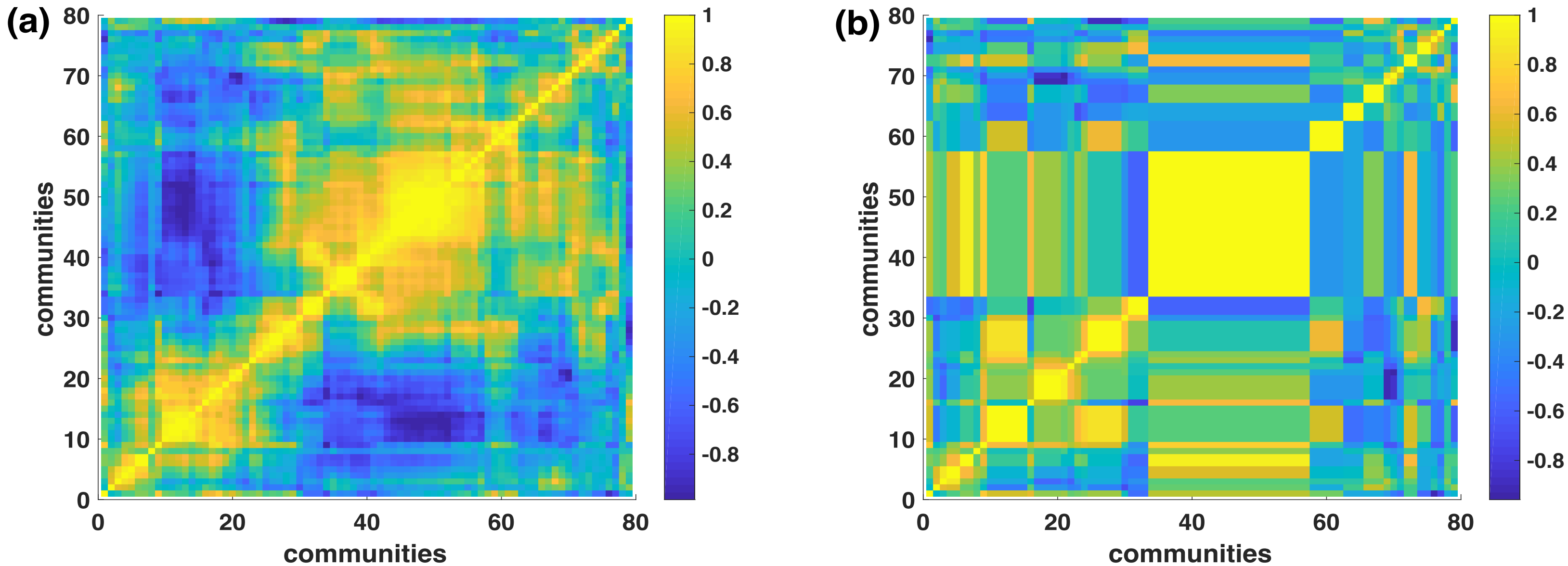

CCS 2018 Satellite: DOOCN

Thessaloniki 27 September 2018

SIMPLICIAL COMPLEXES: EMERGENT HYPERBOLIC NETWORK GEOMETRY AND FRUSTRATED SYNCHRONIZATION

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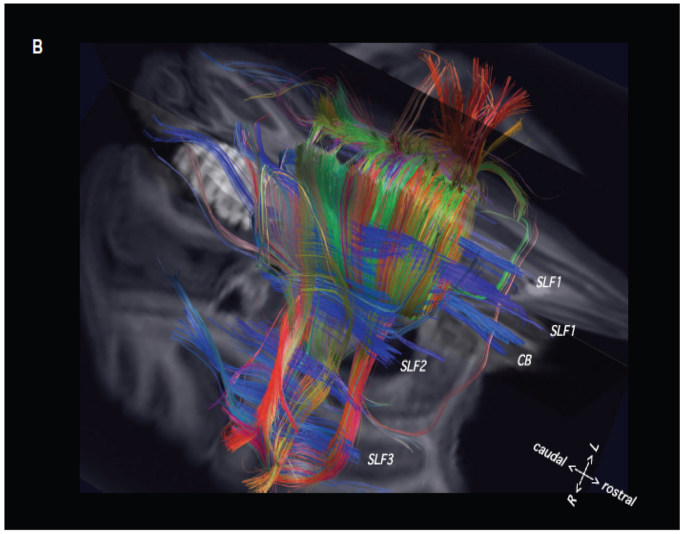
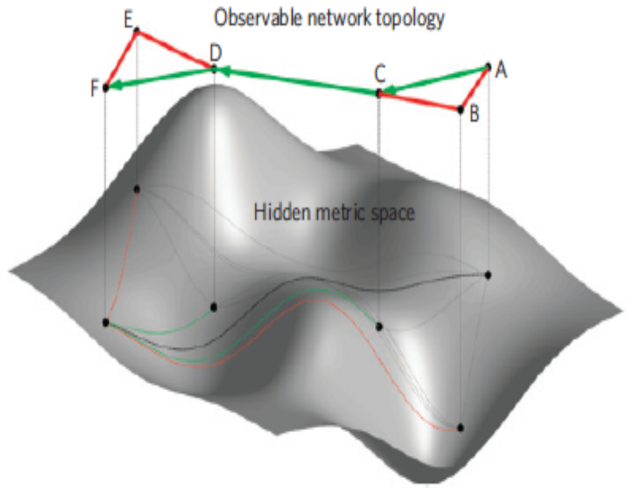
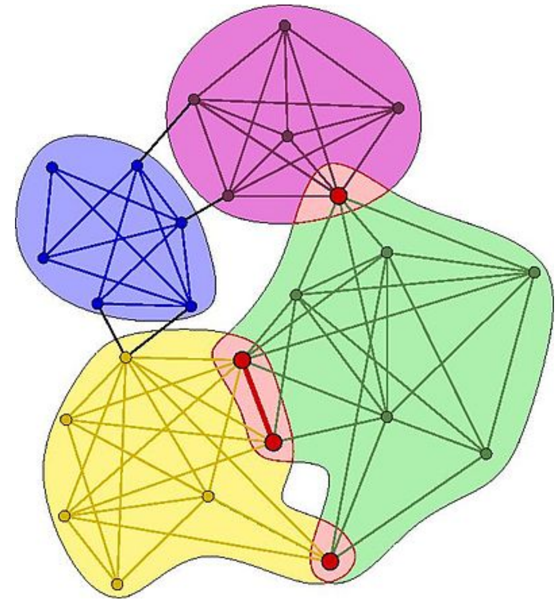
Network Topology and Network Geometry







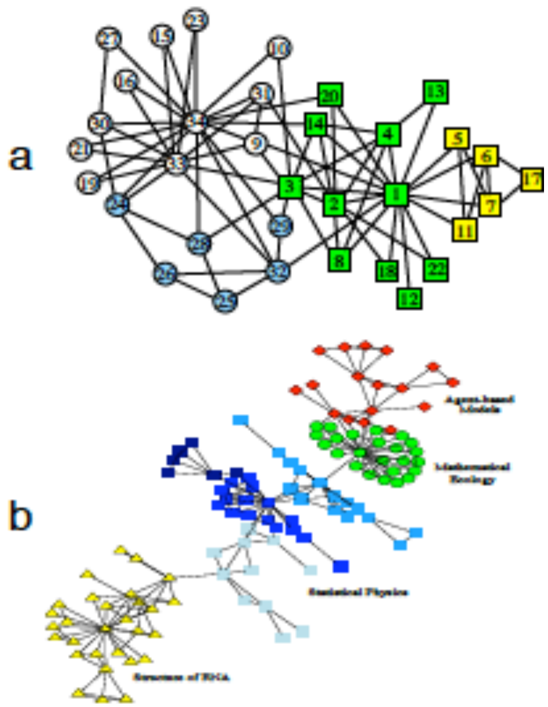
are expected to have impact in a variety of applications,

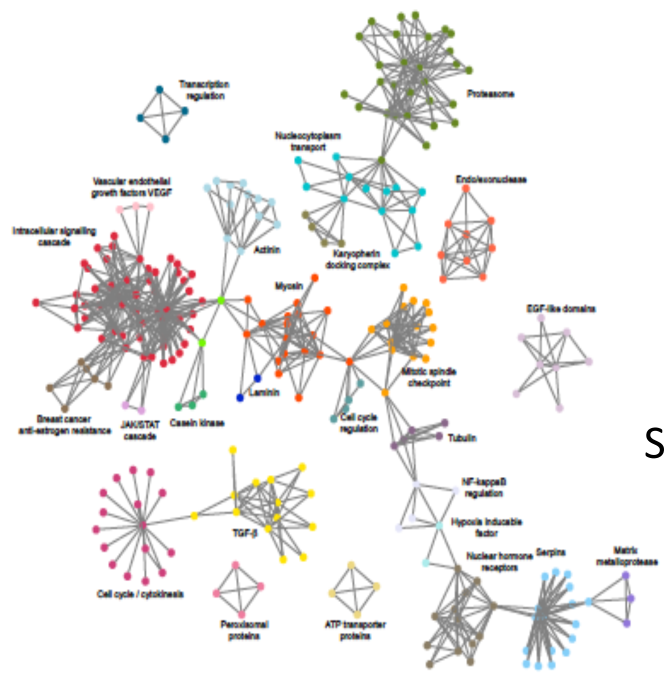
ranging from

brain research to routing protocols in the Internet

Community Structure and Network Topology

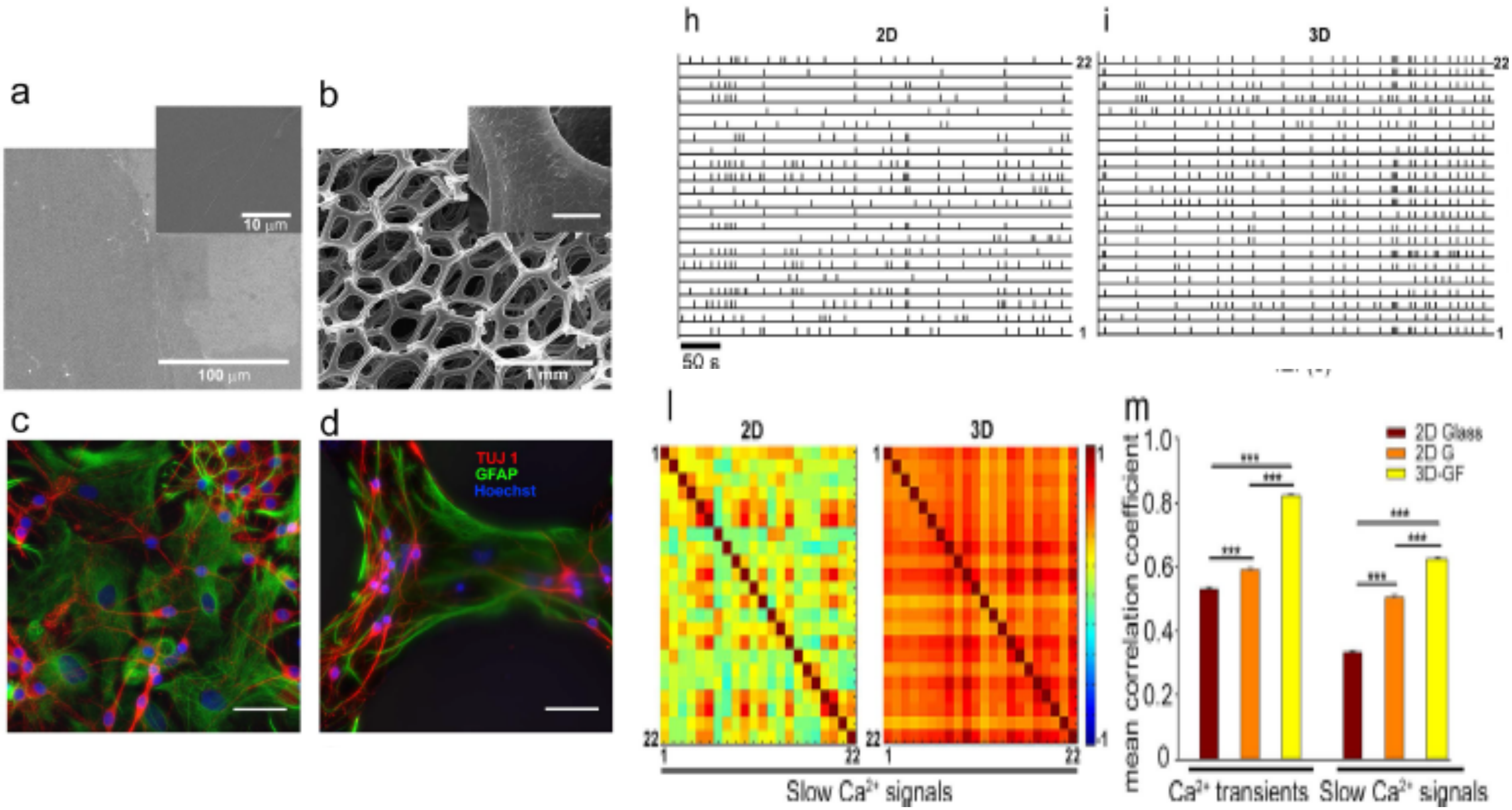
most complex networks
have a
mesoscale structure
which reveal densely connected
communities





