1.1 Probability Spaces & Random Elements

1.1.1 σ-fields & measures

Def $\mathcal{F}$ collection of subsets of a sample space Ω. $\mathcal{F}$ is a σ-field (σ-algebra) iff.

(1) $\phi \in \mathcal{F}$ (2) If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (3) If $A_i \in \mathcal{F} \Rightarrow \cup A_i \in \mathcal{F}$.

$(\Omega, \mathcal{F})$ - "measurable space"

Def. $\mathcal{B} = \sigma(C)$ "Borel σ-field" if $\forall A \in \mathcal{B}$, $A = (a,b) \in \mathbb{R}$.

Def. $(\Omega, \mathcal{F})$ - meas. space. a set function ν on $\mathcal{F}$ is called a "measure" iff

(1) $0 \le \nu(A) \le \infty$ $\forall A \in \mathcal{F}$ (2) $\nu(\phi) = 0$ (3) If $A_i \in \mathcal{F}$ and $A_i \cap A_j = \phi$ then $\nu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i)$

Def. $(\Omega, \mathcal{F}, \nu)$ - "measure space".

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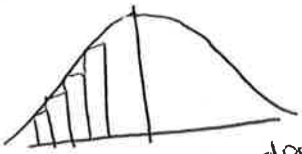
* Lebesgue measure * unique measure m on $(\mathbb{R}, \mathcal{B})$ such that $m([a,b]) = b-a$.

Remark: Lebesgue Integration vs. Riemannian Integration.



Reimann divides the domain

Lebesgue divides the Range.

Lebesgue integration can be approximated by simple functions $\varphi(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x)$

$A_i = \{x \in A : f(x) = a_i\}$

by the following:

$\int_E \varphi(x)\,dx = \sum_{i=1}^{n} a_i \mu(A_i)$

if $\mu(A_i)$ is finite.

$$\int_E \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x)\,dx = \sum_{i=1}^{n} a_i \int_E \mathbb{1}_{A_i}(x)\,dx$$

$$= \sum_{i=1}^{n} a_i \mu(A_i)$$

If f is a nonneg. func. then $\int_E f(x)\,dx = \sup\{\int_E \varphi(x)\,dx\}$

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* ctbl: 1 to 1 corresp. of Ω to $\mathbb{Z}$. *

Prop. properties of meas. $(\Omega, \mathcal{F}, \nu)$ - meas. space.

(1) Monotonicity If $A \subset B$ then $\nu(A) \le \nu(B)$

(2) Subadditive. $\forall$ sequence $A_1, A_2 \ldots$

$$\nu(\cup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \nu(A_i)$$

(3) Continuity. If $A_1 \subset A_2 \subset A_3 \ldots$ (($A_1 \supset A_2 \supset A_3 \ldots$) and $\nu(A_1) < \infty$) where $\lim A_n = \cup_{i=1}^{\infty} A_i$ ($\cap_{i=1}^{\infty} A_i$)

then, $\nu(\lim A_n) = \lim \nu(A_n)$

Prop. CDF, F on $\mathbb{R}$

(a) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$ (b) $F(\infty) = \lim_{x \to \infty} F(x) = 1$

(c) F is nondecreasing ($F(x) \le F(y)$ if $x \le y$) (d) $\lim_{y \to x, y > x} F(y) = F(x)$

Remark: Product Sets, product σ-field $\sigma(\Pi_{i \in I} \mathcal{F}_i)$, product space $\Pi_{i \in I} \Omega_i$. and product measures:

Ex. Lebesgue measure. $[a_1, b_1] \times [a_2, b_2]$

$$(b_1 - a_1)(b_2 - a_2) = m([a_1, b_1])\, m([a_2, b_2])$$

Prop. (product measure thrm.)
$(\Omega_i, \mathcal{F}_i, \nu_i)$ - meas. space(s) s.t. ν_i σ-finite $\forall i$, then $\exists$ unique σ-finite meas. on the product σ-field $\sigma(\mathcal{F}_1 \times \ldots \times \mathcal{F}_k)$ called the product measure, $\nu_1 \times \ldots \times \nu_k$ s.t.

$$\nu_1 \times \ldots \times \nu_k (A_1 \times \ldots \times A_k) = \nu_1(A_1) \cdot \ldots \cdot \nu_k(A_k).$$

Ex. $F(x_1, \ldots x_k) = \mathbb{P}((-\infty, x_1] \times \ldots \times (-\infty, x_k])$ the joint CDF.
then $\mathbb{P}$ is the unique product measure iff. iff independent.

$$F(x_1, \ldots x_k) = F_1(x_1) \cdot \ldots \cdot F_k(x_k)$$

confusing example since this is a "nonexample"

* The CDF is NOT ALWAYS THE PRODUCT MEASURE.

1.1.2 Measurable functions & Distributions

$f: \Omega \to \Lambda$. $f^{-1}(B) = \{f \in B\} = \{\omega \in \Omega : f(\omega) \in B\}$.

