$Var(X)$ is non-negative definite $\Rightarrow (Cov(X_i, X_j))^2 \leq Var(X_i) Var(X_j)$ $i \neq j$

$( Cov(X_i, X_j) = E(X_i - EX_i)(X_j - EX_j) )$

correlation coefficient $\rho_{X_i, X_j} = Cov(X_i, X_j) / \sqrt{Var(X_i) Var(X_j)}$

if $\rho_{X_i, X_j} = 0 \rightarrow$ uncorrelated

if $X_1, \dots, X_k$ are independent (or at least pairwise independent) $\rightarrow$ uncorrelated (converse is not true)

$\Rightarrow$ also if $E|X_1 \dots X_k| < \infty \rightarrow E(X_1 \dots X_k) = EX_1 \dots EX_k$

<u>Useful inequalities</u>: $(E(XY))^2 \leq EX^2 EY^2$

$E|XY| \leq (E|X|^p)^{1/p} (E|Y|^q)^{1/q}$, $p > 1$ & $\frac{1}{p} + \frac{1}{q} = 1$

$(E|X|^r)^{1/r} \leq (E|X|^s)^{1/s}$, $1 \leq r \leq s$

$(E|X+Y|^p)^{1/p} \leq (E|X|^p)^{1/p} + (E|Y|^p)^{1/p}$, $p \geq 1$

Def.: $A \subset \mathbb{R}^k$ is convex iff for $x, y \in A \Rightarrow tx + (1-t)y \in A$ $\forall t \in [0,1]$

$f: A \subset \mathbb{R}^k \rightarrow \mathbb{R}$ is convex iff $f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$

<u>Jensen's inequality</u>: $f(EX) \leq Ef(X)$ (if $EX$ finite and $P(X \in A) = 1$)

<u>Ex</u>: $(EX)^{-1} < E(X^{-1})$, $E(\log X) < \log(EX)$,

<u>Moment generating and characteristic functions</u>:

Def.: random $k$-vector $X$: i) moment generating function (mgf) $\psi_X(t) = Ee^{t^T X}$

ii) characteristic function (chf): $\phi_X(t) = Ee^{i t^T X}$

$t \in \mathbb{R}^k$

properties: • $\psi_X(0) = \phi_X(0) = 1$

• $\phi_X(t)$ is bounded by 1 and uniformly continuous on $\mathbb{R}^k$

• $\psi_X(t)$ is nonnegative but may be $\infty$ everywhere except at $t=0$

• $Y = A^T X + c \rightarrow \psi_Y(u) = e^{c^T u} \psi_X(Au)$, $\phi_Y(u) = e^{i c^T u} \phi_X(Au)$

- If $\Psi_X(t)$ & $\Psi_X(-t)$ are finite, then $X$ has finite moments of any order (also absolute moments)
- If $\Psi_X(t)$ is finite in a neighborhood of 0, then $\mu_{r_1,\dots,r_k} = E(X_1^{r_1} \dots X_k^{r_k})$ is finite for non-negative integers $r_i$ and $\Psi_X(t) = \sum_{r_1,\dots,r_k} \frac{\mu_{r_1,\dots,r_k} t_1^{r_1} \dots t_k^{r_k}}{r_1! \dots r_k!}$ for $t_i$ in the neighborhood of 0.

$$\Rightarrow E(X_1^{r_1} \dots X_k^{r_k}) = \left. \frac{\partial^{r_1+\dots+r_k} \Psi_X(t)}{\partial t_1^{r_1} \dots \partial t_k^{r_k}} \right|_{t=0}$$

- If $0 < \Psi_X(t) < \infty$, then $\varkappa_X(t) = \log \Psi_X(t)$ is the cumulant generating function.

$$\varkappa_X(t) = \sum_{r_1 \dots r_k} \frac{\varkappa_{r_1,\dots,r_k} t_1^{r_1} \dots t_k^{r_k}}{r_1! \dots r_k!}, \quad \varkappa_i \text{ cumulants of } X$$

(one-to-one correspondence w/ moments)

- If $\Psi_X(t)$ is not finite, we can use $\phi_X(t)$: If $E|X_1^{r_1} \dots X_k^{r_k}| < \infty$

$$\rightarrow \left. \frac{\partial^{r_1+\dots+r_k} \phi_X(t)}{\partial t_1^{r_1} \dots \partial t_k^{r_k}} \right|_{t=0} = (-1)^{\frac{r_1+\dots+r_k}{2}} E(X_1^{r_1} \dots X_k^{r_k})$$

<u>Theorem</u>: $X, Y$ random $k$-vectors:

i) If $\phi_X(t) = \phi_Y(t)$ $\forall t \in \mathbb{R}^k$ then $P_X = P_Y$

ii) If $\Psi_X(t) = \Psi_Y(t) < \infty$ $\forall t$ in a neighborhood of 0 then $P_X = P_Y$

If $X, Y$ independent: $\Psi_{X+Y}(t) = \Psi_X(t)\, \Psi_Y(t)$ & $\phi_{X+Y}(t) = \phi_X(t)\, \phi_Y(t)$

$X$ is symmetric about 0 iff $\phi_X(t)$ is real-valued