

SINGULAR LEARNING THEORY FOR FACTOR ANALYSIS

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ABSTRACT. Watanabe’s singular learning theory provides a framework for asymptotic analysis of Bayesian model selection for statistical models with singularities, where traditional statistical regularity assumptions fail. Learning coefficients, also known as real log canonical thresholds, play a central role in singular learning, as they govern the asymptotic behavior of Bayesian marginal likelihood integrals in settings where the Laplace approximations used for regular statistical models are not applicable. Learning coefficients are algebraic invariants that quantify the geometric complexity of a model and reveal how the singular structure impacts the model’s generalization properties. In this paper, we apply algebraic methods to study the learning coefficients of factor analysis models, which are widely used latent variable models for continuously distributed data. Our main results provide a general upper bound for the learning coefficients as well as exact formulas for specific cases.

1. INTRODUCTION

A Bayesian approach to statistical model selection involves the evaluation of the *marginal likelihood*, which is obtained by integrating the likelihood function against the prior distribution [Rob07, Chap. 7]. In particular, the posterior probability of a considered model equals the normalization of the marginal likelihood under weighting by a prior model probability. As an exact value for the marginal likelihood integral is difficult to obtain in general, different approximation schemes have been developed [FW12]. The theme taken up in this paper is asymptotic approximation and, specifically, the widely used approach of *Bayesian information criteria* that originated in the work of [Sch78]. Bayesian information criteria (BIC) provide a proxy for the logarithm of the marginal likelihood that is formed by penalizing the maximum log-likelihood achievable in a model. The BIC penalty for a d -dimensional statistical model equals $\frac{d}{2} \log(n)$, where n is the sample size. Under regularity conditions, this penalty coincides with the Laplace approximation of the log-marginal likelihood up to a remainder that is bounded in probability [Hau88].

A *singularity* of a parametric statistical model is a parameter vector at which the model’s Fisher-information matrix is singular. At singularities, the BIC approximation need no longer hold. However, seminal work of Watanabe shows, under mild analyticity assumptions on the model and its parametrization, that even at singularities, the logarithm of the marginal likelihood integral can be approximated with the help of penalties of the form $\ell \log(n)$, where ℓ is a geometric invariant referred to as a *learning coefficient* [Wat09]. Watanabe’s work uses resolution of singularities of real algebraic and analytic varieties to build a stochastic version of the theory of singular Laplace integrals [AGZV88]. Geometrically, the learning coefficient at a given parameter vector is the *real log canonical threshold*

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of a function defining the real fiber of the model's parametrization at the considered point. Although computing a resolution of singularities is infeasible at the scale of practical statistical problems, learning coefficients have been derived for several families of models. This information, possibly in the form of bounds, can then be exploited for statistical model selection via the *singular BIC (sBIC)* methodology proposed in [DP17].

The prime source of singular statistical models is latent variable models such as mixture models [RM11, YW03, Aoy10], reduced rank regression models [AW05], neural networks [Aoy09], or latent tree models [Zwi11, DLWZ17]. A key example of a latent variable model that has not yet been studied from the perspective of Watanabe's singular learning theory is the factor analysis model, which is arguably the most fundamental model with continuous latent variables [Har76]. In this paper, we seek to address this gap and initiate the study of learning coefficients for factor analysis.

Factor analysis models explain dependence among a number of observed random variables by a smaller number of latent random variables, called factors. Suppose we observe a sample of independent and identically distributed (i.i.d.) random vectors X_1, \dots, X_n , each vector being p -dimensional and, without loss of generality, centered to have zero mean vector. In the model with k factors, we assume that the observed random vectors are generated as

$$X_i = \Lambda F_i + \epsilon_i, \quad i = 1, \dots, n,$$

where F_i is a k -vector comprised of i.i.d. standard normal random variables, the matrix Λ is a real-valued matrix in $\mathbb{R}^{p \times k}$ referred to as the *factor loading matrix*, and ϵ_i is a p -vector that constitutes noise and has independent coordinates with j -th entry ϵ_{ij} normally distributed with mean zero and variance $\psi_j > 0$. It follows that the observations X_1, \dots, X_n are multivariate normal with a covariance matrix that is parameterized as

$$(1.1) \quad \Sigma_k(\psi, \Lambda) = \text{diag}(\psi) + \Lambda \Lambda^T$$

for a parameter vector $\psi = (\psi_1, \dots, \psi_p) \in \mathbb{R}_+^p = (0, \infty)^p$ and a parameter matrix $\Lambda = (\lambda_{ij}) \in \mathbb{R}^{p \times k}$. The factor analysis model can be identified with its set of covariance matrices, that is, the image of the parametrization map Σ_k . The dimension of this image is the minimum of

$$(1.2) \quad d_k = \dim(\Sigma_k(\mathbb{R}_+^p, \mathbb{R}^{p \times k})) = (k+1)p - \binom{k}{2},$$

and the dimension $p(p+1)/2$ of the ambient space of symmetric $(p \times p)$ -matrices, compare [DSS07].

At regular points, the model dimension yields the BIC-penalty term $\frac{d_k}{2} \log(n)$. However, the factor analysis model also has singularities that are important for the problem of selecting the number of factors k [Drt09, DP17]. The singularities correspond to the points (ψ, Λ) at which the rank of the Jacobian of $\Sigma_k(\psi, \Lambda)$ drops. Singularities of the factor analysis model are studied in [AR56, Theorem 5.9], which gives a general sufficient condition for a point being singular. In addition, a more explicit description of the singular points of the one factor model is given in [Drt09, LG12].

The main theorem of this paper gives a bound on the learning coefficients for the factor analysis model. The learning coefficients depend on the covariance matrix Σ_0 that defines the distribution of the observations X_1, \dots, X_n as well as a prior distribution on the parameters (ψ, Λ) . We will consider the default case where the prior is a smooth and everywhere positive function, in which case the precise form of the prior has no further effect on learning coefficients. We show:

Theorem 1.1. Let $\ell_k(\Sigma_0)$ be the learning coefficient of the factor analysis model with k latent factors at a fixed covariance matrix Σ_0 in this model. Let $r \in \{0, \dots, k\}$ be the minimum rank of any matrix $\Lambda \in \mathbb{R}^{p \times k}$ such that $\Sigma_0 = \psi + \Lambda\Lambda^T$ for some $\psi \in \mathbb{R}_+^p$. If $d_r \leq p(p+1)/2$, then

$$\ell_k(\Sigma_0) \leq \frac{p(k+2) + r(p-k+1)}{4}.$$

The case $d_r > p(p+1)/2$ is covered in Lemma 3.4(2). We remark that, in the considered setup, the covariance matrix Σ_0 also belongs to the factor analysis model with r factors. Obtaining a bound or, a fortiori, an exact value of the learning coefficient depending on r is, thus, crucial for improved statistical model selection via the criterion from [DP17].

In this spirit, we conjecture the bound in Theorem 1.1 to be tight for $k < p$ and matrices $\Sigma_0 = \psi + \Lambda\Lambda^T$ given by a generic rank r matrix Λ . (The case $k = p$ is special, see Theorem 4.6.) This conjecture is supported by further results in Section 4. In particular, we prove the following.

Proposition 1.2. *The bound of Theorem 1.1 is tight in the following cases:*

- (1) $r = 0$;
- (2) $r = 1$ and Λ is generic;
- (3) $r = k$ and Λ is generic.

In a full analysis of the case $k = 1$, we show that there do exist special singularities at which the learning coefficient is smaller than the bound from Theorem 1.1.

This manuscript is organized as follows: Section 2 contains preliminaries on real log canonical thresholds (aka learning coefficients) which form a general framework for the geometric theory of singular statistical models. Section 3 reviews factor analysis models through the lens of singular learning theory. In Section 4.2, we prove Theorem 1.1 which is based on a detailed investigation of learning coefficients at diagonal covariance matrices (Section 4.1). Finally, we treat the case of generic one-factor covariance matrices (Section 4.3) and classify the singularity types of the one-factor model (Section 5).

2. PRELIMINARIES ON REAL LOG CANONICAL THRESHOLDS

We recall some facts on the algebraic theory of learning coefficients from Shaowei Lin's paper [Lin17]. Throughout, let $\Omega \subseteq \mathbb{R}^d$ be a compact set, and let $\mathcal{A}(\Omega)$ be the ring of real-valued functions that are analytic at every point in Ω .

2.1. Real log canonical thresholds of ideals. Let $\mathcal{I} = \langle f_1, \dots, f_r \rangle \subseteq \mathcal{A}(\Omega)$ be an ideal generated by functions f_i not identically 0, and let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth, nearly analytic function, that is, ϕ is the (point-wise) product of a function in $\mathcal{A}(\Omega)$ and a smooth function that is strictly positive on Ω . Then the zeta function

$$\zeta(z) = \int_{\Omega} (f_1(\omega)^2 + \dots + f_r(\omega)^2)^{-z/2} |\phi(\omega)| d\omega$$

has an analytic continuation to the whole complex plane and the poles of the continuation are positive rational numbers, see, for instance, [Ati70]. Let $\ell_{\Omega}(\mathcal{I}; \phi)$ be the smallest of these poles and $m_{\Omega}(\mathcal{I}; \phi)$ its multiplicity as a pole of $\zeta(z)$. The pair

$$\text{RLCT}_{\Omega}(\mathcal{I}; \phi) = (\ell_{\Omega}(\mathcal{I}; \phi), m_{\Omega}(\mathcal{I}; \phi))$$

is called the *real log canonical threshold* of the ideal \mathcal{I} with respect to the *amplitude function* ϕ over Ω . With slight abuse of notation, we will also refer to $\ell_{\Omega}(\mathcal{I}; \phi)$ as the *real log canonical*

threshold and to $m_\Omega(\mathcal{I}; \phi)$ as its *multiplicity* or *order*. The pair $\text{RLCT}_\Omega(\mathcal{I}; \phi)$ is independent of the choice of generators f_1, \dots, f_r [Lin17, Proposition 6]. Moreover, it can be computed locally, in the following sense. We can order pairs $(\lambda, m), (\lambda', m') \in \mathbb{R}^2$ by the total ordering

$$(2.1) \quad (\lambda, m) \leq (\lambda', m') \text{ if and only if } (\lambda < \lambda' \text{ or } (\lambda = \lambda' \text{ and } m \geq m')),$$

that is, by lexicographic order with reversed order in the second component. The following is [Lin17, Proposition 4] with $f(\omega) = (f_1(\omega)^2 + \dots + f_r(\omega)^2)^{-1/2}$.

Fact 2.1 (Locality of the RLCT). *For every $x \in \Omega$, there exists a compact neighborhood $\Omega_x \subseteq \Omega$ of x such that*

$$\text{RLCT}_x(\mathcal{I}; \phi) := \text{RLCT}_{\Omega_x}(\mathcal{I}; \phi) = \text{RLCT}_U(\mathcal{I}; \phi)$$

for all compact neighborhoods $U \subseteq \Omega_x$ of x . Moreover,

$$\text{RLCT}_\Omega(\mathcal{I}; \phi) = \min_{x \in \Omega} \text{RLCT}_x(\mathcal{I}; \phi),$$

where the minimum is with respect to the order of (2.1) and it suffices to take the minimum over all x in the analytic variety $\mathcal{V}(\mathcal{I}) = \{\omega \in \Omega \mid \forall g \in \mathcal{I} \ g(\omega) = 0\}$.

For homogeneous ideals, we can directly use this local notion to determine the real log canonical threshold, see [Aoy13, Theorem 2]. Recall that a function f on Ω is homogeneous in the subset $\omega_1, \dots, \omega_j$ of the variables if there exists a non-negative integer δ such that $f(a\omega_1, \dots, a\omega_j, \omega_{j+1}, \dots, \omega_d) = a^\delta f(\omega)$ for all $\omega = (\omega_1, \dots, \omega_d) \in \Omega$.

