

Proximal Approximate Inference in State-Space Models

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Abstract

We present a class of algorithms for state estimation in nonlinear, non-Gaussian state-space models. Our approach is based on a variational Lagrangian formulation that casts Bayesian inference as a sequence of entropic trust-region updates subject to dynamic constraints. This framework gives rise to a family of forward-backward algorithms, whose structure is determined by the chosen factorization of the variational posterior. By focusing on Gauss–Markov approximations, we derive recursive schemes with favorable computational complexity. For general nonlinear, non-Gaussian models we close the recursions using generalized statistical linear regression and Fourier–Hermite moment matching.

1 Introduction

Accurate state estimation in partially observable dynamical phenomena is a fundamental problem across scientific and engineering disciplines, including robotics (Barfoot, 2024), economics (Jacquier et al., 2002), and biology (Murray, 2002). Bayesian inference in state-space models (SSMs) provides a principled framework for this task, encompassing techniques such as Kalman filtering, sequential Monte Carlo, and message-passing algorithms. Additionally, an important link between inference and optimization has been established early in the field (Kálmán, 1960; Cox, 1964; Bell, 1994), giving rise to a suite of inference-as-optimization algorithms. Recently, this relationship has been further deepened by the advent of approximate methods, such as variational inference (VI, Wainwright & Jordan, 2008) and expectation propagation (EP, Minka, 2001), which generalize the optimization perspective beyond point estimates to arbitrary full posterior approximations.

For state-space models with conjugate prior-likelihood pairs, the corresponding Bayesian posterior distribution admit closed-form solutions by a variety of efficient recursive algorithms, such as the Rauch–Tung–Striebel (RTS) smoother for linear-Gaussian SSMs (Rauch et al., 1965). However, extensions to general nonlinear, non-Gaussian settings are not trivial. While sampling-based methods such as sequential Monte Carlo (Chopin & Papaspiliopoulos, 2020) and Markov Chain Monte Carlo (Brooks et al., 2011) provide asymptotically exact solutions, they do so at a price of high computational complexity and limited scalability, particularly for sequences in high-dimensional spaces and long time horizons (Beskos et al., 2014). Approximate Bayesian techniques, though often suboptimal, can offer a trade-off that sacrifices exactness for a significant computational advantage. This aspect motivates our focus on this class of methods.

Variational inference and expectation propagation methods have been adapted for Bayesian inference in state-space models (Deisenroth & Mohamed, 2012; Chang et al., 2020; Wilkinson et al., 2020). However, despite the strong connection of both frameworks to numerical optimization (Wilkinson et al., 2023), many existing approaches extend VI and EP to state-space representations in an ad-hoc manner. Such methods often integrate variational principles into the structure of a Rauch-Tung-Striebel smoother (Rauch et al.,

1965) template rather than deriving algorithms from a foundational optimization-based perspective. As a result, they may not fully leverage the potential benefits of a systematic approach to state estimation in the most general settings.

In this work, we cast approximate Bayesian inference in nonlinear, non-Gaussian state-space models as an iterative *dynamic* optimization problem over the space of candidate posteriors. Our key contributions are:

- *Dynamic optimization formulation*: We introduce a principled variational framework for Bayesian inference that formulates inference as a sequence of entropic trust-region updates in probability-density space, using structured entropic proximal regularization.
- *Unified recursive algorithms*: We derive a family of efficient forward–backward recursion schemes whose structure is determined entirely by the factorization of the variational posterior — forward-, reverse-, or hybrid-Markov factorizations.
- *Adaptive step size*: We show that the step size between iterative updates corresponds to a Lagrangian multiplier whose optimal value is determined by a dual Rényi divergence objective.
- *Generalization of classical smoothers*: Our method generalizes algorithms in the linear-Gaussian setting, thereby providing a bridge between classical and modern approximate inference techniques.
- *Flexible posterior approximations*: We provide practical realizations using Gauss–Markov approximations, instantiated via generalized statistical linear regression and Fourier–Hermite moment matching, allowing for efficient inference in nonlinear and non-Gaussian models.

The paper is organized as follows. Section 2 introduces the approximate inference problem in state-space models. Section 3 provides a brief review of proximal variational optimization, which serves as the foundation for our method. In Section 4, we present our dynamic Lagrangian framework for approximate Bayesian inference. Section 5 specializes this framework to the Gaussian setting and derives practical recursive algorithms. We then connect our approach to related work in Section 6.

2 Problem Statement

We consider the problem of Bayesian state estimation in nonlinear, non-Gaussian state-space models. Let $\{x_k \in \mathbb{R}^d\}_{k=0}^T$ be a latent discrete-time Markov process and $\{y_k \in \mathbb{R}^m\}_{k=1}^T$ a corresponding sequence of noisy observations. The system is governed by the state-space dynamics:

$$x_0 \sim p_0(\cdot), \quad x_{k+1} \sim f_k(\cdot | x_k), \quad y_{k+1} \sim h_{k+1}(\cdot | x_{k+1}), \quad (1)$$

where $p_0(x_0)$ is the initial prior probability density, $f_k(x_{k+1} | x_k)$ is the Markovian prior probability of the latent stochastic dynamics, and $h_k(y_k | x_k)$ is the conditional probability density of the stochastic measurements. State estimation refers to the problems of online filtering and offline smoothing, which aim to reconstruct the posterior distributions $p_k(x_k | y_{1:k})$ and $p_k(x_k | y_{1:T})$, for all $k > 0$, respectively. For arbitrary forms of the latent dynamics $f_k(x_{k+1} | x_k)$ and the measurement model $h_k(y_k | x_k)$, these filtering and smoothing posterior densities are generally intractable.

In this work, we focus on finding the best approximation of the smoothing posterior density within a certain variational family of joint distributions

$$q(x_{0:T} | y_{1:T}) \approx p(x_{0:T} | y_{1:T}) \propto p_0(x_0) \prod_{k=0}^{T-1} f_k(x_{k+1} | x_k) h_{k+1}(y_{k+1} | x_{k+1}).$$

More specifically, to leverage the structure of state-space models, we focus on Gauss–Markov approximate posterior densities, leading to the following possibilities:

Assumption 1 (Forward-Markov factorization) *For state-space models of the form (1) the Bayesian posterior $p(x_{0:T} | y_{1:T})$ can be approximated by a forward-Markov decomposition*

$$\overrightarrow{q}(x_{0:T} | y_{1:T}) = \overrightarrow{q}_0(x_0 | y_{1:T}) \prod_{k=0}^{T-1} \overrightarrow{q}_k(x_{k+1} | x_k, y_{k+1:T}).$$

Assumption 2 (Reverse-Markov factorization) *For state-space models of the form (1), the Bayesian posterior $p(x_{0:T} | y_{1:T})$ can be approximated by a reverse-Markov decomposition*

$$\overleftarrow{q}(x_{0:T} | y_{1:T}) = \overleftarrow{q}_T(x_T | y_{1:T}) \prod_{k=1}^T \overleftarrow{q}_k(x_{k-1} | x_k, y_{1:k-1}).$$

We use \overrightarrow{q} and \overleftarrow{q} to distinguish forward- and reverse-Markov approximations, respectively. Moreover, while Assumption 1 and 2 explicitly state the dependency of the (conditional) posteriors on the measurements, we omit it in subsequent sections for a simpler notation.

3 Entropic Proximal Variational Optimization

Before introducing our approach to variational smoothing in state-space models, we briefly review the principles of entropic proximal variational optimization in a simpler, static setting. Consider a hidden variable $x \in \mathbb{R}^d$, observed data $y \in \mathbb{R}^m$, a likelihood $p(y | x)$, and a prior $p(x)$. The goal is to approximate the posterior $p(x | y)$ by a distribution $q(x)$ that minimizes the Kullback–Leibler (KL) divergence:

$$q^*(x) = \arg \min_{q(x)} \mathbb{D}_{\text{KL}}[q(x) || p(x | y)].$$

Since the exact posterior $p(x | y)$ is generally intractable, variational inference optimizes the evidence lower bound (ELBO) instead (Blei et al., 2017):

$$q^*(x) = \arg \max_{q(x)} \mathcal{L}(q) = \mathbb{E}_q[\log p(x, y)] - \mathbb{E}_q[\log q(x)] \leq \log p(y), \quad (2)$$

where $p(y)$ is the marginal likelihood or evidence. In conjugate models, the ELBO can be optimized exactly (Bishop, 2006), but for more general settings, approximate solutions are required. Black-box variational inference (Ranganath et al., 2014) provides a flexible framework for such cases. However, alternative methods such as natural and Riemannian gradient approaches (Honkela et al., 2010) and proximal variational inference (Chrétien & Hero, 2002; Khan et al., 2015; Theis & Hoffman, 2015) explicitly account for the geometry of the variational family and often lead to more stable and efficient optimization (Amari, 1998).

In this work, we focus on the entropic proximal variational optimization framework to derive recursive algorithms for structured approximate inference in nonlinear non-Gaussian state-space models. However, before directing our attention to the case of state-space models, we use the static case in (2) to illustrate the mechanics of entropic proximal variational optimization. We start by formulating the approximate inference problem within the iterative entropic proximal optimization framework proposed by Teboulle (1992); Iusem et al. (1994). Starting from the ELBO in (2), we introduce an entropic-proximal constraint on subsequent posterior iterate in the form of a Kullback–Leibler divergence, leading to the following nonlinear program:

$$\begin{aligned} & \underset{q(x)}{\text{maximize}} \quad \mathbb{E}_q[\log p(x, y)] - \mathbb{E}_q[\log q(x)], \\ & \text{subject to} \quad \mathbb{D}_{\text{KL}}[q(x) || q^{[i]}(x)] \leq \varepsilon \quad \text{and} \quad \int q(x) dx = 1, \end{aligned} \quad (3)$$

where $q^{[i]}(x)$ is approximate posterior at iteration i and $\varepsilon \geq 0$ is a hyperparameter that controls the information bottleneck between two iterations. Additionally, problem (3) includes a distributional normalization constraint for $q(x)$, whereas the positivity of $q(x)$ is implied by the logarithmic function embedded within the KL constraint, which enforces a trust region around the current iterate $q^{[i]}(x)$. The following lemma derives the solution to (3), the variational distribution iterate $q^{[i+1]}(x)$, by constructing the Lagrangian.

Proposition 1 (Damped Gibbs posterior) *The Bayesian posterior characterized by a likelihood $p(y | x)$ and a prior $p(x)$ and constrained to lie within a Kullback–Leibler ε -ball centered at $q^{[i]}(x)$ is the maximizer of the constrained nonlinear program (3) and takes the form of the following Gibbs posterior*

$$q^{[i+1]}(x) = \left[\mathcal{Z}^{[i+1]}(\beta) \right]^{-1} \left[p(y | x) p(x) \right]^{1-\beta} \left[q^{[i]}(x) \right]^\beta,$$

with a damping parameter $\beta \in [0, 1)$ and a normalizing constant

$$\mathcal{Z}^{[i+1]}(\beta) = \int \left[p(y \mid x) p(x) \right]^{1-\beta} \left[q^{[i]}(x) \right]^\beta dx.$$

The damping β is a proxy of the Lagrangian multiplier $\alpha \geq 0$ associated with the Kullback–Leibler divergence constraint in (3), so that $\beta = \alpha/(1 + \alpha)$. Furthermore, the optimal β is a minimizer of the dual problem

$$\underset{\beta}{\text{minimize}} \quad \mathcal{G}(\beta) = \frac{\beta\varepsilon}{1-\beta} + \frac{1}{1-\beta} \log \mathcal{Z}^{[i+1]}(\beta), \quad \text{subject to } 0 \leq \beta < 1.$$

Proof. See Appendix A.

Remark 1 The posterior iterate $q^{[i+1]}(x)$ is as an interpolation in the space of densities between the true posterior $p(x \mid y) \propto p(y \mid x) p(x)$ and the previous iterate $q^{[i]}(x)$ in the space of probability densities.

Remark 2 The dual objective over β in Proposition 1 can be interpreted as an optimization over a family of statistical divergences, as revealed by the variational Rényi bound (Li & Turner, 2016):

$$\frac{1}{1-\beta} \log \mathcal{Z}^{[i+1]}(\beta) = \log p(y) - \mathbb{D}_\beta \left[q^{[i]}(x) \parallel p(x \mid y) \right],$$

where $\mathbb{D}_\beta [\cdot \parallel \cdot]$ denotes the Rényi divergence with order β . As the damping parameter β varies, the corresponding Rényi divergence traces a continuum of α -divergence geometries, each emphasizing different regions of the posterior distribution. Thus, varying β smoothly alters the global geometric structure of the optimization problem but preserves the local notion of distance defined by the Fisher information metric induced by primal Kullback–Leibler divergence objective (Amari, 2016).

Having reviewed the principles of entropic proximal optimization, we now turn to state-space models. In the next section, we extend this framework into the dynamic setting and derive recursive algorithms for approximate Bayesian inference that exploit the structure of the assumed posterior.

4 Entropic Proximal Bayesian Smoothing

For a state-space model of the form (1), we can adapt the ELBO from (2) as follows

$$\mathcal{L}(q) = \mathbb{E}_q \left[\log p(x_{0:T}, y_{1:T}) \right] - \mathbb{E}_q \left[\log q(x_{0:T}) \right] \leq \log p(y_{1:T}),$$

where $p(x_{0:T}, y_{1:T})$ is the joint state-measurement distribution given by

$$p(x_{0:T}, y_{1:T}) = p_0(x_0) \prod_{k=0}^{T-1} f_k(x_{k+1} \mid x_k) h_{k+1}(y_{k+1} \mid x_{k+1}).$$

We now formulate the entropic proximal optimization problem over the joint posterior $q(x_{0:T})$:

$$\begin{aligned} & \underset{q(x_{0:T})}{\text{maximize}} \quad \mathbb{E}_q \left[\log p(x_{0:T}, y_{1:T}) \right] - \mathbb{E}_q \left[\log q(x_{0:T}) \right], \\ & \text{subject to} \quad \mathbb{D}_{\text{KL}} \left[q(x_{0:T}) \parallel q^{[i]}(x_{0:T}) \right] \leq \varepsilon \quad \text{and} \quad \int q(x_{0:T}) dx_{0:T} = 1. \end{aligned} \tag{4}$$

This formulation does not yet impose any assumptions on the structure of the approximate posterior $q(x_{0:T})$. While (4) can be solved in a manner similar to (3), such an approach can lead to significant computational complexity due to the high dimensionality of $q(x_{0:T})$ as it extends over the state and time dimensions. To address this challenge, we introduce variations to (4) that take advantage of sparsity induced by the forward- and reverse Markov structure from Assumption 1 and Assumption 2, enabling recursive inference algorithms with linear time complexity in the horizon T .

4.1 Damped Forward-Markov Posterior

Here, we assume a forward-Markov decomposition of $q(x_{0:T})$ as stated in Assumption 1. This decomposition of the posterior leads to a generic *backward-forward* recursive algorithm for computing the approximate Bayesian smoothing posterior.

Proposition 2 (Optimal forward-Markov posterior) *For state-space models of the form (1), and under Assumption 1, the approximate forward-Markov smoothing posterior that solves problem (4) is characterized by the following set of tilted (conditional) distributions:*

$$\overrightarrow{q}_0^{[i+1]}(x_0) = \left[\overrightarrow{\mathcal{Z}}_0^{[i+1]} \right]^{-1} \left[\overrightarrow{q}_0^{[i]}(x_0) \right]^\beta \left[\exp \left\{ \overrightarrow{V}_0^{[i+1]}(x_0) \right\} \right]^{1-\beta}, \quad (5)$$

$$\begin{aligned} \overrightarrow{q}_k^{[i+1]}(x_{k+1} | x_k) &= \left[\overrightarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \\ &\quad \times \left[f_k(x_{k+1} | x_k) \exp \left\{ \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta}, \end{aligned} \quad (6)$$

where $\beta \in [0, 1]$ is the damping associated with the Lagrangian multiplier $\alpha \geq 0$ so that $\beta = \alpha/(1+\alpha)$, while $\overrightarrow{\mathcal{Z}}_0^{[i+1]}$ and $\overrightarrow{\psi}_k^{[i+1]}(x_k)$ are the corresponding normalizing factors

$$\begin{aligned} \overrightarrow{\mathcal{Z}}_0^{[i+1]} &= \int \left[\overrightarrow{q}_0^{[i]}(x_0) \right]^\beta \left[\exp \left\{ \overrightarrow{V}_0^{[i+1]}(x_0) \right\} \right]^{1-\beta} dx_0, \\ \overrightarrow{\psi}_k^{[i+1]}(x_k) &= \int \left[\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \left[f_k(x_{k+1} | x_k) \exp \left\{ \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} dx_{k+1}. \end{aligned} \quad (7)$$

The potential functions $\overrightarrow{V}_k^{[i+1]}(x_k)$, for all $0 \leq k \leq T$, are computed recursively backwards via

$$\overrightarrow{V}_k^{[i+1]}(x_k) = \begin{cases} \log h_T(y_T | x_T) & \text{if } k = T, \\ \log h_k(y_k | x_k) + 1/(1-\beta) \log \overrightarrow{\psi}_k^{[i+1]}(x_k) & \text{if } 0 < k < T, \\ \log p_0(x_0) + 1/(1-\beta) \log \overrightarrow{\psi}_0^{[i+1]}(x_0) & \text{if } k = 0, \end{cases} \quad (8)$$

bearing in mind that $\overrightarrow{\psi}_k^{[i+1]}(x_k)$ are functions of $\overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1})$. Finally, the optimal damping β is the minimizer of the dual objective

$$\underset{\beta}{\text{minimize}} \quad \overrightarrow{\mathcal{G}}(\beta) = \frac{\beta \varepsilon}{1-\beta} + \frac{1}{1-\beta} \log \overrightarrow{\mathcal{Z}}_0^{[i+1]}(\beta), \quad \text{subject to } 0 \leq \beta < 1. \quad (9)$$

Proof. See Appendix B.