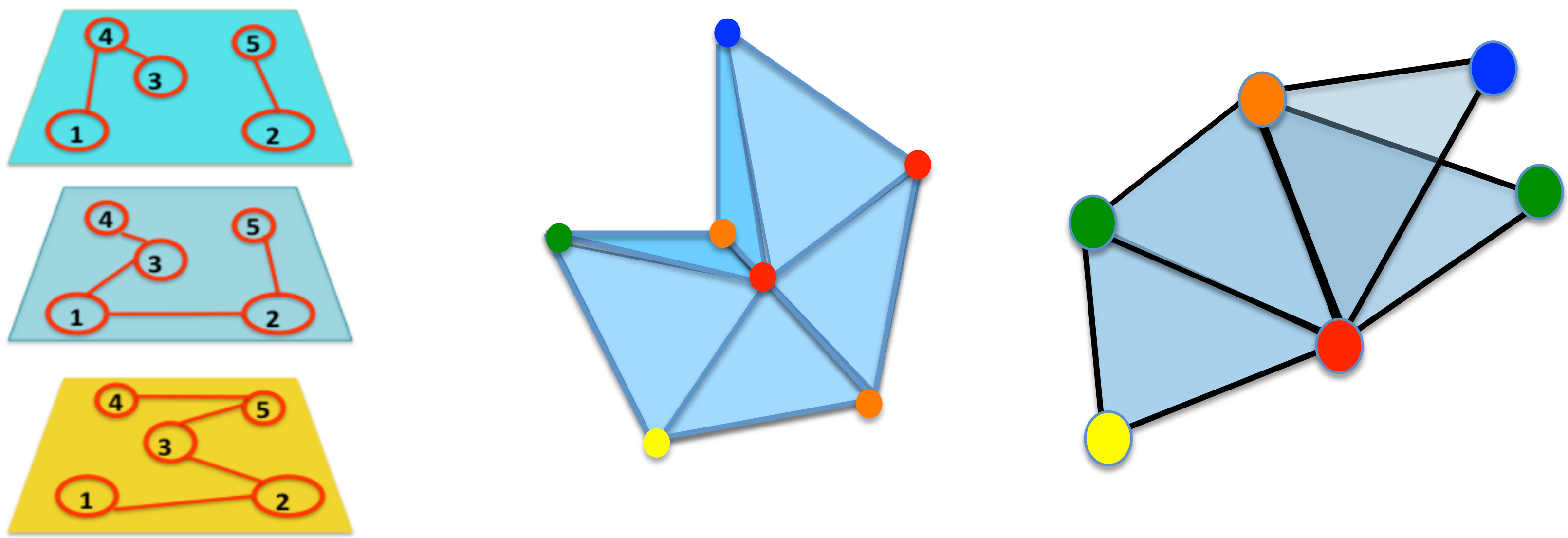
From
S. Fortunato
RMP

The role of dimensionality in neuronal dynamics



Uloa Severino et al. Scientific Reports (2016)

Generalized network structures



Going beyond the framework of simple networks

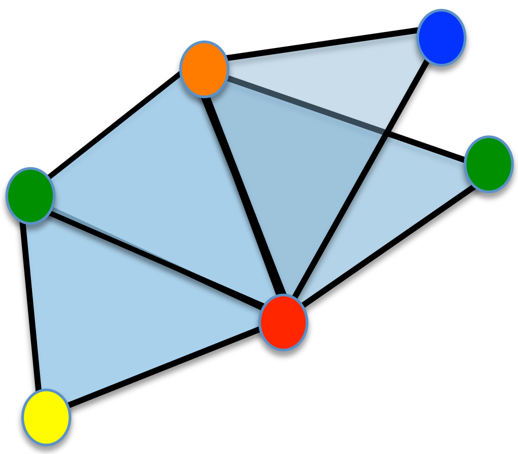
is of fundamental importance

for understanding the relation between structure and dynamics in

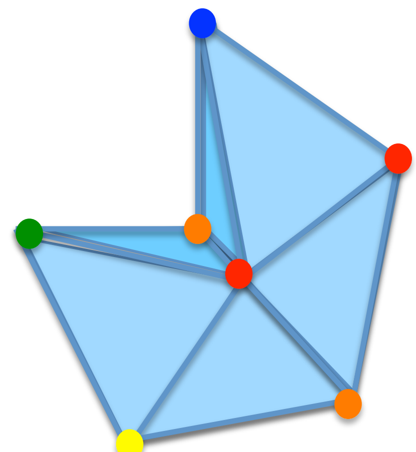
complex systems

Simplicial Complexes

Simplicial complexes are characterizing the interaction between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.

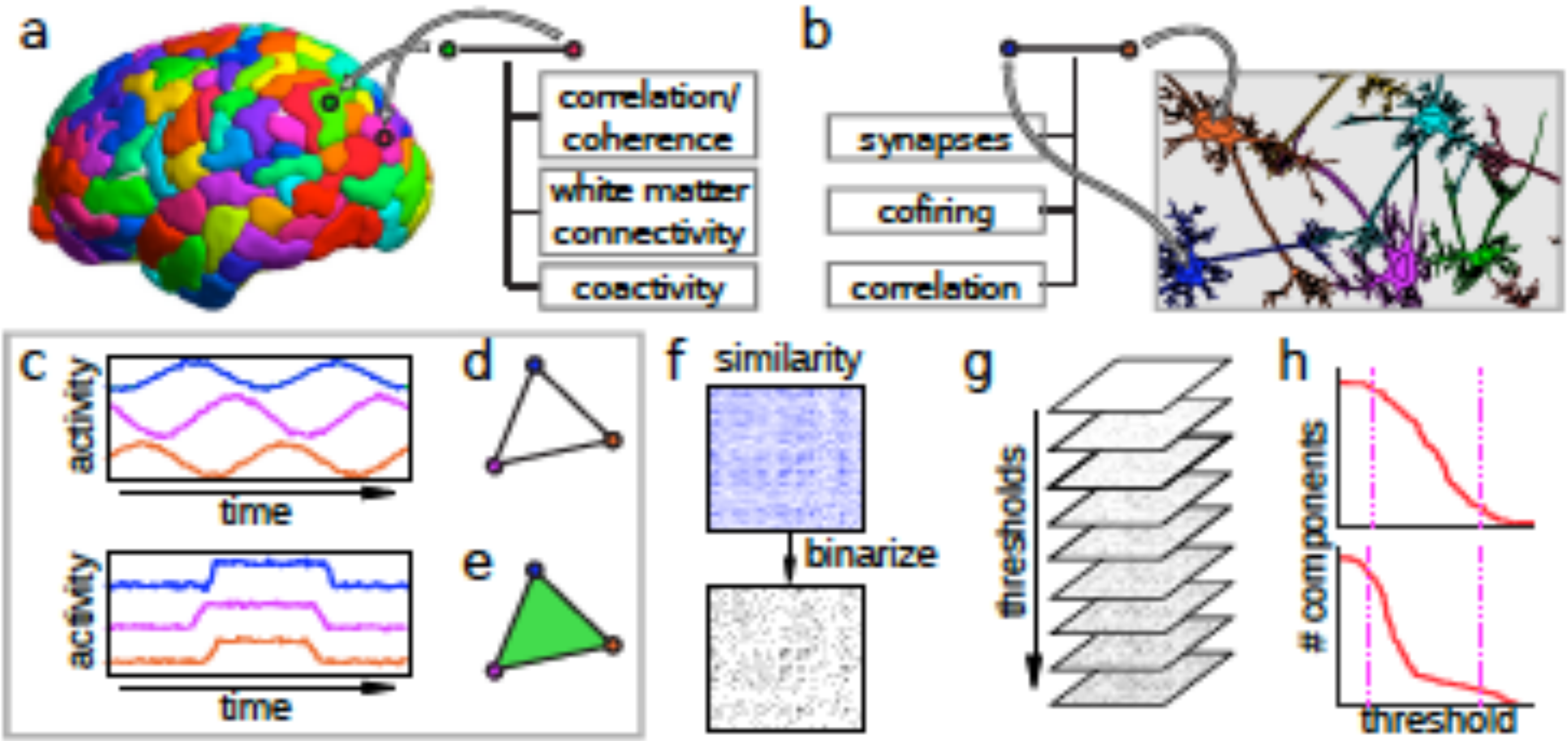


d=2 simplicial complex



d=3 simplicial complex

Brain data as simplicial complexes

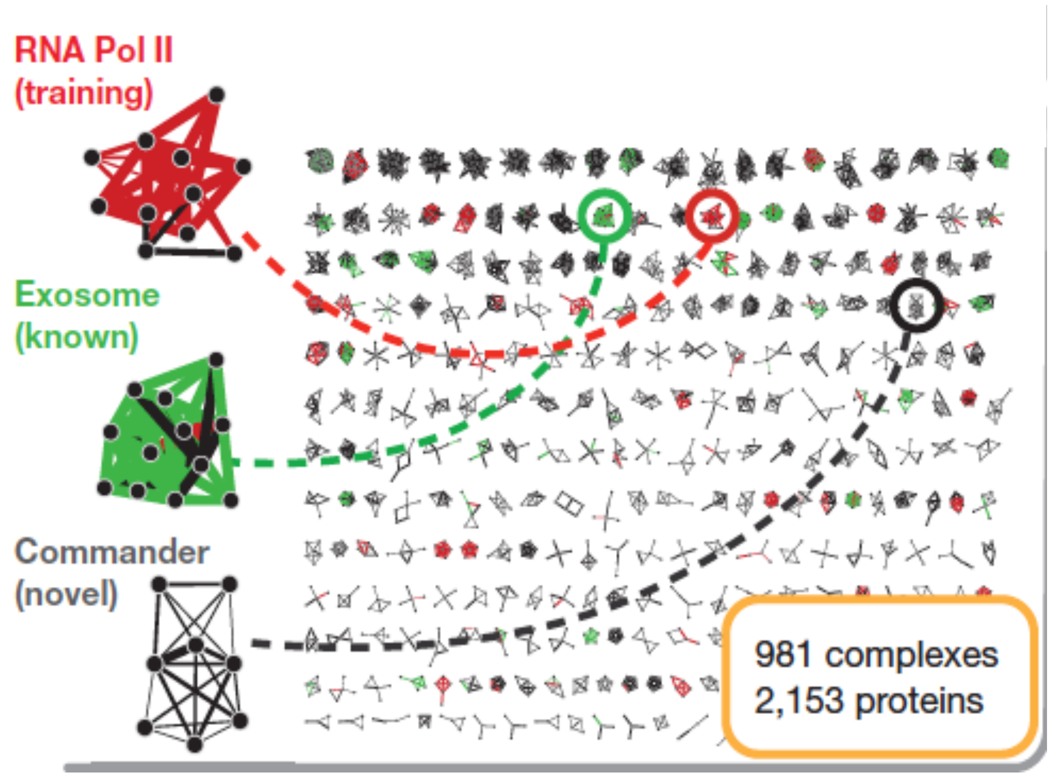


Giusti et al (2016)

Protein interaction networks as simplicial complexes

Protein interaction networks

- Nodes: proteins
- Simplices: protein complexes



Wan et al. Nature 2015

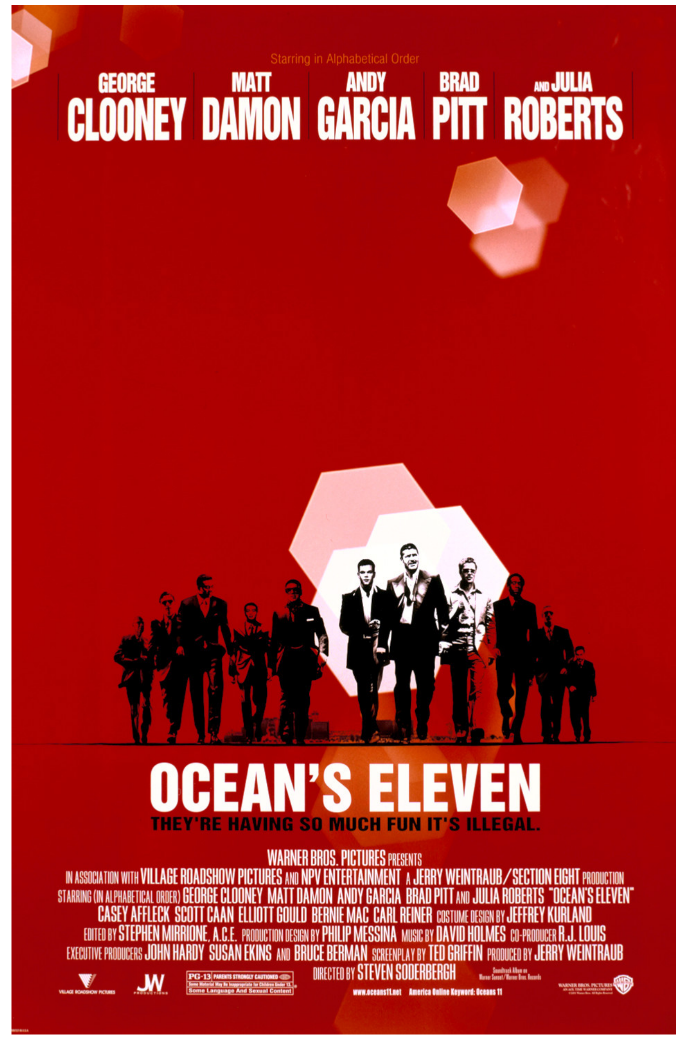
Collaboration networks as simplicial complexes

Actor collaboration networks

- Nodes: Actors
- Simplicies: Co-actors of a movie

Scientific collaboration networks

- Nodes: Scientists
- Simplicies: Co-authors



The hidden metric

of complex networks

It is believed that most complex networks have
an hidden metric
such that
the nodes
close in the hidden metric
are more likely to be linked to each other.

