side block will maintain its dimension of $K-r$ tones. Therefore, in this new case, we have to prove that $N_{A_3}(K+1, r+1) = N_{A_3}(K, r) + \Delta N_{A_3}(r+1)$. To calculate $\Delta N_{A_3}(r+1)$, we must recognize that now only ω_{k1} can be made equal to the new ω_1 tone and, thus,

$$\omega_0+\omega_0+(k_2-1)\Delta\omega-\omega_0-(k_3-1)\Delta\omega = \omega_0+(r)\Delta\omega \text{ or } k_2-k_3=r.$$

Once again, there will be only one possible k_3 for each k_2, which means that the number of new mixing products will equal the number of k_2 or $K-r$. Thus, $\Delta N_{A_3}(r+1) = K-r$, which, in fact, implies $N_{A_3}(K+1, r+1) = N_{A_3}(K, r) + \Delta N_{A_3}(r+1)$.

2) Co-Channel Mixing Products of Type B: The number of co-channel mixing components of type B were again derived as ACP products of type B, whose expression was already proven above.

3) Co-Channel Mixing Products of Type C: Equation (21) represents the number of co-channel mixing products falling on $\omega_r = \omega_{k1} + \omega_{k2} - \omega_{k2}$, where $k_1 = r$ and $k_2 (\neq k_1) \leq K$. The insertion of a new tone ω_{K+1} will contribute with six new products to ω_r. Therefore, $\Delta N_C(r) = 6$ and, in fact, $N_C(K+1, r) = N_C(K, r) + \Delta N_C(r)$.

4) Co-Channel Mixing Products of Type D: As can be seen from (22), the number of products that fall on $\omega_r = \omega_{k1} + \omega_{k2} - \omega_{k2}$ $(k_1 = k_2)$ is three, independent on K. Accordingly, $\Delta N_D(r)$ should be zero, and really, the insertion of a new tone ω_{K+1} cannot contribute to this type of mixing product since it involves only the tone ω_r.

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José Carlos Pedro (S'90–M'95), for photograph and biography, see this issue, p. 2374.

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