Fact 2.2 (Homogenous ideals). *Let $1 \leq j \leq d$ and suppose that the space $U = \{\omega \in \Omega \mid \omega_1 = 0, \dots, \omega_j = 0\}$ is non-empty. Fix $\omega_0 \in U$. Moreover, assume that generators f_1, \dots, f_r of \mathcal{I} and ϕ are homogeneous functions of $\omega_1, \dots, \omega_j$, and that $\phi(\omega_0) \geq c\phi(\omega)$ for a constant $c > 0$ and all ω in a neighborhood of ω_0 . Then*

$$\text{RLCT}_{\omega_0}(\mathcal{I}; \phi) \leq \text{RLCT}_\omega(\mathcal{I}; \phi)$$

for all $\omega \in \Omega$ with $\omega_i = \omega_{0i}$ for all $i > j$. In particular,

$$\text{RLCT}_\Omega(\mathcal{I}; \phi) = \min_{\omega \in U} \text{RLCT}_\omega(\mathcal{I}; \phi).$$

2.2. Newton polyhedra and monomial ideals. We will now revisit how to compute RLCTs of monomial ideals via Newton polyhedra as described, for instance, in Lin's paper [Lin17]; see also [AGZV88, Chapter 8]. Keeping with the above notation, suppose that $0 \in \Omega$ and that $\mathcal{I} = \langle \omega^{a_1}, \dots, \omega^{a_r} \rangle \subseteq \mathcal{A}(\Omega)$ is an ideal generated by monomials, where $\omega^b = \omega_1^{b_1} \cdots \omega_d^{b_d}$ for $b \in \mathbb{Z}_{\geq 0}^d$. As \mathcal{I} is homogeneous, we know by Fact 2.2 that, concerning the RLCT, the only point of interest is the origin.

A combinatorial method to determine $\text{RLCT}_0(\mathcal{I}; \omega^\tau)$ for a fixed $\tau = (\tau_1, \dots, \tau_d) \in \mathbb{Z}_{\geq 0}^d$ is given as follows. The *Newton polyhedron* of \mathcal{I} is the convex polyhedron

$$\mathcal{P}(\mathcal{I}) = \text{conv}\{a_i + \xi \mid i \in [r], \xi \in \mathbb{R}_{\geq 0}^d\}.$$

The τ -distance δ_τ of \mathcal{I} is the smallest $t \in \mathbb{R}_{\geq 0}$ such that $t \cdot (\tau_1 + 1, \dots, \tau_d + 1) \in \mathcal{P}(\mathcal{I})$ and its τ -multiplicity μ_τ is the codimension of the face F of $\mathcal{P}(\mathcal{I})$ with $t \cdot (\tau_1 + 1, \dots, \tau_d + 1) \in F$.

Fact 2.3. [Lin17, Theorem 3] *The real log canonical threshold at the origin of a monomial ideal is given by the τ -distance and τ -multiplicity as*

$$\text{RLCT}_0(\langle \omega^{a_1}, \dots, \omega^{a_r} \rangle; \omega^\tau) = (1/\delta_\tau, \mu_\tau).$$

Example 2.4. We compute the real log canonical threshold of a linear subspace of codimension c in \mathbb{R}^d as given by the monomial ideal $\mathcal{I} = \langle \omega_1, \dots, \omega_c \rangle$. As phase function, we use $1 = \omega^0$, so that $\tau = 0$. The Newton polyhedron of \mathcal{I} is

$$\mathcal{P}(\mathcal{I}) = \text{conv}\{e_i + \xi \mid i \in [c], \xi \in \mathbb{R}_{\geq 0}^d\} \subseteq \mathbb{R}^d,$$

where e_i denotes the i -th unit vector. Clearly, all points $a \in \mathcal{P}(\mathcal{I})$ satisfy $\sum_{i=1}^c a_i \geq 1$. Therefore, the set $F = \{(a_1, \dots, a_d) \in \mathbb{R}_{\geq 0}^d \mid \sum_{i=1}^c a_i = 1\}$ forms a facet of $\mathcal{P}(\mathcal{I})$ and a point of the form (t, \dots, t) lies in F if and only if $t = 1/c$, and this point is contained in the relative interior of F . Hence, $\delta_0 = 1/c$ and $\mu_0 = 1$ which implies $\text{RLCT}_0(\langle \omega_1, \dots, \omega_c \rangle; 1) = (c, 1)$ by Fact 2.3.

2.3. Calculation rules. We now gather further techniques to manipulate ideals and RLCTs. The first result, referred to as “chain rule” in [Lin11] is central to many arguments in this paper. We derive this slightly more general statement from [Lin17, Proposition 8] by shifting the point $x \in \Omega$ to the origin by a linear translation.

Fact 2.5 (Chain Rule). *Let $x \in \Omega$ and $W \subseteq \Omega$ be a compact neighbourhood of x . Let M be a real analytic manifold and $\rho : M \rightarrow W$ a real analytic map whose restriction $\rho^{-1}(W \setminus \mathcal{V}(\mathcal{I})) \rightarrow W \setminus \mathcal{V}(\mathcal{I})$ is a real analytic isomorphism, that is, bijective with real analytic inverse. Then*

$$\text{RLCT}_x(\mathcal{I}; \phi) = \min_{y \in \rho^{-1}(x)} \text{RLCT}_y(\rho^*\mathcal{I}; (\phi \circ \rho) \cdot \det \text{Jac } \rho),$$

where $\rho^*\mathcal{I} = \{g \circ \rho \mid g \in \mathcal{I}\} = \langle f_1 \circ \rho, \dots, f_r \circ \rho \rangle \subseteq \mathcal{A}(M)$ is the pullback of \mathcal{I} under ρ and $\text{Jac } \rho$ is the Jacobian matrix of ρ .

In this paper, M will always be a subset of a real algebraic variety that is locally (algebraically) isomorphic to an affine space and ρ will be a polynomial map.

We review two further calculation rules.

Fact 2.6 (Sum and Product Rule). [Lin17, Proposition 7] *Let $\Omega_1 \subseteq \mathbb{R}^{d_1}$ and $\Omega_2 \subseteq \mathbb{R}^{d_2}$ be compact subsets and let $\mathcal{I} \subseteq \mathcal{A}(\Omega_1)$ and $\mathcal{J} \subseteq \mathcal{A}(\Omega_2)$ be finitely generated ideals. Then, composing with the canonical projections $\Omega_1 \times \Omega_2 \rightarrow \Omega_i$, we can consider \mathcal{I} and \mathcal{J} as ideals of $\mathcal{A}(\Omega_1 \times \Omega_2)$. Let $\phi_1 : \Omega_1 \rightarrow \mathbb{R}$ and $\phi_2 : \Omega_2 \rightarrow \mathbb{R}$ be nearly analytic. Denote $\text{RLCT}_{\Omega_1}(\mathcal{I}; \phi_1) = (\ell_1, m_1)$ and $\text{RLCT}_{\Omega_2}(\mathcal{J}; \phi_2) = (\ell_2, m_2)$. Then*

- (1) $\text{RLCT}_{\Omega_1 \times \Omega_2}(\mathcal{I} + \mathcal{J}; \phi_1 \cdot \phi_2) = (\ell_1 + \ell_2, m_1 + m_2 - 1)$ and
- (2) $\text{RLCT}_{\Omega_1 \times \Omega_2}(\mathcal{I}\mathcal{J}; \phi_1 \cdot \phi_2) = \begin{cases} (\ell_1, m_1) & \text{if } \ell_1 < \ell_2, \\ (\ell_2, m_2) & \text{if } \ell_1 > \ell_2, \\ (\ell_1, m_1 + m_2) & \text{if } \ell_1 = \ell_2. \end{cases}$

2.4. Blow-up at a point. Blow-ups (over the real numbers) are a standard tool to construct maps ρ that satisfy the conditions of the chain rule (Fact 2.5) and can, hence, be used for computations concerning RLCTs.

In this paper, we will only use the blow-up at a point, namely the origin of \mathbb{R}^d with d some fixed positive integer. This blow-up is a real algebraic map $\rho : M \rightarrow \mathbb{R}^d$ from a d -dimensional real algebraic variety M . The map is an isomorphism outside the origin in \mathbb{R}^d and pulls back the origin to a real projective space $\mathbb{P}_{\mathbb{R}}^{d-1}$. The manifold M can be covered by d copies of \mathbb{R}^d which we call charts and, as the RLCT is a local concept, it suffices to

consider the restrictions of ρ to these charts. For $i \in [d]$, the restriction to the i -th chart can be described as

$$\begin{aligned} \rho_i : \mathbb{R}^d &\rightarrow \mathbb{R}^d \\ (x_1, \dots, x_d) &\mapsto (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_d). \end{aligned}$$

Its Jacobian determinant is given by $\det \text{Jac } \rho_i = x_i^{d-1}$.

3. SINGULAR LEARNING THEORY FOR FACTOR ANALYSIS

In this section, we review key aspects of singular learning theory through the lens of the factor analysis model.

3.1. Marginal likelihood and learning coefficients. Let X_1, \dots, X_n be a sample of independent and identically distributed random vectors, with X_i taking values in \mathbb{R}^p . Without loss of generality, we assume the expectation vectors $\mathbb{E}[X_i]$ to be zero. Let

$$S_n = \frac{1}{n} \sum_{i=1}^n X_i X_i^T$$

be the $p \times p$ sample covariance matrix, and let PD_p denote the cone of positive definite $p \times p$ matrices. Define the function

$$(3.1) \quad \ell(\Sigma | S_n) = \frac{p}{2} \log(2\pi) + \frac{1}{2} \log \det(\Sigma) + \frac{1}{2} \text{tr}(\Sigma^{-1} S_n), \quad \Sigma \in PD_p.$$

Then $-n\ell(\Sigma | S_n)$ is the Gaussian log-likelihood function, which maps a matrix $\Sigma \in PD_p$ to the logarithm of the joint density of (X_1, \dots, X_n) when the X_i are i.i.d. multivariate normal with covariance matrix Σ .

In the factor analysis model with k factors, the covariance matrix of the observations is given by the parametrization map from (1.1), so $\Sigma = \Sigma_k(\psi, \Lambda) = \text{diag}(\psi) + \Lambda \Lambda^T$ with $\psi = (\psi_1, \dots, \psi_p) \in \mathbb{R}_+^p$ and $\Lambda = (\lambda_{ij}) \in \mathbb{R}^{p \times k}$. Suppose that, for a Bayesian treatment, we have chosen a prior distribution for (ψ, Λ) and that this prior distribution has Lebesgue density $\varphi_k(\psi, \Lambda)$ on $\mathbb{R}_+^p \times \mathbb{R}^{p \times k}$. In this paper, we assume that the prior density φ_k is everywhere positive, bounded, and smooth; compare, e.g., [LD16]. The marginal likelihood of the k -factor model is now the integral

$$(3.2) \quad L_{k,n}(S_n) = \int_{\mathbb{R}^{p \times k}} \int_{\mathbb{R}_+^p} e^{-n\ell(\Sigma_k(\psi, \Lambda) | S_n)} \varphi_k(\psi, \Lambda) d\psi d\Lambda.$$

While exact values of $L_{k,n}(S_n)$ are challenging to obtain, general results from [Wat09, Chap. 6] describe the asymptotics of $L_{k,n}(S_n)$ as n increases. Let $\Sigma_0 = \Sigma_k(\psi_0, \Lambda_0)$ with $\psi_0 \in \mathbb{R}_+^p$ and $\Lambda_0 \in \mathbb{R}^{p \times k}$ be the true covariance matrix of the multivariate normal observations X_1, \dots, X_n . The negated and scaled log-likelihood function in (3.1) is minimized uniquely by $\Sigma = S_n$ and its minimal value is

$$(3.3) \quad \ell(S_n) := \ell(S_n | S_n) = \frac{p}{2} \log(2\pi) + \frac{1}{2} \log \det(S_n) + \frac{p}{2}.$$

From [Wat09, Chap. 6], it follows that the marginal likelihood sequence obtained by varying n satisfies

$$(3.4) \quad -\log L_{k,n}(S_n) = n\ell(S_n) + \ell_k(\Sigma_0) \log(n) - [m_k(\Sigma_0) - 1] \log \log(n) + O_{\text{prob}}(1),$$

where $O_{prob}(1)$ denotes a sequence of random variables that is bounded in probability. In (3.4), $\ell_k(\Sigma_0)$ is a positive rational number and $m_k(\Sigma_0)$ is an integer, with the following terminology.