Emergent geometry

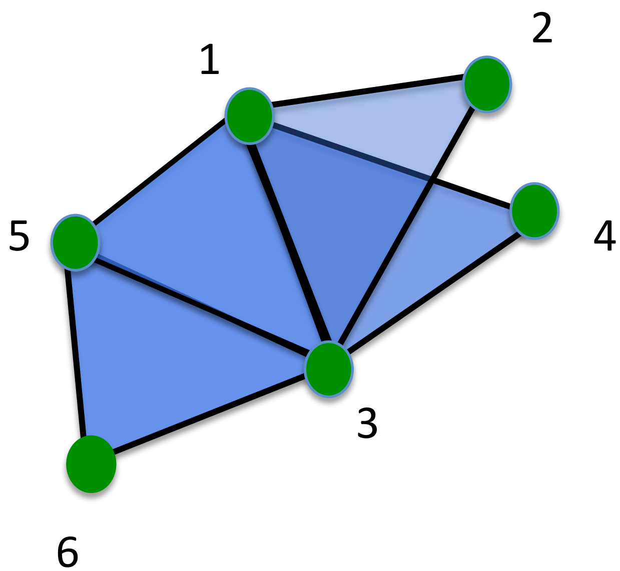
In the framework of emergent geometry
networks with hidden geometry
are generated
by equilibrium or non-equilibrium dynamics
that makes no use of the
hidden geometry

Growing networks describe the emergence of scale-free networks

Would growing simplicial complexes describe
the emergence of hyperbolic complex network geometry?

Generalized degree

The generalized degree $k_{d,\delta}(\mu)$ of a δ-face μ in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the δ-face μ.

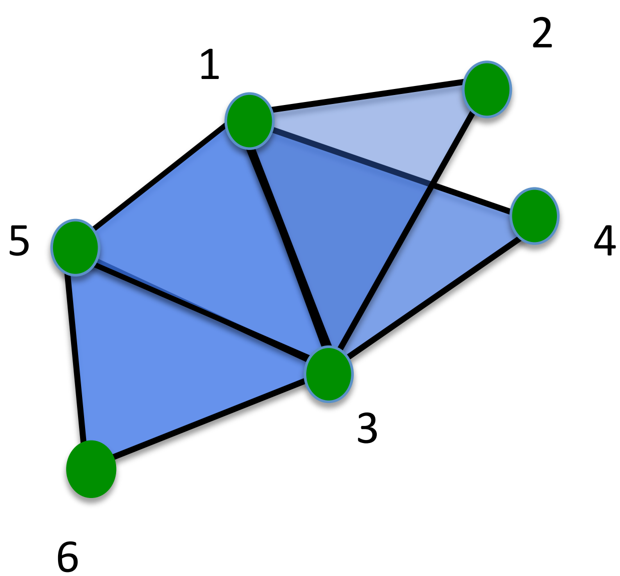


$k_{2,0}(\mu)$ Number of triangles incident to the node μ

$k_{2,1}(\mu)$ Number of triangles incident to the link μ

Generalized degree

The generalized degree $k_{d,\delta}(\mu)$ of a δ-face μ in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the δ-face μ.



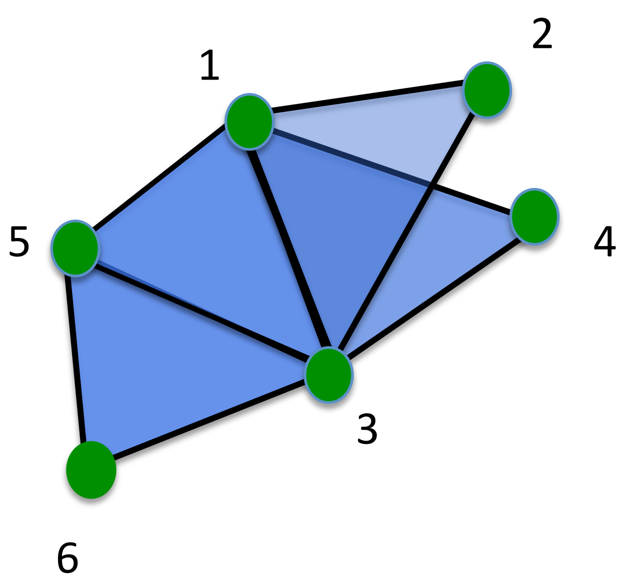
i	$k_{2,0}(i)$
1	3
2	1
3	4
4	1
5	2
6	1

(i,j)	$k_{2,1}(i,j)$
(1,2)	1
(1,3)	3
(1,4)	1
(1,5)	1
(2,3)	1
(3,4)	1
(3,5)	2
(3,6)	1
(5,6)	1

Incidence number

To each (d-1)-face μ we associate the incidence number

$$n_\mu = k_{d,d-1}(\mu) - 1$$

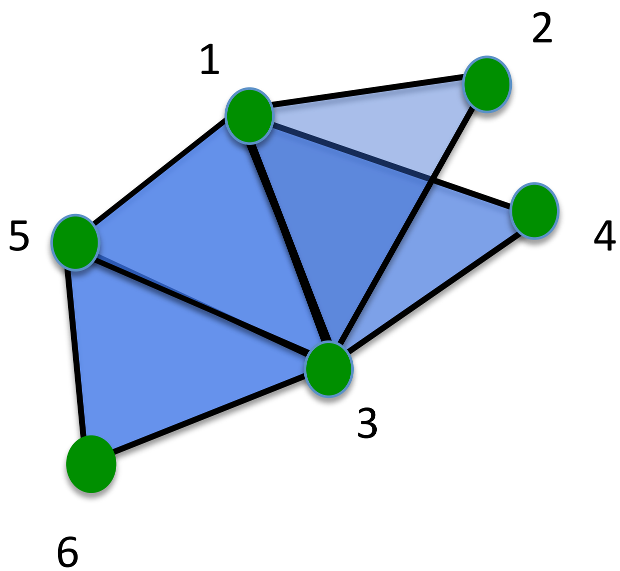


(i,j)	$n_{(i,j)}$
(1,2)	0
(1,3)	2
(1,4)	0
(1,5)	0
(2,3)	0
(3,4)	0
(3,5)	1
(3,6)	0
(5,6)	0

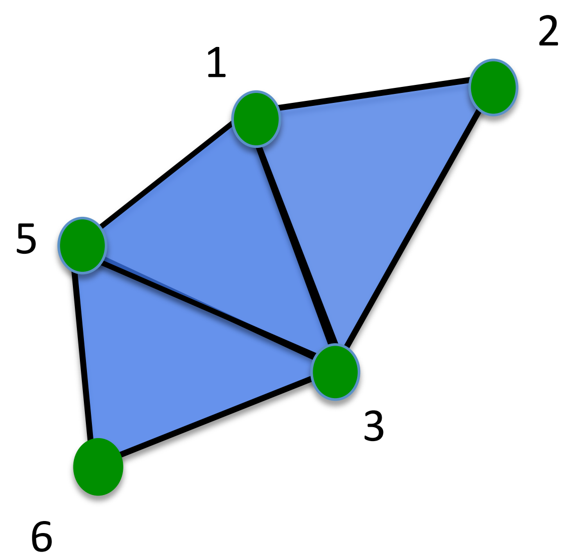
Manifolds

If n_μ takes only values $n_\mu=0,1$ each (d-1)-face is incident at most to two d-dimensional simplices.

In this case the simplicial complex is a discrete manifold.



NOT A MANIFOLD



MANIFOLD

Network Geometry with Flavor

Starting from a single d-dimensional simplex

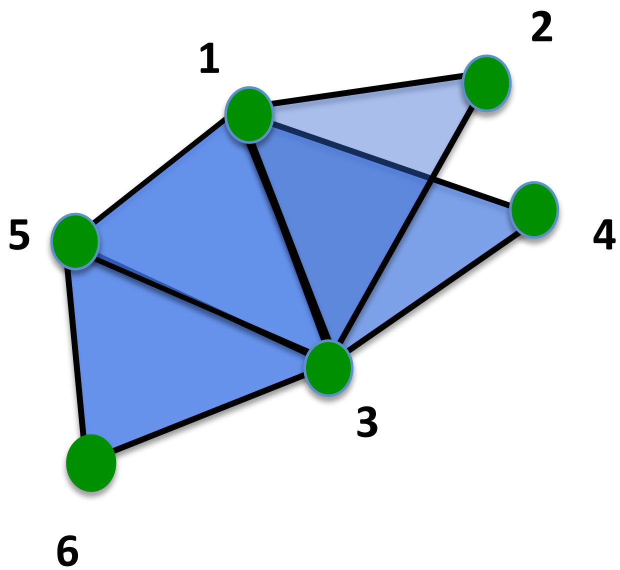
(1) GROWTH :

At every timestep we add a new d simplex

(formed by one new node and an existing (d-1)-face).

(2) ATTACHMENT:

The probability that a new node will be connected to a face μ depends on the flavor s=-1,0,1 and is given by



$$\Pi_{\mu}^{[s]} = \frac{1 + s n_{\mu}}{\sum_{\mu'} (1 + s n_{\mu'})}$$

Bianconi & Rahmede (2016)

Attachment probability

$$\Pi_{\mu}^{[s]} = \frac{(1 + s\, n_{\mu})}{\sum_{\mu' \in Q_{d,d-1}} (1 + s\, n_{\mu'})} = \begin{cases} \dfrac{(1 - n_{\mu})}{Z^{[-1]}}, & s = -1 \\ \dfrac{1}{Z^{[0]}}, & s = 0 \\ \dfrac{k_{\mu}}{Z^{[1]}}, & s = 1 \end{cases}$$

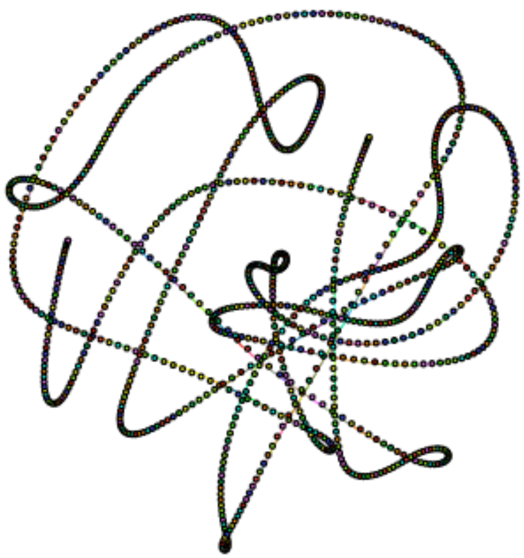
s=-1 Manifold n_{μ}**=0,1**

s=0 Uniform attachment n_{μ}**=0,1,2,3,4…**

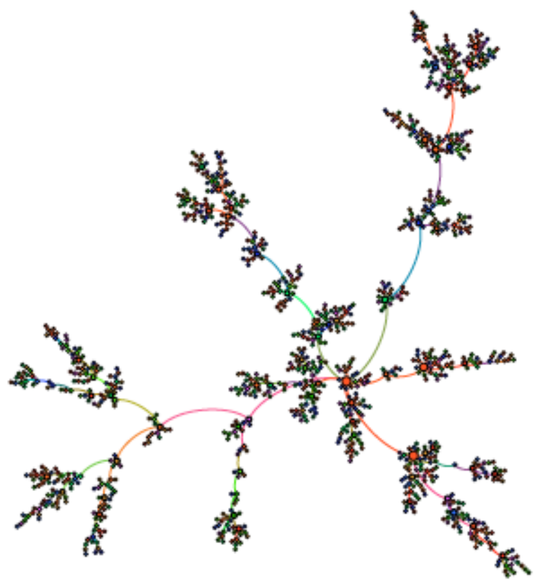
s=1 Preferential attachment n_{μ}**=0,1,2,3,4…**

Dimension d=1

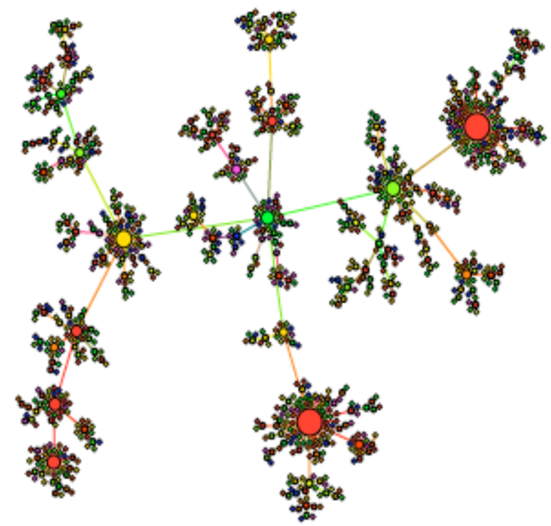
Manifold | **Uniform attachment** | **Preferential attachment**