Def. $f: (\Omega, \mathcal{F}) \to (\Lambda, \mathcal{G})$ then f is a "measurable function" iff $f^{-1}(\mathcal{G}) \subset \mathcal{F}$.

* If $\Lambda = \mathbb{R}$ and $\mathcal{G} = \mathcal{B}$ then f is "Borel meas." or "Borel function".
* If $X: (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ then X is a "random variable".

sub-sigma algebra: σ-alge. that is also a subset of $\mathcal{F}$.

Def. "simple function" $\varphi(\omega) = \sum_{i=1}^{k} a_i \mathbb{1}_{A_i}(\omega)$, $A_1 \ldots A_k$ - meas. sets $\in \Omega$ and $a_1 \ldots a_k \in \mathbb{R}$ then φ is a Borel function.

Prop. $(\Omega, \mathcal{F})$ - meas. space.
(1) f is Borel iff $f^{-1}(a, \infty) \in \mathcal{F}$ $\forall a \in \mathbb{R}$. (2) If f and g are Borel then fg and $af + bg$, f/g are Borel.
(3) If $f_1 \ldots f_n$ are Borel then so are $\limsup f_n$, $\liminf f_n$, $\sup f_n$ & $\inf f_n$.
(4) $f: (\Omega, \mathcal{F}) \to (\Lambda, \mathcal{G})$ and $g: (\Lambda, \mathcal{G}) \to (\Delta, \mathcal{H})$ meas. then $f \circ g: (\Omega, \mathcal{F}) \to (\Delta, \mathcal{H})$ is meas.
(5) Let Ω be Borel in $\mathbb{R}^p$. If $f: \Omega \to \mathbb{R}^p$ is cts. then f is meas.

Remark: f - non neg. Borel on $(\Omega, \mathcal{F})$ then $\exists \{\varphi_n\}$ s.t. $0 \le \varphi_1 \le \varphi_2 \ldots \le f$ s.t. $\lim \varphi_n = f$.

Def. $(\Omega, \mathcal{F}, \nu)$ - meas. space, and f - meas. func. $f: (\Omega, \mathcal{F}) \to (\Lambda, \mathcal{G})$ the "induced measure" by f, denoted $\nu \circ f^{-1}$ is a measure on $\mathcal{G}$:

$$\nu \circ f^{-1}(B) = \nu(f \in B) = \nu(f^{-1}(B)), \quad B \in \mathcal{G}.$$

Remark: $\mathbb{P}$ - prob. meas. then $\mathbb{P} \circ X^{-1}$ is the dist. of X. (p.d.f)

1.2 Integration & Differentiation.

1.2.1 Integration

Def. The integral of a nonneg. simple function φ wrt ν is,

$$\int \varphi \, d\nu = \sum_{i=1}^{k} a_i \nu(A_i).$$

$$\left(\varphi(\omega) = \sum_{i=1}^{k} a_i \mathbb{1}_{A_i}(\omega) \right). \quad (*)$$

Def. f be a nonneg. Borel function and S_f the collection of all nonneg. simple functions $(*)$ s.t. $\varphi(\omega) \le f(\omega)$ $\forall \omega \in \Omega$.

$$\int f \, d\nu = \sup \left\{ \int \varphi \, d\nu : \varphi \in S_f \right\}$$

Hence $\forall$ Borel $f \ge 0$ $\exists \{\varphi_n(\omega)\}$ s.t. $0 \le \varphi_i \le f$ and $\lim_{n\to\infty} \int \varphi_n \, d\nu = \int f \, d\nu$.

Ex. Lebesgue integral.
If $\Omega = \mathbb{R}$ and m - lebesgue then

$$\int_{[a,b]} f \, dm(x) = \int_a^b f(x) \, dx$$

Prop. $(\Omega, \mathcal{F}, \nu)$ - meas. space and f & g - Borel functions
(1) If $\int f d\nu$ exists and $a \in \mathbb{R}$ then $\int (af) d\nu$ exists and $= a \int f d\nu$.
(2) If $\int f d\nu$ & $\int g d\nu$ $\exists$. and $\int f d\nu + \int g d\nu$ is well-def, then $\int (f+g) d\nu$ $\exists$ = " ".

Prop $(\Omega, \mathcal{F}, \nu)$ - meas. space & f & g Borel.
(1) If $f \le g$ a.e. then $\int f d\nu \le \int g d\nu$
(2) $f \ge 0$ a.e. and $\int f d\nu = 0$ then $f = 0$ a.e.

Remark: Need to establish: $\int \lim f_n d\nu = \lim \int f_n d\nu$.