Remark 3 The potential functions $\overrightarrow{V}_k(x_k)$ can be interpreted as the log-space accumulation of the backward filter message $\log q(y_{k:T} | x_k)$, where the log-normalizers $\log \overrightarrow{\psi}_k(x_k)$ and $\log \overrightarrow{\mathcal{Z}}_0$ correspond to $\log q(y_{k+1:T} | x_k)$ and $\log q(y_{1:T})$, respectively.

Remark 4 Under Assumption 1, the marginal smoothing distributions $\overrightarrow{q}_{k+1}^{[i+1]}(x_{k+1})$, for all $0 < k \leq T$, are computed via forward propagation starting from $\overrightarrow{q}_0^{[i+1]}(x_0)$

$$\overrightarrow{q}_{k+1}^{[i+1]}(x_{k+1}) = \int \overrightarrow{q}_k^{[i+1]}(x_k) \overrightarrow{q}_k^{[i+1]}(x_{k+1} | x_k) dx_k,$$

where $\overrightarrow{q}_k^{[i+1]}(x_{k+1} | x_k)$ and $\overrightarrow{q}_0^{[i+1]}(x_0)$ are given by Proposition 2.

4.2 Damped Reverse-Markov Posterior

Next, we assume a reverse-Markov decomposition of $q(x_{0:T})$ as described in Assumption 2. This decomposition, in contrast to the forward-Markov assumption, leads to a *forward-backward* recursive algorithm.

Proposition 3 (Optimal reverse-Markov posterior) *For state-space models of the form (1), and under Assumption 2, the approximate reverse-Markov smoothing posterior that solves problem (4) is characterized by the following set of tilted (conditional) distributions:*

$$\overleftarrow{q}_k^{[i+1]}(x_{k-1} \mid x_k) = \left[\overleftarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overleftarrow{q}_k^{[i]}(x_{k-1} \mid x_k) \right]^\beta \quad (10)$$

$$\times \left[f_{k-1}(x_k \mid x_{k-1}) \exp \left\{ \overleftarrow{V}_{k-1}^{[i+1]}(x_{k-1}) \right\} \right]^{1-\beta},$$

$$\overleftarrow{q}_T^{[i+1]}(x_T) = \left[\overleftarrow{\mathcal{Z}}_T^{[i+1]} \right]^{-1} \left[q_T^{[i]}(x_T) \right]^\beta \left[\exp \left\{ \overleftarrow{V}_T^{[i+1]}(x_T) \right\} \right]^{1-\beta}, \quad (11)$$

where $\beta \in [0, 1]$ is the damping associated with the Lagrangian multiplier $\alpha \geq 0$ so that $\beta = \alpha/(1+\alpha)$, while $\overleftarrow{\psi}_k^{[i+1]}(x_k)$ and $\overleftarrow{\mathcal{Z}}_T^{[i+1]}$ are the corresponding normalizing factors

$$\begin{aligned} \overleftarrow{\psi}_k^{[i+1]}(x_k) &= \int \left[\overleftarrow{q}_k^{[i]}(x_{k-1} \mid x_k) \right]^\beta \left[f_{k-1}(x_k \mid x_{k-1}) \exp \left\{ \overleftarrow{V}_{k-1}^{[i+1]}(x_{k-1}) \right\} \right]^{1-\beta} dx_{k-1}, \\ \overleftarrow{\mathcal{Z}}_T^{[i+1]} &= \int \left[\overleftarrow{q}_T^{[i]}(x_T) \right]^\beta \left[\exp \left\{ \overleftarrow{V}_T^{[i+1]}(x_T) \right\} \right]^{1-\beta} dx_T. \end{aligned} \quad (12)$$

The potential functions $\overleftarrow{V}_k^{[i+1]}(x_k)$, for all $0 \leq k < T$, are computed recursively forward via

$$\overleftarrow{V}_k^{[i+1]}(x_k) = \begin{cases} \log p_0(x_0) & \text{if } k = 0, \\ \log h_k(y_k \mid x_k) + 1/(1-\beta) \log \overleftarrow{\psi}_k^{[i+1]}(x_k) & \text{if } 0 < k \leq T, \end{cases} \quad (13)$$

bearing in mind that $\overleftarrow{\psi}_k^{[i+1]}(x_k)$ are functions of $\overleftarrow{V}_{k-1}^{[i+1]}(x_{k-1})$. Finally, the optimal damping β is the minimizer of the dual objective

$$\underset{\beta}{\text{minimize}} \quad \overleftarrow{\mathcal{G}}(\beta) = \frac{\beta \varepsilon}{1-\beta} + \frac{1}{1-\beta} \log \overleftarrow{\mathcal{Z}}_T^{[i+1]}(\beta), \quad \text{subject to } 0 \leq \beta < 1. \quad (14)$$

Proof. See Appendix C.

Remark 5 The potential functions $\overleftarrow{V}_k(x_k)$ can be interpreted as the log-space accumulation of the forward filter message $\log q(x_k \mid y_{1:k})$, where the log-normalizers $\log \overleftarrow{\psi}_k(x_k)$ and $\log \overleftarrow{\mathcal{Z}}_T$ correspond to $\log q(x_k \mid y_{1:k-1})$ and $\log q(y_{1:T})$, respectively.

Remark 6 Under Assumption 2, the marginal smoothing distributions $\overleftarrow{q}_k^{[i+1]}(x_k)$, for all $0 \leq k < T$, are computed via backward propagation starting from $\overleftarrow{q}_T^{[i+1]}(x_T)$

$$\overleftarrow{q}_{k-1}^{[i+1]}(x_{k-1}) = \int \overleftarrow{q}_k^{[i+1]}(x_k) \overleftarrow{q}_k^{[i+1]}(x_{k-1} \mid x_k) dx_k,$$

where $\overleftarrow{q}_k^{[i+1]}(x_{k-1} \mid x_k)$ and $\overleftarrow{q}_T^{[i+1]}(x_T)$ are given by Proposition 3.

4.3 Hybrid Posterior Factorization

The recursions in Proposition 2 and Proposition 3 are a direct result of the forward- and reverse-Markov decompositions from Assumption 1 and Assumption 2. While these schemes lead to two distinct smoothing algorithms, it is possible to combine elements of both to construct a hybrid smoothing solution that leverages both decompositions in one algorithm.

Corollary 1 (Optimal hybrid marginals) *For state-space models of the form 1, given the forward-Markov conditionals $\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k)$ and reverse-Markov conditionals $\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)$ associated with the same joint smoothing distribution $q^{[i]}(x_{0:T} | y_{1:T})$, we compute the marginals $q_k^{[i+1]}(x_k)$ associated with problem 4 by leveraging the interpretation from Remark 3 and Remark 5, we combine the recursions from Proposition 2 and Proposition 3*

$$q_k^{[i+1]}(x_k) \propto \begin{cases} \left[q_0^{[i]}(x_0)\right]^\beta \left[\exp\left\{\overrightarrow{V}_0^{[i+1]}(x_0)\right\}\right]^{1-\beta} & \text{if } k = 0, \\ \left[q_k^{[i]}(x_k)\right]^\beta \left[\exp\left\{\overleftarrow{V}_k^{[i+1]}(x_k)\right\} \overrightarrow{\psi}_k^{[i+1]}(x_k)\right]^{1-\beta} & \text{if } 0 < k < T, \\ \left[q_T^{[i]}(x_T)\right]^\beta \left[\exp\left\{\overleftarrow{V}_T^{[i+1]}(x_T)\right\}\right]^{1-\beta} & \text{if } k = T. \end{cases}$$

The forward–backward recursion schemes derived in Proposition 2, Proposition 3, and Corollary 1 share a general algorithmic structure that does not rely on specific assumptions about the dynamics $f_k(x_{k+1} | x_k)$, the measurement model $h_k(y_k | x_k)$, or the forms of the approximate posteriors $\overrightarrow{q}(x_{0:T})$ and $\overleftarrow{q}(x_{0:T})$. In the sections that follow, we present concrete implementations of these schemes by adopting conditionally Gaussian approximations for the forward- and reverse-Markov posterior distributions.

5 Practical Recursive Inference Algorithms

To transform the recursive schemes introduced in Section 4.1 and Section 4.2 into concrete and tractable algorithms, we impose additional structure on the forward and backward updates. This is accomplished by specializing the conditional posteriors defined in (6) and (10), along with the corresponding potential functions from (8) and (13). In particular, we focus on a class of approximate smoothing algorithms where the joint posterior is restricted to be Gaussian, and the Markov conditionals are constrained to the Gauss–Markov family. This restriction enables recursive formulations that admit closed-form updates.

Assumption 3 (Forward Gauss–Markov approximation) *Given a state-space model (1) and a forward-Markov factorization in Assumption 1, we restrict the approximate posterior $\overrightarrow{q}(x_{0:T} | y_{1:T})$ to the family of forward Gauss–Markov densities, leading to the following parameterization*

$$\overrightarrow{q}(x_{0:T} | y_{1:T}) = \mathcal{N}(x_0 | \overrightarrow{m}_0, \overrightarrow{P}_0) \prod_{k=0}^{T-1} \mathcal{N}(x_{k+1} | \overrightarrow{F}_k x_k + \overrightarrow{d}_k, \overrightarrow{\Sigma}_k).$$

Assumption 4 (Reverse Gauss–Markov approximation) *Given a state-space model (1) and a reverse-Markov decomposition in Assumption 2, we restrict the approximate posterior $\overleftarrow{q}(x_{0:T} | y_{1:T})$ to the family of reverse Gauss–Markov densities, leading to the following parameterization*

$$\overleftarrow{q}(x_{0:T} | y_{1:T}) = \mathcal{N}(x_T | \overleftarrow{m}_T, \overleftarrow{P}_T) \prod_{k=1}^T \mathcal{N}(x_{k-1} | \overleftarrow{F}_k x_k + \overleftarrow{d}_k, \overleftarrow{\Sigma}_k).$$

In the following sections, we develop tractable recursive inference schemes by constructing local quadratic approximations to the potential functions and deriving update rules that preserve the Gauss–Markov structure of the approximate posterior across iteration.

5.1 Statistical Function Approximations

A key step in constructing efficient smoothing algorithms is to approximate the potential functions in (8) and (13) with tractable forms that preserve the recursive structure and admit closed-form updates. We achieve this by introducing second-order *statistical expansions* of the log-density functions associated with the latent dynamics $\log f_k(x_{k+1} | x_k)$ and the measurement model $\log h_k(y_k | x_k)$.

Given the iterative nature of the proposed optimization procedure (4), it is natural to construct these expansions locally, around the current iterate $q^{[i]}(x_{0:T})$. This ensures that the approximation is tailored to the current posterior belief and captures relevant structure in the vicinity of the current iterate. Crucially, the KL-based trust-region constraint used in the entropic proximal updates plays an important role in controlling the step size of each iteration. It ensures that the distribution $q^{[i+1]}$ does not deviate too far from $q^{[i]}$, thereby keeping the updates within the region where the local expansions remain valid. This interaction between local approximations and bounded updates stabilizes the optimization and maintains the fidelity of the recursive inference scheme (Teboulle, 1992; Iusem et al., 1994; Chrétien & Hero, 2002).

Definition 1 (Statistical second-order expansion) Let $z \sim \mathcal{N}(m, P)$ and let $g(z)$ be a twice-differentiable scalar function. A second-order statistical expansion of $g(z)$ with respect to the random variable z takes a quadratic form $g(z) \approx -\frac{1}{2} z^\top U z + z^\top u + \eta$.

Definition 1 specifies the structure but not the computation of the expansion parameters. We return to this in later sections, where we describe two approximation strategies for computing (U, u, η) .

Assumption 5 (Quadratic expansion of log-densities) Let $(x_{k+1}, x_k) \sim q_k^{[i]}(x_{k+1}, x_k)$, we assume the statistical expansion of $\ell_f^{[i]}(x_{k+1}, x_k) \approx \log f_k(x_{k+1} | x_k)$, for all $0 \leq k < T$, is parameterized by

$$\ell_f^{[i]}(x_{k+1}, x_k) = -\frac{1}{2} \begin{bmatrix} x_{k+1}^\top & x_k^\top \end{bmatrix} \begin{bmatrix} C_{\bar{x}\bar{x},k}^{[i]} & -C_{\bar{x}x,k}^{[i]} \\ -C_{x\bar{x},k}^{[i]} & C_{xx,k}^{[i]} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} + \begin{bmatrix} x_{k+1}^\top & x_k^\top \end{bmatrix} \begin{bmatrix} c_{\bar{x},k}^{[i]} \\ c_{x,k}^{[i]} \end{bmatrix} + \kappa_k^{[i]},$$

and the statistical expansion of $\ell_h^{[i]}(x_k) \approx \log h_k(y_k | x_k)$, for all $0 < k \leq T$, is given as

$$\ell_h^{[i]}(x_k) = -\frac{1}{2} x_k^\top L_k^{[i]} x_k + x_k^\top l_k^{[i]} + \nu_k^{[i]}.$$

Finally, for $x_0 \sim q_0^{[i]}(x_0)$, we define the statistical expansion of $\ell_p^{[i]}(x_0) \approx \log p_0(x_0)$ as

$$\ell_p^{[i]}(x_0) = -\frac{1}{2} x_0^\top L_0^{[i]} x_0 + x_0^\top l_0^{[i]} + \nu_0^{[i]}.$$

5.2 Recursive Quadratic Potentials

In this section, we derive tractable schemes for computing the potential functions defined in (8) and (13), leveraging the statistical approximations introduced in Assumption 5. We show that these approximations lead to tractable recursions over quadratic forms of $\bar{V}_k(x_k)$ and $\bar{V}_k(x_k)$ from Proposition 2 and Proposition 3. The use of quadratic potentials is consistent with the interpretation in Remark 3 and Remark 5, where these functions are interpreted as log-space forward and backward filtering messages. Under the Gaussian approximation, such messages are naturally parameterized by quadratic functions, which justifies and supports the form adopted in our smoothing framework.

Before introducing these recursive itself, let us first start by defining the following parametric forms for the potential functions and associated log-normalizing functions

$$V_k^{[i+1]}(x_k) = -\frac{1}{2} x_k^\top R_k^{[i+1]} x_k + x_k^\top r_k^{[i+1]} + \rho_k^{[i+1]}, \quad (15)$$

$$\log \psi_k^{[i+1]}(x_k) = -\frac{1}{2} x_k^\top S_k^{[i+1]} x_k + x_k^\top s_k^{[i+1]} + \xi_k^{[i+1]}. \quad (16)$$

These quadratic forms enable efficient message-passing updates while preserving the Gaussian structure of the approximate posterior.