Chain

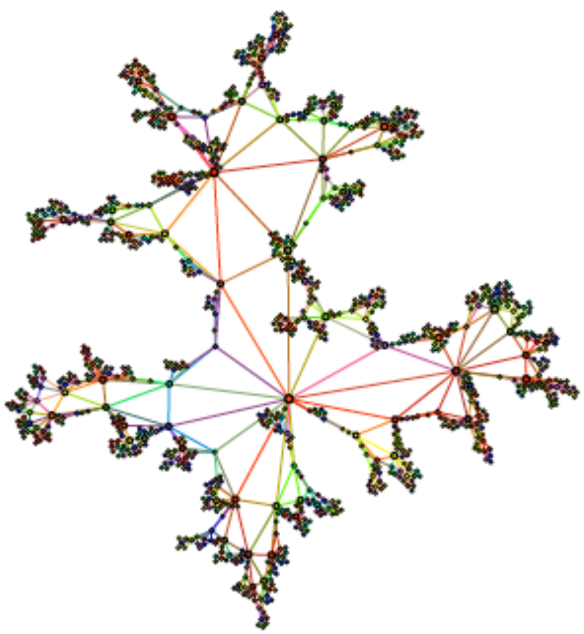
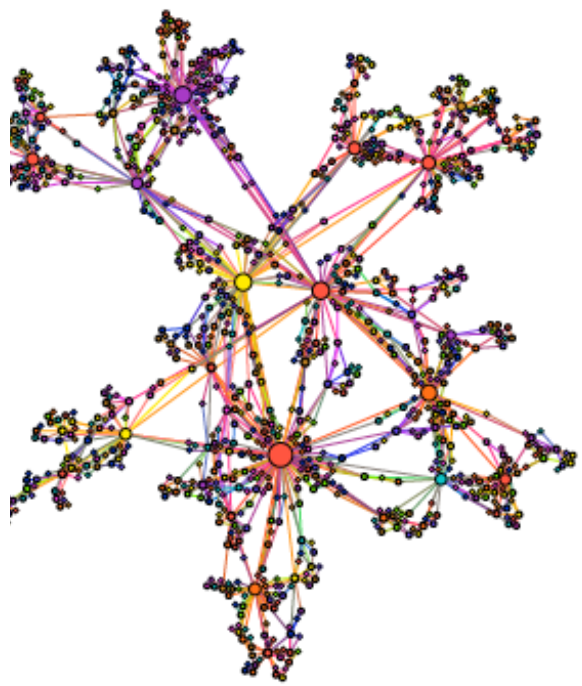
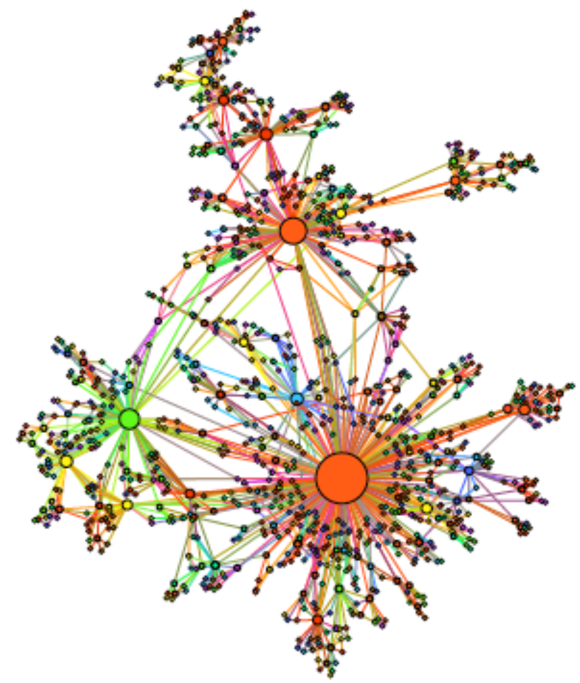


Exponential



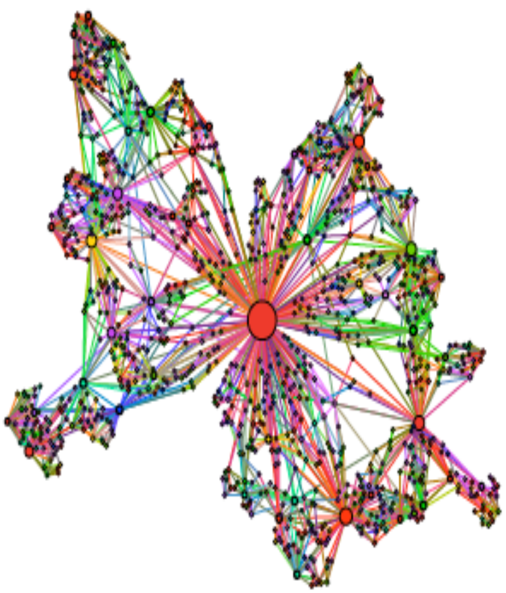
Scale-free BA model

Dimension d=2

Manifold	Uniform attachment	Preferential attachment
		
Exponential	Scale-free	Scale-free

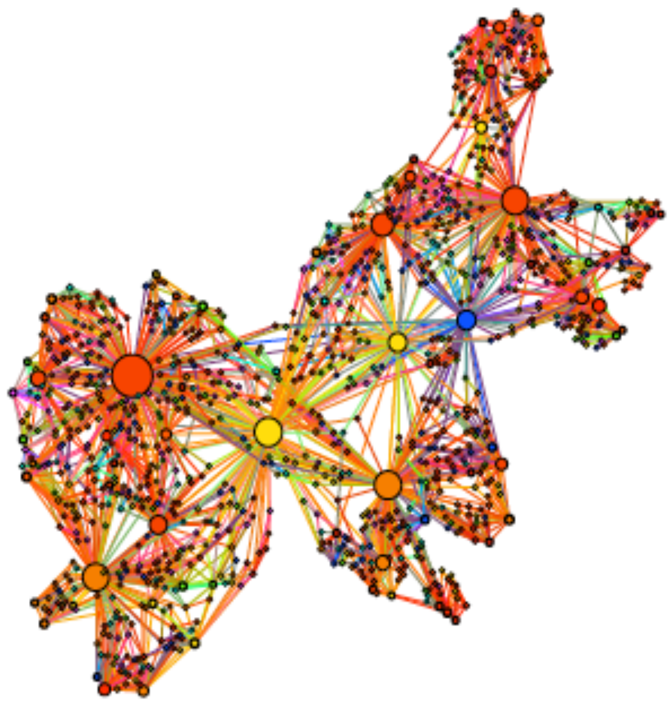
Dimension d=3

Manifold



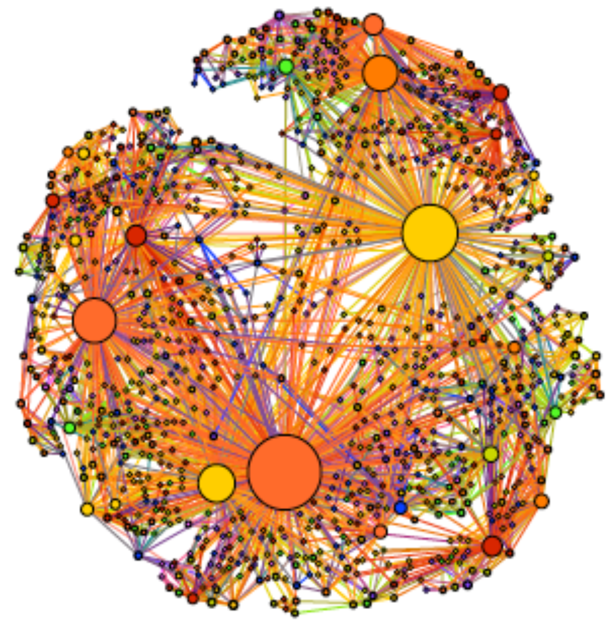
Scale-free

Uniform attachment



Scale-free

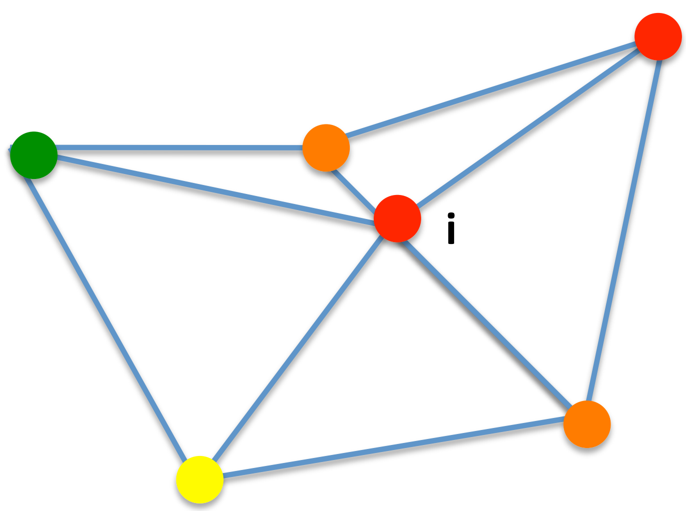
Preferential attachment



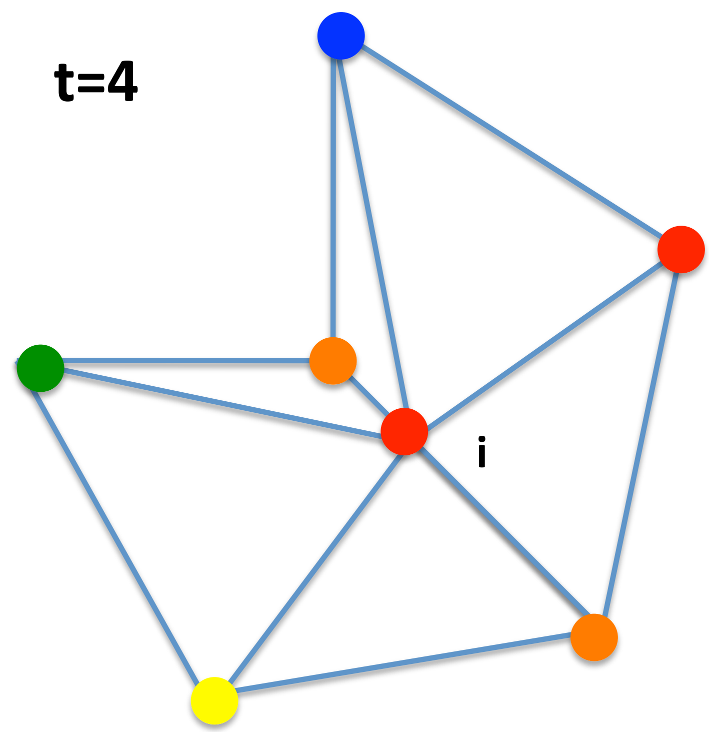
Scale-free

Effective preferential attachment in d=3

t=3



t=4



Node i has generalized degree 3

Node i is incident to 5 unsaturated faces

Node i has generalized degree 4

Node i is incident to 6 unsaturated faces

Degree distribution

For d+s=1

$$P_d(k) = \left(\frac{d}{d+1}\right)^{k-d} \frac{1}{d+1}$$

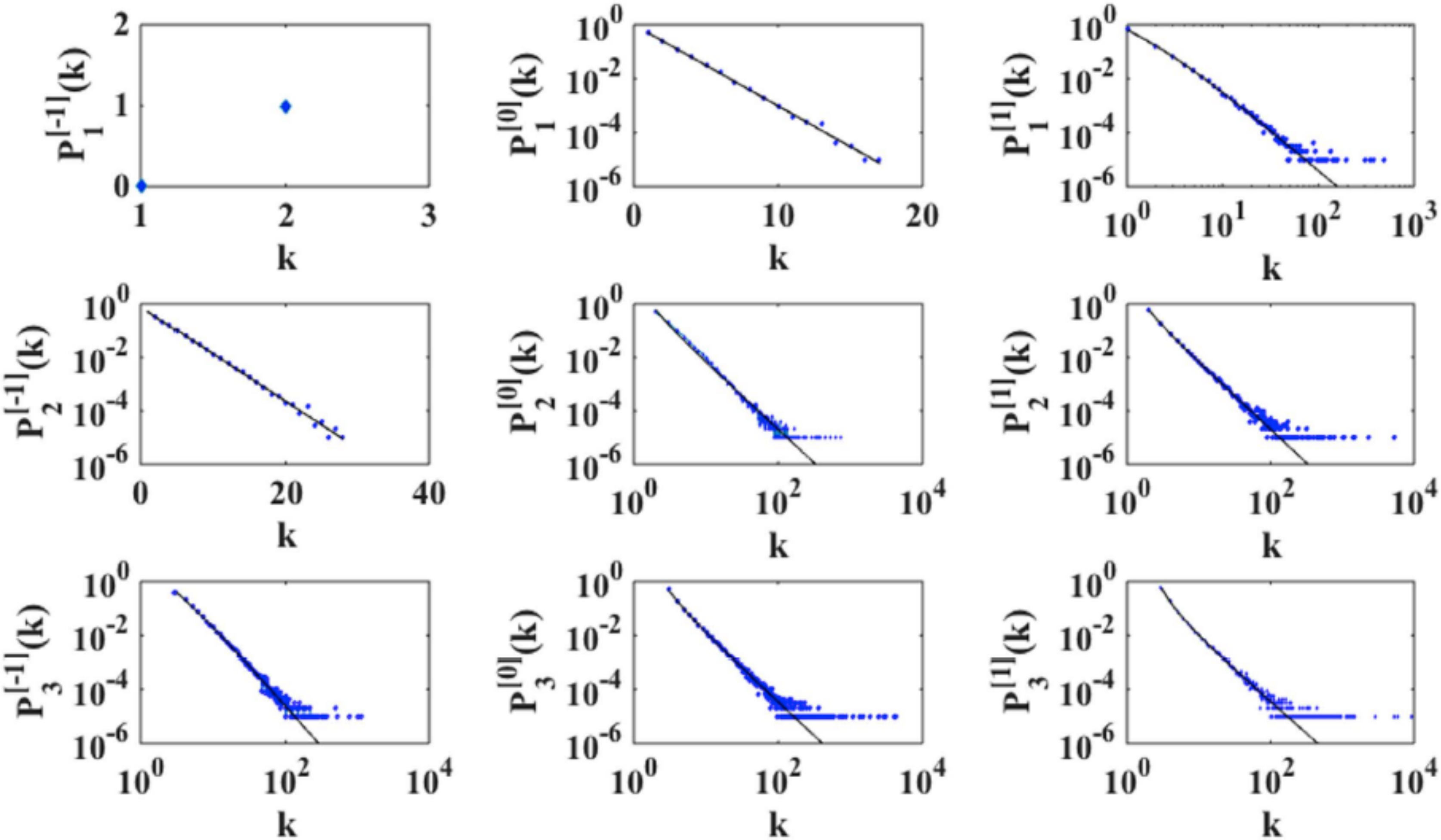
For d+s>1

$$P_d(k) = \frac{d+s}{2d+s} \frac{\Gamma(1+(2s+s)(d+s-1))}{\Gamma(d/(d+s-1))} \frac{\Gamma(k-d+d/(d+s-1))}{\Gamma(k-d+(2d+s)(d+s-1))}$$

