

Convergence and stability of Q-learning in Hierarchical Reinforcement Learning

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Abstract

Hierarchical Reinforcement Learning promises, among other benefits, to efficiently capture and utilize the temporal structure of a decision-making problem and to enhance continual learning capabilities, but theoretical guarantees lag behind practice. In this paper, we propose a Feudal Q-learning scheme and investigate under which conditions its coupled updates converge and are stable. By leveraging the theory of Stochastic Approximation and the ODE method, we present a theorem stating the convergence and stability properties of Feudal Q-learning. This provides a principled convergence and stability analysis tailored to Feudal RL. Moreover, we show that the updates converge to a point that can be interpreted as an equilibrium of a suitably defined game, opening the door to game-theoretic approaches to Hierarchical RL. Lastly, experiments based on the Feudal Q-learning algorithm support the outcomes anticipated by theory.

Keywords: Hierarchical Reinforcement Learning, Feudal Q-learning, Two-Timescale Stochastic Approximation, ODE method.

1 Introduction

Decision-making architectures have played a central role for decades [1] both in engineering and other domains, e.g., guidance, navigation and control of Apollo missions [2], chemical plants [3], smart grids [4], unmanned aerial vehicles [5], recommender systems [6], and algorithms [7]. Moreover, architectures are ubiquitous in nature, e.g., diversity in the nervous system enables humans to have fast and accurate sensorimotor control [8]. Reinforcement Learning (RL) is a framework in which an agent learns to make sequential decisions through interaction with an environment in order to maximize cumulative reward [9]. Decision-making architectures have also been proposed and studied in RL. Hierarchical Reinforcement Learning (HRL) is a subfield of RL that deals with hierarchical structures for decision-making agents. Prospective advantages include improved long-term credit assignment, continual learning, interpretability, and the integration of preexisting policies [10], [11]. Many approaches have been proposed in HRL, e.g., Feudal RL [12], [13], Options [14], [15], MAXQ [16], and HAM [17]. Moreover, HRL has been shown to be effective in practice [18], [19], [20], [13], [21]. Nonetheless, the theoretical foundations of HRL remain underdeveloped compared with those of RL. Moreover, it is known that in HRL, non-stationarity issues can arise [18], [19]. The main goal of our work is to contribute to the Feudal RL theory by proposing a Feudal Q-learning algorithm with proven theoretical guarantees.

In a two-level Feudal RL algorithm, two policies are interacting with each other. The high-level policy learns which goals to set for the low-level policy, which, in turn, learns how to achieve

them. Broadly speaking, Feudal Q-learning consists of two interconnected Q-learning update rules that influence each other using data generated from a stochastic process. To rigorously analyze such intertwined updates under stochastic data, we rely on the Stochastic Approximation (SA) framework [22], [23], and in particular on its Two-Timescale variant (TTSA) [24]. By leveraging the ODE method [24], [25], we study the update rules as coupled dynamical systems, which directly tackles the non-stationarity problem. We consider infinite-horizon Markov Decision Processes (MDPs) with finite state and action spaces and, by analyzing Feudal Q-learning through the lens of TTSA, we are able to prove convergence and stability of the updates. To the best of the authors' knowledge, our work is the first to prove stability of an HRL algorithm and convergence for Feudal Q-learning updates that are interdependent. The closest works in the literature are [26], [27]. In [26], the authors define a high-level infinite-horizon MDP and a low-level episodic MDP that is independent from the high-level policy. Assuming convergence in the low-level MDP, convergence in the high-level MDP is proven [26, Theorem 2]. Our work differs from [26] as two interconnected infinite-horizon MDPs are considered, and a coupled two-timescale analysis is performed, showing both convergence and stability of the updates. Our setup can bring some benefits, for example, it is easy to consider an extrinsic reward in the low-level infinite-horizon problem, which is shown to be beneficial in practice [13]. Moreover, in our framework, we do not need to assume that the duration of the low-level episode is sampled from a geometric distribution, which can potentially result in arbitrary long episodes. Another drawback of the setup in [26] is that the choice of the low-level discount factor is tied to the choice of the expected episode duration. In [27], the work in [26] is extended by considering the convergence of Q-learning deep RL methods with non-stationary features that are convergent. Another benefit of the approach we propose is that, by directly linking Feudal RL to the theory of TTSA, one could leverage results present in the vast literature of TTSA, e.g., finite-time guarantees [28], [29], or multi-timescales [30] that would enable deeper hierarchies. Thus, TTSA theory is not only employed to establish convergence and stability, but also to introduce a new perspective on HRL, by demonstrating that a Feudal RL problem can be rigorously cast within the TTSA framework. This is an important step towards closing the gap between theory and practice in HRL, as seeing the HRL problem as a TTSA one can help in designing new and efficient algorithms with theoretical guarantees.

An additional contribution of our work is to formalize a game theoretic interpretation of this HRL problem. Game theory and RL are closely related fields, as both deal with optimal decision-making. Probably the most popular connection is between game theory and multi-agent RL (MARL) [31], as MARL involves multiple agents interacting in a shared environment. It is interesting to note a more subtle connection between HRL and game theory: even though the original problem setup involves finding only one agent, we are decomposing the original MDP into multiple levels, thus we can see each level as an agent in a multi-agent system. We highlight that our analysis shows that the updates converge to a point that can be interpreted as a Nash and Stackelberg equilibrium, opening the door to game-theoretic approaches to HRL.

Notation Consider a function f defined over a finite set X , we indicate with $\|\cdot\|_\infty$ the infinity norm, i.e., $\|f\|_\infty = \max_{x \in X} \{|f(x)|\}$ and with $\|\cdot\|_2$ the 2-norm, i.e., $\|f\|_2 = \sqrt{\sum_{x \in X} (f(x))^2}$. Consider an $m \times n$ matrix A with element a_{ij} in the i -th row j -th column, we indicate with $\text{vec}(\cdot)$ the vectorization of the matrix obtained by stacking its columns, i.e., $[a_{1,1}, \dots, a_{m,1}, \dots, a_{1,n}, \dots, a_{m,n}]^\top = \text{vec}(A)$. The floor function $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ is defined as $\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\}$.

2 Problem Formulation

Consider an MDP, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, P, r, \gamma \rangle$, where \mathcal{S} is a finite state space, \mathcal{A} is a finite action space, $P(s_{k+1}|s_k, a_k) : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \rightarrow [0, 1]$ is the dynamic, defining the probability of transitioning to the next state $s_{k+1} \in \mathcal{S}$ given the current state $s_k \in \mathcal{S}$ and action $a_k \in \mathcal{A}$, $r : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ is the deterministic reward function, and $\gamma \in (0, 1)$ is the discount factor. A general goal in RL is to find a policy $\pi : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$ that maximizes, for every state s , $\mathbb{E}_{a \sim \pi}[Q_\pi(s, a)]$, where $Q_\pi : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ is defined as:

$$Q_\pi(s, a) := \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{k=0}^N \gamma^k r(s_k, a_k) \middle| s_0 = s, a_0 = a \right] \quad (1)$$

and the expectation is taken according to $s_{k+1} \sim P(\cdot | s_k, a_k)$, $a_k \sim \pi(\cdot | s_k)$.

Feudal RL solves the MDP by decomposing the flat policy π into a hierarchical structure, see Figure 1. To mathematically formalize the problem, we propose two fictitious MDPs: the high-level MDP, $\mathcal{M}^h := \langle \mathcal{S}^h, \Omega, P_{\pi^h}^h, r_{\pi^h}^h, \gamma^h \rangle$, and the low-level MDP, $\mathcal{M}^l := \langle \mathcal{S}^l, \mathcal{A}, P_{\pi^h}^l, r^l, \gamma^l \rangle$. Every T time steps, the high-level policy, π^h , selects a goal, $\omega_t^h \in \Omega$, that the low-level policy, π^l , should achieve. To achieve the goal, the low-level policy interacts with the system at every time step by selecting actions $a_k \in \mathcal{A}$.

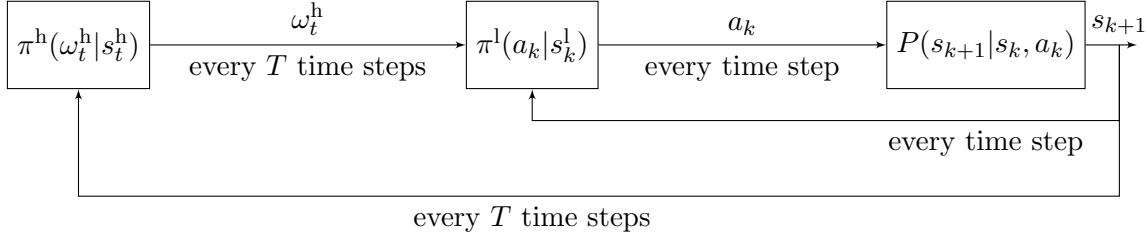


Figure 1: Block diagram of the Feudal RL scheme. Instead of having a flat policy interacting with the system, there is a high-level policy, π^h , selecting goals for a low-level policy, π^l , that interacts with the system P .

A few general considerations on \mathcal{M}^h and \mathcal{M}^l are in order. Firstly, the designer has the freedom of choice in the goal space Ω and in the low-level reward r^l . For example, Ω can coincide with \mathcal{S} (or, more generally, a subset of the power set of \mathcal{S}) and r^l can be a combination of r and of a distance function describing how close the system is to the goal. While there is some freedom of choice in $r_{\pi^h}^h$ as well, it is often convenient to define it as a function of the extrinsic reward r . Secondly, elements of one MDP, e.g., $P_{\pi^h}^h$ and $r_{\pi^h}^h$, depend on the policy of the other MDP. This interdependence is key to capture the non-stationarity issues arising in HRL. Lastly, \mathcal{M}^h and \mathcal{M}^l are running in parallel with potentially different time-scales: one time step in the high-level MDP corresponds to T time steps in the low-level MDP. For this reason, we use t to indicate the time step of the high-level MDP and k for the time step of the low-level MDP, as shown in Figure 2.

Two standard assumptions in theoretical RL, boundedness of the rewards and finiteness of the sets [32], [33], are needed to prove the convergence of the proposed Feudal Q-learning updates.

Assumption 1 (Bounded rewards). *The reward function r and the low-level reward r^l are bounded, i.e., $\exists \bar{r}, \bar{r}^l > 0$ such that $\|r(s, a)\|_\infty \leq \bar{r}$ and $\|r^l(s^l, a)\|_\infty \leq \bar{r}^l$.*

Assumption 2 (Finite sets). *The sets \mathcal{S} , \mathcal{A} , Ω are finite.*

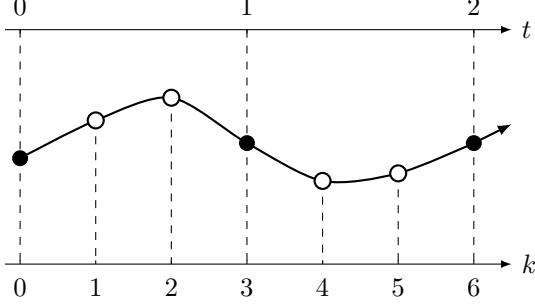


Figure 2: Time scales of the high-level and low-level MDPs. Circles represent the states of the real system, while black circles indicate the states where the high-level MDP makes a decision. In this example, the high-level MDP makes a decision every $T = 3$ time steps of the low-level MDP.

2.1 High-level MDP

Let us now introduce the quantities composing the high-level MDP $\mathcal{M}^h = \langle \mathcal{S}^h, \Omega, P_{\pi^1}^h, r_{\pi^1}^h, \gamma^h \rangle$. We choose $\mathcal{S}^h = \mathcal{S}$, and, for ease of exposition and without loss of generality, $\Omega = \mathcal{S}$. The variable $s_t^h \in \mathcal{S}^h$ represents the state of the high-level MDP at time t and corresponds to the original MDP state s_{tT} , whereas $\omega_t^h \in \Omega$ is the action of the high-level MDP at time t , i.e., the goal for the low-level MDP. $\pi^h : \mathcal{S}^h \times \Omega \rightarrow [0, 1]$ represents the high-level policy selecting goals. $P_{\pi^1}^h(s_{t+1}^h | s_t^h, \omega_t^h)$ and $r_{\pi^1}^h(s_t^h, \omega_t^h)$ describe, respectively, the dynamic and the reward in the high-level MDP. Before describing in more detail $P_{\pi^1}^h$ and $r_{\pi^1}^h$, let us introduce some useful quantities.