Definition 3.1. The coefficient $\ell_k(\Sigma_0)$ in (3.4) is the *learning coefficient* of the k -factor model at the covariance matrix Σ_0 , and $m_k(\Sigma_0)$ is its *order* or *multiplicity*.

The reader not so familiar with learning coefficients might take Fact 3.2 below (using the notation from Sections 2.1 and 3.3) as a definition of $\ell_k(\Sigma_0)$ and $m_k(\Sigma_0)$.

As we revisit in Lemma 3.4, it holds that $\ell_k(\Sigma_0) \in (0, d_k/2]$, see [Wat09, Theorem 7.2]. For the order, it holds by definition that $m_k(\Sigma_0) \in [d_k] := \{1, \dots, d_k\}$, see [Wat09, p. 32]. Here, d_k is the dimension from (1.2). Our notation suppresses any dependence of the learning coefficient and its order on the prior density φ_k . Indeed, the assumed smoothness and positivity ensures that φ_k is bounded above and bounded away from zero on every compact subset, which implies that $\ell_k(\Sigma_0)$ and $m_k(\Sigma_0)$ do not depend on the specific form of the prior [Lin11, Lemma 3.8].

3.2. Setup towards sBIC. The factor analysis model \mathcal{M}_r with r latent factors can be identified with a submodel of the k -factor model \mathcal{M}_k as follows: As above, identify \mathcal{M}_k with its space of covariance matrices $\Sigma_k(\psi, \Lambda) = \Lambda\Lambda^T + \text{diag}(\psi)$, where we vary $\Lambda \in \mathbb{R}^{p \times k}$ and $\psi \in \mathbb{R}_+^p$. Now, the following conditions are equivalent for a covariance matrix $\Sigma_0 \in \mathcal{M}_k$, and, if they are satisfied, Σ_0 lies in the r -factor model \mathcal{M}_r :

- (1) $\Sigma_0 = \Sigma_k(\psi, \Lambda)$ for some $\Lambda \in \mathbb{R}^{p \times k}$ of rank at most r and some $\psi \in \mathbb{R}_+^p$;
- (2) $\Sigma_0 = \Sigma_r(\psi, \Lambda) = \Sigma_k(\psi, [\Lambda \ 0])$ for some $\Lambda \in \mathbb{R}^{p \times r}$ and some $\psi \in \mathbb{R}_+^p$.

So, we have a (totally) ordered system of statistical models

$$\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \dots \subseteq \mathcal{M}_k$$

and are, hence, in the situation of marginal likelihood estimation and improved model selection via the *singular Bayesian Information Criterion* (sBIC) of Drton and Plummer, see [DP17]. For all choices $r \leq k$ for which we can compute the learning coefficients exactly, there exists a Zariski open subset U_{kr} of the space of $(p \times k)$ -matrices of rank at most r such that, for all $\Sigma_1 = \Sigma_k(\psi_1, \Lambda_1), \Sigma_2 = \Sigma_k(\psi_2, \Lambda_2) \in \mathcal{M}_r$ with $\Lambda_1, \Lambda_2 \in U_{kr}$, we have $\ell_k(\Sigma_1) = \ell_k(\Sigma_2)$ and $m_k(\Sigma_1) = m_k(\Sigma_2)$. We denote this (*generic*) learning coefficient and order of \mathcal{M}_k along \mathcal{M}_r by ℓ_{kr} and m_{kr} , respectively, that is, $\ell_{kr} = \ell_k(\Sigma_0)$ and $m_{kr} = m_k(\Sigma_0)$, where $\Sigma_0 = \Lambda\Lambda^T + \psi \in \mathcal{M}_r$ with $\Lambda \in U_{kr}$. Note that the definition of ℓ_{kr} and m_{kr} is independent of the data X_1, \dots, X_n . In order to apply the sBIC to model selection among the models $\mathcal{M}_0, \dots, \mathcal{M}_k$, it suffices to compute the numbers ℓ_{sr} and m_{sr} for all $s, r \in \{0, \dots, k\}$.

3.3. Fiber ideals for factor analysis. The *fiber ideal* of the k -factor model with p -dimensional observations at the covariance matrix $\Sigma_0 = (\sigma_{ij})$ is defined as

$$\bar{\mathcal{I}}_{p,k}(\Sigma_0) = \mathcal{I}_{p,k}(\Sigma_0) + \mathcal{I}'_{p,k}(\Sigma_0),$$

where

$$\begin{aligned} \mathcal{I}_{p,k}(\Sigma_0) &= \langle \lambda_i \lambda_j^T - \sigma_{ij} \mid 1 \leq i < j \leq p \rangle, \\ \mathcal{I}'_{p,k}(\Sigma_0) &= \langle \lambda_i \lambda_i^T + \psi_i - \sigma_{ii} \mid 1 \leq i \leq p \rangle, \end{aligned}$$

Λ is a $(p \times k)$ -matrix of indeterminates with row vectors λ_i , and ψ is a p -vector of indeterminates. Note that the *reduced fiber ideal* $\mathcal{I}_{p,m}(\Sigma_0)$ is an ideal of the ring $\mathcal{A}(\mathbb{R}^{p \times k})$ of functions that are analytic in every point of $\mathbb{R}^{p \times k}$. Using the calculation rules for real log canonical

thresholds of ideals from Section 2 will allow us to reduce the problem of determining $\ell_k(\Sigma_0)$ and $m_k(\Sigma_0)$ to computing real log canonical thresholds of $\mathcal{I}_{p,m}(\Sigma_0)$.

First, we will transform the fiber ideal $\bar{\mathcal{I}}_{p,k}(\Sigma_0)$ by the following map:

$$\rho : (\psi'_1, \dots, \psi'_p, \lambda_1, \dots, \lambda_p) \mapsto (\psi'_1 - \lambda_1 \lambda_1^T + \sigma_{11}, \dots, \psi'_p - \lambda_p \lambda_p^T + \sigma_{pp}, \lambda_1, \dots, \lambda_p)$$

The Jacobian matrix of ρ is an upper triangular matrix with ones along the diagonal, so $\det \text{Jac } \rho = 1$. We compute the pullback of $\bar{\mathcal{I}}_{p,k}(\Sigma_0)$ as

$$\rho^* \bar{\mathcal{I}}_{p,k}(\Sigma_0) = \mathcal{I}_{p,k}(\Sigma_0) + \mathcal{J},$$

where $\mathcal{J} = \langle \psi'_1, \dots, \psi'_p \rangle$. Now, let $\phi = \varphi_k$ be the prior density which, as noted before, can be assumed identical to 1 [Lin17, Lemma 1]. It follows from the Chain Rule (Fact 2.5) that

$$\text{RLCT}_x(\bar{\mathcal{I}}_{p,k}(\Sigma_0); \varphi_k) = \text{RLCT}_x(\bar{\mathcal{I}}_{p,k}(\Sigma_0); 1) = \min_{y \in \rho^{-1}(x)} \text{RLCT}_y(\mathcal{I}_{p,k}(\Sigma_0) + \mathcal{J}; 1)$$

for every $x = (\psi, \Lambda) \in \mathbb{R}_+^p \times \mathbb{R}^{p \times k}$. Moreover, Fact 2.2 and Example 2.4 show that the number $\text{RLCT}_{(\psi'_1, \dots, \psi'_p)}(\mathcal{J}; 1)$ takes $(2\frac{p}{2}, 1) = (p, 1)$ as its minimal value. Note for this, that \mathcal{J} is a homogeneous ideal and that $(\psi'_1, \dots, \psi'_p, \Lambda) = (0, \dots, 0, \Lambda)$ is always in the fiber $\rho^{-1}(\psi, \Lambda)$ for some Λ because the diagonal entries σ_{ii} of the positive semi-definite matrix Σ_0 are non-negative. Applying Fact 2.6(1) to the sum $\mathcal{I}_{p,k}(\Sigma_0) + \mathcal{J}$, we infer

$$(3.5) \quad \text{RLCT}_{(\psi, \Lambda)}(\bar{\mathcal{I}}_{p,k}(\Sigma_0); \varphi_k) = \text{RLCT}_\Lambda(\mathcal{I}_{p,k}(\Sigma_0); 1) + (p, 0)$$

for all $(\psi, \Lambda) \in \mathbb{R}_+^p \times \mathbb{R}^{p \times k}$. Combining this with [Lin17, Theorem 2], we can now reduce the problem of computing learning coefficients and orders for factor analysis models to determining the RLCT of the ideal $\mathcal{I}_{p,k}(\Sigma_0)$. This observation is crucial to all proofs in the paper.

Fact 3.2. *Let $\Sigma_0 \in \mathbb{R}^{p \times p}$ be a symmetric positive definite matrix lying in the k -factor model \mathcal{M}_k with p -dimensional observations. Then the following holds for the learning coefficient $\ell_k(\Sigma_0)$ and its order $m_k(\Sigma_0)$ of \mathcal{M}_k :*

$$(2\ell_k(\Sigma_0), m_k(\Sigma_0)) = \min_\Lambda \text{RLCT}_\Lambda(\mathcal{I}_{p,k}(\Sigma_0); 1) + (p, 0),$$

where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times k}$ with $\Sigma_k(\psi, \Lambda) = \Sigma_0$ for some $\psi \in \mathbb{R}_+^p$.

Another calculation rule is specific for factor analysis models and deals with multiplication of the fixed covariance matrix by a diagonal matrix.

Lemma 3.3. *Let $\Gamma = \text{diag}(\gamma)$ be a diagonal matrix given by a vector $\gamma \in (\mathbb{R} \setminus \{0\})^p$ with all entries non-zero. If $\Sigma_0 \in \mathbb{R}^{p \times p}$ is in the k -factor model then so is $\Gamma \Sigma_0 \Gamma$,*

$$\ell_k(\Sigma_0) = \ell_k(\Gamma \Sigma_0 \Gamma) \text{ and } m_k(\Sigma_0) = m_k(\Gamma \Sigma_0 \Gamma).$$

Proof. As Γ is invertible, the off-diagonal entries of $\Gamma(\Lambda \Lambda^T - \Sigma_0)\Gamma = \Gamma \Lambda \Lambda^T \Gamma - \Gamma \Sigma_0 \Gamma$ generate the same ideal as the off-diagonal entries of $\Lambda \Lambda^T - \Sigma_0$ in the ring $\mathcal{A}(\mathbb{R}^{p \times k})$, where Λ is a $(p \times k)$ -matrix of indeterminates, namely the ideal $\mathcal{I}_{p,k}(\Sigma_0)$. The Jacobian determinant of the map $\rho : \Lambda' \mapsto \Lambda = \Gamma^{-1} \Lambda' \Gamma^{-1}$ is the inverse of a monomial in the diagonal entries of Γ and, hence, a non-zero constant. The result follows by the chain rule (Fact 2.5) because $\rho^* \mathcal{I}_{p,k}(\Sigma_0) = \mathcal{I}_{p,k}(\Gamma \Sigma_0 \Gamma)$. \square

Lemma 3.4. *Consider a k -factor model, where p is the dimension of the observations.*

- (1) If $d_k \leq p(p+1)/2$ and the covariance matrix Σ_0 is chosen generically from the k -factor model, then the learning coefficient is $\ell_{kk} = \ell_k(\Sigma_0) = d_k/2$ and the order is $m_{kk} = m_k(\Sigma_0) = 1$.
- (2) If $r \leq k$ such that $d_r > p(p+1)/2$ and Σ_0 is chosen generically from the k -factor model, then $\ell_{kr} = \ell_{kk} = \ell_k(\Sigma_0) = p(p+1)/4$ and $m_{kr} = m_{kk} = m_k(\Sigma_0) = 1$.