Proposition 4 (Recursive forward Gauss–Markov potentials) Let $\bar{q}_k^{[i]}(x_{k+1} | x_k)$ be a forward Gauss–Markov conditional as defined in Assumption 3, and let $\ell_f^{[i]}(x_{k+1}, x_k)$, $\ell_h^{[i]}(x_k)$, and $\ell_p^{[i]}(x_0)$ be the

second-order approximations of the log-probabilities from Assumption 5, then the potentials $\vec{V}_k^{[i+1]}(x_k)$ in (8) are quadratic functions of the form (15), computed recursively backwards starting from $k = T$ as:

$$\vec{R}_T^{[i+1]} = L_T^{[i]}, \quad \vec{r}_T^{[i+1]} = l_T^{[i]}.$$

For $0 \leq k < T$, the updates follow:

$$\begin{aligned}\vec{R}_k^{[i+1]} &= L_k^{[i]} + \frac{1}{1-\beta} \vec{S}_k^{[i+1]}, \\ \vec{r}_k^{[i+1]} &= l_k^{[i]} + \frac{1}{1-\beta} \vec{s}_k^{[i+1]},\end{aligned}$$

where the log-normalizing function $\log \vec{\psi}_k^{[i+1]}(x_k)$ has quadratic form (16) with parameters:

$$\begin{aligned}\vec{S}_k^{[i+1]} &= \vec{G}_{xx,k}^{[i+1]} - \left[\vec{G}_{\bar{x}x,k}^{[i+1]} \right]^\top \left[\vec{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \vec{G}_{\bar{x}x,k}^{[i+1]}, \\ \vec{s}_k^{[i+1]} &= \vec{g}_{x,k}^{[i+1]} + \left[\vec{G}_{\bar{x}x,k}^{[i+1]} \right]^\top \left[\vec{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \vec{g}_{\bar{x},k}^{[i+1]},\end{aligned}$$

with intermediate quantities defined as:

$$\begin{aligned}\vec{G}_{\bar{x}\bar{x},k}^{[i+1]} &:= (1-\beta) \left[C_{\bar{x}\bar{x},k}^{[i]} + \vec{R}_{k+1}^{[i+1]} \right] + \beta \left[\vec{\Sigma}_k^{[i]} \right]^{-1}, \\ \vec{G}_{xx,k}^{[i+1]} &:= (1-\beta) C_{xx,k}^{[i]} + \beta \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{F}_k^{[i]}, \\ \vec{G}_{\bar{x}x,k}^{[i+1]} &:= (1-\beta) C_{\bar{x}x,k}^{[i]} + \beta \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{F}_k^{[i]}, \\ \vec{g}_{\bar{x},k}^{[i+1]} &:= (1-\beta) \left[c_{\bar{x},k}^{[i]} + \vec{r}_k^{[i+1]} \right] + \beta \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{d}_k^{[i]}, \\ \vec{g}_{x,k}^{[i+1]} &:= (1-\beta) c_{x,k}^{[i]} - \beta \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{d}_k^{[i]}.\end{aligned}$$

Finally, given a quadratic potential function $\vec{V}_0^{[i+1]}(x_0)$ of the form (15) and Gaussian posterior density $\vec{q}_0^{[i]}(x_0) = \mathcal{N}(\vec{m}_0^{[i]}, \vec{P}_0^{[i]})$, the log-normalizing constant $\log \vec{\mathcal{Z}}_0^{[i+1]}$ is computed according to (7)

$$\log \vec{\mathcal{Z}}_0^{[i+1]} = -\frac{1}{2} \left[\vec{m}_0^{[i]} \right]^\top \vec{U}^{[i+1]} \vec{m}_0^{[i]} + \left[\vec{m}_0^{[i]} \right]^\top \vec{u}^{[i+1]} + \vec{\eta}^{[i+1]}, \quad (17)$$

where

$$\begin{aligned}\vec{U}^{[i+1]} &= \vec{J}_{mm}^{[i+1]} - \left[\vec{J}_{xm}^{[i+1]} \right]^\top \left[\vec{J}_{xx}^{[i+1]} \right]^{-1} \vec{J}_{xm}^{[i+1]}, \\ \vec{u}^{[i+1]} &= - \left[\vec{J}_{xm}^{[i+1]} \right]^\top \left[\vec{J}_{xx}^{[i+1]} \right]^{-1} \vec{j}_x^{[i+1]},\end{aligned}$$

and we have defined

$$\begin{aligned}\vec{J}_{xx}^{[i+1]} &:= (1-\beta) \vec{R}_0^{[i+1]} + \beta \left[\vec{P}_0^{[i]} \right]^{-1}, & \vec{J}_{xm}^{[i+1]} &:= \beta \left[\vec{P}_0^{[i]} \right]^{-1}, \\ \vec{J}_{mm}^{[i+1]} &:= \beta \left[\vec{P}_0^{[i]} \right]^{-1}, & \vec{j}_x^{[i+1]} &:= (1-\beta) \vec{r}_0^{[i+1]}.\end{aligned}$$

Proof. See Appendix D.

Proposition 5 (Recursive reverse Gauss–Markov potentials) Let $\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)$ be a reverse Gauss–Markov conditional as defined in Assumption 4) and let $\ell_f^{[i]}(x_k, x_{k-1})$, $\ell_h^{[i]}(x_k)$, and $\ell_p^{[i]}(x_0)$ be the second-order approximations of the log-probabilities in Assumption 5, then the potentials $\vec{V}_k^{[i+1]}(x_k)$ from (13) are quadratic functions of the form (15), computed recursively forwards starting from $k = 0$ with

$$\overleftarrow{R}_0^{[i+1]} = L_0^{[i]}, \quad \overleftarrow{r}_0^{[i+1]} = l_0^{[i]},$$

For $0 < k \leq T$, the updates follow:

$$\begin{aligned}\overleftarrow{R}_k^{[i+1]} &= L_k^{[i]} + \frac{1}{1-\beta} \overleftarrow{S}_k^{[i+1]}, \\ \overleftarrow{r}_k^{[i+1]} &= l_k^{[i]} + \frac{1}{1-\beta} \overleftarrow{s}_k^{[i+1]},\end{aligned}$$

where the log-normalizing function $\log \overleftarrow{\psi}_k^{[i+1]}(x_k)$ has quadratic form (16) with parameters:

$$\begin{aligned}\overleftarrow{S}_k^{[i+1]} &= \overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} - \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]}, \\ \overleftarrow{s}_k^{[i+1]} &= \overleftarrow{g}_{\bar{x},k}^{[i+1]} + \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overleftarrow{g}_{x,k}^{[i+1]},\end{aligned}$$

with intermediate quantities defined as:

$$\begin{aligned}\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} &:= (1-\beta) C_{\bar{x}\bar{x},k-1}^{[i]} + \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{F}_k^{[i]}, \\ \overleftarrow{G}_{xx,k}^{[i+1]} &:= (1-\beta) \left[C_{xx,k-1}^{[i]} + \overleftarrow{R}_{k-1}^{[i+1]} \right] + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1}, \\ \overleftarrow{G}_{x\bar{x},k}^{[i+1]} &:= (1-\beta) C_{x\bar{x},k-1}^{[i]} + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{F}_k^{[i]}, \\ \overleftarrow{g}_{\bar{x},k}^{[i+1]} &:= (1-\beta) c_{\bar{x},k-1}^{[i]} - \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]}, \\ \overleftarrow{g}_{x,k}^{[i+1]} &:= (1-\beta) \left[c_{x,k-1}^{[i]} + \overleftarrow{r}_{k-1}^{[i+1]} \right] + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]}.\end{aligned}$$

Finally, given a quadratic potential function $\overleftarrow{V}_T^{[i+1]}(x_T)$ of the form (15) and Gaussian posterior density $\overleftarrow{q}_T^{[i]}(x_T) = \mathcal{N}(\overleftarrow{m}_T^{[i]}, \overleftarrow{P}_T^{[i]})$, the log-normalizing constant $\log \overleftarrow{\mathcal{Z}}_T^{[i+1]}$ is computed according to (12)

$$\log \overleftarrow{\mathcal{Z}}_T^{[i+1]} = -\frac{1}{2} \left[\overleftarrow{m}_T^{[i]} \right]^\top \overleftarrow{U}^{[i+1]} \overleftarrow{m}_T^{[i]} + \left[\overleftarrow{m}_T^{[i]} \right]^\top \overleftarrow{u}^{[i+1]} + \overleftarrow{\eta}^{[i+1]}, \quad (18)$$

where

$$\begin{aligned}\overleftarrow{U}^{[i+1]} &= \overleftarrow{J}_{xx}^{[i+1]} - \left[\overleftarrow{J}_{mx}^{[i+1]} \right]^\top \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1} \overleftarrow{J}_{mx}^{[i+1]}, \\ \overleftarrow{u}^{[i+1]} &= - \left[\overleftarrow{J}_{mx}^{[i+1]} \right]^\top \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1} \overleftarrow{j}_m^{[i+1]},\end{aligned}$$

and we have defined

$$\begin{aligned}\overleftarrow{J}_{xx}^{[i+1]} &:= \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}, & \overleftarrow{J}_{mx}^{[i+1]} &:= \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}, \\ \overleftarrow{J}_{mm}^{[i+1]} &:= (1-\beta) \overleftarrow{R}_T^{[i+1]} + \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}, & \overleftarrow{j}_m^{[i+1]} &:= (1-\beta) \overleftarrow{r}_T^{[i+1]}.\end{aligned}$$

Proof. See Appendix E.

The results in Proposition 4 and Proposition 5 are derived using general second-order statistical expansions of the log-transition and log-likelihood functions. We now turn to the practical question of how to construct these approximations and present two distinct strategies for realizing them.

5.2.1 Generalized Statistical Linear Regression

Our first approach to realizing the second-order statistical expansions is based on generalized statistical linear regression (GSLR, García-Fernández et al., 2015; Tronarp et al., 2018). These *enabling approximations* (Särkkä & Svensson, 2023) produce affine-Gaussian surrogates of the dynamics $f_k(x_{k+1} | x_k)$ and the measurement model $h_k(y_k | x_k)$, yielding functional forms that directly satisfy the required quadratic structure for potentials and log-normalizing functions from Assumption 5.

Definition 2 (Generalized statistical linear regression) Let x and y be two random variables and supposed that the first and second moments $\mathbb{E}[x]$, $\mathbb{V}[x]$, $\mathbb{E}[y]$, $\mathbb{V}[y]$, and $\mathbb{C}[y, x]$ are known. Then, statistical linear regression approximates $p(y | x)$ by an affine-Gaussian model $y \approx Ax + b + \omega$, $\omega \sim \mathcal{N}(0, \Omega)$, with

$$A = \mathbb{C}[y, x] \mathbb{V}[x]^{-1}, \quad b = \mathbb{E}[y] - A \mathbb{E}[x], \quad \Omega = \mathbb{V}[y] - A \mathbb{V}[x] A^\top,$$

where the parameters $\{A, b\}$ minimize the mean squared error objective $\text{MSE}(A, b) = \mathbb{E}[(y - Ax - b)^\top (y - Ax - b)]$ and the covariance matrix satisfies $\Omega = \mathbb{V}[y - Ax - b]$. When only the conditional moments $\mathbb{E}[y | x]$ and $\mathbb{V}[y | x]$ are available, generalized statistical linear regression estimates marginal moments via the law of total expectation, so that:

$$\mathbb{E}[y] = \mathbb{E}\left[\mathbb{E}[y | x]\right], \quad \mathbb{V}[y] = \mathbb{E}\left[\mathbb{V}[y | x]\right] + \mathbb{V}\left[\mathbb{E}[y | x]\right], \quad \mathbb{C}[y, x] = \mathbb{C}\left[\mathbb{E}[y | x], x\right],$$

where the outer expectations are evaluated with respect to a Gaussian marginal distribution $p(x)$ using numerical quadrature integration rules (Särkkä & Svensson, 2023).

Based on the recipe from Definition 2, and given the conditional moments $\mathbb{E}[x_{k+1} | x_k]$, $\mathbb{V}[x_{k+1} | x_k]$, $\mathbb{E}[y_k | x_k]$, and $\mathbb{V}[y_k | x_k]$, we can replace the conditional densities $f_k(x_{k+1} | x_k)$ and $h_k(y_k | x_k)$, at each iteration $[i+1]$, by their affine-Gaussian approximations

$$\begin{aligned} x_{k+1} &\approx A_k^{[i]} x_k + b_k^{[i]} + \omega_k^{[i]}, & \omega_k^{[i]} &\sim \mathcal{N}(0, \Omega_k^{[i]}), & 0 \leq k < T, \\ y_k &\approx H_k^{[i]} x_k + e_k^{[i]} + \delta_k^{[i]}, & \delta_k^{[i]} &\sim \mathcal{N}(0, \Delta_k^{[i]}), & 0 < k \leq T, \end{aligned} \tag{19}$$

which corresponds to generalized statistical linear regression with respect to the Gaussian marginal distributions $q_k^{[i]}(x_k)$ from iteration $[i]$

$$\begin{array}{lll} A_k^{[i]} &= \mathbb{C}^{[i]}[x_{k+1}, x_k] \mathbb{V}^{[i]}[x_k]^{-1}, & H_k^{[i]} &= \mathbb{C}^{[i]}[y_k, x_k] \mathbb{V}^{[i]}[x_k]^{-1}, \\ b_k^{[i]} &= \mathbb{E}^{[i]}[x_{k+1}] - A_k^{[i]} \mathbb{E}^{[i]}[x_k], & e_k^{[i]} &= \mathbb{E}^{[i]}[y_k] - H_k^{[i]} \mathbb{E}^{[i]}[x_k], \\ \Omega_k^{[i]} &= \mathbb{V}^{[i]}[x_{k+1}] - A_k^{[i]} \mathbb{V}^{[i]}[x_k] \left[A_k^{[i]}\right]^\top, & \Delta_k^{[i]} &= \mathbb{V}^{[i]}[y_k] - H_k^{[i]} \mathbb{V}^{[i]}[x_k] \left[H_k^{[i]}\right]^\top. \end{array}$$

Finally, the prior density $p_0(x_0)$ is likewise approximated by a Gaussian density by matching the first and second moments

$$p_0(x_0) \approx \mathcal{N}(\mu_0, \Lambda_0), \quad \text{with } \mu_0 = \mathbb{E}[x_0], \quad \Lambda_0 = \mathbb{V}[x_0]. \tag{20}$$

Given the GSLR approximations (19) and (20), we obtain the quadratic log-density expansions required by Assumption 5, enabling closed-form computation in Proposition 4 and Proposition 5. Specifically, the resulting coefficients for $\ell_f^{[i]}(x_{k+1}, x_k)$ are:

$$\begin{aligned} C_{\bar{x}\bar{x}, k}^{[i]} &= \left[\Omega_k^{[i]}\right]^{-1}, & C_{\bar{x}x, k}^{[i]} &= \left[\Omega_k^{[i]}\right]^{-1} A_k^{[i]}, & c_{\bar{x}, k}^{[i]} &= \left[\Omega_k^{[i]}\right]^{-1} b_k^{[i]}, \\ C_{xx, k}^{[i]} &= \left[A_k^{[i]}\right]^\top \left[\Omega_k^{[i]}\right]^{-1} A_k^{[i]}, & C_{\bar{x}\bar{x}, k}^{[i]} &= \left[A_k^{[i]}\right]^\top \left[\Omega_k^{[i]}\right]^{-1}, & c_{x, k}^{[i]} &= -\left[A_k^{[i]}\right]^\top \left[\Omega_k^{[i]}\right]^{-1} b_k^{[i]}. \end{aligned}$$

for all $0 \leq k < T$, while the coefficients of $\ell_h^{[i]}(x_k)$, for all $0 < k \leq T$, are:

$$L_k^{[i]} = \left[H_k^{[i]}\right]^\top \left[\Delta_k^{[i]}\right]^{-1} H_k^{[i]}, \quad l_k^{[i]} = \left[H_k^{[i]}\right]^\top \left[\Delta_k^{[i]}\right]^{-1} \left[y_k - e_k^{[i]}\right],$$

and, finally, the coefficients associated with $\ell_p^{[i]}(x_0)$ are:

$$L_0^{[i]} = \Lambda_0^{-1}, \quad l_0^{[i]} = \Lambda_0^{-1} \mu_0.$$

5.2.2 Fourier–Hermite Series Expansion

The statistical approximation provided by statistical linear regression comes with two main limitations: it imposes an explicit additive Gaussian noise model on the approximated dynamics and measurements, and it inherently neglects second-order information. Alternatively, we can retrieve second-order statistical approximations by using Fourier–Hermite series (Sarmavuori & Särkkä, 2011; Hassan & Särkkä, 2023), which leverage Hermite polynomial bases in a Hilbert space \mathcal{H} (Malliavin, 2015) to capture higher-order effects.

Definition 3 Let $g \in \mathcal{H}$ be a scalar-valued function and let $s \sim \mathcal{N}(0, I)$ be a standard Gaussian random variable. A second-order Fourier–Hermite expansion of $g(s)$ is given by

$$g(s) \approx \mathbb{E}[g(s)] + \mathbb{E}[g(s) H_1(s)]^\top H_1(s) + \frac{1}{2} \text{tr} \left\{ \mathbb{E}[g(s) H_2(s)] H_2(s) \right\},$$

where $H_1(s)$ and $H_2(s)$ are first- and second-order Hermite polynomials defined as

$$H_1(s) = s, \quad H_2(s) = ss^\top - I.$$

This expansion generalizes to any Gaussian $\mathcal{N}(m, P)$ by letting $z = Rs + m$ and $P = RR^\top$, so that:

$$\begin{aligned} g(z) &\approx \mathbb{E}[g(z)] + \mathbb{E}\left[g(z) H_1(R^{-1}(z - m))\right]^\top H_1(R^{-1}(z - m)) \\ &\quad + \frac{1}{2} \text{tr} \left\{ \mathbb{E}\left[g(z) H_2(R^{-1}(z - m))\right] \left[R^{-1}(z - m)(z - m)^\top R^{-\top} - I\right] \right\} \\ &= -\frac{1}{2} z^\top U z + z^\top u + \eta. \end{aligned}$$

The coefficients of this quadratic form are:

$$\begin{aligned} U &:= -\mathbb{E}[G_{zz}(z)], \quad u := \mathbb{E}[G_z(z)] - \mathbb{E}[G_{zz}(z)] m, \\ \eta &:= \mathbb{E}[g(z)] - \mathbb{E}[G_z(z)] m + \frac{1}{2} m^\top \mathbb{E}[G_{zz}(z)] m - \frac{1}{2} \text{tr} \left\{ R^\top \mathbb{E}[G_{zz}(z)] R \right\}. \end{aligned}$$

where $G_z(z)$ and $G_{zz}(z)$ are the Jacobian and Hessian of $g(z)$, respectively, which, for $z \sim \mathcal{N}(m, P)$, satisfy the following identities via integration by parts (Hassan & Särkkä, 2023)

$$\mathbb{E}\left[g(z) H_1(R^{-1}(z - m))\right] = R^\top \mathbb{E}[G_z(z)], \quad \mathbb{E}\left[g(z) H_2(R^{-1}(z - m))\right] = R^\top \mathbb{E}[G_{zz}(z)] R.$$

The expectations involved in this approximation can be efficiently evaluated using numerical quadrature integration rules (Särkkä & Svensson, 2023).

