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An Introduction to Partial Differential Equations

Daniel J. Arrigo

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An Introduction to Partial Differential Equations

Daniel J. Arrigo
Universital of Central Arkansas

SYNTHESIS LECTURES ON MATHEMATICS AND STATISTICS #21

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ABSTRACT

This book is an introduction to methods for solving partial differential equations (PDEs). After the introduction of the main four PDEs that could be considered the cornerstone of Applied Mathematics, the reader is introduced to a variety of PDEs that come from a variety of fields in the Natural Sciences and Engineering and is a springboard into this wonderful subject. The chapters include the following topics: First-order PDEs, Second-order PDEs, Fourier Series, Separation of Variables, and the Fourier Transform. The reader is guided through these chapters where techniques for solving first- and second-order PDEs are introduced. Each chapter ends with a series of exercises illustrating the material presented in each chapter.

The book can be used as a textbook for any introductory course in PDEs typically found in both science and engineering programs and has been used at the University of Central Arkansas for over ten years.

KEYWORDS

advection equation, heat equation, wave equation and Laplace's equation, method of characteristics, separation of variables, Fourier series, and the Fourier transform

Contents

Preface

This is an introductory book about obtaining exact solutions to partial differential equations (PDEs). It is based on my lecture notes from a course I have taught almost every year since 2001 at the University of Central Arkansas (UCA).

When I began teaching the course, I tried several textbooks. There are many fine textbooks on the market but they just seemed to miss the mark for students at UCA. Even though the average ACT scores of incoming freshmen at UCA are among the highest in the state, the textbooks that were available were too sophisticated for my students. I also felt that the way most books taught the subject matter could be improved upon. For example, a lot of the books start with the separation of variables, a technique used for solving second-order linear PDEs. As most books on ordinary differential equations start with solving first-order ODEs before considering second-order ODEs, I felt the same order would be beneficial in solving PDEs. This naturally led to the presentation in this book.

In Chapter 1, I introduce four basic PDEs which some would consider the cornerstone of Applied Mathematics: the advection equation, the heat equation, Laplace's equation, and the wave equation. After this, I list 12 PDEs (systems of PDEs) that appear in science and engineering and provide a springboard into the subject matter. Most of these PDEs are nonlinear in nature since our world is inherently nonlinear. However, one must first know how to solve linear PDEs before entering the nonlinear world.

In Chapter 2, I introduce the student to first-order PDEs. Through a change of variables, we solve constant coefficient and linear PDEs and are led to the method of characteristics. We continue with solving quasilinear and higher-dimensional PDEs, and then progress to fully nonlinear first-order PDES. The chapter ends with Charpit's method, a method that seeks compatibility between two first-order PDEs.

In Chapter 3, we focus on second-order PDEs, and in particular, three standard forms: (i) parabolic standard form, (ii) hyperbolic standard form, and (iii) elliptic standard form. Students learn how to transform to each standard form. The chapter ends with a derivation of the classic d'Alembert solution.

In Chapter 4, after a brief introduction to separation of variables for the heat equation, I introduce Fourier series. I introduce both the regular Fourier series and the Fourier Sine and Cosine series. Several examples are considered showing various standard functions and their Fourier series representations. At this point, I return the students to solving PDEs.

In Chapter 5, we continue our discussion with the separation of variables where we consider the heat equation, Laplace's equation and the wave equation. We start with the heat equation and consider several types of problems. One example has fixed homogeneous boundary conditions, no flux boundary conditions and radiating boundary conditions; then we consider nonhomogeneous boundary conditions. Next, we consider nonhomogeneous equations, equations with solution dependent source terms, then solution dependent convective terms. We move on to Laplace's equation and, finally, to the wave equation.

The final chapter, Chapter 6, involves the Fourier (Sine/Cosine) transform. It is a generalization of the Fourier series, where the length of the interval approaches infinity. It is through these transforms that we are able to solve a variety of PDEs on the infinite and half infinite domain.

The book is self-contained; the only requirements are a solid foundation in calculus and elementary differential equations. Chapters 1–5 have been the basis of a one-semester course at the University of Central Arkansas for over a decade. The material in Chapter 6 could certainly be included. For the times that I have taught the course, I have omitted Chapter 6 in favor of student seminars. I ask students to pick topics, extensions or applications of the material covered in class, and present oral seminars to the class with formal write-ups on their topic being due by the end of the course. My goal is that at the end of the course the students understand why studying this subject is important.

Daniel J. Arrigo
January 2018

Acknowledgments

I first would like to thank my wife Peggy, who once again became a book widow. My love and thanks. Second, I would like to thank Professors West Vayo (University of Toledo) and Jill Guerra (University of Arkansas – Fort Smith) who used earlier versions of this book and gave valuable feed back. Third, I would like to thank all of my students who, over the past 10+ years, volunteered to read the book and gave much-needed input on both the presentation of material and on the complexity of the examples given. Finally, I would like to thank Susanne Filler of Morgan & Claypool Publishers. Once again, she made the process a simple and straightforward one.

Daniel J. Arrigo
January 2018

CHAPTER 1

Introduction

An ordinary differential equation (ODE) is an equation which involves ordinary derivatives. For example, if $y = y(x)$, then the following are ODEs:

$$y' = 0, \quad y' = y, \quad y'' - y = 0, \quad y'' - y^2 = 0. \tag{1.1}$$

In an introductory course in ODEs, one learns techniques to solve such equations. The first in (1.1) is probably the simplest as

$$y' = 0 \tag{1.2}$$

gives rise to the solution

$$y = c, \tag{1.3}$$

where c is an arbitrary constant. The second in (1.1)

$$y' = y \tag{1.4}$$

gives rise to the solution

$$y = ce^x, \tag{1.5}$$

where again c is an arbitrary constant. Figure 1.1 shows solution curves as we vary the constant c. In order to pick out a particular curve we would have to add to the ODE some sort of initial condition. For example, if we were to say that $y(0) = 1$, then the single curve highlighted in blue would be our solution.

Partial differential equations (PDEs), on the other hand, are equations which involve partial derivatives. For example, if $u = u(x, y)$ then

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1.6}$$

are all PDEs. The first in (1.6)

$$\frac{\partial u}{\partial x} = 0 \tag{1.7}$$

is one of the easiest to solve and one can verify that

$$u = c, \quad u = y, \quad u = 3\sin y, \quad \text{and} \quad u = y^2 + \ln|y| \tag{1.8}$$

are all solutions of (1.7). In fact, the most general solution of (1.7) is

$$u = f(y), \tag{1.9}$$

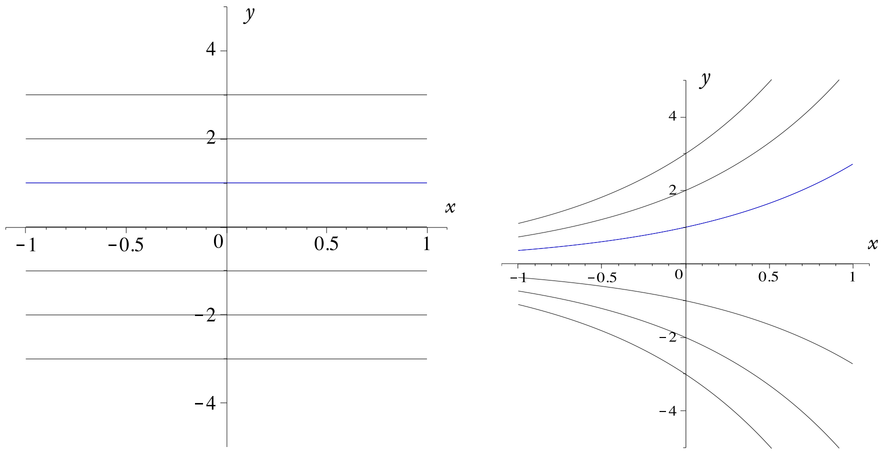

Figure 1.1: The solutions of (1.2) and (1.4).

where $f(y)$ is an arbitrary function for y. Similarly, the most general solution of

$$\frac{\partial u}{\partial y} = 0 \tag{1.10}$$

is

$$u = g(x), \tag{1.11}$$

where $g(x)$ is an arbitrary function for x. One of the subtle differences in the solutions of ODEs and solutions of PDEs is that in ODEs, the solutions contain constants of integration whereas in PDEs, the solutions contain functions of integration.

But, what about the third PDE in (1.6)? As this is just the addition of (1.7) and (1.10), one might think that the solution would be

$$u = f(y) + g(x). \tag{1.12}$$

However, substituion into

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1.13}$$

shows it is not identically satisfied! The reader might want to show that the actual solution is $u = f(x - y)$, where f is an arbitray function of its argument. So, a natural question is how did this solution come about? This is a good question and one that will be answered in this book.

To motivate the study of PDEs, we begin with a standard formulation of the four basic models which lie at the cornerstone of applied mathematics—the advection equation, the diffusion equation, Laplace's equation, and the wave equation.

1.1 MODEL EQUATIONS

1.1.1 ADVECTION EQUATION

Suppose a certain amount of a chemical spills into a river and flows downstream. Let us suppose that the concentration is given by $u = u(x,t)$. Then the total amount of chemical is given by

$$\int_0^b u(x,t)dx. \tag{1.14}$$

If the river flows with constant speed c, then this spill will flow downstream and if we assume that the chemical does not diffuse, then at $t = h$ we have

$$\int_{ch}^{b+ch} u(x,t+h)dx \tag{1.15}$$

and these two are equal.

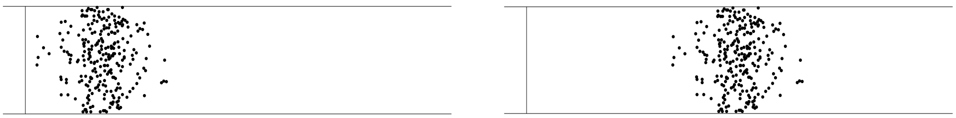

Figure 1.2: Chemical spill.

Thus,

$$\int_{ch}^{b+ch} u(x,t+h)dx = \int_0^b u(x,t)dx. \tag{1.16}$$

Differentiating with respect to h gives

$$\int_{ch}^{b+ch} u_t(x,t+h)dx + cu(b+ch,t+h) - cu(ch,t+h) = 0 \tag{1.17}$$

and using the fundamental theorem of Calculus, the last two terms can be written as

$$\int_{ch}^{b+ch} u_t(x,t+h)dx + c\int_{ch}^{b+ch} u_x(x,t+h)dx = 0 \tag{1.18}$$

or

$$\int_{ch}^{b+ch} \Big(u_t(x,t+h) + cu_x(x,t+h)\Big)dx = 0 \tag{1.19}$$

and since b is arbitrary the integrand is zero and in the limit as $h \to 0$ gives

$$u_t + cu_x = 0. \tag{1.20}$$

If we knew the initial concentration at $t = 0$, say

$$u(x,0) = f(x) \tag{1.21}$$

we would want to solve (1.20) subject to the initial condition (1.21).

1.1.2 DIFFUSION EQUATION

The diffusion equation is used to describe the flow of heat and is often called the heat equation. The flow of heat is due to a transfer of thermal energy caused by an agitation of molecular matter. The two basic processes that take place in order for thermal energy to move is (1) conduction—collisions of neighboring molecules not moving appreciably, and (2) convection—vibrating molecules changing locations. We consider the diffusion in a one-dimensional rod.

Consider a rod of length L, cross-section area A and density $\rho(x)$.

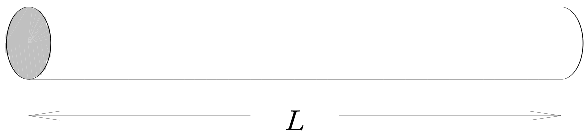

Figure 1.3: (a) The typical cross-section.

We introduce the thermal energy density function $e(x,t)$ defined by

$$e(x,t) = c(x)\rho(x)u(x,t), \tag{1.22}$$

where $c(x)$ is the specific heat, $\rho(x)$ is the density along the rod, and $u(x,t)$ is the temperature in the rod at the location x and at time t. We define the total heat $H(x,t)$ as

$$H(x,t) = \int_0^L c(x)\rho(x)u(x,t)A\,dx. \tag{1.23}$$

The amount of thermal energy per unit time flowing to the right per unit surface area is called "heat flux" defined by $\phi(x,t)$.

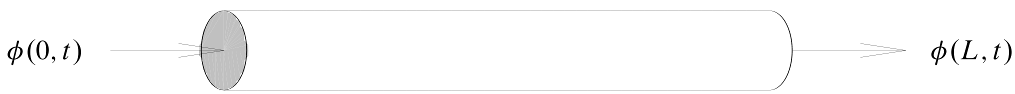

Figure 1.3: (b) Change in the flux in a rod of length L.

If heat is generated within the rod and its heat density $Q(x,t)$ is given, then the total heat generated is

$$\int_0^L Q(x,t)A\,dx. \tag{1.24}$$

The conservation of heat energy states:

$$\begin{matrix}\text{rate of change} \\ \text{of total heat}\end{matrix} = \begin{matrix}\text{heat flowing} \\ \text{across boundaries}\end{matrix} + \begin{matrix}\text{heat energy} \\ \text{generated inside}\end{matrix}$$

or, mathematically,

$$\frac{dH}{dt} = (\phi(0,t) - \phi(L,t))\,A + \int_0^L Q(x,t)Adt,$$

so

$$\frac{d}{dt}\int_0^L c(x)\rho(x)u(x,t)Adx = (\phi(0,t) - \phi(L,t))\,A + \int_0^L Q(x,t)Adt. \tag{1.25}$$

Using Leibniz' rule and the fundamental theorem of calculus, Eq. (1.25) becomes

$$\int_0^L c(x)\rho(x)\frac{\partial u(x,t)}{\partial t}Adx = -\int_0^L \frac{\partial \phi}{\partial x}Adx + \int_0^L Q(x,t)Adt,$$

which gives

$$\int_0^L \left(c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x,t)\right)Adx = 0. \tag{1.26}$$

Since this applies to any length L, then

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} + \frac{\partial \phi}{\partial x} - Q(x,t) = 0,$$

or

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t). \tag{1.27}$$

Flux and Fourier's Law

Fourier's law is a description on how heat flows in a temperature field and is based on the following.

1. If the temperature is constant, there is no heat flow.
2. If there is a temperature difference, there will be flow and it will flow hot to cold.
3. The greater the temperature difference, the greater the heat flow.
4. Heat flows differently for different materials.

With these in mind, Fourier suggested the following form for heat flux:

$$\phi = -k\frac{\partial u}{\partial x}, \tag{1.28}$$

where k is the thermal conductivity. This then gives (1.27) as

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x,t,u)\,\frac{\partial u}{\partial x}\right) + Q(x,t). \tag{1.29}$$

In the case where the density, the specific heat and thermal conductivity are all constant, (1.29) becomes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{1.30}$$

where $D = k/\rho c$, is the coefficient of diffusion, and $f = Q/\rho c$, is the source term. Equation (1.30) is often referred to as the $1-D$ diffusion or heat equation with constant diffusion D with source f.

Boundary and Initial Conditions

In solving Eq. (1.30) for the temperature $u(x,t)$, it is necessary to prescribe information at the endpoints of the rod as well as a initial temperature distribution. These are referred to as boundary conditions (BCs) and initial conditions (ICs).

Boundary Conditions

These are conditions at the end of the rod. There are several types and the following are the most common.

(i) *Prescribed Temperature*

Usually given as

$$u(0,t) = u_l, \qquad u(L,t) = u_r, \tag{1.31}$$

where u_l and u_r are constant, or

$$u(0,t) = u_l(t), \qquad u(L,t) = u_r(t), \tag{1.32}$$

with u_l and u_r varying with respect to time.

(ii) *Prescribed Temperature Flux*

If we were to prescribe the heat flow at the boundary, we would impose

$$-k_0\frac{\partial u}{\partial x} = \phi(x,t), \tag{1.33}$$

so

$$-k_0\frac{\partial u}{\partial x}(0,t) = \phi(0,t), \qquad -k_0\frac{\partial u}{\partial x}(L,t) = \phi(L,t), \tag{1.34}$$

where $\phi(0,t)$ and $\phi(L,t)$ are given functions of t. In the case of insulated boundaries, i.e., zero flux, then these would be

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0. \tag{1.35}$$

(iii) *Radiating Boundary Conditions*

In the case where one or both ends are such that the temperature change is proportional to the temperature itself, we would use

$$\frac{\partial u}{\partial x}(0,t) = k_1 u(0,t), \qquad \frac{\partial u}{\partial x}(L,t) = k_2 u(L,t), \tag{1.36}$$

where $k_i > 0$ is gain and $k_i < 0$ is loss.

Initial Conditions

In addition to describing the temperature at the boundary, it is also necessary to describe the temperature initially (at $t = 0$). This is typically given as

$$u(x,0) = f(x). \tag{1.37}$$

1.1.3 LAPLACE'S EQUATION

The solutions of Laplace's equation are used in many important fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. In the study of heat conduction, the Laplace equation is the steady-state heat equation.

Here we will derive the equation in context of 2D irrotational fluid flows. We consider a small control element emersed in a fluid. We denote the sides of this rectangular region by dx and dy. The flow through the region is given by $\vec{v} =< u, v >$, where u and v are velocities in the x and y directions, respectively. We also assume the density of the fluid ρ is constant throughout the entire fluid. We will assume for the sake of discussion that the flow is left to right and bottom to top. The mass flow in from the left and bottom of the control element is

$$\dot{m}_{\text{in}} = \rho u\Big|_x dy + \rho v\Big|_y dx \tag{1.38}$$

and mass flow out from the right and top is

$$\dot{m}_{\text{out}} = \rho u\Big|_{x+dx} dy + \rho v\Big|_{y+dy} dx. \tag{1.39}$$

If $M = \rho dA = \rho dxdy$ is the overall mass, which is constant (we are assuming the density is constant), then the change in mass is $\frac{dM}{dt} = \dot{m}_{\text{in}} - \dot{m}_{\text{out}} = 0$. So from (1.38) and (1.39)

$$\rho u\Big|_x dy + \rho v\Big|_y dx - \left(\rho u\Big|_{x+dx} dy + \rho v\Big|_{y+dy} dx\right) = 0. \tag{1.40}$$

Dividing through by $dxdy$ gives

$$-\frac{\rho u\Big|_{x+dx} - \rho u\Big|_x}{dx} - \frac{\rho v\Big|_{y+dy} - \rho v\Big|_y}{dy} = 0 \tag{1.41}$$

and in the limit as $dx, dy \to 0$ gives

$$(\rho u)_x + (\rho v)_y = 0 \tag{1.42}$$

which is known as the continuity equation. When ρ is constant, then (1.42) becomes

$$u_x + v_y = 0. \tag{1.43}$$

We now consider the deformation of the rectangular element given in Fig. 1.4a. At time t, the velocity in the x direction at the point O is u and at point B is $u + \frac{\partial u}{\partial y} dy$. Also, the velocity in the y direction at the point O is v and at point A is $v + \frac{\partial v}{\partial x} dx$.

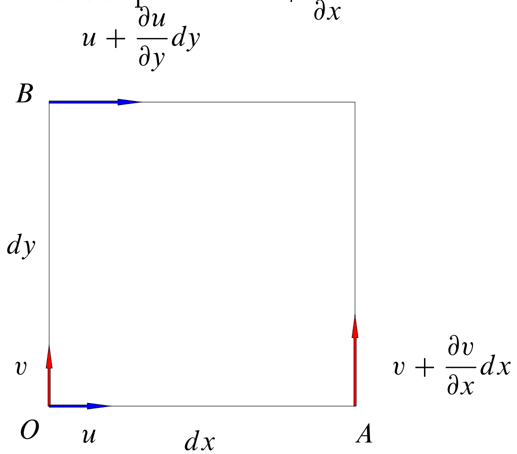

Figure 1.4: (a) A control element.

So at time $t + dt$, the rectangle has deformed to that shown in Fig. 1.4b.
For small angles

$$d\alpha = \tan d\alpha = \frac{\partial v}{\partial x} dt, \quad d\beta = \tan d\beta = \frac{\partial u}{\partial y} dt \tag{1.44}$$

from which we obtain the angular velocities $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ as

$$\frac{d\alpha}{dt} = \frac{\partial v}{\partial x}, \quad \frac{d\beta}{dt} = \frac{\partial u}{\partial y}. \tag{1.45}$$

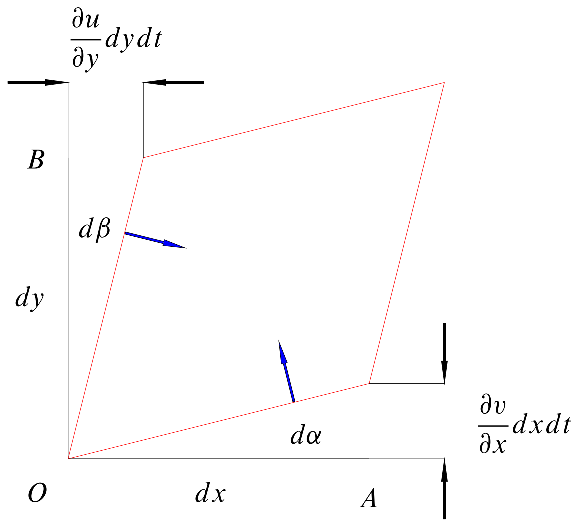

Figure 1.4: (b) Deformed control element.

If we denote ω as the average angular velocity of the rectangle then we obtain

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{1.46}$$

If the fluid is irrotational, then $\omega = 0$ giving

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{1.47}$$

If we introduce a potential ϕ such that

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \tag{1.48}$$

then we see that (1.47) is automatically satisfied whereas (1.43) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1.49}$$

which is known as Laplace's equation.

1.1.4 WAVE EQUATION

Consider the motion of a perfectly elastic string in which the horizontal motion is negligible and the vertical motion is small. Let us represent this vertical displacement by $y = u(x,t)$. The string will move according to a change in tension throughout the string and, in particular, on $[x, x+\Delta x]$ where the tension at the endpoints is $T(x,t)$ and $T(x+\Delta x, t)$, respectively (see Figs. 1.5a and 1.5b).

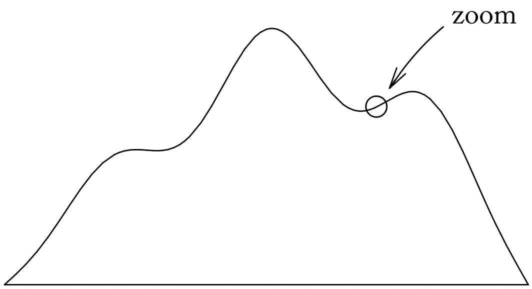

Figure 1.5: (a) A plucked string.

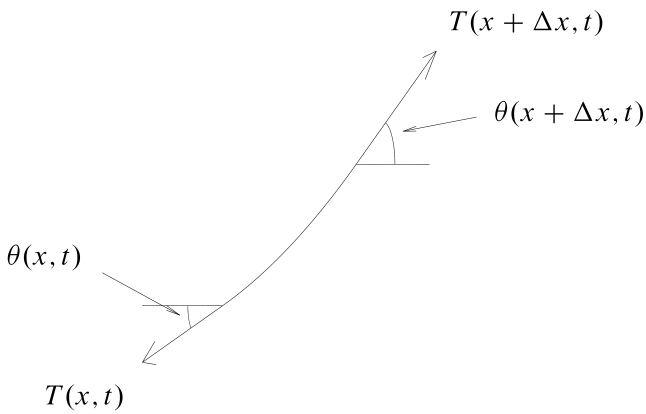

Figure 1.5: (b) Tension in a small length of string.

If the mass throughout the string is denoted by $\rho = \rho(x)$, using Newton's second law $F = ma$ gives

$$F = ma = \rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}. \tag{1.50}$$

This must be balanced by the resultant tension, namely, $T\uparrow - T\downarrow$. At the endpoints we have

$$T\downarrow = T(x,t)\ \sin\ \theta(x,t), \tag{1.51a}$$

$$T\uparrow = T(x+\Delta x,t)\ \sin\ \theta(x+\Delta x,t) \tag{1.51b}$$

and the difference gives

$$F = T(x+\Delta x,t)\sin\ \theta(x+\Delta x,t) - T(x,t)\sin\ \theta(x,t). \tag{1.52}$$

The balance of forces, namely Eqs. (1.50) and (1.52), gives rise to the following

$$\rho(x)\Delta x\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\ \theta(x+\Delta x,t) - T(x,t)\sin\ \theta(x,t),$$

and in the limit as $\Delta x \to 0$ then

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \lim_{\Delta x\to 0}\left(\frac{T(x+\Delta x,t)\sin\ \theta(x+\Delta x,t) - T(x,t)\sin\ \theta(x,t)}{\Delta x}\right),$$

from which we obtain

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\ \sin\ \theta(x,t)\Big). \tag{1.53}$$

Since we are assuming that displacements are small, then $\theta(x,t) \simeq 0$, so that

$$\sin\ \theta(x,t) \simeq\ \tan\ \theta(x,t)$$

leads us to replace Eq. (1.53) with

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\ \tan\ \theta(x,t)\Big). \tag{1.54}$$

Since

$$\tan\ \theta(x,t) = \frac{\partial u}{\partial x} \tag{1.55}$$

this gives Eq. (1.54) as

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\ \frac{\partial u}{\partial x}\right). \tag{1.56}$$

This PDE is commonly known as the "wave equation."
If we assume that the string is homogeneous and perfectly elastic, then $\rho(x) = \rho_0$ and $T(x,t) = T_0$, both constant. This, in turn, gives the wave equation as

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2}. \tag{1.57}$$

We note that the units of T_0 is $kg\ m/s^2$ and the units of ρ_0 is kg/m giving the units of T_0/ρ_0 as m^2/s^2 – the units of speed. Thus, we introduce the term *wave speed* which we denote by the variable c where $c^2 = T_0/\rho_0$. This gives the wave equation as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{1.58}$$

Boundary Conditions

As in the case of the heat equation, it is necessary that boundary conditions be prescribed, i.e., conditions at the end points of the string. For example, if the endpoints are fixed at zero, then the following BCs would be used:

$$u(0,t) = 0, \qquad u(L,t) = 0. \tag{1.59}$$

If the ends were forced to move, then we would impose the following BCs

$$u(0,t) = f(t), \qquad u(L,t) = g(t), \tag{1.60}$$

where f and g would be specified. If the ends were free to move, then we would impose the following BCs:

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0. \tag{1.61}$$

Similarly, as with the heat equation, we must also impose initial conditions, however, unlike the case of the heat equation, we must prescribe two initial conditions—both position and speed. For example, we might impose

$$u\,(x,0) = f(x), \quad \text{position}, \tag{1.62a}$$

$$\frac{\partial u}{\partial t}\,(x,0) = g(x), \quad \text{velocity}. \tag{1.62b}$$

1.2 PDES ARE EVERYWHERE

In the previous section we derived the advection equation, heat equation, Laplace's equation, and the wave equation. The list does not stop there. In fact, there are literally hundreds of PDEs that can be found in various areas of science and engineering. Here I list just a few. It is my intent to attribute motivation to why one would construct methods to solve PDEs.

1. Burgers' Equation

$$u_t + u u_x = \nu u_{xx} \tag{1.63}$$

is a fundamental PDE that incorporates both nonlinearity and diffusion. It was first introduced as a simplified model for turbulence [1] and appears in various areas of applied mathematics, such as soil-water flow [2], nonlinear acoustics [3], and traffic flow [4].

2. Fisher's equation

$$u_t = u_{xx} + u(1-u) \tag{1.64}$$

is a model proposed for the wave of advance of advantageous genes [5] and also has applications in early farming [6], chemical wave propagation [7], nuclear reactors [8], chemical kinetics [9], and in theory of combustion [10].

3. The Fitzhug–Nagumo equation

$$u_t = u_{xx} + u(1-u)(u+k) \tag{1.65}$$

models the transmission of nerve impulses [11], [12], and arises in population genetics models [13].

4. The Korteweg deVries equation (KdV)

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.66}$$

describes the evolution of long water waves down a canal of rectangular cross section. It has also been shown to model longitudinal waves propagating in a one-dimensional lattice, ion-acoustic waves in a cold plasma, waves in elastic rods, and used to describe the axial component of velocity in a rotating fluid flow down a tube [14] .

5. The Eikonial equation

$$|\nabla u| = F(x), \quad x \in \mathbb{R}^n \tag{1.67}$$

appearing in ray optics [15].

6. Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar}{2m}\nabla^2 u + V(x)\psi \tag{1.68}$$

is an equation that is at the cornerstone of Quantum Mechanics. Here $i = \sqrt{-1}$ is the imaginary unit, ψ is the time-dependent wave function, $\hbar$ reduced Planck's constant, and $V(x)$ is the potential [16].

7. The Gross–Pitaevskii equation

$$i\psi_t = -\nabla^2\psi + (V(x) + |\psi|^2)\psi \tag{1.69}$$

is a model for the single-particle wavefunction in a Bose–Einstein condensate [17], [18].

8. Plateau's equation

$$(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 0 \tag{1.70}$$

arises in the study minimal surfaces [19].

9. The Sine–Gordon equation

$$u_{xy} = \sin u \tag{1.71}$$

arises in the study of surfaces of constant negative curvature [20], and in the study of crystal dislocations [21].

10. Equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \end{aligned} \tag{1.72}$$

arises in linear elasticity. Here, σ_{xx}, σ_{xy}, and σ_{yy} are normal and shear stresses, and F_x and F_y are body forces [22].

11. The Navier–Stokes equations

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u} \end{aligned} \tag{1.73}$$

describe the velocity field and pressure of incompressible fluids. Here ν is the kinematic viscosity, $\mathbf{u}$ is the velocity of the fluid parcel, P is the pressure, and ρ is the fluid density [23].

12. Maxwell's Equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \end{aligned} \tag{1.74}$$

which appear in Electricity and Magnetism. Here $\mathbf{E}$ denotes the electric field, $\mathbf{B}$ denotes the magnetic field, $\mathbf{D}$ denotes the electric displacement field, $\mathbf{J}$ denotes the free current density, ρ denotes the free electric charge density, ε_0 the permittivity of free space, and μ_0 the permeability of free space [24] .

1.3 EXERCISES

1.1. Match each solution to its corresponding PDE

$$
\begin{array}{llll}
(i) & u = x^2 + y^2, & (a) & 2u_x - u_y = 1 \\
(ii) & u = x^2 + y & (b) & xu_x + yu_y = 2u \\
(iii) & u = \sqrt{x^2 + y^2} & (c) & u_x + uu_y = u + 2x \\
(iv) & u = x + y, & (d) & u_x^2 + u_y^2 = 1
\end{array}
$$

1.2. Which of the following is not a solution of $u_{tt} = 4u_{xx}$

$$
\begin{array}{llll}
(i) & u = x - 2t, & (ii) & u = \dfrac{1}{x + 2t} \\
(iii) & u = \dfrac{x - 2t}{x + 2t}, & (iv) & u = (x - 2t)(x + 2t)
\end{array}
$$

1.3. Show that

$$u = e^x f(2x - y),$$

where f is an arbitrary function of its argument is a solution of

$$u_x + 2u_y = u.$$

1.4. Show that

$$u = f(x) + g(y),$$

where f and g are arbitrary functions is a solution of

$$u_{xy} = 0.$$

1.5. Find constant a and b such that $u = e^{at} \sin bx$, and $u = e^{at} \cos bx$ are solutions of

$$u_t = u_{xx}.$$

1.6. Show that $u = \dfrac{e^{-x^2/4t}}{\sqrt{t}}$ is also a solution of

$$u_t = u_{xx}.$$

1.7. Consider the PDE

$$u_t = u_{xx} + 2\,\mathrm{sech}^2 x\, u. \tag{E1}$$

If $v = e^{k^2 t} \sinh kt$ or $v = e^{k^2 t} \cosh kt$ (k is an arbitrary constant), then

$$u = v_x - \tanh x \cdot v$$

satisfies (E1).

1.8. Show

$$u = \ln \left| \frac{2f'(x)g'(y)}{(f(x) + g(y))^2} \right|,$$

where $f(x)$ and $g(y)$ are arbitrary functions satisfies Liouville's equation

$$u_{xy} = e^u.$$

1.9. Show

$$u = 4 \tan^{-1} \left(e^{ax + a^{-1} y} \right),$$

where a is an arbitrary nonzero constant satisfies the Sine–Gordon equation

$$u_{xy} = \sin u.$$

1.10. Show that if

$$u = f(x + ct)$$

satisfies the KdV equation (1.66) then f satisfies

$$cf' + 6ff' + f''' = 0, \tag{E2}$$

where prime denotes differentiation with respect to the argument of f. Show there is one value of c such that $f(r) = 2\,\mathrm{sech}^2 r$ is a solution of (E2).

1.11. The PDE

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

is known as the modified Korteweg de Vries (mKdV) equation. Show that if v is a solution of the mKdV, then

$$u = v_x - v^2$$

is a solution of the KdV (1.66).

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CHAPTER 2

First-Order PDEs

Partial differential equations (PDEs) of the form

$$u_x - 2u_y = 0, \quad u_x + yu_u = 0, \quad u_x + uu_y = 1, \quad u_x^2 + u_y^2 = 1 \tag{2.1}$$

are all examples of first-order PDEs. The first equation is constant coefficient, the second equation is linear, the third equation quasilinear, and the last equation nonlinear. In general, equations of the form

$$F(x, y, u, u_x, u_y) = 0, \tag{2.2}$$

are first-order PDEs. This chapter deals with techniques to construct general solutions of (2.2) when they are constant coefficient, linear, quasilinear, and fully nonlinear equations.

2.1 CONSTANT COEFFICIENT EQUATIONS

PDEs of the form

$$au_x + bu_y = cu, \tag{2.3}$$

where a, b, and c are all constant are called constant coefficient PDEs. For example, consider

$$u_x - u_y = 0. \tag{2.4}$$

Our goal is to find a solution $u = u(x, y)$ of this equation. If we introduce the change of variables

$$r = x + y, \quad s = x - y, \tag{2.5}$$

then the derivatives u_x and u_y transform as

$$u_x = u_r + u_s, \quad u_y = u_r - u_s, \tag{2.6}$$

and Eq. (2.4) becomes

$$u_r + u_s - u_r + u_s = 0$$

or

$$u_s = 0. \tag{2.7}$$

Integration of (2.7) gives

$$u = f(r),$$

or

$$u = f(x+y). \tag{2.8}$$

Substituting (2.8) into Eq. (2.4) shows that it is satisfied and thus is the solution. Let us introduce a new set of variables, say

$$r = x + y, \quad s = x^2 - y^2. \tag{2.9}$$

The derivatives transform as

$$u_x = u_r + \left(\frac{s}{r} + r\right) u_s, \quad u_y = u_r + \left(\frac{s}{r} - r\right) u_s, \tag{2.10}$$

and Eq. (2.4) becomes

$$2ru_s = 0 \quad \Rightarrow \quad u_s = 0. \tag{2.11}$$

As (2.11) is the same (2.7), integration yields again

$$u = f(r),$$

or

$$u = f(x+y), \tag{2.12}$$

the same solution as given in (2.8).

Finally, consider a third change of variables

$$r = x - y, \quad s = x^2 - y^2. \tag{2.13}$$

The derivatives transform as

$$u_x = u_r + \left(r + \frac{s}{r}\right) u_s, \quad u_y = -u_r + \left(r - \frac{s}{r}\right) u_s, \tag{2.14}$$

and Eq. (2.4) becomes, after simplification,

$$ru_r + su_s = 0, \tag{2.15}$$

a new first-order PDE. This equation is actually more complicated than the one we started with! As the changes of variables (2.5) and (2.9) transformed the original PDE to an equation that is simple and the change of variables (2.13) transformed the original equation that is more complicated, a natural question is: What is common in the change of variables (2.5) and (2.9) that is not in (2.13)? The answer is that one of the variables is $r = x + y$. In fact, if we choose $r = x + y$ and $s = s(x, y)$, arbitrary, then

$$u_x = u_r + s_x u_s, \quad u_y = u_r + s_y u_s, \tag{2.16}$$

and Eq. (2.4) becomes, after simplification

$$\left(s_x - s_y\right) u_s = 0, \tag{2.17}$$

from which we deduce that $u_s = 0$, recovering the solution found in (2.8). We note that $s_x - s_y \neq 0$ since $s_x - s_y = 0$ would make the Jacobian of the transformation, $r_x s_y - r_y s_x = 0$.

The question is, how did we know how to choose $r = x + y$ as one of the new variables? For example, suppose we consider

$$2u_x - u_y = 0, \tag{2.18}$$

or

$$u_x + 5u_y = 0, \tag{2.19}$$

what would be the right choice of $r(x, y)$ that leads to a simple equation like $u_s = 0$? In an attempt to answer this, let us introduce the change of variables $r = r(x, y)$ and $s = s(x, y)$ and try to find r so that the original PDE becomes $u_s = 0$. Under a general changes of variables

$$u_x = r_x u_r + s_x u_s, \quad u_y = r_y u_r + s_y u_s, \tag{2.20}$$

Equation (2.4) becomes

$$\left(r_x - r_y\right) u_r + \left(s_x - s_y\right) u_s = 0 \tag{2.21}$$

and in order to obtain our target PDE, it is necessary to choose

$$r_x - r_y = 0. \tag{2.22}$$

However, to solve (2.22) is to solve (2.4)! In fact, to solve any PDE in the form of (2.3), using a general change of variables, it would be necessary to have one solution to

$$ar_x - br_y = 0. \tag{2.23}$$

So without knowing one of the variables is $r = x + y$, our first attempt at trying to solve (2.4) would have failed!