Proof. Throughout, we will use ideas from the proof of Theorem 2 in [DSS07]. First note that, if $r \leq k$ with $d_r > p(p+1)/2$, then $d_k > p(p+1)/2$ and the images of the parametrization maps Σ_r and Σ_k are Zariski dense in the cone of positive definite $(p \times p)$ -matrices. Therefore, $\ell_{kr} = \ell_{kk}$ and $m_{kr} = m_{kk}$ in this case.

Now, we go to the general case, where $k \in \{0, \dots, p\}$. The parametrization map Σ_k extends to a morphism $\Sigma_k : \mathbb{A}_{\mathbb{C}}^p \times \mathbb{A}_{\mathbb{C}}^{p \times k} \rightarrow \overline{\mathcal{M}_k}$ of schemes of finite type over \mathbb{C} , where $\overline{\mathcal{M}_k}$ is the complex Zariski closure of the k -factor model, which has dimension $D_k := \min\{d_k, p(p+1)/2\}$. By [Har77, Corollary 10.7] and [Har77, Theorem 10.2], the fiber of generic $\Sigma_0 \in \overline{\mathcal{M}_k}$ is smooth over \mathbb{C} of codimension D_k . Clearly, for $\Sigma_0 \in \mathcal{M}_k$, the corresponding fiber has a real point in $\mathbb{R}_+^p \times \mathbb{R}^{p \times k}$, just by definition of \mathcal{M}_k . So, if $\Sigma_0 \in \mathcal{M}_k$ is chosen generically, this real point is a smooth point of the fiber, which implies that the dimension of the set of points of the fiber that lie in $\mathbb{R}_+^p \times \mathbb{R}^{p \times k}$ also has (real) dimension D_k [Man20, Theorem 2.2.9], and it is smooth over \mathbb{R} . Consequently, using [Art68, Corollary 1.6], for each point in $\mathbb{R}_+^p \times \mathbb{R}^{p \times k}$ on the fiber, there exists a (Euclidean) neighborhood U and a real analytic isomorphism $\mathbb{R}^p \times \mathbb{R}^{p \times k} \rightarrow U$ that transforms the defining ideal $\overline{\mathcal{I}}_{p,k}(\Sigma_0)$ to an ideal generated by D_k variables. The statement now follows by Example 2.4, (3.5) and Fact 3.2. \square

3.4. LQ-decomposition. We use a specific type of transformation for $\mathcal{I}_{p,k}(\Sigma_0)$ to compute the local RLCTs from Fact 3.2. Denote by $\mathcal{L}_{r,+}^{p \times k}$ the space of real $(p \times k)$ -matrices of the form

$$(3.6) \quad \Lambda' = \begin{bmatrix} \Lambda'_{11} & 0 \\ \Lambda'_{21} & \Lambda'_{22} \end{bmatrix}$$

with Λ'_{11} lower diagonal with positive diagonal entries.

Fact 3.5. Let $\Sigma_0 \in \mathbb{R}^{p \times p}$ be in the k -factor model \mathcal{M}_k and let $r \in \{0, \dots, k\}$ be the minimum rank of any matrix $\Lambda \in \mathbb{R}^{p \times k}$ such that $\Sigma_0 = \psi + \Lambda \Lambda^T$ for some $\psi \in \mathbb{R}_+^p$. Then

$$\min_{\Lambda} \text{RLCT}_{\Lambda}(\mathcal{I}_{p,k}(\Sigma_0); 1) = \min_{\Lambda'} \text{RLCT}_{\Lambda'}(\mathcal{I}_{p,k}(\Sigma_0); 1),$$

where the minima range over all $\Lambda \in \mathbb{R}^{p \times k}$ and $\Lambda' \in \mathcal{L}_{r,+}^{p \times k}$, respectively, with $\Sigma_k(\psi, \Lambda) = \Sigma_0 = \Sigma_k(\psi', \Lambda')$ for some $\psi, \psi' \in \mathbb{R}_+^p$, and the elements of $\mathcal{I}_{p,k}(\Sigma_0)$ on the right side of the equation are considered as functions on $\mathcal{L}_{r,+}^{p \times k}$.

Proof. First, we fix $\Lambda_* \in \mathbb{R}^{p \times k}$ with $\Sigma_0 = \psi + \Lambda_* \Lambda_*^T$, so that $\text{rank } \Lambda_* \in \{r, r+1, \dots, k\}$ and, hence, some $(r \times r)$ -minor of Λ_* is non-zero. The linear transformation that permutes the rows and columns of Λ_* such that the non-vanishing $(r \times r)$ -minor is transformed into the determinant of the matrix consisting of the first r rows and columns of a $(p \times k)$ -matrix has non-zero constant Jacobian determinant. So, without changing $\text{RLCT}_{\Lambda_*}(\mathcal{I}_{p,k}(\Sigma_0))$, we can assume that

$$\Lambda_* = \begin{bmatrix} \Lambda_{*11} & \Lambda_{*12} \\ \Lambda_{*21} & \Lambda_{*22} \end{bmatrix},$$

where $\Lambda_{*11} \in \mathbb{R}^{r \times r}$ is of full rank. Furthermore, as the non-vanishing of a minor defines a Zariski open set, we can find a (compact) neighbourhood $\Omega \subseteq \mathbb{R}^{p \times k}$ of Λ_* such that

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

with $\Lambda_{11} \in \mathbb{R}^{r \times r}$ of full rank, for all $\Lambda \in \Omega$. The considerations of [AMS08, Example 4.1.2] show that QR-decomposition is an analytic isomorphism of full-rank square matrices, so that we can write $\Lambda_{11}^T = Q_{11}^T(\Lambda'_{11})^T$ with $Q_{11} \in O(r)$, the space of real $(r \times r)$ orthonormal matrices, and $\Lambda'_{11} = (\lambda'_{ij})$ is a real $(r \times r)$ lower triangular matrix with $\lambda_{ii} > 0$ for all $i \in \{1, \dots, k\}$, where Q and Λ'_{11} are uniquely determined and depend analytically on Λ . After transposition, $\Lambda_{11} = \Lambda'_{11}Q_{11}$. Define $Q_{12} = (\Lambda'_{11})^{-1}\Lambda_{12} \in \mathbb{R}^{r \times (k-r)}$, so that

$$[\Lambda_{11} \quad \Lambda_{12}] = \Lambda'_{11} [Q_{11} \quad Q_{12}].$$

The matrix $[Q_{11} \quad Q_{12}]$ is still orthonormal by [Mui82, Theorem A9.8], and note that Q_{12} depends analytically on Λ . Now fix any analytic way of finding an orthonormal basis of the orthogonal complement of a given r -dimensional subspace of \mathbb{R}^k and, in this way, complete to

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in O(k).$$

Set

$$\Lambda' = \begin{bmatrix} \Lambda'_{11} & 0 \\ \Lambda'_{21} & \Lambda'_{22} \end{bmatrix} \in \mathcal{L}_{r,+}^{p \times k}.$$

where Λ'_{21} and Λ'_{22} are uniquely determined by $\Lambda' = \Lambda Q^T = \Lambda Q^{-1}$. We found a decomposition $\Lambda = \Lambda'Q$ or $\Lambda \in \Omega$, where $\Lambda' \in \mathcal{L}_{r,+}^{p \times k}$ and $Q \in O(k)$ are uniquely determined and depend analytically on Λ . Denote this decomposition as

$$\delta : \Omega \rightarrow \mathcal{L}_{r,+}^{p \times k} \times O(k), \Lambda \mapsto (\Lambda', Q),$$

and note the first projection of $\delta(\Omega)$ is a full dimensional set because the projection is an open map. This δ is the analytic inverse of the real analytic isomorphism

$$(3.7) \quad \mu : \delta(\Omega) \rightarrow \Omega, (\Lambda', Q) \mapsto \Lambda'Q$$

The Jacobian determinant is a non-zero constant by the very fact that it is a real analytic isomorphism. As $\Lambda'(\Lambda')^T = \Lambda Q Q^T \Lambda^T = \Lambda \Lambda^T$ for each $\Lambda \in \Omega$, we see that the pullback $\mu^* \mathcal{I}_{p,k}(\Sigma_0)$ is generated by the same functions as $\mathcal{I}_{p,k}(\Sigma_0)$ itself replacing λ_{ij} by λ'_{ij} . Using that the Jacobian determinant of μ is strictly positive on a neighbourhood of (Λ', Q) , that Q is already determined by Λ' , and that $\delta(\Omega)$ is full-dimensional yields the result. \square

4. LEARNING COEFFICIENTS FOR FACTOR ANALYSIS MODELS

4.1. Diagonal covariance matrices. We begin by studying the instance when the given covariance matrix Σ_0 is diagonal, that is, lies in the zero factor model. In this case, the reduced fiber ideal is independent of Σ_0 and given by

$$\mathcal{I}_{p,k,0} = \mathcal{I}_{p,k}(\Sigma_0) = \langle \lambda_i \lambda_j^T \mid 1 \leq i < j \leq p \rangle,$$

see Section 3.3. Specializing to the case $k = 1$ gives

$$\mathcal{I}_{p,1,0} = \langle \lambda_i \lambda_j \mid 1 \leq i < j \leq p \rangle$$

which is a monomial ideal.

Lemma 4.1. *Let $\Sigma_0 \in \mathbb{R}^{p \times p}$ be a diagonal matrix with positive diagonal entries. Then*

$$\min_{\Lambda} \text{RLCT}_{\Lambda}(\mathcal{I}_{p,1,0}; 1) = \text{RLCT}_0(\mathcal{I}_{p,1,0}; 1) = (p/2, 1),$$

where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times 1}$ with $\psi + \Lambda \Lambda^T = \Sigma_0$ for some $\psi \in \mathbb{R}_+^p$.

Proof. The first equality follows by Fact 2.2. For the second, we consider the Newton polyhedron method that we recalled in Section 2.2. We write $\mathcal{I} = \mathcal{I}_{p,1,0}$. The Newton polyhedron of \mathcal{I} is the convex positive real cone

$$\mathcal{P}(\mathcal{I}) = \text{conv} \left\{ e_i + e_j + \xi \mid 1 \leq i < j \leq p, \xi \in \mathbb{R}_{\geq 0}^p \right\}$$

generated by the exponent vectors $e_i + e_j$ of the monomial generators of \mathcal{I} , where e_i denotes the i -th unit vector. Clearly, all points in $x \in \mathcal{P}(\mathcal{I})$ satisfy the inequality

$$\sum_{i=1}^m x_i \geq 2.$$

The set of points where equality holds, that is, the convex hull of the vectors $e_i + e_j$, forms a facet of $\mathcal{P}(\mathcal{I})$. This facet contains the point $(2/p, \dots, 2/p)$ in its relative interior. Hence, for $\tau = 0$ which is the exponent vector of the face function 1 considered as a monomial, the τ -distance of $\mathcal{P}(\mathcal{I})$ is $2/p$ and the τ -multiplicity is one. The statement of the lemma now follows from Fact 2.3. \square

Proposition 4.2. *The learning coefficient and its order of the factor analysis model with p -dimensional observations and $k = 1$ latent factor along the submodel with $r = 0$ latent factors satisfy*

$$\ell_{10} = \ell_1(\Sigma_0) = \frac{3p}{4} \quad \text{and} \quad m_{10} = m_1(\Sigma_0) = 1,$$

where Σ_0 is any diagonal matrix with positive real diagonal entries.

