Conditional distribution:

To defini cond. distribution function when there is no p.d.f.

Theorem 1.7:

(i) X is a random n-vector on $\overbrace{(\Omega, \mathcal{F}, P)}^{\text{probability space}}$, and $\mathcal{A}$ is a sub-σ-field of $\mathcal{F}$ — Then there is a function $P(B, \omega)$ on $\mathcal{B}^n \times \Omega$ such that

(a) $P(B,\omega) = P[X^{-1}(B) \mid \mathcal{A}]$ for any fixed $B \in \mathcal{B}^n$

(b) $P(\cdot, \omega)$ is a probability measure on $(R^n, \mathcal{B}^n)$ for any fixed $\omega \in \Omega$

Let $(\Omega, \mathcal{F}, P) \xrightarrow[Y]{} (\Lambda, \mathcal{G})$ then $P_{X|Y}(B|y)$ such that

(a) $P_{X|Y}(B|y) = P[X^{-1}(B) \mid Y=y] \rightarrow P_Y$ for any fixed $B \in \mathcal{B}^n$

(b) $P_{X|Y}(\cdot|y)$ is a probability measure on $(R^n, \mathcal{B}^n)$ for any fixed $y \in \Lambda$

if $E|g(X,Y)| < \infty$ with a Borel function g then

$$E[g(X,Y) \mid Y=y] = E[g(X,y) \mid Y=y] = \int_{R^n} g(x,y)\, dP_{X|Y}(x|y)$$

(ii) Let $(\Lambda, \mathcal{G}, P_1)$ a probability space. P_2 is a function from $\mathcal{B}^n \times \Lambda$ to R and satisfies:

(a) $P_2(\cdot, y)$ a Prob. measure on $(R^n, \mathcal{B}^n)$ for any $y \in \Lambda$

(b) $P_2(B, \cdot)$ is Borel for any $B \in \mathcal{B}^n$

Then P is a unique Prob. measure on $(R^n \times \Lambda, \sigma(\mathcal{B}^n \times \mathcal{G}))$ for $B \in \mathcal{B}^n$ and $C \in \mathcal{G}$

$$P(B \times C) = \int_C P_2(B, y)\, dP_1(y)$$

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if $(\Omega, G) = (R^m, \mathcal{B}^m)$ and $X(x,y) = x$, $Y(x,y) = y$ define the coordinate random vectors, then $P_Y = P_1$,

$P_{X|Y}(\cdot|y) = P_2(\cdot|y)$

Then $P(B \times C) = \int_C P_2(B, y)\, dP_1(y)$ is the joint dist. of (X, Y) $\rightarrow$

$$F(x,y) = \int_{(-\infty, y]} P_{X|Y}\left((-\infty, x] \mid z\right) dP_Y(z), \quad x \in R^n, y \in R^m$$

Markov chains and ~~martingales~~ martingale (The relation between Markov chain and correlation)

Markov chain: A sequence of random vectors $\{X_n : n = 1, 2, \dots\}$ is a Markov chain process if and only if:

$$P(B \mid X_1, \dots, X_n) = P(B \mid X_n), \quad B \in \sigma(X_{n+1})$$

$n = 2, 3, \dots$

why not 1 ?

compare this relation with $P(A \mid Y_1, Y_2) = P(A \mid Y_1)$ $A \in \sigma(X)$ says that X_{n+1} is conditionally independent of $(X_1, \dots, X_{n-1})$. But $(X_1, \dots, X_{n-1})$ is not necessarily independent of (X_n, X_{n+1})

$\boxed{\text{Example 1.24}}$: First order autoregressive processes

$\varepsilon_1, \varepsilon_2 \equiv$ independent random variables in Prob. space

$X_1 = \varepsilon_1$

$X_{n+1} = \underset{\text{const in } R}{\underline{\rho}} X_n + \varepsilon_{n+1}$ $\longrightarrow$ $\{X_n\} \equiv$ first-order autoregressive process

for any $B \in \mathcal{B}$

$$P(X_{n+1} \in B \mid X_1, \dots, X_n) = P_{\varepsilon_{n+1}}(B - \rho X_n) = P(X_{n+1} \in B \mid X_n)$$

Markov chain characteristics:

Proposition 1.12: A sequence of random vectors $\{X_n\}$ is a Markov chain if and only if one of the following three conditions hold:

(a) for any $n = 2, 3, \ldots$ and any integrable $h(X_{n+1})$ simple function with a Borel function h, $E[h(X_{n+1}) | X_1, \ldots, X_n] = E[h(X_{n+1}) | X_n]$

(b) for any $n = 1, 2, \ldots$, $B \in \sigma(X_{n+1}, X_{n+2}, \ldots)$, $P(B | X_1, \ldots, X_n) = P(B | X_n)$

(c) for any $n = 2, 3, \ldots$, $A \in \sigma(X_1, \ldots X_n)$ and $B \in \sigma(X_{n+1}, X_{n+2}, \ldots)$

$$P(A \cap B | X_n) = P(A | X_n) \, P(B | X_n)$$

if (c) holds then $\to$ let $A_1 \in \sigma(X_n)$, $A_2 \in \sigma(X_1, \ldots, X_{n-1})$, $B \in \sigma(X_{n+1}, X_{n+2}, \ldots)$

$$\to \int_{A_1 \cap A_2} E(I_B | X_n) \, dP = \int_{A_1} I_{A_2} \, E(I_B | X_n) \, dP$$

$$= \int_{A_1} E[I_{A_2} E(I_B | X_n) | X_n] \, dP$$

$$= \int_{A_1} E(I_{A_2} | X_n) \, E(I_B | X_n) \, dP$$

$$= \int_{A_1} E(I_{A_2} I_B | X_n) \, dP$$

$$= P(A_1 \cap A_2 \cap B)$$

Martingales: let $\{X_n\}$ is a sequence of integrable random variables on probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$

$\sigma(X_n) \subset \mathcal{F}_n \longrightarrow$

The sequence $\{X_n, \mathcal{F}_n : n = 1, 2, \ldots\}$ is said to be martingale if and only if

$E(X_{n+1}|F_n) = X_n$

$\quad " \quad \geq X_n \longrightarrow$ sub martingale

$\quad " \quad \leq X_n \longrightarrow$ super martingale

$\{X_n\}$ is said to be martingale if and only if

$\{X_n, \sigma\{X_1, \ldots, X_n\}\}$ is a martingale.

if $\{X_n, F_n\}$ is martingale then $\{X_n\}$ is martingale

$EX_1 = EX_j$ (or $EX_1 \leq EX_2 \leq \ldots$) is also martingale

+ Another way to construct martingale is to use a sequence of independent integrable random variable $\{\varepsilon_n\}$ by letting $X_n = \varepsilon_1 + \ldots + \varepsilon_n$

$$E(X_{n+1} | X_1, \ldots, X_n) = E(X_n + \varepsilon_{n+1} | X_1, \ldots X_n) = X_n + E\varepsilon_{n+1}$$

* To check the properties of martingales in pp. 29 in

Proposition 1.13 – 1.16