For our second attempt, we will try and work backward. We will start with the answer, $u_s = 0$, and try and target our original PDE, Eq. (2.4). Therefore, if we start with

$$u_s = 0, \tag{2.24}$$

and use a general chain rule

$$u_s = u_x x_s + u_y y_s, \tag{2.25}$$

then (2.24) becomes

$$u_x x_s + u_y y_s = 0. \tag{2.26}$$

Choosing

$$x_s = 1, \quad y_s = -1, \tag{2.27}$$

gives the original Eq. (2.4). Integrating (2.27) gives

$$x = s + a(r), \quad y = -s + b(r), \tag{2.28}$$

where $a(r)$ and $b(r)$ are arbitrary functions of r. From (2.24), we obtain $u = c(r)$, another arbitrary function, and eliminating s from (2.28) gives

$$x + y = a(r) + b(r) = d(r) \quad \Rightarrow \quad r = d^{-1}(x + y) \tag{2.29}$$

so

$$u = c\left(d^{-1}(x + y)\right) \quad \Rightarrow \quad u = f(x + y) \tag{2.30}$$

since the addition, the inverse, and composition of arbitrary functions is arbitrary. The next two examples illustrate this technique further.

Example 2.1
Consider

$$2u_x + u_y = 0. \tag{2.31}$$

If

$$u_s = u_x x_s + u_y y_s, \tag{2.32}$$

choosing

$$x_s = 2, \quad y_s = 1 \tag{2.33}$$

gives (2.32) as

$$u_s = 2u_x + u_y, \tag{2.34}$$

and via (2.31) gives (2.34) as

$$u_s = 0. \tag{2.35}$$

Integrating (2.33) and (2.35) gives

$$x = 2s + a(r), \quad y = s + b(r), \quad u = c(r). \tag{2.36}$$

Eliminating s from (2.36) gives $x - 2y = a(r) - 2b(r) = d(r)$ (some arbitrary function) and solving for r gives $r = d^{-1}(x - 2y)$. Using this, from (2.36) we obtain

$$u = f(x - 2y) \tag{2.37}$$

as the solution of (2.31) noting that $c(d^{-1}) = f$.

Example 2.2
Consider

$$u_x + 4u_y = u. \tag{2.38}$$

If

$$u_s = u_x x_s + u_y y_s, \tag{2.39}$$

and we choose

$$x_s = 1, \quad y_s = 4, \tag{2.40}$$

then (2.39) becomes

$$u_s = u_x + 4u_y, \tag{2.41}$$

and via (2.38) becomes

$$u_s = u. \tag{2.42}$$

Integrating (2.40) and (2.42) gives

$$x = s + a(r), \quad y = 4s + b(r), \quad u = c(r)\mathrm{e}^s. \tag{2.43}$$

Eliminating s from (2.43) gives

$$4x - y = 4a(r) - b(r) = d(r) \tag{2.44}$$

and

$$u = c(r)\mathrm{e}^s = c(r)\mathrm{e}^{x-a(r)} = c(r)\mathrm{e}^{-a(r)}\mathrm{e}^x = e(r)\mathrm{e}^x \tag{2.45}$$

(where $d(r) = 4a(r) - b(r)$ and $c(r)\mathrm{e}^{-a(r)} = e(r)$) and eliminating r between (2.44) and (2.45) gives

$$u = \mathrm{e}^x f(4x - y), \tag{2.46}$$

the solution of the PDE (2.38).

2.2 LINEAR EQUATIONS

We now turn our attention to first-order PDEs of the form

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u, \tag{2.47}$$

where a, b, and c are now functions of the variables x and y. These type of equations are called linear PDEs.

Example 2.3
Consider

$$x u_x - y u_y = 0. \tag{2.48}$$

Does the method from the preceding section work here? The answer—yes! If we let

$$u_s = u_x x_s + u_y y_s, \tag{2.49}$$

and choose

$$x_s = x, \quad y_s = -y, \tag{2.50}$$

then (2.48) becomes

$$u_s = 0. \tag{2.51}$$

We are required to solve (2.50) and (2.51). The solution of these are

$$x = a(r)e^s, \quad y = b(r)e^{-s}, \quad u = c(r), \tag{2.52}$$

where a, b, and c[2] are arbitrary function of integration. Eliminating s from the first two in (2.52) gives

$$xy = a(r)b(r) \quad \Rightarrow \quad r = A(xy), \tag{2.53}$$

and eliminating r from the third of (2.52) and (2.53) gives

$$u = c\left(A(xy)\right) \quad \Rightarrow \quad u = f(xy). \tag{2.54}$$

In general, for linear equations in the form (2.47), if we introduce a general chain rule

$$u_s = u_x x_s + u_y y_s, \tag{2.55}$$

then we can target this PDE by choosing

$$x_s = a(x, y), \quad y_s = b(x, y), \quad u_s = c(x, y)u. \tag{2.56}$$

Example 2.4
Consider

$$x u_x + y u_y = u. \tag{2.57}$$

Here, we must solve

$$x_s = x, \quad y_s = y, \quad u_s = u. \tag{2.58}$$

[2]Please note that these are arbitrary functions of integration and are not the same as a, b, and c as given in (2.47).

The solution of these are

$$x = a(r)e^s, \quad y = b(r)e^s, \quad u = c(r)e^s. \tag{2.59}$$

Eliminating s from the first and second and first and third of (2.59) gives

$$\frac{y}{x} = \frac{b(r)}{a(r)}, \quad \text{and} \quad \frac{u}{x} = \frac{c(r)}{a(r)}, \tag{2.60}$$

and further elimination of r gives

$$\frac{u}{x} = f\left(\frac{y}{x}\right) \quad \text{or} \quad u = xf\left(\frac{y}{x}\right). \tag{2.61}$$

Example 2.5
Consider

$$u_t + xu_x = 1, \quad u(x,0) = -x^2. \tag{2.62}$$

We note that in this example the independent variables have changed from (x, y) to (x, t). It is also an initial value problem, meaning that the arbitrary function that will appear in the final solution will have some prescribed form given by the initial condition. Here we must solve

$$t_s = 1, \quad x_s = x, \quad u_s = 1. \tag{2.63}$$

The solution of these are

$$t = s + a(r), \quad x = b(r)e^s, \quad u = s + c(r). \tag{2.64}$$

Eliminating the variable s in (2.64) gives

$$xe^{-t} = b(r)e^{-a(r)}, \quad u - t = c(r) - a(r) \tag{2.65}$$

and further elimination of r gives

$$u = t + f(xe^{-t}). \tag{2.66}$$

Now we impose the initial condition $u(x,0) = -x^2$ on (2.66). In doing so, $f(x) = -x^2$ and thus, the final solution is

$$u = t - \left(xe^t\right)^2. \tag{2.67}$$

We now want to solve this problem in a slightly different way. As we are essentially changing from (x,t) to (r,s), we wish to create a boundary condition in the (r,s) plane that corresponds to the boundary condition $u(x,0) = -x^2$ in the (x,t) plane. In the (x,t) plane, the line $t = 0$ is the boundary where u is defined. We must associate a curve in the (r,s) plane. Here, we choose $s = 0$ (most will work) and connect the two boundaries by $x = r$.

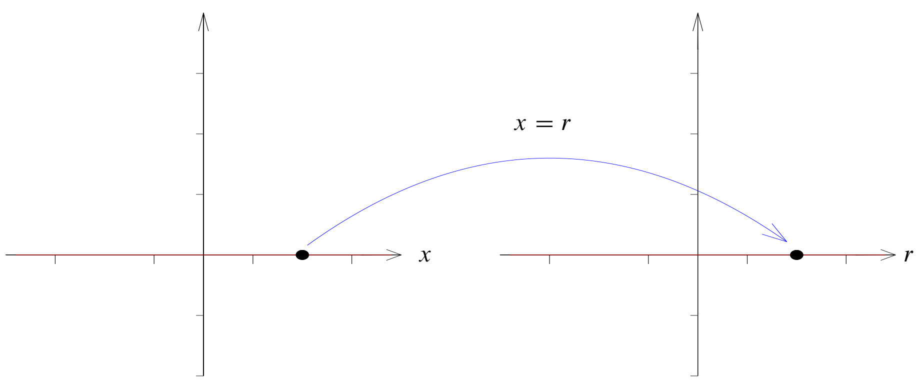

Figure 2.1: Change in the boundary from the (x, t) plane to the (r, s) plane.

Thus, the new boundary conditions are

$$t = 0, \quad x = r, \quad u = -r^2, \quad \text{when} \;\; s = 0. \tag{2.68}$$

Solving (2.63) still gives (2.64). However, when we impose the new boundary conditions (2.68) we find that $a(r) = 0$, $b(r) = r$ and $c(r) = -r^2$ giving

$$t = s, \quad x = r\mathrm{e}^s, \quad u = s - r^2. \tag{2.69}$$

Elimination of r and s in (2.69) gives

$$u = t - x^2\mathrm{e}^{-2t}, \tag{2.70}$$

the solution presented in (2.67).

Example 2.6
Solve

$$xu_x + 2yu_y = u + x^2, \quad u(x, x) = 0. \tag{2.71}$$

Here, the characteristic equations are

$$x_s = x, \quad y_s = 2y, \quad u_s = u + x^2. \tag{2.72}$$

In the (x, y) plane, the boundary is $y = x$. In the (r, s), we will choose it to be $s = 0$. Again, we will connect the boundaries via $x = r$ giving the boundary conditions

$$x = r, \quad y = r, \quad u = 0, \quad \text{when} \;\; s = 0. \tag{2.73}$$

It is important to note that in order to solve for u in (2.72), we will need x first. The solution of the first two of (2.72) is

$$x = a(r)e^s, \quad y = b(r)e^{2s}. \tag{2.74}$$

The boundary condition in (2.73) give $a(r) = r$ and $b(r) = r$, giving

$$x = re^s, \quad y = re^{2s}. \tag{2.75}$$

Using x from (2.75) enables us to solve the remaining equation in (2.72) for u. This gives

$$u = r^2e^{2s} + c(r)e^s. \tag{2.76}$$

Imposing the last boundary condition from (2.73) gives $c(r) = -r^2$ and thus,

$$u = r^2e^{2s} - r^2e^s. \tag{2.77}$$

Eliminating r and s from (2.75) and (2.77) gives

$$u = x^2 - \frac{x^3}{y}. \tag{2.78}$$

Example 2.7
Consider

$$yu_x + xu_y = u. \tag{2.79}$$

Here, we must solve

$$x_s = y, \quad y_s = x, \quad u_s = u. \tag{2.80}$$

As the first two of (2.80) is a coupled system, their solution will be coupled. The solution of these are

$$x = a(r)e^s + b(r)e^{-s}, \quad y = a(r)e^s - b(r)e^{-s}, \quad u = c(r)e^s. \tag{2.81}$$

In the previous example, eliminating s was easy. Here this is not the case. Noting that

$$x + y = 2a(r)e^s \quad \text{and} \quad x - y = 2b(r)e^{-s} \tag{2.82}$$

and multiplying these gives

$$(x+y)(x-y) = 4a(r)b(r) \quad \Rightarrow \quad r = A(x^2 - y^2) \tag{2.83}$$

and, further, using (2.83) in conjunction with (2.81) leads finally to the solution

$$u = (x+y)f\left(x^2 - y^2\right). \tag{2.84}$$

This example clearly shows that even though this technique works, trying to eliminate the variables r and s can be quite tricky. In the next section we will bypass the introduction of the variables r and s.

2.3 METHOD OF CHARACTERISTICS

In solving first-order PDEs of the form

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)u, \tag{2.85}$$

it was necessary to solve the system of ODEs

$$x_s = a(x,y), \quad y_s = b(x,y), \quad u_s = c(x,y)u. \tag{2.86}$$

As seen in Example 2.2, in solving

$$u_x + 4u_y = u \tag{2.87}$$

we associated the system

$$x_s = 1, \quad y_s = 4, \quad u_s = u, \tag{2.88}$$

or

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = 4, \quad \frac{\partial u}{\partial s} = u. \tag{2.89}$$

If we separate (2.89), then

$$\partial x = \partial s, \quad \frac{\partial y}{4} = \partial s, \quad \frac{\partial u}{u} = \partial s. \tag{2.90}$$

We can rewrite system (2.90) as

$$\partial x - \frac{\partial y}{4} = 0, \quad \frac{\partial u}{u} = \partial x, \tag{2.91}$$

thereby eliminating the variable s. Further, integrating leads to

$$4\,x - y = A(r), \quad \ln|u| - x = B(r) \tag{2.92}$$

and eliminating of r leads to

$$\ln|u| - x = f(4\,x - y) \tag{2.93}$$

or

$$u = \mathrm{e}^x f(4\,x - y) \tag{2.94}$$

noting that after exponentiation, we replaced e^f with f. If we treat r in (2.92) as constant, then we can treat the partial derivatives as ordinary derivatives in (2.91), then

$$4\,dx - dy = 0, \quad \frac{du}{u} = dx. \tag{2.95}$$

If we integrate (2.95) we obtain

$$4\,x - y = c_1, \quad \ln|u| - x = c_2. \tag{2.96}$$

Comparing (2.96) and (2.92) shows that the constants c_1 and c_2 play the role of $A(r)$ and $B(r)$, and since we wish to eliminate r, this is equivalent to having $c_2 = f(c_1)$. This would be the solution of the PDE. We typically write Eq. (2.95) as

$$\frac{dx}{1} = \frac{dy}{4} = \frac{du}{u}. \tag{2.97}$$

These are called characteristics equations, and the method is called the method of characteristics. In general, for linear PDEs of the form (2.85), the method of characteristics requires us to solve

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{c(x,y)u}. \tag{2.98}$$

Example 2.8
Here, we revisit Example 2.7, Eq. (2.79) considered earlier. The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}. \tag{2.99}$$

Solving the first pair in (2.99) we obtain

$$x^2 - y^2 = c_1. \tag{2.100}$$

For the second pair in (2.99), we use a result derived in #6 in the exercises that

$$\frac{d(x+y)}{x+y} = \frac{du}{u}, \tag{2.101}$$

which leads to

$$\frac{u}{x+y} = c_2, \tag{2.102}$$

and the general solution is $c_2 = f(c_1)$, which leads to

$$u = (x+y)\, f(x^2 - y^2). \tag{2.103}$$

Example 2.9
Consider

$$2x\, u_x + y\, u_y = 2x, \quad u(x,x) = x^2 + x. \tag{2.104}$$

The characteristic equations are

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{du}{2x}. \tag{2.105}$$

Solving the first and second and first and third in (2.105) gives

$$\frac{y^2}{x} = c_1, \quad u - x = c_2, \tag{2.106}$$

giving the general solution as

$$u = x + f\left(\frac{y^2}{x}\right). \tag{2.107}$$

Using the initial condition in (2.104) gives

$$x + f(x) = x + x^2 \quad \Rightarrow \quad f(x) = x^2, \tag{2.108}$$

thus giving the solution

$$u = x + \left(\frac{y^2}{x}\right)^2. \tag{2.109}$$

2.4 QUASILINEAR EQUATIONS

First-order PDEs of the form

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u), \tag{2.110}$$

where a, b, and c are all functions of the variables x, y, and u, are called quasilinear PDEs. The method of characteristics can also be used here giving

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}, \tag{2.111}$$

noting the difference in this and the linear case is the resulting ODEs are fully coupled. The following example illustrate this case.

Example 2.10
Consider

$$y u_x + (x - u) u_y = y. \tag{2.112}$$

The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{x - u} = \frac{du}{y}. \tag{2.113}$$

From the first and third of (2.113), we obtain $u - x = c_1$; using this in the first and second of (2.113), we obtain

$$\frac{dx}{y} = -\frac{dy}{c_1}, \tag{2.114}$$

which yields

$$c_1 x + \frac{1}{2}y^2 = c_2, \tag{2.115}$$

or

$$(u - x)x + \frac{1}{2}y^2 = c_2. \tag{2.116}$$

Eliminating the constants gives rise to the solution

$$(u - x)x + \frac{1}{2}y^2 = f\,(u - x)\,. \tag{2.117}$$

Example 2.11
Consider

$$(x + u)\,u_x + (y + u)\,u_y = x - y. \tag{2.118}$$

The characteristic equations are

$$\frac{dx}{x + u} = \frac{dy}{y + u} = \frac{du}{x - y}. \tag{2.119}$$

To solve these, we re-write them as

$$\frac{dx}{du} = \frac{x + u}{x - y}, \quad \frac{dy}{du} = \frac{y + u}{x - y}. \tag{2.120}$$

Subtracting gives

$$\frac{d(x - y)}{du} = \frac{x - y}{x - y} = 1, \tag{2.121}$$

which integrates, giving

$$x - y = u + c_1. \tag{2.122}$$

Using this in the first and third of (2.119) gives

$$\frac{dx}{x + u} = \frac{du}{u + c_1}, \tag{2.123}$$

which is a linear ODE. It has as its solution

$$\frac{x}{u + c_1} - \ln|u + c_1| - \frac{c_1}{u + c_1} = c_2, \tag{2.124}$$

and using c_1 above gives

$$\frac{y+u}{x-y} - \ln|x-y| = c_2. \tag{2.125}$$

Therefore, the solution of the original PDE is given by

$$\frac{y+u}{x-y} - \ln|x-y| = f(x-y-u). \tag{2.126}$$

As the final example in this section, we consider the PDE

$$u_t + c(x,t,u)u_x = 0. \tag{2.127}$$

Equation (2.127) is commonly referred to the first-order wave equation and $c = c(t,x,u)$ is usually referred to as the "wave speed." In particular, we will consider the following three equations

$$u_t + 2u_x = 0, \quad u_t + 2xu_x = 0, \quad u_t + 2uu_x = 0, \tag{2.128}$$

all subject to the initial condition $u(x,0) = \operatorname{sech} x$. Each can be solved using the method of characteristics giving

$$u = f(x-2t), \quad u = f\left(x\mathrm{e}^{-2t}\right), \quad u = f(x-2tu), \tag{2.129}$$

respectively, and imposing the initial conditions gives the solutions

$$u = \operatorname{sech}(x-2t), \quad u = \operatorname{sech}\left(x\mathrm{e}^{-2t}\right), \quad u = \operatorname{sech}(x-2tu). \tag{2.130}$$

Figures 2.2a, 2.2b, and 2.2c show their respective solutions.

It is interesting to note that in Fig. 2.2a, the wave moves to the right without changing its shape, in Fig. 2.2b, the wave spread out and, in Fig. 2.2c, the wave moves to the right with its speed changing according to height.

2.5 HIGHER-DIMENSIONAL EQUATIONS

The method of characteristics easily extends to PDEs with more than two independent variables. The following examples demonstrate this.

Example 2.12
Consider

$$x\,u_x + y\,u_y - z\,u_z = u. \tag{2.131}$$

The characteristic equations for this would be

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-z} = \frac{du}{u}. \tag{2.132}$$

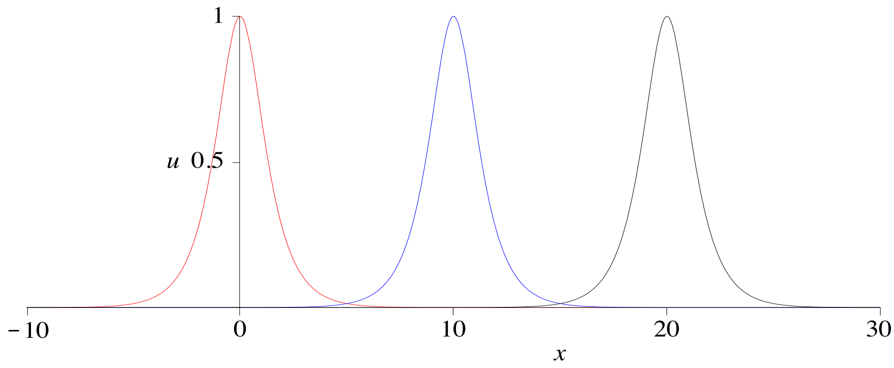

Figure 2.2: (a) The solution (2.130)(i) at times t = 0 (red), 5 (blue), and 10 (black).

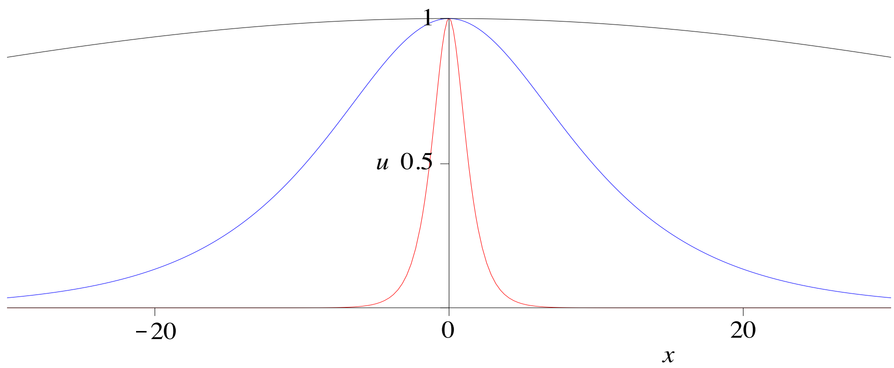

Figure 2.2: (b) The solution (2.130)(ii) at times t = 0 (red), 1 (blue), and 2 (black).

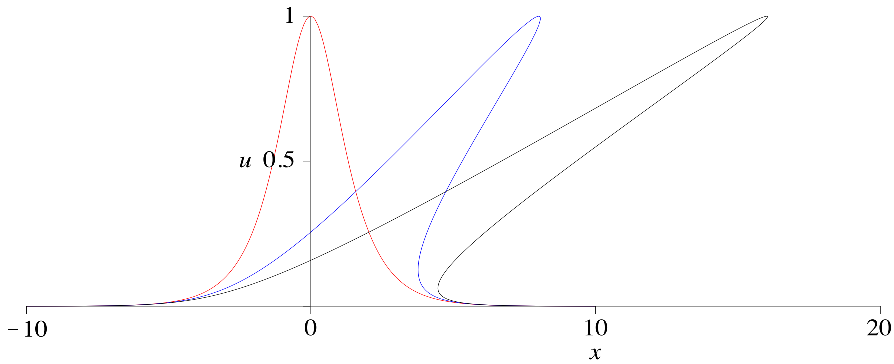

Figure 2.2: (c) The solution (2.130)(iii) at times t = 0 (red), 4 (blue), and 8 (black).

We now pick in pairs

$$(i)\ \frac{dx}{x} = \frac{dy}{y} \qquad (ii)\ \frac{dx}{x} = \frac{dz}{-z} \qquad (iii)\ \frac{dx}{x} = \frac{du}{u}. \tag{2.133}$$

Integrating each, we obtain

$$\frac{y}{x} = c_1, \quad xz = c_2, \quad \frac{u}{x} = c_3. \tag{2.134}$$

Extending the result when we have two independent variables, the general solution is $c_3 = f(c_1, c_2)$. For this problem, the solution is

$$u = xf\left(\frac{y}{x}, xz\right). \tag{2.135}$$

Example 2.13
Consider

$$u_t + u_x - xu_y = 0, \ u(x, y, 0) = \left(x^2 + 2y\right) e^{-x}. \tag{2.136}$$

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{1} = \frac{dy}{-x}, \quad du = 0. \tag{2.137}$$

Solving the first and second and second and third and the last of (2.137) gives

$$x - t = c_1, \quad \frac{x^2}{2} + y = c_2, \quad u = c_3. \tag{2.138}$$

Thus, we have so far, the solution as

$$u = f\left(x - t, \frac{x^2}{2} + y\right). \tag{2.139}$$

Using the initial condition in (2.136) gives

$$f\left(x, \frac{x^2}{2} + y\right) = \left(x^2 + 2y\right) \mathrm{e}^{-\mathrm{x}} \tag{2.140}$$

which we identify $f(a, b) = 2b\mathrm{e}^{-\mathrm{a}}$, giving the final solution

$$u = \left(x^2 + 2y\right) e^{-(x-t)}. \tag{2.141}$$

2.6 FULLY NONLINEAR FIRST-ORDER EQUATIONS

In this section we introduce two methods to obtain exact solutions to first-order fully nonlinear PDEs—the method of characteristics and Charpit's method.

2.6.1 METHOD OF CHARACTERISTICS

In order to develop the method of characteristics for a fully nonlinear first-order equation

$$F(x, y, u, u_x, u_y) = 0 \tag{2.142}$$

we return to the case of quasilinear equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{2.143}$$

and define F as

$$F(x, y, u, u_x, u_y) = a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0. \tag{2.144}$$

Recall that the characteristic equations for Eq. (2.143) are

$$\frac{dx}{ds} = a(x, y, u), \quad \frac{dy}{ds} = b(x, y, u), \quad \frac{du}{ds} = c(x, y, u). \tag{2.145}$$

If we treat x, y, u, p, and q as independent variables (where $p = u_x$ and $q = u_y$), we see that (2.145) can also be obtained by

$$\frac{dx}{ds} = F_p, \quad \frac{dy}{ds} = F_q, \quad \frac{du}{ds} = pF_p + qF_q, \tag{2.146}$$

where, as usual, subscripts refere to partial differentiation. The system (2.146) is three equations in five unknowns. To complete the system, we will need two more equations – one for $\dfrac{dp}{ds}$ and one for $\dfrac{dq}{ds}$. Differentiating (2.142) with respect to x and y gives

$$F_x + pF_u + p_x F_p + q_x F_q = 0, \tag{2.147a}$$
$$F_y + qF_u + p_y F_p + q_y F_q = 0. \tag{2.147b}$$

Using the fact that

$$q_x = p_y, \qquad p_y = q_x, \tag{2.148}$$

gives

$$F_x + pF_u + p_x F_p + p_y F_q = 0, \tag{2.149a}$$
$$F_y + qF_u + q_x F_p + q_y F_q = 0. \tag{2.149b}$$

If we consider $\dfrac{dp}{ds}$, then from the chain rule (and (2.149a))

$$\begin{aligned} \frac{dp}{ds} &= \frac{dp}{dx}\frac{dx}{ds} + \frac{dp}{dy}\frac{dy}{ds} \\ &= p_x F_p + p_y F_q \\ &= -F_x - pF_u. \end{aligned} \tag{2.150}$$

Similarly, if we consider $\dfrac{dq}{ds}$, then from the chain rule (and (2.149b))

$$\begin{aligned} \frac{dq}{ds} &= \frac{dq}{dx}\frac{dx}{ds} + \frac{dq}{dy}\frac{dy}{ds}, \\ &= q_x F_p + q_y F_q \\ &= -F_y - qF_u. \end{aligned} \tag{2.151}$$

Thus, we have the following characteristic equations:

$$\begin{aligned} \frac{dx}{ds} = F_p, \quad \frac{dy}{ds} = F_q, \quad \frac{du}{ds} = pF_p + qF_q, \\ \frac{dp}{ds} = -F_x - pF_u, \quad \frac{dq}{ds} = -F_y - qF_u. \end{aligned} \tag{2.152}$$

We will now consider two examples.

Example 2.14
Solve

$$u_x = u_y^2, \qquad u(0, y) = -\frac{y^2}{2}. \tag{2.153}$$

Here, we define F as

$$F = p - q^2, \tag{2.154}$$

so that the characteristic equations (2.152) become

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = -2q, \quad \frac{du}{ds} = p - 2q^2, \quad \frac{dp}{ds} = 0 \quad \frac{dq}{ds} = 0. \tag{2.155}$$

Trying to solve (2.155), eliminate the variables r and s, and then impose the initial condition is quite a task. As we did previously in this chapter, we will identify a boundary and boundary conditions in the (r, s) plane, solve (2.155) subject to these new conditions, and then eliminate the variables (r, s). In the (x, y) plane, the line $x = 0$ is the boundary where u is defined. To this, we associate a boundary in the (r, s) plane. Given the flexibility, we choose $s = 0$ and connect the two boundaries via $r = y$. Therefore, we have

$$x = 0, \quad y = r, \quad u = -\frac{r^2}{2} \quad \text{when} \quad s = 0 \tag{2.156}$$

To determine p and q on $s = 0$, it is necessary to consider the initial condition $u(0, y) = -\frac{y^2}{2}$. Differentiating with respect to y gives $u_y(0, y) = -y$, and from the original PDE (2.153), $u_x(0, y) = u_y^2(0, y)$. As we know u_y on the boundary, this then gives

$$p = r^2, \quad q = -r \text{ when } s = 0. \tag{2.157}$$

We now solve (2.155). From the last two equations of (2.155) we obtain

$$p = a(r), \quad q = b(r), \tag{2.158}$$

where a and b are arbitrary functions. From the initial condition (2.157), we find that

$$p = r^2, \quad q = -r, \tag{2.159}$$

for all s. Further, using these (2.159), we have from (2.155)

$$\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 2r, \quad \frac{du}{ds} = -r^2. \tag{2.160}$$

These integrate to give

$$x = s + c(r), \quad y = 2rs + d(r), \quad u = -r^2 s + e(r). \tag{2.161}$$

Using the initial conditions (2.156) gives

$$x = s, \quad y = 2rs + r, \quad u = -r^2 s - \frac{r^2}{2}. \tag{2.162}$$

On elimination of r and s in (2.162), we obtain

$$u = -\frac{y^2}{2(2x + 1)}. \tag{2.163}$$

Example 2.15
Solve

$$u_x u_y = u, \quad u(x, 1 - x) = 1. \tag{2.164}$$

Here, we define F as

$$F = pq - u, \tag{2.165}$$

so that the characteristic equations (2.152) become

$$\frac{dx}{ds} = q, \quad \frac{dy}{ds} = p, \quad \frac{du}{ds} = 2pq, \quad \frac{dp}{ds} = p, \quad \frac{dq}{ds} = q. \tag{2.166}$$

As in the previous example, we will choose a new boundary in the (r, s) plane. For convenience, we will choose $s = 0$ where we identify that

$$x = r, \quad y = 1 - r, \quad u = 1. \tag{2.167}$$

As we will need two more boundary conditions, one for p and q, we will use both the original PDE and boundary condition given in (2.164). Differentiating the boundary condition with respect to x gives

$$u_x(x, 1-x) - u_y(x, 1-x) = 0 \tag{2.168}$$

while on the boundary, the original PDE gives

$$u_x(x, 1-x)u_y(x, 1-x) = 1. \tag{2.169}$$

If we denote $u_x = p$ and $u_y = q$, then from (2.168) and (2.169), we have the following conditions on the boundary

$$p - q = 0, \tag{2.170a}$$
$$pq = 1. \tag{2.170b}$$

From these we find that $p = \pm 1, q = \pm 1$ – two cases. Each will be considered separately.

Case 1 $p = q = 1$
We first solve the last two equations in (2.166) as the first three equations need p and q. Both are easily solved giving

$$p = a(r)\mathrm{e}^s, \quad q = b(r)\mathrm{e}^s. \tag{2.171}$$

Imposing the boundary condition shows that $a(r) = 1$ and $b(r) = 1$, giving

$$p = \mathrm{e}^s, \quad q = \mathrm{e}^s. \tag{2.172}$$

From the first two equations in (2.166), we now have

$$\frac{dx}{ds} = q = \mathrm{e}^s, \quad \frac{dy}{ds} = p = \mathrm{e}^s. \tag{2.173}$$

Again, these are easily solved, giving

$$x = \mathrm{e}^s + c(r), \quad y = \mathrm{e}^s + d(r). \tag{2.174}$$

Imposing the boundary condition in (2.167) gives $c(r) = r - 1$ and $d(r) = -r$, leading to

$$x = \mathrm{e}^s + r - 1, \quad y = \mathrm{e}^s - r. \tag{2.175}$$

The final equation to be solved is $\dfrac{du}{ds} = 2pq$. At this point we two options: (1) bring in the solutions from (2.172); or (2), use will do this and solve $\dfrac{du}{ds} = 2u$. Again, this easily integrates, giving $u = e(r)\mathrm{e}^{2s}$ and the boundary condition in (2.167) gives $e(r) = 1$, leading to

$$x = \mathrm{e}^s + r - 1, \quad y = \mathrm{e}^s - r, \quad u = \mathrm{e}^{2s}. \tag{2.176}$$

Eliminating the parameters r and s in (2.176) gives the exact solution

$$u = \left(\frac{x+y+1}{2}\right)^2. \tag{2.177}$$

Case 2 $p = q = -1$
As the procedure is identical to that presented in Case 1, we simply give the results. The solutions to the characteristic equations subject to the boundary conditions are

$$x = r + 1 - \mathrm{e}^s, \quad y = 2 - r - \mathrm{e}^s, \quad u = \mathrm{e}^{2s} \quad , p = -\mathrm{e}^s, \quad q = -\mathrm{e}^s. \tag{2.178}$$

Eliminating the parameters r and s in (2.178) gives the exact solution.

$$u = \left(\frac{3-x-y}{2}\right)^2 \tag{2.179}$$

It is interesting to note that the same PDE subject to the same initial conditions gives rise to two independent solutions. This if often the case with nonlinear PDEs.

2.6.2 CHARPIT'S METHOD

An alternate method for deriving exact solutions of first-order nonlinear partial differential equations is known as *Charpit's Method*. Consider the nonlinear PDE

$$u_x = u_y^2 \tag{2.180}$$

and the linear PDE

$$u_x - 2x\,u_y = y. \tag{2.181}$$

The solution of (2.181) is found to be

$$u = xy + \frac{2}{3}x^3 + f\left(x^2 + y\right). \tag{2.182}$$

Substitution of the solution (2.182) into the nonlinear PDE (2.180) and simplifying gives

$$f'^2(\lambda) = \lambda, \tag{2.183}$$

where $\lambda = x^2 + y$. We solve (2.183) giving $f(\lambda) = c \pm \frac{2}{3}\,\lambda^{3/2}$, which leads to the exact solution $u = c + xy + \frac{2}{3}x^3 \pm \frac{2}{3}\,\left(x^2 + y\right)^{3/2}$.

Consider the first-order PDE

$$y u_x - 2u u_y = 0, \tag{2.184}$$

whose solution is found to be

$$4xu + y^2 - f(u) = 0. \tag{2.185}$$

Substitution of the solution (2.185) into the nonlinear PDE (2.180) gives

$$uf'(u) - f(u) = 0 \tag{2.186}$$

whose general solution is

$$f = cu, \tag{2.187}$$

giving an exact solution to the PDE (2.180) as

$$4xu + y^2 = cu, \tag{2.188}$$

or

$$u = -\frac{y^2}{4x - c}. \tag{2.189}$$

A natural question is: How did we know how to pick a second PDE whose solution would lead to a solution of the original nonlinear PDE? Before we answer this question it is interesting to consider the following pairs of equations:

$$(i)\quad u_x = u_y^2, \quad u_x - 2x\,u_y = y, \tag{2.190a}$$
$$(ii)\quad u_x = u_y^2, \quad y u_x - 2u u_y = 0. \tag{2.190b}$$

In the first pair, we augmented the original nonlinear PDE with one that is linear (and easily solvable). We substituted the solution of the linear equation into the nonlinear PDE and obtained an ODE, which we solved. This gave rise to an exact solution to the original equation. We also did this for the second pair. Thus, we were able to show that each pair of PDEs shared a common solution. If two PDEs share a common solution, they are said to be *compatible*. However, it should be noted that not all first-order PDEs will be compatible. Consider, for example,

$$u_y = 2y. \tag{2.191}$$

Integrating gives

$$u = y^2 + f(x), \tag{2.192}$$

and substituting (2.192) into (2.180) gives

$$f'(x) = 4y^2, \tag{2.193}$$

and clearly no function $f(x)$ will work.

So we ask: Is it possible to determine whether two PDEs are compatible before trying to find their common solution? We consider the first pair in (2.190) and construct higher order PDEs by differentiating with respect to x and y. This leads to the following:

$$u_{xx} = 2u_y u_{xy}, \tag{2.194a}$$
$$u_{xy} = 2u_y u_{yy}, \tag{2.194b}$$
$$u_{xx} - 2xu_{xy} - 2u_y = 0, \tag{2.194c}$$
$$u_{xy} - 2xu_{yy} = 1. \tag{2.194d}$$

Solving the first three of (2.194) gives

$$u_{xx} = \frac{2u_y^2}{u_x - x}, \quad u_{xy} = \frac{u_y}{u_y - x}, \quad u_{yy} = \frac{1}{2(u_y - x)}, \quad (u_y \neq x) \tag{2.195}$$

and substitution in the last of (2.194) shows it is identically satisifed. Thus, we have a way to check whether two equations are compatible. This method is known as Charpit's method.