Proof. This follows directly from Lemma 4.1 and Fact 3.2. \square

Lemma 4.3. *If $p \geq k + 2$, then*

$$\min_{\Lambda} \text{RLCT}_{\Lambda}(\mathcal{I}_{p,k,0}; 1) = \text{RLCT}_0(\mathcal{I}_{p,k,0}; 1) = (pk/2, 1),$$

where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times k}$ with $\psi + \Lambda \Lambda^T = \Sigma_0$ for some $\psi \in \mathbb{R}_+^p$.

Proof. Again, the first equality follows by Fact 2.2. For the second, we proceed by induction on k . If $k = 0$, then the claim follows from Lemma 3.4 and, although technically unnecessary, the claim also follows for $k = 1$ by Lemma 4.1.

Now, for the induction step, fix $k > 0$. The first transformation we use is a blow-up at the origin, see Section 2.4. By symmetry, we only need to consider one chart and we can write the transformation as

$$\begin{aligned} \lambda_{pk} &= \lambda'_{pk} \text{ and} \\ \lambda_{ij} &= \lambda'_{pk} \lambda'_{ij} \text{ for } (i, j) \neq (p, k), \end{aligned}$$

which has Jacobian determinant equal to $(\lambda'_{pk})^{pk-1}$. Omitting the primes on the new variables and letting $\tilde{\lambda}_i = (\lambda_{i1}, \dots, \lambda_{i(k-1)})$, the pullback of the ideal $\mathcal{I}_{p,k,0}$ is

$$\langle \lambda_{pk}^2 \rangle \cdot (\mathcal{J}_1 + \mathcal{J}_2),$$

where

$$\begin{aligned}\mathcal{J}_1 &= \langle \lambda_{jk} + \tilde{\lambda}_p \tilde{\lambda}_j^T \mid 1 \leq j \leq p-1 \rangle, \\ \mathcal{J}_2 &= \langle \lambda_{ik} \lambda_{jk} + \tilde{\lambda}_i \tilde{\lambda}_j^T \mid 1 \leq i < j \leq p-1 \rangle.\end{aligned}$$

It follows from the very definition of the RLCT that

$$\text{RLCT}_0(\langle \lambda_{pk}^2 \rangle; \lambda_{pk}^{pk-1}) = \frac{pk}{2},$$

see also [LUSB14, Theorem 7.1]. Using the form of the generators for \mathcal{J}_1 , we infer

$$\begin{aligned}\mathcal{J}_1 + \mathcal{J}_2 &= \mathcal{J}_1 + \langle (-\tilde{\lambda}_p \tilde{\lambda}_i^T)(-\tilde{\lambda}_p \tilde{\lambda}_j^T) + \tilde{\lambda}_i \tilde{\lambda}_j^T \mid 1 \leq i < j \leq p-1 \rangle \\ &= \mathcal{J}_1 + \langle \tilde{\lambda}_i (\tilde{\lambda}_p^T \tilde{\lambda}_p + I) \tilde{\lambda}_j^T \mid 1 \leq i < j \leq p-1 \rangle,\end{aligned}$$

where I denotes the identity matrix of dimension $p-1$. The first equality follows by subtracting the product of the i -th and the j -th generator of \mathcal{J}_1 from the corresponding generator of \mathcal{J}_2 while the second is simply matrix multiplication. Indeed, $(-\tilde{\lambda}_p \tilde{\lambda}_i^T)(-\tilde{\lambda}_p \tilde{\lambda}_j^T) = (\tilde{\lambda}_i \tilde{\lambda}_p^T)(\tilde{\lambda}_p \tilde{\lambda}_j^T) = \tilde{\lambda}_i (\tilde{\lambda}_p^T \tilde{\lambda}_p) \tilde{\lambda}_j^T$.

Now, transform the variables λ_{jk} for $1 \leq j \leq p-1$ as

$$\lambda_{jk} = \lambda'_{jk} - \tilde{\lambda}_p \tilde{\lambda}_j^T,$$

leaving all other variables fixed. This map has Jacobian determinant equal to 1 and shows that the RLCT_0 of $\mathcal{J}_1 + \mathcal{J}_2$ is equal to the RLCT_0 of

$$(4.1) \quad \langle \lambda_{jk} : 1 \leq j \leq p-1 \rangle + \langle \tilde{\lambda}_i (I + \tilde{\lambda}_p^T \tilde{\lambda}_p) \tilde{\lambda}_j^T : 1 \leq i < j \leq p-1 \rangle.$$

By Example 2.4

$$\text{RLCT}_0(\langle \lambda_{jk} : 1 \leq j \leq p-1 \rangle; 1) = (p-1, 1).$$

Since the matrix $I + \tilde{\lambda}_p^T \tilde{\lambda}_p$ is positive definite and thus has all of its eigenvalues real, positive and bounded away from 0, we may change coordinates as

$$\tilde{\lambda}_i = \tilde{\lambda}'_i (I + \tilde{\lambda}_p^T \tilde{\lambda}_p)^{-1/2}, \quad 1 \leq i \leq p-1,$$

where the positive definite square root is given by

$$I + \left(\frac{1}{\sqrt{1 + \tilde{\lambda}_p^T \tilde{\lambda}_p}} - 1 \right) \cdot \frac{\tilde{\lambda}_p^T \tilde{\lambda}_p}{\tilde{\lambda}_p^T \tilde{\lambda}_p}$$

and, hence, the Jacobian determinant of this transformation is positive on its domain and can be ignored. Dropping the primes, this gives

$$\begin{aligned}&\text{RLCT}_0(\langle \tilde{\lambda}_i (I + \tilde{\lambda}_p^T \tilde{\lambda}_p) \tilde{\lambda}_j^T : 1 \leq i < j \leq p-1 \rangle; 1) \\ &= \text{RLCT}_0(\langle \tilde{\lambda}_i \tilde{\lambda}_j^T : 1 \leq i < j \leq p-1 \rangle; 1) = \text{RLCT}_0(I_{p-1, k-1, 0}; 1).\end{aligned}$$

Using the induction hypothesis and the sum rule (Fact 2.6(1)), we obtain that the RLCT_0 of the ideal in (4.1) is

$$(4.2) \quad \left(p-1 + \frac{(p-1)(k-1)}{2}, 1 \right) = \left(\frac{(p-1)(k+1)}{2}, 1 \right).$$

If $p \geq k+2$, as we assume, then

$$\frac{(p-1)(k+1)}{2} > \frac{pk}{2}.$$

By the product rule (Fact 2.6(2)), we conclude that

$$\text{RLCT}_0(\mathcal{I}_{p,k,0}) = \text{RLCT}_0(\langle \lambda_{pk}^2 \rangle; \lambda_{pk}^{pk-1}) = \left(\frac{pk}{2}, 1 \right). \quad \square$$

Lemma 4.4. *In the case $k = p - 1 \geq 1$,*

$$\min_{\Lambda} \text{RLCT}_{\Lambda}(\mathcal{I}_{p,p-1,0}; 1) = \text{RLCT}_0(\mathcal{I}_{p,p-1,0}; 1) = (pk/2, p - 1) = \left(\frac{p(p-1)}{2}, p - 1 \right),$$

where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times k}$ with $\psi + \Lambda \Lambda^T = \Sigma_0$ for some $\psi \in \mathbb{R}_+^p$.

Proof. The argument runs along the same lines as in the proof of Lemma 4.3. Similar as there, the first equality follows by Fact 2.2, and the case $k = 1$ follows from Lemma 4.1. Again, we proceed by induction on $k > 1$. The proof is identical until (4.2) which is replaced by

$$(p-1, 1) + \left(\frac{(p-1)(p-2)}{2}, p-2 \right) - (0, 1) = \left(\frac{p(p-1)}{2}, p-2 \right) = (pk/2, p-2).$$

Since the first entry of this last pair equals $\text{RLCT}_0(\langle \lambda_{pk}^2 \rangle; \lambda_{pk}^{pk-1}) = pk/2$, the product rule (Fact 2.6(2)) yields

$$\text{RLCT}_0(\mathcal{I}_{p,p-1,0}; 1) = (pk/2, 1 + p - 2) = \left(\frac{p(p-1)}{2}, p - 1 \right).$$

□

Note that, in the case $k = p - 1 = 0$, the fiber ideal $\mathcal{I}_{1,0,0}$ equals $\langle 1 \rangle$ and, hence, we cannot apply our machinery. The correct values for the learning coefficient and the multiplicity in Theorem 4.6 are derived differently.

Lemma 4.5. *In the case $k = p \geq 1$,*

$$\min_{\Lambda} \text{RLCT}_{\Lambda}(\mathcal{I}_{p,p,0}; 1) = \text{RLCT}_0(\mathcal{I}_{p,p,0}; 1) = \left(\frac{p^2 - p + 1}{2}, 1 \right),$$

where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times k}$ with $\psi + \Lambda \Lambda^T = \Sigma_0$ for some $\psi \in \mathbb{R}_+^p$.

Proof. The proof runs along the same lines as the one of Lemma 4.3. The case $p = k = 1$ is covered by Lemma 4.1 and we proceed by induction on $p = k > 1$. The proof is identical to the one of Lemma 4.3 up to (4.2) which is replaced by

$$\left(p - 1 + \frac{(p-1)^2 - (p-1) + 1}{2}, 1 \right) = \left(\frac{p^2 - p + 1}{2}, 1 \right).$$

For $p > 1$, the first entry of this last pair is strictly smaller than $p^2/2 = \text{RLCT}_0(\langle \lambda_{pk}^2 \rangle; \lambda_{pk}^{pk-1})$. So, the product rule (Fact 2.6) yields

$$\text{RLCT}_0(\mathcal{I}_{p,p,0}; 1) = \left(\frac{p^2 - p + 1}{2}, 1 \right).$$

□

Theorem 4.6. *The learning coefficient and its order of the factor analysis model with p -dimensional observations and k latent factors along the submodel with $r = 0$ latent factors satisfy*

$$\ell_{k0} = \ell_k(\Sigma_0) = \begin{cases} \frac{p(k+2)}{4} & \text{if } k \leq p-1, \\ \frac{p^2+p+1}{4} & \text{if } k = p \end{cases}$$

and

$$m_{k0} = m_k(\Sigma_0) = \begin{cases} 1 & \text{if } k \leq p-2 \text{ or } k = p-1 = 0 \text{ or } k = p, \\ p-1 & \text{if } k = p-1 > 0, \end{cases}$$

where Σ_0 is any diagonal matrix.

Proof. This follows immediately from Lemma 4.3, Lemma 4.4, Lemma 4.5, and Fact 3.2. Note that the case $p = 1$ and $k = 0$ follows directly from Lemma 3.4. \square

4.2. A general upper bound. In the following lemma, we denote the first entry of the RLCT by $\text{rlct}_\Lambda(\mathcal{I}_{p,k}(\Sigma_0); 1)$. Note that the cases of Lemma 4.7 and Theorem 4.8 with $d_r > p(p+1)/2$ are covered by Lemma 3.4.