Thrm $f_1 \dots f_n$ be sequence of Borel functions on $(\Omega, \mathcal{F}, \nu)$ then
(1) Fatou's lemma: If $f_n \ge 0$
$\int \liminf f_n d\nu \le \liminf \int f_n d\nu$.
(2) DC Thrm: If $\lim f_n = f$ a.e. and $\exists g$ integrable. s.t. $|f_n| \le g$ a.e.
then $\int \lim f_n d\nu = \lim \int f_n d\nu$.
(3) MC Thrm: If $0 \le f_1 \le \dots \le f_n$ and $\lim f_n = f$ a.e. then
$\int \lim f_n d\nu = \lim \int f_n d\nu$.

Consequences of Thrm above:
(a) Interchange of diff & integration: Let $(\Omega, \mathcal{F}, \nu)$ be meas. space f - Borel, $\left| \frac{\partial f}{\partial \theta} \right| \le g(\omega)$ a.e. then $\frac{d}{d\theta} \int f \, d\nu = \int \frac{\partial f}{\partial \theta} d\nu$. } DC Thrm.
(b) change of vars) f meas. from $(\mathcal{B}, \mathcal{F}, \nu)$

$\boxed{\text{Thrm}}$ (Change of variables) Let f be meas. $f : (\Omega, \mathcal{F}, \nu) \to (\Lambda, \mathcal{G})$ and g Borel on $(\Lambda, \mathcal{G})$ Then

$$\int_\Omega g \circ f \, d\nu = \int_\Lambda g \, d(\nu \circ f^{-1})$$

$\boxed{\text{Motivation:}}$

$E(X) = \int x f \, dx$ (prob.)

① $E(X) = \int_\Omega X \, dP = \int_{\mathbb{R}} x \, dP_X$

function measure. value $P \circ X^{-1}$ dist of X (p.d.f)

② & we can rewrite prev. prop. of Borel fuctions :

Ex. $Y = (X_1, X_2)$ $g(Y) = X_1 + X_2$

$E(X_1 + X_2) = EX_1 + EX_2 = \int_{\mathbb{R}} x \, dP_{X_1} + \int_{\mathbb{R}} x \, dP_{X_2}$

which is easier to compute than $\int_{\mathbb{R}} x \, dP_{(X_1+X_2)}$

$\boxed{\text{Thrm}}$ (Fubini) Let ν_i be σ-finite on $(\Omega_i, \mathcal{F}_i)$ and either $f \geq 0$ or f is integ. wrt $\nu_1 \times \nu_2$

f-Borel on $\prod_i (\Omega_i, \mathcal{F}_i)$

then $g(\omega_2) = \int_{\Omega_1} f(\omega_1, \omega_2) \, d\nu_1$ exists a.e. ν_2 defines

Integr. a Borel function s.t.

$$\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) \, d\nu_1 \times \nu_2 = \int_{\Omega_2} \left[\int_{\Omega_1} f(\omega_1, \omega_2) \, d\nu_1 \right] d\nu_2$$

1.2.2 Radon - Nikodym derivative

$(\Omega, \mathcal{F}, \nu)$ - meas. space, $f \geq 0$ Borel then

$$\lambda(A) = \int_A f \, d\nu, \quad A \in \mathcal{F}$$

is a measure. $(\nu(A) = 0 \Rightarrow \lambda(A) = 0)^* \Rightarrow \lambda << \nu$

$\boxed{\text{Thm}}$ (Radon-Nikodym) Let ν & λ be two meas. on $(\Omega, \mathcal{F})$ and ν σ-finite. If $\lambda << \nu$ the $\exists$ $f \geq 0$ Borel on Ω s.t.

$$\lambda(A) = \int_A f \, d\nu, \quad A \in \mathcal{F}$$

and f is called the "Radon Niko." derive or "density" of λ wrt ν denoted $d\lambda / d\nu$

Further, f is ! a.e. ν.

Ex. F c.d.f and $\mathbb{P}$ prob. meas. corresp. to F then m-lebesgue. $\forall A \in \mathcal{B}$ (Hence f is the p.d.f of $\mathbb{P}$ wrt. lebesgue)

$\mathbb{P}(A) = \int_A f \, dm$,

and $f = \dfrac{d\mathbb{P}}{dm} = \dfrac{dF}{dx}$

$\cancel{E(x) = \int_\Omega X^{(x)} dP(x) = \int_A X \, dP \circ X \circ X^{-1} = \int_\Lambda X \, dP}$

$$EX = \int_\Omega X \, dP = \int_{\mathbb{R}} x \, d(P \circ X^{-1})$$

$$\begin{cases} g(x) = x \\ f(x) = X(x) \end{cases} \Rightarrow \boxed{\cancel{f \circ g = X \circ X}}$$

$$f(g(x)) = X(g(x)) = X(x)$$

$$g(f(x)) = g(X(x)) = X(x)$$

$$\int_\Omega g \circ f(x) \, dP = \int_\Omega X(x) \, dP$$

$$\| $$

$$\int g \, dP(f^{-1}(x))$$