Consider the compatibility of the following first-order PDEs:

$$\begin{aligned} F(x, y, u, p, q) &= 0, \\ G(x, y, u, p, q) &= 0, \end{aligned} \tag{2.196}$$

where $p = u_x$ and $q = u_y$. Calculating the x and y derivatives of (2.196) gives

$$\begin{aligned} F_x + pF_u + u_{xx}F_p + u_{xy}F_q &= 0, \\ F_y + qF_u + u_{xy}F_p + u_{yy}F_q &= 0, \\ G_x + pG_u + u_{xx}G_p + u_{xy}G_q &= 0, \\ G_y + qG_u + u_{xy}G_p + u_{yy}G_q &= 0. \end{aligned} \tag{2.197}$$

Solving the first three (2.197) for u_{xx}, u_{xy}, and u_{yy} gives

$$u_{xx} = \frac{-F_x\,G_q - p\,F_u\,G_q + F_q\,G_x + p\,F_q\,G_u}{F_p\,G_q - F_q\,G_p}, \tag{2.198a}$$
$$u_{xy} = \frac{-F_p\,G_x - p\,F_p\,G_u + F_x\,G_p + p\,F_u\,G_p}{F_p\,G_q - F_q\,G_p}, \tag{2.198b}$$
$$u_{yy} = \frac{\begin{array}{c} F_p^2\,G_x + p\,F_p^2\,G_u - F_y\,F_p\,G_q - q\,F_u\,F_p\,G_q \\ +q\,F_u\,F_q\,G_p - F_x\,F_p\,G_p - p\,F_u\,F_p\,G_p + F_y\,F_q\,G_p \end{array}}{(F_p\,G_q - F_q\,G_p)F_q}. \tag{2.198c}$$

Substitution into the last of (2.197) gives

$$F_p\,G_x + F_q\,G_y + (p\,F_p + q\,F_q)G_u - (F_x + p\,F_u)G_p - (F_y + q\,F_u)G_q = 0, \tag{2.199}$$

conveniently written as

$$\begin{vmatrix} D_x F & F_p \\ D_x G & G_p \end{vmatrix} + \begin{vmatrix} D_y F & F_q \\ D_y G & G_q \end{vmatrix} = 0, \tag{2.200}$$

where $D_x\ F = F_x + p\ F_u$, $D_y\ F = F_y + q\ F_u$, and $|\cdot|$ the usual determinant.

Example 2.16
Consider

$$u_x = u_y^2. \tag{2.201}$$

This is the example we considered already; now we will determine all classes of equation that are compatible with this one. Denoting

$$G = u_x - u_y^2 = p - q^2, \tag{2.202}$$

where $p = u_x$ and $q = u_y$, then

$$G_x = 0,\ \ G_y = 0,\ \ G_u = 0,\ \ G_p = 1,\ \ G_q = -2q, \tag{2.203}$$

and the Charpit equations are

$$\begin{vmatrix} D_x F & F_p \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} D_y F & F_q \\ 0 & -2q \end{vmatrix} = 0, \tag{2.204}$$

or, after expansion

$$F_x - 2qF_y + \left(p - 2q^2\right) F_u = 0, \tag{2.205}$$

noting that the third term can be replaced by $-pF_u$ due to the original equation. Solving this linear PDE by the method of characteristics gives the solution as

$$F = F(2xu_y + y, xu_x + u,\ u_x,\ u_y). \tag{2.206}$$

If we set F in (2.206)

$$F(a, b, c, d) = c - a, \tag{2.207}$$

we obtain (2.181) whereas, if we choose F as

$$F(a, b, c, d) = ac - 2bd, \tag{2.208}$$

we obtain (2.184). Other choices would lead to new compatible equations which could give rise to new exact solutions.

Example 2.17
Consider

$$u_x^2 + u_y^2 = u^2. \tag{2.209}$$

Denoting $p = u_x$ and $q = u_y$, then

$$G = u_x^2 + u_y^2 - u^2 = p^2 + q^2 - u^2. \tag{2.210}$$

Thus,

$$G_x = 0, \quad G_y = 0, \quad G_u = -2u, \quad G_p = 2p, \quad G_q = 2q, \tag{2.211}$$

and the Charpit equation's are

$$\begin{vmatrix} D_x F & F_p \\ -2pu & 2p \end{vmatrix} + \begin{vmatrix} D_y F & F_q \\ -2qu & 2q \end{vmatrix} = 0, \tag{2.212}$$

or, after expansion,

$$pF_x + qF_y + \left(p^2 + q^2\right) F_u + puF_p + quF_q = 0, \tag{2.213}$$

noting that the third term can be replaced by $u^2 F_u$ due to the original equation. Solving (2.213), a linear PDE, by the method of characteristics gives the solution as

$$F = F\left(x - \frac{p}{u}\ln u, y - \frac{q}{u}\ln u, \frac{p}{u}, \frac{q}{u}\right). \tag{2.214}$$

Consider the following particular example:

$$x - \frac{p}{u}\ln u + y - \frac{q}{u}\ln u = 0, \tag{2.215}$$

or

$$u_x + u_y = (x + y)\frac{u}{\ln u}. \tag{2.216}$$

If we let $u = \mathrm{e}^{\sqrt{v}}$, then this becomes

$$v_x + v_y = 2(x + y), \tag{2.217}$$

which, by the method of characteristics, has the solution

$$v = 2xy + f(x - y). \tag{2.218}$$

This, in turn, gives the solution for u as

$$u = \mathrm{e}^{\sqrt{2xy + f(x-y)}}. \tag{2.219}$$

Substitution into the original Eq. (2.209) gives the following ODE:

$$f'^2 - 2\lambda f' - 2f + 2\lambda^2 = 0, \tag{2.220}$$

where $f = f(\lambda)$ and $\lambda = x - y$. If we let $f = g + \frac{1}{2}\lambda^2$, then we obtain

$$g'^2 - 2g = 0, \tag{2.221}$$

whose solution is given by

$$g = \frac{(\lambda + c)^2}{2}, \quad g = 0, \tag{2.222}$$

where c is an arbitrary constant of integration. This, in turn, gives

$$f = \lambda^2 + c\lambda + \frac{1}{2}c^2, \quad f = \frac{1}{2}\lambda^2 \tag{2.223}$$

and substitution into (2.219) gives

$$u = \mathrm{e}^{\sqrt{x^2+y^2+c(x-y)+\frac{1}{2}c^2}}, \quad u = \mathrm{e}^{\sqrt{2xy+(x-y)^2/2}} \tag{2.224}$$

as exact solutions to the original PDE.

It is interesting to note that when we substitute the solution of the compatible equation into the original it reduces to an ODE. A natural question is: Does this always happen? This was proven to be true in two independent variables by the author [1].

2.7 EXERCISES

2.1. Solve the following first-order PDE using a change of coordinates $(x, y) \to (r, s)$

(i) $u_x - 2u_y = -u$,
(ii) $2xu_x + 3y\,u_y = x, \quad u(x, x) = 1$,
(iii) $2u_t - u_x = 4$,
(iv) $u_x + u_y = 6y, \quad u(x, 1) = 2 + x$,
(v) $xu_x - 2uu_y = x$,
(vi) $u_t - xu_x = 2t, \quad u(x, 0) = \sin x$.

2.2. Solve the following first-order PDEs using the method of characteristics

(i) $xyu_x + (x^2 + y^2)u_y = yu$,
(ii) $u_x + (y + 1)u_y = u + x$,
(ii) $x^2u_x - y^2u_y = u^2$,
(iv) $xu_x + (x + y)u_y = x$,
(v) $yu_x + xu_y = xy, \quad u(x, 0) = x^2$.

2.3. Solve the following first-order PDEs subject to the initial condition $u(x,0) = \operatorname{sech}(x)$. Explain the behavior of each solution for $t > 0$

$$\begin{array}{ll} \text{(i)} & u_t + (x+1)u_x = 0, \\ \text{(ii)} & u_t + (u-2)u_x = 0, \\ \text{(iii)} & u_t + (u-3x)u_x = 0, \\ \text{(iv)} & u_t + (1+2x+3u)u_x = 0. \end{array}$$

2.4. Use the method characteristics to solve the following:

$$\begin{array}{rrl} \text{(i)} & u_x^2 - 3u_y^2 - u = 0, & u(x,0) = x^2, \\ \text{(ii)} & u_t + u_x^2 + u = 0, & u(x,0) = x, \\ \text{(iii)} & u_x^2 + u_y^2 = 1, & u(x,1) = \sqrt{x^2+1}, \\ \text{(iv)} & u_x^2 - u\,u_y = 0, & u(x,y) = 1 \text{ along } y = 1-x. \end{array}$$

2.5. Use Charpit's method to find compatible first-order equations to the one given. Use any one of your equations to find an exact solution.

$$\begin{array}{rr} \text{(i)} & u_x^2 + u_y^2 = x^2, \\ \text{(ii)} & u_t + uu_x^2 = 0, \\ \text{(iii)} & u_x u_y = 1. \end{array}$$

2.6. If the characteristic equations are

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)},$$

show for some constant α, β, and γ that

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} = \frac{d(\alpha a + \beta b + \gamma c)}{\alpha a + \beta b + \gamma c}.$$

2.7. Generalize Charpit's method for nonlinear first-order PDEs of the form

$$F\left(x_1, x_2, \cdots x_n, u_{x_1}, u_{x_2}, \cdots u_{x_n}\right) = 0.$$

REFERENCES

[1] D. J. Arrigo, Nonclassical contact symmetries and Charpit's method of compatibility, *Journal of Nonlinear Mathematical Physics*, 12(3), pp. 321–329, 2005. DOI: 10.2991/jnmp.2005.12.3.1. 44

CHAPTER 3

Second-Order Linear PDEs

3.1 INTRODUCTION

The general class of second-order linear PDEs are of the form

$$\begin{aligned} a(x,y)u_{xx} + b(x,y)u_{xy} &+ c(x,y)u_{yy} \\ + d(x,y)u_x &+ e(x,y)u_y + f(x,y)u = g(x,y). \end{aligned} \tag{3.1}$$

The three PDEs that lie at the cornerstone of applied mathematics are: the heat equation, the wave equation, and Laplace's equation:

$$\begin{array}{rll} (i) & u_t = u_{xx}, & \text{the heat equation} \\ (ii) & u_{tt} = u_{xx}, & \text{the wave equation} \\ (iii) & u_{xx} + u_{yy} = 0, & \text{Laplace's equation;} \end{array} \tag{3.2}$$

or, using the same independent variables, x and y

$$(i) \quad u_{xx} - u_y = 0, \qquad \text{the heat equation} \tag{3.3a}$$

$$(ii) \quad u_{xx} - u_{yy} = 0, \qquad \text{the wave equation} \tag{3.3b}$$

$$(iii) \quad u_{xx} + u_{yy} = 0, \qquad \text{Laplace's equation.} \tag{3.3c}$$

Analogous to characterizing quadratic equations

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{3.4}$$

as either hyperbolic, parabolic, or elliptic determined by

$$\begin{array}{rcll} b^2 - 4ac & > & 0, & \text{hyperbolic,} \\ b^2 - 4ac & = & 0, & \text{parabolic,} \\ b^2 - 4ac & < & 0, & \text{elliptic,} \end{array} \tag{3.5}$$

we do the same for PDEs. So, for the heat equation (3.3a) $a = 1,\ b = 0,\ c = 0$, so $b^2 - 4ac = 0$ and the heat equation is parabolic. Similarly, the wave equation (3.3b) is hyperbolic and Laplace's equation (3.3c) is elliptic. This leads to a natural question. Is it possible to transform one PDE to another where the new PDE is simpler? Namely, under a change of variable

$$r = r(x,y), \qquad s = s(x,y), \tag{3.6}$$

can we transform to one of the following *standard* forms:

$$u_{rr} - u_{ss} + \text{l.o.t.s.} = 0, \quad \text{hyperbolic,} \tag{3.7a}$$

$$u_{ss} + \text{l.o.t.s.} = 0, \quad \text{parabolic,} \tag{3.7b}$$

$$u_{rr} + u_{ss} + \text{l.o.t.s.} = 0, \quad \text{elliptic,} \tag{3.7c}$$

where the term "l.o.t.s" stands for lower-order terms. For example, consider the PDE

$$2u_{xx} - 2u_{xy} + 5u_{yy} = 0. \tag{3.8}$$

This equation is elliptic since the elliptic $b^2 - 4ac = 4 - 4(2)(5) = -36 < 0$. If we introduce new coordinates,

$$r = 2x + y, \qquad s = x - y, \tag{3.9}$$

then by a change of variable using the chain rule

$$\begin{aligned} u_x &= u_r r_x + u_s s_x, \\ u_y &= u_r r_y + u_s s_y, \\ u_{xx} &= u_{rr} r_x^2 + 2u_{rs} r_x s_x + u_{ss} s_x^2 + u_r r_{xx} + u_s s_{xx}, \\ u_{xy} &= u_{rr} r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy}, \\ u_{yy} &= u_{rr} r_y^2 + 2u_{rs} r_y s_y + u_{ss} s_y^2 + u_r r_{yy} + u_s s_{yy}, \end{aligned} \tag{3.10}$$

gives

$$\begin{aligned} u_{xx} &= 4u_{rr} + 4u_{rs} + u_{ss}, \\ u_{xy} &= 2u_{rr} - u_{rs} - u_{ss}, \\ u_{yy} &= u_{rr} - 2u_{rs} + u_{ss}. \end{aligned} \tag{3.11}$$

Under (3.9), Eq. (3.8) becomes

$$2\left(4u_{rr} + 4u_{rs} + u_{ss}\right) - 2\left(2u_{rr} - u_{rs} - u_{ss}\right) + 5\left(u_{rr} - 2u_{rs} + u_{ss}\right) = 0$$

which simplifies to

$$u_{rr} + u_{ss} = 0, \tag{3.12}$$

which is Laplace's equation (also elliptic). Before we consider transformations for PDEs in general, it is important to determine whether the equation type could change under transformation. Consider the general class of PDEs

$$au_{xx} + bu_{xy} + cu_{yy} + \text{lots} = 0 \tag{3.13}$$

where a, b, and c are functions of x and y. Note that we have grouped the lower-order terms into the term *lots* as they will not affect the type. Under a change of variable $(x, y) \to (r, s)$ with the change of variable formulas (3.10), we obtain

$$\begin{aligned} & a\left(u_{rr} r_x^2 + 2u_{rs} r_x s_x + u_{ss} s_x^2 + u_r r_{xx} + u_s s_{xx}\right) \\ + \; & b\left(u_{rr} r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy}\right) \\ + \; & c\left(u_{yy} + u_{rr} r_y^2 + 2u_{rs} r_y s_y + u_{ss} s_y^2 + u_r r_{yy} + u_s s_{yy}\right) = 0. \end{aligned} \tag{3.14}$$

Rearranging (3.14), and neglecting lower-order terms, gives

$$\begin{aligned}(ar_x^2 + br_xr_y + cr_y^2)u_{rr} &+ (2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y)u_{rs} \\ &+ (as_x^2 + bs_xs_y + cs_y^2)u_{ss} = 0.\end{aligned} \tag{3.15}$$

Setting

$$\begin{aligned}A &= ar_x^2 + br_xr_y + cr_y^2, \\ B &= 2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y, \\ C &= as_x^2 + bs_xs_y + cs_y^2,\end{aligned} \tag{3.16}$$

gives from (3.15)

$$Au_{rr} + Bu_{rs} + Cu_{ss} = 0, \tag{3.17}$$

whose type is given by

$$B^2 - 4AC = (b^2 - 4ac)\left(r_xs_y - r_ys_x\right)^2, \tag{3.18}$$

from which we deduce that

$$\begin{aligned}b^2 - 4ac &> 0 \quad\Rightarrow\quad B^2 - 4AC > 0 \\ b^2 - 4ac &= 0 \quad\Rightarrow\quad B^2 - 4AC = 0 \\ b^2 - 4ac &< 0 \quad\Rightarrow\quad B^2 - 4AC < 0,\end{aligned} \tag{3.19}$$

showing that the equation type is unchanged under transformation. We note that $r_xs_y - r_ys_x \neq 0$ since the Jacobian of the transformation is nonzero.

We now consider transformations to standard form. As there are three standard forms—hyperbolic, parabolic, and elliptic—we will deal with each type separately.

3.2 STANDARD FORMS

If we introduce the change of coordinates

$$r = r(x, y), \qquad s = s(x, y), \tag{3.20}$$

with the derivatives changing as (3.10), we can substitute (3.10) into the general linear equation (3.1) and rearrange, obtaining

$$\begin{aligned}(ar_x^2 &+ br_xr_y + cr_y^2)u_{rr} + \left(2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y\right)u_{rs} \\ &+ (as_x^2 + bs_xs_y + cs_y^2)u_{ss} + \text{lots} = 0.\end{aligned} \tag{3.21}$$

Our goal now is to target a given standard form and solve a set of equations for the new variables r and s.

3.2.1 PARABOLIC STANDARD FORM

Comparing (3.21) with the parabolic standard form (3.7b) leads to choosing

$$ar_x^2 + br_xr_y + cr_y^2 = 0, \tag{3.22a}$$
$$2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y = 0. \tag{3.22b}$$

Since in the parabolic case $b^2 - 4ac = 0$, then substituting $c = \dfrac{b^2}{4a}$,[2] we find both equations of (3.22) are satisfied if

$$2ar_x + br_y = 0, \tag{3.23}$$

with the choice of $s(x, y)$ arbitrary. The following examples demonstrate this.

Example 3.1
Consider

$$u_{xx} + 6u_{xy} + 9u_{yy} = 0. \tag{3.24}$$

Here, $a = 1$, $b = 6$ and $c = 9$, showing that $b^2 - 4ac = 0$, so the PDE is parabolic. Solving

$$r_x + 3r_y = 0, \tag{3.25}$$

gives

$$r = R(3x - y). \tag{3.26}$$

As we wish to find new coordinates as to transform the original equation to standard form, we choose

$$r = 3x - y, \qquad s = y. \tag{3.27}$$

We calculate second derivatives

$$u_{xx} = 9u_{rr}, \quad u_{xy} = -3u_{rr} + 3u_{rs}, \quad u_{yy} = u_{rr} - 2u_{rs} + u_{ss}. \tag{3.28}$$

Substituting (3.28) into (3.24) gives

$$u_{ss} = 0. \tag{3.29}$$

As it turns out, we can explicitly solve (3.29) giving

$$u = f(r)s + g(r), \tag{3.30}$$

where f and g are arbitrary functions of r. In terms of the original variables (3.27), we obtain

$$u = yf(3x - y) + g(3x - y), \tag{3.31}$$

[2] If $c = 0$, then $b = 0$ and the PDE can be brought into standard form by dividing by a.

the general solution of (3.24).

Example 3.2
Consider

$$x^2u_{xx} - 4xyu_{xy} + 4y^2u_{yy} + xu_x = 0. \tag{3.32}$$

Here, $a = x^2$, $b = -4xy$ and $c = 4y^2$, showing that $b^2 - 4ac = 0$, so the PDE is parabolic. Solving

$$x^2r_x - 2xyr_y = 0, \tag{3.33}$$

or

$$xr_x - 2yr_y = 0, \tag{3.34}$$

gives

$$r = R(x^2y). \tag{3.35}$$

As we wish to find new variables r and s, we choose

$$r = x^2y, \qquad s = y. \tag{3.36}$$

Calculating the first derivative gives

$$u_x = 2xyu_r, \tag{3.37}$$

and second derivatives

$$\begin{aligned} u_{xx} &= 4x^2y^2u_{rr} + 2yu_r, \\ u_{xy} &= 2x^3yu_{rr} + 2xyu_{rs} + 2xu_r, \\ u_{yy} &= x^4u_{rr} + 2x^2u_{rs} + u_{ss}. \end{aligned} \tag{3.38}$$

Substituting (3.37) and (3.38) into (3.32) gives

$$4y^2u_{ss} - 4x^2yu_r = 0, \tag{3.39}$$

or, in terms of the new variables r and s,

$$u_{ss} - \frac{r}{s^2}u_r = 0. \tag{3.40}$$

An interesting question is whether different choices of the arbitrary function R and the variable s would lead to a different standard form. For example, if we chose

$$r = 2\ln x + \ln y, \qquad s = \ln y, \tag{3.41}$$

then (3.32) would become

$$u_{ss} - u_r - u_s = 0, \tag{3.42}$$

a constant coefficient parabolic equation, whereas choosing

$$r = 2\ln x + \ln y, \qquad s = 2\ln x, \tag{3.43}$$

then (3.32) would become

$$u_{ss} - u_r = 0, \tag{3.44}$$

the heat equation.

3.2.2 HYPERBOLIC STANDARD FORM

In order to obtain the standard form for the hyperbolic type

$$u_{rr} - u_{ss} + \text{lots} = 0, \tag{3.45}$$

from (3.21), we find it is necessary to choose

$$ar_x^2 + br_xr_y + cr_y^2 = -\left(as_x^2 + bs_xs_y + cs_y^2\right), \tag{3.46a}$$

$$2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y = 0. \tag{3.46b}$$

The problem is that this system, (3.46), is still a very hard problem to solve (both PDEs are nonlinear and coupled!). Therefore, we introduce a modified hyperbolic form that is much easier to work with before returning to this case.

3.2.3 MODIFIED HYPERBOLIC FORM

The modified hyperbolic standard form is defined as

$$u_{rs} + \text{lots} = 0, \tag{3.47}$$

noting that $a = 0$, $b = 1$, and $c = 0$, so that $b^2 - 4ac > 0$. In order to target the modified hyperbolic form, from (3.21) it is now necessary to choose

$$ar_x^2 + br_xr_y + cr_y^2 = 0, \qquad as_x^2 + bs_xs_y + cs_y^2 = 0. \tag{3.48}$$

If we rewrite (3.48)

$$a\left(\frac{r_x}{r_y}\right)^2 + 2b\,\frac{r_x}{r_y} + c = 0, \qquad a\left(\frac{s_x}{s_y}\right)^2 + 2b\,\frac{s_x}{s_y} + c = 0 \tag{3.49}$$

(assuming that $r_ys_y \neq 0$), we can solve equations (3.49) for $\dfrac{r_x}{r_y}$ and $\dfrac{s_x}{s_y}$. This leads to two first-order linear PDEs for r and s. The solutions of these then give rise to the correct standard variables. The following examples demonstrate this.

Example 3.3
Consider

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0. \tag{3.50}$$

Here, $a = 1$, $b = -5$, and $c = 6$, showing that $b^2 - 4ac = 1 > 0$, so the PDE is hyperbolic. Thus, (3.48) becomes

$$r_x^2 - 5r_x r_y + 6r_y^2 = 0, \qquad s_x^2 - 5s_x r_y + 6s_y^2 = 0, \tag{3.51}$$

and factoring gives

$$\left(r_x - 2r_y\right)\left(r_x - 3r_y\right) = 0, \qquad \left(s_x - 2s_y\right)\left(s_x - 3s_y\right) = 0, \tag{3.52}$$

from which we choose

$$r_x - 2r_y = 0, \qquad s_x - 3s_y = 0, \tag{3.53}$$

giving rise to solutions

$$r = R(2x + y), \qquad s = S(3x + y), \tag{3.54}$$

where R and S are arbitrary functions of their arguments. As we wish to find new coordinates to transform the original equation to standard form, we choose

$$r = 2x + y, \qquad s = 3x + y. \tag{3.55}$$

Calculating second derivatives gives

$$\begin{aligned} u_{xx} &= 4u_{rr} + 12u_{rs} + 9u_{ss}, \\ u_{xy} &= 2u_{rr} + 5u_{rs} + 3u_{ss}, \\ u_{yy} &= u_{rr} + 2u_{rs} + u_{ss}. \end{aligned} \tag{3.56}$$

Substituting (3.56) into (3.50) gives

$$u_{rs} = 0. \tag{3.57}$$

Solving (3.57) gives

$$u = f(r) + g(s), \tag{3.58}$$

where f and g are arbitrary functions. In terms of the original variables, we obtain the solution to (3.50) as

$$u = f(2x + y) + g(3x + y). \tag{3.59}$$

Example 3.4
Consider

$$xu_{xx} - (x + y)u_{xy} + yu_{yy} = 0. \tag{3.60}$$

Here, $a = x$, $b = -(x + y)$, and $c = y$, showing that $b^2 - 4ac = (x - y)^2 > 0$, so the PDE is hyperbolic. We thus need to solve

$$xr_x^2 - (x + y)r_x r_y + yr_y^2 = 0, \tag{3.61}$$

or, upon factoring,

$$\left(xr_x - yr_y\right)\left(r_x - r_y\right) = 0. \tag{3.62}$$

As s satisfies the same equation, we choose the first factor for r and the second for s

$$xr_x - yr_y = 0, \quad s_x - s_y = 0. \tag{3.63}$$

Upon solving (3.63), we obtain

$$r = f(xy), \qquad s = g(x + y). \tag{3.64}$$

The simplest choice for the new variables r and s are

$$r = xy, \qquad s = x + y. \tag{3.65}$$

Calculating second derivatives gives

$$\begin{aligned} u_{xx} &= y^2 u_{rr} + 2y u_{rs} + u_{ss}, \\ u_{xy} &= xy u_{rr} + (x + y)u_{rs} + u_{ss} + u_r, \\ u_{yy} &= x^2 u_{rr} + 2x u_{rs} + u_{ss}, \end{aligned} \tag{3.66}$$

and substituting (3.66) into (3.60) gives

$$\left(2xy - x^2 - y^2\right) u_{rs} - (x + y)u_r = 0, \tag{3.67}$$

or, in terms of the new variables, r and s, gives the standard form

$$u_{rs} + \frac{s}{s^2 - 4r} u_r = 0. \tag{3.68}$$

3.2.4 REGULAR HYPERBOLIC FORM

We now wish to transform a given hyperbolic PDE to its regular standard form

$$u_{rr} - u_{ss} + \text{lots} = 0. \tag{3.69}$$

First, let us consider the following example:

$$x^2 u_{xx} - y^2 u_{yy} = 0. \tag{3.70}$$

If we were to transform to modified standard form, we would solve

$$xr_x - yr_y = 0, \quad xs_x + ys_y = 0, \tag{3.71}$$

which gives

$$r = f(xy), \qquad s = g(x/y). \tag{3.72}$$

If we choose

$$r = \ln x + \ln y, \qquad s = \ln x - \ln y, \tag{3.73}$$

then the original PDE becomes

$$u_{rs} - u_s = 0. \tag{3.74}$$

However, if we introduce new variables α and β such that

$$\alpha = \frac{r+s}{2}, \qquad \beta = \frac{r-s}{2}, \tag{3.75}$$

then

$$\alpha = \ln x, \qquad \beta = \ln y, \tag{3.76}$$

and the PDE (3.70) becomes

$$u_{\alpha\alpha} - u_{\beta\beta} - u_\alpha + u_\beta = 0, \tag{3.77}$$

a PDE in regular hyperbolic form. In fact, one can show (see the exercises) that if

$$\alpha = \frac{r+s}{2}, \qquad \beta = \frac{r-s}{2}, \tag{3.78}$$

where r and s satisfies (3.48), then α and β satisfies

$$a\alpha_x^2 + b\alpha_x\alpha_y + c\alpha_y^2 = -\left(a\beta_x^2 + b\beta_x\beta_y + c\beta_y^2\right), \tag{3.79a}$$

$$2a\alpha_x\beta_x + b(\alpha_x\beta_y + \alpha_y\beta_x) + 2c\alpha_y\beta_y = 0, \tag{3.79b}$$

which is essentially (3.46). This gives a convenient way to go directly to the variables that lead to the regular hyperbolic form. We note that

$$\alpha, \beta = \frac{r \pm s}{2}, \tag{3.80}$$

so we can consider (3.49) again, but instead of factoring, we treat each as a quadratic equation in r_x/r_y and s_x/s_y and solve accordingly. It is important to note the placing of the $\pm$. We demonstrate with the following examples.

Example 3.5
Consider

$$8u_{xx} - 6u_{xy} + u_{yy} = 0. \tag{3.81}$$

The corresponding equations for r and s are

$$8r_x^2 - 6r_x r_y + r_y^2 = 0, \qquad 8s_x^2 - 6s_x s_y + s_y^2 = 0, \tag{3.82}$$

but as they are identical it suffices to only consider one. Dividing the first of (3.82) by r_y^2 gives

$$8\left(\frac{r_x}{r_y}\right)^2 - 6\frac{r_x}{r_y} + 1 = 0. \tag{3.83}$$

Solving (3.83) by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{6 \pm 2}{16} \tag{3.84}$$

or

$$8r_x - (3 \pm 1)r_y = 0. \tag{3.85}$$

The method of characteristics gives

$$r = R\left((3 \pm 1)x + 8y\right) = R(3x + 8y \pm x) \tag{3.86}$$

which leads to the choice

$$r = 3x + 8y, \qquad s = x. \tag{3.87}$$

Under this transformation, the original Eq. (3.81) becomes

$$u_{rr} - u_{ss} = 0, \tag{3.88}$$

the desired standard form.

Example 3.6
Consider

$$xy^3u_{xx} - x^2y^2u_{xy} - 2x^3yu_{yy} - y^3u_x + 2x^3u_y = 0. \tag{3.89}$$

The corresponding equations for r and s are

$$xy^3r_x^2 - x^2y^2r_xr_y - 2x^3yr_y^2 = 0, \qquad xy^3s_x^2 - x^2y^2s_xr_y - 2x^3ys_y^2 = 0, \tag{3.90}$$

and choosing the first gives

$$y^2\left(\frac{r_x}{r_y}\right)^2 - xy\frac{r_x}{r_y} - 2x^2 = 0. \tag{3.91}$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{(1 \pm 3)x}{2y} \tag{3.92}$$

or

$$2yr_x - (1 \pm 3)\,xr_y = 0. \tag{3.93}$$

Solving gives

$$r = R\left(x^2 + 2y^2 \pm 3x^2\right). \tag{3.94}$$

If we choose R to be simple and split according to the $\pm$, we obtain

$$r = x^2 + 2y^2, \qquad s = 3x^2. \tag{3.95}$$

Under this transformation, the original Eq. (3.89) becomes

$$u_{rr} - u_{ss} = 0, \tag{3.96}$$

the desired standard form.

3.2.5 ELLIPTIC STANDARD FORM

In order to obtain the standard form for the elliptic type, i.e.,

$$u_{rr} + u_{ss} + \text{lots} = 0, \tag{3.97}$$

then from (3.21), it is necessary to choose

$$ar_x^2 + br_xr_y + cr_y^2 = \left(as_x^2 + bs_xs_y + cs_y^2\right), \tag{3.98a}$$
$$2ar_xs_x + b(r_xs_y + r_ys_x) + 2cr_ys_y = 0. \tag{3.98b}$$

This is almost identical to the regular hyperbolic case but instead of choosing

$$\alpha, \beta = \frac{r \pm s}{2}, \tag{3.99}$$

we choose

$$\alpha, \beta = \frac{r \pm is}{2}, \tag{3.100}$$

and the procedure is the same. The next few examples illustrate this.

Example 3.7
Consider

$$u_{xx} - 4u_{xy} + 5u_{yy} = 0. \tag{3.101}$$

The corresponding equations for r and s are

$$r_x^2 - 4r_x r_y + 5r_y^2 = 0, \qquad s_x^2 - 4s_x s_y + 5s_y^2 = 0 \tag{3.102}$$

but as they are identical it suffices to only consider one. Dividing (3.102) by r_y^2 gives

$$\left(\frac{r_x}{r_y}\right)^2 - 4\frac{r_x}{r_y} + 5 = 0. \tag{3.103}$$

Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = 2 \pm i, \tag{3.104}$$

or

$$r_x - (2 \pm i)r_y = 0. \tag{3.105}$$

The method of characteristics gives

$$r = R\,(2x + y \pm ix)\,; \tag{3.106}$$

we choose

$$R(\lambda) = \lambda, \tag{3.107}$$

and taking the real and imaginary parts leads to the choice

$$r = 2x + y, \qquad s = x. \tag{3.108}$$

Under this transformation, the original Eq. (3.101) becomes

$$u_{rr} + u_{ss} = 0, \tag{3.109}$$

the desired standard form.

Example 3.8
Consider

$$2\left(1 + x^2\right)^2 u_{xx} - 2\left(1 + x^2\right)\left(1 + y^2\right) u_{xy} + \left(1 + y^2\right)^2 u_{yy} + 4x\left(1 + x^2\right) u_x = 0. \tag{3.110}$$

The corresponding equations for r and s are

$$2\left(1+x^2\right)^2 r_x^2 - 2\left(1+x^2\right)\left(1+y^2\right) r_x r_y + \left(1+y^2\right)^2 r_y^2 = 0, \tag{3.111a}$$
$$2\left(1+x^2\right)^2 s_x^2 - 2\left(1+x^2\right)\left(1+y^2\right) s_x s_y + \left(1+y^2\right)^2 s_y^2 = 0, \tag{3.111b}$$

but as they are identical it suffices to only consider one. Solving by the quadratic formula gives

$$\frac{r_x}{r_y} = \frac{(2 \pm i)\left(1+y^2\right)}{1+x^2}, \tag{3.112}$$

or

$$2\left(1+x^2\right) r_x - (1 \pm i)\left(1+y^2\right) r_y = 0. \tag{3.113}$$

The method of characteristics gives the solution as

$$r = R\left(\tan^{-1} x + 2\tan^{-1} y \pm i \tan^{-1} x\right); \tag{3.114}$$

we choose $R(\lambda) = \lambda$, and choosing gives r and s

$$r = \tan^{-1} x + 2\tan^{-1} y, \qquad s = \tan^{-1} x. \tag{3.115}$$

Under this transformation, the original Eq. (3.110) becomes

$$u_{rr} + u_{ss} - 2yu_r = 0, \tag{3.116}$$

and upon using the original transformation gives

$$u_{rr} + u_{ss} - 2\tan\frac{r-s}{2} u_r = 0, \tag{3.117}$$

the desired standard form.

3.3 THE WAVE EQUATION

We now re-visit the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \ -\infty < x < \infty \tag{3.118}$$

subject to the initial conditions

$$\begin{aligned} u(x,0) &= f(x), \quad -\infty < x < \infty, \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \quad -\infty < x < \infty. \end{aligned} \tag{3.119}$$

As seen previously, we can target the modified hyperbolic equation by making the change of variables: $r = x - ct$ and $s = x + ct$. Equation (3.118) becomes

$$u_{rs} = 0, \tag{3.120}$$

which was solved in Example 3.3. There, we obtained the solution

$$u = F(r) + G(s) \tag{3.121}$$

or in terms of x and t

$$u = F(x - ct) + G(x + ct). \tag{3.122}$$

Before we impose the initial conditions, we want to understand what this solution means. We look at the following examples: Let $u = F(x - ct)$ (that is, $G = 0$) and let $F(x - ct) = \operatorname{sech}(x - ct)$. When $c = 1$, this simplifies to $u = \operatorname{sech}(x - t)$. This is just a wave traveling to the right. Similarly, if we look at $u = \operatorname{sech}(x + ct)$ this is a wave traveling to the left. Graphs for this at various times for t are given in Figs. 3.1a and 3.1b.

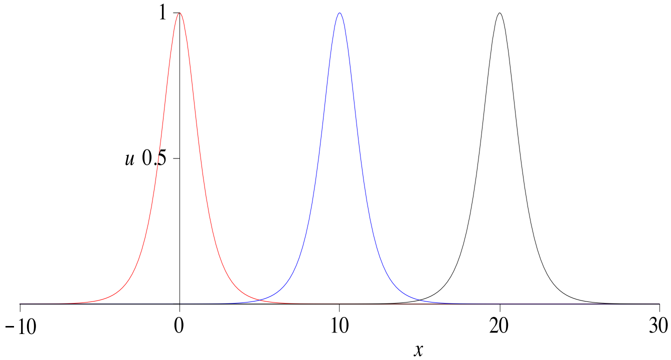

Figure 3.1: (a) Traveling wave solutions $u = \operatorname{sech}(x - t)$ at times t = 0 (red), 10 (blue), and 20 (black).

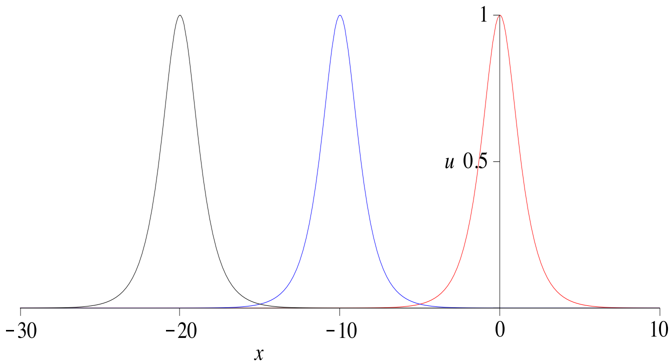

Figure 3.1: (b) Traveling wave solutions $u = \operatorname{sech}(x + t)$ at times t = 0 (red), 10 (blue), and 20 (black).

Thus, in general

$$u = F(x-ct) + G(x+ct), \tag{3.123}$$

consists of two waves, one traveling right and one traveling left.

We now incorporate the initial conditions (3.119) on the solution (3.123). In doing so, we obtain

$$F(x) + G(x) = f(x), \quad -cF'(x) + cG'(x) = g(x). \tag{3.124}$$

From (3.124), we obtain

$$F'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x), \quad G'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x), \tag{3.125}$$

and upon integration

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds, \tag{3.126a}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds. \tag{3.126b}$$

Thus, we obtain

$$u(x,t) = \frac{1}{2}\left\{f(x-ct) + f(x+ct)\right\} + \frac{1}{2c}\left\{\int_{x_0}^{x+ct} g(s)ds - \int_{x_0}^{x-ct} g(s)ds\right\}, \tag{3.127}$$

which simplifies to

$$u(x,t) = \frac{1}{2}\left\{f(x-ct) + f(x+ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{3.128}$$

This is known as the d'Alembert solution.