Lemma 4.7. *Let $\Sigma_0 \in \mathbb{R}^{p \times p}$ be a fixed covariance matrix that is chosen generically from the r -factor model for some $r \in \{0, \dots, k\}$. If $d_r \leq p(p+1)/2$, then*

$$\min_{\Lambda} \text{rlct}_\Lambda(\mathcal{I}_{p,k}(\Sigma_0); 1) \leq \frac{pk + r(p-k+1)}{2},$$

where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times k}$ with $\Sigma_k(\psi, \Lambda) = \Sigma_0$ for some $\psi \in \mathbb{R}_+^p$.

Proof. By the considerations in Section 3.4, we may assume that $\Sigma_0 = LL^T + \psi$, where $\psi \in \mathbb{R}_+^p$ and $L \in \mathbb{R}^{p \times r}$ of the form

$$L = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix}$$

with L_{11} a lower triangular $(r \times r)$ -matrix of full rank. Moreover, by Fact 3.5, we can replace in the definition of $\mathcal{I}_{p,k}(\Sigma_0)$ the $(p \times k)$ -matrix of indeterminates Λ by a $(p \times k)$ -matrix of indeterminates of the form

$$\Lambda = \begin{pmatrix} \Lambda_{11} & 0 \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix},$$

where Λ_{11} is a lower triangular $(r \times r)$ -matrix on whose diagonal entries we impose that they be positive. We call this representation *LQ-coordinates*. In these, the generators of $\mathcal{I}_{m,k}(\Sigma_0)$ are the off-diagonal entries of the matrix

$$(4.3) \quad \begin{aligned} & \begin{pmatrix} \Lambda_{11} & 0 \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} \Lambda_{11} & 0 \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}^T - \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix}^T \\ &= \left[\begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix} \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix}^T - \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix}^T \right] + \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{22} \Lambda_{22}^T \end{pmatrix}. \end{aligned}$$

In particular, each generator is the sum of a term formed from the entries of $(\Lambda_{11}, \Lambda_{21})$ and a second term formed from the entries of Λ_{22} .

We have that the off-diagonal entries of the first matrix on the right-hand side of (4.3), in square brackets, generate the ideal $\mathcal{I}_{p,r}(\Sigma_0)$ in LQ-coordinates and the off-diagonal entries of the second matrix generate $\mathcal{I}_{p-r,k-r}(0)$. Applying the inequality

$$(x+y)^2 \leq 2x^2 + 2y^2$$

to each generator in turn, we see that the sum of squares of the generators of $\mathcal{I}_{p,k}(\Sigma_0)$ is bounded above by two times the sum of the two sums of squares that are associated with $\mathcal{I}_{p,r}(\Sigma_0)$ and $\mathcal{I}_{p-r,k-r}(0)$, respectively. We deduce that

$$\text{rlct}_\Lambda(\mathcal{I}_{p,k}(\Sigma_0); 1) \leq \text{rlct}_{\Lambda_{11}}(\mathcal{I}_{p,r}(\Sigma_0); 1) + \text{rlct}_0(\mathcal{I}_{p-r,k-r}(0); 1)$$

for every $\Lambda \in \mathbb{R}^{p \times k}$ in LQ-coordinates. The claimed upper bound now follows, because

$$\text{rlct}_{\Lambda_{11}}(\mathcal{I}_{p,r}(\Sigma_0); 1) = pr - \frac{r(r-1)}{2}$$

by Lemma 3.4 and

$$\text{rlct}_0(\mathcal{I}_{p-r,k-r}(0); 1) \leq \frac{(p-r)(k-r)}{2}$$

by Lemmas 4.3, 4.4 and 4.5. \square

Theorem 4.8. *Let $\ell_k(\Sigma_0)$ be the learning coefficient of the factor analysis model with k latent factors at a fixed generic covariance matrix Σ_0 in the r -factor model, where $r \in \{0, \dots, k\}$. If $d_r = p(r+1) - r(r-1)/2 \leq p(p+1)/2$, then*

$$\ell_k(\Sigma_0) \leq \frac{p(k+2) + r(p-k+1)}{4}.$$

Note that the inequality just given is an equality for $r = 0$ (Theorem 4.6) and $r = k$ (Lemma 3.4). The values for $1 \leq r \leq m-1$ are equally spaced between those extremes. The case $d_r > p(p+1)/2$ is covered by Lemma 3.4.

Proof. The theorem follows immediately from Lemma 4.7 and Fact 3.2. \square

Conjecture 4.9. *The bound from Theorem 4.8 is tight for all $p > k$ and r satisfying the assumptions of the theorem. What is more, $\ell_{kr} = (p(k+2) + r(p-k+1))/4$ for all such p , k and r .*

Remark 4.10. We continue the line of thought from the proof of Theorem 4.8 by applying one further coordinate transformation and manipulations on the generators of the resulting ideals. This will be used in Section 4.3 below to handle the case $r = 1$. Note that

$$(4.4) \quad \begin{aligned} & \left[\begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix} \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix}^T - \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix}^T \right] + \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{22}\Lambda_{22}^T \end{pmatrix} \\ &= \begin{pmatrix} \Lambda_{11}\Lambda_{11}^T - L_{11}L_{11}^T & \Lambda_{11}\Lambda_{21}^T - L_{11}L_{21}^T \\ \Lambda_{21}\Lambda_{11}^T - L_{21}L_{11}^T & \Lambda_{22}\Lambda_{22}^T + \Lambda_{21}\Lambda_{21}^T - L_{21}L_{21}^T \end{pmatrix}. \end{aligned}$$

Since Λ_{11} is invertible in LQ-coordinates, we may apply the real analytic isomorphism using

$$\Lambda'_{21} = \Lambda_{21}\Lambda_{11}^T - L_{21}L_{11}^T$$

leaving the variables in Λ_{11} and Λ_{22} fixed, whose inverse on the Λ_{21} -coordinates is

$$\Lambda_{21} = (\Lambda'_{21} + L_{21}L_{11}^T)\Lambda_{11}^{-T}.$$

In the new coordinates, after dropping primes, our ideal is generated by the off-diagonal entries of

$$\begin{pmatrix} \Lambda_{11}\Lambda_{11}^T - L_{11}L_{11}^T & \Lambda_{21}^T \\ \Lambda_{21} & \Lambda_{22}\Lambda_{22}^T + (\Lambda_{21} + L_{21}L_{11}^T)\Lambda_{11}^{-T}\Lambda_{11}^{-1}(\Lambda_{21}^T + L_{11}L_{21}^T) - L_{21}L_{21}^T \end{pmatrix}.$$

We may use the generators in Λ_{21} to remove terms from the generators in the lower right block of the matrix. Doing so, we obtain that the ideal is generated by the off-diagonal entries of

$$(4.5) \quad \begin{pmatrix} \Lambda_{11}\Lambda_{11}^T - L_{11}L_{11}^T & \Lambda_{21}^T \\ \Lambda_{21} & \Lambda_{22}\Lambda_{22}^T + L_{21}L_{11}^T\Lambda_{11}^{-T}\Lambda_{11}^{-1}L_{11}L_{21}^T - L_{21}L_{21}^T \end{pmatrix}.$$

Since Λ_{21} has $(p-r)r$ entries, it follows by Fact 2.6(1) that the RLCT of the ideal is $((p-r)r, 0)$ plus the RLCT of the ideal generated by the off-diagonal entries of the two matrices

$$(4.6) \quad \Lambda_{11}\Lambda_{11}^T - L_{11}L_{11}^T$$

and

$$(4.7) \quad \Lambda_{22}\Lambda_{22}^T + L_{21}L_{11}^T \left[(\Lambda_{11}\Lambda_{11}^T)^{-1} - (L_{11}L_{11}^T)^{-1} \right] L_{11}L_{21}^T.$$

The former matrix is of size $r \times r$ and the latter is of size $(p-r) \times (p-r)$.

4.3. Generic one-factor covariance matrices.

Lemma 4.11. *Let $p > 1$ and let $\Sigma_0 \in \mathbb{R}^{p \times p}$ in the 1-factor model with p -dimensional observations be chosen generically (in the sense that all of its off-diagonal entries are non-zero). If $p > 3$ then*

$$\text{RLCT}_\Lambda(\mathcal{I}_{p,2}(\Sigma_0); 1) = \left(\frac{3p-1}{2}, 1 \right)$$

and

$$\text{RLCT}_\Lambda(\mathcal{I}_{2,2}(\Sigma_0); 1) = (1, 1),$$

$$\text{RLCT}_\Lambda(\mathcal{I}_{3,2}(\Sigma_0); 1) = (3, 1),$$

where the minima range over all $\Lambda \in \mathbb{R}^{p \times 2}$ with $\psi + \Lambda\Lambda^T = \Sigma_0$ for some $\psi \in \mathbb{R}_+^p$ and analogously for $p \in \{2, 3\}$, respectively.

Proof. The case $p = 2$ is covered by Lemma 3.4(2). So, suppose $p \geq 3$. The covariance matrix Σ_0 being in \mathcal{M}_1 means that its off-diagonal entries are products $\gamma_i\gamma_j$ of the entries of a vector $\gamma \in \mathbb{R}^m$. The genericity assumption in the statement of the lemma translates to all entries of γ being non-zero. By Lemma 3.3, we may assume that γ is the vector all of whose entries are equal to 1, that is, all off-diagonal entries of Σ_0 are equal to 1.

By Fact 3.5, we may only consider factor loading matrices of the form

$$\Lambda = \begin{pmatrix} \lambda_{11} & 0 \\ \lambda_{21} & \lambda_{22} \\ \vdots & \vdots \end{pmatrix}.$$

The reduced fiber ideal then is

$$\mathcal{I} = \mathcal{I}_{p,2}(\Sigma_0) = \langle \lambda_{11}\lambda_{i1} - 1, \lambda_{i1}\lambda_{j1} + \lambda_{i2}\lambda_{j2} - 1 \mid 2 \leq i, j \leq p, i \neq j \rangle.$$

Since $\lambda_{j1} \cdot (\lambda_{11}\lambda_{i1} - 1) - \lambda_{i1} \cdot (\lambda_{11}\lambda_{j1} - 1) = \lambda_{i1} - \lambda_{j1}$ we obtain

$$\mathcal{I} = \langle \lambda_{11}\lambda_{p1} - 1, \lambda_{i1} - \lambda_{p1}, \lambda_{p1}^2 + \lambda_{i2}\lambda_{j2} - 1 : 2 \leq i, j \leq p, i \neq j \rangle.$$

We apply the real analytic isomorphism

$$\lambda_{11} = \lambda'_{11}, \quad \lambda_{i1} = \lambda'_{i1} + \lambda'_{p1} \quad (2 \leq i \leq p-1), \quad \lambda_{p1} = \lambda'_{p1}$$

whose Jacobian determinant equals 1. After dropping primes, the pullback of the ideal \mathcal{I} is

$$(4.8) \quad \mathcal{J} + \langle \lambda_{i1} \mid 2 \leq i \leq p-1 \rangle,$$

with

$$(4.9) \quad \mathcal{J} = \langle \lambda_{11}\lambda_{p1} - 1, \lambda_{p1}^2 + \lambda_{i2}\lambda_{j2} - 1 \mid 2 \leq i < j \leq p \rangle.$$

In the sequel, we compute the RLCT associated with \mathcal{J} whose second component will be computed as 1. Using the sum rule (Fact 2.6(1)) and Example 2.4, the first component of the RLCT for \mathcal{I} is larger than the one of \mathcal{J} by $p-2$ and its second component is 1.