NGF are always scale-free for d>1-s

- **For s=1 NGF are always scale free**
- **For s=0 and d>1 the NGF are scale-free**
- **For s=-1 and d>2 the NGF are scale-free**

Degree distribution of NGF



Generalized degree distributions

flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Bimodal	Exponential	Power-law
$\delta = d-2$	Exponential	Power-law	Power-law
$\delta \leq d-3$	Power-law	Power-law	Power-law

The power-law

generalized degree distribution

are scale-free for

$$\boxed{d \geq d_c^{[\delta,s]} = 2(\delta+1)+s}$$

Emergent community structure

Modularity

and

Clustering coefficient

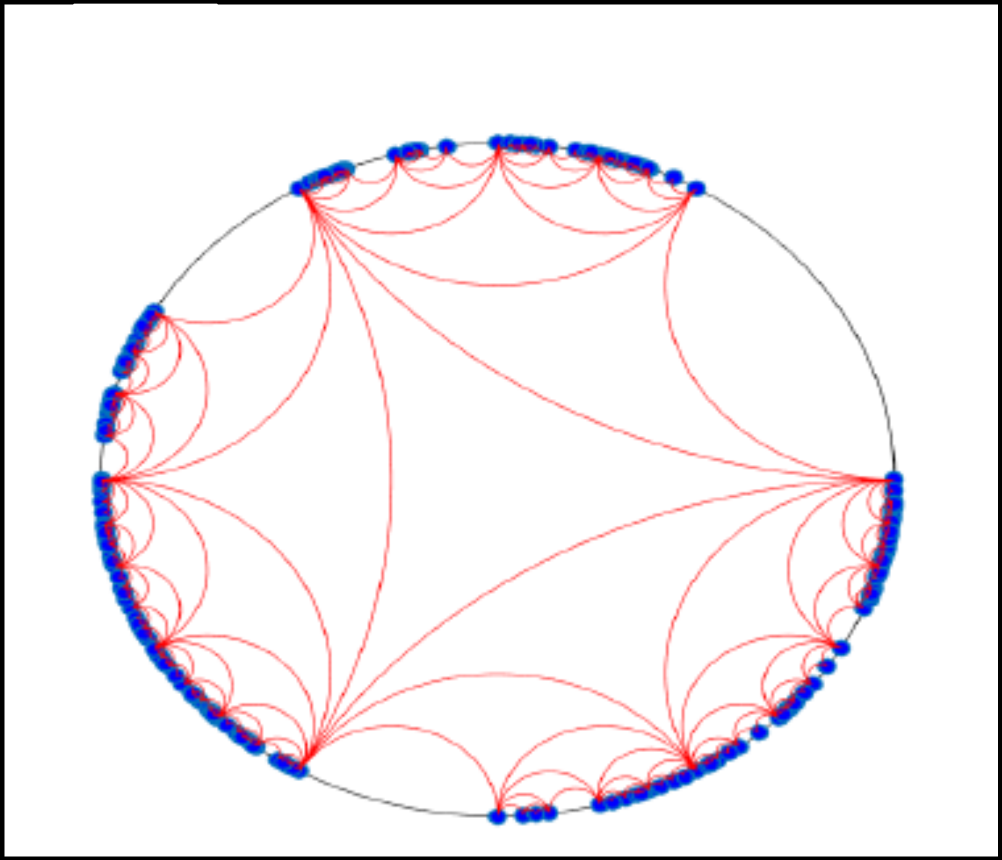
of NGF

M	$s=-1$	$s=0$	$s=1$	C	$s=-1$	$s=0$	$s=1$
$d=2$	0.97	0.94	0.90	$d=2$	0.65	0.74	0.79
$d=3$	0.91	0.85	0.80	$d=3$	0.77	0.81	0.84

Emergent Hyperbolic geometry

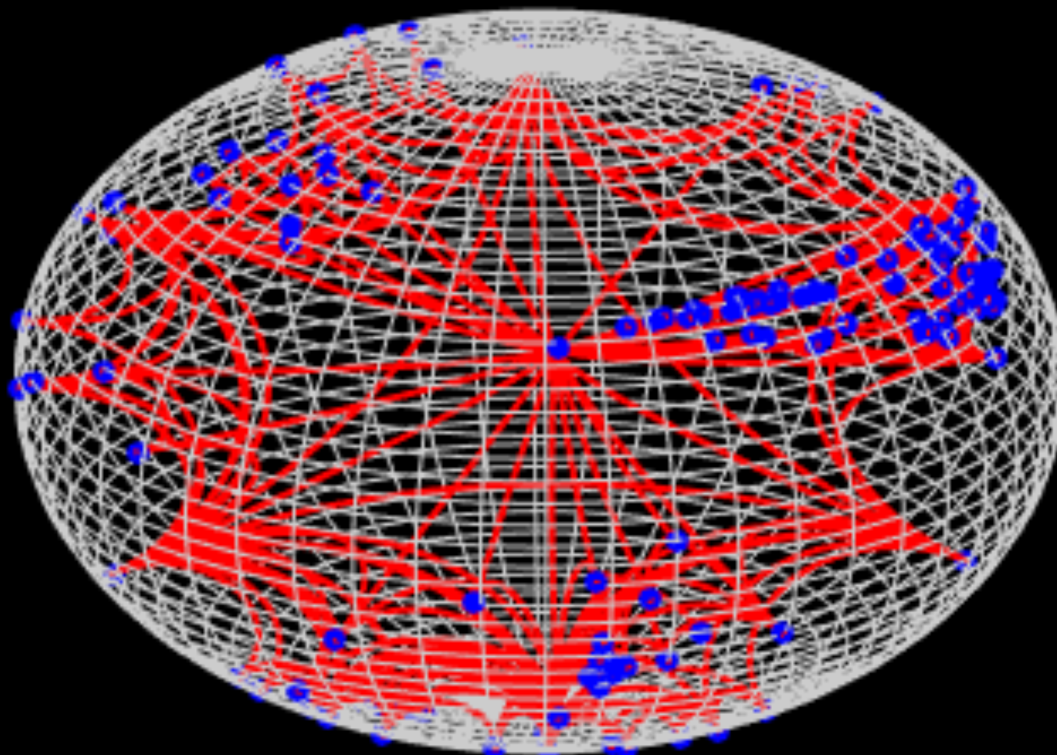
The emergent hidden geometry is the hyperbolic H^d space

Here all the links have equal length



d=2

Emergent hyperbolic geometry

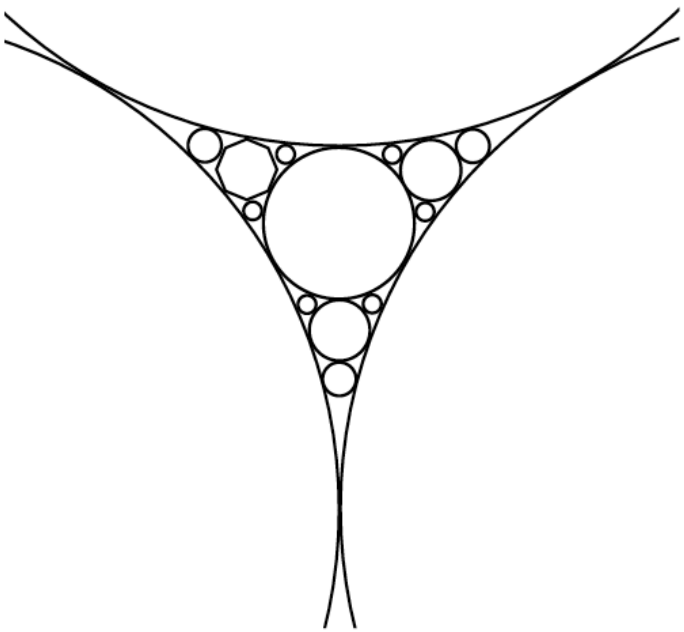


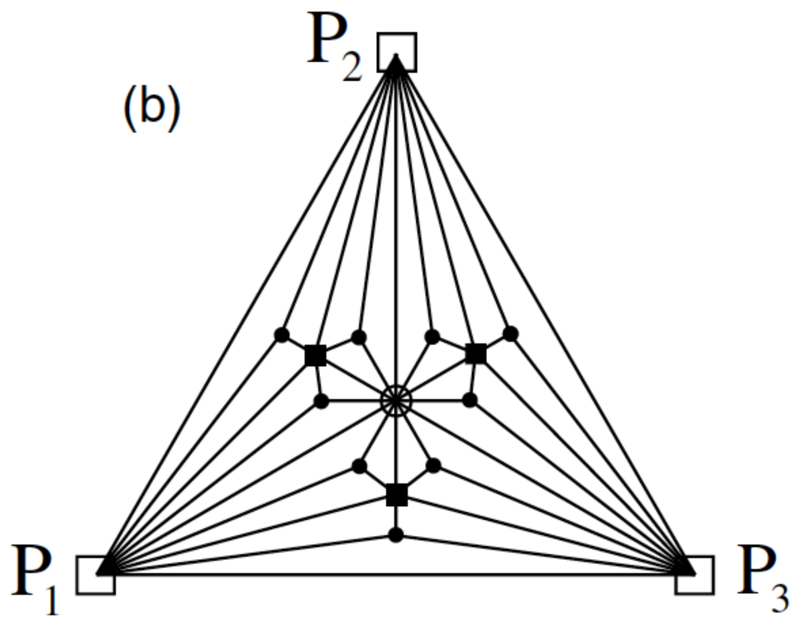
d=3

Apollonian networks

Apollonian networks are formed by linking the centers of an Apollonian sphere packing