Example 3.9
Consider

$$u_{tt} = u_{xx},$$

subject to the initial conditions:

$$u(x,0) = \begin{cases} 2 & \text{if } -1 < x < 1, \\ 0 & \text{if otherwise,} \end{cases} \quad u_t(x,0) = 0. \tag{3.129}$$

If we let

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0, \end{cases} \tag{3.130}$$

then (3.129) can be rewritten as

$$u(x, 0) = 2\,(H(x+1) - H(x-1)) \tag{3.131}$$

so the solution for this problem is given by:

$$u(x, t) = H(x+1+t) - H(x-1+t) + H(x+1-t) - H(x-1-t). \tag{3.132}$$

Figure 3.2a illustrates the solution at times $t = 0, .5, 1, 1.5$, and 2.

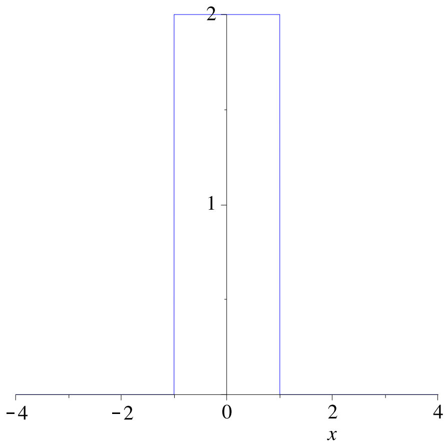

Figure 3.2: (a) The initial condition at $t = 0$.

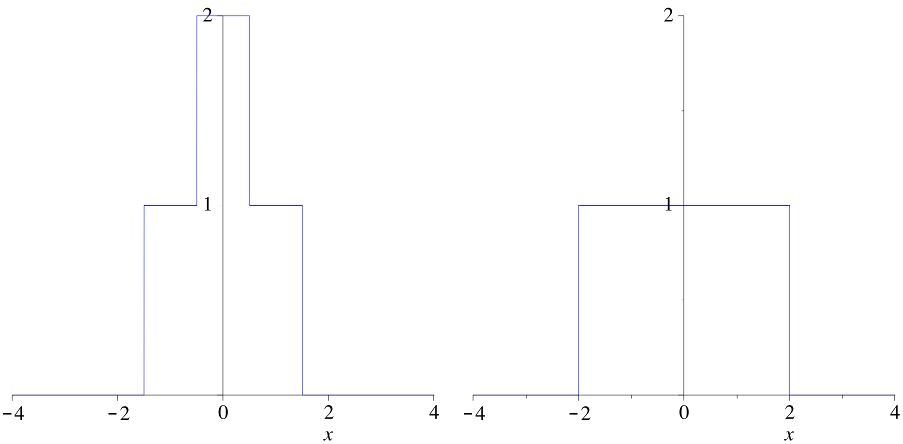

Figure 3.2: (b) Traveling wave solutions at times t = 0.5 and 1.

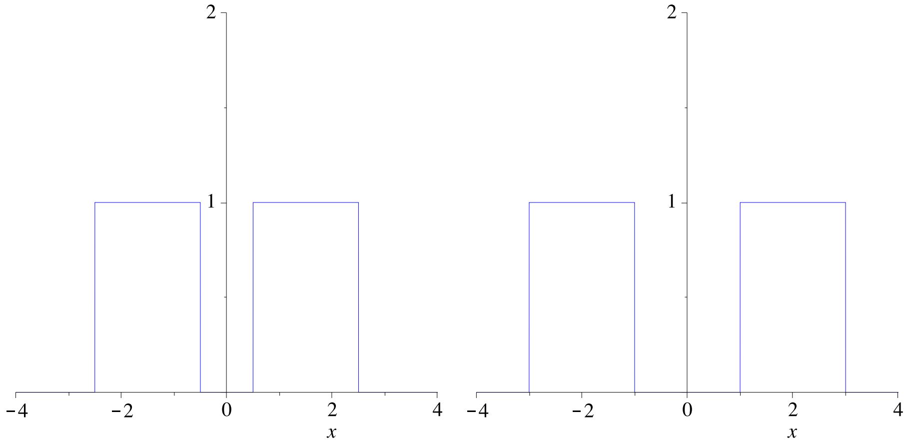

Figure 3.2: (c) Traveling wave solutions at times t = 1.5 and 2.

3.4 EXERCISES

3.1. Determine the type of the following second-order PDEs

(i) $(1+x^2)u_{xx} - (1+y^2)u_{yy} = u_x + u_y$
(ii) $u_{xx} + 2u_{xy} + u_{yy} = 0$
(iii) $y^2 u_{xx} + 2y u_{xy} + u_{yy} = 0$
(iv) $xy u_{xy} + u = u_x + u_y, \quad xy \neq 0$
(v) $2u_{xx} - 5u_{xy} - 3u_{yy} = 1$
(vi) $u_{xx} - 4y u_{xy} + 13u_{yy} = u$
(vii) $(1+y^2)u_{xx} + 2xyy u_{xy} + (1+x^2)u_{yy} = 1$

3.2. Transform the following PDEs to standard form. Find the general solution if possible.

(i) $u_{xx} + 2u_{xy} + u_{yy} = 0$
(ii) $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = 0$
(iii) $y^2 u_{xx} + 2xy u_{xy} + x^2 u_{yy} + x u_x + y u_y = 0$
(iv) $2u_{xx} - 3u_{xy} + u_{yy} = u_x + u_y$
(v) $x^2 u_{xx} - 3xy u_{xy} + 2y^2 u_{yy} = 0$
(vi) $2y^2 u_{xx} - 3y u_{xy} + u_{yy} = 0$
(vii) $13u_{xx} - 6u_{xy} + u_{yy} = 0$
(viii) $u_{xx} - 2\cos x u_{xy} + 2\cos^2 x u_{yy} = 0$
(ix) $8y^2 u_{xx} + 4y u_{xy} + 5u_{yy} - 5y^{-1} u_y = 0$

3.3. Earlier in this chapter, we saw that in order to target hyperbolic form it was necessary to solve the system:

$$\begin{aligned} ar_x^2 + br_x r_y + cr_y^2 &= -\left(as_x^2 + bs_x s_y + cs_y^2\right), \\ 2ar_x s_x + b(r_x s_y + r_y s_x) + 2cr_y s_y &= 0. \end{aligned} \tag{3.133}$$

Show that if r and s are defined as

$$r = \alpha - \beta \quad s = \alpha + \beta,$$

then (3.133) is satisfied if α and β satisfy

$$\begin{aligned} a\alpha_x^2 + 2b\alpha_x\alpha_y + c\alpha_y^2 &= 0, \\ a\beta_x^2 + 2b\beta_x\beta_y + c\beta_y^2 &= 0. \end{aligned} \tag{3.134}$$

Furthermore, to target elliptic form it was necessary to solve the system:

$$\begin{aligned} ar_x^2 + br_x r_y + cr_y^2 &= \left(as_x^2 + bs_x s_y + cs_y^2\right), \\ 2ar_x s_x + b(r_x s_y + r_y s_x) + 2cr_y s_y &= 0. \end{aligned} \tag{3.135}$$

Show that if r and s are defined as

$$r = \alpha - i\beta \quad s = \alpha + i\beta,$$

where $i = \sqrt{-1}$, then (3.135) is satisfied if α and β satisfy again, system (3.134).

3.4. The PDE

$$x^2 u_{xx} - 4xyu_{xy} + 4y^2 u_{yy} + xu_x = 0$$

is parabolic. Introducing new coordinates

$$r = x^2 y, \quad s = y,$$

reduces the PDE to

$$u_{ss} - \frac{r}{s^2} u_r = 0.$$

In fact, any choice of

$$r = R\left(x^2 y\right), \quad s = S(x, y),$$

will transform the original PDE to one that is in parabolic standard form (provided the Jacobian of the transformation is not zero). Can the choice of R and S be made such that we can transform to

$$u_{ss} = 0 \quad \text{or} \quad u_{ss} = u_r?$$

CHAPTER 4

Fourier Series

At this point, we will consider the major technique for solving standard boundary value equations. Consider, for example, the heat equation

$$u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0 \tag{4.1}$$

subject to

$$u(0,t) = u(\pi,t) = 0, \tag{4.2a}$$
$$u(x,0) = 2\sin x. \tag{4.2b}$$

Here, we will assume that solutions are of the form

$$u(x,t) = X(x)T(t). \tag{4.3}$$

Substituting into the heat equation (4.1) gives

$$XT' = X''T, \tag{4.4}$$

or, after dividing by TX

$$\frac{T'}{T} = \frac{X''}{X}. \tag{4.5}$$

Since each side of (4.5) is a function of a different variable, they therefore must be independent of each other, and thus

$$\frac{T'}{T} = \lambda = \frac{X''}{X}, \tag{4.6}$$

for some constant λ. The boundary conditions in (4.2a) become, accordingly,

$$X(0) = X(\pi) = 0. \tag{4.7}$$

From (4.6) we have $X'' = \lambda X$, which we integrate. In doing so, we have three cases, depending on the sign of λ. These are

$$X(x) = \begin{cases} c_1 e^{nx} + c_2 e^{-nx} & \text{if } \lambda = n^2, \\ c_1 x + c_2 & \text{if } \lambda = 0, \\ c_1 \sin nx + c_2 \cos nx & \text{if } \lambda = -n^2, \end{cases} \tag{4.8}$$

where n is a constant. We will consider each case separately. In the first case, where $\lambda = n^2$ imposing the boundary conditions (4.7) gives

$$c_1 e^0 + c_2 e^0 = 0, \quad c_1 e^{n\pi} + c_2 e^{-n\pi} = 0, \tag{4.9}$$

and solving for c_1 and c_2 shows that both are identically zero. This shows that $u = 0$ identically, and the initial condition (4.2b) cannot be satisfied.

In the second case where $\lambda = 0$, imposing the boundary conditions (4.7) gives that

$$c_1 0 + c_2 = 0, \quad c_1 \pi + c_2 = 0, \tag{4.10}$$

and solving for c_1 and c_2 shows again that both are identically zero, and again $u = 0$.

In the third case, $\lambda = -n^2$, using the boundary conditions (4.7) gives

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 \sin n\pi + c_2 \cos n\pi = 0, \tag{4.11}$$

which leads to

$$c_2 = 0, \quad \sin n\pi = 0 \quad \Rightarrow \quad n = 0, \pm 1, \pm 2, \ldots. \tag{4.12}$$

However, it suffices to consider only the positive values of n. From (4.6) with $\lambda = -n^2$ gives

$$T' = -n^2 T, \tag{4.13}$$

from which we find that

$$T(t) = c_3 e^{-n^2 t} \tag{4.14}$$

where c_3 is a constant of integration and thus, gives us a solution to the original PDE as

$$u = TX = ce^{-n^2 t} \sin nx, \tag{4.15}$$

where we have set $c = c_1 c_3$. Finally, imposing the initial condition $u(x, 0) = 2 \sin x$ gives

$$u(x, 0) = ce^0 \sin nx = 2 \sin x, \tag{4.16}$$

from which we obtain $c = 2$ and $n = 1$. Therefore, the solution to the PDE subject to the initial and boundary conditions is

$$u(x, t) = 2e^{-t} \sin x. \tag{4.17}$$

If the initial condition was different, say $u(x, 0) = 4 \sin 3x$, then $c = 4$ and $n = 3$ and the solution would be

$$u(x, t) = 4e^{-9t} \sin 3x. \tag{4.18}$$

However, if the initial condition was $u(x, 0) = 2 \sin x + 4 \sin 3x$, it would be impossible to choose n and c in (4.15) to satisfy both. However, if we were to solve the heat equation with each initial condition separately, we can simply add the solutions together. In this case, this would be

$$u(x, t) = 2e^{-t} \sin x + 4e^{-9t} \sin 3x. \tag{4.19}$$

This is called the **principle of superposition**.

Theorem 4.1 Principle of Superposition. *If u_1 and u_2 are two solutions to the heat equation, then $u = c_1u_1 + c_2u_2$ is also a solution.*

The principle of superposition easily extends to more than two solutions. Thus, if

$$u(x,t) = (a_n \cos nx + b_n \sin nx)\, e^{-n^2 t}, \quad n = 0, 1, 2, 3, 4, \ldots \tag{4.20}$$

are solutions to the heat equation (a_n and b_n are constants), then so is

$$u(x,t) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) e^{-n^2 t}. \tag{4.21}$$

If the initial conditions were such that they only involved sine and cosine functions, we could choose the integers n and constants a_n and b_n accordingly to match the terms in the initial condition. However, if the initial condition was, for example, $u(x,0) = \pi x - x^2$, then it is not obvious how to proceed, as (4.21) contains no x or x^2 terms. However, consider the following

$$\begin{aligned}
u_1 &= \frac{8}{\pi} e^{-t} \sin x, \\
u_2 &= \frac{8}{\pi} \left(e^{-t} \sin x + \frac{1}{27} e^{-9t} \sin 3x \right), \\
u_3 &= \frac{8}{\pi} \left(e^{-t} \sin x + \frac{1}{27} e^{-9t} \sin 3x + \frac{1}{125} e^{-25t} \sin 5x \right).
\end{aligned} \tag{4.22}$$

From Fig. 4.1, one will notice that with each additional term added in (4.22), the solution is a better match to the initial condition. In fact, if we consider

$$u = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-(2n-1)^2 t} \sin(2n-1)x, \tag{4.23}$$

we get an excellent match to the initial condition. Thus, we are led to ask: How are the integers n and constants a_n and b_n chosen as to match the initial condition?

4.1 FOURIER SERIES

It is well known that many functions can be represented by a power series

$$f(x) = \sum_{i=0}^{\infty} a_n (x - x_0)^n, \tag{4.24}$$

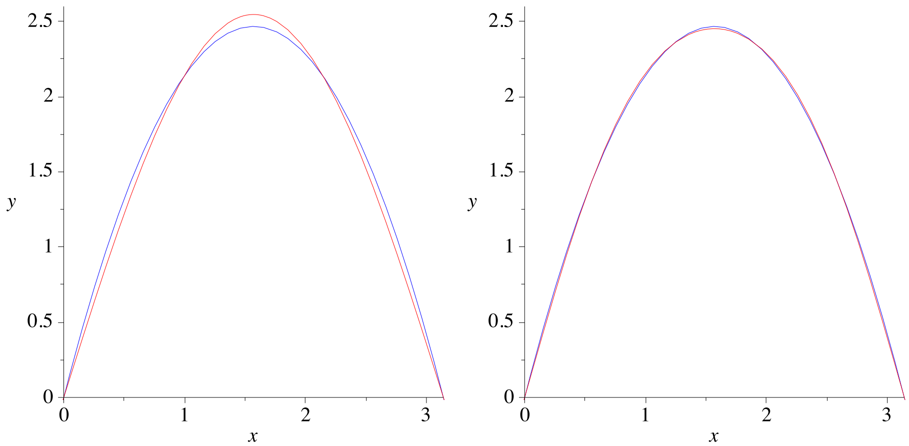

Figure 4.1: The solutions (4.22) with one and two terms at $t = 0$ (blue) compared with $u(x, 0) = \pi x - x^2$ (red).

where x_0 is the center of the series and a_n, constants determined by

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \quad i = 0, 1, 2, 3, \ldots. \tag{4.25}$$

For functions that require different properties, say for example, fixed points at the endpoints of an interval, a different type of series is required. An example of such a series is called Fourier series. For example, suppose that $f(x) = \pi x - x^2$ has a Fourier series

$$\pi x - x^2 = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x \ldots. \tag{4.26}$$

How do we choose b_1, b_2, b_3, etc., such that the Fourier series looks like the function? Notice that if we multiply (4.26) by $\sin x$ and integrate from 0 to π

$$\int_0^\pi (\pi x - x^2) \sin x \, dx = b_1 \int_0^\pi \sin^2 x \, dx + b_2 \int_0^\pi \sin x \sin 2x \, dx + \ldots, \tag{4.27}$$

then we obtain

$$4 = b_1 \frac{\pi}{2} \quad \Rightarrow \quad b_1 = 8/\pi, \tag{4.28}$$

since

$$\int_0^\pi (\pi x - x^2) \sin x \, dx = 4, \quad \int_0^\pi \sin^2 x \, dx = \frac{\pi}{2},$$
$$\text{and} \quad \int_0^\pi \sin x \sin nx \, dx = 0, \quad n = 2, 3, 4, \ldots. \tag{4.29}$$

Similarly, if multiply (4.26) by $\sin 2x$ and integrate from 0 to π

$$\int_0^\pi (\pi x - x^2)\sin 2x\,dx = b_1 \int_0^\pi \sin x \sin 2x\,dx + b_2 \int_0^\pi \sin^2 2x\,dx + \ldots. \tag{4.30}$$

then we obtain

$$0 = b_2\,\frac{\pi}{2} \quad\Rightarrow\quad b_2 = 0. \tag{4.31}$$

Multiply (4.26) by $\sin 3x$ and integrate from 0 to π

$$\int_0^\pi (\pi x - x^2)\sin 3x\,dx = b_1 \int_0^\pi \sin x \sin 3x\,dx + b_2 \int_0^\pi \sin 2x \sin 3x\,dx + \ldots, \tag{4.32}$$

then we obtain

$$\frac{4}{27} = b_3\,\frac{\pi}{2} \quad\Rightarrow\quad b_3 = \frac{8}{27\pi}. \tag{4.33}$$

Continuing in this fashion, we would obtain

$$b_4 = 0,\;\; b_5 = \frac{8}{125\pi},\;\; b_6 = 0,\;\; b_7 = \frac{8}{343\pi}. \tag{4.34}$$

Substitution of the constants $b_1 - b_5$ gives (4.22).

4.2 FOURIER SERIES ON $[-\pi,\pi]$

Consider the series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{4.35}$$

where a_0, a_n, and b_n are constant coefficients. The question is: How do we choose the coefficients as to give an accurate representation of $f(x)$? We use the following integral identities for $\cos n\pi x$ and $\sin n\pi x$:

$$\int_{-\pi}^{\pi} \cos nx\,dx = 0, \qquad \int_{-\pi}^{\pi} \sin nx\,dx = 0, \qquad \int_{-\pi}^{\pi} \sin nx\ \cos mx\,dx = 0, \quad \forall m,n$$

$$\int_{-\pi}^{\pi} \cos nx\ \cos mx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{4.36}$$

$$\int_{-\pi}^{\pi} \sin nx\ \sin mx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n. \end{cases}$$

First, if we integrate (4.35) from $-\pi$ to π, then by the properties in (4.36), we are left with

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx = 2\pi a_0, \tag{4.37}$$

from which we deduce

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx. \tag{4.38}$$

Next, we multiply the series (4.35) by $\cos mx$, giving

$$f(x)\cos mx = \frac{1}{2}a_0 \cos mx + \sum_{n=1}^{\infty}\left(a_n \cos nx\ \cos mx + b_n \sin nx\ \cos mx\right). \tag{4.39}$$

Again, integrate from $-\pi$ to π. From (4.36), the integration of $a_0 \cos mx$ is zero; from (4.36), the integration of $\cos nx\ \cos mx$ is zero except when $n = m$; and further, from (4.36) the integration of $\sin nx\ \cos mx$ is zero for all m and n. This leaves

$$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi a_n, \tag{4.40}$$

or

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx. \tag{4.41}$$

Similarly, if we multiply the series (4.35) by $\sin mx$ then we obtain

$$f(x)\sin mx = \frac{1}{2}a_0 \sin mx + \sum_{n=1}^{\infty}\left(a_n \cos nx\ \sin mx + b_n \sin nx\ \sin mx\right), \tag{4.42}$$

which we integrate from $-\pi$ to π. From (4.36), the integration of $a_0 \sin m\pi x$ is zero; from (4.36) the integration of $\sin nx\ \cos mx$ is zero for all m and n; and further, from (4.36) the integration of $\sin nx\ \sin mx$ is zero except when $n = m$. This leaves

$$\int_{-\pi}^{\pi} f(x)\sin nx\, dx = b_n \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi b_n, \tag{4.43}$$

or

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx. \tag{4.44}$$

Therefore, the Fourier series representation of a function $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \tag{4.45}$$

where the coefficients a_0, a_n, and b_n are chosen such that

$$\begin{gathered} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \quad \text{for } \ n = 1, 2, \ldots. \end{gathered} \tag{4.46}$$

As the integral formulas for a_0 and a_n are so similar, it is more convenient for us to shift the factor of $1/2$ in a_0 into the Fourier series, thus

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right), \tag{4.47}$$

where all of the a_n's are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx. \tag{4.48}$$

Example 4.2
Consider

$$f(x) = x^2, \quad [-\pi, \pi]. \tag{4.49}$$

From (4.48) we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{2\pi^2}{3} \tag{4.50}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx \\ &= \frac{1}{\pi}\left.\left[\frac{x^2 \sin nx}{n} + 2\frac{x\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]\right|_{-\pi}^{\pi} \\ &= \frac{4(-1)^n}{n^2}, \end{aligned} \tag{4.51}$$

and from (4.46)

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx \\ &= \frac{1}{\pi}\left.\left[-\frac{x^2 \cos nx}{n} + 2\frac{x\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]\right|_{-\pi}^{\pi} \\ &= 0. \end{aligned} \tag{4.52}$$

Thus, the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$ is

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}. \tag{4.53}$$

Figure 4.2 shows consecutive plots of the Fourier series (4.53) with 5 and 10 terms on the interval $[-\pi,\pi]$ and the function (4.49).

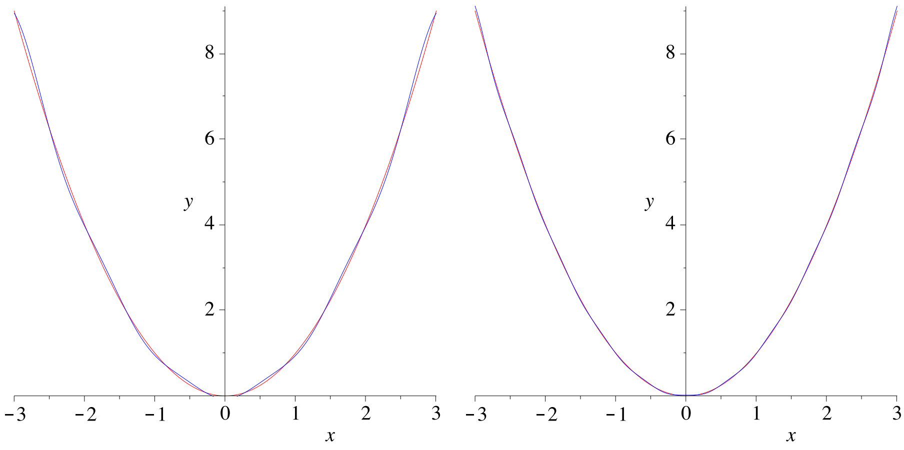

Figure 4.2: The Fourier series (4.53) with 5 and 10 terms (blue) and (4.49) (red).

Example 4.3
Consider

$$f(x) = x, \quad [-\pi, \pi]. \tag{4.54}$$

From (4.48) we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left.\frac{x^2}{2}\right|_{-\pi}^{\pi} = 0, \tag{4.55}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \\ &= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]\Bigg|_{-\pi}^{\pi} \\ &= 0, \end{aligned} \tag{4.56}$$

and from (4.46)

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\ &= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]\Bigg|_{-\pi}^{\pi} \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned} \tag{4.57}$$

Thus, the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}. \tag{4.58}$$

Figure 4.3 shows consecutive plots of the Fourier series (4.58) with 5 and 50 terms on the interval $[-\pi,\pi]$ and the function (4.54).

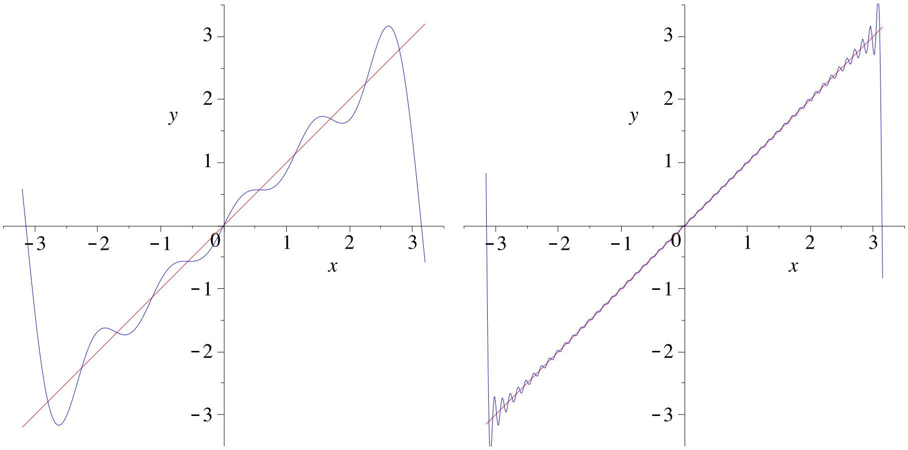

Figure 4.3: The solutions (4.58) with 5 and 50 terms and (4.54).

Example 4.4
Consider

$$f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0, \\ x+1 & \text{if } \quad 0 < x < \pi. \end{cases} \tag{4.59}$$

From (4.48) we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{0}^{\pi} x+1\,dx = \frac{1}{\pi}\,x\Big|_{-\pi}^{0} + \frac{1}{\pi}\,\frac{x^2}{2}+x\Big|_{0}^{\pi} = \frac{\pi}{2}+2, \tag{4.60}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (x+1)\cos nx\,dx \\ &= \frac{1}{\pi}\,\frac{\sin nx}{n}\Big|_{-\pi}^{0} + \frac{1}{\pi}\left[(x+1)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]\Big|_{0}^{\pi} \\ &= \frac{1}{\pi}\,\frac{(-1)^n-1}{n^2}, \end{aligned} \tag{4.61}$$

and from (4.46)

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{0} \sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (x+1)\sin nx\,dx \\ &= \frac{1}{\pi}\,-\frac{\cos nx}{n}\Big|_{-\pi}^{0} + \frac{1}{\pi}\left[-(x+1)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]\Big|_{0}^{\pi} \\ &= \frac{(-1)^{n+1}}{n}. \end{aligned} \tag{4.62}$$

Thus, the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$f(x) = \frac{\pi}{4} + 1 + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \frac{(-1)^n - 1}{n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right). \tag{4.63}$$

Figure 4.4 shows a plot of the original function (4.59) and the Fourier series (4.63) with 10 terms.

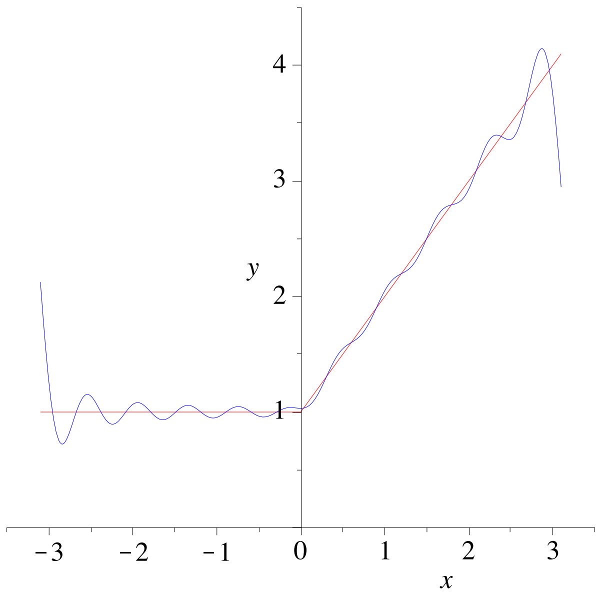

Figure 4.4: The solutions (4.63) with 10 terms (blue) and (4.59) (red).

As many problems are on intervals more general than $[-\pi, \pi]$, it is natural for us to extend to more general intervals.

4.3 FOURIER SERIES ON $[-L, L]$

Consider the series

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{4.64}$$

where L is a positive number and $a_0, a_n,$ and b_n constant coefficients. The question is: How are the coefficients chosen as to give an accurate representation of $f(x)$? Here, we use the following properties of $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$:

$$\int_{-L}^{L}\cos\frac{n\pi x}{L}dx = 0,\qquad \int_{-L}^{L}\sin\frac{n\pi x}{L}dx = 0,\qquad \int_{-L}^{L}\sin\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L}dx = 0\quad \forall m,n$$

$$\int_{-L}^{L}\cos\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L}dx = \begin{cases}0 & \text{if } m\neq n\\ L & \text{if } m=n\end{cases} \tag{4.65}$$

$$\int_{-L}^{L}\sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L}dx = \begin{cases}0 & \text{if } m\neq n\\ L & \text{if } m=n\ .\end{cases}$$

If we integrate (4.64) from $-L$ to L, then by the identities in (4.65), we are left with

$$\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-L}^{L} a_0 dx = La_0, \tag{4.66}$$

from which we deduce

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx. \tag{4.67}$$

Next, we multiply the series (4.64) by $\cos\frac{m\pi x}{L}$, giving

$$f(x)\cos\frac{m\pi x}{L} = \frac{1}{2}a_0\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right). \tag{4.68}$$

Again, we integrate from $-L$ to L and use the identities in (4.65). This gives

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = a_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L}dx = La_n, \tag{4.69}$$

or

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx. \tag{4.70}$$

Similarly, if we multiply the series (4.64) by $\sin\frac{m\pi x}{L}$, then we obtain

$$f(x)\sin\frac{m\pi x}{L} = \frac{1}{2}a_0\sin\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right), \tag{4.71}$$

which we integrate from $-L$ to L. Using the identities in (4.65) gives

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = b_n\int_{-L}^{L}\sin^2\frac{n\pi x}{L}dx = La_n, \tag{4.72}$$

or

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{4.73}$$

Therefore, the Fourier series representation of a function $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right), \tag{4.74}$$

where the coefficients a_n and b_n are chosen such that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{4.75}$$

for $n = 0, 1, 2, \ldots$.

Example 4.5

Consider

$$f(x) = 9 - x^2, \quad [-3, 3]. \tag{4.76}$$

In this case, $L = 3$, so from (4.75) we have

$$a_0 = \frac{1}{3}\int_{-3}^{3} 9 - x^2\,dx = \frac{1}{3}\left[9x - \frac{x^3}{3}\right]\Bigg|_{-3}^{3} = 12, \tag{4.77}$$

$$\begin{aligned} a_n &= \frac{1}{3}\int_{-3}^{3}(9-x^2)\cos\frac{n\pi x}{3}dx \\ &= \frac{1}{3}\left[\left(\frac{27}{n\pi} - \frac{3x^2}{n\pi} + \frac{54}{n^3\pi^3}\right)\sin\frac{n\pi x}{3} - \frac{18x}{\pi^2 n^2}\cos\frac{n\pi x}{3}\right]\Bigg|_{-3}^{3} \\ &= \frac{36(-1)^{n+1}}{n^2\pi^2}, \end{aligned} \tag{4.78}$$

and

$$\begin{aligned} b_n &= \frac{1}{3}\int_{-3}^{3}(9-x^2)\sin\frac{n\pi x}{3}dx \\ &= \frac{1}{3}\left[-\left(\frac{27}{n\pi} - \frac{3x^2}{n\pi} + \frac{54}{n^3\pi^3}\right)\cos\frac{n\pi x}{3} - \frac{18x}{\pi^2 n^2}\sin\frac{n\pi x}{3}\right]\Bigg|_{-3}^{3} \\ &= 0. \end{aligned} \tag{4.79}$$

Thus, the Fourier series for (4.76) on $[-3, 3]$ is

$$f(x) = 6 + \sum_{n=1}^{\infty}\frac{36(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{3}. \tag{4.80}$$

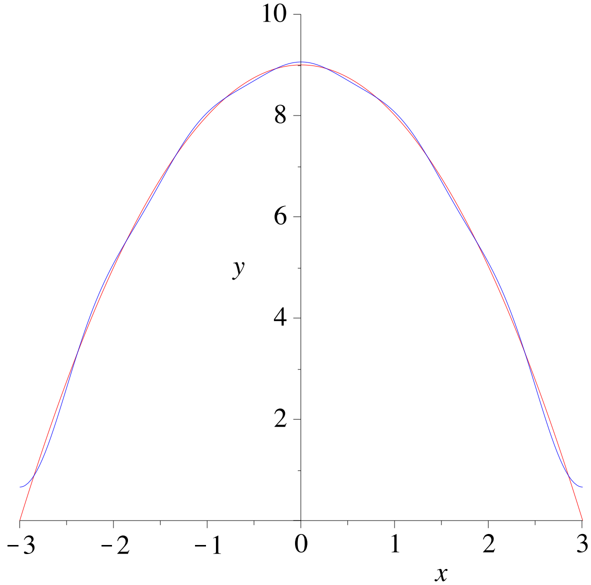

Figure 4.5: The Fourier series (4.80) using the first 20 terms (blue) and (4.76) (red).

Figure 4.5 shows the graph of this Fourier series (4.80) with the first 20 terms (blue) and (4.59) (red).

Example 4.6
Consider

$$f(x) = \begin{cases} -2-x & \text{if } -2 < x < -1, \\ x & \text{if } -1 < x < 1, \\ 2-x & \text{if } 1 < x < 2. \end{cases} \tag{4.81}$$

In this case $L = 2$ so from (4.75) we have

$$\begin{aligned} a_0 &= \frac{1}{2}\left\{\int_{-2}^{-1}(-2-x)\,dx + \int_{-1}^{1} x\,dx + \int_{1}^{2}(2-x)\,dx\right\} \\ &= \frac{1}{2}\left\{\left[-2x - \frac{x^2}{2}\right]\Bigg|_{-2}^{-1} + \frac{x^2}{2}\Bigg|_{-1}^{1} + \left[2x - \frac{x^2}{2}\right]\Bigg|_{1}^{2}\right\} = 0, \end{aligned} \tag{4.82}$$

$$\begin{aligned}
a_n &= \frac{1}{2}\left\{\int_{-2}^{-1}(-2-x)\cos\frac{n\pi x}{2}\,dx + \int_{-1}^{1} x\cos\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\cos\frac{n\pi x}{2}\,dx\right\} \\
&= \left[-\frac{(x+2)}{n\pi}\sin\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Bigg|_{-2}^{-1} \\
&+ \left[\frac{x}{n\pi}\sin\frac{n\pi x}{2} + \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Bigg|_{-1}^{1} \\
&+ \left[-\frac{(x-2)}{n\pi}\sin\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]\Bigg|_{1}^{2} \\
&= 0,
\end{aligned} \tag{4.83}$$

and

$$\begin{aligned}
b_n &= \frac{1}{2}\left\{\int_{-2}^{-1}(-2-x)\sin\frac{n\pi x}{2}\,dx + \int_{-1}^{1} x\sin\frac{n\pi x}{2}\,dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}\,dx\right\} \\
&= \left[\frac{(x+2)}{n\pi}\cos\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{-2}^{-1} \\
&+ \left[-\frac{x}{n\pi}\cos\frac{n\pi x}{2} + \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{-1}^{1} \\
&+ \left[\frac{(x-2)}{n\pi}\cos\frac{n\pi x}{2} - \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_{1}^{2} \\
&= \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}.
\end{aligned} \tag{4.84}$$

Thus, the Fourier series for (4.81) on $[-2, 2]$ is

$$\begin{aligned}
f(x) &= \frac{16}{\pi^2}\left\{\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - + \dots\right\} \\
&= \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}.
\end{aligned} \tag{4.85}$$

Figure 4.6 shows the graph of this Fourier series (4.85) with the first five terms (blue) and (4.81) (blue).

Example 4.7
Consider

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases} \tag{4.86}$$

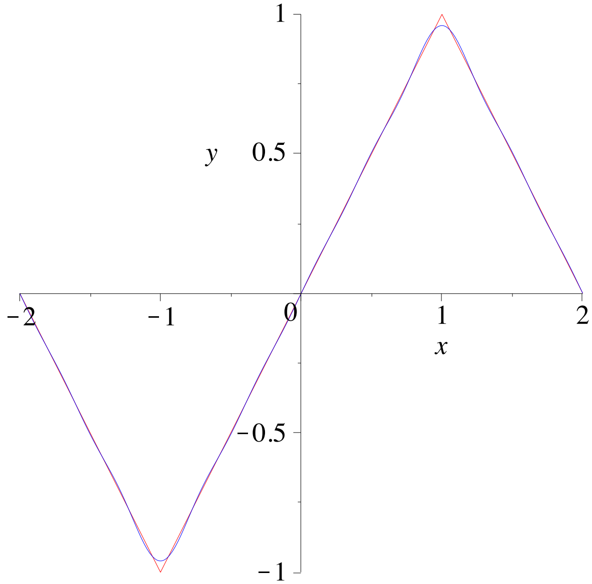

Figure 4.6: The Fourier series (4.85) with 5 terms (blue) and (4.81) (red).

In this case, $L = 1$, so from (4.75) we have

$$a_0 = \int_0^1 dx = 1, \tag{4.87}$$

$$a_n = \int_0^1 \cos n\pi x \, dx = \frac{1}{n\pi} \sin n\pi x \Big|_0^1 = 0, \tag{4.88}$$

and

$$b_n = \int_0^1 \sin n\pi x \, dx - \frac{1}{n\pi} \cos\{n\pi x \Big|_0^1 = \frac{1-(-1)^n}{n\pi}. \tag{4.89}$$

Thus, the Fourier series for (4.86) on $[-1, 1]$ is

$$\begin{aligned} f(x) &= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin n\pi x \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1}. \end{aligned} \tag{4.90}$$

Figure 4.7 shows the graph of this Fourier series (4.90) with the first 10 and 50 terms (blue) and (4.86) (red).