A decision epoch is a time step in the low-level MDP where the high-level MDP has made a decision, i.e., all the low-level time steps belonging to the set $\{0, T, 2T, \dots\}$. In order to take into account the time spent between decision epochs, we introduce the discrete variable $s_k^{\text{de}} \in \mathcal{S}^{\text{de}}$, where $\mathcal{S}^{\text{de}} := \{0, \dots, T - 1\}$. The variable s_k^{de} is recursively defined as:

$$s_{k+1}^{\text{de}} = \begin{cases} s_k^{\text{de}} + 1, & \text{if } s_k^{\text{de}} \neq T - 1 \\ 0, & \text{if } s_k^{\text{de}} = T - 1 \end{cases}, \quad \text{with } s_0^{\text{de}} = 0.$$

Given the deterministic evolution of s^{de} , one can easily find the probability $P^{\text{de}}(s_{k+1}^{\text{de}} | s_k^{\text{de}})$ of transitioning to s_{k+1}^{de} given s_k^{de} . At time step k , the low-level policy $\pi^1 : \mathcal{S}^l \times \mathcal{A} \rightarrow [0, 1]$ selects an action $a_k \in \mathcal{A}$ to apply to the real system. We are now ready to present $P_{\pi^1}^h$ and $r_{\pi^1}^h$.

The high-level dynamic is:

$$P_{\pi^1}^h(s_{t+1}^h | s_t^h, \omega_t^h) = \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i | s_{i-1}, a_{i-1}) \pi^1(a_{i-1} | s_{i-1}, \omega_{i-1}^h, s_{i-1}^{\text{de}}) \quad (2)$$

where $s_0 = s_t^h$, $s_T = s_{t+1}^h$, and $\omega_i^h = \omega_t^h$, $s_i^{\text{de}} = i$, $\forall i \in \{0, \dots, T - 1\}$. The steps for obtaining Eq. (2) can be found in Appendix B. It is clear from Eq. (2) that the dynamic of the high-level MDP depends on the low-level policy, we highlight this dependence by placing the subscript π^1 in $P_{\pi^1}^h$.

Since the high-level policy should learn how to solve the original MDP \mathcal{M} , the reward $r_{\pi^1}^h$ should depend on r , the reward of the original MDP. The high-level reward is:

$$r_{\pi^1}^h(s^h, \omega^h) = \mathbb{E}_{\pi^1} \left[\sum_{k=0}^{T-1} (\gamma^h)^k r(s_k, a_k) | s_0 = s^h \right] \quad (3)$$

where $\gamma^h \in (0, 1)$.

The high-level Q-function $Q_{\pi^h, \pi^l}^h : \mathcal{S}^h \times \Omega \rightarrow \mathbb{R}$ is defined as:

$$Q_{\pi^h, \pi^l}^h(s^h, \omega^h) := \lim_{N \rightarrow \infty} \mathbb{E}_{\pi^h, \pi^l} \left[\sum_{t=0}^N (\gamma^h)^t r_{\pi^l}^h(s_t^h, \omega_t^h) \mid s_0^h = s^h, \omega_0^h = \omega^h \right] \quad (4)$$

With the definitions of $r_{\pi^l}^h$ and Q_{π^h, π^l}^h , and by setting $\gamma^h = \gamma$, if we consider a trajectory of the system, the experienced value of the high-level Q-function in Eq. (4) corresponds to the one of the original MDP in Eq. (1). In this way, it is possible to compare the performance of a flat RL algorithm and a Feudal one.

2.2 Low-level MDP

Recall that the low-level MDP is $\mathcal{M}^l := \langle \mathcal{S}^l, \mathcal{A}, P_{\pi^h}^l, r^l, \gamma^l \rangle$. Here, the state space is $\mathcal{S}^l = \mathcal{S} \times \Omega \times \mathcal{S}^{de}$. The low-level state $s_k^l := (s_k, \omega_k^l, s_k^{de}) \in \mathcal{S}^l$ is composed by the state of the real system s_k , the goal to be tracked $\omega_k^l = \omega_{\lfloor \frac{k}{T} \rfloor}^h$, and s_k^{de} . The low-level dynamic is:

$$P_{\pi^h}^l(s_{k+1}^l | s_k^l, a_k^l) = \begin{cases} 0, & \text{if } s_k^{de} \neq T-1 \text{ and } \omega_{k+1}^l \neq \omega_k^l \\ P^{de}(s_{k+1}^{de} | s_k^{de}) P(s_{k+1} | s_k, a_k), & \text{if } s_k^{de} \neq T-1 \text{ and } \omega_{k+1}^l = \omega_k^l \\ P^{de}(s_{k+1}^{de} | s_k^{de}) P(s_{k+1} | s_k, a_k) \pi^h(\omega_{k+1}^l | s_k), & \text{if } s_k^{de} = T-1 \end{cases} \quad (5)$$

The steps for obtaining Eq. (5) can be found in Appendix B. Analogously to the high-level dynamic, also here we note that $P_{\pi^h}^l$ depends on the high-level policy.

As mentioned earlier, there is no fixed choice of the low-level reward r^l , which is a free design parameter. The low-level Q-function $Q_{\pi^h, \pi^l}^l : \mathcal{S}^l \times \mathcal{A} \rightarrow \mathbb{R}$ is defined as:

$$Q_{\pi^h, \pi^l}^l(s^l, a) := \lim_{N \rightarrow \infty} \mathbb{E}_{\pi^h, \pi^l} \left[\sum_{k=0}^N (\gamma^l)^k r^l(s_k^l, a_k) \mid s_0^l = s^l, a_0 = a \right] \quad (6)$$

where $\gamma^l \in (0, 1)$.

2.3 Problem statement

Having defined the high- and low-level MDPs, we can now state the problem we want to solve. Our goal is to find the optimal high- and low-level Q-functions, $Q^{h,*} : \mathcal{S}^h \times \Omega \rightarrow \mathbb{R}$ and $Q^{l,*} : \mathcal{S}^l \times \mathcal{A} \rightarrow \mathbb{R}$, that satisfy, for every state-action pair, the Bellman equations:

$$Q^{h,*}(s_t^h, \omega_t^h) = r_{\pi^{l,*}}^h(s_t^h, \omega_t^h) + (\gamma^h)^T \mathbb{E}_{\pi^{l,*}} \left[\max_{\omega_{t+1}^h} Q^{h,*}(s_{t+1}^h, \omega_{t+1}^h) \right] \quad (7a)$$

$$Q^{l,*}(s_k^l, a_k) = r^l(s_k^l, a_k) + \gamma^l \mathbb{E}_{\pi^{h,*}} \left[\max_{a_{k+1}} Q^{l,*}(s_{k+1}^l, a_{k+1}) \right] \quad (7b)$$

$$\text{with } \pi^{h,*} \in \arg \max_{\pi^h} Q_{\pi^h, \pi^{l,*}}^h \text{ and } \pi^{l,*} \in \arg \max_{\pi^l} Q_{\pi^{h,*}, \pi^l}^l \quad (7c)$$

It is interesting to note that the Bellman equation for one level, e.g., Eq. (7a), depends on the policy of the other level, e.g, $\pi^{l,*}$. This is a non-stationarity issue typical of HRL algorithms that complicates their analysis [18], [19].

The goal that we want to reach is to jointly learn optimal Q-functions, while the other level is using the optimal policy.

Table 1 summarizes the main variables and symbols presented so far.

Symbol	Meaning
a_k	Low-level action at time step k .
\mathcal{A}	Action space of the original MDP.
$\gamma, \gamma^h, \gamma^l$	Discount factors.
$\mathcal{M}, \mathcal{M}^h, \mathcal{M}^l$	Original, high-, and low-level MDP.
ω_t^h	High-level action at time step t .
ω_k^l	Goal, at time step k , that the low-level policy should achieve.
Ω	Action space for the high-level MDP.
$P, P_{\pi^l}^h, P_{\pi^h}^l$	Dynamic of the original, high-, and low-level MDP.
P^{de}	Dynamic of s^{de} .
π, π^h, π^l	Flat, high-, and low-level policy.
$\pi^{h,*}, \pi^{l,*}$	Optimal high- and low-level policy (Eq. (7c)).
Q_π	Q-function of a (flat) policy π .
$Q_{\pi^h, \pi^l}^h, Q_{\pi^h, \pi^l}^l$	High- and low-level Q-function.
$Q^{h,*}, Q^{l,*}$	Optimal high- and low-level Q-function (Eq. (7)).
r	Reward of the original MDP.
$r_{\pi^l}^h, r^l$	High- and low-level reward.
\bar{r}, \bar{r}^l	Bounds on the reward functions.
$\mathcal{S}, \mathcal{S}^h, \mathcal{S}^l$	Original, high-, and low-level state space.
s_k	State of the original MDP at time step k .
s_t^h	High-level state at time t (corresponds to s_{tT}).
s_k^l	Low-level state at time k (corresponds to $(s_k, \omega_k^l, s_k^{\text{de}})$).
s_k^{de}	Variable considering the time spent between decision epochs.
S^{de}	Set $\{0, \dots, T - 1\}$ to which s_k^{de} belongs.
T	Number of low-level time steps between consecutive high-level decisions.

Table 1: Table summarizing the main symbols.

3 Convergence and stability guarantees of Feudal Q-learning

In this Section, in order to solve the problem in Eq. (7), we will propose an update rule based on Q-learning. Since the TTSAs theory constitutes our primary theoretical framework, we will briefly recall its main elements and employ it to derive insights into the behaviour of the proposed Feudal Q-learning updates. The Feudal Q-learning updates, $Q^{h,n}$ and $Q^{l,n}$, where $n > 0$ represents the iteration, will be shown to converge to the optimal high- and low-level Q-functions, $Q^{h,*}$ and $Q^{l,*}$ in Eq. (7). Moreover, we will prove that the learning process is stable, thus $Q^{h,n}$ and $Q^{l,n}$ will always have finite values.

3.1 Feudal Q-learning updates

As the Feudal Q-learning updates will depend on trajectories collected through experiments or simulations, before presenting the update rule, we will make an assumption on how the data is collected.

Assumption 3 (On-policy updates and Lipschitz continuous policies). *The policies, $\pi^{h,n}$ and $\pi^{l,n}$, that collect trajectories used to update $Q^{h,n}$ and $Q^{l,n}$ depend respectively on $Q^{h,n}$ and $Q^{l,n}$, i.e., $\pi^{h,n} = \mathcal{F}^{h,n}(Q^{h,n})$ and $\pi^{l,n} = \mathcal{F}^{l,n}(Q^{l,n})$, where the operators $\mathcal{F}^{h,n} : \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \rightarrow \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|}$ and $\mathcal{F}^{l,n} : \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \rightarrow \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|}$ denote the policy extraction methods.*

Moreover, the high-level policy is Lipschitz continuous with respect to $Q^{h,n}$, i.e., $\forall n, \exists L_{\pi^{h,n}}^{Q^h} \in \mathbb{R}_{\geq 0} : \forall (Q_1^h, Q_2^h) \in \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \times \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} :$

$$|\mathcal{F}^{h,n}(Q_1^h) - \mathcal{F}^{h,n}(Q_2^h)| \leq L_{\pi^{h,n}}^{Q^h} \|Q_1^h - Q_2^h\|_{\infty}$$

An analogous assumption holds for the low-level policy, with a constant $L_{\pi^l}^{Q^l}$.

Concerning Assumption 3, a Boltzmann exploration policy satisfies the Lipschitz continuous assumption [34].

We are now ready to present the Feudal Q-learning updates:

$$\begin{aligned} Q^{h,n+1}(s_t^h, \omega_t^h) = & Q^{h,n}(s_t^h, \omega_t^h) + \\ & + \beta(n) \left[\sum_{k=0}^{T-1} (\gamma^h)^k r(s_{t+k}, a_{t+k}) + (\gamma^h)^T \max_{\omega} Q^{h,n}(s_{t+1}^h, \omega) - Q^{h,n}(s_t^h, \omega_t^h) \right] \end{aligned} \quad (8a)$$

$$\begin{aligned} Q^{l,n+1}(s_k^l, a_k) = & Q^{l,n}(s_k^l, a_k) + \\ & + \alpha(n) [r^l(s_k^l, a_k) + \gamma^l \max_a Q^{l,n}(s_{k+1}^l, a) - Q^{l,n}(s_k^l, a_k)] \end{aligned} \quad (8b)$$

with step-sizes $\alpha(n), \beta(n) > 0$.

There is no requirement on how to initialize the iterative updates, e.g., $Q^{h,0}$ and $Q^{l,0}$ can be set to zero or they can be optimistically initialized [35].