Given a set of marginal distributions $q_k^{[i]}(x_k)$, we can apply the expansion in Definition 3 to the log-transition and log-measurement functions to obtain second-order approximations consistent with Assumption 5. This approach provides an alternative approximation to GLSR, which does not impose an explicit additive Gaussian noise assumption and inherently incorporates second-order information of the state-space model, offering potentially higher fidelity in the approximation.

5.3 Gauss–Markov Posterior Updates

Having established the quadratic approximations that enable recursive computation of the potential functions, we now turn our attention to the computation of the tilted distributions introduced in Proposition 2 and Proposition 3. As specified in Assumption 3 and Assumption 4, our goal is to maintain the Gauss–Markov structure of the forward and reverse variational posteriors throughout the iterative optimization process. To ensure that the updated variational distributions at iteration $[i+1]$ remain within the Gauss–Markov family, we must project the corresponding tilted distributions onto a (conditional) Gaussian form.

We begin by deriving a general moment-matching rule for tilted marginal distributions of the form in equations (5) and (11). These results then directly yield closed-form updates for the parameters $\overrightarrow{m}_0^{[i+1]}$, $\overrightarrow{P}_0^{[i+1]}$, $\overleftarrow{m}_T^{[i+1]}$, and $\overleftarrow{P}_T^{[i+1]}$, preserving the tractability and structure of the overall smoothing algorithm.

Lemma 1 (Tilted Gaussian moment matching) Let $q^{[i+1]}(x)$ be a tilted distribution of the form

$$q^{[i+1]}(x) = \left[\mathcal{Z}^{[i+1]} \right]^{-1} \left[q^{[i]}(x) \right]^{\beta} \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta},$$

with $q^{[i]}(x) = \mathcal{N}(x \mid m^{[i]}, P^{[i]})$ and a normalizing constant

$$\mathcal{Z}^{[i+1]} = \int \left[q^{[i]}(x) \right]^{\beta} \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta} dx,$$

then the first and second moments of $q^{[i+1]}(x)$ are given by

$$\begin{aligned} \mathbb{E}^{[i+1]}[x] &= m^{[i]} + \frac{1}{\beta} P^{[i]} \frac{\partial \log \mathcal{Z}^{[i+1]}}{\partial m^{[i]}}, \\ \mathbb{V}^{[i+1]}[x] &= \frac{1}{\beta} P^{[i]} + \frac{1}{\beta^2} P^{[i]} \frac{\partial^2 \log \mathcal{Z}^{[i+1]}}{\partial m^{[i]} \partial [m^{[i]}]^\top} P^{[i]}. \end{aligned}$$

Proof. See Appendix F.

Corollary 2 (Forward Gauss–Markov boundary) Let $\vec{q}_0^{[i]}(x_0)$ be a Gaussian marginal and let $\log \vec{\mathcal{Z}}_0^{[i+1]}$ be the log-normalizing constant of the tilted distribution in (17). Then, applying Lemma 1, the optimal Gaussian approximation to the updated tilted marginal $\vec{q}_0^{[i+1]}(x_0)$ from (5) is given by:

$$\vec{P}_0^{[i+1]} = \left[\vec{J}_{xx}^{[i+1]} \right]^{-1}, \quad \vec{m}_0^{[i+1]} = \left[\vec{J}_{xx}^{[i+1]} \right]^{-1} \left[\vec{j}_x^{[i+1]} + \vec{J}_{xm}^{[i+1]} \vec{m}_0^{[i]} \right].$$

By substituting in the explicit expressions, the update simplifies to:

$$\begin{aligned} \vec{P}_0^{[i+1]} &= \left[(1 - \beta) \vec{R}_0^{[i+1]} + \beta \left[\vec{P}_0^{[i]} \right]^{-1} \right]^{-1}, \\ \vec{m}_0^{[i+1]} &= \vec{P}_0^{[i+1]} \left[(1 - \beta) \vec{r}_0^{[i+1]} + \beta \left[\vec{P}_0^{[i]} \right]^{-1} \vec{m}_0^{[i]} \right]. \end{aligned}$$

Corollary 3 (Reverse Gauss–Markov boundary) Let $\overleftarrow{q}_T^{[i]}(x_T)$ be a Gaussian marginal and let $\log \overleftarrow{\mathcal{Z}}_T^{[i+1]}$ be the log-normalizing constant of the tilted distribution in (18). Then, applying Lemma 1, the optimal Gaussian approximation to the updated tilted marginal $\overleftarrow{q}_T^{[i+1]}(x_T)$ from (11) is given by:

$$\overleftarrow{P}_T^{[i+1]} = \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1}, \quad \overleftarrow{m}_T^{[i+1]} = \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1} \left[\overleftarrow{j}_m^{[i+1]} + \overleftarrow{J}_{mx}^{[i+1]} \overleftarrow{m}_T^{[i]} \right].$$

By substituting in the explicit expressions, the update simplifies to:

$$\begin{aligned} \overleftarrow{P}_T^{[i+1]} &= \left[(1 - \beta) \overleftarrow{R}_T^{[i+1]} + \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1} \right]^{-1}, \\ \overleftarrow{m}_T^{[i+1]} &= \overleftarrow{P}_T^{[i+1]} \left[(1 - \beta) \overleftarrow{r}_T^{[i+1]} + \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1} \overleftarrow{m}_T^{[i]} \right]. \end{aligned}$$

Projecting the tilted conditionals in (6) and (10) onto the affine-Gaussian parametric family defined in Assumption 3 and Assumption 4 is more involved, as it requires deriving the conditional moments of the forward and reverse distributions $\mathbb{E}[x_{k+1} \mid x_k]$, $\mathbb{V}[x_{k+1} \mid x_k]$, $\mathbb{E}[x_{k-1} \mid x_k]$, and $\mathbb{V}[x_{k-1} \mid x_k]$. In what follows, we derive closed-form expressions for these moments and use them to update the parameters of the forward and reverse Gauss–Markov posteriors.

Lemma 2 (Forward Gauss–Markov conditionals) Let $\vec{q}_k^{[i]}(x_{k+1} \mid x_k)$ be a forward Gauss–Markov conditional as defined in Assumption 3. Given a second-order expansion of $\ell_f^{[i]}(x_{k+1}, x_k)$ following Assumption 5, then, the conditional mean and covariance of the updated tilted conditional $\vec{q}_k^{[i+1]}(x_{k+1} \mid x_k)$

are given by:

$$\begin{aligned}\mathbb{E}^{[i+1]}[x_{k+1} | x_k] &= \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]}\right]^{-1} \left[\overrightarrow{G}_{xx,k}^{[i+1]} x_k - \overrightarrow{g}_{x,k}^{[i+1]} + \frac{\partial \log \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k}\right], \\ \mathbb{V}^{[i+1]}[x_{k+1} | x_k] &= \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]}\right]^{-1} \left[\overrightarrow{G}_{xx,k}^{[i+1]} + \frac{\partial^2 \log \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top}\right] \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]}\right]^{-1}.\end{aligned}$$

When $\log \overrightarrow{\psi}_k^{[i+1]}(x_k)$ is approximated quadratically as in Proposition 4, then parameters of the affine-Gaussian conditional $\overrightarrow{q}_k^{[i+1]}(x_{k+1} | x_k)$ become:

$$\overrightarrow{\Sigma}_k^{[i+1]} = \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]}\right]^{-1}, \quad \overrightarrow{F}_k^{[i+1]} = \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]}\right]^{-1} \overrightarrow{G}_{\bar{x}x,k}^{[i+1]}, \quad \overrightarrow{d}_k^{[i+1]} = \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]}\right]^{-1} \overrightarrow{g}_{\bar{x},k}^{[i+1]},$$

and can be further simplified to:

$$\begin{aligned}\overrightarrow{\Sigma}_k^{[i+1]} &= \left[(1-\beta) \left[C_{\bar{x}\bar{x},k}^{[i]} + \overrightarrow{R}_{k+1}^{[i+1]}\right] + \beta \left[\overrightarrow{\Sigma}_k^{[i]}\right]^{-1}\right]^{-1}, \\ \overrightarrow{F}_k^{[i+1]} &= \overrightarrow{\Sigma}_k^{[i+1]} \left[(1-\beta) C_{\bar{x}x,k}^{[i]} + \beta \left[\overrightarrow{\Sigma}_k^{[i]}\right]^{-1} \overrightarrow{F}_k^{[i]}\right], \\ \overrightarrow{d}_k^{[i+1]} &= \overrightarrow{\Sigma}_k^{[i+1]} \left[(1-\beta) [c_{\bar{x},k}^{[i]} + \overrightarrow{r}_k^{[i+1]}] + \beta \left[\overrightarrow{\Sigma}_k^{[i]}\right]^{-1} \overrightarrow{d}_k^{[i]}\right].\end{aligned}$$

Proof. See Appendix G.

Lemma 3 (Reverse Gauss–Markov conditionals) Let $\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)$ be a forward Gauss–Markov conditional as defined in Assumption 4. Given a second-order expansion of $\ell_f^{[i]}(x_k, x_{k-1})$ following Assumption 5, then, the conditional mean and covariance of the updated tilted conditional $\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)$ are given by:

$$\begin{aligned}\mathbb{E}^{[i+1]}[x_{k-1} | x_k] &= \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]}\right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} x_k - \overleftarrow{g}_{\bar{x},k}^{[i+1]} + \frac{\partial \log \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k}\right], \\ \mathbb{V}^{[i+1]}[x_{k-1} | x_k] &= \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]}\right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} + \frac{\partial^2 \log \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top}\right] \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]}\right]^{-1}.\end{aligned}$$

When $\log \overleftarrow{\psi}_k^{[i+1]}(x_k)$ is approximated quadratically as in Proposition 5, then the parameters of the affine-Gaussian conditional $\overleftarrow{q}_k^{[i+1]}(x_{k-1} | x_k)$ become:

$$\overleftarrow{\Sigma}_k^{[i+1]} = \left[\overleftarrow{G}_{xx,k}^{[i+1]}\right]^{-1}, \quad \overleftarrow{F}_k^{[i+1]} = \left[\overleftarrow{G}_{xx,k}^{[i+1]}\right]^{-1} \overleftarrow{G}_{x\bar{x},k}^{[i+1]}, \quad \overleftarrow{d}_k^{[i+1]} = \left[\overleftarrow{G}_{xx,k}^{[i+1]}\right]^{-1} \overleftarrow{g}_{x,k}^{[i+1]},$$

and can be further simplified to:

$$\begin{aligned}\overleftarrow{\Sigma}_k^{[i+1]} &= \left[(1-\beta) \left[C_{xx,k-1}^{[i]} + \overleftarrow{R}_{k-1}^{[i+1]}\right] + \beta \left[\overleftarrow{\Sigma}_k^{[i]}\right]^{-1}\right]^{-1}, \\ \overleftarrow{F}_k^{[i+1]} &= \overleftarrow{\Sigma}_k^{[i+1]} \left[(1-\beta) C_{x\bar{x},k-1}^{[i]} + \beta \left[\overleftarrow{\Sigma}_k^{[i]}\right]^{-1} \overleftarrow{F}_k^{[i]}\right], \\ \overleftarrow{d}_k^{[i+1]} &= \overleftarrow{\Sigma}_k^{[i+1]} \left[(1-\beta) [c_{x,k-1}^{[i]} + \overleftarrow{r}_{k-1}^{[i+1]}] + \beta \left[\overleftarrow{\Sigma}_k^{[i]}\right]^{-1} \overleftarrow{d}_k^{[i]}\right].\end{aligned}$$

Proof. See Appendix H.

5.4 Recursive Gaussian Marginals

Given the Gauss–Markov parameterization of the posterior in Assumption 3 and Assumption 4 and the update rules introduced in Section 5.3, the smoothing posterior marginals at each iteration $[i]$ can be computed

efficiently. Specifically, we use a forward recursion to evaluate the marginals of the forward Gauss–Markov posterior as described in Remark 4 and a backward recursion for the reverse Gauss–Markov posterior from Remark 6. These marginals play a central role in our proposed iterative smoothing framework, as they are used to perform second-order statistical expansions of the log-densities associated with the dynamics and measurement model, as discussed in Section 5.2.

Corollary 4 (Forward Gauss–Markov marginals) *Given the updated forward Gauss–Markov posterior from Corollary 2 and Lemma 2, the forward marginal smoothing distributions $\overrightarrow{q}_k^{[i+1]}(x_k)$, for all $0 < k \leq T$, are computed in closed form via the forward recursion:*

$$\begin{aligned}\overrightarrow{m}_{k+1}^{[i+1]} &= \overrightarrow{F}_k^{[i+1]} \overrightarrow{m}_k^{[i+1]} + \overrightarrow{d}_k^{[i+1]}, \\ \overrightarrow{P}_{k+1}^{[i+1]} &= \overrightarrow{F}_k^{[i+1]} \overrightarrow{P}_k^{[i+1]} \left[\overrightarrow{F}_k^{[i+1]} \right]^\top + \overrightarrow{\Sigma}_k^{[i+1]},\end{aligned}$$

with the initial condition $\overrightarrow{q}_0^{[i+1]} = \mathcal{N}(\overrightarrow{m}_0^{[i+1]}, \overrightarrow{P}_0^{[i+1]})$.

Corollary 5 (Reverse Gauss–Markov marginals) *Given the updated reverse Gauss–Markov posterior from Corollary 3 and Lemma 3, the reverse marginal smoothing distributions $\overleftarrow{q}_k^{[i+1]}(x_k)$, for all $0 \leq k < T$, are computed in closed form via the backward recursion:*

$$\begin{aligned}\overleftarrow{m}_{k-1}^{[i+1]} &= \overleftarrow{F}_k^{[i+1]} \overleftarrow{m}_k^{[i+1]} + \overleftarrow{d}_k^{[i+1]}, \\ \overleftarrow{P}_{k-1}^{[i+1]} &= \overleftarrow{F}_k^{[i+1]} \overleftarrow{P}_k^{[i+1]} \left[\overleftarrow{F}_k^{[i+1]} \right]^\top + \overleftarrow{\Sigma}_k^{[i+1]},\end{aligned}$$

with the terminal condition $\overleftarrow{q}_T^{[i+1]} = \mathcal{N}(\overleftarrow{m}_T^{[i+1]}, \overleftarrow{P}_T^{[i+1]})$.

Corollary 6 (Hybrid marginals) *Given the recursion from Proposition 4 and Proposition 5, the marginal smoothing distributions $q_k^{[i+1]}(x_k)$, for all $0 < k < T$, are computed according to Corollary 1 in closed form by completing the squares:*

$$\begin{aligned}P_k^{[i+1]} &= \left[(1 - \beta) \left[\overleftarrow{R}_k^{[i+1]} + \overrightarrow{S}_k^{[i+1]} \right] + \beta \left[P_k^{[i]} \right]^{-1} \right]^{-1}, \\ m_k^{[i+1]} &= P_k^{[i+1]} \left[(1 - \beta) \left[\overleftarrow{r}_k^{[i+1]} + \overrightarrow{s}_k^{[i+1]} \right] + \beta \left[P_k^{[i]} \right]^{-1} m_k^{[i]} \right],\end{aligned}$$

where $q_0^{[i+1]}(x_0) = \mathcal{N}(\overrightarrow{m}_0^{[i+1]}, \overrightarrow{P}_0^{[i+1]})$ and $q_T^{[i+1]}(x_T) = \mathcal{N}(\overleftarrow{m}_T^{[i+1]}, \overleftarrow{P}_T^{[i+1]})$.

5.5 Optimal Damping Parameter

We now address the selection of the damping parameter β , which plays a critical role in the entropic proximal update. As described in (4), β arises from the Kullback–Leibler constraint and varies across iterations. Propositions 2 and 3 show that the optimal value of β at each iteration is obtained by minimizing the corresponding dual objective associated with the forward or reverse factorization.