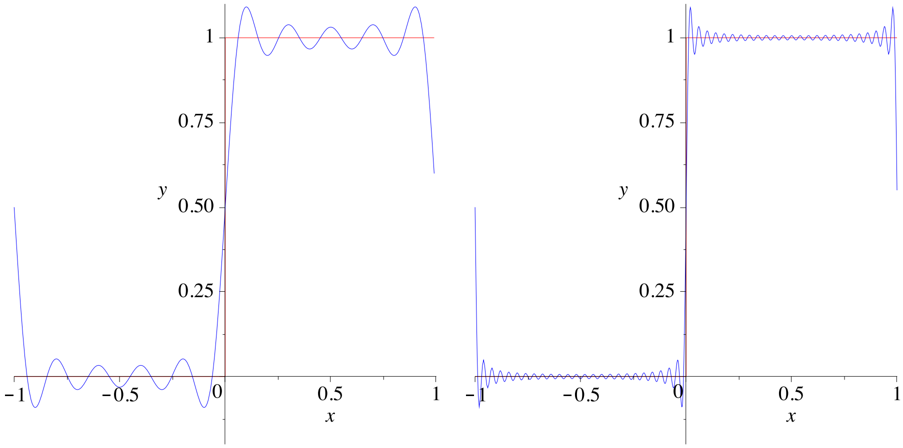

Figure 4.7: The solutions (4.90) with 10 and 50 terms (blue) and (4.86) (red).

It is interesting to note that regardless of the number of terms we have in the Fourier series, we cannot eliminate the spikes at $x = -1, 0, 1$ etc. This phenomenon is know as **Gibb's phenomena**.

4.4 ODD AND EVEN EXTENSIONS

Consider $f(x) = x$ on $[0,\pi]$. Here the interval is half the interval $[-\pi,\pi]$. Can we still construct a Fourier series for this? Well, it really depends on what $f(x)$ looks like on the interval $[-\pi,0]$. For example, if $f(x) = x$ on $[-\pi,0]$, then the answer is yes. If $f(x) = -x$ on $[-\pi,0]$, then the answer is also yes. In either case, as long as we are given $f(x)$ on $[-\pi,0]$, the answer is yes. If we are just given $f(x)$ on $[0,\pi]$, then it is natural to extend $f(x)$ to $[-\pi,0]$ as either an odd extension or even extension. Recall that a function is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. For example, if

$$f(x) = x, \quad \text{then} \quad f(-x) = -x = -f(x) \tag{4.91}$$

so $f(x) = x$ is odd. Similarly, if

$$f(x) = x^2, \quad \text{then} \quad f(-x) = (-x)^2 = x^2 = f(x) \tag{4.92}$$

so $f(x) = x^2$ is even. For each extension, the Fourier series will contain only sine terms or cosine terms. These series respectively are called *Sine series* and *Cosine series*. Before we consider each series separately, it is necessary to establish the following lemmas.

Lemma 4.8 *If $f(x)$ is an odd function then*

$$\int_{-l}^{l} f(x)\, dx = 0,$$

and if $f(x)$ is an even function, then

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx.$$

The proofs are left as an exercise for the reader.

At this point, we are ready to consider each series separately.

4.4.1 SINE SERIES

If $f(x)$ is given on $[0,L]$, and we assume that $f(x)$ is an odd function, that gives us $f(x)$ on the interval $[-L,0]$. We now consider the Fourier coefficients a_n and b_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{4.93}$$

Since $f(x)$ is odd and $\cos\frac{n\pi x}{L}$ is even, then their product is odd, and by Lemma 4.8

$$a_n = 0, \quad \forall n. \tag{4.94}$$

Similarly, since $f(x)$ is odd and $\sin\frac{n\pi x}{L}$ is odd, then their product is even, and by Lemma 4.8

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{4.95}$$

The Fourier series is therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{4.96}$$

where b_n is given in (4.95).

Example 4.9

Find a Fourier sine series for

$$f(x) = x^2, \quad [0,1]. \tag{4.97}$$

The coefficient b_n is given by

$$\begin{aligned} b_n &= 2\int_0^1 x^2 \sin n\pi x\,dx \\ &= 2\left[-\frac{x^2\cos n\pi\, x}{n\pi} + 2\frac{x\sin n\pi\, x}{n^2\pi^2} + 2\frac{\cos n\pi\, x}{n^3\pi^3}\right]\Bigg|_0^1 \\ &= 2\left[2\frac{(-1)^n - 1}{n^3\pi^3} - \frac{(-1)^n}{n\pi}\right] \end{aligned} \tag{4.98}$$

giving the Fourier Sine series as

$$f = 2\sum_{n=1}^{\infty}\left(2\frac{(-1)^n - 1}{n^3\pi^3} - \frac{(-1)^n}{n\pi}\right)\sin n\pi x. \tag{4.99}$$

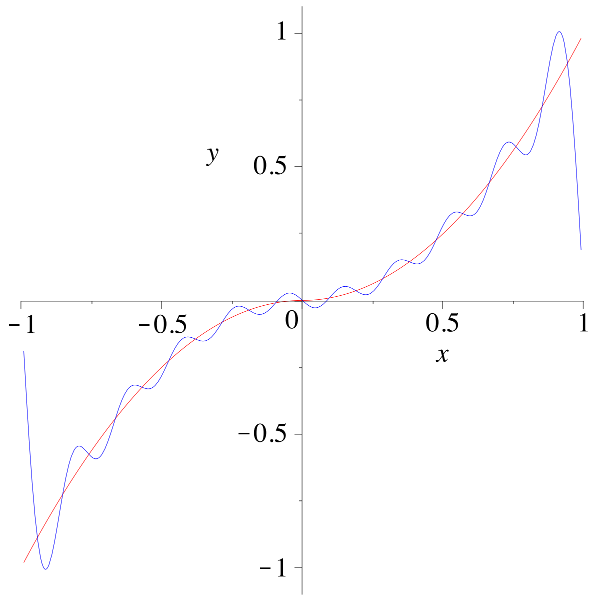

Figure 4.8: The function (4.97) with its odd extension (red) and its Fourier Sine series (4.99) with 10 terms (blue).

Example 4.10
Find a Fourier Sine series for

$$f(x) = \cos x, \quad [0, \pi]. \tag{4.100}$$

In calculating the coefficient b_n, the case when $n = 1$ is special and must be handled separately. When $n = 1$, then

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x \, dx = 0. \tag{4.101}$$

When $n > 1$, then

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \cos x \sin nx\, dx \\ &= \left[-\frac{1}{2}\frac{\cos(n-1)x}{n-1} - \frac{1}{2}\frac{\cos(n+1)x}{n+1}\right]\Bigg|_0^\pi \\ &= \frac{n\,(1+(-1)^n)}{n^2-1}. \end{aligned} \tag{4.102}$$

The Fourier Sine series is then given by

$$\begin{aligned} f &= \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n\,(1+(-1)^n)}{n^2-1}\sin nx \\ &= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx. \end{aligned} \tag{4.103}$$

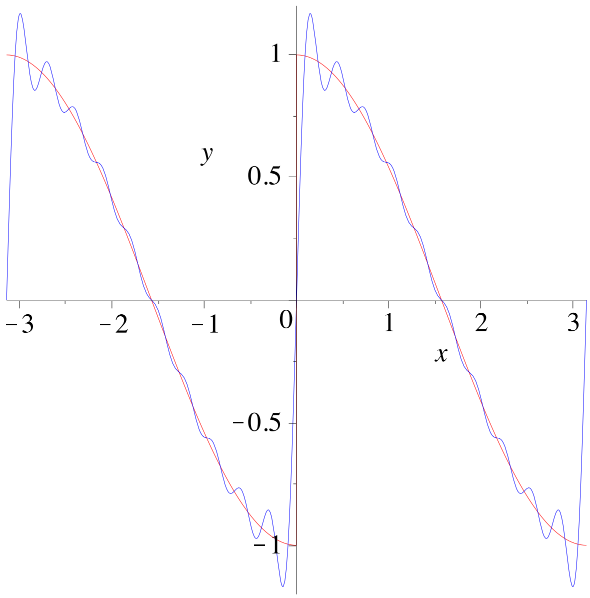

Figure 4.9: The function (4.100) with its odd extension (red) and its Fourier Sine series (4.103) with 20 terms (blue).

4.4.2 COSINE SERIES

If $f(x)$ is given on $[0,L]$, we assume that $f(x)$ is an even function, which gives us $f(x)$ on the interval $[-L,L]$. We now consider the Fourier coefficients a_n and b_n. For a_n,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \tag{4.104}$$

Since $f(x)$ is even and $\cos\frac{n\pi x}{L}$ is even, then their product is even, and by Lemma 4.8

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \tag{4.105}$$

Similarly

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{4.106}$$

Since $f(x)$ is even and $\sin\frac{n\pi x}{L}$ is odd, then their product is odd, and by Lemma 4.8

$$b_n = 0, \quad \forall\, n. \tag{4.107}$$

The Fourier series is therefore

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}, \tag{4.108}$$

where a_n is given in (4.105).

Example 4.11
Find a Fourier cosine series for

$$f(x) = x, \quad [0,2]. \tag{4.109}$$

The coefficient a_0 is given by

$$a_0 \;=\; \frac{2}{2}\int_0^2 x\,dx = \left.\frac{x^2}{2}\right|_0^2 = 2. \tag{4.110}$$

The coefficient a_n is given by

$$\begin{aligned} a_n &= \int_0^2 x\cos\frac{n\pi}{2}x\,dx \\ &= \left[\frac{2x}{n\pi}\sin\frac{n\pi\, x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi\, x}{2}\right]\Bigg|_0^2 \\ &= \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) \end{aligned} \tag{4.111}$$

giving the Fourier Cosine series as

$$f = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi x}{2}. \tag{4.112}$$

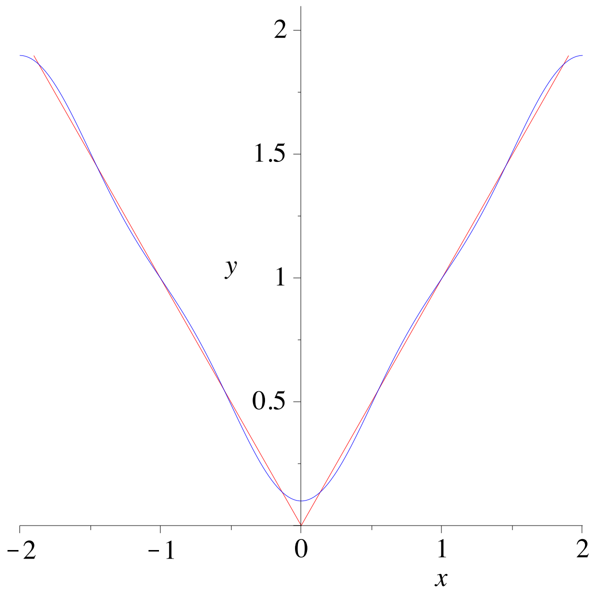

Figure 4.10: The function (4.109) with its even extension (red) and its Fourier Sine series (4.112) with 5 terms (blue).

Example 4.12
Find a Fourier cosine series for

$$f(x) = \sin x, \quad [0, \pi]. \tag{4.113}$$

The coefficient a_0 is given by

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} \sin x \, dx \\ &= \frac{2}{\pi} \left[-\cos x \right]\Big|_0^{\pi} \\ &= \frac{4}{\pi}. \end{aligned} \tag{4.114}$$

For the remaining coefficients a_n, the case a_1 again needs to be considered separately. For a_1,

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x \, dx = 0, \tag{4.115}$$

and the coefficient $a_n, \ \ n \geq 2$ is given by

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos nx \, dx \\ &= -\frac{2}{\pi}\left[\frac{1}{2}\frac{\cos(n-1)x}{n-1} - \frac{1}{2}\frac{\cos(n+1)x}{n+1}\right]\Bigg|_0^\pi \\ &= -\frac{2}{\pi}\frac{(1+(-1)^n)}{n^2-1}. \end{aligned} \tag{4.116}$$

Thus, the Fourier series is

$$f = -\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+(-1)^n)}{n^2-1}\cos nx. \tag{4.117}$$

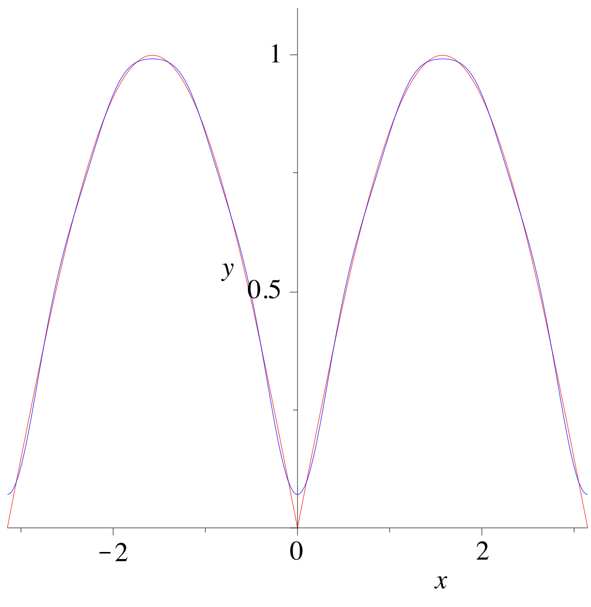

Figure 4.11: The function (4.113) with its even extension (red) and its Fourier sine series (4.117) with 5 terms (blue).

Example 4.13
Find a Fourier Sine and Cosine series for

$$f(x) = \begin{cases} 4x - x^2 & \text{for } 0 \le x \le 2 \\ 8 - 2x & \text{for } 2 < x < 4. \end{cases} \tag{4.118}$$

For the Fourier Sine series, b_n is obtained by

$$\begin{aligned} b_n &= \frac{2}{4}\int_0^2 (4x - x^2)\sin\frac{n\pi x}{4}\,dx + \frac{2}{4}\int_2^4 (8-2x)\sin\frac{n\pi x}{4}\,dx \\ &= \left[\left(\frac{32-16x}{n^2\pi^2}\right)\sin\frac{n\pi x}{4} + \left(\frac{2x^2-8x}{n\pi} - \frac{64}{n^3\pi^3}\right)\cos\frac{n\pi x}{4}\right]\Bigg|_0^2 \\ &+ \left[-\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4} + \frac{4x-16}{n\pi}\cos\frac{n\pi x}{4}\right]\Bigg|_2^4 \\ &= \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{64}{n^3\pi^3}\left(1 - \cos\frac{n\pi}{2}\right). \end{aligned} \tag{4.119}$$

Thus, the Fourier Sine series is given by

$$f = \sum_{n=1}^{\infty}\left(\frac{16}{\pi^2 n^2}\sin\frac{n\pi}{2} + \frac{64}{n^3\pi^3}\left(1-\cos\frac{n\pi}{2}\right)\right)\sin\frac{n\pi x}{4}. \tag{4.120}$$

For the Fourier Cosine series, a_0 and a_n are given by

$$\begin{aligned} a_0 &= \frac{2}{4}\int_0^2 4x - x^2\,dx + \frac{2}{4}\int_2^4 8 - 2x\,dx \\ &= \left[x^2 - \frac{x^3}{6}\right]\Bigg|_0^2 + \left[4x - \frac{x^2}{2}\right]\Bigg|_2^4 \\ &= \frac{8}{3} + 2 = \frac{14}{3}, \end{aligned} \tag{4.121}$$

and

$$\begin{aligned} a_n &= \frac{2}{4}\int_0^2 (4x - x^2)\cos\frac{n\pi x}{4}\,dx + \frac{2}{4}\int_2^4 (8-2x)\cos\frac{n\pi x}{4}\,dx \\ &= \left[\left(\frac{32-16x}{n^2\pi^2}\right)\cos\frac{n\pi x}{4} + \left(\frac{8x-2x^2}{n\pi} + \frac{64}{n^3\pi^3}\right)\sin\frac{n\pi x}{4}\right]\Bigg|_0^2 \\ &+ \left[-\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4} + \frac{16-4x}{n\pi}\sin\frac{n\pi x}{4}\right]\Bigg|_2^4 \\ &= \frac{16}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - \cos n\pi - 2\right) + \frac{64}{n^3\pi^3}\sin\frac{n\pi}{2}. \end{aligned} \tag{4.122}$$

Thus, the Fourier Cosine series is given by

$$f = \frac{7}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\left(\cos\frac{n\pi}{2} - \cos n\pi - 2\right) + \frac{4}{n^3\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{4}. \tag{4.123}$$

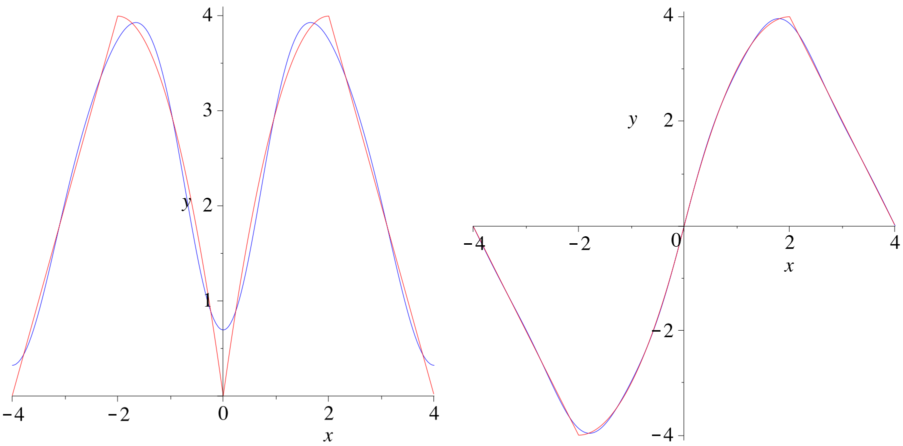

Figure 4.12: Fourier Sine and Cosine series (with 5 terms) (blue) and odd and even extensions for the function (4.118) (red).

As shown in the examples in this chapter, often only a few terms are needed to obtain a fairly good representation of the function. It is interesting to note that if discontinuity is encountered on the extension, Gibb's phenomenon occurs. In the next chapter, we return to solving the heat equation, Laplace's equation, and the wave equation, using separation of variables as introduced at the beginning of this chapter.

4.5 EXERCISES

4.1. Find Fourier series for the following:

(i) $f(x) = \mathrm{e}^{-x}$ on $[-1, 1]$

(ii) $f(x) = |\,x\,|$ on $[-2, 2]$

(iii) $f(x) = \begin{cases} 1 & \text{if } -1 < x < 0, \\ 1 - x & \text{if } \quad 0 < x < 1. \end{cases}$

(iv) $f(x) = (4 - |x|)^2$ on $[-2, 2]$.

4.2. Find Fourier Cosine and Sine series for the following and illustrate the function and its corresponding series on $[-L, L]$.

(i) $f(x) = e^x$ on $[0, 1]$

(ii) $f(x) = x - x^2$ on $[0, 2]$

(iii) $f(x) = \begin{cases} x + 1 & \text{if } 0 < x < 1, \\ 4 - 2x & \text{if } 1 < x < 2. \end{cases}$

(iv) $f(x) = \begin{cases} 2x^2 & \text{if } 0 < x < 1, \\ 3 - x & \text{if } 1 < x < 3. \end{cases}$

4.3. Find the first 10 terms numerically of the Fourier (Sine) series of the following:

(i) $f(x) = \mathrm{e}^{-\mathrm{x}^2}$, on $[-1, 1]$

(ii) $f(x) = \sqrt{x}$, on $[0, 4]$ (Sine)

(iii) $f(x) = -x \ln x$ on $[0, 1]$ (Sine).

CHAPTER 5

Separation of Variables

We are now ready to resume our work on solving the three main equations—the heat equation, Laplace's equation, and the wave equation—using the method of separation of variables.

5.1 THE HEAT EQUATION

Consider the heat equation

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{5.1}$$

subject to the initial and boundary conditions

$$u(0,t) = u(1,t) = 0 \tag{5.2a}$$
$$u(x,0) = x - x^2. \tag{5.2b}$$

Assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{5.3}$$

the heat equation (5.1) becomes

$$XT' = X''T, \tag{5.4}$$

which, after dividing by XT and expanding, gives

$$\frac{T'}{T} = \frac{X''}{X}. \tag{5.5}$$

As T is a function of t only and X a function of x only, this implies that

$$\frac{T'}{T} = \frac{X''}{X} = \lambda, \tag{5.6}$$

where λ is a constant. This gives

$$T' = \lambda T, \quad X'' = \lambda X. \tag{5.7}$$

From (5.2a) and (5.3), the boundary conditions become

$$X(0) = X(1) = 0. \tag{5.8}$$

Integrating the X equation in (5.7) gives rise to three cases, depending on the sign of λ. As seen at the beginning of the last chapter, only the case where $\lambda = -k^2$ for some constant k is applicable. This has the solution

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.9}$$

Imposing the boundary conditions (5.8) shows that (5.9) becomes

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 \sin k + c_2 \cos k = 0, \tag{5.10}$$

which leads to

$$c_2 = 0, \quad c_1 \sin k = 0 \quad \Rightarrow \quad k = 0, \pi, 2\pi, \ldots n\pi, \ldots \tag{5.11}$$

where n is an integer. From (5.7), we further deduce that

$$T(t) = c_3 e^{-n^2\pi^2 t}, \tag{5.12}$$

giving the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x, \tag{5.13}$$

where we have set $c_1 c_3 = b_n$. Using the initial condition (5.2b) gives

$$u(x,0) = x - x^2 = \sum_{n=1}^{\infty} b_n \sin n\pi x. \tag{5.14}$$

From the last chapter, we recognize this as a Fourier Sine series and know that the coefficients b_n are chosen such that

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2) \sin n\pi x \, dx \\ &= 2\left[\frac{1-2x}{n^2\pi^2} \sin n\pi x + \left(\frac{x^2 - x}{n\pi} - \frac{2}{n^3\pi^3}\right) \cos n\pi x\right]\Bigg|_0^1 \\ &= \frac{4}{n^3\pi^3}(1 - (-1)^n). \end{aligned} \tag{5.15}$$

Thus, the solution of the PDE is

$$u(x,t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x. \tag{5.16}$$

Figure 5.1 shows the solution at times $t = 0$ (red), $t = 0.1$ (blue), and $t = 0.2$ (black).

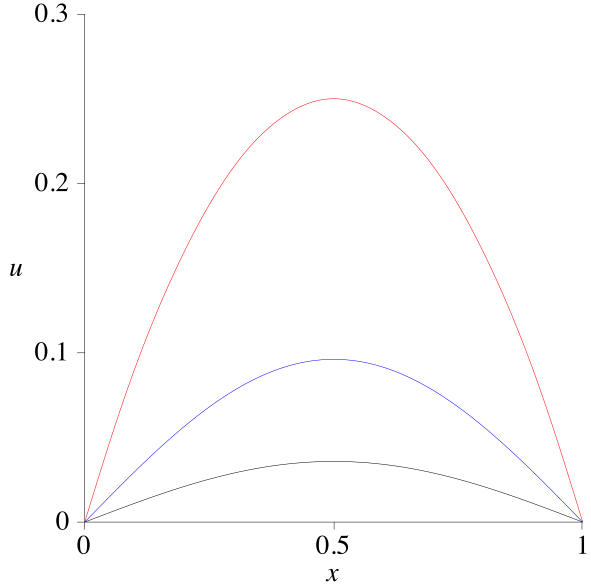

Figure 5.1: The solution of the heat equation with fixed boundary conditions at various times.

Example 5.1
Solve

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{5.17}$$

subject to

$$u_x(0,t) = u_x(1,t) = 0 \tag{5.18a}$$
$$u(x,0) = x - x^2. \tag{5.18b}$$

This problem is similar to the proceeding problem, except the boundary conditions are different. The last problem had the boundaries fixed at zero whereas in this problem, the boundaries are insulated (no flux across the boundary). Again, assuming that the solutions are separable

$$u(x,t) = X(x)T(t), \tag{5.19}$$

we obtain the following from the heat equation:

$$T' = \lambda T, \quad X'' = \lambda X, \tag{5.20}$$

where λ is a constant. The boundary conditions in (5.18a) become

$$X'(0) = X'(1) = 0. \tag{5.21}$$

Integrating the X equation in (5.20) gives rise again to three cases, depending on the sign of λ. As seen in the last chapter, only the case where $\lambda = -k^2$ for some constant k is relevant. Thus, we have

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.22}$$

Imposing the boundary conditions (5.18a) shows that

$$c_1 k \cos 0 - c_2 k \sin 0 = 0, \quad c_1 k \cos k - c_2 k \sin k = 0, \tag{5.23}$$

which leads to

$$c_1 = 0, \quad c_2 \sin k = 0 \quad \Rightarrow \quad k = 0, \pi, 2\pi, \ldots, n\pi, \tag{5.24}$$

where n is an integer. From (5.20), we also deduce that

$$T(t) = c_3 e^{-n^2\pi^2 t}, \tag{5.25}$$

giving the solution

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x, \tag{5.26}$$

where we have set $c_1 c_3 = a_n$. Using the initial condition gives

$$\begin{aligned} u(x,0) = x - x^2 \quad &= \sum_{n=0}^{\infty} a_n \cos n\pi x \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x. \end{aligned} \tag{5.27}$$

From Chapter 4, we recognize this as a Fourier cosine series and that the coefficients a_n are chosen such that

$$a_0 = 2\int_0^1 (x - x^2)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \frac{1}{3}, \tag{5.28}$$

$$\begin{aligned} a_n &= 2\int_0^1 (x - x^2) \cos n\pi x \, dx \\ &= 2\left[\frac{1-2x}{n^2\pi^2}\cos n\pi x + \left(\frac{x-x^2}{n\pi} + \frac{2}{n^3\pi^3}\right)\sin n\pi x\right]\Bigg|_0^1 \\ &= -\frac{2}{n^2\pi^2}(1 + (-1)^n). \end{aligned} \tag{5.29}$$

Thus, the solution of the PDE is

$$u(x,t) = \frac{1}{6} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} - 1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x. \tag{5.30}$$

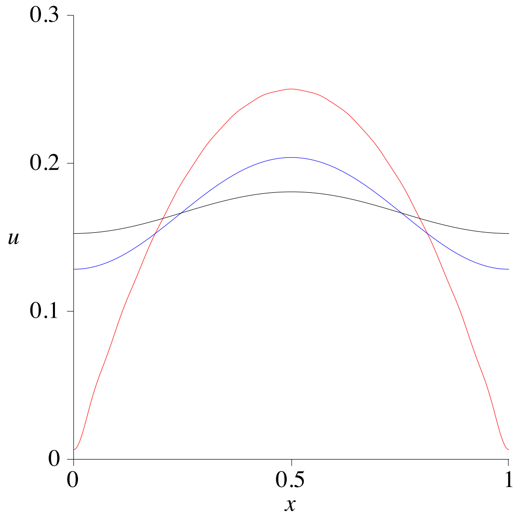

Figure 5.2: The solution of the heat equation with no flux boundary conditions at various times.

Figure 5.2 shows the solution at times $t = 0$ (red), $t = 0.25$ (blue), and $t = 0.5$ (black). It is interesting to note that even though that same initial conditions are used for each of the two problems, fixing the boundaries and insulating them gives rise to two totally different behaviors for $t > 0$.

Example 5.2
Solve

$$u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0 \tag{5.31}$$

subject to

$$u(0,t) = u_x(2,t) = 0 \tag{5.32a}$$

$$u(x,0) = \begin{cases} x & \text{if } 0 < x < 1, \\ 2 - x & \text{if } 1 < x < 2. \end{cases} \tag{5.32b}$$

In this problem, we have a mixture of both fixed and no flux boundary conditions. Again, assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{5.33}$$

gives rise to

$$T' = \lambda T, \quad X'' = \lambda X, \tag{5.34}$$

where λ is a constant. The boundary conditions in (5.32a) accordingly become

$$X(0) = X'(1) = 0. \tag{5.35}$$

Integrating the X equation in (5.34) with $\lambda = -k^2$ for some constant k gives

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.36}$$

Imposing the boundary conditions (5.35) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0, \quad c_1 k \cos 2k - c_2 k \sin 2k = 0, \tag{5.37}$$

which leads to

$$c_2 = 0, \quad \cos 2k = 0 \;\Rightarrow\; k = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots, \frac{(2n-1)\pi}{4}, \tag{5.38}$$

for integer n. From (5.34), we then deduce that

$$T(t) = c_3 e^{-\frac{(2n-1)^2}{16}\pi^2 t}, \tag{5.39}$$

giving the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{(2n-1)}{4}\pi x, \tag{5.40}$$

where we have set $c_1 c_3 = b_n$. Using the initial condition (5.32b) gives

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)}{4}\pi x. \tag{5.41}$$

Recognizing that we have a Fourier sine series, we obtain the coefficients b_n as

$$\begin{aligned} b_n &= 2\int_0^1 x \sin \frac{(2n-1)}{4}\pi x \, dx + \int_1^2 (2-x) \sin \frac{(2n-1)}{4}\pi x \, dx \\ &= \left[\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x - \frac{8x}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x \right]\Bigg|_0^1 \\ &+ \left[-\frac{32}{(2n-1)^2\pi^2} \sin \frac{(2n-1)}{4}\pi x + \frac{8(x-2)}{(2n-1)\pi} \cos \frac{2n-1}{4}\pi x \right]\Bigg|_1^2 \\ &= \frac{32}{(2n-1)^2\pi^2} \left(2 \sin \frac{(2n-1)\pi}{4} + \cos n\pi \right). \end{aligned} \tag{5.42}$$

Hence, the solution of the PDE is

$$u(x,t) = \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{\left(2 \sin \frac{(2n-1)\pi}{4} + \cos n\pi \right)}{(2n-1)^2} e^{-\frac{(2n-1)^2}{16}\pi^2 t} \sin \frac{2n-1}{4}\pi x. \tag{5.43}$$

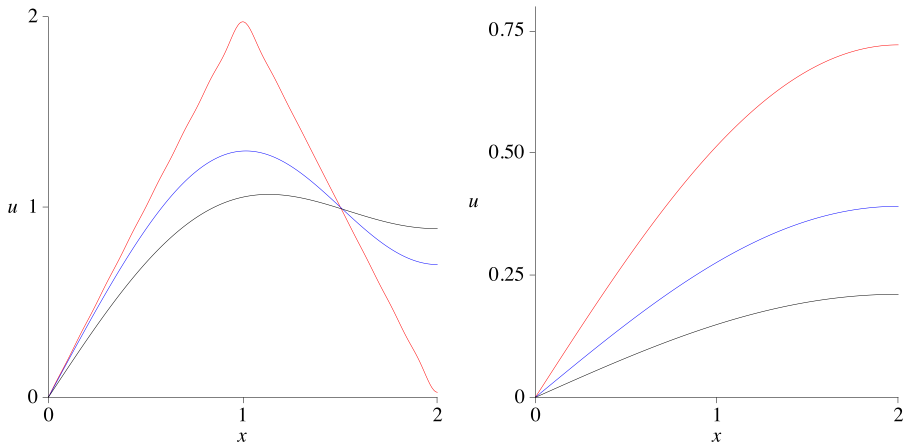

Figure 5.3: Short and later time behavior of the solution (5.43).

Figure 5.3 shows the solution both in short time $t = 0$ (red), $t = 0.1$ (blue), and $t = 0.2$ (black) and later times $t = 1$ (red), $t = 2$ (blue), and $t = 3$ (black).

Example 5.3
Solve

$$u_t = u_{xx}, \quad 0 < x < 2, \quad t > 0 \tag{5.44}$$

subject to

$$u(0,t) = 0, \;\; u_x(2,t) = -u(2,t) \tag{5.45a}$$
$$u(x,0) = 2x - x^2. \tag{5.45b}$$

In this problem, we have a fixed left endpoint and a radiating right endpoint. Assuming separable solutions

$$u(x,t) = X(x)T(t), \tag{5.46}$$

gives rise to

$$T' = \lambda T, \quad X'' = \lambda X, \tag{5.47}$$

where λ is a constant. The boundary conditions in (5.45a) accordingly become

$$X(0) = 0, \quad X'(2) = -X(2). \tag{5.48}$$

Integrating the X equation in (5.47) with $\lambda = -k^2$ for some constant k gives

$$X(x) = c_1 \sin kx + c_2 \cos kx. \tag{5.49}$$

Imposing the first boundary condition of (5.48) shows that

$$c_1 \sin 0 + c_2 \cos 0 = 0 \quad \Rightarrow \quad c_2 = 0 \tag{5.50}$$

and the second boundary condition of (5.48) gives

$$\tan 2k = -k. \tag{5.51}$$

It is important that we recognize that the solutions of (5.51) are not equally spaced as seen in earlier problems. In fact, there are an infinite number of solutions of this equation. Figure 5.4 graphically shows the curves $y = -k$ and $y = \tan 2k$. The first three intersection points are the first three solutions of (5.51).

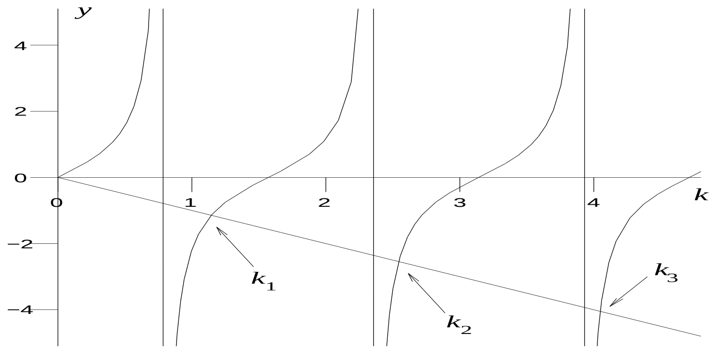

Figure 5.4: The graph of $y = \tan 2k$ and $y = -k$.

Thus, it is necessary that we solve for k numerically. The first 10 solutions are given in Table 5.1.

Table 5.1: The first ten solutions of $\tan 2k = -k$

n	k_n	n	k_n
1	1.144465	6	8.696622
2	2.54349	7	10.258761
3	4.048082	8	11.823162
4	5.586353	9	13.389044
5	7.138177	10	14.955947

Therefore, we have

$$X(x) = c_1 \sin k_n x. \tag{5.52}$$

Further, integrating (5.47) for T gives

$$T(t) = c_3 e^{-k_n^2 t} \tag{5.53}$$

and together with X, we have the solution to the PDE as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k_n^2 t} \sin k_n x. \tag{5.54}$$

Imposing the boundary conditions (5.45a) gives

$$u(0,t) = 2x - x^2 = \sum_{n=1}^{\infty} c_n \sin k_n x. \tag{5.55}$$

It is important to know that the c_n's are **not** given by the formula

$$c_n = \frac{2}{2}\int_0^2 (2x - x^2)\sin k_n x \, dx \tag{5.56}$$

because the k_n's are not equally spaced. So it is necessary to examine (5.55) on its own. Multiplying by $\sin k_m x$ and integrating over [0,2] gives

$$\int_0^2 (2x - x^2)\sin k_m x \, dx = \sum_{n=1}^{\infty} c_n \int_0^2 \sin k_m x \, \sin k_n x \, dx. \tag{5.57}$$

For $n \neq m$, we have

$$\begin{aligned}\int_0^2 \sin k_m x \, \sin k_n x \, dx &= \frac{k_m \sin 2k_n \cos 2k_m - k_n \sin 2k_m \cos 2k_n}{k_n^2 - k_m^2} \\ &= \frac{k_m k_n \cos 2k_m \cos 2k_n}{k_n^2 - k_m^2}\left(\frac{\sin 2k_n}{k_n \cos 2k_n} - \frac{\sin 2k_m}{k_m \cos 2k_m}\right)\end{aligned} \tag{5.58}$$

and imposing (5.51) for each of k_m and k_n shows (5.58) to be identically zero. Therefore, we obtain the following when $n = m$

$$\int_0^2 (2x - x^2)\sin k_n x \, dx = c_n \int_0^2 \sin^2 k_n x \, dx, \tag{5.59}$$

or

$$c_n = \frac{\displaystyle\int_0^2 (2x - x^2)\sin k_n x \, dx}{\displaystyle\int_0^2 \sin^2 k_n x \, dx}. \tag{5.60}$$

Table 5.2 gives the first ten c_n's that correspond to each k_n.

Having obtained k_n and c_n, the solution to the problem is found in (5.54). Figure 5.5 show plots at time $t = 0$ (red), $t = 1$ (blue), and 2 (black) when 20 terms are used.

Table 5.2: The coefficients c_n from (5.60)

n	c_n	n	c_n
1	0.873214	6	0.028777
2	0.341898	7	-.016803
3	-.078839	8	0.015310
4	0.071427	9	-.010202
5	-.032299	10	0.009458

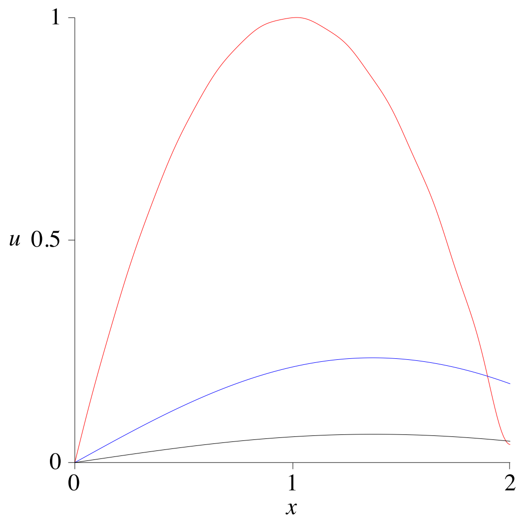

Figure 5.5: The solution (5.54) at various times.