3.2 TTSAs and its connection to Feudal Q-learning

Pioneered by [22], SA algorithms are a family of iterative methods to solve root finding problems, e.g., finding $z^* \in \mathbb{R}^d : f(z^*) = 0$, where only noisy data is available, e.g., we only have access to $f(z_n) + M_{n+1}$, where M_{n+1} is a martingale difference sequence playing the role of the noise. A classic Single Timescale SA scheme can be written as:

$$z_{n+1} = z_n + \alpha(n)[f(z_n) + M_{n+1}] \quad (9)$$

where $z_n \in \mathbb{R}^d$ are the iterates of the algorithm. When referring to Two-Timescale SA schemes, the updates are:

$$x_{n+1} = x_n + \alpha(n)[h(x_n, y_n) + M_{n+1}^{(1)}], \quad (10a)$$

$$y_{n+1} = y_n + \beta(n)[g(x_n, y_n) + M_{n+1}^{(2)}], \quad (10b)$$

where x_n and y_n can be thought of as the Two-Timescale counterpart of z_n .

Two fundamental properties of an SA scheme are convergence and stability of the algorithm. Convergence refers to having the iterates converging almost surely (a.s.) to z^* , stability indicates that the iterates remain bounded (preferably without the need of projecting the iterates into a predefined set) [23]. A notable method to analyze the convergence and stability of SA schemes is the so-called ODE method [25]. The central idea is to show that the iterate z_n "tracks" the continuous time ODE $\dot{z}(t) = f(z(t))$, whose solutions converge to z^* . The ODE method is supported by a rich theoretical foundation both for the Single Timescale case and the Two-Timescale case [23], [24], [25], [36], [37].

Concerning our Feudal Q-learning problem, we firstly cast the updates in Eq. (8) as Eq. (10), with x_n and y_n respectively representing $Q^{l,n}$ and $Q^{h,n}$. Secondly, similar to [23, Chapter 8], we study the related singularly perturbed ODEs:

$$\dot{x}(t) = \frac{1}{\epsilon} h(x(t), y(t)), \quad (11a)$$

$$\dot{y}(t) = g(x(t), y(t)) \quad (11b)$$

that the iterates x_n and y_n should track. The vector-valued function h has as component $h_{s^l, a}(x_n, y_n)$, related to the low-level state and action pair (s^l, a) , the low-level Bellman error:

$$h_{s^l, a}(x_n, y_n) := r^l(s^l, a) + \gamma^l \mathbb{E}_{P_{\pi^h}^l} [\max_{a^+} Q^{l,n}(s^{l,+}, a^+)] - Q^{l,n}(s^l, a) \quad (12)$$

while the vector-valued function g has as component $g_{s^h, \omega^h}(x_n, y_n)$, related to the high-level state and action pair (s^h, ω^h) , the high-level Bellman error:

$$g_{s^h, \omega^h}(x_n, y_n) := r_{\pi^l}^h(s^h, \omega^h) + (\gamma^h)^T \mathbb{E}_{P_{\pi^l}^h} [\max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+)] - Q^{h,n}(s^h, \omega^h) \quad (13)$$

For more details, see Appendix A. The parameter $\epsilon > 0$ is the singular perturbation parameter and, by having $\epsilon \downarrow 0$, we impose a faster, $x(t)$, and a slower, $y(t)$, dynamic. This timescale separation is obtained in the Feudal Q-learning algorithm by imposing the condition $\frac{\beta(n)}{\alpha(n)} \xrightarrow{n \rightarrow \infty} 0$ on the step-sizes, which motivates part of the following assumption; the remaining parts are standard Robbins-Monro conditions [22].

Assumption 4 (Step-sizes). *The step-sizes satisfy the following conditions:*

a) *the step-sizes are not summable:*

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N \alpha(n) = \lim_{N \rightarrow \infty} \sum_{n=0}^N \beta(n) = \infty,$$

b) *the step-sizes are square-summable:*

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N (\alpha(n)^2 + \beta(n)^2) < \infty,$$

c) in the limit, the ratio between the step-sizes satisfies:

$$\lim_{n \rightarrow \infty} \frac{\beta(n)}{\alpha(n)} = 0.$$

Within the TTSA framework, the timescale separation ensures that the iterates of the algorithm in Eq. (10) "track" the solutions of the following ODEs:

$$\frac{dx}{dt} = h(x(t), y) \quad (14)$$

where y is held fixed as a constant parameter, and:

$$\frac{dy}{dt} = g(\lambda(y(t)), y(t)) \quad (15)$$

where $\lambda : \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \rightarrow \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|}$.

One can easily see from Eqs. (12) and (13) that equilibria of the ODEs (14) and (15) correspond to solutions of the Bellman equations. Thus, studying the asymptotic behaviour of the ODEs, as done in Lemmas 1 and 2, is of great interest and sheds light on the behaviour of the algorithm.

Lemma 1 (Existence and property of the globally asymptotically stable equilibrium of h). *The ODE (14) has a globally asymptotically stable equilibrium $\lambda(y)$. Moreover, λ is a Lipschitz map.*

Lemma 2 (Existence of the globally asymptotically stable equilibrium of g). *The ODE (15) has a globally asymptotically stable equilibrium y^* .*

For a proof of Lemmas 1 and 2, which will be used to prove Theorem 1, see Appendix C.

An interpretation of Lemma 1 is that, for a fixed high-level Q-function, y , the low-level Q-function, x , converges to the solution of the low-level Bellman equation. Thus, while the high-level Q-function is slowly varying, the low-level Q-function quickly converges to the optimal low-level Q-function. Even though it is quasi-static from the low-level Q-function perspective, the high-level Q-function is not constant. Lemma 2 is telling us that the high-level Q-function converges to the solution of the high-level Bellman equation, where the low-level Q-function is assumed to be always at its optimal value.

3.3 Theoretical guarantees

After establishing the remaining assumptions, this subsection presents the main theoretical result of the paper.

Since we are using an on-policy algorithm, we would like to be greedy in the limit, in order to converge to the optimal policies. Furthermore, we would like to assure infinite state-action visitation, which is a standard requirement in Q-learning [38]. For this reason, we require the following assumptions, which are stated for the sake of readability only for the high-level policy, but apply to the low-level one as well.

Assumption 5 (Greedy in the limit policies). *Assume, for the sake of simplicity, that for a given state s^h , $Q^{h,n}(s^h, \omega)$ has a unique maximizer ω^h . The high-level policy is greedy in the limit, i.e., as $n \rightarrow \infty$:*

$$\pi^{h,n}(\omega^h | s^h) = \begin{cases} 1, & \omega^h = \arg \max_{\omega \in \Omega} Q^{h,n}(s^h, \omega) \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

Assumption 6 (Infinite exploration policies). Consider a high-level trajectory $(s_t^h, \omega_t^h)_{t \geq 0}$ and denote with $n_t(s^h, \omega^h)$ the number of times action ω^h has been chosen in state s^h during the first t time steps, i.e., $n_t(s^h, \omega^h) := \sum_{i=0}^{t-1} \mathbb{1}(s_i^h = s^h, \omega_i^h = \omega^h)$, where $\mathbb{1}(\cdot)$ represents the indicator function. We assume that, $\forall (s^h, \omega^h) \in \mathcal{S}^h \times \Omega$, $\lim_{t \rightarrow \infty} n_t(s^h, \omega^h) = \infty$ a.s.

Policies satisfying Assumptions 5 and 6 are referred to as Greedy in the Limit with Infinite Exploration (GLIE), see [39] for a Boltzmann exploration policy and an ϵ -greedy policy that are GLIE.

An additional assumption is needed to prove the stability of the algorithm [36].

Assumption 7 (Compact convergence). The policies $\pi_c^h := \mathcal{F}^{h,n}(cQ^h)$, $c \geq 1$, satisfy $\pi_c^h \rightarrow \pi_\infty^h$ as $c \rightarrow \infty$, uniformly on compacts for some π_∞^h .

See [36, Proposition 11] for a Boltzmann policy that satisfies Assumption 7. Moreover, an analogous assumption applies for the low-level policy.

Having laid out the necessary assumptions, we are now in a position to state the main theorem.

Theorem 1 (Convergence and stability of Feudal Q-learning). If Assumptions 1-7 hold, then:

a) Feudal Q-learning converges to the optimal values:

$$\lim_{n \rightarrow \infty} (Q^{h,n}, Q^{l,n}) = (Q^{h,*}, Q^{l,*}), \text{ a.s.}; \quad (17)$$

b) the iterates $(Q^{h,n}, Q^{l,n})$ are stable:

$$\sup_n (||Q^{h,n}|| + ||Q^{l,n}||) < \infty, \text{ a.s.} \quad (18)$$

Proof. See Appendix C. □

This result provides the first convergence and stability guarantees on Q-learning for Feudal RL. While relying on classical techniques used for analysis of RL algorithms [25], [33], [40], this result represents an important addition to the existing literature. Firstly, in order to leverage the existing theory on TTSA, the updates had to be cast as on-policy updates (Assumption 3). Otherwise, the terms $M_{n+1}^{(1)}$ and $M_{n+1}^{(2)}$ would not be martingale difference sequences, resulting in a nonvanishing bias. Secondly, having cast the Feudal Q-learning updates within the TTSA framework and interpreted the algorithm as coupled slow and fast updates, one can view the design of HRL algorithms as a problem of synthesizing the underlying ODEs. In particular, modifying the mean fields h and g or introducing structured gains [41] offer a principled route to derive variants with improved transient behaviour or asymptotic properties. Lastly, as reported in Section 3.2, our result sheds light on the behaviour of HRL algorithms. Qualitatively, faster updates in the low-level, which translates in our setup in bigger step-sizes for the low-level, are beneficial for convergence.

4 Game-Theoretic Interpretation

In this section, we highlight the connection between HRL and game theory. In particular, we focus on the proposed Feudal Q-learning scheme and discuss how it can be interpreted as a two-player game, whose solution is, in this special case, both a Nash equilibrium and a Stackelberg equilibrium.

Learning in the high- and low-level MDP can be seen as a two-player game, where the high-level player is trying to maximize Q_{π^h, π^l}^h and the low-level player is trying to maximize Q_{π^h, π^l}^l . The agents are learning Q-functions $Q^{h,n}$ and $Q^{l,n}$ and, through mappings $\mathcal{F}^{h,n}$ and $\mathcal{F}^{l,n}$, use them

to derive their policies, which will be greedy in the limit as in Assumption 5. For this reason, with a slight abuse of notation, we sometimes replace the policy symbol with a Q-function. We then define the optimal Bellman operators $\mathcal{T}_{Q^h}^h : \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \rightarrow \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|}$ and $\mathcal{T}_{Q^l}^l : \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \rightarrow \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|}$ as:

$$(\mathcal{T}_{Q^h}^h Q^h)(s_t^h, \omega_t^h) := r_{Q^h}^h(s_t^h, \omega_t^h) + (\gamma^h)^T \mathbb{E}_{Q^h} \left[\max_{\omega_{t+1}^h} Q^h(s_{t+1}^h, \omega_{t+1}^h) \right], \quad \forall (s_t^h, \omega_t^h) \in \mathcal{S}^h \times \Omega \quad (19)$$

$$(\mathcal{T}_{Q^l}^l Q^l)(s_k^l, a_k) := r^l(s_k^l, a_k) + \gamma^l \mathbb{E}_{Q^l} \left[\max_{a_{k+1}} Q^l(s_{k+1}^l, a_{k+1}) \right], \quad \forall (s_k^l, a_k) \in \mathcal{S}^l \times \mathcal{A} \quad (20)$$

A standard result in RL states that a greedy policy with respect to the optimal Q-function, i.e., the Q-function that is a fixed point of the optimal Bellman operator, represents an optimal policy [9, Chapter 3]. This motivates the following definitions [42]. We call high-level rational reaction set, D^h , and low-level rational reaction set, D^l , the sets:

$$D^h := \{(Q^h, Q^l) \in \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \times \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \mid Q^h = \mathcal{T}_{Q^h}^h Q^h\} \quad (21)$$

$$D^l := \{(Q^h, Q^l) \in \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \times \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \mid Q^l = \mathcal{T}_{Q^l}^l Q^l\} \quad (22)$$

For completeness, we introduce the definitions of Nash equilibrium and Stackelberg equilibrium [42], [43].

Definition 1 (Nash equilibrium). *A pair (\bar{Q}^h, \bar{Q}^l) such that $(\bar{Q}^h, \bar{Q}^l) \in D^h \cap D^l$ is a Nash equilibrium.*

Definition 2 (Stackelberg equilibrium). *A pair $(\bar{Q}^h, \bar{Q}^l) \in D^l$ such that:*

$$Q_{\bar{Q}^h, \bar{Q}^l}^h \geq Q_{Q^h, Q^l}^h, \quad \forall (Q^h, Q^l) \in D^l. \quad (23)$$

is a Stackelberg equilibrium with player \bar{Q}^h as a leader and player \bar{Q}^l as a follower.

Now, we are ready to state and prove Lemma 3, connecting our HRL updates to game theory.