The dual problems defined in (9) and (14) are nonlinear in β and can be approached using standard numerical optimization techniques (Nocedal & Wright, 2006). However, in practice, we found that off-the-shelf solvers may struggle with feasibility issues in highly nonlinear regimes. These difficulties typically stem from violations of the convexity assumptions implicitly required for the quadratic potentials defined in Propositions 4 and 5, and can lead to instability unless positive definiteness is carefully handled.

To circumvent these challenges, we adopt a simple yet robust alternative. We apply a bisection method to the Lagrange multiplier α , which implicitly determines the damping parameter via the relation $\beta = \alpha/(1 + \alpha)$. This approach searches for the root of the gradient of the Lagrangian $\mathcal{R}(\alpha)$, ensuring that the KL constraint is satisfied at each iteration. The root-finding condition is given by:

$$\frac{\partial \mathcal{R}(\alpha^*)}{\partial \alpha^*} = \varepsilon - \mathbb{D}_{\text{KL}} \left[q^{[i+1]}(x_{0:T}; \alpha^*) \parallel q^{[i]}(x_{0:T}) \right] = 0, \quad (21)$$

where $\mathcal{R}(\cdot)$ represents the Lagrangian associated with Problem (4)

$$\mathcal{R}(\alpha) = \mathbb{E}_{q^{[i+1]}_\alpha} \left[\log p(x_{0:T}, y_{1:T}) - \log q^{[i+1]}(x_{0:T}; \alpha) \right] + \alpha \left[\varepsilon - \mathbb{D}_{\text{KL}} \left[q^{[i+1]}(x_{0:T}; \alpha) \parallel q^{[i]}(x_{0:T}) \right] \right] + \text{const.}$$

Here, $q^{[i+1]}(x_{0:T}; \alpha)$ denotes the updated variational posterior that depends implicitly on β , and thus on α . Detailed derivations of the forward and reverse Lagrangian formulations are provided in Appendix B and Appendix C, respectively.

An outline of the bisection procedure for determining the optimal damping parameter is presented in Algorithm 3 and Algorithm 6, corresponding to the forward and reverse Gauss–Markov smoothers. Importantly, the root condition (21) guarantees that the optimal solution $q^{[i+1]}(x_{0:T})$ lies on the boundary of the KL-ball centered at $q^{[i]}(x_{0:T})$ with radius ε . This induces an automatic damping adaptation mechanism that dynamically enforces the trust-region constraint across iterations.

5.6 Proximal Variational Bayesian Smoothing Algorithms

The preceding sections provide all the necessary components to instantiate the general results of Proposition 2, Proposition 3, and Corollary 1 for the case of Gaussian variational posterior approximations. In this section, we present an overview of the resulting three iterative Bayesian smoothing algorithms.

The *Forward Proximal Variational Smoother (FPVS)*, described in Algorithm 7, proceeds by iteratively refining the posterior estimate in key steps. First, a backward recursion is used to compute the potential functions and the forward conditional smoothing distributions, as detailed in Algorithm 1. This is followed by a forward recursion that infers the smoothed marginal distributions, as outlined in Algorithm 2. At each iteration, the damping parameter is updated via the procedure described in Algorithm 3, which ensures satisfaction of the proximal constraint imposed by the entropic regularization.

The *Reverse Proximal Variational Smoother (RPVS)*, presented in Algorithm 8, follows a similar structure but inverts the order of recursion. Specifically, a forward recursion computes the potential functions and the reverse conditional smoothing distributions, as given in Algorithm 4. This is followed by a backward recursion to compute the smoothed marginal distributions, according to Algorithm 5. As in the forward smoother, the damping parameter is adaptively chosen at each iteration using the method outlined in Algorithm 6.

The *Hybrid Proximal Variational Smoother (HPVS)*, shown in Algorithm 10, combines elements of both the forward and reverse approaches to jointly leverage their structural advantages. In this scheme, the potential functions and log-normalizing terms are computed using both the backward recursion from the forward smoother in Algorithm 1 and the forward recursion from the reverse smoother in Algorithm 4. The resulting forward and reverse representations are then fused to compute the smoothed marginal distributions in parallel, as described in Corollary 1. The damping parameter can be adapted following either the forward or reverse update strategy, depending on implementation preferences.

All of the proposed methods share several favorable properties. First, the recursions are formulated in log-space, which improves numerical stability and mitigates issues related to underflow in long time horizons. Second, each recursion is inherently damped through a trust-region constraint, ensuring that updates remain well-behaved on the statistical manifold and avoiding erratic jumps in the posterior estimates. Finally, all algorithms exhibit linear time complexity with respect to the temporal dimension, making them scalable and practical for long sequences.

6 Connection to Existing Bayesian Inference Algorithms

This section provides an overview of recent research that helps situate our contribution. We highlight related research in signal processing, Markovian Gaussian processes, and approximate Bayesian inference.

6.1 Forward-Backward Smoothing Algorithms

For the forward Gauss–Markov decomposition introduced in Proposition 2 and Proposition 4, a direct conceptual connection can be made to the classical smoother proposed by Cox (1964). In his work, Cox formulates

Bayesian smoothing in state-space models with additive Gaussian noise as a maximum a posteriori (MAP) optimization problem. Subject to certain non-singularity conditions, he derives a dynamic programming solution that propagates adjoint state functions backward in time, which aligns closely with the backward recursion over potential functions used in our framework. This method was later extended to nonlinear systems via iterative linearization techniques (Mortensen, 1968). Our approach generalizes Cox’s formulation by moving beyond MAP estimation to target a full Gaussian approximation of the smoothing posterior. In doing so, we accommodate a broader class of nonlinear, non-Gaussian state-space models. Moreover, by incorporating entropic regularization, our method introduces a principled mechanism for trust-region control, ensuring stable and well-posed updates over the space of variational densities.

In contrast, the reverse Gauss–Markov decomposition described in Proposition 3 and Proposition 5 yields a recursion that closely parallels the structure of the Rauch–Tung–Striebel (RTS) smoother (Rauch et al., 1965). The RTS smoother operates by first performing a forward filtering pass to accumulate measurement information, followed by a backward recursion that propagates smoothed marginals through the implicit computation of reverse posterior conditionals. Our iterated reverse Gauss–Markov smoother generalizes this two-pass structure to a significantly broader class of models. It lifts the standard assumptions of linear dynamics and Gaussian noise, enabling application to nonlinear, non-Gaussian systems. Additionally, by operating directly in the log-domain of the filtered densities and incorporating entropic proximal regularization, our method enhances numerical stability and provides a principled means of controlling the update step size throughout the recursion.

Finally, the hybrid posterior decomposition introduced in Corollary 1 and Corollary 6 bears a strong resemblance to log-space two-filter smoothers (Mayne, 1966; Fraser & Potter, 1969). Originally developed for MAP-based smoothing in linear-Gaussian state-space models, these methods were later extended to nonlinear settings through Gaussian-sum filters (Kitagawa, 1987). The canonical structure involves independent forward and backward filtering recursions that are subsequently combined to form the smoothing solution. Our hybrid smoother adopts this two-filter architecture by jointly leveraging both the forward and reverse Gauss–Markov decompositions of the variational posterior. In doing so, it generalizes the classical two-filter approach to accommodate nonlinear and non-Gaussian models within a variational inference framework. To the best of our knowledge, this is the first attempt to formulate an iterated two-filter smoother that integrates proximal updates and operates in the log-domain of the densities, akin to the forward and reverse variants. A notable advantage of this hybrid scheme is the structural independence of the forward and backward recursions, which allows for parallel execution. Although this design increases the overall computational cost relative to the single-pass algorithms, it offers practical benefits in terms of speed and modularity when implemented on modern parallel hardware.

6.2 Posterior-Linearization Bayesian Smoothing

Our work is primarily inspired by posterior-linearization algorithms (García-Fernández et al., 2016; Tronarp et al., 2018), which perform approximate Gaussian Bayesian smoothing in nonlinear state-space models by alternating between posterior linearization and a Rauch-Tung-Striebel smoothing pass. These algorithms can be viewed as generalizations of the maximum-a-posteriori iterated smoother proposed by Bell (1994). While these methods offer advantages over traditional extended and unscented smoothers (Särkkä & Svensson, 2023), they have two significant limitations.

First, they impose linearity and additive noise assumptions on the dynamics and measurement models, which can be problematic in state-space models with multiplicative noise, as highlighted by Corenflos & Abdulsamad (2023). Second, these algorithms, in their original form, fall within the class of undamped Gauss–Newton optimization methods, as they rely on linear approximations of the dynamics and measurement models. Gauss–Newton methods, however, require a full-rank Jacobian to ensure convergence (Nocedal & Wright, 2006), a condition that is not generally met in nonlinear settings.

To address the latter issue, a partial remedy was proposed by Raitoharju et al. (2018), who introduced an ad-hoc damping mechanism for the mean updates in the iterated posterior-linearization filter, albeit at the cost of a computationally expensive nested optimization loop. In contrast, Lindqvist et al. (2021) proposed a posterior-linearization smoother that implements damping through Levenberg–Marquardt regularization

and line-search procedures with convergence guarantees. Both approaches of Raitoharju et al. (2018) and Lindqvist et al. (2021), however, fail to exploit the information-geometric structure of the statistical manifold on which the approximate smoothing distribution is assumed to reside.

Our approach generalizes these algorithms and resolves both weaknesses in a theoretically grounded manner. By using Fourier–Hermite expansions, we overcome the limitations of linear-Gaussian approximations, while the introduction of entropic proximal constraints provides a principled way to damp optimization over the space of densities, leading to techniques akin to trust-region approaches (Nocedal & Wright, 2006; Teboulle, 1992). This improvement, however, comes at the cost of increased algorithmic complexity.

6.3 Approximate Bayesian Inference in State-Space Models

The field of approximate Bayesian inference has seen significant progress in adapting standard techniques to exploit the temporal structure of state-space models, resulting in specialized algorithms for Bayesian smoothing. One influential approach builds on expectation propagation (EP), introduced by Minka (2001). In particular, Deisenroth & Mohamed (2012) applied EP to nonlinear state-space models with additive Gaussian noise. Their method iteratively linearizes the dynamics and measurement models around the current posterior estimate, yielding tractable forward–backward message-passing recursions. However, the construction is largely heuristic and lacks a clearly defined global objective, which complicates convergence analysis, especially in nonlinear or non-Gaussian regimes. In contrast, our method is grounded in a well-posed variational optimization problem, enabling us to draw on a body of theoretical analysis from convex and information-theoretic optimization (Teboulle, 1992; Chrétien & Hero, 2002), and ensuring a more principled treatment of smoothing in complex models.

A parallel line of work has explored variational inference (VI)(Wainwright & Jordan, 2008) as a foundation for approximate smoothing. For instance, Courts et al. (2021) proposed a variational Gaussian smoother that optimizes the evidence lower bound using off-the-shelf constrained nonlinear solvers. Their formulation relies on a particular parameterization of twin-marginal distributions, subject to feasibility constraints. A key limitation, however, is that the resulting inference procedure is non-recursive, and optimization is carried out directly in the parameter space of the posterior, rather than over the statistical manifold where natural gradient methods can be more efficient(Amari, 1998). In contrast, Barfoot et al. (2020), building on the work of Opper & Archambeau (2009), derived a natural gradient-based variational Gaussian smoother that, while also non-recursive, leverages structured parameterizations to exploit the sparsity of the state-space model, significantly improving computational efficiency. Recently, Tronarp (2025) introduced a recursive variational framework that derives forward-backward algorithms similar to ours, albeit without proximal regularization.

Our framework is closely related to this line of work. Like the structured natural gradient approach of Barfoot et al. (2020), our method falls within the class of information-theoretic variational optimizers identified by Khan et al. (2015). However, it differs in two important respects. First, it is inherently recursive, producing a family of forward–backward smoothing algorithms that scale linearly in the time horizon. Second, it is agnostic to the parameterization of the variational posterior, allowing for flexible approximation families. By casting inference as a dynamic optimization problem constrained by KL-based trust regions, our method naturally exploits the temporal structure of state-space models and inherits convergence properties from proximal optimization theory.

6.4 Approximate Bayesian Inference in Temporal Gaussian Processes

There is a well-established connection between temporal Gaussian processes (GP) and *linear* state-space models (O’Hagan, 1978). Hartikainen & Särkkä (2010) showed that temporal GP models with Matérn kernels can be reformulated exactly as linear-Gaussian SSMS. This enables using Kalman filtering and smoothing with linear time complexity to perform inference in temporal GPs, a significant improvement over the cubic complexity of standard GP inference.

Building on this foundation, Chang et al. (2020) proposed a variational method that combines conjugate-computation variational inference (Khan & Lin, 2017) with Rauch–Tung–Striebel recursions, enabling principled inference in models with linear dynamics and nonlinear measurement models. Wilkinson et al. (2020) ex-

tended this framework by unifying variational inference and expectation propagation within Kalman smoothing. Wilkinson et al. (2023) further generalized these approaches via the Bayes–Newton framework, which interprets VI, EP, and posterior linearization as optimization algorithms akin to Newton’s method. Despite these advances, a common critique of this class of algorithms is their ad-hoc treatment of the temporal dynamics. While inference is formulated from an optimization perspective, the temporal structure is handled implicitly using linear RTS smoothing, rather than being explicitly integrated into the inference objective.

In the continuous-time setting, several variational methods have been developed. Archambeau et al. (2007) proposed one of the earliest approaches for inference in partially-observed diffusion processes with linear dynamics by approximating the posterior with a linear stochastic differential equation. This approximation can be viewed as the continuous-time limit of a Gauss–Markov factorization. Ala-Luhtala et al. (2015) extended this line of work to nonlinear dynamics through sigma-point approximations. More recently, Wildner & Koepll (2021) and Verma et al. (2024) proposed natural-gradient-based generalizations of the method introduced by Archambeau et al. (2007). Wildner & Koepll (2021) developed a moment-based recursive inference algorithm that reformulates the smoothing problem as an optimal control problem, thereby explicitly incorporating the dynamic structure of the setting. In contrast, Verma et al. (2024) adopts a site-based approach inspired by Minka (2001), introducing a non-recursive algorithm that leverages iterative posterior linearization to handle nonlinear dynamics. Finally, Bartosh et al. (2025) recently proposed a continuous-time simulation-free approach based on modern flow and score matching techniques that scales to high dimensional problem.

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A Proof of Proposition 1

To find the critical point of (3) with respect to distribution $q(x)$ given the constraints, we first construct the corresponding Lagrangian functional

$$\begin{aligned}\mathcal{R}(q, \lambda, \alpha) &= \mathbb{E}_q \left[\log p(y, x) \right] - \mathbb{E}_q \left[\log q(x) \right] \\ &\quad + \lambda \left[1 - \int q(x) dx \right] + \alpha \left[\varepsilon - \mathbb{D}_{\text{KL}} \left[q(x) \parallel q^{[i]}(x) \right] \right],\end{aligned}\tag{22}$$

where λ and $\alpha \geq 0$ are the Lagrangian multipliers associated with the constraints. Next, we set the functional derivative of $\mathcal{R}(\cdot)$ with respect to $q(x)$ to zero

$$\frac{\partial \mathcal{R}(q, \lambda, \alpha)}{\partial q(x)} = \log p(y, x) - \log q(x) - 1 - \lambda - \alpha \log q(x) + \alpha \log q^{[i]}(x) - \alpha := 0,$$

which in turn leads to the optimal iterate $q^{[i+1]}(x)$

$$\begin{aligned}q^{[i+1]}(x) &= \left[\exp \{ \lambda + (1 + \alpha) \} \right]^{-1/(1+\alpha)} \left[p(y \mid x) p(x) \right]^{1/(1+\alpha)} \left[q^{[i]}(x) \right]^{\alpha/(1+\alpha)} \\ &\propto \left[p(y \mid x) p(x) \right]^{1/(1+\alpha)} \left[q^{[i]}(x) \right]^{\alpha/(1+\alpha)}.\end{aligned}\tag{23}$$

Plugging the result from (23) back into the Lagrangian (22), we get the dual problem

$$\underset{\lambda, \alpha}{\text{minimize}} \quad \mathcal{G}(\lambda, \alpha) = \alpha \varepsilon + \lambda + (1 + \alpha) \int q^{[i+1]}(x) dx, \quad \text{subject to } \alpha \geq 0.\tag{24}$$