5.1.1 NONHOMOGENEOUS BOUNDARY CONDITIONS

In the preceding examples, the boundary conditions were either fixed to zero, insulted, or radiating. Often, we encounter boundary conditions which are nonstandard or nonhomogeneous. For example, the boundary may be fixed to a particular constant or the flux is maintained at a constant value. The following examples illustrate this.

Example 5.4
Solve

$$u_t = u_{xx}, \quad 0 < x < 3, \quad t > 0 \tag{5.61}$$

subject to

$$u(0,t) = 0, \;\; u(3,t) = 3, \tag{5.62a}$$
$$u(x,0) = 4x - x^2. \tag{5.62b}$$

If we seek separable solutions $u(x,t) = X(x)T(t)$, then from (5.62a) we have

$$X(0)T(t) = 0, \quad X(3)T(t) = 3, \tag{5.63}$$

and we have a problem! The second boundary condition doesn't separate. To overcome this we try and transform this problem to one that we know how to solve. As the right boundary condition is not zero, we try a transformation to fix the boundary to zero. If we try $u = v + 3$, this cures the problem, however, in the process it transforms the left boundary condition to -3. The next simplest transformation is to introduce $u = v + ax + b$ and ask: Can we choose the constants a and b so that both boundary conditions are zero? Upon substitution of both boundary conditions (5.62a), we obtain

$$0 = v(0,t) + a \times 0 + b, \;\; 3 = v(3,t) + 3a + b. \tag{5.64}$$

Now we require that $v(0,t) = 0$ and $v(3,t) = 0$, which implies that we must choose $a = 1$ and $b = 0$. Therefore, we have

$$u = v + x. \tag{5.65}$$

We notice that under the transformation (5.65), the original PDE (5.61) doesn't change form since

$$u_t = u_{xx} \quad \Rightarrow \quad v_t = v_{xx}.$$

However, the initial condition does change. At $t = 0$, then

$$\begin{array}{rrcl} & u(x,0) & = & v(x,0) + x \\ \Rightarrow & 4x - x^2 & = & v(x,0) + x \\ \Rightarrow & v(x,0) & = & 3x - x^2. \end{array} \tag{5.66}$$

Thus, we have the new problem to solve

$$v_t = v_{xx}, \quad 0 < x < 3, \quad t > 0 \tag{5.67}$$

subject to

$$v(x,0) = 3x - x^2, \quad v(0,t) = 0, \;\; v(3,t) = 0. \tag{5.68}$$

As usual, we seek separable solutions $v(x,t) = X(x)T(t)$ which lead to the systems $X'' = -k^2 X$ and $T' = -k^2 T$ with the boundary conditions $X(0) = 0$ and $X(3) = 0$. Solving for X gives

$$X(x) = c_1 \sin kx + c_2 \cos kx, \tag{5.69}$$

and imposing both boundary conditions gives

$$X(x) = c_1 \sin \frac{n\pi}{3} x, \tag{5.70}$$

and

$$T(t) = c_3 e^{-\frac{n^2\pi^2}{9}t},$$

where n is an integer. Therefore, we have the solution of (5.67) subject to (5.68) as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x. \tag{5.71}$$

Recognizing that we have a Fourier sine series, we obtain the coefficients b_n as

$$\begin{aligned} b_n &= \frac{2}{3} \int_0^3 (3x - x^2) \sin \frac{n\pi}{3} x \, dx \\ &= \left[-\frac{6(2x-3)}{n^2\pi^2} \sin \frac{n\pi}{3} x + \frac{2(n^2\pi^2x^2 - 3n^2\pi^2 x - 18)}{n^3\pi^3} \cos \frac{n\pi}{3} x \right]\Bigg|_0^3 \\ &= \frac{36(1-(-1)^n)}{n^3\pi^3}. \end{aligned} \tag{5.72}$$

This gives

$$v(x,t) = \frac{36}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x \tag{5.73}$$

and since $u = v + x$, we obtain the solution for u as

$$u(x,t) = x + \frac{36}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} e^{-\frac{n^2\pi^2}{9}t} \sin \frac{n\pi}{3} x. \tag{5.74}$$

Example 5.5
Solve

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{5.75}$$

subject to

$$u_x(0,t) = -1, \;\; u_x(1,t) = 0, \tag{5.76a}$$
$$u(x,0) = 0. \tag{5.76b}$$

Unfortunately, the trick $u = v + ax + b$ won't work since $u_x = v_x + a$ and choosing a to fix the right boundary condition to zero only makes the left boundary condition nonzero. To overcome this we might try $u = v + ax^2 + bx$, but the original Eq. (5.75) changes

$$u_t = u_{xx} \quad \Rightarrow \quad v_t = v_{xx} + 2a. \tag{5.77}$$

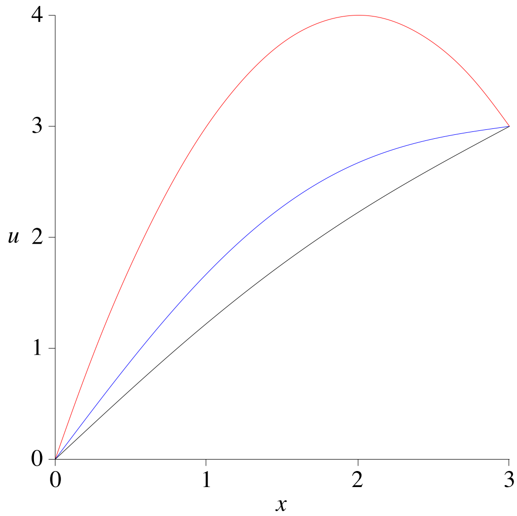

Figure 5.6: The solution (5.74) at time $t = 0$ (red), $t = 1$ (blue), and $t = 2$ (black).

As a second attempt, we try

$$u = v + a(x^2 + 2t) + bx, \tag{5.78}$$

so now

$$u_t = u_{xx} \quad \Rightarrow \quad v_t = v_{xx}. \tag{5.79}$$

Since $u_x = v_x + 2ax + b$, then choosing $a = 1/2$ and $b = -1$ gives the the new boundary conditions as $v_x(0,t) = 0$ and $v_x(1,t) = 0$ and the transformation becomes

$$u = v + \frac{1}{2}(x^2 + 2t) - x. \tag{5.80}$$

Finally, we consider the initial condition (5.76). From (5.80), we have

$$v(x,0) = u(x,0) - \frac{1}{2}x^2 + x = x - \frac{x^2}{2}$$

and our problem is transformed to the new problem

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{5.81}$$

subject to

$$v_x(0,t) = v_x(1,t) = 0, \tag{5.82a}$$

$$v(x,0) = x - \frac{1}{2}x^2. \tag{5.82b}$$

A separation of variables $v = XT$ leads to $X'' = -k^2 X$ and $T' = -k^2 T$ from which we obtain

$$X = c_1 \sin kx + c_2 \cos kx, \quad X' = c_1 k \cos kx - c_2 k \sin kx, \tag{5.83}$$

and imposing the boundary conditions (5.82a) gives

$$c_1 k \cos 0 - c_2 k \sin 0 = 0, \quad c_1 k \cos k - c_2 k \sin k = 0, \tag{5.84}$$

from which we obtain

$$c_1 = 0, \quad k = n\pi, \tag{5.85}$$

where n is an integer. This then leads to

$$X(x) = c_2 \cos n\pi x \tag{5.86}$$

and further

$$T(t) = c_3 e^{-n^2\pi^2 t}. \tag{5.87}$$

Finally, we arrive at

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x, \tag{5.88}$$

noting that we have chosen $a_n = c_1 c_3$. Upon substitution of $t = 0$ and using the initial condition (5.82b), we have

$$x - \frac{1}{2}x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \tag{5.89}$$

a Fourier cosine series. The coefficients are obtained by

$$a_0 = \frac{2}{1}\int_0^1 \left(x - \frac{1}{2}x^2\right) dx = x^2 - \frac{1}{3}x^3 \Big|_0^1 = \frac{2}{3} \tag{5.90}$$

$$\begin{aligned} a_n &= \frac{2}{1}\int_0^1 (x - \frac{1}{2}x^2) \cos n\pi x \, dx \\ &= \left[\frac{2(1-x)}{n^2\pi^2} \cos n\pi x + \left(\frac{(2x - x^2)}{n\pi} + \frac{2}{n^3\pi^3} \right) \sin n\pi x \right] \Bigg|_0^1 \\ &= -\frac{2}{n^2\pi^2}. \end{aligned} \tag{5.91}$$

Thus, we obtain the solution for v as

$$v(x,t) = \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2 t} \cos n\pi x \tag{5.92}$$

and this, together with the transformation (5.80), gives

$$u(x,t) = \frac{1}{2}(x^2 + 2t) - x + \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2 t}\cos n\pi x. \tag{5.93}$$

Figure 5.7 shows plots at time $t = 0.5$ (red), $t = 1.0$ (blue), and $t = 1.5$ (black). It is interesting to note that at the left boundary $u_x = -1$ and since the flux $\phi = -ku_x$ implies that $\phi = k > 0$, that means heat is being added at the left boundary. Hence, the profile increases at the left while the right boundary is still insulated (i.e., no flux).

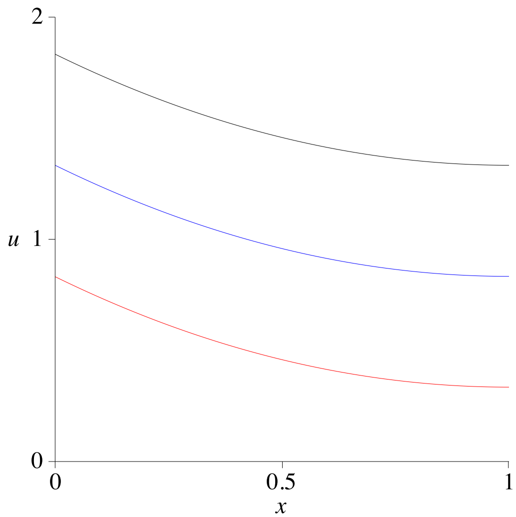

Figure 5.7: The solution (5.93) at time $t = 0.5$ (red), $t = 1$ (blue), and $t = 1.5$ (black).

5.1.2 NONHOMOGENEOUS EQUATIONS

We now focus our attention to solving the heat equation with a source term

$$u_t = u_{xx} + Q(x), \quad 0 < x < L, \quad t > 0, \tag{5.94}$$

subject to

$$u(0,t) = 0, \quad u(L,t) = 0, \tag{5.95a}$$
$$u(x,0) = f(x). \tag{5.95b}$$

To investigate this problem, we will consider a particular example where $L = 2$, $f(x) = 2x - x^2$, and $Q(x) = 1 - |x - 1|$. If we were to consider this problem without a source term and use the separation of variables technique, we would obtain the solution

$$u(x,t) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^3}e^{-\frac{n^2\pi^2}{4}t}\sin\frac{n\pi x}{2}. \tag{5.96}$$

Suppose that we looked for solutions of (5.94) with $Q = 0$ in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{2}, \tag{5.97}$$

noting that both boundary conditions in (5.95a) are satisfied. Substituting into the heat equation, and isolating coefficients of $\sin\frac{n\pi x}{2}$, would lead to

$$T_n'(t) = -\frac{n^2\pi^2}{4}T_n(t). \tag{5.98}$$

Solving this would give

$$T_n(t) = c_n e^{-\frac{n^2\pi^2}{4}t} \tag{5.99}$$

for some constant c_n and (5.97), (5.99), and the initial condition (5.95b), would lead to (5.96). With this idea, we try and solve the heat equation with a source term (5.94) by looking for solutions of the form (5.97). However, in order for this technique to work, it is also necessary to expand the source term in terms of a Fourier Sine series

$$Q(x) = \sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{2}. \tag{5.100}$$

For

$$Q(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2-x & \text{if } 1 < x < 2, \end{cases} \tag{5.101}$$

where

$$\begin{aligned} q_n &= \int_0^2 Q(x)\sin\frac{n\pi x}{2}dx \\ &= \int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}dx \\ &= \left[\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} - \frac{2x}{n\pi}\cos\frac{n\pi x}{2}\right]\Bigg|_0^1 \\ &+ \left[-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} + \frac{2x-4}{n\pi}\cos\frac{n\pi x}{2}\right]\Bigg|_1^2 \\ &= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned} \tag{5.102}$$

Substituting both (5.97) and (5.100) into (5.94) gives

$$\sum_{n=1}^{\infty} T_n'(t)\sin\frac{n\pi x}{2} = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{2}\right)^2 T_n(t)\sin\frac{n\pi x}{2} + \sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{2} \tag{5.103}$$

and re-grouping and isolating the coefficients of $\sin\frac{n\pi x}{2}$ gives

$$T_n'(t) + \frac{n^2\pi^2}{4}T_n(t) = q_n, \tag{5.104}$$

a linear ODE in $T_n(t)$! On solving (5.104) we obtain

$$T_n(t) = \frac{4}{n^2\pi^2}q_n + b_n e^{-(\frac{n\pi}{2})^2 t}, \tag{5.105}$$

where b_n is a constant of integration, giving the final solution

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}q_n + b_n e^{-(\frac{n\pi}{2})^2 t}\right)\sin\frac{n\pi x}{2}. \tag{5.106}$$

Imposing the initial condition (5.95b) with $f(x) = 2x - x^2$ gives

$$2x - x^2 = \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}q_n + b_n\right)\sin\frac{n\pi x}{2}. \tag{5.107}$$

If we set

$$c_n = \frac{4}{n^2\pi^2}q_n + b_n, \tag{5.108}$$

then we have

$$2x - x^2 = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2}, \tag{5.109}$$

a regular Fourier Sine series. Therefore,

$$\begin{aligned} c_n &= \int_0^2 (2x - x^2)\sin\frac{n\pi x}{2}dx \\ &= \frac{16}{n^3\pi^3}(1 - \cos n\pi), \end{aligned} \tag{5.110}$$

which, in turn, gives

$$b_n = c_n - \frac{4}{n^2\pi^2}q_n \tag{5.111}$$

and finally, the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}q_n + \left(c_n - \frac{4}{n^2\pi^2}q_n\right)e^{-(\frac{n\pi}{2})^2 t}\right)\sin\frac{n\pi x}{2}, \tag{5.112}$$

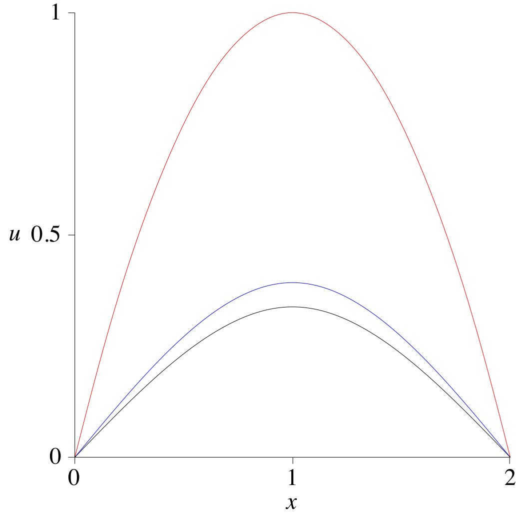

Figure 5.8: The solution (5.112) at time $t = 0$ (red), $t = 1$ (blue), and $t = 2$ (black).

where q_n and c_n are given in (5.102) and (5.110), respectively. Plots are given in Fig. 5.8 at times $t = 0$ (red), $t = 1$ (blue), and $t = 2$ (black).

It is interesting to note that if we let $t \to \infty$, the solution approaches a single curve. This is what is called steady state (no changes in time). It is natural to ask: Can we find this steady state solution? The answer is yes. For the steady state, $u_t \to 0$ as $t \to \infty$ and the original PDE becomes

$$u_{xx} + Q(x) = 0. \tag{5.113}$$

Integrating twice with $Q(x)$ given in (5.101) gives

$$u = \begin{cases} -\frac{x^3}{6} + c_1 x + c_2 & \text{if } 0 < x < 1, \\ \frac{x^3}{6} - x^2 + k_1 x + k_2 & \text{if } 1 < x < 2, \end{cases} \tag{5.114}$$

where c_1, c_2, k_1, and k_2 are constants of integration. Imposing that the solution and its first derivative are continuous at $x = 1$ and that the solution is zero at the endpoints gives

$$c_1 = \frac{1}{2}, \quad c_2 = 0, \quad k_1 = \frac{3}{2}, \quad k_2 = -\frac{1}{3}. \tag{5.115}$$

This, in turn, gives the steady state solution

$$u = \begin{cases} -\frac{x^3}{6} + \frac{x}{2} & \text{if } 0 < x < 1, \\ \frac{x^3}{6} - x^2 + \frac{3x}{2} - \frac{1}{3} & \text{if } 1 < x < 2. \end{cases} \tag{5.116}$$

5.1.3 EQUATIONS WITH A SOLUTION-DEPENDENT SOURCE TERM

We now consider the heat equation with a solution dependent source term. For simplicity, we will consider a source term that is linear. Take, for example,

$$u_t = u_{xx} + \alpha u, \quad 0 < x < 1, \ \ t > 0, \tag{5.117}$$

subject to

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = x - x^2, \tag{5.118}$$

where α is some constant. We could try a separation of variables to obtain solutions for this problem, but we will try a different technique. We will try and transform the PDE to one that has no source term. In attempting to do so, we seek a transformation of the form

$$u(x,t) = A(x,t)v(x,t) \tag{5.119}$$

and ask: Is it possible to find A such that the source term in (5.117) can be eliminated? Substituting (5.119) in (5.117) gives

$$Av_t + A_t v = Av_{xx} + 2A_x v_x + A_{xx} v + \alpha A v. \tag{5.120}$$

Dividing by A and expanding and regrouping gives

$$v_t = v_{xx} + 2\frac{A_x}{A} v_x + \left(\frac{A_{xx}}{A} - \frac{A_t}{A} + \alpha \right) v. \tag{5.121}$$

In order to target the standard heat equation $v_t = v_{xx}$, we choose

$$A_x = 0, \quad \frac{A_{xx}}{A} - \frac{A_t}{A} + \alpha = 0. \tag{5.122}$$

From the first equation in (5.122), we see that $A = A(t)$, and from the second we obtain $A' = \alpha A$, which has the solution $A(t) = A_0 e^{\alpha t}$, for some constant A_0 leading to $u = A_0 e^{\alpha t} v$. The boundary conditions become

$$\begin{aligned} u(0,t) = 0 \quad &\Rightarrow \quad A_0 e^{\alpha t} v(0,t) = 0 \quad \Rightarrow \quad v(0,t) = 0, \\ u(1,t) = 0 \quad &\Rightarrow \quad A_0 e^{\alpha t} v(1,t) = 0 \quad \Rightarrow \quad v(1,t) = 0, \end{aligned} \tag{5.123}$$

so the boundary conditions are unchanged. Next, we consider the initial condition, so

$$u(x,0) = x - x^2 \quad \Rightarrow \quad A_0 e^{\alpha 0} v(x,0) = x - x^2 \quad \Rightarrow \quad A_0 v(x,0) = x - x^2. \tag{5.124}$$

To leave the initial condition unchanged we choose $A_0 = 1$. Thus, under the transformation

$$u = e^{\alpha t} v \tag{5.125}$$

the problem (5.117) and (5.118) becomes

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{5.126}$$

$$v(x,0) = 0, \quad v(1,t) = 0, \quad v(x,0) = x - x^2. \tag{5.127}$$

This particular problem was considered at the beginning of this chapter in Example 5.1, where the solution was given in (5.16) by

$$v(x,t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x, \tag{5.128}$$

and so, from (5.125), we obtain the solution of (5.117) subject to (5.118) as

$$u(x,t) = \frac{4}{\pi^3} e^{\alpha t} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} e^{-n^2\pi^2 t} \sin n\pi x. \tag{5.129}$$

Figure 5.9 shows plots at times $t = 0$ (red), $t = 0.1$ (blue), and $t = 0.2$ (black) when $\alpha = 5$ and 12. It is interesting to note that in the case where $\alpha = 5$, the diffusion is slower in comparison with no source term (for $\alpha = 0$, see Fig. 5.1), whereas there is no diffusion at all when $\alpha = 12$.

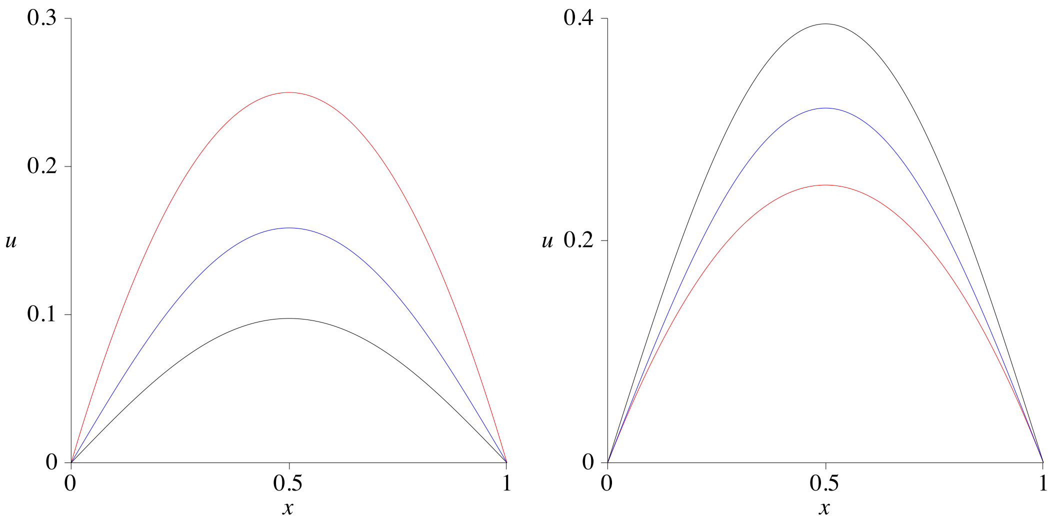

Figure 5.9: The solution of the heat equation with a source (5.117) with $\alpha = 5$ and 12 at various times.

We may ask: For what value of α do we achieve a steady state solution? To answer this, consider the first few terms of the solution (5.129)

$$u = \frac{8}{\pi^3} e^{\alpha t} \left(e^{-\pi^2 t} \sin \pi x + \frac{1}{27} e^{-9\pi^2 t} \sin 3\pi x + \dots \right). \tag{5.130}$$

Clearly, the exponential terms in the brackets in (5.130) will decay to zero, with the first term decaying the slowest. Therefore, it is the balance between $e^{\alpha t}$ and $e^{-\pi^2 t}$ which determines whether the solution will decay to zero or not. It is equality $\alpha = \pi^2$ that leads to the steady state solution

$$u_\infty = \frac{8 \sin \pi x}{\pi^3}. \tag{5.131}$$

Example 5.6
Solve

$$u_t = u_{xx} + \alpha u, \quad 0 < x < 1, \quad t > 0 \tag{5.132}$$

subject to

$$u_x(0,t) = 0, \;\; u_x(2,t) = 0, \tag{5.133a}$$
$$u(x,0) = 4x - x^3. \tag{5.133b}$$

It was already established that the transformation (5.125) will transform the heat equation with a solution dependent source term to the plain heat equation and will leave the initial condition and fixed boundary conditions unchanged. It is now necessary to determine what happens to no flux boundary conditions (5.133a). Using (5.125) we have

$$\begin{aligned} u_x(0,t) = 0 \;&\Rightarrow\; A_0 e^{\alpha t} v_x(0,t) = 0 \;\Rightarrow\; v_x(0,t) = 0, \\ u_x(2,t) = 0 \;&\Rightarrow\; A_0 e^{\alpha t} v_x(2,t) = 0 \;\Rightarrow\; v_x(2,t) = 0, \end{aligned} \tag{5.134}$$

and so the boundary conditions remain the same! Thus, problem (5.132) reduces to

$$v_t = v_{xx}, \quad 0 < x < 2, \;\; t > 0 \tag{5.135}$$

$$v(x,0) = 2x - x^2, \quad v_x(0,t) = 0, \quad v_x(2,t) = 0. \tag{5.136}$$

Using a separation of variables and imposing the boundary conditions gives (see Example 5.2)

$$v = \frac{2}{3} - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(1 + (-1)^n)}{n^2} e^{-\frac{n^2\pi^2}{4} t} \cos \frac{n\pi}{2} x, \tag{5.137}$$

and together with the transformation (5.125), the solution of (5.132) and (5.133) is

$$u = e^{\alpha t} \left(\frac{2}{3} - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(1 + (-1)^n)}{n^2} e^{-\frac{n^2\pi^2}{4} t} \cos \frac{n\pi}{2} x, \right). \tag{5.138}$$

Figure 5.10 shows plots at $t = 0$ (red), $t = 0.2$ (blue), and $t = 0.4$ (black) when $\alpha = -2$ and 2. It is interesting to note that the sign of α will determine whether the solution will grow or decay exponentially.

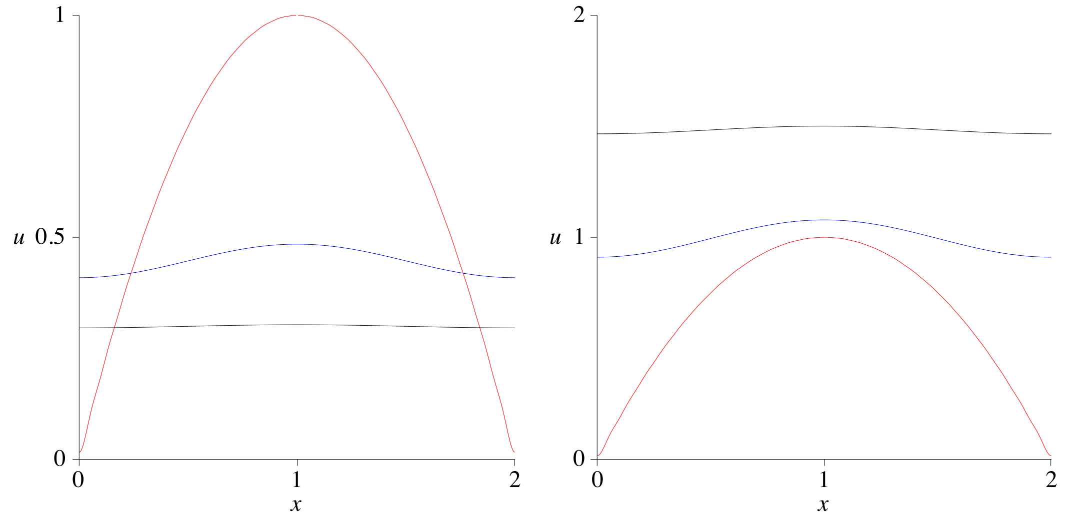

Figure 5.10: The solution of the heat equation with a source (5.132) with no flux boundary condition with $\alpha = -2, 2$ at various times.

5.1.4 EQUATIONS WITH A SOLUTION-DEPENDENT CONVECTIVE TERM

We now consider the heat equation with a solution-dependent linear convection term

$$u_t = u_{xx} + \beta u_x, \quad 0 < x < 1, \quad t > 0, \tag{5.139}$$

subject to

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{5.140a}$$
$$u(x,0) = x - x^2, \tag{5.140b}$$

where β is some constant. We consider the same initial and boundary conditions as in the previous section, as it provides a means of comparing the two respective problems. Again, we could try a separation of variables to obtain solutions for this problem, but again we try and transform the PDE to one that has no convection term using a transformation of the form

$$u(x,t) = A(x,t)v. \tag{5.141}$$

Substituting (5.141) in (5.139) gives

$$Av_t + A_t v = Av_{xx} + 2A_x v_x + A_{xx} v + \beta \left(Av_x + A_x v \right). \tag{5.142}$$

Dividing by A and expanding and regrouping gives

$$v_t = v_{xx} + \frac{2A_x + \beta A}{A} v_x + \frac{A_{xx} - A_t + \beta A_x}{A} v. \tag{5.143}$$

In order to obtain the standard heat equation, we choose

$$2A_x + \beta A = 0, \quad A_{xx} - A_t + \beta A_x = 0. \tag{5.144}$$

From the first of (5.144) we obtain that $A(x,t) = C(t)e^{-\frac{1}{2}\beta x}$, where $C(t)$ is an arbitrary function of integration; from the second of (5.144), we obtain $C' + \frac{\beta^2}{4}C = 0$, which has the solution $C(t) = C_0 e^{-\frac{1}{4}\beta^2 t}$ for some constant C_0. This then gives

$$A(x,t) = C_0 e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} \tag{5.145}$$

which in turn gives

$$u(x,t) = C_0 e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v. \tag{5.146}$$

The boundary conditions from (5.140a) becomes

$$\begin{aligned} u(0,t) = 0 \quad &\Rightarrow \quad C_0 e^{-\frac{1}{4}\beta^2 t} v(0,t) = 0 \quad \Rightarrow v(0,t) = 0, \\ u(1,t) = 0 \quad &\Rightarrow \quad C_0 e^{-\frac{1}{2}-\frac{1}{4}\beta^2 t} v(1,t) = 0 \quad \Rightarrow v(1,t) = 0, \end{aligned} \tag{5.147}$$

so the boundary conditions are unchanged. Next, we consider the initial condition (5.140b), so

$$u(x,0) = x - x^2 \quad \Rightarrow \quad v(x,0) = (x - x^2)e^{\frac{1}{2}\beta x}, \tag{5.148}$$

where we have chosen $C_0 = 1$. Here, the initial condition actually changes. Thus, under the transformation

$$u = e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} v, \tag{5.149}$$

the problem (5.139) and (5.140) become

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{5.150}$$

and

$$v(x,0) = 0, \quad v(1,t) = 0, \tag{5.151a}$$

$$v(x,0) = (x - x^2)e^{\frac{1}{2}\beta x}. \tag{5.151b}$$

Following the solution procedure outlined in the previous section, the solution of (5.150) and (5.151) is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x, \tag{5.152}$$

where b_n is now given by

$$b_n = \frac{2}{1}\int_0^1 (x - x^2)e^{\frac{1}{2}\beta x} \sin n\pi x \, dx, \tag{5.153}$$

and from (5.149)

$$u(x,t) = e^{-\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t} \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x. \tag{5.154}$$

At this point, we will consider two particular examples: $\beta = 6$ and $\beta = -12$.
For $\beta = 6$,

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2)e^{3x} \sin n\pi x \, dx \\ &= -4n\pi \frac{27 + n^2\pi^2 + 2n^2\pi^2 e^3 \cos n\pi}{(9 + n^2\pi^2)^3}. \end{aligned} \tag{5.155}$$

For $\beta = -12$,

$$\begin{aligned} b_n &= 2\int_0^1 (x - x^2)e^{-6x} \sin n\pi x \, dx \\ &= 2n\pi \frac{7n^2\pi^2 + 108 + (5n^2\pi^2 + 324)e^{-6} \cos n\pi}{(36 + n^2\pi^2)^3}. \end{aligned} \tag{5.156}$$

The respective solutions for each are

$$\begin{aligned} u(x,t) &= -4\pi\, e^{-3x-9t} \\ &\times \sum_{n=1}^{\infty} n \frac{27 + n^2\pi^2 + 2n^2\pi^2 e^3 \cos n\pi}{(9 + n^2\pi^2)^3} e^{-n^2\pi^2 t} \sin n\pi x, \end{aligned} \tag{5.157}$$

$$\begin{aligned} u(x,t) &= 2\pi\, e^{6x+36t} \\ &\times \sum_{n=1}^{\infty} n \frac{7n^2\pi^2 + 108 + (5n^2\pi^2 + 324)e^{-6} \cos n\pi}{(36 + n^2\pi^2)^3} e^{-n^2\pi^2 t} \sin n\pi x. \end{aligned} \tag{5.158}$$

Figure 5.11 shows graphs at a variety of times for $\beta = 6$ and $\beta = -12$.

5.2 LAPLACE'S EQUATION

The 2D Laplace's equation is

$$u_{xx} + u_{yy} = 0. \tag{5.159}$$

To this we attach the boundary conditions

$$u(x,0) = 0, \quad u(x,1) = x - x^2 \tag{5.160a}$$

$$u(0,y) = 0, \quad u(1,0) = 0. \tag{5.160b}$$

We will show that separation of variables also works for this equation. If we assume solutions of the form

$$u(x,y) = X(x)Y(y), \tag{5.161}$$

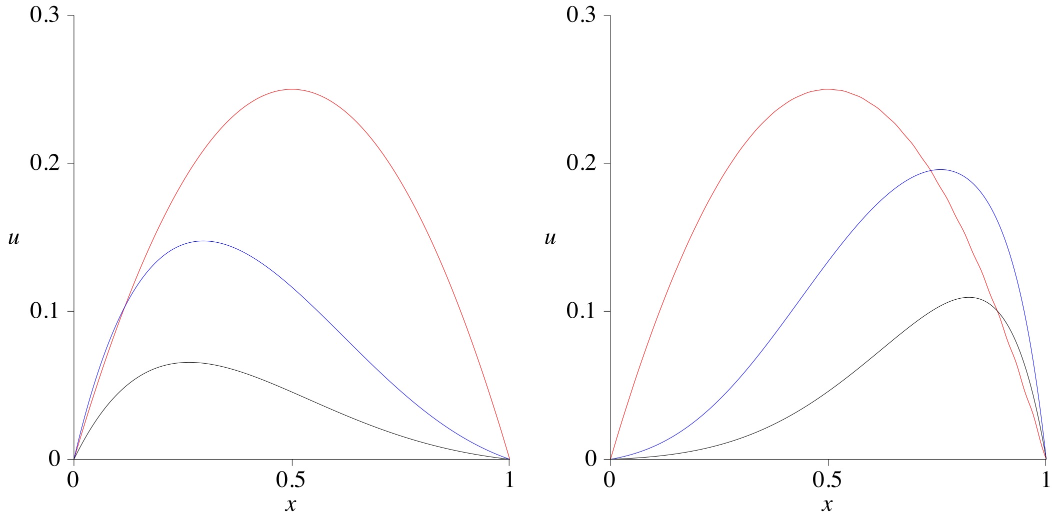

Figure 5.11: The solution of the heat equation with convection with fixed boundary conditions with $\beta = 6, -12$.

then substituting this into (5.159) gives

$$X''Y + XY'' = 0. \tag{5.162}$$

Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \tag{5.163}$$

and since each term in (5.163) is only a function of x or y, then each must be constant giving

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda. \tag{5.164}$$

From the first of (5.160a) and both of (5.160b) we deduce the boundary conditions

$$Y(0) = 0, \quad X(0) = 0, \quad X(1) = 0. \tag{5.165}$$

The remaining boundary condition in (5.160a) will be used later. As we saw in a previous section, in order to solve the X equation in (5.164) subject to the boundary conditions in (5.165), it is necessary to set $\lambda = -k^2$. The X equation (5.164) has the general solution

$$X = c_1 \sin kx + c_2 \cos kx. \tag{5.166}$$

To satisfy the boundary conditions in (5.165) it is necessary to have $c_2 = 0$ and $k = n\pi, \quad k \in \mathbb{Z}^+$ so

$$X(x) = c_1 \sin n\pi x. \tag{5.167}$$

From (5.164), we obtain the solution to the Y equation

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y. \tag{5.168}$$

Since $Y(0) = 0$, this implies $c_4 = 0$ so

$$X(x)Y(y) = c_n \sin n\pi x \sinh n\pi y, \tag{5.169}$$

where we have chosen $c_n = c_1 c_3$. Therefore, we obtain the solution to (5.159) subject to three of the four boundary conditions in (5.160)

$$u = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh n\pi y. \tag{5.170}$$

The remaining boundary condition in (5.160a) now needs to be satisfied, thus

$$u(x,1) = x - x^2 = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh n\pi. \tag{5.171}$$

This looks like a Fourier Sine series and if we let $b_n = c_n \sinh n\pi$, this becomes

$$\sum_{n=1}^{\infty} b_n \sin n\pi x = x - x^2, \tag{5.172}$$

which is precisely a Fourier sine series. The coefficients b_n are given by

$$\begin{aligned} b_n &= \frac{2}{1} \int_0^1 \left(x - x^2\right) \sin n\pi x \, dx \\ &= \frac{4}{n^3 \pi^3} (1 - \cos n\pi), \end{aligned} \tag{5.173}$$

and since $b_n = c_n \sinh n\pi$, this gives

$$c_n = \frac{4(1 - (-1)^n)}{n^3 \pi^3 \sinh n\pi}. \tag{5.174}$$

Thus, the solution to Laplace's equation (5.159) with the boundary conditions given in (5.160) is

$$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}. \tag{5.175}$$

Figure 5.12 shows both a top view and a 3D view of the solution.