Lemma 3. *The pair $(Q^{h,*}, Q^{l,*})$ found by the Feudal Q-learning updates is both a Nash equilibrium and a Stackelberg equilibrium where the high-level is the leader and the low-level is the follower.*

Proof. In order to show that $(Q^{h,*}, Q^{l,*})$ is a Nash equilibrium, we need to show that $(Q^{h,*}, Q^{l,*}) \in D^h \cap D^l$. It is immediate to verify it noticing that $Q^{h,*}$ and $Q^{l,*}$ are satisfying Eqs. (7a) and (7b), i.e., they are verifying the conditions $Q^{h,*} = \mathcal{T}_{Q^{l,*}}^h Q^{h,*}$ and $Q^{l,*} = \mathcal{T}_{Q^{h,*}}^l Q^{l,*}$.

We now want to show that the solution found by the Feudal Q-learning updates is a Stackelberg equilibrium, where the high-level is the leader and the low-level is the follower. In order to show it, we need to show that:

1. $(Q^{h,*}, Q^{l,*}) \in D^l$,
2. $Q_{Q^{h,*}, Q^{l,*}}^h \geq Q_{Q^h, Q^l}^h, \quad \forall (Q^h, Q^l) \in D^l$.

Point 1. is satisfied using the result stated in the paragraph on Nash equilibrium, since $(Q^{h,*}, Q^{l,*}) \in D^h \cap D^l$ implies that $(Q^{h,*}, Q^{l,*}) \in D^l$.

Point 2. can be shown by contradiction. Assume that there exists one or more pairs $(Q^h, Q^l) \in D^l$ such that $Q_{Q^h, Q^l}^h > Q_{Q^{h,*}, Q^{l,*}}^h$ and $(Q^h, Q^l) \neq (Q^{h,*}, Q^{l,*})$. Let us call \mathcal{C} the non-empty set of such pairs.

Let us consider the pair (\bar{Q}^h, \bar{Q}^l) that is maximizing Q_{Q^h, Q^l}^h , i.e., $Q_{\bar{Q}^h, \bar{Q}^l}^h \geq Q_{Q^h, Q^l}^h \quad \forall (Q^h, Q^l) \in \mathcal{C}$.

The condition $(\bar{Q}^h, \bar{Q}^l) \in D^l$ implies satisfying Eq. (7b), i.e., finding the unique fixed point satisfying $\bar{Q}^l = \mathcal{T}_{Q^h}^l \bar{Q}^l$.

For a given low-level Q-function, Q^l , the optimal high-level Q-function is unique and it is the only fixed point of the operator $\mathcal{T}_{Q^h}^l$. Thus, $\bar{Q}^h = \mathcal{T}_{\bar{Q}^l}^h \bar{Q}^h$. Thus we have found the unique pair (\bar{Q}^h, \bar{Q}^l) satisfying $(\bar{Q}^h, \bar{Q}^l) \in D^h \cap D^l$. Being the unique pair satisfying $(\bar{Q}^h, \bar{Q}^l) \in D^h \cap D^l$ and since $(Q^{h,*}, Q^{l,*})$ satisfies the condition, then $Q_{\bar{Q}^h, \bar{Q}^l}^h = Q_{Q^{h,*}, Q^{l,*}}^h$ and $(\bar{Q}^h, \bar{Q}^l) = (Q^{h,*}, Q^{l,*})$, reaching a contradiction.

Moreover, the other pairs (Q^h, Q^l) in \mathcal{C} either lead to $Q_{\bar{Q}^h, \bar{Q}^l}^h = Q_{Q^{h,*}, Q^{l,*}}^h$ or $Q_{\bar{Q}^h, \bar{Q}^l}^h > Q_{Q^{h,*}, Q^{l,*}}^h$. On the one hand, the equality case can be ruled out with arguments similar to the ones used for the pair (\bar{Q}^h, \bar{Q}^l) . On the other hand, the inequality case would result in (Q^h, Q^l) pairs for which $Q_{Q^{h,*}, Q^{l,*}}^h < Q_{\bar{Q}^h, \bar{Q}^l}^h$.

Thus, there is no other pair $(Q^h, Q^l) \in D^l$ such that $Q_{Q^{h,*}, Q^{l,*}}^h > Q_{Q^{h,*}, Q^{l,*}}^h$. \square

5 Numerical Experiments

Based on the proposed Feudal Q-learning updates in Eqs. 8a and 8b, we test our HRL algorithm in the Four Rooms Minigrid environment [44] shown in Figure 3. The goal is to make the agent, starting always in the same initial state s_0 (the red triangle in Figure 3 represents the initial pose), enter a specific position on the map (green tile). At each time step, the agent can either turn 90° clockwise, 90° counterclockwise, or move forward. The high-level policy chooses tiles from a subset of \mathcal{S} as the goal for the low-level policy, and it receives a positive reward when the agent moves from a black tile to the green tile; otherwise, it gets no reward. The low-level policy receives a positive reward when the agent has reached the goal position set by the high-level policy.

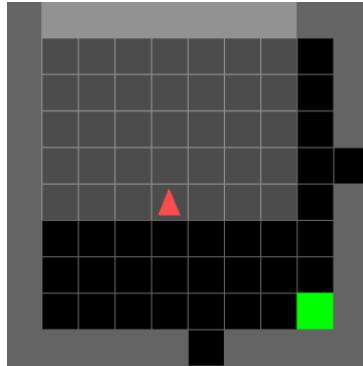


Figure 3: First room of the Four Rooms Minigrid environment. The red triangle represents the initial pose of the agent, while the green square is the tile to reach. Grey tiles represent walls, while black tiles are walkable areas. Lit tiles help recognize the direction the agent is facing.

In the first experiment, we train the HRL agent for 5,000 episodes, where the green tile is positioned as in Figure 3. An episode consists of the agent interacting with the environment for 300 low-level time steps. After $\approx 1,870$ episodes, the agent has learnt a high-level policy and a low-level one that enable it to achieve a cumulative reward of 49 consistently. A standard Watkins' Q-learning agent [38] also converges to a policy that achieves a cumulative reward of 49, see Figure 4.

In the second experiment, we retrain the same agent, i.e., the agent initialized with the high- and low-level policies obtained at the end of the first experiment, for 5,000 episodes. For this experiment, we moved the green tile to the bottom-left corner of the same room. In this case,

the HRL agent converges after $\approx 1,450$ episodes. It consistently achieves a cumulative reward of 49, which, in our experiments, represents the maximum cumulative reward achieved by Watkins' Q-learning algorithm in the environment of the second experiment.

We plot in Figure 4 the cumulative reward per episode of the two HRL trainings and of a Watkins' Q-learning training, with the plots smoothed using a moving average that considers 50 episodes. We want to highlight that the hyperparameters of the experiments have not been optimized; thus, we foresee that better performances, e.g., faster convergence, can be obtained by properly tuning the hyperparameters. A study of the algorithm's finite-time performances is out of the scope of the paper; it should consider hyperparameter tuning and averaging of different trainings.

The experiments show the convergent behaviour predicted by Theorem 1. Moreover, from the two experiments, we see how Feudal Q-learning helps with continual learning: retraining the agent with a different goal (as in the second experiment) requires $\approx 22\%$ fewer episodes for convergence. This likely reflects that the low-level policy retains useful goal-tracking behaviour, easing adaptation.

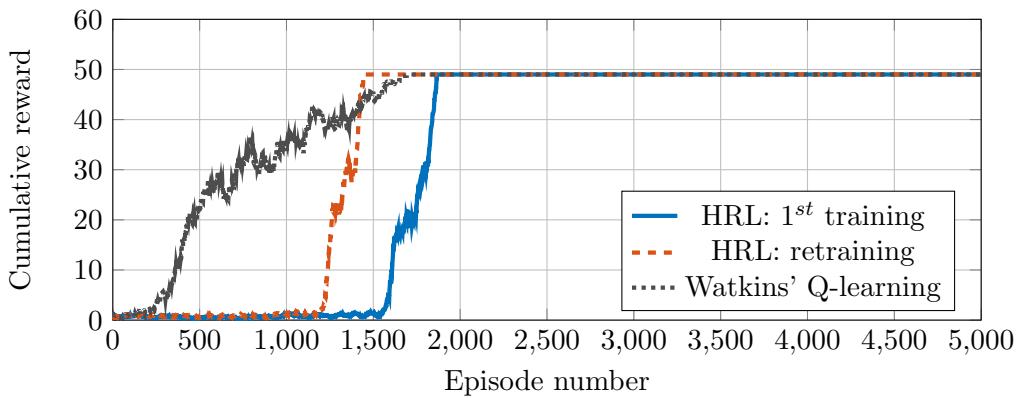


Figure 4: Cumulative reward per episode in the two trainings. The experiment clearly shows the convergent behaviour predicted by Theorem 1. Moreover, the HRL agent is converging to the same cumulative reward as the standard Q-learning agent.

6 Conclusions and Outlook

We presented an HRL scheme, termed Feudal Q-learning, and theoretical results stating its convergence and stability. To the best of the authors' knowledge this is the first work guaranteeing stability of the hierarchical algorithm.

The non-stationarity problems typical of HRL are directly tackled using a timescale separation stemming from the theory of singularly perturbed ODEs. The dynamical system interpretation of the algorithm enabled a general analysis that delivered new theoretical insights on sufficient conditions for achieving desired properties of the algorithm. Future work could include analyzing HRL algorithms with other system theoretic techniques, e.g., the small gain theorem, or designing alternative HRL algorithms by employing the ODE method as described in [37, Chapter 8].

Additionally, the proposed scheme naturally connects to the field of game theory: we have multiple agents interacting, π^h and π^l , while having potentially different goals, i.e., respectively maximizing Q_{π^h, π^l}^h and Q_{π^h, π^l}^l . Future work should further explore the connections between HRL and game theory.

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A Details on the formulation of Feudal Q-learning as TTSA iterates

As reported in Section 3.2, we rewrite the Feudal Q-learning updates in Eq. (8) as Eq. 10, in a form consistent with the TTSA literature's notation [23], [36]. Being in a finite sets setting, we treat $Q^{h,n}$ and $Q^{l,n}$ as matrices and we define x_n and y_n as the vectorized versions of $Q^{l,n}$ and $Q^{h,n}$, respectively. In particular, for the high-level:

- $y_n := \text{vec}(Q^{h,n})$
- We define $\rho_{s^h}^h \in \{1, \dots, |\mathcal{S}^h|\}$ the row number that state s^h occupies in $Q^{h,n}$. Analogously, we define $c_{\omega^h}^h \in \{1, \dots, |\Omega|\}$ the column number that goal ω^h occupies in $Q^{h,n}$.
 $g(x_n, y_n) : \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \times \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \rightarrow \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|}$ is the vector-valued function whose $((c_{\omega^h}^h - 1)|\mathcal{S}^h| + \rho_{s^h}^h)$ -th component is $g_{s^h, \omega^h}(x_n, y_n)$, where:

$$g_{s^h, \omega^h}(x_n, y_n) = r_{\pi^l}^h(s^h, \omega^h) + (\gamma^h)^T \mathbb{E}_{P_{\pi^l}^h} [\max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+)] - Q^{h,n}(s^h, \omega^h)$$
- $M_{n+1}^{(2)}(x_n, y_n)$ is the vector-valued function whose $((c_{\omega^h}^h - 1)|\mathcal{S}^h| + \rho_{s^h}^h)$ -th component is $M_{n+1, s^h, \omega^h}^{(2)}(x_n, y_n)$
- $M_{n+1, s^h, \omega^h}^{(2)}(x_n, y_n) := (\gamma^h)^T \max_{\omega} Q^{h,n}(s_{t+1}^h, \omega) - (\gamma^h)^T \mathbb{E}_{s_{t+1}^h \sim P_{\pi^l}^h} [\max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+)] + \sum_{k=0}^{T-1} (\gamma^h)^k r(s_{t+k}, a_{t+k}) - r_{\pi^l}^h(s^h, \omega^h)$, with $s_{t+1}^h \sim P_{\pi^l}^h(\cdot | s^h, \omega^h)$, $a_{t+k} \sim \pi^l(\cdot | s_{t+k}^l)$, $s_{t+k+1}^l \sim P_{\pi^h}^l(\cdot | s_{t+k}^l, a_{t+k})$.