If we compute and set the gradient of $\mathcal{G}(\cdot)$ with respect to λ to zero, we get

$$\begin{aligned}\lambda^{[i+1]} &= (1 + \alpha) \left[-1 + \log \int \left[p(y \mid x) p(x) \right]^{1/(1+\alpha)} \left[q^{[i]}(x) \right]^{\alpha/(1+\alpha)} dx \right] \\ &= (1 + \alpha) \left[-1 + \log \mathcal{Z}^{[i+1]}(\alpha) \right]\end{aligned}\tag{25}$$

which when plugged back into (23) leads to the normalized approximate Gibbs posterior

$$q^{[i+1]}(x) = \left[\mathcal{Z}^{[i+1]}(\beta) \right]^{-1} \left[p(y \mid x) p(x) \right]^{1-\beta} \left[q^{[i]}(x) \right]^\beta.\tag{26}$$

where have defined $\beta := \alpha/(1 + \alpha)$, so that $\beta = 0$ when $\alpha = 0$ and $\beta \rightarrow 1$ as $\alpha \rightarrow \infty$. By using the results from (25) and (26), we can rewrite the dual problem (24) as follows

$$\underset{\beta}{\text{minimize}} \quad \mathcal{G}(\beta) = \frac{\beta \varepsilon}{1 - \beta} + \frac{1}{1 - \beta} \log \mathcal{Z}^{[i+1]}(\beta), \quad \text{subject to } 0 \leq \beta < 1.$$

B Proof of Proposition 2

Given the forward-Markov decomposition from Assumption 1, we can factorize problem (4) over time. Starting with the ELBO objective, we can write

$$\begin{aligned}\mathcal{L}(\vec{q}) &= \mathbb{E}_{\vec{q}} \left[\log p(x_{0:T}, y_{1:T}) \right] - \mathbb{E}_{\vec{q}} \left[\log \vec{q}(x_{0:T}) \right] \\ &= \int \vec{q}_0(x_0) \left[\log p(x_0) - \log \vec{q}_0(x_0) \right] dx_0 + \sum_{k=1}^T \int \vec{q}_k(x_k) \log h_k(y_k \mid x_k) dx_k \\ &\quad + \sum_{k=0}^{T-1} \int \vec{q}_k(x_k) \vec{q}_k(x_{k+1} \mid x_k) \left[\log f_k(x_{k+1} \mid x_k) - \log \vec{q}_k(x_{k+1} \mid x_k) \right] dx_k dx_{k+1}.\end{aligned}$$

Additionally, the Kullback–Leibler divergences factorizes forward as follows

$$\begin{aligned}\varepsilon &\geq \mathbb{D}_{\text{KL}} \left[\vec{q}(x_{0:T}) \parallel \vec{q}^{[i]}(x_{0:T}) \right] \\ \varepsilon &\geq \int \vec{q}_0(x_0) \prod_{k=0}^{T-1} \vec{q}_k(x_{k+1} \mid x_k) \log \frac{\vec{q}_0(x_0) \prod_{k=0}^{T-1} \vec{q}_k(x_{k+1} \mid x_k)}{\vec{q}_0^{[i]}(x_0) \prod_{k=0}^{T-1} \vec{q}_k^{[i]}(x_{k+1} \mid x_k)} dx_{0:T} \\ \varepsilon &\geq \int \vec{q}_0(x_0) \log \frac{\vec{q}_0(x_0)}{\vec{q}_0^{[i]}(x_0)} dx_0 \\ &\quad + \sum_{k=0}^{T-1} \int \vec{q}_k(x_k) \int \vec{q}_k(x_{k+1} \mid x_k) \log \frac{\vec{q}_k(x_{k+1} \mid x_k)}{\vec{q}_k^{[i]}(x_{k+1} \mid x_k)} dx_{k+1} dx_k.\end{aligned}$$

Finally, the normalization constraint also factorizes to

$$1 = \int \vec{q}_0(x_0) dx_0, \quad 1 = \int \vec{q}_k(x_{k+1} \mid x_k) dx_{k+1}, \quad \forall x_k, 0 \leq k < T.$$

The information-theoretic proximal optimization problem (4) now has the form

$$\begin{aligned}&\underset{\substack{\vec{q}_k(x_{k+1} \mid x_k), \\ \vec{q}_0(x_0)}}{\text{maximize}} \quad \int \vec{q}_0(x_0) [\log p(x_0) - \log \vec{q}_0(x_0)] dx_0 + \sum_{k=1}^T \int \vec{q}_k(x_k) \log h_k(y_k \mid x_k) dx_k \\ &\quad + \sum_{k=0}^{T-1} \int \vec{q}_k(x_k) \vec{q}_k(x_{k+1} \mid x_k) [\log f_k(x_{k+1} \mid x_k) - \log \vec{q}_k(x_{k+1} \mid x_k)] dx_k dx_{k+1}, \\ &\text{subject to} \quad \vec{q}_{k+1}(x_{k+1}) = \int \vec{q}_k(x_k) \vec{q}_k(x_{k+1} \mid x_k) dx_k, \quad \forall x_{k+1}, 0 \leq k < T, \\ &\quad 1 = \int \vec{q}_0(x_0) dx_0, \quad 1 = \int \vec{q}_k(x_{k+1} \mid x_k) dx_{k+1}, \quad \forall x_k, 0 \leq k < T, \\ &\quad \varepsilon \geq \int \vec{q}_0(x_0) \log \frac{\vec{q}_0(x_0)}{\vec{q}_0^{[i]}(x_0)} dx_0 \\ &\quad + \sum_{k=0}^{T-1} \int \vec{q}_k(x_k) \int \vec{q}_k(x_{k+1} \mid x_k) \log \frac{\vec{q}_k(x_{k+1} \mid x_k)}{\vec{q}_k^{[i]}(x_{k+1} \mid x_k)} dx_{k+1} dx_k.\end{aligned}$$

Next, we construct the Lagrangian functional $\vec{\mathcal{R}}(\vec{q}_0, \vec{q}_k, \vec{\gamma}_0, \vec{\lambda}_k, \vec{V}_k, \alpha)$

$$\begin{aligned}\vec{\mathcal{R}}(\cdot) &= \int \vec{q}_0(x_0) [\log p(x_0) - \log \vec{q}_0(x_0)] dx_0 + \sum_{k=1}^T \int \vec{q}_k(x_k) \log h_k(y_k \mid x_k) dx_k \\ &\quad + \sum_{k=0}^{T-1} \int \vec{q}_k(x_k) \vec{q}_k(x_{k+1} \mid x_k) [\log f_k(x_{k+1} \mid x_k) - \log \vec{q}_k(x_{k+1} \mid x_k)] dx_k dx_{k+1} \\ &\quad + \sum_{k=0}^{T-1} \int \vec{V}_{k+1}(x_{k+1}) \left[\int \vec{q}_k(x_k) \vec{q}_k(x_{k+1} \mid x_k) dx_k - \vec{q}_{k+1}(x_{k+1}) \right] dx_{k+1} \\ &\quad + \sum_{k=0}^{T-1} \int \vec{\lambda}_k(x_k) \left[1 - \int \vec{q}_k(x_{k+1} \mid x_k) dx_{k+1} \right] dx_k + \vec{\gamma}_0 \left[1 - \int \vec{q}_0(x_0) dx_0 \right] \\ &\quad + \alpha \left[\varepsilon - \int \vec{q}_0(x_0) \log \frac{\vec{q}_0(x_0)}{\vec{q}_0^{[i]}(x_0)} dx_0 \right. \\ &\quad \left. - \sum_{k=0}^{T-1} \int \vec{q}_k(x_k) \int \vec{q}_k(x_{k+1} \mid x_k) \log \frac{\vec{q}_k(x_{k+1} \mid x_k)}{\vec{q}_k^{[i]}(x_{k+1} \mid x_k)} dx_{k+1} dx_k \right],\end{aligned}\tag{27}$$

where $\vec{V}_k(x_k)$, $\vec{\lambda}_k(x_k)$, $\vec{\gamma}_0$, and $\alpha \geq 0$ are Lagrangian multipliers. To find the critical point, we set the functional derivative with respect to $\vec{q}_k(x_{k+1} | x_k)$ to zero

$$\begin{aligned}\vec{q}_k^{[i+1]}(x_{k+1} | x_k) &= \left[\exp \left\{ \vec{\lambda}_k(x_k) / \vec{q}_k(x_k) + (1 + \alpha) \right\} \right]^{-1/(1+\alpha)} \\ &\quad \times \left[\vec{q}_k^{[i]}(x_{k+1} | x_k) \right]^{\alpha/(1+\alpha)} \left[f_k(x_{k+1} | x_k) \exp \left\{ \vec{V}_{k+1}(x_{k+1}) \right\} \right]^{1/(1+\alpha)}.\end{aligned}\quad (28)$$

Additionally, we set the functional derivative with respect to $\vec{q}_0(x_0)$ to zero and get

$$\vec{q}_0^{[i+1]}(x_0) = \left[\exp \left\{ \vec{\gamma}_0 + (1 + \alpha) \right\} \right]^{-1/(1+\alpha)} \left[\vec{q}_0^{[i]}(x_0) \right]^{\alpha/(1+\alpha)} \left[\exp \left\{ \vec{V}_0(x_0) \right\} \right]^{1/(1+\alpha)}, \quad (29)$$

where, for convenience of notation, we define

$$\vec{V}_0(x_0) = \log p_0(x_0) + (1 + \alpha) \log \int \left[\vec{q}_0^{[i]}(x_1 | x_0) \right]^{\alpha/(1+\alpha)} \left[f_0(x_1 | x_0) \exp \left\{ \vec{V}_1(x_1) \right\} \right]^{1/(1+\alpha)} dx_1. \quad (30)$$

Notice that, it is not clear at this point that the solutions (28) and (29) are normalized densities. Establishing that requires solving for the (functional) multipliers $\vec{\lambda}_k(x_k)$ and $\vec{\gamma}_0$, which are associated with the normalization constraints. Plugging the solution (28) into the Lagrangian (27) results in the dual functional

$$\begin{aligned}\vec{\mathcal{G}}(\cdot) &= \alpha \varepsilon + \int \vec{q}_0(x_0) \left[\log p(x_0) - \log \vec{q}_0(x_0) \right] dx_0 + \sum_{k=1}^T \int \vec{q}_k(x_k) \log h_k(y_k | x_k) dx_k \\ &\quad + \alpha \int \vec{q}_0(x_0) \left[\log \vec{q}_0^{[i]}(x_0) - \log \vec{q}_0(x_0) \right] dx_0 + \vec{\gamma}_0 \left[1 - \int \vec{q}_0(x_0) dx_0 \right] \\ &\quad + \sum_{k=0}^{T-1} \int \vec{\lambda}_k(x_k) dx_k - \sum_{k=0}^{T-1} \int \vec{V}_{k+1}(x_{k+1}) \vec{q}_{k+1}(x_{k+1}) dx_{k+1} \\ &\quad + (1 + \alpha) \sum_{k=0}^{T-1} \int \vec{q}_k(x_k) \int \vec{q}_k^{[i+1]}(x_{k+1} | x_k) dx_{k+1} dx_k.\end{aligned}$$

Now, we can solve for the multipliers $\vec{\lambda}_k(x_k)$ by setting the associated derivatives to zero

$$\begin{aligned}\vec{\lambda}_k^{[i+1]}(x_k) &= (1 + \alpha) \vec{q}_k(x_k) \left[-1 + \log \int \left[\vec{q}_k^{[i]}(x_{k+1} | x_k) \right]^{\alpha/(1+\alpha)} \right. \\ &\quad \times \left. \left[f_k(x_{k+1} | x_k) \exp \left\{ \vec{V}_{k+1}(x_{k+1}) \right\} \right]^{1/(1+\alpha)} dx_{k+1} \right] \\ &= (1 + \alpha) \vec{q}_k(x_k) \left[-1 + \log \vec{\psi}_k^{[i+1]}(x_k) \right].\end{aligned}\quad (31)$$

Plugging (31) into (28) returns the normalized tilted conditional

$$\begin{aligned}\vec{q}_k^{[i+1]}(x_{k+1} | x_k) &= \left[\vec{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\vec{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \\ &\quad \times \left[f_k(x_{k+1} | x_k) \exp \left\{ \vec{V}_{k+1}(x_{k+1}) \right\} \right]^{1-\beta},\end{aligned}\quad (32)$$

where have defined $\beta := \alpha/(1 + \alpha)$, so that $\beta = 0$ when $\alpha = 0$, and $\beta \rightarrow 1$ as $\alpha \rightarrow \infty$. Next, we use the results from (29), (31), and (32) to simplify the functional $\vec{\mathcal{G}}(\cdot)$ further

$$\begin{aligned}\vec{\mathcal{G}}(\cdot) &= \alpha \varepsilon + \sum_{k=1}^T \int \vec{q}_k(x_k) \log h_k(y_k | x_k) dx_k - \sum_{k=1}^T \int \vec{V}_k(x_k) \vec{q}_k(x_k) dx_k \\ &\quad + (1 + \alpha) \sum_{k=1}^{T-1} \int \vec{q}_k(x_k) \log \vec{\psi}_k^{[i+1]}(x_k) dx_k + \vec{\gamma}_0 + (1 + \alpha) \int \vec{q}_0^{[i+1]}(x_0) dx_0,\end{aligned}$$

and solve for the optimal multiplier $\overrightarrow{\gamma}_0$ by zeroing the associated derivative, leading to

$$\begin{aligned}\overrightarrow{\gamma}_0^{[i+1]} &= (1 + \alpha) \left[-1 + \log \int \left[\overrightarrow{q}_0^{[i]}(x_0) \right]^{\alpha/(1+\alpha)} \left[\exp \left\{ \overrightarrow{V}_0(x_0) \right\} \right]^{1/(1+\alpha)} dx_0 \right] \\ &= (1 + \alpha) \left[-1 + \log \overrightarrow{\mathcal{Z}}_0^{[i+1]} \right],\end{aligned}\quad (33)$$

which after plugging into (29) delivers

$$\overrightarrow{q}_0^{[i+1]}(x_0) = \left[\overrightarrow{\mathcal{Z}}_0^{[i+1]} \right]^{-1} \left[\overrightarrow{q}_0^{[i]}(x_0) \right]^\beta \left[\exp \left\{ \overrightarrow{V}_0(x_0) \right\} \right]^{1-\beta}. \quad (34)$$

Plugging (34) and (33) back into $\overrightarrow{\mathcal{G}}(\cdot)$ leads to further simplification of the dual

$$\begin{aligned}\overrightarrow{\mathcal{G}}(\cdot) &= \alpha \varepsilon + \sum_{k=1}^T \int \overrightarrow{q}_k(x_k) \log h_k(y_k | x_k) dx_k - \sum_{k=1}^T \int \overrightarrow{V}_k(x_k) \overrightarrow{q}_k(x_k) dx_k \\ &\quad + (1 + \alpha) \sum_{k=1}^{T-1} \int \overrightarrow{q}_k(x_k) \log \overrightarrow{\psi}_k^{[i+1]}(x_k) dx_k + (1 + \alpha) \log \overrightarrow{\mathcal{Z}}_0^{[i+1]}.\end{aligned}$$

Finally, to find the optimal potentials, we zero the derivative of $\overrightarrow{\mathcal{G}}(\cdot)$ with respect to $\overrightarrow{q}_k(x_k), \forall 0 < k \leq T$. Combined with the results from (30) and (31), we can write

$$\overrightarrow{V}_k^{[i+1]}(x_k) = \begin{cases} \log h_T(y_T | x_T) & \text{if } k = T, \\ \log h_k(y_k | x_k) + 1/(1-\beta) \log \overrightarrow{\psi}_k^{[i+1]}(x_k) & \text{if } 0 < k < T, \\ \log p_0(x_0) + 1/(1-\beta) \log \overrightarrow{\psi}_0^{[i+1]}(x_0) & \text{if } k = 0,\end{cases}$$

which, when plugged back into the dual $\overrightarrow{\mathcal{G}}(\cdot)$, leads to the final simplification

$$\overrightarrow{\mathcal{G}}(\beta) = \frac{\beta \varepsilon}{1-\beta} + \frac{1}{1-\beta} \log \overrightarrow{\mathcal{Z}}_0^{[i+1]}(\beta).$$

C Proof of Proposition 3

This proof follows a similar scheme to that in Section B. However, the reverse-Markov decomposition results in novel recursions. Starting from the reverse-Markov decomposition in Assumption 2, we factorize problem (4) over time. Thus, we can write the ELBO as

$$\begin{aligned}\mathcal{L}(\overleftarrow{q}) &= \mathbb{E}_{\overleftarrow{q}} \left[\log p(x_{0:T}, y_{1:T}) \right] - \mathbb{E}_{\overleftarrow{q}} \left[\log \overleftarrow{q}(x_{0:T}) \right] \\ &= \int \overleftarrow{q}_0(x_0) \log p(x_0) dx_0 - \int \overleftarrow{q}_T(x_T) \log \overleftarrow{q}_T(x_T) dx_T + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log h_k(y_k | x_k) dx_k \\ &\quad + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \overleftarrow{q}_k(x_{k-1} | x_k) \left[\log f_{k-1}(x_k | x_{k-1}) - \log \overleftarrow{q}_k(x_{k-1} | x_k) \right] dx_{k-1} dx_k.\end{aligned}$$