To do so, we compute a decomposition of the variety $V(\mathcal{J})$ (whose ambient space is a $(p+1)$ -dimensional affine space) into smaller varieties. Note that the extension of \mathcal{J} to the ring $\mathbb{R}[\lambda_{11}, \lambda_{p1}, \lambda_{22}, \dots, \lambda_{p2}][\lambda_{p1}^2 - 1]$ equals the ideal

$$Q = \langle \lambda_{11}\lambda_{p1} - 1, \lambda_{i2} - \lambda_{p2}, \lambda_{p1}^2 + \lambda_{p2}^2 - 1 \mid 2 \leq i \leq p-1 \rangle.$$

Its $(p+1) \times p$ Jacobian matrix has the following form:

$$\begin{bmatrix} \lambda_{p1} & 0 & \cdots & 0 & 0 \\ \lambda_{11} & 0 & \cdots & 0 & 2\lambda_{\mu_1} \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & -1 & \cdots & -1 & 2\lambda_{p2} \end{bmatrix},$$

This matrix has full rank p whenever $\lambda_{11}\lambda_{p1} \neq 0$ which is never the case on the variety defined by Q as $\lambda_{11}\lambda_{p1} - 1 \in Q$. So, Q defines a (complex) smooth curve.

The intersection of $V(\mathcal{J})$ with the two hyperplanes $\lambda_{p1} = 1$ and $\lambda_{p1} = -1$ decompose into the lines defined by the ideals

$$I_l = \langle \lambda_{11} - 1, \lambda_{p1} - 1, \lambda_{i2} \mid 2 \leq i \leq p, i \neq l \rangle, \quad l = 2, \dots, p$$

and

$$\langle \lambda_{11} + 1, \lambda_{p1} + 1, \lambda_{i2} \mid 2 \leq i \leq p, i \neq l \rangle, \quad l = 2, \dots, p,$$

respectively. Consequently, \mathcal{J} is the intersection of these ideals and Q , and $V(\mathcal{J})$ is the union of the smooth varieties defined by them. In particular, the singular locus of $V(\mathcal{J})$ consists of the pairwise intersections of these components. Only two points lie in such intersections, namely,

$$w = (\lambda_{11}, \lambda_{p1}, \lambda_{22}, \dots, \lambda_{p2}) = (1, 1, 0, \dots, 0)$$

and $w' = -w$. Outside these points, the RLCT of \mathcal{J} equals $(p, 1)$ by Example 2.4. By symmetry, $\text{RLCT}_w(\mathcal{J}; 1) = \text{RLCT}_{-w}(\mathcal{J}; 1)$, and we will see, a posteriori, that this pair is smaller than $(p, 1)$. So, for the remainder of this proof, it suffices to consider the point w .

We begin by translating w to the origin by changing λ_{i1} -variables as $\lambda_{11} = \lambda'_{11} + 1$ and $\lambda_{p1} = \lambda'_{p1} + 1$. The pullback of \mathcal{J} under this map is

$$\mathcal{J}_w = \langle \lambda_{11}\lambda_{p1} + \lambda_{11} + \lambda_{p1}, \lambda_{p1}^2 + 2\lambda_{p1} + \lambda_{i2}\lambda_{j2} \mid 2 \leq i < j \leq p \rangle.$$

We further apply the real analytic map defined by

$$\lambda'_{11} = \lambda_{11}\lambda_{p1} + \lambda_{11} + \lambda_{p1}, \quad \lambda'_{p1} = \lambda_{p1}^2 + 2\lambda_{p1} + \lambda_{22}\lambda_{32}, \quad \lambda'_{i2} = \lambda_{i2} \quad \text{for } 2 \leq i \leq p,$$

whose Jacobian determinant equals $2(\lambda_{11} + 1)^2$ which is non-zero on the real variety defined by \mathcal{J}_w and can, hence, be ignored. Dropping the primes and simplifying, the resulting ideal is

$$(4.10) \quad \langle \lambda_{11}, \lambda_{p1} \rangle + \mathcal{J}_1,$$

where

$$(4.11) \quad \mathcal{J}_1 = \langle \lambda_{i2}\lambda_{j2} - \lambda_{22}\lambda_{32} : 2 \leq i < j \leq p \rangle$$

$$(4.12) \quad = \langle \lambda_{i2}\lambda_{j2} - \lambda_{k2}\lambda_{l2} : 2 \leq i < j \leq p, 2 \leq k < l \leq p \rangle.$$

Again, by Fact 2.6(1) and Example 2.4, we further reduced the problem to computing $\text{RLCT}_0(\mathcal{J}_1; 1)$ by adding 2 to its first component.

For $p = 3$, the ideal \mathcal{J}_1 equals 0 and, hence, we compute the desired minimum of RLCTs as $\text{RLCT}_0(\langle \lambda_{11}, \lambda_{p1} \rangle; 1) + (p-2, 0) = (2, 1) + (1, 0) = (3, 1)$.

So, suppose that $p > 3$. Here, we apply the blow-up at the origin to \mathcal{J}_1 , see Section 2.4. The generators of the ideal \mathcal{J}_1 in (4.12) are invariant under every permutation of the variables. So, all charts of the blow-up are the same and we may just consider one of them, say the chart corresponding to λ_{22} . Using the generators in (4.11), the pullback of \mathcal{J}_1 in this chart is

$$\langle \lambda_{22}^2(\lambda_{i2} - \lambda_{32}), \lambda_{22}^2(\lambda_{j2}\lambda_{k2} - \lambda_{32}) : 3 < i, 2 < j < k \leq p \rangle,$$

and we have the Jacobian determinant λ_{22}^{p-2} . Making the change of variables

$$\lambda'_{22} = \lambda_{22}, \quad \lambda'_{i2} = \lambda_{i2} - \lambda_{32} \quad \text{for } 3 < j \leq p,$$

yields the ideal

$$(4.13) \quad \langle \lambda_{22}^2\lambda_{32}(\lambda_{32} - 1), \lambda_{22}^2\lambda_{i2} : 3 < i \leq p \rangle$$

There are two intersection points between the strict transform defined by $\langle \lambda_{32}(\lambda_{32} - 1), \lambda_{i2} : 3 < i \leq p \rangle$ and the exceptional divisor defined by $\langle \lambda_{22}^2 \rangle$, namely

$$(\lambda_{22}, \lambda_{32}, \lambda_{42}, \dots, \lambda_{m2}) = (0, 0, 0, \dots, 0)$$

$$(\lambda_{22}, \lambda_{32}, \lambda_{42}, \dots, \lambda_{m2}) = (0, 1, 0, \dots, 0).$$

By symmetry, the RLCT at both points are the same, so we consider the first point. As $\lambda_{32} - 1$ is non-zero at this point and, hence, is a unit in the ring of real analytic functions on a small neighborhood, we may instead consider the ideal

$$\mathcal{J}_0 = \langle \lambda_{22}^2 \rangle \cdot \langle \lambda_{32}, \dots, \lambda_{p2} \rangle.$$

By Example 2.4, we can now compute

$$\text{RLCT}_0(\langle \lambda_{32}, \dots, \lambda_{p2} \rangle; 1) = (p-2, 1).$$

Moreover, using for instance [LUSB14, Theorem 7.1] or the Newton polyhedron method from Section 2.2, we infer

$$\text{RLCT}_0(\langle \lambda_{22}^2 \rangle; \lambda_{22}^{p-2}) = ((p-1)/2, 1).$$

As $p > 3$, we get that $(p-1)/2 < p-2$ and, so, the product rule (Fact 2.6(2)) yields

$$\text{RLCT}_0(\mathcal{J}_0; \lambda_{22}^{p-2}) = ((p-1)/2, 1).$$

Finally, using the sum rule (Fact 2.6(1)), the desired minimum of RLCTs is $((p-1)/2, 1) + (2, 0) + (p-2, 0) = ((3p-1)/2, 1)$, where the second summand comes from (4.10) and the third summand comes from (4.8). \square

Proposition 4.12. *Let $p > 3$. The learning coefficient and its order of the factor analysis model with observations of dimension $p > 3$ and $k = 2$ latent factors along the submodel with $r = 1$ latent factor satisfy*

$$\ell_{21} = \frac{5p - 1}{4} \quad \text{and} \quad m_{21} = 1.$$

In the case of 2-dimensional observations, the invariants are $\ell_{21} = 3/2$ and $m_{21} = 1$. In the case of 3-dimensional observations, the invariants are $\ell_{21} = 3$ and $m_{21} = 1$.

Proof. The proposition follows immediately from Lemma 4.11 and Fact 3.2. \square

Remark 4.13. We specialize the set of generators from Remark 4.10 to the case $r = 1$. The matrix $\Lambda_{11}\Lambda_{11}^T - L_{11}L_{11}^T$ from (4.6) is of size 1×1 here and, hence, has no off-diagonal entries. So, we are left with the off-diagonal entries of the following matrix from (4.7):

$$\Lambda_{22}\Lambda_{22}^T + L_{21}L_{11}^T \left[(\Lambda_{11}\Lambda_{11}^T)^{-1} - (L_{11}L_{11}^T)^{-1} \right] L_{11}L_{21}^T$$

Note that $\Lambda_{11} > 0$. Also, L_{11} consist of a single entry and that L_{21} is a vector. As before, we assume that Σ_0 is generic in the sense that all of its off-diagonal entries are non-zero. This is equivalent to saying that $L_{11} \neq 0$ and all entries of L_{21} are non-zero. Using Lemma 3.3, we may assume that actually $L_{11} = 1$ and L_{21} is a vector consisting of ones. So, the above matrix simplifies to

$$\Lambda_{22}\Lambda_{22}^T + \left(\frac{1}{\Lambda_{11}^2} - 1 \right) \cdot \mathbf{1},$$

where $\mathbf{1}$ is a $(p-1) \times (p-1)$ -matrix of ones. Leaving the Λ_{22} -coordinates fixed and applying $a = \frac{1}{\Lambda_{11}^2} - 1$ whose inverse is $\Lambda_{11} = \frac{1}{\sqrt{1+a}}$ gives a real analytic isomorphism. It pulls back the ideal generated by the off-diagonal entries of the matrix above to

$$\mathcal{J}_{p,k} = \langle a + \tilde{\lambda}_i \tilde{\lambda}_j^T \mid 2 \leq i < j \leq p \rangle,$$

where $\tilde{\lambda}_i$ is the i -th row of Λ_{22} which is of length $k-1$. Following Remark 4.10, we then have

$$\text{RLCT}_\Lambda(\mathcal{I}_{p,k}(\mathbf{1}); 1) = \text{RLCT}_{(a, \Lambda_{22})}(\mathcal{J}_{p,k}; 1) + (p-1, 0).$$

Lemma 4.14. *Let $k \geq 1$ and $p \geq k+2$. Then*

$$\min_{(a, \Lambda_{22})} \text{RLCT}_{(a, \Lambda_{22})}(\mathcal{J}_{p+1,k+1}; 1) = \left(\frac{pk}{2} + 1, 1 \right),$$

where the minimum ranges over all (a, Λ_{22}) whose image under the transformation in Remark 4.13 is $\Lambda \in \mathbb{R}^{p \times k}$ in LQ -coordinates such that the off-diagonal entries of $\Lambda\Lambda^T$ equal 1. In particular, if Σ_0 is chosen generically from the 1-factor model then

$$\min_{\Lambda} \text{RLCT}_\Lambda(\mathcal{I}_{p,k}(\Sigma_0); 1) = \left(\frac{pk + p - k + 1}{2}, 1 \right),$$

where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times k}$ such that $\Sigma_0 = \psi + \Lambda\Lambda^T$ for some $\psi \in \mathbb{R}_+^p$.