Whereas, for the low-level:

- $x_n := \text{vec}(Q^{l,n})$
- We define $\rho_{s^l}^l \in \{1, \dots, |\mathcal{S}^l|\}$ the row number that state s^l occupies in $Q^{l,n}$. Analogously, we define $c_a^l \in \{1, \dots, |\mathcal{A}|\}$ the column number that action a^l occupies in $Q^{l,n}$.
 $h(x_n, y_n) : \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \times \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \rightarrow \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|}$ is the vector-valued function whose $((c_a^l - 1)|\mathcal{S}^l| + \rho_{s^l}^l)$ -th component is $h_{s^l, a}(x_n, y_n)$, where:

$$h_{s^l, a}(x_n, y_n) = r^l(s^l, a) + \gamma^l \mathbb{E}_{P_{\pi^h}^l} [\max_a Q^{l,n}(s^{l,+}, a^+)] - Q^{l,n}(s^l, a)$$
- $M_{n+1}^{(1)}(x_n, y_n)$ is the vector-valued function whose $((c_a^l - 1)|\mathcal{S}^l| + \rho_{s^l}^l)$ -th component is $M_{n+1, s^l, a}^{(1)}(x_n, y_n)$
- $M_{n+1, s^l, a}^{(1)}(x_n, y_n) := \gamma^l \max_a Q^{l,n}(s_{k+1}^l, a) - \gamma^l \mathbb{E}_{s_{k+1}^l \sim P_{\pi^h}^l} [\max_a Q^{l,n}(s^{l,+}, a^+)]$, where $s_{k+1}^l \sim P_{\pi^h}^l(\cdot | s^l, a)$.

In this way, we have shown how to rewrite the Feudal Q-learning updates in Eq. (8) as the TTSA updates in Eq. (10).

B Derivation of high- and low-level dynamics

For ease in exposition, we will use $P()$ to indicate a probability that is clear from the context. An high-level trajectory between two time steps $t_1 \leq t_2$ is $\tau_{[t_1, t_2]}^h := (s_{t_1}^h, \omega_{t_1}^h, s_{t_1+1}^h, \omega_{t_1+1}^h, \dots, s_{t_2}^h, \omega_{t_2}^h)$. Moreover, we indicate with τ_t^h an high-level trajectory starting from the initial time, i.e., $\tau_t^h := \tau_{[0, t]}^h$.

Similarly, for the low-level MDP we define $\tau_{[k_1, k_2]}^l := (s_{k_1}^l, a_{k_1}, \dots, s_{k_2}^l, a_{k_2})$ and $\tau_k^l := \tau_{[0, k]}^l$.

High-level dynamics Since the evolution of s^{de} inside of a decision epoch is known and deterministic, and since the goal ω_k^l is constant between decision epochs, the probability $P_{\pi^l}^h(s_1^h|s_0^h, \omega_0^h)$ is equal to $P(s_T|s_0, \omega_0^l = \dots = \omega_{T-1}^l = \omega_t^h, s_0^{\text{de}} = 0, \dots, s_{T-1}^{\text{de}} = T-1)$. We will indicate with E the event where $\omega_0^l = \dots = \omega_{T-1}^l = \omega_t^h$ and $s_0^{\text{de}} = 0, \dots, s_{T-1}^{\text{de}} = T-1$. Then, $P(s_T|s_0, E)$ can be written as:

$$\begin{aligned}
& P(s_T|s_0, E) = \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} P(s_T|s_0, E, a_0, s_1, a_1, \dots, s_{T-1}, a_{T-1}) P(a_0, s_1, a_1, \dots, s_{T-1}, a_{T-1}|s_0, E) \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} P(s_T|\tau_{T-1}^l) P(\tau_{T-1}^l|s_0, E) \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} P(s_T|s_{T-1}, a_{T-1}) P(\tau_{T-1}^l|s_0, E) \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} P(s_T|s_{T-1}, a_{T-1}) P(a_{T-1}|s_{T-1}, \tau_{T-2}^l) P(s_{T-1}, \tau_{T-2}^l|s_0, E) \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} P(s_T|s_{T-1}, a_{T-1}) \pi^l(a_{T-1}|s_{T-1}^l) P(s_{T-1}, \tau_{T-2}^l|s_0, E) \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} P(s_T|s_{T-1}, a_{T-1}) \pi^l(a_{T-1}|s_{T-1}^l) P(s_{T-1}|s_{T-2}, a_{T-2}) P(\tau_{T-2}^l|s_0, E) \\
&= \dots \text{ (iteratively perform steps above)} \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \pi^l(a_{i-1}|s_{i-1}, \omega_{i-1}^l, s_{i-1}^{\text{de}})
\end{aligned}$$

Where the equalities can be found by applying the law of total probability, the chain rule, and properly managing independent variables.

We have shown how to derive $P_{\pi^l}^h(s_1^h|s_0^h, \omega_0^h)$ for the sake of exposition, similar steps readily apply to derive $P_{\pi^l}^h(s_{t+1}^h|s_t^h, \omega_t^h)$.

Low-level dynamics By applying the chain rule and properly managing independent variables, we can write the low-level dynamics as:

$$\begin{aligned}
& P_{\pi^h}^l(s_{k+1}, \omega_{k+1}^l, s_{k+1}^{\text{de}}|s_k, \omega_k^l, s_k^{\text{de}}, a_k) = \\
&= P(s_{k+1}^{\text{de}}|s_{k+1}, \omega_{k+1}^l, s_k, \omega_k^l, s_k^{\text{de}}, a_k) P(s_{k+1}, \omega_{k+1}^l|s_k, \omega_k^l, s_k^{\text{de}}, a_k) \\
&= P(s_{k+1}^{\text{de}}|s_k^{\text{de}}) P(s_{k+1}, \omega_{k+1}^l|s_k, \omega_k^l, s_k^{\text{de}}, a_k) \\
&= P(s_{k+1}^{\text{de}}|s_k^{\text{de}}) P(s_{k+1}|\omega_{k+1}^l, s_k, \omega_k^l, s_k^{\text{de}}, a_k) P(\omega_{k+1}^l|s_k, \omega_k^l, s_k^{\text{de}}, a_k) \\
&= P(s_{k+1}^{\text{de}}|s_k^{\text{de}}) P(s_{k+1}|s_k, a_k) P(\omega_{k+1}^l|s_k, \omega_k^l, s_k^{\text{de}})
\end{aligned} \tag{24}$$

Knowing that the goal is not changing unless we are at a decision epoch and that the high-level policy π^h selects goals, we have that:

$$P(\omega_{k+1}^l|s_k, \omega_k^l, s_k^{\text{de}}) = \begin{cases} 0, & \text{if } s_k^{\text{de}} \neq T-1 \text{ and } \omega_{k+1}^l \neq \omega_k^l \\ 1, & \text{if } s_k^{\text{de}} \neq T-1 \text{ and } \omega_{k+1}^l = \omega_k^l \\ \pi^h(\omega_{k+1}^l|s_k), & \text{if } s_k^{\text{de}} = T-1 \end{cases} \tag{25}$$

We get the desired result by combining Eq. (24) and Eq. (25).

C Proof of Theorem 1

Lemma 4 (Lipschitz continuity of g and h). *The functions g and h are Lipschitz in the infinity norm, i.e., there exist $L_g, L_h \geq 0$ such that, for any $(x^1, y^1), (x^2, y^2) \in \mathbb{R}^{|\mathcal{S}^1| \times |\mathcal{A}|} \times \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|}$, it holds that:*

$$\begin{aligned}\|g(x^1, y^1) - g(x^2, y^2)\|_\infty &\leq L_g \|(x^1, y^1) - (x^2, y^2)\|_\infty \\ \|h(x^1, y^1) - h(x^2, y^2)\|_\infty &\leq L_h \|(x^1, y^1) - (x^2, y^2)\|_\infty\end{aligned}$$

Proof. We will prove that g is Lipschitz, with similar steps one can prove Lipschitz continuity of h . We show that each component of g , i.e., g_{s^h, ω^h} is Lipschitz continuous, then Lipschitz continuity of g follows.

Recall that $g_{s^h, \omega^h}(Q^l, Q^h) = r_{\pi^l}^h(s^h, \omega^h) + (\gamma^h)^T \mathbb{E}_{s^{h,+} \sim P_{\pi^l}^h} [\max_{\omega^+} Q^h(s^{h,+}, \omega^+)] - Q^h(s^h, \omega^h)$.

To show the Lipschitz continuity of g_{s^h, ω^h} , we will proceed considering the three addends composing g_{s^h, ω^h} . Then, since the sum of Lipschitz continuous functions is itself Lipschitz, g_{s^h, ω^h} is Lipschitz continuous with respect to (Q^l, Q^h) . The steps are listed in the following, and the more involved steps will be detailed below.

1. $r_{\pi^l}^h$ is Lipschitz continuous with respect to (Q^l, Q^h) :
 - (a) We will show that $r_{\pi^l}^h$ is Lipschitz continuous with respect to π^l
 - (b) We will show that π^l is Lipschitz continuous with respect to Q^l
 - (c) Since the composition of Lipschitz continuous functions is Lipschitz continuous, it is easy to show that $r_{\pi^l}^h(s^h, \omega^h)$ is Lipschitz continuous with respect to (Q^l, Q^h)
2. $G_{Q^h, P_{\pi^l}^h}^2 := (\gamma^h)^T \mathbb{E}_{s^{h,+} \sim P_{\pi^l}^h} [\max_{\omega^+} Q^h(s^{h,+}, \omega^+)]$ is Lipschitz continuous with respect to (Q^l, Q^h) :
 - (a) We will show that $G_{Q^h, P_{\pi^l}^h}^2$ is Lipschitz continuous with respect to Q^h
 - (b) We will show that $G_{Q^h, P_{\pi^l}^h}^2$ is Lipschitz continuous with respect to $P_{\pi^l}^h$
 - (c) We will show that $P_{\pi^l}^h$ is Lipschitz continuous with respect to π^l
 - (d) We have already shown that π^l is Lipschitz continuous with respect to Q^l
 - (e) Since the composition of Lipschitz continuous functions is Lipschitz continuous, it is easy to show that $G_{Q^h, P_{\pi^l}^h}^2(s^h, \omega^h)$ is Lipschitz continuous with respect to (Q^l, Q^h)
3. $-Q^h(s^h, \omega^h)$ is clearly Lipschitz continuous with respect to (Q^l, Q^h)

1a) Lipschitz continuity of $r_{\pi^l}^h$ with respect to π^l . Firstly, the probability of a low-level trajectory, $P_{\pi^l, P^l}^{\tau_k^l}$, is:

$$P_{\pi^l, P^l}^{\tau_k^l} = \prod_{i=0}^k \pi^l(a_i | s_i^l) P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i)$$

and the high-level reward can be rewritten as:

$$r_{\pi^l}^h(s^h, \omega^h) = \sum_{k=0}^{T-1} (\gamma^h)^k \sum_{\tau_k^l} P_{\pi^l, P^l}^{\tau_k^l} r(s_k, a_k)$$

Thus:

$$\begin{aligned}
& |r_{\pi^{l,1}}^h(s^h, \omega^h) - r_{\pi^{l,2}}^h(s^h, \omega^h)| \\
&= \left| \sum_{k=0}^{T-1} (\gamma^h)^k \sum_{\tau_k^l} P_{\pi^{l,1}, P^l}^{\tau_k^l} r(s_k, a_k) - \sum_{k=0}^{T-1} (\gamma^h)^k \sum_{\tau_k^l} P_{\pi^{l,2}, P^l}^{\tau_k^l} r(s_k, a_k) \right| \\
&= \left| \sum_{k=0}^{T-1} (\gamma^h)^k \sum_{\tau_k^l} (P_{\pi^{l,1}, P^l}^{\tau_k^l} - P_{\pi^{l,2}, P^l}^{\tau_k^l}) r(s_k, a_k) \right| \\
&\leq \sum_{k=0}^{T-1} (\gamma^h)^k \sum_{\tau_k^l} |(P_{\pi^{l,1}, P^l}^{\tau_k^l} - P_{\pi^{l,2}, P^l}^{\tau_k^l})| |r(s_k, a_k)| \\
&\leq \bar{r} \sum_{k=0}^{T-1} (\gamma^h)^k \sum_{\tau_k^l} |(P_{\pi^{l,1}, P^l}^{\tau_k^l} - P_{\pi^{l,2}, P^l}^{\tau_k^l})| \tag{26}
\end{aligned}$$

where the first inequality follows from the triangle inequality and the second one from the boundedness of the reward function in Assumption 1. We will now find a bound on $|(|P_{\pi^{l,1}, P^l}^{\tau_k^l} - P_{\pi^{l,2}, P^l}^{\tau_k^l})|$ that depends on $\|\pi^{l,1} - \pi^{l,2}\|_\infty$.