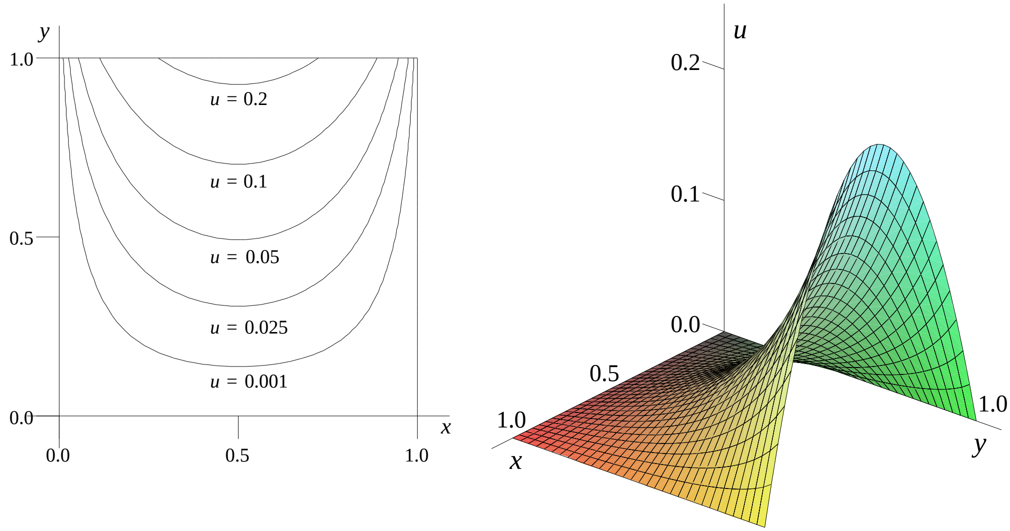

Figure 5.12: The solution of (5.159) with the boundary conditions (5.160).

Example 5.7
Solve

$$u_{xx} + u_{yy} = 0, \tag{5.176}$$

subject to

$$u(x,0) = 0, \quad u(x,1) = 0, \tag{5.177a}$$
$$u(0,y) = 0, \quad u(1,y) = y - y^2. \tag{5.177b}$$

Assume separable solutions of the form

$$u(x,y) = X(x)Y(y). \tag{5.178}$$

Then substituting this into (5.176) gives

$$X''Y + XY'' = 0. \tag{5.179}$$

Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \tag{5.180}$$

from which we obtain

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda. \tag{5.181}$$

From (5.177) we deduce the boundary conditions

$$X(0) = 0, \quad Y(0) = 0, \quad Y(1) = 0. \tag{5.182}$$

The remaining boundary condition in (5.177b) will be used later. As seen in the previous problem, in order to solve the Y equation in (5.181) subject to the boundary conditions in (5.182), it is necessary to set $\lambda = k^2$. The Y Eq. (5.181) has the general solution

$$Y = c_1 \sin ky + c_2 \cos ky \tag{5.183}$$

To satisfy the boundary conditions in (5.182), it is necessary to have $c_2 = 0$ and $k = n\pi$ so

$$Y(y) = c_1 \sin n\pi y. \tag{5.184}$$

From (5.181), we obtain the solution to the X equation

$$X(x) = c_3 \sinh n\pi x + c_4 \cosh n\pi x. \tag{5.185}$$

Since $X(0) = 0$, this implies $c_4 = 0$. This gives

$$X(x)Y(y) = c_n \sinh n\pi x \sin n\pi y, \tag{5.186}$$

where we have chosen $c_n = c_1 c_3$. Therefore, we obtain

$$u = \sum_{n=1}^{\infty} c_n \sinh n\pi x \sin n\pi y. \tag{5.187}$$

The remaining boundary condition in (5.177b) now needs to be satisfied, thus

$$u(1, y) = y - y^2 = \sum_{n=1}^{\infty} c_n \sinh n\pi \sin n\pi y. \tag{5.188}$$

If we let $b_n = c_n \sinh n\pi$, this becomes

$$\sum_{n=1}^{\infty} b_n \sin n\pi y = y - y^2. \tag{5.189}$$

Comparing with the previous problem, we find that interchanging x and y interchanges the two problems, and so we conclude that

$$b_n = \frac{16}{n^3\pi^3}(1 - \cos n\pi). \tag{5.190}$$

Therefore, the solution to Laplace's equation (5.176) subject to (5.177) is

$$u = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \frac{\sinh n\pi x}{\sinh n\pi} \sin n\pi y. \tag{5.191}$$

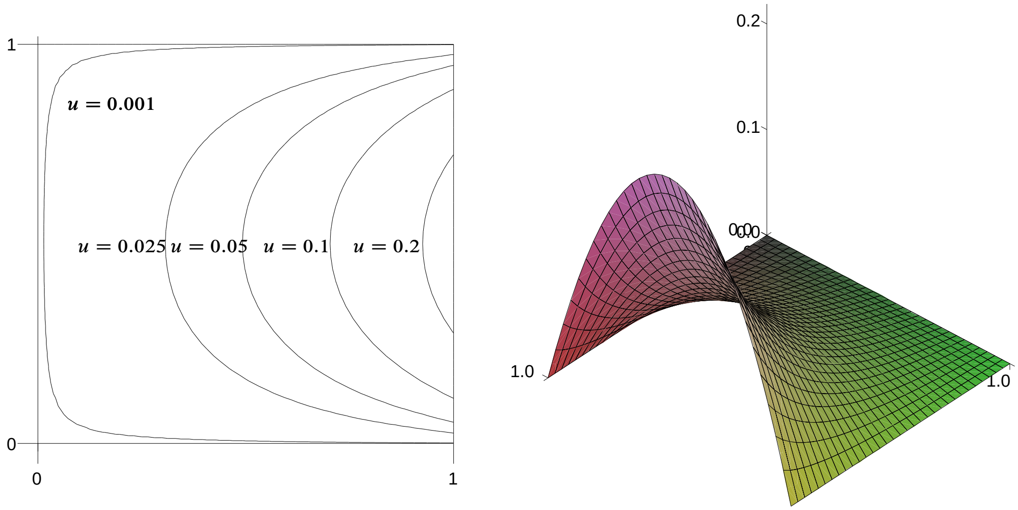

Figure 5.13: The solution of (5.176) with the boundary conditions (5.177).

Figure 5.13 shows both a top view and a 3D view of the solution.

In comparing the solutions (5.175) and (5.191) we find that if we interchange x and y, they are the same. This should not be surprising because if we consider Laplace's equation with the boundary conditions given in (5.160) and (5.177), that if we interchange x and y, the problems are transformed to each other.

Example 5.8
Solve

$$u_{xx} + u_{yy} = 0 \tag{5.192}$$

subject to

$$u(x, 0) = x - x^2, \quad u(x, 1) = 0 \tag{5.193a}$$
$$u(0, y) = 0, \quad u(1, y) = 0. \tag{5.193b}$$

Again, assuming separable solutions of the form

$$u(x, y) = X(x)Y(y) \tag{5.194}$$

leads to

$$X''Y + XY'' = 0, \tag{5.195}$$

when substituted into (5.192). Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0. \tag{5.196}$$

Since each term is only a function of x or y then each must be constant giving

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda. \tag{5.197}$$

From the second of (5.193a) and both of (5.193b), we deduce the boundary conditions

$$X(0) = 0, \quad X(1) = 0, \quad Y(1) = 0, \tag{5.198}$$

noting that the last boundary condition is different than the boundary condition considered at the beginning of this section (i.e., Example 5.7 where $Y(0) = 0$). The remaining boundary condition in (5.193a) will be used later. In order to solve the X equation in (5.197) subject to the boundary conditions (5.198), it is necessary to set $\lambda = -k^2$. The X Eq. (5.197) has the general solution

$$X = c_1 \sin kx + c_2 \cos kx. \tag{5.199}$$

To satisfy the boundary conditions in (5.198), it is necessary to have $c_2 = 0$ and $k = n\pi$ (n is still a positive integer), so

$$X(x) = c_1 \sin n\pi x. \tag{5.200}$$

From (5.197), we obtain the solution to the Y equation

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y. \tag{5.201}$$

Since $Y(1) = 0$, this implies

$$c_3 \sinh n\pi + c_4 \cosh n\pi = 0 \quad \Rightarrow \quad c_4 = -c_3 \frac{\sinh n\pi}{\cosh n\pi}. \tag{5.202}$$

From (5.201) we have

$$\begin{aligned} Y(y) &= c_3 \sinh n\pi y - c_3 \cosh n\pi y \frac{\sinh n\pi}{\cosh n\pi} \\ &= -c_3 \frac{\sinh n\pi(1-y)}{\cosh n\pi}, \end{aligned} \tag{5.203}$$

so

$$X(x)Y(y) = c_n \sin n\pi x \sinh n\pi(1-y), \tag{5.204}$$

where we have chosen $c_n = -c_1 c_3 / \cosh n\pi$. Therefore, we obtain

$$u = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh n\pi(1-y). \tag{5.205}$$

The remaining boundary condition is (5.193a) now needs to be satisfied, thus

$$u(x,0) = x - x^2 = \sum_{n=1}^{\infty} c_n \sin n\pi x \sinh n\pi. \tag{5.206}$$

At this point, we recognize that this problem is now identical to the first problem in this section where we obtained

$$c_n = \frac{4(1-(-1)^n)}{n^3\pi^3 \sinh n\pi}, \tag{5.207}$$

so the solution to Laplace's equation with the boundary conditions given in (5.193) is

$$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi(1-y)}{\sinh n\pi}. \tag{5.208}$$

Figure 5.14 shows the solution.

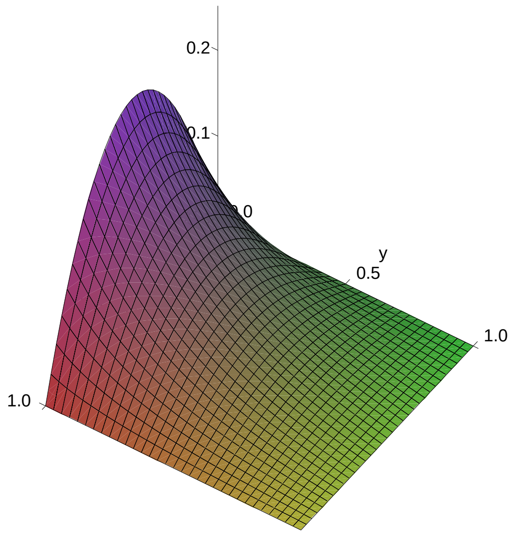

Figure 5.14: The solution of Laplace's equation with the boundary conditions (5.193).

Example 5.9
As the final example, we consider

$$u_{xx} + u_{yy} = 0 \tag{5.209}$$

subject to

$$u(x,0) = 0, \quad u(x,1) = 0 \tag{5.210a}$$
$$u(0,y) = y - y^2, \quad u(1,y) = 0. \tag{5.210b}$$

We could go through a separation of variables to obtain the solution but we can avoid many of the steps by considering the previous three problems and using symmetry arguments. Since interchanging the variables x and y transforms (5.175) to (5.191), then interchanging x and y in (5.208) will give the solution to this problem, namely

$$u(x,y) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\frac{\sinh n\pi(1-x)}{\sinh n\pi}\sin n\pi y. \tag{5.211}$$

Figure 5.15 shows the solution.

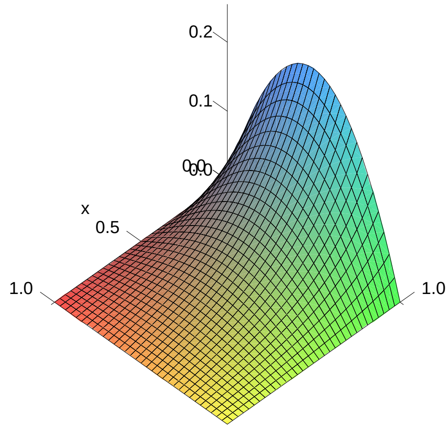

Figure 5.15: The solution of Laplace's equation with the boundary conditions (5.210).

5.2.1 LAPLACE'S EQUATION ON AN ARBITRARY RECTANGULAR DOMAIN

In this section, we solve Laplace's equation on an arbitrary domain $[0, L_x] \times [0, L_y]$ when all the boundaries are zero except one. Thus, we will solve each of the 4 different problems depending on whether the top, bottom, right or left boundary is nonzero. We consider each separately.

Top Boundary

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.212}$$

subject to

$$u(x,0) = 0, \quad u(x, L_y) = f(x), \tag{5.213a}$$
$$u(0, y) = 0, \quad u(L_x, y) = 0, \tag{5.213b}$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L_x} \frac{\sinh \dfrac{n\pi y}{L_x}}{\sinh \dfrac{n\pi L_y}{L_x}}, \tag{5.214}$$

where

$$b_n = \frac{2}{L_x} \int_0^{L_x} f(x) \sin \frac{n\pi x}{L_x}\, dx. \tag{5.215}$$

Bottom Boundary

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.216}$$

subject to

$$u(x,0) = f(x), \quad u(x, L_y) = 0 \tag{5.217a}$$
$$u(0, y) = 0, \quad u(L_x, y) = 0, \tag{5.217b}$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L_x} \frac{\sinh \dfrac{n\pi (L_y - y)}{L_x}}{\sinh \dfrac{n\pi L_y}{L_x}} \tag{5.218}$$

where

$$b_n = \frac{2}{L_x} \int_0^{L_x} f(x) \sin \frac{n\pi x}{L_x}\, dx. \tag{5.219}$$

Right Boundary

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.220}$$

subject to

$$u(x,0) = 0, \quad u(x, L_y) = 0 \tag{5.221a}$$
$$u(0, y) = 0, \quad u(L_x, y) = g(y), \tag{5.221b}$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{L_y} \frac{\sinh \dfrac{n\pi x}{L_y}}{\sinh \dfrac{n\pi L_x}{L_y}} \tag{5.222}$$

where

$$b_n = \frac{2}{L_y} \int_0^{L_y} g(y) \sin \frac{n\pi y}{L_y}\, dy. \tag{5.223}$$

Left Boundary

In general, using separation of variables, the solution of

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \tag{5.224}$$

subject to

$$u(x,0) = 0, \ \ u(x, L_y) = 0 \tag{5.225a}$$
$$u(0,y) = g(y), \ \ u(L_x, y) = 0, \tag{5.225b}$$

is

$$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{L_y} \frac{\sinh \dfrac{n\pi (L_x - x)}{L_y}}{\sinh \dfrac{n\pi L_x}{L_y}}, \tag{5.226}$$

where

$$b_n = \frac{2}{L_y} \int_0^{L_y} g(y) \sin \frac{n\pi y}{L_y}\, dy. \tag{5.227}$$

Often, we are asked to solve Laplace's equation subject to four different nonzero boundary condition. For example,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.228}$$

subject to

$$\begin{aligned} u(x,2) &= x - x^2, \qquad u(x,0) = \begin{cases} 4x^2, & 0 < x \le \frac{1}{2} \\ 4(1-x)^2 & \frac{1}{2} < x < 1 \end{cases} \\ u(0,y) &= 2y - y^2, \qquad u(1,y) = \begin{cases} y & 0 < y \le 1 \\ 2 - y & 1 < y < 2. \end{cases} \end{aligned} \tag{5.229}$$

We consider four subproblems. In each subproblem we take zero on three of the four boundaries and have the remaining boundary one of the four in (5.229). If we denote the solution of each subproblem as u_1, u_2, u_3, and u_4, then the solution of the entire problem is given by

$$u = u_1 + u_2 + u_3 + u_4. \tag{5.230}$$

Subproblem 1—Top Boundary

$$u_{1xx} + u_{1yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.231}$$

subject to

$$\begin{aligned} &u_1(x,0) = 0, \ \ u_1(x,2) = x - x^2, \\ &u_1(0,y) = 0, \ \ u_1(1,y) = 0. \end{aligned} \tag{5.232}$$

From (5.215) we have

$$b_n = \frac{4(1-(-1)^n)}{n^3\pi^3}, \tag{5.233}$$

and from (5.214) we have

$$u_1 = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} \sin n\pi x \, \frac{\sinh n\pi y}{\sinh 2n\pi}. \tag{5.234}$$

Subproblem 2—Bottom Boundary

$$u_{2xx} + u_{2yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.235}$$

subject to

$$\begin{aligned} &u_2(x,0) = \begin{cases} 4x^2, & 0 < x \le \frac{1}{2} \\ 4(1-x)^2 & \frac{1}{2} < x < 1 \end{cases}, \ \ u_2(x,2) = 0, \\ &u_2(0,y) = 0, \ \ u_2(1,y) = 0. \end{aligned} \tag{5.236}$$

From (5.219) we obtain

$$b_n = \frac{16\left(\cos n\pi - 1 + n\pi \sin \frac{n\pi}{2}\right)}{n^3\pi^3} \tag{5.237}$$

and from (5.218) we obtain

$$u_2 = \sum_{n=1}^{\infty} \frac{16\left(\cos n\pi - 1 + n\pi \sin \frac{n\pi}{2}\right)}{n^3\pi^3} \, \frac{\sinh n\pi(2-y)}{\sinh 2n\pi}. \tag{5.238}$$

Subproblem 3—Right Boundary

$$u_{3xx} + u_{3yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.239}$$

subject to

$$\begin{aligned} &u_3(x,0) = 0, \ \ u_3(x,2) = 0, \\ &u_3(0,y) = 0, \ \ u_3(1,y) = \begin{cases} y & 0 < y \le 1 \\ 2-y & 1 < y < 2. \end{cases} \end{aligned} \tag{5.240}$$

From (5.223) we obtain

$$b_n = \frac{8\sin\frac{n\pi}{2}}{n^2\pi^2} \tag{5.241}$$

and from (5.222) we obtain

$$u_3 = \sum_{n=1}^{\infty} \frac{8\sin\frac{n\pi}{2}}{n^2\pi^2} \frac{\sinh n\pi x}{\sinh n\pi} \sin\frac{n\pi y}{2}. \tag{5.242}$$

Subproblem 4—Left Boundary

$$u_{4xx} + u_{4yy} = 0, \quad 0 < x < 1, \quad 0 < y < 2 \tag{5.243}$$

subject to

$$\begin{aligned} &u_4(x,0) = 0, \;\; u_4(x,2) = 0, \\ &u_4(0,y) = 2y - y^2, \;\; u_4(1,y) = 0. \end{aligned} \tag{5.244}$$

From (5.227) we obtain

$$b_n = \frac{16(1-(-1)^n)}{n^3\pi^3} \tag{5.245}$$

and from (5.226) we obtain

$$u_4 = \sum_{n=1}^{\infty} \frac{16(1-(-1)^n)}{n^3\pi^3} \frac{\sinh\frac{n\pi(1-x)}{2}}{\sinh\frac{n\pi}{2}} \sin\frac{n\pi y}{2}. \tag{5.246}$$

Adding (5.234), (5.238), (5.242), and (5.246) gives the solution of our problem. Figure 5.16 shows the solution.

5.3 THE WAVE EQUATION

We end the chapter with the wave equation. We saw in the previous chapter the D'Alembert solution and in this section we obtain separable solutions and show the D'Alembert solution emerging. Here, we will consider a particular example

$$u_{tt} - u_{xx} = 0, \tag{5.247}$$

with the boundary conditions

$$u(0,t) = 0, \;\; u(3,t) = 0, \tag{5.248a}$$

$$u(x,0) = \begin{cases} 0 & \text{if } 0 < x < 1, \\ -(x-1)(x-2) & \text{if } 1 \le x \le 2, \\ 0 & \text{if } 2 < x < 3, \end{cases} \tag{5.248b}$$

$$u_t(x,0) = 0. \tag{5.248c}$$

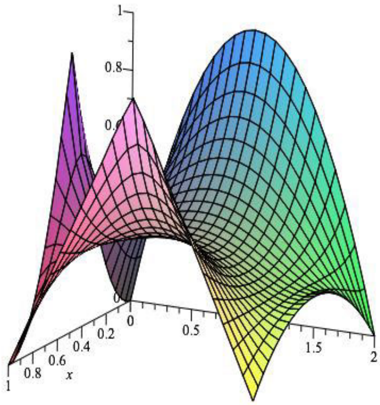

Figure 5.16: The solution of Laplace's equation with the boundary conditions (5.229).

Assuming solutions of the form

$$u(x,t) = X(x)T(t), \tag{5.249}$$

then substituting this into (5.247) gives

$$T''X - TX'' = 0. \tag{5.250}$$

Dividing by TX and expanding gives

$$\frac{T''}{T} - \frac{X''}{X} = 0, \tag{5.251}$$

and since each term is only a function of x or t, then each must be constant giving

$$\frac{T''}{T} = \lambda, \qquad \frac{X''}{X} = \lambda. \tag{5.252}$$

From (5.248a) and (5.249),

$$X(0) = 0, \quad X(3) = 0. \tag{5.253}$$

As we have seen previously, of the three possible cases for λ ($\lambda < 0$, $\lambda = 0$ or $\lambda > 0$) the only case that can satisfy the boundary conditions (5.253) is $\lambda < 0$. Therefore, the X solution of (5.252) subject to (5.253) is

$$X = c_1 \sin \frac{n\pi x}{3} \tag{5.254}$$

noting that $\lambda = -\frac{n^2\pi^2}{9}$. From (5.252) we solve for T giving

$$T = c_2\, sin\frac{n\pi t}{3} + c_3\, cos\frac{n\pi t}{3}. \tag{5.255}$$

Thus, the solution of (5.247) is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \sin\frac{n\pi t}{3} + b_n \cos\frac{n\pi t}{3}\right)\sin\frac{n\pi x}{3}, \tag{5.256}$$

where we have chosen $a_n = c_1c_2$ and $b_n = c_1c_3$. In order to find a_n and b_n, it is necessary to use the initial conditions (5.248b) and (5.248c). Substituting these into (5.256) gives

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3} = \begin{cases} 0 & \text{if } 0 < x < 1, \\ -(x-1)(x-2) & \text{if } 1 \le x \le 2, \\ 0 & \text{if } 2 < x < 3, \end{cases} \tag{5.257a}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} a_n \frac{n\pi}{3} \sin\frac{n\pi x}{3} = 0. \tag{5.257b}$$

These we recognize as Fourier Sine series from which we obtain $a_n = 0$ and

$$b_n = \frac{36}{n^3\pi^3}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right) - \frac{6}{n^2\pi^2}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right) \tag{5.258}$$

and thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi t}{3}\sin\frac{n\pi x}{3}. \tag{5.259}$$

Different time snap shots are shown in Fig. 5.17.

It is interesting to note that a closer examination of (5.259) shows that if we use the identity

$$2\sin A\cos B = \sin A + B + \sin A - B, \tag{5.260}$$

then we can re-write (5.259)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{b_n}{2}\left(\sin\frac{n\pi(x+t)}{3} + \sin\frac{n\pi(x-t)}{3}\right) \tag{5.261}$$

which can be seen as the addition of two waves—one moving left and one moving right. If we recall, the D'Alembert solution for the wave equation with

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \tag{5.262}$$

is

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) \tag{5.263}$$

which we can see emerging from (5.261).

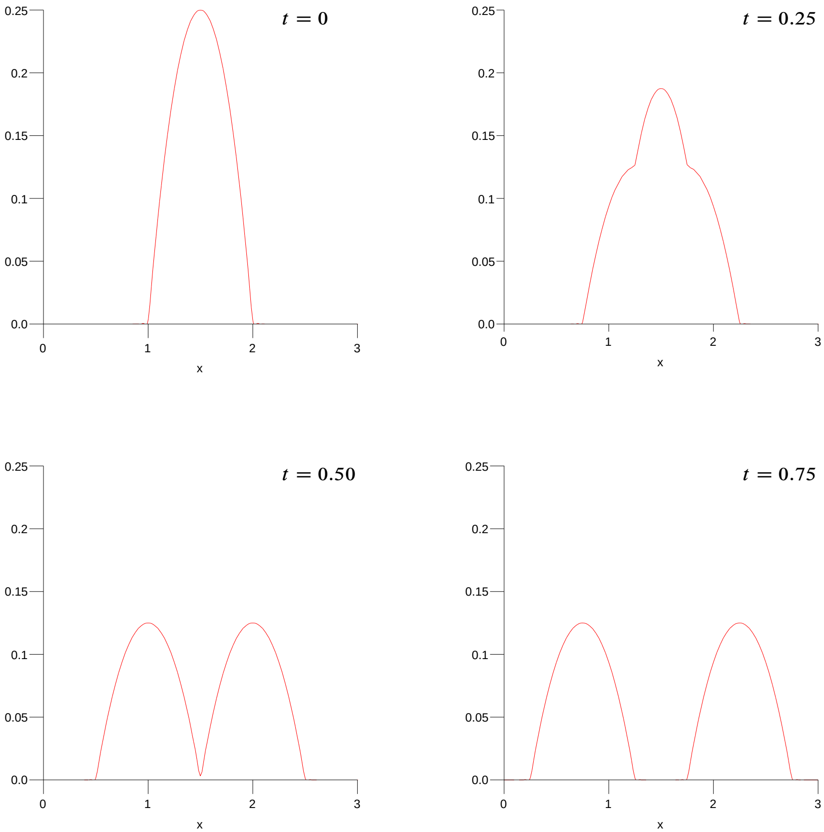

Figure 5.17: The solution of (5.259) at times $t = 0$, $t = 0.25$, $t = .5$, and $t = .75$.

5.4 EXERCISES

5.1. Solve the heat equation

$$u_t = u_{xx}, \qquad 0 < x < 2,$$

subject to the initial condition

$$u(x,0) = 2x - x^2,$$

and subject to the following boundary conditions:

$$\begin{array}{rll} (i) & u(0,t)=0, & u(2,t)=0, \\ (ii) & u_x(0,t)=0, & u_x(2,t)=0, \\ (iii) & u(0,t)=0, & u_x(2,t)=0, \\ (iv) & u_x(0,t)=0, & u(2,t)=0. \end{array}$$

5.2. Solve the heat equation

$$u_t = u_{xx}, \qquad 0 < x < 3,$$

subject to the following initial and boundary conditions:

$$\begin{array}{rlll} (i) & u(x,0)=x^2-5x+7, & u(0,t)=7, & u(3,t)=1, \\ (ii) & u(x,0)=2x^2-4x, & u_x(0,t)=-1, & u_x(3,t)=5. \end{array}$$

Give a sketch of the solutions.

5.3. Solve Laplace's equation

$$u_{xx}+u_{yy}=0, \qquad 0<x<1, \quad 0<y<1,$$

subject to the boundary conditions

$$\begin{array}{rllll} (i) & u(x,0)=0 & u(0,y)=0 & u(x,1)=x & u(1,y)=0 \\ (ii) & u(x,0)=0 & u(0,y)=0 & u(x,1)=0 & u(1,y)=1. \end{array}$$

Give a sketch of the solution.

5.4. Determine conditions on $P(x,t)$ and $Q(x,t)$ such that the PDE

$$u_t = u_{xx} + P(x,t)u_x + Q(x,t)u$$

can be transformed to standard form ($v_t = v_{xx}$) by a transformation of the form

$$u = F(x,t)v.$$

Use this to solve

$$u_t = u_{xx} + \frac{4x}{x^2+1}u_x + \frac{2u}{x^2+1} \quad 0<x<1, \quad t>0,$$

subject to

$$u(x,0) = \frac{x-x^2}{x^2+1}, \quad u(0,t)=0, \quad u(1,t)=0.$$

5.5. Consider

$$u_t = u_{xx}, \quad 0 < x < L, \quad t > 0,$$

subject to

$$u(x,0) = f(x), \quad u(0,t) = p(t), \quad u(L,t) = q(t).$$

Show a transformation of the form

$$u = v + A(t)x + B(t)$$

transforms the problem to

$$v_t = v_{xx} - q'(t)\frac{x}{L} - p'(t)\frac{(L-x)}{L},$$

subject to

$$v(x,0) = f(x) - q(0)\frac{x}{L} - p(0)\frac{(L-x)}{L}, \quad v(0,t) = 0, \quad v(1,t) = 0.$$

5.6. Show

$$u_t = u_{xx} + \beta u_x, \quad 0 < x < 2, \quad t > 0$$

(β constant) subject to

$$u(x,0) = 4x - x^3, \quad u_x(0,t) = 0, \quad u_x(2,t) = 0,$$

becomes, under the transformation $u = e^{at+bx}v$ for some suitable a and b

$$v_t = v_{xx}, \quad 0 < x < 2, \quad t > 0,$$

subject to

$$v_x(0,t) - \frac{\beta}{2}v(0,t) = 0, \quad v_x(2,t) - \frac{\beta}{2}v(2,t) = 0, \quad v(x,0) = (4x - x^3)e^{\frac{1}{2}\beta x}.$$

CHAPTER 6

Fourier Transform

Separation of variables is a powerful technique for solving some partial differential equations on finite domains. When the domain become infinite (or semi infinite) this technique doesn't work and we need a new technique—the Fourier transform. The Fourier transform can be seen as a generalization of the Fourier series when the length of the interval becomes infinite.

6.1 FOURIER TRANSFORM

We define the Fourier transform of a function $f(x)$ as

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = F(\omega), \tag{6.1}$$

and the Inverse Fourier transform of $F(w)$ as

$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega = f(x). \tag{6.2}$$

Example 6.1
Find the Fourier transform of

$$f(x) = \begin{cases} 1 & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Solution. From (6.1) we have

$$\begin{aligned} \mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} dx \\ &= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i\omega x}}{-i\omega} \right|_{-1}^{1} \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \qquad (\text{using } e^{i\omega} = \cos\omega + i \sin\omega) \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega} \end{aligned} \tag{6.4}$$

and so $F(w) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin \omega}{\omega}$. This also give the following inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{\frac{\sin \omega}{\omega}\right\} = \sqrt{\frac{\pi}{2}} f(x), \tag{6.5}$$

where $f(x)$ is given in (6.3).

Example 6.2
Find the Fourier transform of

$$f(x) = e^{-a|x|}, \quad a > 0. \tag{6.6}$$

Solution. From (6.1) we have

$$\begin{aligned}
\mathcal{F}\left\{e^{-a|x|}\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{(a-i\omega)x} dx + \int_{0}^{\infty} e^{-(a+i\omega)x} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\left. \frac{e^{(a-i\omega)x}}{a-i\omega} \right|_{-\infty}^{0} - \left. \frac{e^{-(a+i\omega)x}}{a+i\omega} \right|_{0}^{\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-i\omega} + \frac{1}{a+i\omega} \right) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\omega^2}
\end{aligned} \tag{6.7}$$

and so $F(w) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2a}{a^2+\omega^2}$. This also give the following inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{\frac{1}{\omega^2+a^2}\right\} = \frac{\sqrt{2\pi}}{2a} e^{-a|x|}. \tag{6.8}$$

Example 6.3
Find the Fourier transform of

$$f(x) = e^{-ax^2}, \quad a > 0. \tag{6.9}$$

Solution. From (6.1) we have

$$\begin{aligned}
\mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{-i\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}}e^{-\omega^2/4a}\int_{-\infty}^{\infty} e^{-a(x+i\omega/2a)^2}dx \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2/4a}\int_{-\infty}^{\infty} e^{-ay^2}dy \quad (\text{where } y = x + i\omega/2a) \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2/4a}\,\sqrt{\frac{\pi}{a}}
\end{aligned} \tag{6.10}$$

and so $F(w) = \dfrac{1}{\sqrt{2a}} \cdot e^{-\omega^2/4a}$. As

$$\mathcal{F}\left\{e^{-ax^2}\right\} = \frac{1}{\sqrt{2a}} \cdot e^{-\omega^2/4a}$$

then

$$\mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2a}} \cdot e^{-\omega^2/4a}\right\} = e^{-ax^2}$$

or on letting $a = 1/4a$ gives

$$\mathcal{F}^{-1}\left\{e^{-b\omega^2}\right\} = \frac{1}{\sqrt{2b}}\, e^{-x^2/4b}. \tag{6.11}$$

OPERATION PROPERTIES OF THE FOURIER TRANSFORM

We denote the Fourier Transform of $f(x)$ and $g(x)$ as $F(\omega)$ and $G(\omega)$, respectively, then we have the following properties:

$$(i) \qquad \mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(\omega) + \beta G(\omega), \tag{6.12a}$$

$$(ii) \qquad \mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega), \tag{6.12b}$$

$$(iii) \qquad \mathcal{F}\left\{e^{iax}f(x)\right\} = F(\omega - a), \tag{6.12c}$$

$$(iv) \qquad \mathcal{F}\left\{f^{(n)}(x)\right\} = (i\omega)^n F(\omega), \tag{6.12d}$$

$$(v) \qquad \mathcal{F}\{(-ix)^n f(x)\} = \frac{d^n F(\omega)}{d\omega^n}, \tag{6.12e}$$

$$(iv) \qquad \mathcal{F}^{-1}\{F(w)G(w)\} = \frac{1}{\sqrt{2\pi}}\, f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)ds. \tag{6.12f}$$

Example 6.4
Find the Fourier transform of

$$f(x) = xe^{-x^2}. \tag{6.13}$$

Solution. Since we know from Example 6.3

$$\mathcal{F}\left\{e^{-x^2}\right\} = \frac{1}{\sqrt{2}} \cdot e^{-\omega^2/4} \tag{6.14}$$

then from property (v)

$$\mathcal{F}\left\{xe^{-x^2}\right\} = i\frac{d}{d\omega}\left(\frac{1}{\sqrt{2}} \cdot e^{-\omega^2/4}\right) = -\frac{i\omega}{2\sqrt{2}} e^{-\omega^2/4}. \tag{6.15}$$

Example 6.5
Find the Inverse Fourier transform of

$$F(\omega) = e^{-\omega^2} \cos\omega. \tag{6.16}$$

Solution. Here, we will use property (ii) in the operational properties. Since

$$\mathcal{F}\{f(x-a)\} = e^{-i\omega a} F(\omega),$$

then

$$\mathcal{F}^{-1}\left\{e^{-i\omega a} F(\omega)\right\} = f(x-a). \tag{6.17}$$

If we re-write $\cos\omega = \frac{1}{2}\left(e^{i\omega} + e^{-i\omega}\right)$ then

$$\begin{aligned}
\mathcal{F}^{-1}\{F(\omega)\} &= \mathcal{F}^{-1}\left\{e^{-\omega^2}\cos\omega\right\} \\
&= \mathcal{F}^{-1}\left\{e^{-\omega^2}\frac{1}{2}\left(e^{i\omega} + e^{-i\omega}\right)\right\} \\
&= \frac{1}{2}\mathcal{F}^{-1}\left\{e^{i\omega}e^{-\omega^2}\right\} + \frac{1}{2}\mathcal{F}^{-1}\left\{e^{-i\omega}e^{-\omega^2}\right\} \\
&= \frac{1}{2\sqrt{2}}\left(e^{(x+1)^2/4} + e^{(x-1)^2/4}\right)
\end{aligned} \tag{6.18}$$

by virtue of the results in Example 6.3.

At this point we are ready to show the real power of the Fourier transform where we will use them to solve PDEs. The following examples illustrate.

Example 6.6
Solve

$$u_t = u_{xx}, \quad -\infty < x < \infty \tag{6.19}$$

subject to

$$u(x,0) = f(x), \quad u, u_x \to 0 \text{ as } |x| \to \infty. \tag{6.20}$$

Here we will use the Fourier transform and denote the transform of $u(x,t)$ by $U(\omega,t)$, i.e.,

$$U(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx. \tag{6.21}$$

Taking the Fourier transform of (6.19) gives

$$\mathcal{F}\{u_t\} = \mathcal{F}\{u_{xx}\} \tag{6.22}$$

which gives

$$U_t = (i\omega)^2 U \tag{6.23}$$

whereas the transform of the initial condition (6.20) gives

$$U(\omega,0) = F(\omega), \tag{6.24}$$

where $F(\omega)$ is the transform of $f(x)$. Solving (6.23) gives

$$U(\omega,t) = U_0(\omega)e^{-\omega^2 t}, \tag{6.25}$$

where $U_0(\omega)$ is a function of integration and imposing the initial condition (6.24) gives $U_0(\omega) = F(\omega)$ giving (6.25) as

$$U(\omega,t) = F(\omega)e^{-\omega^2 t}. \tag{6.26}$$

Now we take the inverse Fourier transform of (6.26). Thus,

$$u(x,t) = \mathcal{F}^{-1}\left(F(\omega)e^{-\omega^2 t}\right). \tag{6.27}$$

If we denote $G(\omega) = e^{-\omega^2 t}$, we see that we can use the convolution theorem and since

$$\mathcal{F}^{-1}\left(e^{-\omega^2 t}\right) = \frac{1}{\sqrt{2t}}\,e^{-x^2/4t} \tag{6.28}$$

(from Example 6.3 with $b = t$) then (6.27) becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(s)e^{-\frac{(x-s)^2}{4t}}\,ds. \tag{6.29}$$

As a particular example, let us consider the initial condition

$$f(x) = \begin{cases} 1, & \text{for } -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{6.30}$$

Thus, (6.29) we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{4t}}\, ds \tag{6.31}$$

which, under the change of variables $s = x - 2\sqrt{t}u$, becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} e^{-\sigma^2}\, d\sigma \tag{6.32}$$

which we can write in terms the the usual error function

$$u(x,t) = \frac{1}{2}\operatorname{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{x-1}{2\sqrt{t}}\right). \tag{6.33}$$

Figure 6.1 shows the solution at times $t = 0$ (red), $t = 1$ (blue), and $t = 2$ (black).

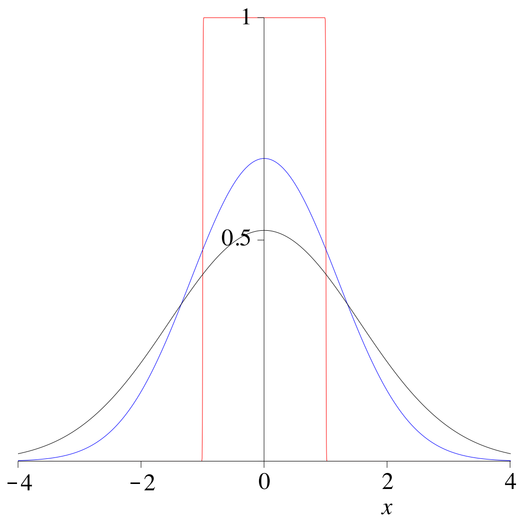

Figure 6.1: The solutions (6.32) at $t = 0$ (red), $t = 1$ (blue), and $t = 2$ (black).