Furthermore, the Kullback–Leibler divergence factorizes reversely to

$$\begin{aligned}\varepsilon &\geq \mathbb{D}_{\text{KL}} \left[\overleftarrow{q}(x_{0:T}) \parallel \overleftarrow{q}^{[i]}(x_{0:T}) \right] \\ &\geq \int \overleftarrow{q}_T(x_T) \prod_{k=1}^T \overleftarrow{q}_k(x_{k-1} | x_k) \log \frac{\overleftarrow{q}_T(x_T) \prod_{k=1}^T \overleftarrow{q}_k(x_{k-1} | x_k)}{\overleftarrow{q}_T^{[i]}(x_T) \prod_{k=1}^T \overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)} dx_{0:T}\end{aligned}$$

$$\begin{aligned}\varepsilon \geq & \int \overleftarrow{q}_T(x_T) \log \frac{\overleftarrow{q}_T(x_T)}{\overleftarrow{q}_T^{[i]}(x_T)} dx_T \\ & + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \int \overleftarrow{q}_k(x_{k-1} | x_k) \log \frac{\overleftarrow{q}_k(x_{k-1} | x_k)}{\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)} dx_k dx_{k-1}\end{aligned}$$

Finally, the normalization constraint also factorizes to

$$1 = \int \overleftarrow{q}_T(x_T) dx_T, \quad 1 = \int \overleftarrow{q}_k(x_{k-1} | x_k) dx_{k-1}, \quad \forall x_k, 0 < k \leq T.$$

Now we can rewrite the optimization problem (4) as follows

$$\begin{aligned}& \underset{\substack{\overleftarrow{q}_k(x_{k-1} | x_k), \\ \overleftarrow{q}_T(x_T)}}{\text{maximize}} \quad \int \overleftarrow{q}_0(x_0) \log p(x_0) dx_0 + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log h_k(y_k | x_k) dx_k \\ & \quad + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \overleftarrow{q}_k(x_{k-1} | x_k) [\log f_{k-1}(x_k | x_{k-1}) - \log \overleftarrow{q}_k(x_{k-1} | x_k)] dx_{k-1} dx_k \\ & \quad - \int \overrightarrow{q}_T(x_T) \log \overleftarrow{q}_T(x_T) dx_T \\ \text{subject to} \quad & \overleftarrow{q}_{k-1}(x_{k-1}) = \int \overleftarrow{q}_k(x_k) \overleftarrow{q}_k(x_{k-1} | x_k) dx_k, \quad \forall x_{k-1}, 0 < k \leq T, \\ & 1 = \int \overleftarrow{q}_T(x_T) dx_T, \quad 1 = \int \overleftarrow{q}_k(x_{k-1} | x_k) dx_{k-1}, \quad \forall x_k, 0 < k \leq T, \\ & \varepsilon \geq \int \overleftarrow{q}_T(x_T) \log \frac{\overleftarrow{q}_T(x_T)}{\overleftarrow{q}_T^{[i]}(x_T)} dx_T \\ & \quad + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \int \overleftarrow{q}_k(x_{k-1} | x_k) \log \frac{\overleftarrow{q}_k(x_{k-1} | x_k)}{\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)} dx_k dx_{k-1},\end{aligned}$$

and the corresponding Lagrangian functional $\overleftarrow{\mathcal{R}}(\overleftarrow{q}_T, \overleftarrow{q}_k, \overleftarrow{\gamma}_T, \overleftarrow{\lambda}_k, \overleftarrow{V}_k, \alpha)$ is

$$\begin{aligned}\overleftarrow{\mathcal{R}}(\cdot) = & \int \overleftarrow{q}_0(x_0) \log p(x_0) dx_0 + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log h_k(y_k | x_k) dx_k - \int \overrightarrow{q}_T(x_T) \log \overleftarrow{q}_T(x_T) dx_T \\ & + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \overleftarrow{q}_k(x_{k-1} | x_k) [\log f_{k-1}(x_k | x_{k-1}) - \log \overleftarrow{q}_k(x_{k-1} | x_k)] dx_{k-1} dx_k \\ & + \sum_{k=1}^T \int \overleftarrow{V}_{k-1}(x_{k-1}) \left[\int \overleftarrow{q}_k(x_k) \overleftarrow{q}_k(x_{k-1} | x_k) dx_k - \overleftarrow{q}_{k-1}(x_{k-1}) \right] dx_{k-1} \\ & + \sum_{k=1}^T \int \overleftarrow{\lambda}_k(x_k) \left[1 - \int \overleftarrow{q}_k(x_{k-1} | x_k) dx_{k-1} \right] dx_k + \overleftarrow{\gamma}_T \left[1 - \int \overleftarrow{q}_T(x_T) dx_T \right] \\ & + \alpha \left[\varepsilon - \int \overleftarrow{q}_T(x_T) \log \frac{\overleftarrow{q}_T(x_T)}{\overleftarrow{q}_T^{[i]}(x_T)} dx_T \right. \\ & \quad \left. - \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \int \overleftarrow{q}_k(x_{k-1} | x_k) \log \frac{\overleftarrow{q}_k(x_{k-1} | x_k)}{\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k)} dx_k dx_{k-1} \right],\end{aligned} \tag{35}$$

where $\overleftarrow{V}_k(x_k)$, $\overleftarrow{\lambda}_k(x_k)$, $\overleftarrow{\gamma}_T$, and $\alpha \geq 0$ are Lagrangian multipliers. To find the solution with respect to $\overleftarrow{q}_k(x_{k-1} | x_k)$, we zero the associated derivatives of $\overleftarrow{\mathcal{R}}(\cdot)$

$$\begin{aligned}\overleftarrow{q}_k^{[i+1]}(x_{k-1} | x_k) = & \left[\exp \left\{ \overleftarrow{\lambda}_k(x_k) / \overleftarrow{q}_k(x_k) + (1 + \alpha) \right\} \right]^{-1/(1+\alpha)} \\ & \times \left[\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k) \right]^{\alpha/(1+\alpha)} \left[f_{k-1}(x_{k+1} | x_k) \exp \left\{ \overleftarrow{V}_{k-1}(x_{k-1}) \right\} \right]^{1/(1+\alpha)}\end{aligned} \tag{36}$$

Moreover, by setting the functional derivative with respect to $\overleftarrow{q}_T(x_T)$ to zero and get

$$\overleftarrow{q}_T^{[i+1]}(x_T) = \left[\exp \left\{ \overleftarrow{\gamma}_T + (1 + \alpha) \right\} \right]^{-1/(1+\alpha)} \left[\overleftarrow{q}_T^{[i]}(x_T) \right]^{\alpha/(1+\alpha)} \left[\exp \left\{ \overleftarrow{V}_T(x_T) \right\} \right]^{1/(1+\alpha)} \quad (37)$$

where, for convenience of notation, we define

$$\begin{aligned} \overleftarrow{V}_T(x_T) &= \log h_T(y_T | x_T) + (1 + \alpha) \log \int \left[\overleftarrow{q}_{T-1}^{[i]}(x_{T-1} | x_T) \right]^{\alpha/(1+\alpha)} \\ &\quad \times \left[f_{T-1}(x_T | x_{T-1}) \exp \left\{ \overleftarrow{V}_{T-1}(x_1) \right\} \right]^{1/(1+\alpha)} dx_{T-1}. \end{aligned} \quad (38)$$

Again, the solutions (36) and (37) are not *yet* normalized densities. We need to solve for the (functional) multipliers associated with the normalization constraints, $\overleftarrow{\lambda}_k(x_k)$ and $\overleftarrow{\gamma}_T$. Plugging the solution (36) into the Lagrangian (35) results in the dual functional

$$\begin{aligned} \overleftarrow{\mathcal{G}}(\cdot) &= \alpha \varepsilon + \int \overleftarrow{q}_0(x_0) \log p(x_0) dx_0 + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log h_k(y_k | x_k) dx_k \\ &\quad + \int \overleftarrow{q}_T(x_T) \left[\alpha \log \overleftarrow{q}_T^{[i]}(x_T) - (1 + \alpha) \log \overleftarrow{q}_T(x_T) \right] dx_T + \overleftarrow{\gamma}_T \left[1 - \int \overleftarrow{q}_T(x_T) dx_T \right] \\ &\quad + \sum_{k=1}^T \int \overleftarrow{\lambda}_k(x_k) dx_k - \sum_{k=1}^T \int \overleftarrow{V}_{k-1}(x_{k-1}) \overleftarrow{q}_{k-1}(x_{k-1}) dx_{k-1} \\ &\quad + (1 + \alpha) \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \int \overleftarrow{q}_k^{[i]}(x_{k-1} | x_k) dx_k dx_{k-1}, \end{aligned}$$

which we use to solve for the multipliers $\overleftarrow{\lambda}_k(x_k)$ by zeroing the associated derivatives

$$\begin{aligned} \overleftarrow{\lambda}_k^{[i+1]}(x_k) &= (1 + \alpha) \overleftarrow{q}_k(x_k) \left[-1 + \log \int \left[\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k) \right]^{\alpha/(1+\alpha)} \right. \\ &\quad \times \left. \left[f_{k-1}(x_k | x_{k-1}) \exp \left\{ \overleftarrow{V}_{k-1}(x_{k-1}) \right\} \right]^{1/(1+\alpha)} dx_{k-1} \right] \\ &= (1 + \alpha) \overleftarrow{q}_k(x_k) \left[-1 + \log \overleftarrow{\psi}_k^{[i+1]}(x_k) \right]. \end{aligned} \quad (39)$$

We retrieve the normalized tilted conditionals by plugging (39) into (36)

$$\begin{aligned} \overleftarrow{q}_k^{[i+1]}(x_{k+1} | x_k) &= \left[\overleftarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overleftarrow{q}_k^{[i]}(x_{k-1} | x_k) \right]^\beta \\ &\quad \times \left[f_{k-1}(x_k | x_{k-1}) \exp \left\{ \overleftarrow{V}_{k-1}(x_{k-1}) \right\} \right]^{1-\beta}, \end{aligned} \quad (40)$$

where have defined $\beta := \alpha/(1 + \alpha)$, so that $\beta = 0$ when $\alpha = 0$, and $\beta \rightarrow 1$ as $\alpha \rightarrow \infty$. Given the results (39) and (40), we simplify the functional $\overleftarrow{\mathcal{G}}(\cdot)$ further

$$\begin{aligned} \overleftarrow{\mathcal{G}}(\cdot) &= \alpha \varepsilon + \int \overleftarrow{q}_0(x_0) \log p(x_0) dx_0 + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log h_k(y_k | x_k) dx_k \\ &\quad - \sum_{k=0}^T \int \overleftarrow{V}_k(x_k) \overleftarrow{q}_k(x_k) dx_k + (1 + \alpha) \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log \overleftarrow{\psi}_k^{[i+1]}(x_k) dx_k \\ &\quad + \overleftarrow{\gamma}_T + (1 + \alpha) \int \overleftarrow{q}_T^{[i+1]}(x_T) dx_T. \end{aligned}$$

Next, we solve for the optimal multiplier $\overleftarrow{\gamma}_T$ by zeroing the associated derivative

$$\overleftarrow{\gamma}_T^{[i+1]} = (1 + \alpha) \left[-1 + \log \int \left[\overleftarrow{q}_T^{[i]}(x_T) \right]^{\alpha/(1+\alpha)} \left[\exp \left\{ \overleftarrow{V}_T(x_T) \right\} \right]^{1/(1+\alpha)} dx_T \right] \quad (41)$$

$$= (1 + \alpha) \left[-1 + \log \overleftarrow{\mathcal{Z}}_T^{[i+1]} \right],$$

leading to the normalization of the tilted distribution (37)

$$\overleftarrow{q}_T^{[i+1]}(x_T) = \left[\overleftarrow{\mathcal{Z}}_T^{[i+1]} \right]^{-1} \left[\overleftarrow{q}_T^{[i]}(x_T) \right]^\beta \left[\exp \left\{ \overleftarrow{V}_T(x_T) \right\} \right]^{1-\beta}. \quad (42)$$

Using (42) and (41) leads to further simplification of the dual $\overleftarrow{\mathcal{G}}(\cdot)$

$$\begin{aligned} \overleftarrow{\mathcal{G}}(\cdot) &= \alpha \varepsilon + \int \overleftarrow{q}_0(x_0) \log p(x_0) dx_0 + \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log h_k(y_k | x_k) dx_k \\ &\quad - \sum_{k=0}^T \int \overleftarrow{V}_k(x_k) \overleftarrow{q}_k(x_k) dx_k + (1 + \alpha) \sum_{k=1}^T \int \overleftarrow{q}_k(x_k) \log \overleftarrow{\psi}_k^{[i+1]}(x_k) dx_k + (1 + \alpha) \log \overleftarrow{\mathcal{Z}}_T^{[i+1]}. \end{aligned}$$

Finally, by taking (38) into consideration and zeroing the derivative of $\overleftarrow{\mathcal{G}}(\cdot)$ with respect to $\overleftarrow{q}_k(x_k), \forall 0 \leq k < T$, we find the optimal potential functions

$$\overleftarrow{V}_k^{[i+1]}(x_k) = \begin{cases} \log p_0(x_0) & \text{if } k = 0, \\ \log h_k(y_k | x_k) + 1/(1 - \beta) \log \overleftarrow{\psi}_k^{[i+1]}(x_k) & \text{if } 0 < k \leq T. \end{cases}$$

If we plug this result back into the dual $\overleftarrow{\mathcal{G}}(\cdot)$, we retrieve the simplest form of the dual

$$\overleftarrow{\mathcal{G}}(\beta) = \frac{\beta \varepsilon}{1 - \beta} + \frac{1}{1 - \beta} \log \overleftarrow{\mathcal{Z}}_T^{[i+1]}(\beta).$$

D Proof of Proposition 4

At $k = T$, Assumption 5 readily delivers a quadratic form for the potential function

$$\begin{aligned} \overrightarrow{V}_T^{[i+1]}(x_T) &= \log h_T(y_T | x_T) = -\frac{1}{2} x_T^\top \overrightarrow{R}_T^{[i+1]} x_T + x_T^\top \overrightarrow{r}_T^{[i+1]} + \overrightarrow{\rho}_T^{[i+1]} \\ &= -\frac{1}{2} x_T^\top L_T^{[i]} x_T + x_T^\top l_T^{[i]} + \nu_T^{[i]}. \end{aligned}$$

For $\overrightarrow{V}_k^{[i+1]}(x_k)$, for all $0 \leq k < T$, we have the backward recursion

$$\begin{aligned} \overrightarrow{V}_k^{[i+1]}(x_k) &= \log h_k(y_k | x_k) + 1/(1 - \beta) \log \overrightarrow{\psi}_k^{[i+1]}(x_k) \\ &= \log h_k(y_k | x_k) + 1/(1 - \beta) \log \int \left[\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \\ &\quad \times \left[f_k(x_{k+1} | x_k) \exp \left\{ \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} dx_{k+1}. \end{aligned}$$

Our aim is to show that this recursion is a tractable reverse propagation of quadratic forms. We start by examining $\overrightarrow{\psi}_k^{[i+1]}(x_k)$, the integral over x_{k+1} . Let us first drag all terms into the exponential, then we can treat the exponent as a quadratic function over x_k and x_{k+1}

$$\begin{aligned} &- \frac{1}{2} \begin{bmatrix} x_{k+1}^\top & x_k^\top \end{bmatrix} \begin{bmatrix} \overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} & -\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \\ -\overrightarrow{G}_{x\bar{x},k}^{[i+1]} & \overrightarrow{G}_{xx,k}^{[i+1]} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} + \begin{bmatrix} x_{k+1}^\top & x_k^\top \end{bmatrix} \begin{bmatrix} \overrightarrow{g}_{\bar{x},k}^{[i+1]} \\ \overrightarrow{g}_{x,k}^{[i+1]} \end{bmatrix} + \overrightarrow{\theta}_k^{[i+1]} \\ &= \beta \log \overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) + (1 - \beta) \log f_k(x_{k+1} | x_k) + (1 - \beta) \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}). \end{aligned}$$

Now, given an affine-Gaussian $\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k)$ (Assumption 3), a quadratic $\log f_k(x_{k+1} | x_k)$ (Assumption 5), and a quadratic potential function $\overrightarrow{V}_{k+1}^{[i+1]}(x_k)$ of the form (15), we can match the quadratic factors between the left- and right-hand sides, leading to

$$\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} = (1 - \beta) \left[C_{\bar{x}\bar{x},k}^{[i]} + \overrightarrow{R}_{k+1}^{[i+1]} \right] + \beta \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1},$$

$$\begin{aligned}
\overrightarrow{G}_{xx,k}^{[i+1]} &= (1 - \beta) C_{xx,k}^{[i]} + \beta \left[\overrightarrow{F}_k^{[i]} \right]^\top \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{F}_k^{[i]}, \\
\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} &= (1 - \beta) C_{\bar{x}x,k}^{[i]} + \beta \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{F}_k^{[i]}, \\
\overrightarrow{G}_{x\bar{x},k}^{[i+1]} &= (1 - \beta) C_{x\bar{x},k}^{[i]} + \beta \left[\overrightarrow{F}_k^{[i]} \right]^\top \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1}, \\
\overrightarrow{g}_{\bar{x},k}^{[i+1]} &= (1 - \beta) \left[c_{\bar{x},k}^{[i]} + \overrightarrow{r}_k^{[i+1]} \right] + \beta \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{d}_k^{[i]}, \\
\overrightarrow{g}_{x,k}^{[i+1]} &= (1 - \beta) c_{x,k}^{[i]} - \beta \left[\overrightarrow{F}_k^{[i]} \right]^\top \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{d}_k^{[i]}, \\
\overrightarrow{\theta}_k^{[i+1]} &= (1 - \beta) \left[\kappa_k^{[i]} + \overrightarrow{\rho}_{k+1}^{[i+1]} \right] - \frac{\beta}{2} \log \left| 2\pi \overrightarrow{\Sigma}_k^{[i]} \right| - \frac{\beta}{2} \left[\overrightarrow{d}_k^{[i]} \right]^\top \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{d}_k^{[i]}.
\end{aligned}$$