$$\begin{aligned}
& |(P_{\pi^{l,1}, P^l}^{\tau_k^l} - P_{\pi^{l,2}, P^l}^{\tau_k^l})| \\
&= \left| \prod_{i=0}^k \pi^{l,1}(a_i | s_i^l) P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) - \prod_{i=0}^k \pi^{l,2}(a_i | s_i^l) P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \right| \\
&= \left| \prod_{i=0}^k \pi^{l,1}(a_i | s_i^l) - \prod_{i=0}^k \pi^{l,2}(a_i | s_i^l) \right| \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \\
&= \left| \sum_{j=0}^k \left(\prod_{i=0}^{j-1} \pi^{l,1}(a_i | s_i^l) \right) \left(\pi^{l,1}(a_j | s_j^l) - \pi^{l,2}(a_j | s_j^l) \right) \left(\prod_{i=j+1}^k \pi^{l,2}(a_i | s_i^l) \right) \cdot \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \right| \\
&\leq \sum_{j=0}^k \left| \left(\prod_{i=0}^{j-1} \pi^{l,1}(a_i | s_i^l) \right) \left(\pi^{l,1}(a_j | s_j^l) - \pi^{l,2}(a_j | s_j^l) \right) \left(\prod_{i=j+1}^k \pi^{l,2}(a_i | s_i^l) \right) \cdot \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \right| \\
&\leq \sum_{j=0}^k \left| \left(\pi^{l,1}(a_j | s_j^l) - \pi^{l,2}(a_j | s_j^l) \right) \right| \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \\
&\leq (k+1) \left\| \pi^{l,1} - \pi^{l,2} \right\|_\infty \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \tag{27}
\end{aligned}$$

where the third equality comes from a rewriting of terms as a telescopic sum.
Now, we can insert (27) in (26):

$$\begin{aligned}
& |r_{\pi^{l,1}}^h(s^h, \omega^h) - r_{\pi^{l,2}}^h(s^h, \omega^h)| \\
&\leq \bar{r} \sum_{k=0}^{T-1} (\gamma^h)^k \sum_{\tau_k^l} \left| (k+1) \left\| \pi^{l,1} - \pi^{l,2} \right\|_\infty \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \right| \\
&\leq \bar{r} \left\| \pi^{l,1} - \pi^{l,2} \right\|_\infty \sum_{k=0}^{T-1} (\gamma^h)^k (k+1) \sum_{\tau_k^l} \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \tag{28}
\end{aligned}$$

Moreover, the term $\sum_{\tau_k^l} \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i)$ can be written as:

$$\begin{aligned}
& \sum_{\tau_k^l} \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \\
&= \sum_{\substack{s_0^l, \dots, s_k^l \in \mathcal{S}^l \\ a_0, \dots, a_k^l \in \mathcal{A}}} \prod_{i=0}^k P_{\pi^h}^l(s_{i+1}^l | s_i^l, a_i) \\
&= \sum_{s_0^l} \sum_{a_0} \sum_{s_1^l} P_{\pi^h}^l(s_1^l | s_0^l, a_0) \sum_{a_1} \sum_{s_2^l} P_{\pi^h}^l(s_2^l | s_1^l, a_1) \dots \sum_{a_{k-1}} \sum_{s_k^l} P_{\pi^h}^l(s_k^l | s_{k-1}^l, a_{k-1}) \\
&= \sum_{s_0^l} \sum_{a_0} \sum_{s_1^l} P_{\pi^h}^l(s_1^l | s_0^l, a_0) \sum_{a_1} \sum_{s_2^l} P_{\pi^h}^l(s_2^l | s_1^l, a_1) \dots \sum_{a_{k-1}} 1 \\
&= \sum_{s_0^l} |\mathcal{A}|^k \\
&= |\mathcal{S}^l| |\mathcal{A}|^k
\end{aligned} \tag{29}$$

Thus, inserting (29) in (28) and defining $L_{r^h}^{\pi^l} := \bar{r} \sum_{k=0}^{T-1} (\gamma^h)^k (k+1) |\mathcal{S}^l| |\mathcal{A}|^k$, we get:

$$\begin{aligned}
& |r_{\pi^{l,1}}^h(s^h, \omega^h) - r_{\pi^{l,2}}^h(s^h, \omega^h)| \\
&\leq \bar{r} \left\| \pi^{l,1} - \pi^{l,2} \right\|_{\infty} \sum_{k=0}^{T-1} (\gamma^h)^k (k+1) |\mathcal{S}^l| |\mathcal{A}|^k \\
&= L_{r^h}^{\pi^l} \left\| \pi^{l,1} - \pi^{l,2} \right\|_{\infty}
\end{aligned}$$

which shows that $r_{\pi^l}^h$ is Lipschitz continuous with respect to π^l .

1b) Lipschitz continuity of π^l with respect to Q_{π^h, π^l}^l . Directly follows from Assumption 3.

2a) Lipschitz continuity of $G_{Q^h, P_{\pi^l}}^2(s^h, \omega^h)$ with respect to Q^h Starting from the definition of $G_{Q^h, P_{\pi^l}}^2(s^h, \omega^h)$:

$$\begin{aligned}
& |G_{Q^{h,1}, P_{\pi^l}^h}^2(s^h, \omega^h) - G_{Q^{h,2}, P_{\pi^l}^h}^2(s^h, \omega^h)| \\
&= |(\gamma^h)^T \mathbb{E}_{s^{h,+} \sim P_{\pi^l}^h} [\max_{\omega^+} Q^{h,1}(s^{h,+}, \omega^+)] - (\gamma^h)^T \mathbb{E}_{s^{h,+} \sim P_{\pi^l}^h} [\max_{\omega^+} Q^{h,2}(s^{h,+}, \omega^+)]| \\
&= |(\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^l}^h(s^{h,+} | s^h, \omega^h) [\max_{\omega^+} Q^{h,1}(s^{h,+}, \omega^+) - \max_{\omega^+} Q^{h,2}(s^{h,+}, \omega^+)]| \\
&\leq (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^l}^h(s^{h,+} | s^h, \omega^h) |\max_{\omega^+} Q^{h,1}(s^{h,+}, \omega^+) - \max_{\omega^+} Q^{h,2}(s^{h,+}, \omega^+)| \\
&\leq (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^l}^h(s^{h,+} | s^h, \omega^h) |\max_{\omega^+} (Q^{h,1}(s^{h,+}, \omega^+) - Q^{h,2}(s^{h,+}, \omega^+))| \\
&\leq (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^l}^h(s^{h,+} | s^h, \omega^h) \max_{\omega^+} |Q^{h,1}(s^{h,+}, \omega^+) - Q^{h,2}(s^{h,+}, \omega^+)| \\
&\leq (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^l}^h(s^{h,+} | s^h, \omega^h) \max_{\bar{s}^h, \omega^+} |Q^{h,1}(\bar{s}^h, \omega^+) - Q^{h,2}(\bar{s}^h, \omega^+)| \\
&\leq (\gamma^h)^T \underbrace{\max_{\bar{s}^h, \omega^+} |Q^{h,1}(\bar{s}^h, \omega^+) - Q^{h,2}(\bar{s}^h, \omega^+)|}_{=\|Q^{h,1} - Q^{h,2}\|_\infty} \underbrace{\sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^l}^h(s^{h,+} | s^h, \omega^h)}_{=1} \\
&= (\gamma^h)^T \|Q^{h,1} - Q^{h,2}\|_\infty
\end{aligned}$$

and the result easily follows.

2b) Lipschitz continuity of $G_{Q^h, P_{\pi^l}}^2$ with respect to $P_{\pi^l}^h$ The desired bound can be found as:

$$\begin{aligned}
& |G_{Q^h, P^{h,1}}^2(s^h, \omega^h) - G_{Q^h, P^{h,2}}^2(s^h, \omega^h)| \\
&= (\gamma^h)^T \left| \sum_{s^{h,+} \in \mathcal{S}^h} (P^{h,1}(s^{h,+} | s^h, \omega^h) - P^{h,2}(s^{h,+} | s^h, \omega^h)) \max_{\omega^+} Q^h(s^{h,+}, \omega^+) \right| \\
&\leq (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} |P^{h,1}(s^{h,+} | s^h, \omega^h) - P^{h,2}(s^{h,+} | s^h, \omega^h)| \max_{\omega^+} Q^h(s^{h,+}, \omega^+) \\
&\leq (\gamma^h)^T \left| \max_{\bar{s}^h, \omega^+} Q^h(\bar{s}^h, \omega^+) \right| \sum_{s^{h,+} \in \mathcal{S}^h} |P^{h,1}(s^{h,+} | s^h, \omega^h) - P^{h,2}(s^{h,+} | s^h, \omega^h)| \\
&\leq (\gamma^h)^T \left| \max_{\bar{s}^h, \omega^+} Q^h(\bar{s}^h, \omega^+) \right| |\mathcal{S}^h| \max_{\bar{s}^h} |P^{h,1}(\bar{s}^h | s^h, \omega^h) - P^{h,2}(\bar{s}^h | s^h, \omega^h)| \\
&\leq (\gamma^h)^T \left| \max_{\bar{s}^h, \omega^+} Q^h(\bar{s}^h, \omega^+) \right| |\mathcal{S}^h| \left\| P^{h,1} - P^{h,2} \right\|_\infty
\end{aligned}$$

Lipschitz continuity easily follows.

2c) Lipschitz continuity of $P_{\pi^l}^h$ with respect to π^l Starting from the high-level dynamics in Eq. (2) with $s^h = s_0$, $s^{h,+} = s_T$, we write:

$$\begin{aligned}
& |P_{\pi^{l,1}}^h(s^{h,+}|s^h, \omega^h) - P_{\pi^{l,2}}^h(s^{h,+}|s^h, \omega^h)| \\
&= \left| \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \pi^{l,1}(a_{i-1}|s_{i-1}, \omega^h, i-1) + \right. \\
&\quad \left. - \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \pi^{l,2}(a_{i-1}|s_{i-1}, \omega^h, i-1) \right| \\
&= \left| \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \left[\prod_{i=1}^T \pi^{l,1}(a_{i-1}|s_{i-1}, \omega^h, i-1) - \prod_{i=1}^T \pi^{l,2}(a_{i-1}|s_{i-1}, \omega^h, i-1) \right] \right| \\
&\leq \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \left| \prod_{i=1}^T \pi^{l,1}(a_{i-1}|s_{i-1}, \omega^h, i-1) - \prod_{i=1}^T \pi^{l,2}(a_{i-1}|s_{i-1}, \omega^h, i-1) \right| \\
&= \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \cdot \\
&\quad \left| \sum_{j=1}^T \prod_{k=1}^{j-1} \pi^{l,1}(a_{k-1}|s_{k-1}, \omega^h, k-1) (\pi^{l,1}(a_{j-1}|s_{j-1}, \omega^h, j-1) - \pi^{l,2}(a_{j-1}|s_{j-1}, \omega^h, j-1)) \prod_{k=j+1}^T \pi^{l,1}(a_{k-1}|s_{k-1}, \omega^h, k-1) \right| \\
&\leq \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \left| \sum_{j=1}^T (\pi^{l,1}(a_{j-1}|s_{j-1}, \omega^h, j-1) - \pi^{l,2}(a_{j-1}|s_{j-1}, \omega^h, j-1)) \right| \\
&\leq \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \sum_{j=1}^T |\pi^{l,1}(a_{j-1}|s_{j-1}, \omega^h, j-1) - \pi^{l,2}(a_{j-1}|s_{j-1}, \omega^h, j-1)| \\
&\leq T \|\pi^{l,1} - \pi^{l,2}\|_\infty \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ a_0, \dots, a_{T-1} \in \mathcal{A}}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1}) \\
&= T \|\pi^{l,1} - \pi^{l,2}\|_\infty \underbrace{\sum_{a_0, \dots, a_{T-1} \in \mathcal{A}} \sum_{\substack{s_1, \dots, s_{T-1} \in \mathcal{S} \\ =1}} \prod_{i=1}^T P(s_i|s_{i-1}, a_{i-1})}_{=1} \\
&= T |\mathcal{A}|^T \|\pi^{l,1} - \pi^{l,2}\|_\infty
\end{aligned}$$

The result easily follows.

Thus combining the results above, we can conclude that g is Lipschitz continuous with respect to both its arguments.

□

Lemma 5 (Properties of $M_{n+1}^{(1)}(x_n, y_n)$ and $M_{n+1}^{(2)}(x_n, y_n)$). $\{M_n^{(1)}\}$ and $\{M_n^{(2)}\}$ are martingale difference sequences with respect to the increasing σ -fields

$$\mathcal{F}_n := \sigma(x_m, y_m, M_{n+1}^{(1)}, M_{n+1}^{(2)}, m \leq n), \quad n \geq 0,$$

satisfying

$$\begin{aligned}\mathbb{E} [\|M_{n+1}^{(1)}\|_2^2 | \mathcal{F}_n] &\leq K^{(1)}(1 + \|x_n\|_2^2 + \|y_n\|_2^2) \\ \mathbb{E} [\|M_{n+1}^{(2)}\|_2^2 | \mathcal{F}_n] &\leq K^{(2)}(1 + \|x_n\|_2^2 + \|y_n\|_2^2)\end{aligned}$$

for $n \geq 0$ and for some constants $K^{(1)}, K^{(2)} > 0$.