They are scale-free and are described by the Lorentz group

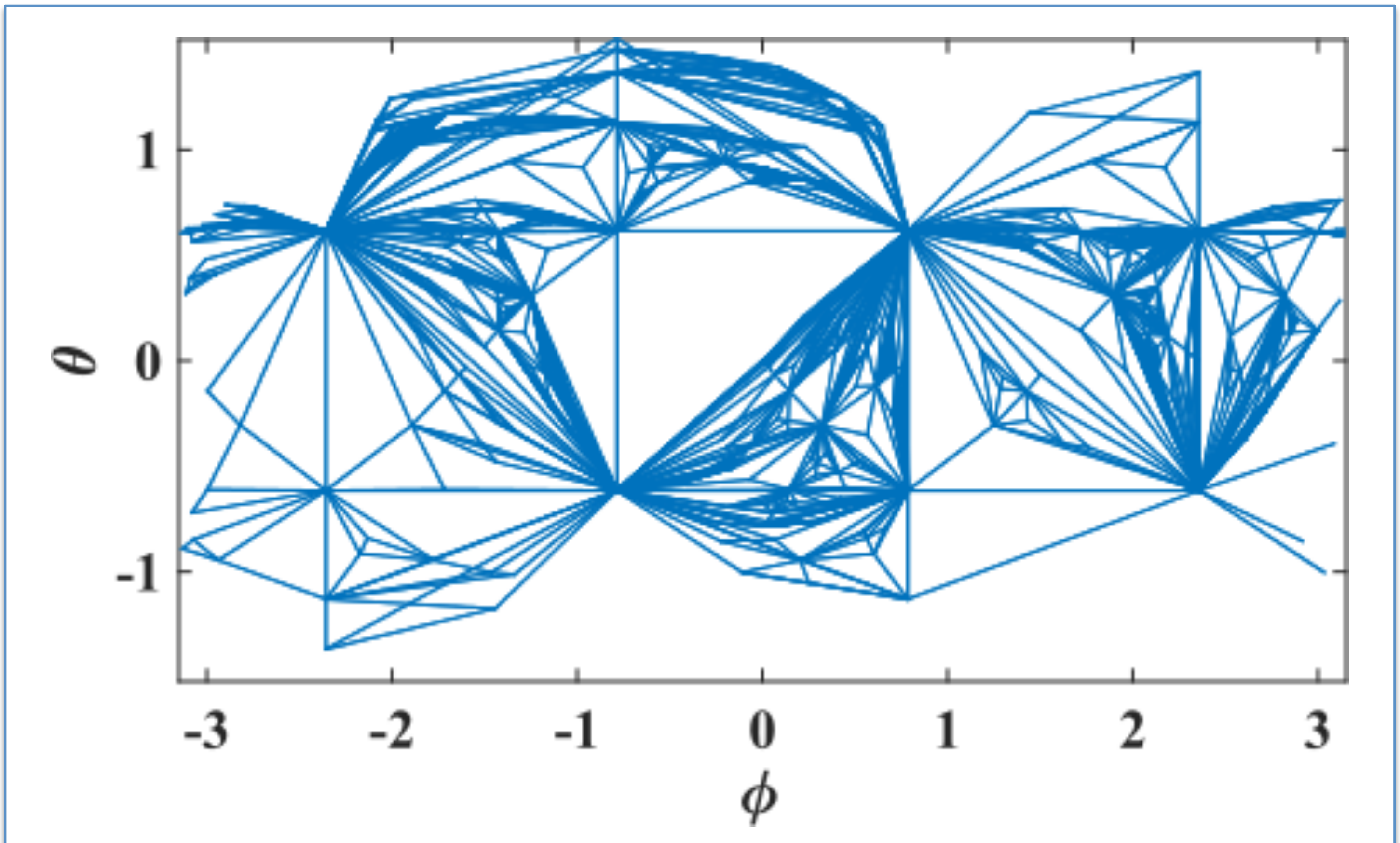




Andrade et al. PRL 2005

Soderberg PRA 1992

Connection with the Apollonian network

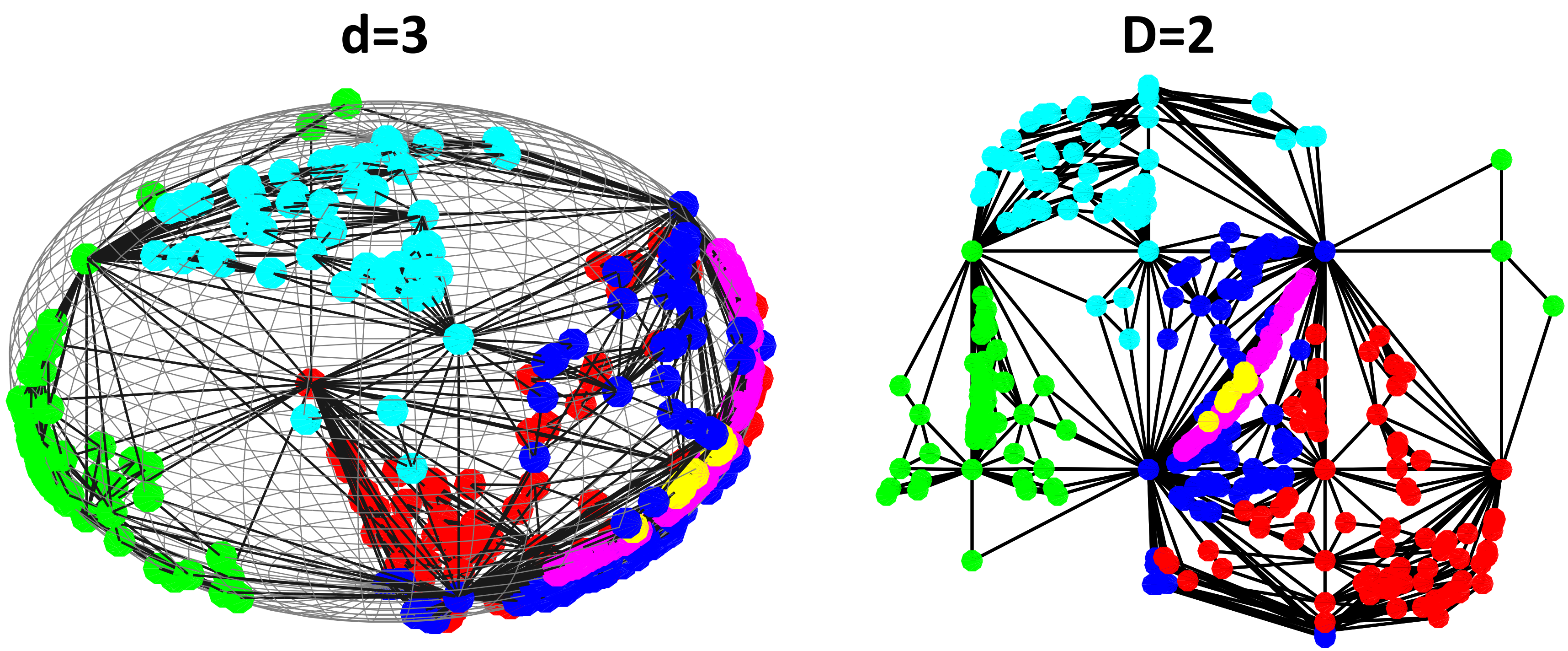


Complex Network Manifolds

And

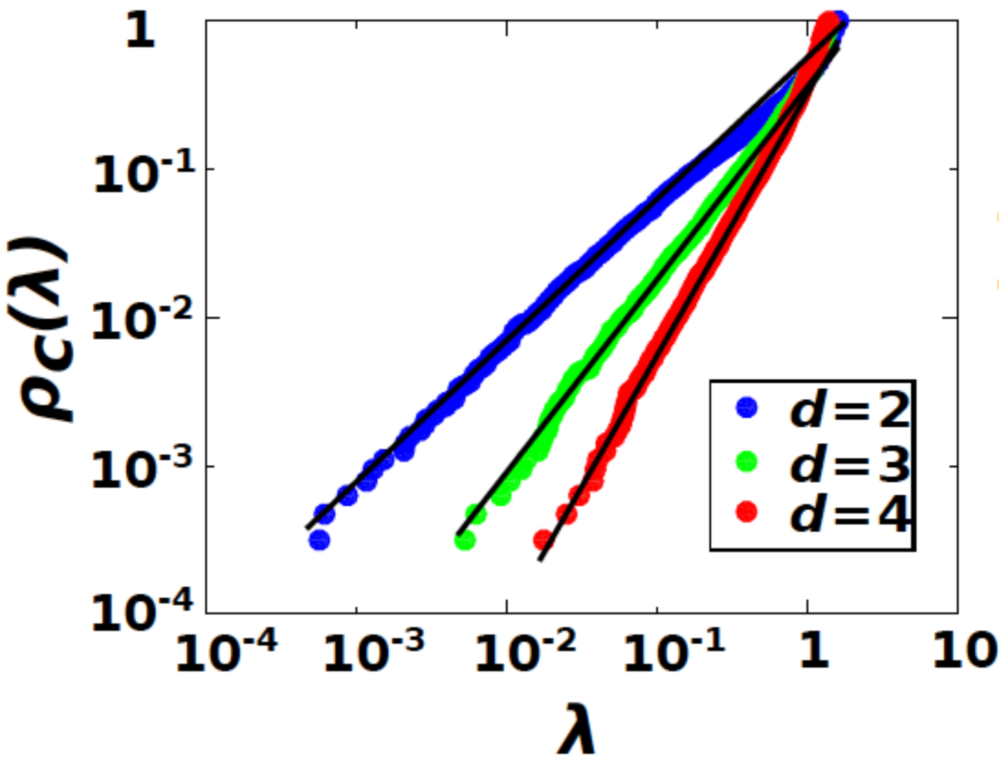
Frustrated Synchronization

Holography of Complex Network Manifolds



d-dimensional Complex Network Manifolds can be interpreted as D-dimensional manifolds with D=d-1

Spectral dimensions of Complex Network Manifolds



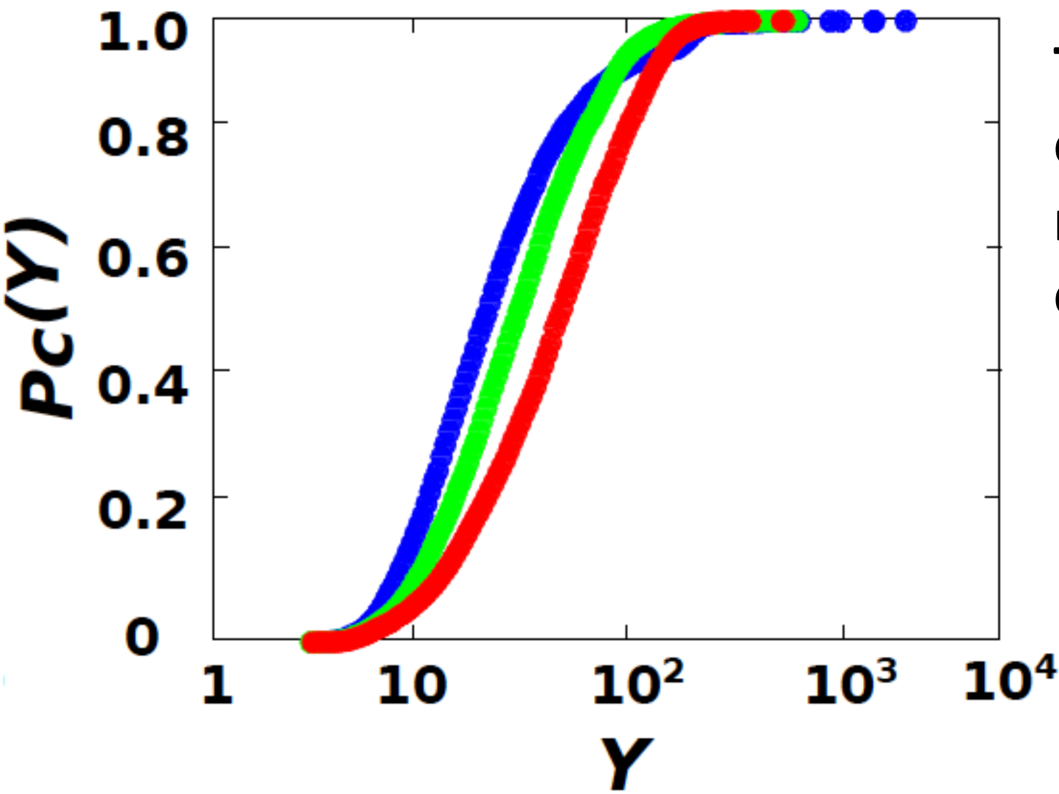
$$L_{ij} = \delta_{ij} - \frac{a_{ij}}{k_i}$$

$$\rho_c(\lambda) \approx \lambda^{-d_s/2}$$

Complex Network Manifolds have finite spectral dimension with

$$d_s \approx d \;\; for \;\; d = 2,3,4$$

Localization of the eigenvectors



The participation ratio evaluates the effective number of nodes on which an eigenmode is localized

$$Y_\lambda = \left[\sum_{i=1}^{N} \left(u_i^\lambda v_i^\lambda \right)^2 \right]^{-1}$$

A large number of eigenmodes are localized

The Kuramoto model

We consider the Kuramoto model

$$\boxed{\frac{d\vartheta_i}{dt} = \omega_i + \sigma \sum_{j=1}^{N} \frac{a_{ij}}{k_i} \sin\left(\vartheta_j - \vartheta_i\right)}$$

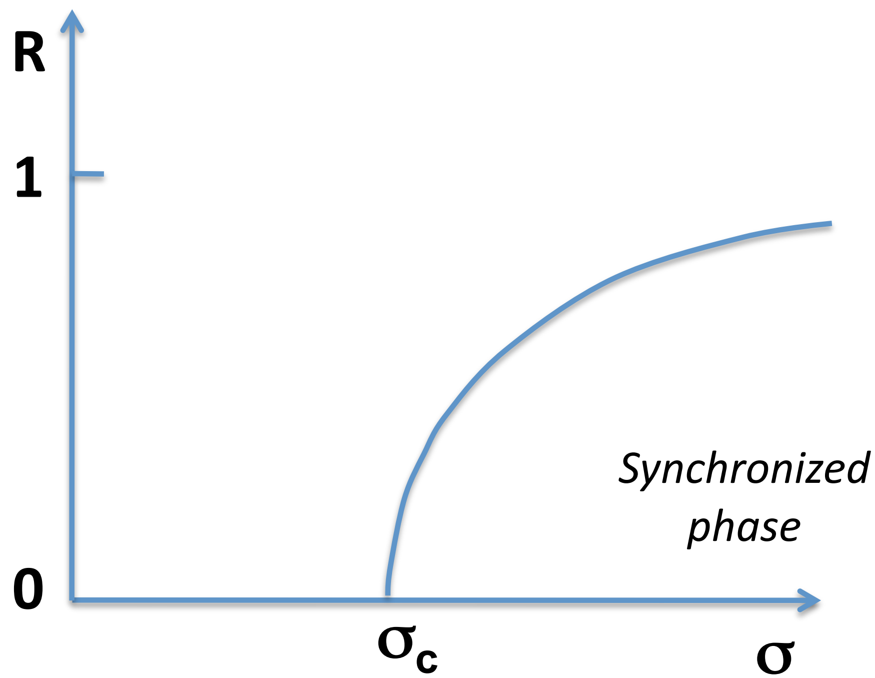
where ω_i is the internal frequency of node i drawn randomly from a Gaussian distribution