With analogous notation for $p \geq 2$ and $k \in \{p-1, p\}$,

$$\min_{(a, \Lambda_{22})} \text{RLCT}_{(a, \Lambda_{22})}(\mathcal{J}_{p+1,k+1}; 1) = \left(\frac{p(p-1)}{2}, 1 \right)$$

and

$$\min_{\Lambda} \text{RLCT}_\Lambda(\mathcal{I}_{p,k}(\Sigma_0); 1) = \left(\frac{p(p-1)}{2}, 1 \right).$$

Proof. We proceed by induction on k . The base case $k = 1$ follows directly from Lemma 4.11 and Remark 4.13 because

$$\min_{(a, \Lambda_{22})} \text{RLCT}_{(a, \Lambda_{22})}(\mathcal{J}_{p+1, 1+1}; 1) = \left(\frac{3(p+1)-1}{2}, 1 \right) - ((p+1)-1, 0) = \left(\frac{p+1}{2} + 1, 1 \right).$$

Now, let $k \geq 2$. For simplicity of notation, we denote $\Lambda_{22} = (\gamma_{ij})$ with $1 \leq i \leq p$ and $1 \leq j \leq k$. We write γ_i for the i -th row of Λ_{22} . Changing generators, we may write $\mathcal{J}_{p+1, k+1} = \mathcal{J}_1 + \mathcal{J}_2$, where $\mathcal{J}_1 = \langle a + \gamma_1 \gamma_2^T \rangle$ and

$$(4.14) \quad \mathcal{J}_2 = \langle \gamma_i \gamma_j^T - \gamma_1 \gamma_2^T \mid 1 \leq i < j \leq p, (i, j) \neq (1, 1) \rangle$$

$$(4.15) \quad = \langle \gamma_i \gamma_j - \gamma_e \gamma_f \mid 1 \leq i < j \leq p, 1 \leq e < f \leq p \rangle.$$

Now we apply the real analytic map with Jacobian determinant equal to 1 given by $a' = a + \gamma_i \gamma_j^T$ that leaves the γ_i -coordinates unchanged. Under this map, \mathcal{J}_2 is left unchanged while \mathcal{J}_1 is transformed to $\langle a' \rangle$ which adds $(1, 0)$ to the RLCT by Fact 2.6(1) and Example 2.4.

So, we only need to consider \mathcal{J}_2 which is homogeneous and hence admits its minimal RLCT at 0. We blow up at this point. The generators of \mathcal{J}_2 in (4.15) are invariant under the action of the symmetric group and, hence, we just have to consider one chart, say the one corresponding to the variable γ_{11} . Its Jacobian determinant is γ_{11}^{pk-1} and it pulls back \mathcal{J}_2 to $\langle \gamma_{11}^2 \rangle \cdot \mathcal{K}$, where

$$\mathcal{K} = \langle \gamma_i \gamma_j^T - \gamma_1 \gamma_2^T \mid \gamma_{11} = 1, 1 \leq i < j \leq p \rangle.$$

We can compute $\text{RLCT}_0(\langle \gamma_{11}^2 \rangle; \gamma_{11}^{pk-1}) = (pk/2, 1)$, see for instance [LUSB14, Theorem 7.1]. By the product rule (Fact 2.6(2)), it remains to compute the minimal RLCT of \mathcal{K} with respect to amplitude 1.

The generators of \mathcal{K} are stable under replacing $\Lambda_{22} = (\gamma_{ij})$ by $Q\Lambda_{22}Q^T$, where Q is a orthonormal matrix. As $\gamma_{11} = 1$, the matrix Λ_{22} varies over a small set inside the space of matrices of rank at least 1. So, we can apply LQ-decomposition (Section 3.4) and assume without changing the RLCT that Λ_{22} varies over $\mathcal{L}_{1,+}^{p,k}$. In particular, we may write

$$\mathcal{K} = \langle \gamma_i \gamma_j^T - \gamma_1 \gamma_2^T \mid \gamma_{12} = \dots = \gamma_{1p} = 0, 1 \leq i < j \leq p \rangle,$$

where γ_{11} only takes values greater than 0 or, in other words, is a unit in the ring of analytic functions the ideal \mathcal{K} is considered in. So, writing $\tilde{\gamma}_i$ for the i -th row of Λ_{22} from which we deleted the first column, \mathcal{K} simplifies to

$$\begin{aligned} \mathcal{K} &= \langle \gamma_{11}(\gamma_{j1} - \gamma_{21}) \mid 3 \leq j \leq p \rangle + \langle \gamma_i \gamma_j^T - \gamma_{11} \gamma_{21} \mid 2 \leq i < j \leq p \rangle \\ &= \langle \gamma_{j1} - \gamma_{21} \mid 3 \leq j \leq p \rangle + \langle \gamma_{i1} \gamma_{j1} + \tilde{\gamma}_i \tilde{\gamma}_j^T - \gamma_{11} \gamma_{21} \mid 2 \leq i < j \leq p \rangle \\ &= \langle \gamma_{j1} - \gamma_{21} \mid 3 \leq j \leq p \rangle + \langle \gamma_{21}^2 - \gamma_{11} \gamma_{21} + \tilde{\gamma}_i \tilde{\gamma}_j^T \mid 2 \leq i < j \leq p \rangle. \end{aligned}$$

We apply the substitution $a = \gamma_{21}^2 - \gamma_{21} \gamma_{11}$, sending all variables except γ_{21} to itself. The Jacobian determinant of this map is $2\gamma_{21} - \gamma_{11}$. As \mathcal{K} is homogeneous in the entries of the γ_i , by Fact 2.2, we can restrict these variables to an arbitrarily small neighborhood of the origin.

Assume to the contrary that $2\gamma_{21} - \gamma_{11} = 0$ for a point on the variety defined by \mathcal{K} which implies, in particular, $\gamma_{21} > 0$. Then, this point satisfies $\tilde{\gamma}_i \tilde{\gamma}_j^T = \gamma_{21}^2$ for all $2 \leq i < j \leq p$. As this relation is homogeneous and, hence, invariant under scaling, γ_{21} can be assumed arbitrarily small. But this is a contradiction to γ_{11} being bounded away from 0 and $\gamma_{11} = 2\gamma_{21}$.

So, this Jacobian determinant can be assumed to be non-zero and, hence, we can ignore it. Transforming \mathcal{K} under this map and applying the substitution $\gamma_{j1} = \gamma_{j1} - \gamma_{21}$ yields

$$\langle \gamma_{j1} \mid 3 \leq j \leq p \rangle + \langle a + \tilde{\gamma}_i \tilde{\gamma}_j^T \mid 2 \leq i < j \leq p \rangle,$$

where the second summand is just $\mathcal{J}_{p,k}$. So, by Fact 2.6(1) and Example 2.4, the minimal RLCT of \mathcal{K} is just $(p-2, 0)$ plus the minimal RLCT of $\mathcal{J}_{p,k}$.

We distinguish two cases. Suppose first that $k \leq p-2$. Then, by the induction hypothesis, the minimal RLCT of $\mathcal{J}_{p,k}$ is $((p-1)(k-1)/2, 1)$. As $pk/2 < p-2 + (p-1)(k-1)/2$ for $k \leq p-2$ and taking into account the summand $(1, 0)$ from the ideal $\langle a' \rangle$, the product rule (Fact 2.6(2)) implies that the minimal RLCT of $\mathcal{J}_{p+1,k+1}$ equals $(1, 0) + (pk/2, 1) = (pk/2 + 1, 1)$ as we claimed.

Now, suppose that $k \in \{p-1, p\}$. In this case, by the induction hypothesis, the minimal RLCT of $\mathcal{J}_{p,k}$ is $((p-2)(p-1)/2, 1)$. As $p-2 + (p-2)(p-1)/2 = (p^2 - p - 2)/2 < pk/2$ for $k \in \{p-1, p\}$, and, again, taking into account the summand $(1, 0)$, we infer that the minimal RLCT of $\mathcal{J}_{p+1,k+1}$ equals $((p^2 - p - 2)/2, 1) + (1, 0) = ((p-1)p/2, 1)$ as asserted. \square

Theorem 4.15. *The learning coefficient and its order of the factor analysis model with p -dimensional observations and $k \leq p-2$ latent factors along the submodel with $r=1$ latent factor satisfy*

$$\ell_{k1} = \frac{pk + 3p - k + 1}{4} \quad \text{and} \quad m_{k1} = 1.$$

If $k \in \{p-1, p\}$ then

$$\ell_{k1} = \frac{p(p+1)}{4} \quad \text{and} \quad m_{k1} = 1.$$

Proof. This follows immediately from Lemma 4.14 and Fact 3.2. \square

5. DISTINCT ONE-FACTOR MODEL SINGULARITIES

In this final section, we show that the genericity assumption on Σ_0 is necessary in general. More precisely, we show that, the 1-factor model \mathcal{M}_1 decomposes into exactly three strata with distinct learning coefficient. So, we now consider models with $k=1$ factors, in which case $\Lambda = (\lambda_1, \dots, \lambda_p)$ varies over vectors in \mathbb{R}^p .

The singularities of the one-factor parametrization Σ_1 are vectors Λ_0 with no more than two non-zero entries, see [AR56]. We distinguish three cases for Λ_0 leading to distinct values of the learning coefficients at the covariance matrices associated to Λ_0 under Σ_1 :

- (a) Λ_0 has more than two non-zero entries;
- (b) Λ_0 has exactly two non-zero entries;
- (c) Λ_0 has at most one non-zero entry.

Case (a) is the generic case (along the 1-factor model) covered by Lemma 3.4 and the learning coefficient equals p . In case (c), a covariance matrix Σ_0 associated with Λ_0 is diagonal which corresponds to the rank 0 case we discussed in Proposition 4.2. The corresponding learning coefficient is $3p/4$. In case (b), Σ_0 has precisely two non-zero off-diagonal entries. This is a singularity that is not covered by the rank 0 case and, as we will show, has different asymptotic behavior:

Proposition 5.1. *If Σ_0 is a positive definite matrix with precisely two non-zero off-diagonal entries then*

$$\ell_1(\Sigma_0) = \frac{2p-1}{2} \quad \text{and} \quad m_1(\Sigma_0) = 1.$$

Proof. Suppose without loss of generality that $\sigma_{12} = \sigma_{21}$ are the two non-zero off-diagonal entries of Σ_0 . The reduced fiber ideal at Σ_0 is

$$\mathcal{I} := \mathcal{I}_{p,1}(\Sigma_0) = \langle \lambda_1\lambda_2 - \sigma_{12}, \lambda_1\lambda_i, \lambda_j\lambda_l \mid 3 \leq i \leq p, 2 \leq j, l \leq p, j \neq l \rangle.$$

The ideal contains

$$\lambda_2 \cdot (\lambda_1\lambda_i) - \lambda_i \cdot (\lambda_1\lambda_2 - \sigma_{12}) = \sigma_{12} \cdot \lambda_i$$

for all $i \geq 3$. It follows that

$$\mathcal{I} = \langle \lambda_1\lambda_2 - \sigma_{12}, \lambda_i \mid 3 \leq i \leq p \rangle.$$

In light of Fact 2.5, we apply the map

$$\lambda_1 = (\lambda'_1 + \sigma_{12})/\lambda'_2, \quad \lambda_i = \lambda'_i \text{ for } i \geq 2.$$

Dropping the primes, this map has Jacobian determinant $1/|\lambda_2|$. As $\sigma_{12} \neq 0$, the ideal \mathcal{I} does not vanish at any point of $U = \{\lambda_2 = 0\}$ and hence we can compute the RLCT away from U . The Jacobian determinant can, therefore, be ignored. The pullback of the ideal \mathcal{I} under this map is

$$\langle \lambda_1, \lambda_i \mid 3 \leq i \leq p \rangle.$$

By Example 2.4, we infer that $\min_{\Lambda} \text{RLCT}_{\Lambda}(\mathcal{I}; 1) = (p-1, 1)$, where the minimum ranges over all $\Lambda \in \mathbb{R}^{p \times 1}$ with $\Sigma_0 = \psi + \Lambda\Lambda^T$ for some $\psi \in \mathbb{R}_+^p$. As usual, Fact 3.2 yields the learning coefficient as conjectured. \square

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