Proof. We will prove Lemma 5 for $M_{n+1}^{(2)}(x_n, y_n)$, similar steps can be applied to $M_{n+1}^{(1)}(x_n, y_n)$. Let us firstly show that $\{M_n^{(2)}\}$ is a martingale difference sequence with respect to \mathcal{F}_n , i.e., $\mathbb{E} [M_{n+1}^{(2)}(x_n, y_n) | \mathcal{F}_n] = 0$. Considering one component of $M_{n+1}^{(2)}(x_n, y_n)$, we write the conditional expectation as:

$$\begin{aligned}&\mathbb{E} [M_{n+1,s^h,\omega^h}^{(2)}(x_n, y_n) | \mathcal{F}_n] \\&= \mathbb{E}_{\substack{s_{t+1}^h \\ \tau[tT, tT+T-1]}} [(\gamma^h)^T \max_{\omega} Q^{h,n}(s_{t+1}^h, \omega) - (\gamma^h)^T \mathbb{E}_{\substack{s^{h,+} \sim P_{\pi^1}^h \\ \omega^+}} [\max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+)]] \\&\quad + \sum_{k=0}^{T-1} (\gamma^h)^k r(s_{tT+k}, a_{tT+k}) - r_{\pi^1}^h(s^h, \omega^h) | \mathcal{F}_n \\&= (\gamma^h)^T \mathbb{E}_{\substack{s_{t+1}^h \sim P_{\pi^1}^h}} [\max_{\omega} Q^{h,n}(s_{t+1}^h, \omega) | \mathcal{F}_n] - (\gamma^h)^T \mathbb{E}_{\substack{s^{h,+} \sim P_{\pi^1}^h \\ \omega^+}} [\max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+) | \mathcal{F}_n] \\&\quad + \mathbb{E}_{\substack{a_{tT+k} \sim \pi^1(\cdot | s_{tT+k}^l) \\ s_{tT+k+1}^l \sim P_{\pi^h}^l(\cdot | s_{tT+k}^l, a_{tT+k})}} \left[\sum_{k=0}^{T-1} (\gamma^h)^k r(s_{tT+k}, a_{tT+k}) | s_{tT} = s^h, \mathcal{F}_n \right] \\&\quad - \mathbb{E}_{\substack{a_k \sim \pi^1(\cdot | s_k^l) \\ s_{k+1}^l \sim P_{\pi^h}^l(\cdot | s_k^l, a_k)}} \left[\sum_{k=0}^{T-1} (\gamma^h)^k r(s_k, a_k) | s_0 = s^h, \mathcal{F}_n \right] \\&= 0\end{aligned}$$

where, considering two time steps $k_1 \leq k_2$, we denote a trajectory of \mathcal{M} with $\tau_{[k_1, k_2]} := (s_{k_1}, a_{k_1}, s_{k_1+1}, a_{k_1+1}, \dots, s_{k_2}, a_{k_2})$.

Let us show now that $\mathbb{E} [\|M_{n+1}^{(2)}\|_2^2 | \mathcal{F}_n] \leq K^{(2)}(1 + \|x_n\|_2^2 + \|y_n\|_2^2)$ for $n \geq 0, K^{(2)} > 0$.

We can bound $|(\gamma^h)^T \max_{\omega} Q^{h,n}(s_{t+1}^h, \omega) - (\gamma^h)^T \mathbb{E}_{s^{h,+} \sim P_{\pi^1}^h} [\max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+)]|$ as:

$$\begin{aligned}&|(\gamma^h)^T \max_{\omega} Q^{h,n}(s_{t+1}^h, \omega) - (\gamma^h)^T \mathbb{E}_{\substack{s^{h,+} \sim P_{\pi^1}^h \\ \omega^+}} [\max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+)]| \\&= |(\gamma^h)^T \max_{\omega} Q^{h,n}(s_{t+1}^h, \omega) - (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^1}^h(s^{h,+} | s^h, \omega^h) \max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+)| \\&\leq (\gamma^h)^T |\max_{\omega} Q^{h,n}(s_{t+1}^h, \omega)| + (\gamma^h)^T \left| \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^1}^h(s^{h,+} | s^h, \omega^h) \max_{\omega^+} Q^{h,n}(s^{h,+}, \omega^+) \right| \\&\leq (\gamma^h)^T |\max_{\bar{s}^h, \omega} Q^{h,n}(\bar{s}^h, \omega)| + (\gamma^h)^T \left| \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^1}^h(s^{h,+} | s^h, \omega^h) \max_{\bar{s}^h, \omega^+} Q^{h,n}(\bar{s}^h, \omega^+) \right| \\&= (\gamma^h)^T \|Q^{h,n}\|_{\infty} + (\gamma^h)^T \|Q^{h,n}\|_{\infty} \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi^1}^h(s^{h,+} | s^h, \omega^h) \\&= 2(\gamma^h)^T \|Q^{h,n}\|_{\infty}\end{aligned}$$

and we can easily show that $|\sum_{k=0}^{T-1}(\gamma^h)^k r(s_{tT+k}, a_{tT+k}) - r_{\pi^l}^h(s^h, \omega^h)| \leq 2T\bar{r}$. Thus we have:

$$\begin{aligned} & \mathbb{E} [|M_{n+1, s^h, \omega^h}^{(2)}(x_n, y_n)|^2 | \mathcal{F}_n] \\ & \leq \mathbb{E} [4(\gamma^h)^{2T} \|Q^{h,n}\|_\infty^2 + 4T^2\bar{r}^2 + 8(\gamma^h)^T T\bar{r} \|Q^{h,n}\|_\infty | \mathcal{F}_n] \\ & = 4(\gamma^h)^{2T} \|Q^{h,n}\|_\infty^2 + 4T^2\bar{r}^2 + 8(\gamma^h)^T T\bar{r} \|Q^{h,n}\|_\infty \\ & \leq 4(\gamma^h)^{2T} \|Q^{h,n}\|_\infty^2 + 4T^2\bar{r}^2 + 1 + (8(\gamma^h)^T T\bar{r} \|Q^{h,n}\|_\infty)^2 \\ & = 4(1 + 16T^2\bar{r}^2)(\gamma^h)^{2T} \|Q^{h,n}\|_\infty^2 + 4T^2\bar{r}^2 + 1 \\ & \leq K_{s^h, \omega^h}(1 + \|x_n\|_2^2 + \|y_n\|_2^2) \end{aligned}$$

for an appropriate $K_{s^h, \omega^h} > 0$. The result then easily follows. \square

In the following, we provide the proofs of Lemma 1 and Lemma 2, which are need to prove Theorem 1.

Proof of Lemma 1 In order to show that the ODE (14) has a globally asymptotically stable equilibrium $\lambda(y)$, it suffices to show that, for a Q^h held fixed as a constant parameter, $r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^l(s^{l,+}, a^+)]$ is a contraction with respect to Q^l . In fact, Theorem 3.1 in [45] applies and guarantees the convergence to the fixed point of $r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^l(s^{l,+}, a^+)]$, which is guaranteed to exist and be unique for the Banach fixed-point theorem.

Contraction of $r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^l(s^{l,+}, a^+)]$ can be easily shown with the following steps:

$$\begin{aligned} & |r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^{l,1}(s^{l,+}, a^+)] - r^l(s^l, a) - \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^{l,2}(s^{l,+}, a^+)]| \\ & = |\gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^{l,1}(s^{l,+}, a^+)] - \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^{l,2}(s^{l,+}, a^+)]| \\ & = \gamma^l \left| \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi^h}^l(s^{l,+} | s^l, a) \left[\max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \max_{a^+} Q^{l,2}(s^{l,+}, a^+) \right] \right| \\ & \leq \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi^h}^l(s^{l,+} | s^l, a) \left| \max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \max_{a^+} Q^{l,2}(s^{l,+}, a^+) \right| \\ & \leq \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi^h}^l(s^{l,+} | s^l, a) \left| \max_{a^+} [Q^{l,1}(s^{l,+}, a^+) - Q^{l,2}(s^{l,+}, a^+)] \right| \\ & \leq \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi^h}^l(s^{l,+} | s^l, a) \left| \max_{\bar{s}^{l,+}, a^+} [Q^{l,1}(\bar{s}^{l,+}, a^+) - Q^{l,2}(\bar{s}^{l,+}, a^+)] \right| \\ & \leq \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi^h}^l(s^{l,+} | s^l, a) \max_{\bar{s}^{l,+}, a^+} |Q^{l,1}(\bar{s}^{l,+}, a^+) - Q^{l,2}(\bar{s}^{l,+}, a^+)| \\ & \leq \gamma^l \|Q^{l,1} - Q^{l,2}\|_\infty \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi^h}^l(s^{l,+} | s^l, a) \\ & = \gamma^l \|Q^{l,1} - Q^{l,2}\|_\infty \end{aligned}$$

Now, we need to prove the Lipschitz continuity of $\lambda(y)$, which requires us to show that the fixed point of $r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^l(s^{l,+}, a^+)]$ is Lipschitz in Q^h . It suffices to show

that the fixed point of $r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^l(s^{l,+}, a^+)]$ is Lipschitz in $P_{\pi^h}^l$, as Lipschitz continuity of $P_{\pi^h}^l$ with respect to Q^h can be easily derived from Eq. (5) and Assumption 3. Firstly, as the reward is bounded by \bar{r} and we are in an infinite horizon discounted setting, we can bound any low-level Q-function as:

$$|Q_{\pi^h, \pi^l}^l| \leq \frac{\bar{r}}{1 - \gamma} =: \bar{Q}^l$$

with $\bar{Q}^l > 0$.

Let us now consider two fixed point of $r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P_{\pi^h}^l} [\max_{a^+} Q^l(s^{l,+}, a^+)]$, namely $Q^{l,1}$ and $Q^{l,2}$, satisfying for all $s^l \in \mathcal{S}^l, a \in \mathcal{A}$:

$$\begin{aligned} Q^{l,1}(s^l, a) &= r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P^{l,1}} [\max_{a^+} Q^{l,1}(s^{l,+}, a^+)] \\ Q^{l,2}(s^l, a) &= r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P^{l,2}} [\max_{a^+} Q^{l,2}(s^{l,+}, a^+)] \end{aligned}$$

Now, for any $s^l \in \mathcal{S}^l, a \in \mathcal{A}$:

$$\begin{aligned} |Q^{l,1}(s^l, a) - Q^{l,2}(s^l, a)| &= |r^l(s^l, a) + \gamma^l \mathbb{E}_{s^{l,+} \sim P^{l,1}} [\max_{a^+} Q^{l,1}(s^{l,+}, a^+)] - r^l(s^l, a) - \gamma^l \mathbb{E}_{s^{l,+} \sim P^{l,2}} [\max_{a^+} Q^{l,2}(s^{l,+}, a^+)]| \\ &= \gamma^l \left| \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,1}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,2}(s^{l,+}, a^+) \right| \\ &\leq \gamma^l \left| \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,1}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) \right| \\ &\quad + \gamma^l \left| \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,2}(s^{l,+}, a^+) \right| \end{aligned} \tag{30}$$

We can now bound the first addend as:

$$\begin{aligned} &\left| \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,1}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) \right| \\ &= \left| \sum_{s^{l,+} \in \mathcal{S}^l} (P^{l,1}(s^{l,+}|s^l, a) - P^{l,2}(s^{l,+}|s^l, a)) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) \right| \\ &\leq \sum_{s^{l,+} \in \mathcal{S}^l} |P^{l,1}(s^{l,+}|s^l, a) - P^{l,2}(s^{l,+}|s^l, a)| \bar{Q}^l \\ &\leq \bar{Q}^l |\mathcal{S}^l| \|P^{l,1} - P^{l,2}\|_\infty \end{aligned} \tag{31}$$

Whereas the second added can be bounded by:

$$\begin{aligned} &\left| \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \max_{a^+} Q^{l,2}(s^{l,+}, a^+) \right| \\ &= \left| \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \left(\max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \max_{a^+} Q^{l,2}(s^{l,+}, a^+) \right) \right| \\ &\leq \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \left| \max_{a^+} Q^{l,1}(s^{l,+}, a^+) - \max_{a^+} Q^{l,2}(s^{l,+}, a^+) \right| \\ &\leq \|Q^{l,1} - Q^{l,2}\|_\infty \sum_{s^{l,+} \in \mathcal{S}^l} P^{l,2}(s^{l,+}|s^l, a) \\ &= \|Q^{l,1} - Q^{l,2}\|_\infty \end{aligned} \tag{32}$$

Then, inserting Eq. (31) and Eq. (32) in Eq. (30) we get for any $s^l \in \mathcal{S}^l, a \in \mathcal{A}$:

$$|Q^{l,1}(s^l, a) - Q^{l,2}(s^l, a)| \leq \gamma^l \bar{Q}^l |\mathcal{S}^l| \left\| P^{l,1} - P^{l,2} \right\|_\infty + \gamma^l \left\| Q^{l,1} - Q^{l,2} \right\|_\infty \quad (33)$$

From which we get:

$$\left\| Q^{l,1} - Q^{l,2} \right\|_\infty \leq \frac{\gamma^l}{1 - \gamma^l} \bar{Q}^l |\mathcal{S}^l| \left\| P^{l,1} - P^{l,2} \right\|_\infty \quad (34)$$

Lipischitz continuity of $\lambda(y)$ follows. ■

Proof of Lemma 2 With similar arguments to the ones used in the proof of Lemma 1, it suffices to show that $r_{\pi^l}^h(s^h, \omega^h) + (\gamma^h)^T \mathbb{E}_{s^{h,+} \sim P_{\pi^l}^h} [\max_{\omega^+} Q^h(s^{h,+}, \omega^+)]$ is a contraction. In that case, Banach fixed-point theorem guarantees the existence and uniqueness of a fixed point and Theorem 3.1 in [45] guarantees the convergence of the ODE (15) to the fixed point.