The global order parameter is

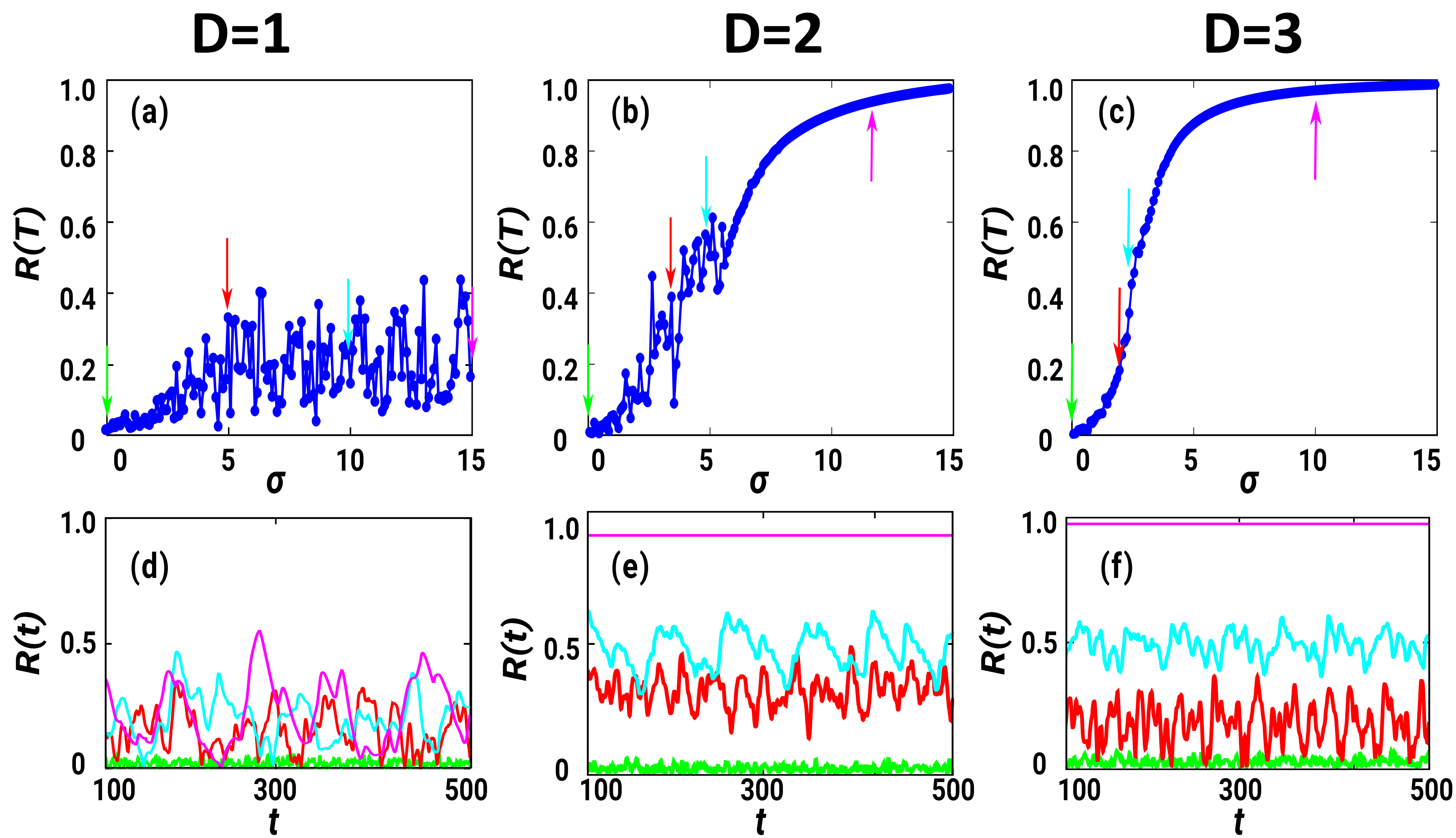
$$\boxed{R = \frac{1}{N} \sum_{j=1}^{N} e^{i\vartheta_j}}$$

Kuramoto Model

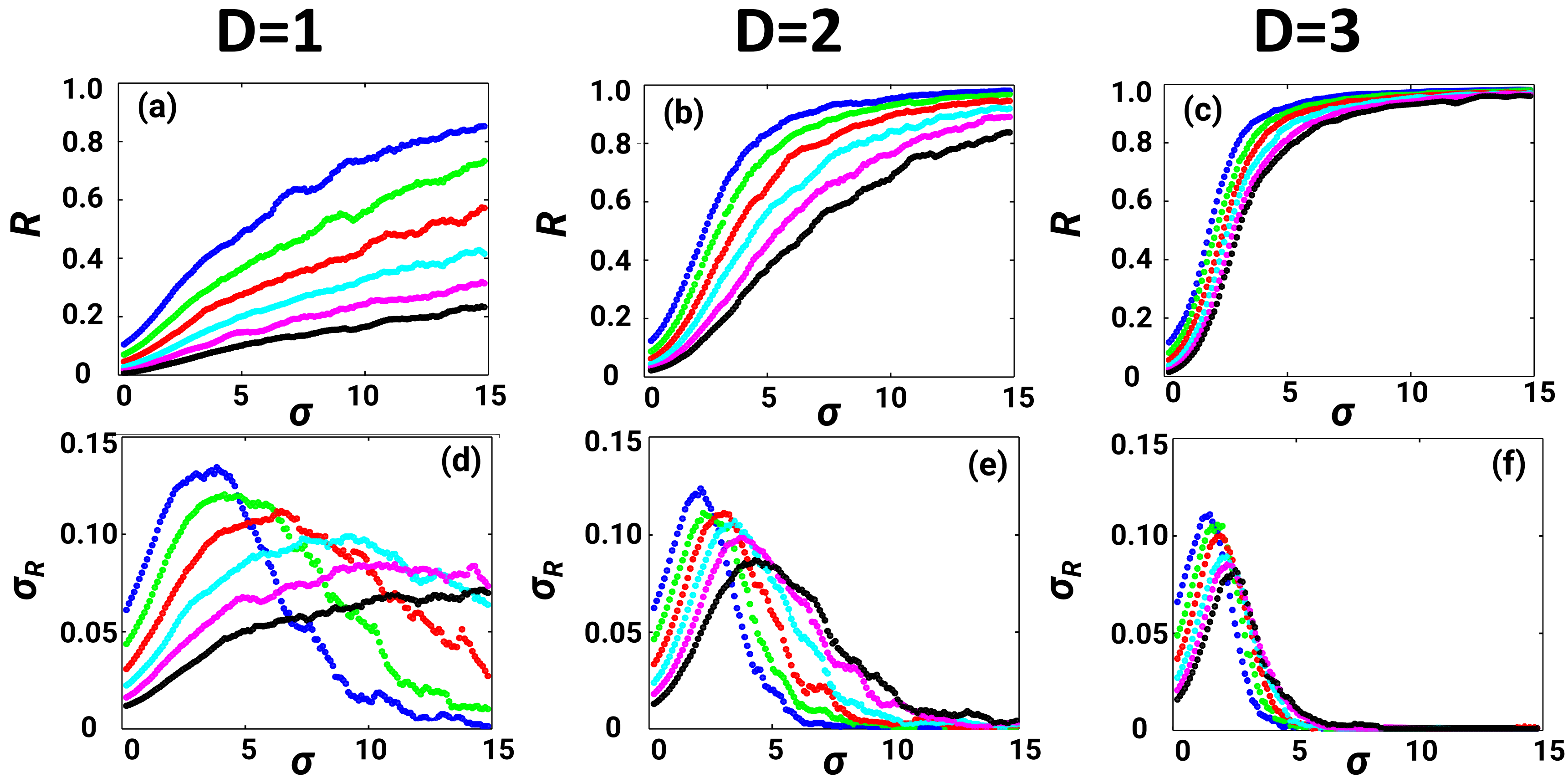
In an infinite fully connected network we have



Frustrated synchronization



Finite size effects



N=100,200,400,800,1600,3200

The finite size effects are less pronounced in larger dimensions

Fully synchronized phase and the spectral dimension

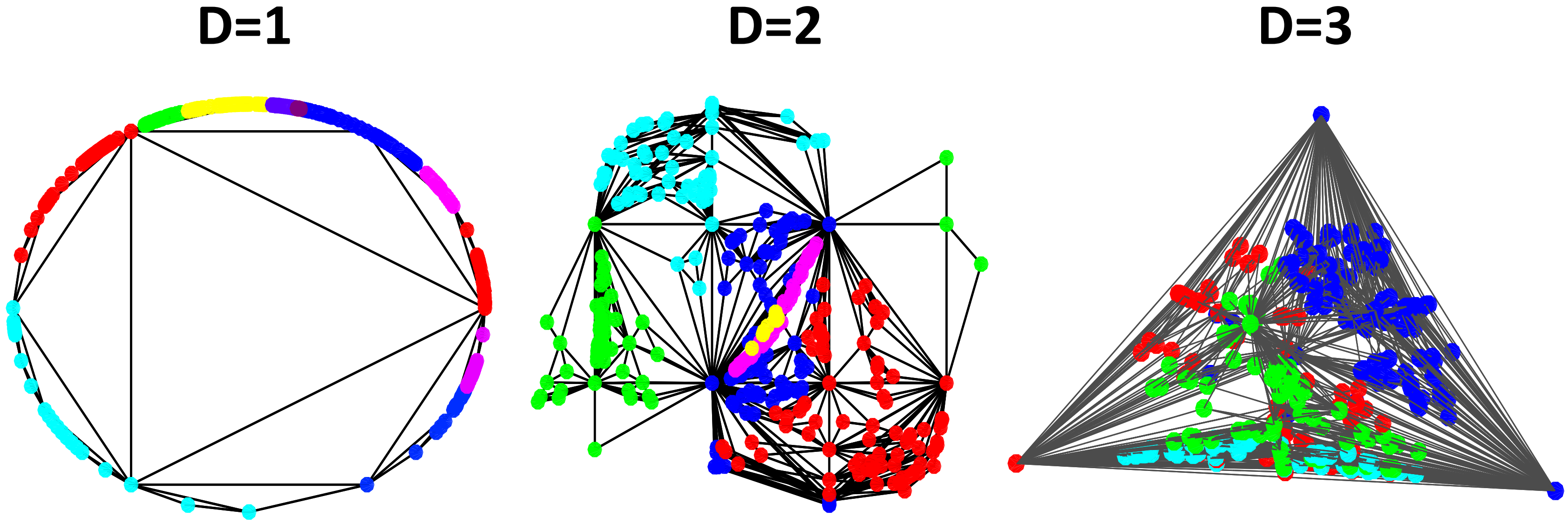
The fully synchronized phase is not thermodynamically achieved

for networks with spectral dimension

$$\boxed{d_s \leq 4}$$

In Complex Network Manifolds with D=3
the fully synchronized state is marginally stable

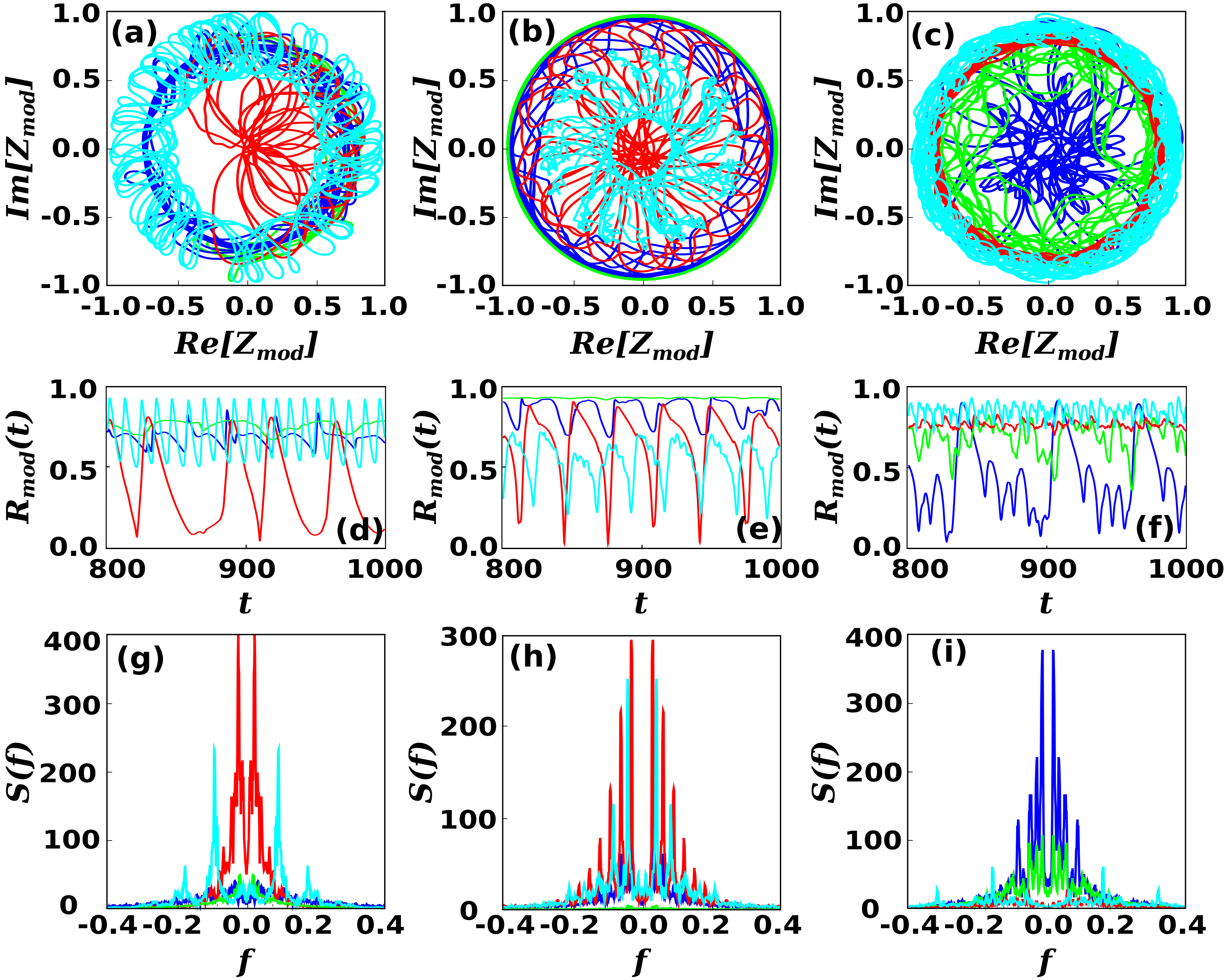
Frustrated synchronization and community structure