Example 6.7
Solve

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0 \tag{6.34}$$

subject to

$$u(x,0) = f(x) \quad u, u_x \to 0 \text{ as } |x| \to \infty. \tag{6.35}$$

Here we will use the Fourier transform. A natural question is: In which variable x or y should we use for our transform? As the interval in x is $(-\infty, \infty)$ and y is $(0, \infty)$ we choose x and denote the transform of $u(x, y)$ by $U(\omega, y)$ i.e.,

$$U(\omega, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-i\omega x} dx. \tag{6.36}$$

Taking the transform of (6.34) gives

$$(i\omega)^2 U + U_{yy} = 0 \tag{6.37}$$

whereas the transform of the initial condition (6.35) gives

$$U(\omega, 0) = F(\omega), \tag{6.38}$$

where $F(\omega)$ is the transform of $f(x)$. Solving (6.37) gives

$$U(\omega, y) = A(\omega)e^{-\omega y} + B(\omega)e^{\omega y}. \tag{6.39}$$

Since $u \to 0$ as $y \to \infty$, then $U \to 0$ as $y \to \infty$. As ω is a real number, if $\omega > 0$ then we would require that $B(\omega) = 0$ in (6.39), whereas if $\omega < 0$ then we would require that $A(\omega) = 0$ in (6.39). To ensure that $U \to 0$ as $y \to \infty$ regardless of ω, we take the solution in (6.39)

$$U(\omega, y) = C(\omega)e^{-|\omega| y} \tag{6.40}$$

and from the IC (6.38), $C(\omega) = F(\omega)$. Thus, (6.40) becomes

$$U(\omega, y) = F(\omega)e^{-|\omega| y}. \tag{6.41}$$

Now we take the inverse Fourier transform of (6.41). Thus,

$$u(x, t) = \mathcal{F}^{-1}\left(F(\omega)e^{-|\omega| y}\right). \tag{6.42}$$

If we denote $G(\omega) = e^{-|\omega| y}$, we see that we can use the convolution theorem and since

$$\mathcal{F}^{-1}\left(e^{-|\omega| y}\right) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2y}{x^2 + y^2} \tag{6.43}$$

then the solution (6.42) becomes

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(s) y}{(x - s)^2 + y^2} ds. \tag{6.44}$$

As an example, we again take $f(x)$ as given in (6.30). Thus, (6.44) can be solved explicitly giving

$$u(x,y) = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{1+x}{y}\right)\right) \tag{6.45}$$

or after some manipulation

$$u(x,y) = \frac{1}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right). \tag{6.46}$$

Figure 6.2 shows different level sets - curves in the xy plane where u is some fixed number.

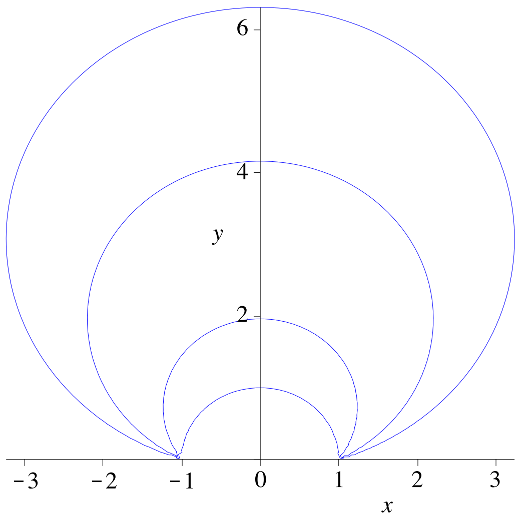

Figure 6.2: Different level sets of (6.46).

6.2 FOURIER SINE AND COSINE TRANSFORMS

Here we define the Fourier Sine and Cosine transform as

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)dx = F_s(\omega) \tag{6.47a}$$

$$\mathcal{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)dx = F_c(\omega) \tag{6.47b}$$

and the Inverse Fourier Sine and Cosine transform as

$$\mathcal{F}_s^{-1}\{F_s(\omega)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\omega)\sin(\omega x)d\omega = f(x) \tag{6.48a}$$

$$\mathcal{F}_c^{-1}\{F_c(\omega)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\omega)\cos(\omega x)d\omega = f(x). \tag{6.48b}$$

Example 6.8
Find the Fourier Sine and Cosine transform of

$$f(x) = e^{-ax}, \quad a > 0. \tag{6.49}$$

We first compute the Fourier Sine transform. From (6.47a)

$$\begin{aligned} \mathcal{F}_s\{f(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(\omega x) dx \\ &= \sqrt{\frac{2}{\pi}} \left\{ -\frac{e^{-ax}}{a} \sin(\omega x) \Big|_0^\infty + \frac{\omega}{a} \int_0^\infty e^{-ax} \cos(\omega x) dx \right\} \\ &= \sqrt{\frac{2}{\pi}} \frac{\omega}{a} \left\{ -\frac{e^{-ax}}{a} \cos(\omega x) \Big|_0^\infty - \frac{\omega}{a} \int_0^\infty e^{-ax} \sin(\omega x) dx \right\} \\ &= \sqrt{\frac{2}{\pi}} \frac{\omega}{a^2} - \frac{\omega^2}{a^2} \mathcal{F}_s\{f(x)\} \end{aligned} \tag{6.50}$$

which leads to $F_s(w) = \sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{a^2+\omega^2}$. By a similar calculation using (6.47b) we find

$$F_c(w) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+\omega^2}. \tag{6.51}$$

As

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{a^2+\omega^2}, \quad \mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+\omega^2} \tag{6.52}$$

then we have the following inverse transforms:

$$\mathcal{F}_s^{-1}\left\{\sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{a^2+\omega^2}\right\} = e^{-ax}, \quad \mathcal{F}_c^{-1}\left\{\sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+\omega^2}\right\} = e^{-ax} \tag{6.53}$$

or

$$\mathcal{F}_s^{-1}\left\{\frac{\omega}{a^2+\omega^2}\right\} = \sqrt{\frac{\pi}{2}} \cdot e^{-ax}, \quad \mathcal{F}_c^{-1}\left\{\frac{1}{a^2+\omega^2}\right\} = \sqrt{\frac{\pi}{2}} \cdot \frac{e^{-ax}}{a}. \tag{6.54}$$

Since interchanging x and ω in the transforms (6.47) give their inverse transforms (6.48) then we obtain the following:

$$\mathcal{F}_s\left\{\frac{x}{a^2+x^2}\right\} = \sqrt{\frac{\pi}{2}} \cdot e^{-a\omega}, \quad \mathcal{F}_c\left\{\frac{1}{a^2+x^2}\right\} = \sqrt{\frac{\pi}{2}} \cdot \frac{e^{-a\omega}}{a}. \tag{6.55}$$

SOME OPERATION PROPERTIES OF THE FOURIER SINE AND COSINE TRANSFORM

We denote the Fourier Transform of $f(x)$ and $g(x)$ as $F(\omega)$ and $G(\omega)$, respectively, then we have the following properties:

$$(i) \qquad \mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(\omega) + \beta G(\omega), \tag{6.56a}$$

$$(ii) \qquad \mathcal{F}_s\left\{f'(x)\right\} = -\omega \mathcal{F}_c\{f(x)\}, \tag{6.56b}$$

$$(iii) \qquad \mathcal{F}_c\left\{f'(x)\right\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \tag{6.56c}$$

$$(iv) \qquad \mathcal{F}_s\left\{f''(x)\right\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \omega\sqrt{\frac{2}{\pi}} f(0), \tag{6.56d}$$

$$(v) \qquad \mathcal{F}_c\left\{f''(x)\right\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0). \tag{6.56e}$$

Example 6.9
We consider calculating the Fourier Sine and Cosine transform of the preceding, Example 6.8, but instead, we will use the properties given above. First we use property (6.56d). If $f(x) = e^{-ax}$, then $f''(x) = a^2 e^{-ax}$ giving $\mathcal{F}_s\{f''(x)\} = a^2 \mathcal{F}_s\{f(x)\}$. Since $f(0) = 1$ then (6.56d) gives

$$a^2 \mathcal{F}_s\{f(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \omega\sqrt{\frac{2}{\pi}} \tag{6.57}$$

and solving for $\mathcal{F}_s$ leads to $F_s(w) = \sqrt{\frac{2}{\pi}} \cdot \frac{\omega}{a^2 + \omega^2}$, the same result given in (6.52). Similarly, for the Cosine transform, we use the result (6.56e). Thus,

$$a^2 \mathcal{F}_c\{f(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} + \sqrt{\frac{2}{\pi}} \tag{6.58}$$

giving $F_c(w) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a^2 + \omega^2}$ again the same result given in (6.52).

One of the benefits of using the Fourier Sine (Cosine) transform is to solve PDEs on the half interval. The following examples illustrate.

Example 6.10
Solve

$$u_t = u_{xx}, \quad 0 < x < \infty \tag{6.59}$$

subject to

$$u(x,0) = f(x), \quad u(0,t) = 0 \quad u, u_x \to 0 \text{ as } |x| \to \infty. \tag{6.60}$$

Here we will use the Fourier Sine transform and denote the transform of $u(x,t)$ by $U(\omega,t)$, i.e.,

$$U(\omega,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin \omega x dx. \tag{6.61}$$

Taking the Fourier Sine transform of (6.59) gives

$$\mathcal{F}\{u_t\} = \mathcal{F}\{u_{xx}\} \tag{6.62}$$

which gives

$$U_t = \sqrt{\frac{2}{\pi}} \omega u(0,t) - \omega^2 U \tag{6.63}$$

and, due to the boundary condition $u(0,t) = 0$

$$U_t = -\omega^2 U \tag{6.64}$$

whereas the transform of the initial condition (6.60) gives

$$U(\omega,0) = F(\omega), \tag{6.65}$$

where $F(\omega)$ is the transform of $f(x)$. Solving (6.64) subject to (6.65) gives

$$U(\omega,t) = F(\omega) e^{-\omega^2 t}. \tag{6.66}$$

Now we take the inverse Fourier Sine transform of (6.66). Thus,

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left(F(\omega) e^{-\omega^2 t}\right) \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) e^{-\omega^2 t} \sin \omega x d\omega. \end{aligned} \tag{6.67}$$

As this is practically too general to solve, we will specify a particular initial condition, that

$$u(x,0) = \begin{cases} A & 0 < x < l \\ 0 & \text{otherwise,} \end{cases} \tag{6.68}$$

where A is constant. Applying the Fourier Sine transform to this initial condition gives

$$F(\omega) = \sqrt{\frac{2}{\pi}} A \frac{1 - \cos \omega l}{\omega} \tag{6.69}$$

and (6.67) becomes

$$u(x,t) = \frac{2A}{\pi} \int_0^\infty e^{-\omega^2 t} \frac{1-\cos\omega l}{\omega} \sin\omega x d\omega \tag{6.70}$$

which we can split into the following:

$$u(x,t) = \frac{2A}{\pi} \int_0^\infty e^{-\omega^2 t} \frac{\sin\omega x}{\omega} - \frac{1}{2} e^{-\omega^2 t} \frac{\sin\omega(x-l)}{\omega} - \frac{1}{2} e^{-\omega^2 t} \frac{\sin\omega(x+l)}{\omega} \, d\omega. \tag{6.71}$$

Using the following result

$$\int_0^\infty e^{-a\omega^2} \frac{\sin b\omega}{\omega} \, d\omega = \frac{\pi}{2} \operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right), \tag{6.72}$$

the solution given in (6.71) becomes

$$u(x,t) = A\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{x+l}{2\sqrt{t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{x-l}{2\sqrt{t}}\right)\right). \tag{6.73}$$

Figure 6.3 shows the solution at times $t = 0$ (red), $t = 0.25$ (blue), $t = 0.5$ (green), $t = 0.75$ (yellow), and $t = 1$ (black) when $A = 3, l = 2$.

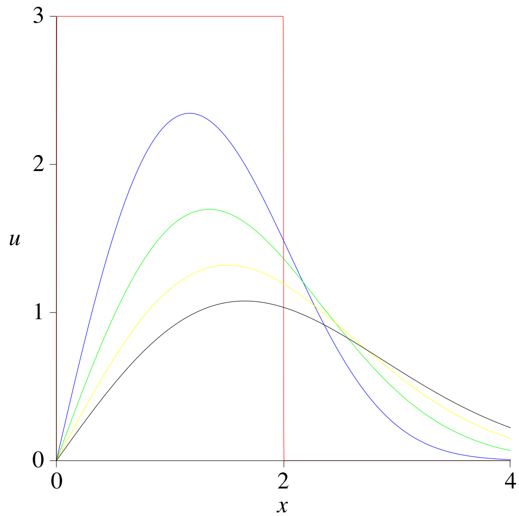

Figure 6.3: The solutions (6.73) at $t = 0, 0.25, 0.5, 0.75, 1$.

Example 6.11
Solve

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \infty, \quad 0 < y < \infty \tag{6.74}$$

subject to

$$u_x(0, y) = 0, \quad u(x, 0) = \frac{1}{1 + x^2}. \tag{6.75}$$

A natural question is: "should one use the Sine transform or the Cosine transform and in which variable, x or y?" If we were to use the Sine transform in the x variable, Laplace's equation would becomes

$$-\sqrt{\frac{2}{\pi}}\omega u(0, y) - \omega^2 U(\omega, y) + U_{yy}(\omega, y) = 0. \tag{6.76}$$

Using the Cosine transform in the x variable, Laplace's equation would become

$$-\sqrt{\frac{2}{\pi}}u_x(0, y) - \omega^2 U(\omega, y) + U_{yy}(\omega, y) = 0 \tag{6.77}$$

using the Sine transform in the y variable, Laplace's equation would become

$$U_{xx}(x, \omega) - \sqrt{\frac{2}{\pi}}\omega u(x, 0) - \omega^2 U(x, \omega) = 0, \tag{6.78}$$

and using the Cosine transform in the y variable, Laplace's equation would become

$$U_{xx}(x, \omega) - \sqrt{\frac{2}{\pi}}u_y(x, 0) - \omega^2 U(x, \omega) = 0. \tag{6.79}$$

So which of these four Equations—(6.76), (6.77), (6.78) or (6.79)—should we use? It really depends on the boundary conditions given. As we are given $u_x(0, y) = 0$ and $u(x, 0) = \dfrac{1}{1 + x^2}$ we would clearly use (6.77) or (6.78) with (6.77) the better choice. Thus, we want to solve

$$-\sqrt{\frac{2}{\pi}}u_x(0, y) - \omega^2 U(\omega, y) + U_{yy}(\omega, y) = 0 \tag{6.80}$$

and with the boundary condition becomes

$$-\omega^2 U(\omega, y) + U_{yy}(\omega, y) = 0. \tag{6.81}$$

Furthermore, the Fourier Cosine transform of the boundary condition $u(x, 0) = \dfrac{1}{1 + x^2}$ give

$$U(\omega, 0) = \sqrt{\frac{\pi}{2}}\, e^{-\omega}. \tag{6.82}$$

Solving (6.80) gives

$$U(\omega, y) = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}. \tag{6.83}$$

Since U is to remain bounded $(y, \omega > 0)$ then with the BC (6.82) we have $A(\omega) = 0$ and $B(\omega) = \sqrt{\frac{\pi}{2}}\, e^{-\omega}$. Thus,

$$U(\omega, y) = \sqrt{\frac{\pi}{2}}\, e^{-\omega(y+1)}. \tag{6.84}$$

Taking the Fourier Cosine inverse gives

$$\begin{aligned} u(x, y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{\pi}{2}}\, e^{-\omega(y+1)} \cos \omega x \, d\omega \\ &= \frac{y+1}{x^2 + (y+1)^2} \end{aligned} \tag{6.85}$$

and one can easily verify that (6.85) satisfies Laplace's equation and the boundary conditions given in (6.75).

6.3 EXERCISES

6.1. Calculate the Fourier transform of the following :

$(i)\quad f(x) = e^{-2|x|}$, $\quad (ii)\quad f(x) = x\, e^{-2x^2}$, $\quad (iii)\quad f(x) = e^{-x^2+2x}$,
$(iv)\quad f(x) = e^{-|x+1|}$, $\quad (v)\quad f(x) = x\, e^{-|x|}$, $\quad (vi)\quad f(x) = x^2 e^{-x^2}$,

$$(vii)\quad f(x) = \begin{cases} 2x - x^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}, \qquad (viii)\quad f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

6.2. If $\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = F(\omega)$ then show the following:

$$(i)\quad \mathcal{F}\{f(ax)\cos bx\} = \frac{1}{2a}\left[F\left(\frac{\omega - b}{a}\right) + F\left(\frac{\omega + b}{a}\right)\right],$$

$$(ii)\quad \mathcal{F}\{f(ax)\sin bx\} = \frac{1}{2ai}\left[F\left(\frac{\omega - b}{a}\right) - F\left(\frac{\omega + b}{a}\right)\right].$$

Use these to find the Fourier transform of the following:

$$(i)\quad f(x) = e^{-|2x|}\cos 3x, \qquad (ii)\quad f(x) = e^{-4x^2}\sin 5x.$$

6.3. Calculate the Fourier Inverse transform of the following:

$(i)\quad f(x) = e^{-x}$, $\quad (ii)\quad f(x) = \sin x$, $\quad (iii)\quad f(x) = \sin x$,
$(iv)\quad f(x) = e^{-x}$, $\quad (v)\quad f(x) = \sin x$, $\quad (vi)\quad f(x) = \sin x$.

6.4. Calculate the Fourier Sine transform of the following:

$$(i) \quad f(x) = e^{-x}, \qquad (ii) \quad f(x) = \frac{1}{\sqrt{x}}, \qquad (iii) \quad f(x) = \frac{4x}{x^2+9},$$

$$(iv) \quad f(x) = \begin{cases} 1 & 0 \le x \le a \\ 0 & \text{otherwise,} \end{cases} \qquad (v) \quad f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

6.5. Calculate the Fourier Cosine transform of the following:

$$(i) \quad f(x) = e^{-x}, \qquad (ii) \quad f(x) = \frac{1}{\sqrt{x}}, \qquad (iii) \quad f(x) = \frac{4}{x^2+9},$$

$$(vii) \quad f(x) = \begin{cases} 1 & 0 \le x \le a \\ 0 & \text{otherwise,} \end{cases} \qquad (viii) \quad f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

6.6. If

$$\begin{aligned} \mathcal{F}_c\{f(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos \omega x dx = F_c(\omega) \\ \mathcal{F}_s\{f(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin \omega x dx = F_s(\omega) \end{aligned}$$

show the following:

$$\begin{aligned} (i) \quad \mathcal{F}_c\{f(ax)\cos bx\} &= \frac{1}{2a}\left[F_c\left(\frac{\omega-b}{a}\right) + F_c\left(\frac{\omega+b}{a}\right)\right], \\ (ii) \quad \mathcal{F}_c\{f(ax)\sin bx\} &= \frac{1}{2a}\left[F_s\left(\frac{\omega+b}{a}\right) - F_s\left(\frac{\omega-b}{a}\right)\right], \\ (iii) \quad \mathcal{F}_s\{f(ax)\cos bx\} &= \frac{1}{2a}\left[F_s\left(\frac{\omega-b}{a}\right) + F_s\left(\frac{\omega+b}{a}\right)\right], \\ (iv) \quad \mathcal{F}_s\{f(ax)\sin bx\} &= \frac{1}{2a}\left[F_c\left(\frac{\omega-b}{a}\right) - F_c\left(\frac{\omega+b}{a}\right)\right]. \end{aligned}$$

Use these to find the Fourier Sine and Cosine transform of the following:

$$(i) \quad f(x) = e^{-x}\cos 3x, \qquad (ii) \quad f(x) = \frac{\sin 5x}{\sqrt{x}}.$$

6.7. Solve

$$u_t = u_{xx}, \quad -\infty < x < \infty, t > 0$$

subject to the following boundary conditions:

$$\begin{aligned} (i) \quad u(x,0) &= \frac{1}{1+x^2}, \quad u, u_x \to 0 \text{ as } |x| \to \infty, \\ (ii) \quad u(x,0) &= e^{-|x|}, \quad u, u_x \to 0 \text{ as } |x| \to \infty. \end{aligned}$$

6.8. Solve

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, y > 0$$

subject to the following boundary conditions:

$$\begin{aligned}(i) \quad & u(x,0) = e^{-x^2}, \quad u, u_x \to 0 \text{ as } |x| \to \infty, \\ (ii) \quad & u(x,0) = \frac{1}{1+x^2}, \quad u, u_x \to 0 \text{ as } |x| \to \infty.\end{aligned}$$

6.9. Solve

$$u_t = u_{xx}, \quad 0 < x < \infty, t > 0$$

subject to the following boundary conditions:

$$\begin{aligned}(i) \quad & u(0,t) = A, \quad u(x,0) = 0, \quad u, u_x \to 0 \text{ as } x \to \infty, \\ (ii) \quad & u(x,0) = e^{-ax} (a > 0), \quad u_x(x,0) = 0, \quad u, u_x \to 0 \text{ as } x \to \infty.\end{aligned}$$

6.10. Solves Laplaces equation $u_{xx} + u_{yy} = 0, 0 < x, y < \infty$ subject to the following boundary conditions:

$$\begin{aligned}(i) \quad & u(0,y) = 0, \; u(x,0) = \frac{x}{1+x^2}, \\ (ii) \quad & u(0,y) = e^{-y}, \; u_y(x,0) = \frac{1}{1+x^2}.\end{aligned}$$

6.11. Solves the wave equation $u_{tt} = c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0$, subject to the following boundary conditions:

$$u(x,0) = f(x), \; u_t(x,0) = 0.$$

CHAPTER 7

Solutions

Chapter 1

1. (i) (b), (ii) (c), (iii) (d), (iv) (a). 2. (i) yes, (ii) yes, (iii) no, (iv) no. 5. $a = -b^2$.
9. $ab = 1$. 10. $c = -4$.

Chapter 2

1. (i) $u = e^{-x} f\,(2x + y)$, (ii) $u = \dfrac{x}{2} - \dfrac{x^3}{2y^2} + 1$, (iii) $u = 2t + f(t + 2x)$,
(iv) $u = 3y^2 + f(x - y)$, (v) $y + 2x + 2(u - x)\ln x + f(u - x) = 0$,
(vi) $u = t^2 + \sin\left(xe^t\right)$.

2. (i) $u = xf\left(\dfrac{y^2}{2x^2} - \ln x\right)$, (ii) $u = -x - 1 + e^x f\left((y+1)e^{-x}\right)$, (iii) $u = \dfrac{2xy}{x + 3y}$,
(iv) $u = x + f(xy - x^2)$, (v) $u = -x^2 - \dfrac{y^2}{2}$.

3. (i) $u = sech\left((x + 1)e^{-t}\right)$, (ii) $u = sech\,(x - (u - 2)t)$,
(iii) $(u - 3x)e^{3t} = u - 3sech^{-1}u$,
(iv) $(1 + 2x + 3y)e^{-2t} = 1 + 2sech^{-1}u + 3u$.

4. (i) $u = xe^y + e^{2y}$, (ii) $u = \sqrt{x^2 + y^2}$ and $u = \sqrt{x^2 + (2 - y)^2}$, (iii) $u = e^{x+y-1}$,
(iv) $u = x^2 + xy$, and $u = x^2 - xy + 2x$.

5. (i) $F = F\left(2xq + q^2, 2yp + p^2, 2u + pq, p/q\right)$,
(ii) $F = F\left(2xuq - y, uq, x - \dfrac{1}{2q^2}, x - \dfrac{1}{2q^2}, p/q\right)$,
(iii) $F = F\,(up - 2y, uq - 2x, p, q)$.

Chapter 3

1. (i) H, (ii) P, (iii) P, (iv) H, (v) H, (vi) E, (vii) E.

2. (i) $r = x - y, \quad s = y, \quad u_{ss} = 0, \quad u = f(x-y)y + g(x-y),$

(ii) $r = x^2 + y^2, \quad s = y, \quad u_{ss} + \dfrac{2r}{r - s^2}\, u_r = 0,$

(iii) $r = x^2 - y^2, \quad s = x^2 + y^2, \quad u_{ss} + \dfrac{s}{s^2 - r^2}\, u_s = 0,$
$u = f(x^2 - y^2)\ln|x + y| + g(x^2 - y^2),$

(iv) $r = x + y, \quad s = x + 2y, \quad u_{rs} + 2u_r + 3u_s = 0,$

(v) $r = \ln x + \ln y, \quad s = 2\ln x + \ln y, \quad u_{rs} + 3u_r + 4u_s = 0,$

(vi) $r = 2x + y^2, \quad s = x + y^2, \quad u_{rs} - \dfrac{u_r + u_s}{2s - r} = 0,$

(vii) $r = x + 3y, \quad s = 2y, \quad u_{rr} + u_{ss} = 0,$

$(viii)$ $r = \sin x, \quad s = \sin x + y, \quad u_{rr} + u_{ss} - \dfrac{r}{1 - r^2}(u_r + u_s) = 0,$

(ix) $r = 3x, \quad s = x - 2y^2, \quad u_{rr} + u_{ss} = 0.$

4. (i) It is not possible.

(ii) $r = 2\ln x + \ln y, \quad s = 2\ln x.$

Chapter 4

1. (i) $$f(x) = \frac{1}{2}\left(e - e^{-1}\right) + \left(e - e^{-1}\right)\sum_{n=1}^{\infty} \frac{\cos n\pi\,(\cos n\pi x + n\pi \sin n\pi x)}{1 + n^2\pi^2},$$

(ii) $$f(x) = 1 + 4\sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2\pi^2} \cos \frac{n\pi x}{2},$$

(iii) $$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n^2\pi^2} \cos n\pi x + \frac{\cos n\pi}{n\pi} \sin n\pi x,$$

(iv) $$f(x) = \frac{28}{3} - 16\sum_{n=1}^{\infty} \frac{\cos n\pi - 2}{n^2\pi^2} \cos \frac{n\pi x}{2}.$$

2. (i) $$f_c(x) = e - 1 + 2\sum_{n=1}^{\infty} \frac{e\cos n\pi - 1}{1 + n^2\pi^2}\cos n\pi x$$

$$f_s(x) = -2\pi \sum_{n=1}^{\infty} \frac{n(e\cos n\pi - 1)}{1 + n^2\pi^2}\sin n\pi x,$$

(ii) $$f_c(x) = -\frac{1}{3} - 4\sum_{n=1}^{\infty} \frac{3\cos n\pi + 1}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$f_s(x) = 4\sum_{n=1}^{\infty} \frac{n^2\pi^2\cos n\pi - 4\cos n\pi + 4}{n^3\pi^3}\sin\frac{n\pi x}{2},$$

(iii) $$f_c(x) = \frac{5}{4} - 4\sum_{n=1}^{\infty} \frac{2\cos n\pi - 3\cos\frac{n\pi}{2} + 1}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$f_s(x) = 2\sum_{n=1}^{\infty} \frac{6\sin\frac{n\pi}{2} + n\pi}{n^2\pi^2}\sin\frac{n\pi x}{2},$$

(iv) $$f_c(x) = \frac{8}{9} - 6\sum_{n=1}^{\infty} \frac{n\pi\cos n\pi - 5n\pi\cos\frac{n\pi}{3} - 12\sin\frac{n\pi}{3}}{n^3\pi^3}\cos\frac{n\pi x}{3}$$

$$f_s(x) = 6\sum_{n=1}^{\infty} \frac{5n\pi\sin\frac{n\pi}{3} - 12\cos\frac{n\pi}{3} - 12}{n^3\pi^3}\sin\frac{n\pi x}{3}.$$

3. (i) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x$$

$a_0 = 1.493648266$	$a_1 = 0.2807893413$	$a_2 = -0.03869202628$
$a_3 = 0.01692345510$	$a_4 = 0.009434378340$	$a_5 = 0.006011684435$
$a_6 = -0.004164711740$	$a_7 = 0.003055298928$	$a_8 = -0.002336974260$
$a_9 = 0.001845282826$	$a_{10} = -0.001493973933$	

3. (ii) $$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{4}$$

$b_1 = 0.8747046386$	$b_2 = -.2406019431$	$b_3 = 0.2560984582$
$b_4 = -.1312665368$	$b_5 = 0.1475827953$	$b_6 = -0.09086250412$
$b_7 = 0.1031454701$	$b_8 = -0.06966160069$	$b_9 = 0.07909381104$
$b_{10} = -0.05656041726$		

3. (iii) $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

$b_1 = 0.3752809082$ $b_2 = 0.07184439810$ $b_3 = 0.03770863282$
$b_4 = 0.01889844269$ $b_5 = 0.01324442018$ $b_6 = 0.008544944317$
$b_7 = 0.006683416220$ $b_8 = 0.004848001062$ $b_9 = 0.004018015103$
$b_{10} = 0.003118724960$

Chapter 5

1. (i) $$u = 16 \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n^3\pi^3} e^{-n^2\pi^2 t/4} \sin \frac{n\pi x}{2}$$

(ii) $$u = \frac{2}{3} - 8 \sum_{n=1}^{\infty} \frac{1 + \cos n\pi}{n^2\pi^2} e^{-n^2\pi^2 t/4} \cos \frac{n\pi x}{2}$$

(iii) $$u = 32 \sum_{n=1}^{\infty} \frac{(2n-1)\pi \cos n\pi + 4}{(2n-1)^3\pi^3} e^{-(2n-1)^2\pi^2 t/4} \sin \frac{(2n-1)\pi x}{4}$$

(iv) $$u = -32 \sum_{n=1}^{\infty} \frac{(2n-1)\pi + 4\cos n\pi}{(2n-1)^3\pi^3} e^{-(2n-1)^2\pi^2 t/4} \cos \frac{(2n-1)\pi x}{4}$$

2. (i) $$u = -2x + 7 + 36 \sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^3\pi^3} e^{-n^2\pi^2 t/9} \sin \frac{n\pi x}{3}$$

(ii) $$u = x^2 + 2t - x - \frac{3}{2} - 18 \sum_{n=1}^{\infty} \frac{1 + \cos n\pi}{n^2\pi^2} e^{-n^2\pi^2 t/9} \cos \frac{n\pi x}{3}$$

3. (i) $$u = -2 \sum_{n=1}^{\infty} \frac{\cos n\pi}{n\pi} \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}$$

(ii) $$u = 2 \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n\pi} \frac{\sinh n\pi x}{\sinh n\pi} \sin n\pi y$$

4. (i) $P_t - PP_x - P_{xx} + 2Q_x = 0$

(ii) $u = \dfrac{v}{x^2+1}$, $u = \dfrac{4}{x^2+1}\displaystyle\sum_{n=1}^{\infty}\frac{\cos n\pi - 1}{n^3\pi^3}e^{-n^2\pi^2 t/4}\sin\frac{n\pi x}{2}$

6. $u = \exp(-\frac{\beta}{2}x - \frac{\beta^2}{4}t)v$

Chapter 6

1. (i) $\dfrac{1}{\sqrt{2\pi}}\cdot\dfrac{4}{\omega^2+4}$, (ii) $\dfrac{-i\omega}{8}\cdot e^{-\omega^2/8}$, (iii) $\dfrac{1}{\sqrt{2}}\cdot e^{1-i\omega-\omega^2/4}$,

(iv) $\dfrac{1}{\sqrt{2\pi}}\cdot\dfrac{2\,e^{i\omega}}{\omega^2+1}$, (v) $-\dfrac{4i}{\sqrt{2\pi}}\cdot\dfrac{\omega}{(\omega^2+1)^2}$, (vi) $\dfrac{2-\omega^2}{4\sqrt{2}}\cdot e^{-\omega^2/4}$,

(vii) $\dfrac{2\,i}{\sqrt{2\pi}\,\omega^3}\cdot\left((i\,\omega-1)+(i\,\omega+1)e^{-2i\omega}\right)$, $(viii)$ $\dfrac{1}{\sqrt{2\pi}\omega}\cdot\left(-1+2\,e^{-i\omega}-e^{-2i\omega}\right)$

2. (i) $\dfrac{4}{\sqrt{2\pi}}\cdot\dfrac{\omega^2+13}{(\omega^2-6\omega+13)\,(\omega^2+6\omega+13)}$, (ii) $-\dfrac{i}{2\sqrt{2}}\cdot e^{-(\omega^2+25)/16}\sinh\left(\dfrac{5\omega}{8}\right)$

3. (i) $\dfrac{1}{\sqrt{2\pi}}\cdot e^{-x^2/4}$, (ii) $\sqrt{\dfrac{\pi}{2}}\cdot(H(x+a)-H(x-a))$, (iii) $\dfrac{\sqrt{2\pi}}{4}\cdot e^{-2|x|}$,

(iv) $\dfrac{2}{\sqrt{2\pi}}\cdot e^{-2ix-|x|}$, (v) $\dfrac{\sqrt{2\pi}}{2i}\cdot\dfrac{d}{dx}e^{-|x|}$, (vi) $\dfrac{\sqrt{2\pi}}{4}\,ix\cdot e^{-|x|}$,

(vii) $\dfrac{1}{\sqrt{2}}\cdot e^{-(x+1)^2/4}$, $(viii)$ $\dfrac{\sqrt{2\pi}}{4}\cdot\left(e^{1+x}\operatorname{erfc}(1+x/2)+e^{1-x}\operatorname{erfc}(1-x/2)\right)$

4. (i) $\sqrt{\dfrac{2}{\pi}}\cdot\dfrac{\omega}{\omega^2+1}$, (ii) $\dfrac{1}{\sqrt{\omega}}$, (iii) $4\sqrt{\dfrac{2}{\pi}}\cdot e^{-3\omega}$, (iv) $\sqrt{\dfrac{2}{\pi}}\cdot\dfrac{1-\cos\omega a}{\omega}$,

(v) $2\sqrt{\dfrac{2}{\pi}}\cdot\dfrac{\sin\omega(1-\cos\omega)}{\omega^2}$

5. (i) $\sqrt{\dfrac{2}{\pi}}\cdot\dfrac{1}{\omega^2+1}$, (ii) $\dfrac{1}{\sqrt{\omega}}$, (iii) $\dfrac{4}{3}\sqrt{\dfrac{2}{\pi}}\cdot e^{-3\omega}$, (iv) $\sqrt{\dfrac{2}{\pi}}\cdot\dfrac{\sin\omega a}{\omega}$,

(v) $2\sqrt{\dfrac{2}{\pi}}\cdot\dfrac{\cos\omega(1-\cos\omega)}{\omega^2}$

6. (i) $\sqrt{\frac{2}{\pi}}\left(\frac{\omega-3}{(\omega-3)^2+1}+\frac{\omega+3}{(\omega+3)^2+1}\right)$, $\sqrt{\frac{2}{\pi}}\left(\frac{1}{(\omega-3)^2+1}+\frac{1}{(\omega+3)^2+1}\right)$,

(ii) $\sqrt{\frac{2}{\pi}}\left(\frac{1}{\sqrt{\omega-5}}-\frac{1}{\sqrt{\omega+5}}\right)$, $\sqrt{\frac{2}{\pi}}\left(\frac{1}{\sqrt{\omega+5}}-\frac{1}{\sqrt{\omega-5}}\right)$

7. (i)
$$u=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\frac{e^{-(x-s)^2/4t}}{1+s^2}\,ds,$$
(ii)
$$u=\frac{1}{2}\left(e^{t+x}\operatorname{erfc}\left(\frac{2t+x}{2\sqrt{t}}\right)+e^{t-x}\operatorname{erfc}\left(\frac{2t-x}{2\sqrt{t}}\right)\right)$$

8. (i) $u=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\,e^{-s^2}}{y^2+(x-s)^2}\,ds$, (ii) $u=\frac{(x^2+y^2-1)y+x^2-y^2+1}{(x^2+y^2)^2+2(x^2-y^2)+1}$

9. (i)
$$u=A\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right),$$
(ii)
$$u=\frac{e^{a^2t}}{2}\left(e^{-ax}\operatorname{erfc}\left(a\sqrt{t}-\frac{x}{2\sqrt{t}}\right)-e^{ax}\operatorname{erfc}\left(a\sqrt{t}+\frac{x}{2\sqrt{t}}\right)\right)$$

10. (i) $u=\frac{x}{x^2+(y+1)^2}$, (ii) $u=\frac{2}{\pi}\int_0^{\infty}\frac{e^{-\omega x}}{1+\omega^2}\cos\omega y\,d\omega$

11. $u=\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)$

Author's Biography

DANIEL J. ARRIGO

Daniel J. Arrigo earned his Ph.D. from the Georgia Institute of Technology in 1991. He has been on staff in the Department of Mathematics at the University of Central Arkansas since 1999 and is currently a professor of mathematics. He has published over 30 journal articles and one book. His research interests include the construction of exact solutions of PDEs; symmetry analysis of nonlinear PDEs; and solutions to physically important equations, such as nonlinear heat equations and governing equations modeling of granular materials and nonlinear elasticity. In 2008, Dr. Arrigo received the Oklahoma-Arkansas Section of the Mathematical Association of America's Award for Distinguished Teaching of College or University Mathematics.