With similar steps to the ones in the proof of Lemma 1, it is easy to show that $r_{\pi^l}^h(s^h, \omega^h) + (\gamma^h)^T \mathbb{E}_{s^{h,+} \sim P_{\pi^l}^h} [\max_{\omega^+} Q^h(s^{h,+}, \omega^+)]$ is a contraction. ■

By assuming boundedness of the iterations, one could use the Lemmas 4, 5, 1, and 2 to verify the assumptions of Theorem 8.1 in [23] and guarantee almost sure convergence of (x_n, y_n) to $(\lambda(y^*), y^*)$, which would translate, for a GLIE policy, in having $(Q^{l,n}, Q^{h,n}) \rightarrow (Q^{l,*}, Q^{h,*})$ a.s. Instead of assuming boundedness of the iterations, we will use the results in [36] to guarantee it.

Lemma 6 (Additional properties of h for stability). *The functions $h_c(x, y) := \frac{h(cx, cy)}{c}$, $c \geq 1$, converge compactly for some h_∞ . The limiting ODE $\frac{dx}{dt} = h_\infty(x(t), y)$ has a unique globally asymptotically stable equilibrium $\lambda_\infty(y)$, where $\lambda_\infty : \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|} \rightarrow \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|}$, is a Lipschitz map and satisfies $\lambda_\infty(0) = 0$.*

Proof. We want to show that, for any compact set $K \subset \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \times \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|}$:

$$\lim_{c \rightarrow \infty} \sup_{(Q^l, Q^h) \in K} |h_c(Q^l, Q^h) - h_\infty(Q^l, Q^h)| = 0 \quad (35)$$

We will work with a component of h_c , i.e., $h_{c,s^l,a} := \frac{h_{s^l,a}(cx, cy)}{c}$, and we will show that:

$$\lim_{c \rightarrow \infty} \sup_{(Q^l, Q^h) \in K} |h_{c,s^l,a}(Q^l, Q^h) - h_{\infty,s^l,a}(Q^l, Q^h)| = 0, \quad \forall s^l \in \mathcal{S}^l, a \in \mathcal{A} \quad (36)$$

It is easy to see that if Eq. (36) holds then Eq. (35) holds as well.

It can be easily shown that h_c and h_∞ are:

$$\begin{aligned} h_{c,s^l,a}(Q^l, Q^h) &= \frac{r^l(s^l, a)}{c} + \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi_c^h}(s^{l,+} | s^l, a) \max_{a^+} Q^l(s^{l,+}, a^+) - Q^l(s^l, a) \\ h_{\infty,s^l,a}(Q^l, Q^h) &:= \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi_\infty^h}(s^{l,+} | s^l, a) \max_{a^+} Q^l(s^{l,+}, a^+) - Q^l(s^l, a) \end{aligned}$$

We want to find a bound on $|h_{c,s^l,a}(Q^l, Q^h) - h_{\infty,s^l,a}(Q^l, Q^h)|$:

$$\begin{aligned}
& |h_{c,s^l,a}(Q^l, Q^h) - h_{\infty,s^l,a}(Q^l, Q^h)| \\
&= \left| \frac{r^l(s^l, a)}{c} + \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi_c^h}^l(s^{l,+}|s^l, a) \max_{a^+} Q^l(s^{l,+}, a^+) - \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi_\infty^h}^l(s^{l,+}|s^l, a) \max_{a^+} Q^l(s^{l,+}, a^+) \right| \\
&\leq \left| \frac{r^l(s^l, a)}{c} + \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} (P_{\pi_c^h}^l(s^{l,+}|s^l, a) - P_{\pi_\infty^h}^l(s^{l,+}|s^l, a)) \max_{a^+} Q^l(s^{l,+}, a^+) \right| \\
&\leq \left| \frac{r^l(s^l, a)}{c} \right| + \gamma^l \|Q^l\|_\infty |\mathcal{S}| |\mathcal{S}^{\text{de}}| \sum_{\omega^+ \in \Omega} |\pi_c^h(\omega^+|s) - \pi_\infty^h(\omega^+|s)|
\end{aligned}$$

where s is part of s^l .

Therefore, since $\lim_{c \rightarrow \infty} \frac{r^l(s^l, a)}{c} = 0$ and due to Assumption 7, $h_{c,s^l,a}$ converges uniformly on compacts.

Showing that $\frac{dx}{dt} = h_\infty(x(t), y)$ has a unique globally asymptotically stable equilibrium $\lambda_\infty(y)$ can be done similarly to the proof of Lemma 1. In this case, for a fixed $y = Q^h$, $\lambda_\infty(y)$ would have as components the unique solution to:

$$0 = \gamma^l \sum_{s^{l,+} \in \mathcal{S}^l} P_{\pi_h^h}^l(s^{l,+}|s^l, a) \max_{a^+} Q^l(s^{l,+}, a^+) - Q^l(s^l, a)$$

One can easily see that $\forall s^l \in \mathcal{S}^l, a \in \mathcal{A}$, $Q^l(s^l, a) = 0$ is the unique solution to the above equation for any fixed $y = Q^h$. This implies the required Lipschitz continuity condition and that $\lambda_\infty(0) = 0$. \square

Lemma 7 (Additional properties of g for stability). *The functions $g_c(x, y) := \frac{g(c\lambda_\infty(y), cy)}{c}$, $c \geq 1$, converge compactly for some g_∞ . The limiting ODE $\frac{dy}{dt} = g_\infty(\lambda_\infty(y(t)), y(t))$ has the origin in $\mathbb{R}^{|\mathcal{S}^h| \times |\Omega|}$ as its unique globally asymptotically stable equilibrium.*

Proof. The proof closely follows the one of Lemma 6.

We want to show that, for any compact set $K \subset \mathbb{R}^{|\mathcal{S}^l| \times |\mathcal{A}|} \times \mathbb{R}^{|\mathcal{S}^h| \times |\Omega|}$:

$$\lim_{c \rightarrow \infty} \sup_{(Q^l, Q^h) \in K} |g_c(Q^l, Q^h) - g_\infty(Q^l, Q^h)| = 0 \quad (37)$$

We will work with a component of g_c , i.e., $g_{c,s^h,\omega} := \frac{g_{s^h,\omega}(cx, cy)}{c}$, and we will show that:

$$\lim_{c \rightarrow \infty} \sup_{(Q^l, Q^h) \in K} |g_{c,s^h,\omega}(Q^l, Q^h) - g_{\infty,s^h,\omega}(Q^l, Q^h)| = 0, \quad \forall s^h \in \mathcal{S}^h, \omega \in \Omega \quad (38)$$

It is easy to see that if Eq. (38) holds then Eq. (37) holds as well.

It can be easily shown that g_c and g_∞ are:

$$\begin{aligned}
g_{c,s^h,\omega}(Q^l, Q^h) &= \frac{r_{\pi_c^h}^h(s^l, a)}{c} + (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi_c^h}^h(s^{h,+}|s^h, \omega) \max_{\omega^+} Q^h(s^{h,+}, \omega^+) - Q^h(s^h, \omega) \\
g_{\infty,s^h,\omega}(Q^l, Q^h) &:= (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi_\infty^h}^h(s^{h,+}|s^h, \omega) \max_{\omega^+} Q^h(s^{h,+}, \omega^+) - Q^h(s^h, \omega)
\end{aligned}$$

We want to find a bound on $|g_{c,s^h,\omega}(Q^l, Q^h) - g_{\infty,s^h,\omega}(Q^l, Q^h)|$:

$$\begin{aligned}
& |h_{c,s^l,a}(Q^l, Q^h) - h_{\infty,s^l,a}(Q^l, Q^h)| \\
&= \left| \frac{r_{\pi_c^l}^h(s^l, a)}{c} + (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi_c^l}^h(s^{h,+}|s^h, \omega) \max_{\omega^+} Q^h(s^{h,+}, \omega^+) + \right. \\
&\quad \left. - (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi_\infty^l}^h(s^{h,+}|s^h, \omega) \max_{\omega^+} Q^h(s^{h,+}, \omega^+) \right| \\
&\leq \left| \frac{r_{\pi_c^l}^h(s^l, a)}{c} \right| + (\gamma^h)^T \|Q^h\|_\infty \sum_{s^{h,+} \in \mathcal{S}^h} |P_{\pi_c^l}^h(s^{h,+}|s^h, \omega) - P_{\pi_\infty^l}^h(s^{h,+}|s^h, \omega)| \\
&\leq \left| \frac{1}{c} T \frac{1 - (\gamma^h)^T}{1 - \gamma^h} \bar{r} \right| + (\gamma^h)^T \|Q^h\|_\infty \sum_{s^{h,+} \in \mathcal{S}^h} |P_{\pi_c^l}^h(s^{h,+}|s^h, \omega) - P_{\pi_\infty^l}^h(s^{h,+}|s^h, \omega)|
\end{aligned}$$

With steps similar to the ones in the proof of Lemma 4 (Lipschitz continuity of $P_{\pi_l}^h$ with respect to π^l), it is possible to furtherly bound $|h_{c,s^l,a}(Q^l, Q^h) - h_{\infty,s^l,a}(Q^l, Q^h)|$ as:

$$\begin{aligned}
& |h_{c,s^l,a}(Q^l, Q^h) - h_{\infty,s^l,a}(Q^l, Q^h)| \\
&\leq \left| \frac{1}{c} T \frac{1 - (\gamma^h)^T}{1 - \gamma^h} \bar{r} \right| + \\
&\quad + (\gamma^h)^T \|Q^h\|_\infty \sum_{s^{h,+}} \sum_{\substack{s_1, \dots, s_{T-1} \\ a_0, \dots, a_{T-1}}} \left| \sum_{j=1}^T (\pi_c^l(a_{j-1}|s_{j-1}, \omega^h, j-1) - \pi_\infty^l(a_{j-1}|s_{j-1}, \omega^h, j-1)) \right|
\end{aligned}$$

where $s_0 = s^h$. Therefore, since the first addend tends to zero as $c \rightarrow \infty$ and due to Assumption 7, $g_{c,s^h,\omega}$ converges uniformly on compacts.

Showing that $\frac{dy}{dt} = g_\infty(\lambda_\infty(y(t)), y(t))$ has the origin as its unique globally asymptotically stable equilibrium can be done similarly to the proof of Lemma 6. In this case, the unique solution to:

$$0 = (\gamma^h)^T \sum_{s^{h,+} \in \mathcal{S}^h} P_{\pi_\infty^l}^h(s^{h,+}|s^h, \omega) \max_{\omega^+} Q^h(s^{h,+}, \omega^+) - Q^h(s^h, \omega)$$

is given by $Q^h(s^h, \omega) = 0, \forall s^h \in \mathcal{S}^h, \omega \in \Omega$. □

Lemmas 6 and 7, together with the other Lemmas 1, 2, 4, and 5 verify the assumptions of Theorem 10 in [36] and guarantee boundedness of the iterations. Having guaranteed that the iterates remain bounded, Lemmas 1, 2, 4, and 5 guarantee that our updates verify the assumptions of Theorem 8.1 in [23] and guarantee almost sure convergence of (x_n, y_n) to $(\lambda(y^*), y^*)$, which would translate, for a GLIE policy, in having $(Q^{l,n}, Q^{h,n}) \rightarrow (Q^{l,*}, Q^{h,*})$ a.s.