For every community with n_C nodes we can define the local order parameter

$$Z_{\mathrm{mod}} = R_{\mathrm{mod}} e^{i\psi_{\mathrm{mod}}} = \frac{1}{n_C} \sum_{j \in C}^{N} e^{i\vartheta_j}$$

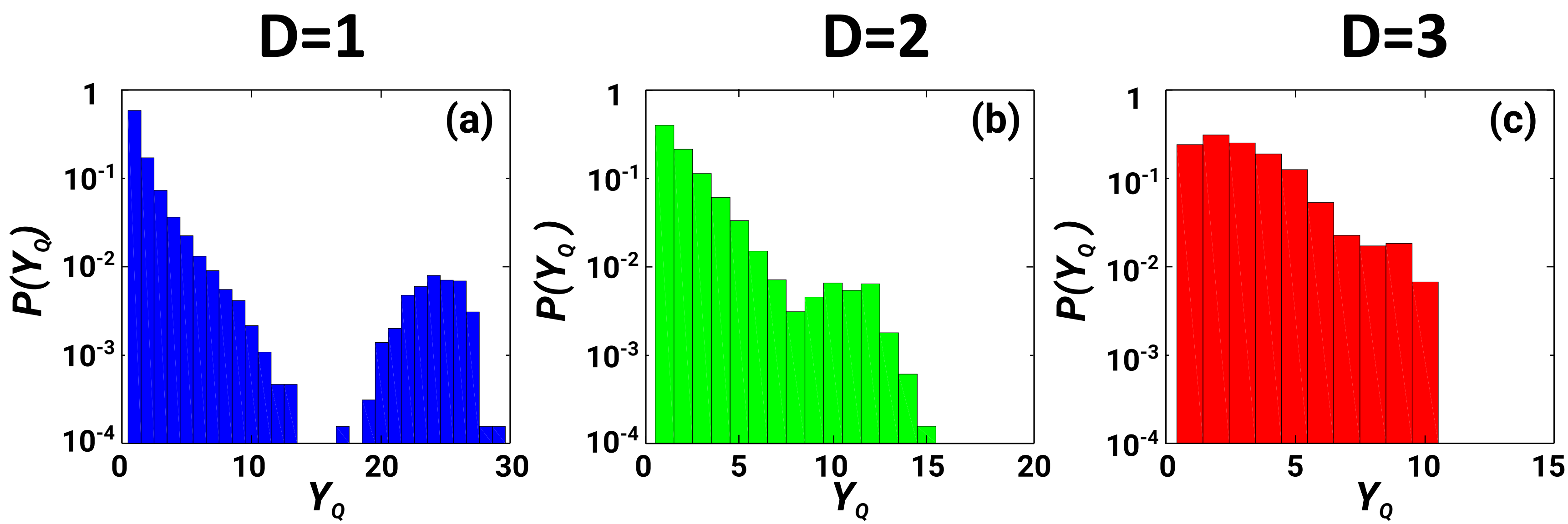
Communities and Frustrated Synchronization



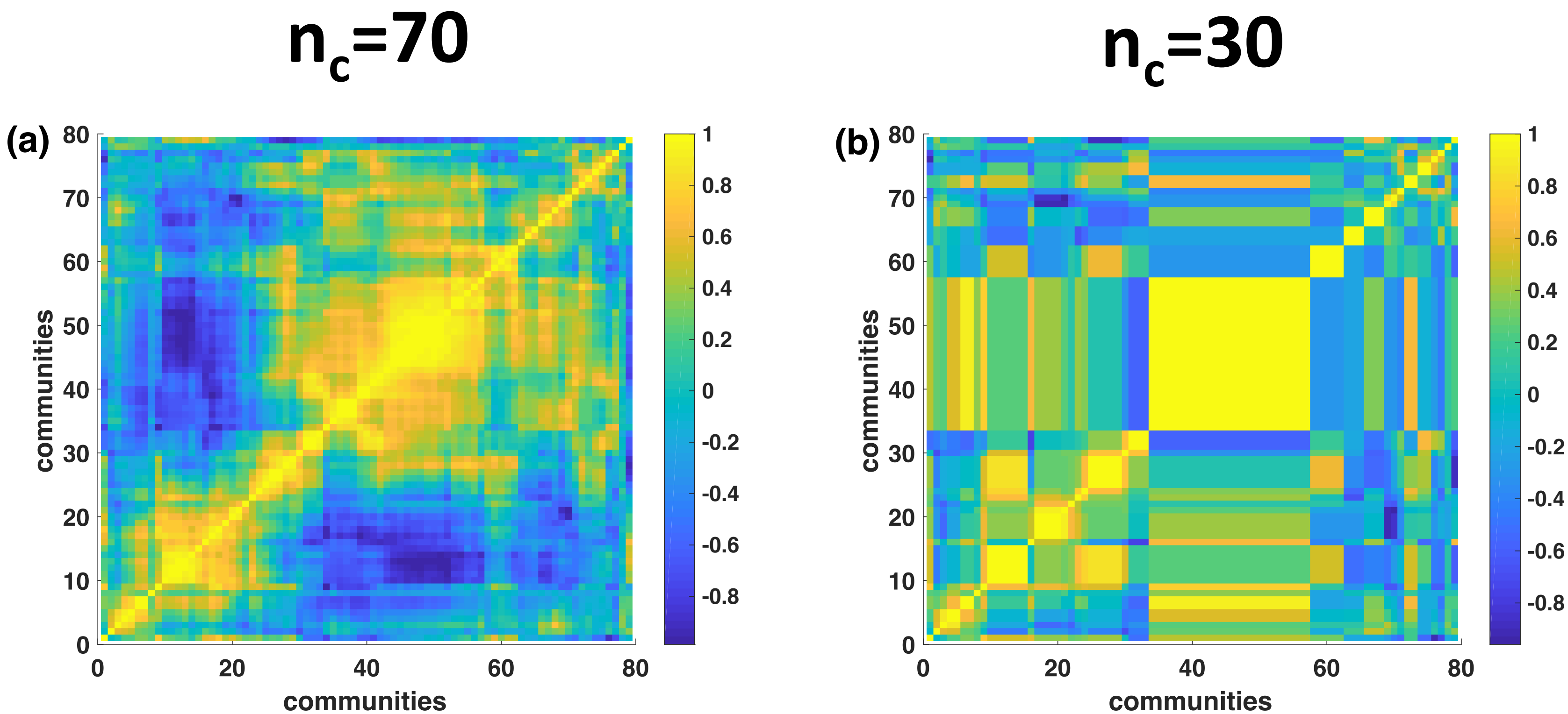
Localization of eigenvector on communities

$$Y_Q = \left[\sum_n \left(\sum_{i \in C_n} u_i^\lambda v_i^\lambda \right)^2 \right]^{-1}$$

measure in how many communities the eigenmode is localized



Correlations among communities and network coarse graining



N=1000, D=2, σ=5

Conclusions

Complex Network Manifolds can
help us understand the interplay between
Network Topology, Network Geometry and Synchronization dynamics

Complex Network Manifolds and Frustrated Synchronization

- *Complex Network Manifolds display Frustrated Synchronization with strong spatio-temporal fluctuations of the order parameter*
- *They combine small-world property and community structure like brain networks*
- *They show a strong dependence on the dimension with the fully synchronized state marginally stable in dimension D=3*

Collaborators and References

Emergent network geometry

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O. T. Courtney and G. Bianconi PRE 95, 062301 (2017)
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Frustrated synchronization in Complex Network Manifolds

A.P. Millan, J. Torres and G. Bianconi Scientific Reports 8, 9910 (2018)

Ensembles of simplicial complexes

O. T. Courtney and G. Bianconi PRE *93*, 062311 (2016)

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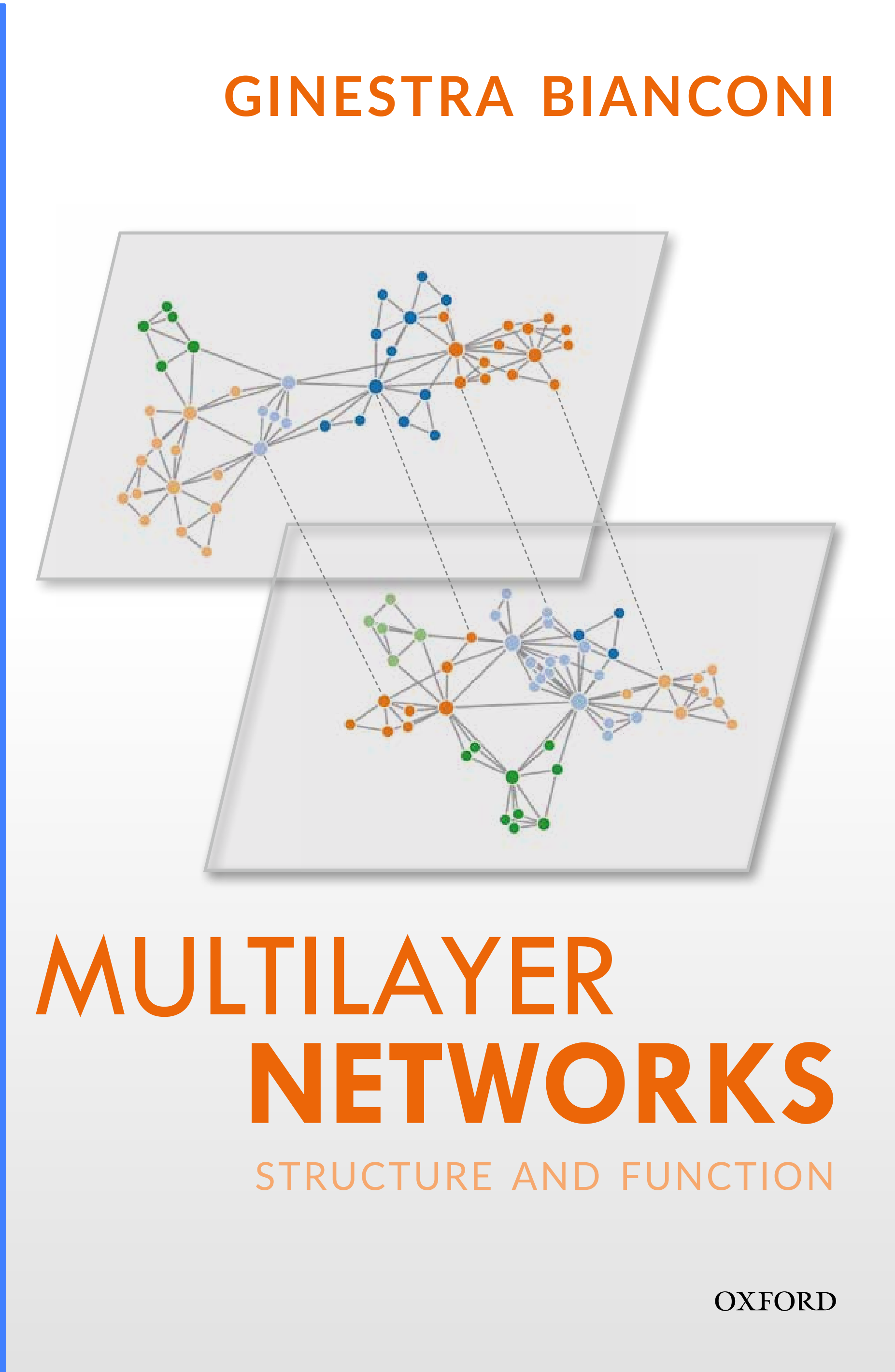

Structure and Function
by
Ginestra Bianconi
Queen Mary University of London

•Pedagogical presentation

•Discussion of general concepts in terms of their impact on interdisciplinary applications