1.3 Distributions and their characteristics

random k-vector $(X_1, \dots, X_k)$ w/ joint pdf f w.r.t. $\nu_1 \times \dots \times \nu_k$ on $\mathcal{B}^k$

$\to$ X_i marginal pdf w.r.t. ν_i: $f_i(x) = \int_{\mathbb{R}^{k-1}} f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k)\, d\nu_1 \dots d\nu_{i-1} d\nu_{i+1} \dots d\nu_k$

$X_1, \dots X_k$ are independent iff $f(x_1, \dots, x_k) = f_1(x_1) \dots f_k(x_k)$

<u>Lemma:</u> $X_1, \dots, X_k$ independent $\to$ $g(X_1, \dots, X_n)$, $h(X_{n+1}, \dots, X_k)$ independent (g, h Borel)

<u>Ex:</u> multivariate normal pdf: $f(x) = (2\pi)^{-k/2} (\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma (x-\mu)\right)$, $x \in \mathbb{R}^k$

$\mu \in \mathbb{R}^k$, Σ positive definite $k \times k$ matrix (if nonnegative definite but singular: X is $N_k(\mu, \Sigma)$ iff $c^T X$ is $N(c^T\mu, c^T \Sigma c)$ $\forall c \in \mathbb{R}^k$)

independent iff Σ is diagonal!

<u>Change of variables:</u> X random k-vector w/ Lebesgue pdf f_X; $Y = g(X)$ w/ g Borel from $(\mathbb{R}^k, \mathcal{B}^k)$ to $(\mathbb{R}^k, \mathcal{B}^k)$; $A_1, \dots, A_m$ disjoint sets in $\mathcal{B}^k$ (s.t. $m(\mathbb{R}^k \setminus (A_1 \cup \dots \cup A_m)) = 0$) and g is one-to-one on A_j w/ non-vanishing Jacobian (i.e. $\det(\partial g(x)/\partial x) \neq 0$ on A_j):

$\Rightarrow$ $f_Y(y) = \sum_{j=1}^{m} \left|\det(\partial h_j(y)/\partial y)\right| f_X(h_j(y))$, $h_j(y) = g^{-1}(y)$ on A_j

<u>Cochran's theorem:</u> $X = N_n(\mu, I_n)$; $X^T X = X^T A_1 X + \dots + X^T A_k X$ (A_i is $n \times n$ symmetric w/ rank n_i)

$X^T A_i X$ have non-central chi-square distribution $\chi^2_{n_i}(\delta_i)$ and are independent iff $n = \sum_{i=1}^{k} n_i$ $\to$ $\delta_i = \mu^T A_i \mu$ & $\mu^T \mu = \sum_{i=1}^{k} \delta_i$

<u>Moments and moment inequalities:</u> Def.: $EX = \int_\Omega X\, dP$ for random variable X on $(\Omega, \mathcal{F}, P)$

if EX^k finite (k positive integer) $\to$ kth moment of X or P_X

if $E|X|^a < \infty$ ($a \in \mathbb{R}$) $\to$ ath absolute moment

if $E|X|^a < \infty$ ($a \in \mathbb{R}_{>0}$) $\to$ EX^k finite $\forall k \leq a$

if $\mu = EX$ & $E(X-\mu)^k$ finite $\to$ kth central moment ($k=2$ called variance)

For vectors $X = (X_1, \dots, X_k)$: $EX = (EX_1, \dots, EX_k)$

variance-covariance matrix: $Var(X) = E((X-EX)(X-EX)^T)$

$Y = c^T X$ ($c \in \mathbb{R}^k$) $\to$ $EY = c^T EX$, $Var(Y) = c^T Var(X) c$