Next, we reformulate the quadratic function in the exponent explicitly as a function of x_{k+1}

$$-\frac{1}{2} x_{k+1}^\top \overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} x_{k+1} + x_{k+1}^\top \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} x_k + \overrightarrow{g}_{\bar{x},k}^{[i+1]} \right] + \left[-\frac{1}{2} x_k^\top \overrightarrow{G}_{xx,k}^{[i+1]} x_k + x_k^\top \overrightarrow{g}_{x,k}^{[i+1]} + \overrightarrow{\theta}_k^{[i+1]} \right].$$

Consequently, the exponential now resembles an unnormalized Gaussian distribution in *information form* with a precision matrix $\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]}$. In this case, we can use the identity

$$\log \int \exp \left\{ -\frac{1}{2} x^\top U x + x^\top u + \sigma \right\} dx = \frac{1}{2} \log |2\pi U^{-1}| + \frac{1}{2} u^\top U^{-1} u + \sigma, \quad (43)$$

to express the log-normalizing function $\log \overrightarrow{\psi}_k^{[i+1]}(x_k)$ as a quadratic function

$$\begin{aligned}
\log \overrightarrow{\psi}_k^{[i+1]}(x_k) &= -\frac{1}{2} x_k^\top \overrightarrow{S}_k^{[i+1]} x_k + x_k^\top \overrightarrow{s}_k^{[i+1]} + \overrightarrow{\xi}_k^{[i+1]} \\
&= \frac{1}{2} \log \left| 2\pi \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \right| + \left[-\frac{1}{2} x_k^\top \overrightarrow{G}_{xx,k}^{[i+1]} x_k + x_k^\top \overrightarrow{g}_{x,k}^{[i+1]} + \overrightarrow{\theta}_k^{[i+1]} \right] \\
&\quad + \frac{1}{2} \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} x_k + \overrightarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} x_k + \overrightarrow{g}_{\bar{x},k}^{[i+1]} \right]
\end{aligned}$$

which, after matching terms, leads to

$$\begin{aligned}
\overrightarrow{S}_k^{[i+1]} &= \overrightarrow{G}_{xx,k}^{[i+1]} - \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \overrightarrow{G}_{\bar{x}x,k}^{[i+1]}, \\
\overrightarrow{s}_k^{[i+1]} &= \overrightarrow{g}_{x,k}^{[i+1]} + \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \overrightarrow{g}_{\bar{x},k}^{[i+1]}, \\
\overrightarrow{\xi}_k^{[i+1]} &= \overrightarrow{\theta}_k^{[i+1]} + \frac{1}{2} \log \left| 2\pi \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \right| + \frac{1}{2} \left[\overrightarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \overrightarrow{g}_{\bar{x},k}^{[i+1]}.
\end{aligned}$$

To get the final form of the *quadratic* potential $\overrightarrow{V}_k^{[i+1]}(x_k)$, we add the contribution of the quadratic log-measurement and log-prior determined by Assumption 5

$$\overrightarrow{R}_k^{[i+1]} = L_k^{[i]} + \frac{1}{1 - \beta} \overrightarrow{S}_k^{[i+1]}, \quad \overrightarrow{r}_k^{[i+1]} = l_k^{[i]} + \frac{1}{1 - \beta} \overrightarrow{s}_k^{[i+1]}, \quad \overrightarrow{\rho}_k^{[i+1]} = \nu_k^{[i]} + \frac{1}{1 - \beta} \overrightarrow{\xi}_k^{[i+1]}.$$

Finally, for a Gaussian $\overrightarrow{q}_0^{[i]}(x_0)$ and a quadratic $\overrightarrow{V}_0^{[i+1]}(x_0)$, we derive a quadratic $\log \overrightarrow{\mathcal{Z}}_0^{[i+1]}$

$$\log \overrightarrow{\mathcal{Z}}_0^{[i+1]} = \log \int \left[\overrightarrow{q}_0^{[i]}(x_0) \right]^\beta \left[\exp \left\{ \overrightarrow{V}_0^{[i+1]}(x_0) \right\} \right]^{1-\beta} dx_0.$$

Again, we formulate a quadratic function over x_0 and $\overrightarrow{m}_0^{[i]}$

$$-\frac{1}{2} \begin{bmatrix} x_0^\top & \left[\overrightarrow{m}_0^{[i]} \right]^\top \end{bmatrix} \begin{bmatrix} \overrightarrow{J}_{xx}^{[i+1]} & -\overrightarrow{J}_{xm}^{[i+1]} \\ -\overrightarrow{J}_{mx}^{[i+1]} & \overrightarrow{J}_{mm}^{[i+1]} \end{bmatrix} \begin{bmatrix} x_0 \\ \overrightarrow{m}_0^{[i]} \end{bmatrix} + \begin{bmatrix} x_0^\top & \left[\overrightarrow{m}_0^{[i]} \right]^\top \end{bmatrix} \begin{bmatrix} \overrightarrow{j}_x^{[i+1]} \\ \overrightarrow{j}_m^{[i+1]} \end{bmatrix} + \overrightarrow{\tau}^{[i+1]}$$

$$= \beta \log \overrightarrow{q}_0^{[i]}(x_0) + (1 - \beta) \overrightarrow{V}_0^{[i+1]}(x_0),$$

where by matching the quadratic factors, we get

$$\begin{aligned} \overrightarrow{J}_{xx}^{[i+1]} &= (1 - \beta) \overrightarrow{R}_0^{[i+1]} + \beta \left[\overrightarrow{P}_0^{[i]} \right]^{-1}, & \overrightarrow{J}_{xm}^{[i+1]} &= \beta \left[\overrightarrow{P}_0^{[i]} \right]^{-1}, \\ \overrightarrow{J}_{mm}^{[i+1]} &= \beta \left[\overrightarrow{P}_0^{[i]} \right]^{-1}, & \overrightarrow{J}_{mx}^{[i+1]} &= \beta \left[\overrightarrow{P}_0^{[i]} \right]^{-1}, \\ \overrightarrow{j}_x^{[i+1]} &= (1 - \beta) \overrightarrow{r}_0^{[i+1]}, & \overrightarrow{j}_m^{[i+1]} &= 0, \\ \overrightarrow{\tau}^{[i+1]} &= (1 - \beta) \overrightarrow{\rho}_0^{[i+1]} - \frac{\beta}{2} \log \left| 2\pi \overrightarrow{P}_0^{[i]} \right|. \end{aligned}$$

Using the identity (43), we get a log-normalizer as a quadratic function over $\overrightarrow{m}_0^{[i]}$

$$\log \overrightarrow{\mathcal{Z}}_0^{[i+1]} = -\frac{1}{2} \left[\overrightarrow{m}_0^{[i]} \right]^\top \overrightarrow{U}^{[i+1]} \overrightarrow{m}_0^{[i]} + \left[\overrightarrow{m}_0^{[i]} \right]^\top \overrightarrow{u}^{[i+1]} + \overrightarrow{\eta}^{[i+1]},$$

where

$$\begin{aligned} \overrightarrow{U}^{[i+1]} &= \overrightarrow{J}_{mm}^{[i+1]} - \left[\overrightarrow{J}_{xm}^{[i+1]} \right]^\top \left[\overrightarrow{J}_{xx}^{[i+1]} \right]^{-1} \overrightarrow{J}_{xm}^{[i+1]}, \\ \overrightarrow{u}^{[i+1]} &= \overrightarrow{j}_m^{[i+1]} - \left[\overrightarrow{J}_{xm}^{[i+1]} \right]^\top \left[\overrightarrow{J}_{xx}^{[i+1]} \right]^{-1} \overrightarrow{j}_x^{[i+1]}, \\ \overrightarrow{\eta}^{[i+1]} &= \overrightarrow{\tau}^{[i+1]} - \frac{1}{2} \left| 2\pi \overrightarrow{J}_{xx}^{[i+1]} \right|^{-1} + \frac{1}{2} \left[\overrightarrow{j}_x^{[i+1]} \right]^\top \left[\overrightarrow{J}_{xx}^{[i+1]} \right]^{-1} \left[\overrightarrow{j}_x^{[i+1]} \right]. \end{aligned}$$

E Proof of Proposition 5

For $k = 0$, the log-prior is a quadratic function per Assumption 5, leading to the following parameterization of the potential

$$\begin{aligned} \overleftarrow{V}_0^{[i+1]}(x_0) &= \log p_0(x_0) = -\frac{1}{2} x_0^\top \overleftarrow{R}_0^{[i+1]} x_0 + x_0^\top \overleftarrow{r}_0^{[i+1]} + \overleftarrow{\rho}_0^{[i+1]} \\ &= -\frac{1}{2} x_0^\top L_0^{[i]} x_0 + x_0^\top l_0^{[i]} + \nu_0^{[i]}. \end{aligned}$$

For $\overleftarrow{V}_k^{[i+1]}(x_k)$, for all $0 < k \leq T$, we have derived a forward recursion

$$\begin{aligned} \overleftarrow{V}_k^{[i+1]}(x_k) &= \log h_k(y_k \mid x_k) + 1/(1 - \beta) \log \overleftarrow{\psi}_k^{[i+1]}(x_k) \\ &= \log h_k(y_k \mid x_k) + 1/(1 - \beta) \log \int \left[\overleftarrow{q}_k^{[i]}(x_{k-1} \mid x_k) \right]^\beta \\ &\quad \times \left[f_{k-1}(x_k \mid x_{k-1}) \exp \left\{ \overleftarrow{V}_{k-1}^{[i+1]}(x_{k-1}) \right\} \right]^{1-\beta} dx_{k-1}. \end{aligned}$$

We show that this recursion is a tractable forward propagation of quadratic forms. We start by moving all terms within the integral into the exponential and treat the exponent as a quadratic function over x_k and x_{k-1}

$$\begin{aligned} &- \frac{1}{2} \begin{bmatrix} x_k^\top & x_{k-1}^\top \end{bmatrix} \begin{bmatrix} \overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} & -\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \\ -\overleftarrow{G}_{x\bar{x},k}^{[i+1]} & \overleftarrow{G}_{xx,k}^{[i+1]} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} + \begin{bmatrix} x_k^\top & x_{k-1}^\top \end{bmatrix} \begin{bmatrix} \overleftarrow{g}_{\bar{x},k}^{[i+1]} \\ \overleftarrow{g}_{x,k}^{[i+1]} \end{bmatrix} + \overleftarrow{\theta}_k^{[i+1]} \\ &= \beta \log \overleftarrow{q}_k^{[i]}(x_k \mid x_{k-1}) + (1 - \beta) \log f_{k-1}(x_k \mid x_{k-1}) + (1 - \beta) \overleftarrow{V}_{k-1}^{[i+1]}(x_{k-1}). \end{aligned}$$

For an affine-Gaussian $\overleftarrow{q}_k^{[i]}(x_{k-1} \mid x_k)$ (Assumption 4), a quadratic $\log f_{k-1}(x_k \mid x_{k-1})$ (Assumption 5), and a quadratic potential function $\overleftarrow{V}_{k+1}^{[i+1]}(x_k)$ of the form (15), we get

$$\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} = (1 - \beta) C_{\bar{x}\bar{x},k-1}^{[i]} + \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{F}_k^{[i]},$$

$$\begin{aligned}
\overleftarrow{G}_{xx,k}^{[i+1]} &= (1 - \beta) \left[C_{xx,k-1}^{[i]} + \overleftarrow{R}_{k-1}^{[i+1]} \right] + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1}, \\
\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} &= (1 - \beta) C_{\bar{x}x,k-1}^{[i]} + \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1}, \\
\overleftarrow{G}_{x\bar{x},k}^{[i+1]} &= (1 - \beta) C_{x\bar{x},k-1}^{[i]} + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{F}_k^{[i]}, \\
\overleftarrow{g}_{\bar{x},k}^{[i+1]} &= (1 - \beta) c_{\bar{x},k-1}^{[i]} - \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]}, \\
\overleftarrow{g}_{x,k}^{[i+1]} &= (1 - \beta) \left[c_{x,k-1}^{[i]} + \overleftarrow{r}_{k-1}^{[i+1]} \right] + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]}, \\
\overleftarrow{\theta}_k^{[i+1]} &= (1 - \beta) \left[\kappa_k^{[i]} + \overleftarrow{\rho}_{k-1}^{[i+1]} \right] - \frac{\beta}{2} \log \left| 2\pi \overleftarrow{\Sigma}_k^{[i]} \right| - \frac{\beta}{2} \left[\overleftarrow{d}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]}.
\end{aligned}$$

By writing this quadratic function explicitly in terms of x_{k-1}

$$-\frac{1}{2} x_{k-1}^\top \overleftarrow{G}_{xx,k}^{[i+1]} x_{k-1} + x_{k-1}^\top \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} x_k + \overleftarrow{g}_{x,k}^{[i+1]} \right] + \left[-\frac{1}{2} x_k^\top \overleftarrow{G}_{x\bar{x},k}^{[i+1]} x_k + x_k^\top \overleftarrow{g}_{\bar{x},k}^{[i+1]} + \overleftarrow{\theta}_k^{[i+1]} \right].$$

we can make use of identity (43) to express the log-normalizing function $\log \overleftarrow{\psi}_k^{[i+1]}(x_k)$ itself as a quadratic function in x_k

$$\begin{aligned}
\log \overleftarrow{\psi}_k^{[i+1]}(x_k) &= -\frac{1}{2} x_k^\top \overleftarrow{S}_k^{[i+1]} x_k + x_k^\top \overleftarrow{s}_k^{[i+1]} + \overleftarrow{\xi}_k^{[i+1]} \\
&= \frac{1}{2} \log \left| 2\pi \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \right| + \left[-\frac{1}{2} x_k^\top \overleftarrow{G}_{\bar{x}x,k}^{[i+1]} x_k + x_k^\top \overleftarrow{g}_{\bar{x},k}^{[i+1]} + \overleftarrow{\theta}_k^{[i+1]} \right] \\
&\quad + \frac{1}{2} \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} x_k + \overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} x_k + \overrightarrow{g}_{x,k}^{[i+1]} \right]
\end{aligned}$$

which, after matching terms, leads to

$$\begin{aligned}
\overleftarrow{S}_k^{[i+1]} &= \overleftarrow{G}_{\bar{x}x,k}^{[i+1]} - \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overleftarrow{G}_{x\bar{x},k}^{[i+1]}, \\
\overleftarrow{s}_k^{[i+1]} &= \overleftarrow{g}_{\bar{x},k}^{[i+1]} + \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overleftarrow{g}_{x,k}^{[i+1]}, \\
\overleftarrow{\xi}_k^{[i+1]} &= \overleftarrow{\theta}_k^{[i+1]} + \frac{1}{2} \log \left| 2\pi \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \right| + \frac{1}{2} \left[\overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overrightarrow{g}_{x,k}^{[i+1]}.
\end{aligned}$$

Given $\log \overleftarrow{\psi}_k^{[i+1]}(x_k)$, we can now construct the *quadratic* potential function $\overleftarrow{V}_k^{[i+1]}(x_k)$ that accounts for the log-measurement contribution according to Assumption 5

$$\overleftarrow{R}_k^{[i+1]} = L_k^{[i]} + \frac{1}{1 - \beta} \overleftarrow{S}_k^{[i+1]}, \quad \overleftarrow{r}_k^{[i+1]} = l_k^{[i]} + \frac{1}{1 - \beta} \overleftarrow{s}_k^{[i+1]}, \quad \overleftarrow{\rho}_k^{[i+1]} = \nu_k^{[i]} + \frac{1}{1 - \beta} \overleftarrow{\xi}_k^{[i+1]}.$$

Finally, for a Gaussian $\overleftarrow{q}_T^{[i]}(x_T)$ and a quadratic $\overleftarrow{V}_T^{[i+1]}(x_T)$, we derive a quadratic $\log \overleftarrow{\mathcal{Z}}_T^{[i+1]}$

$$\log \overleftarrow{\mathcal{Z}}_T^{[i+1]} = \log \int \left[\overleftarrow{q}_T^{[i]}(x_T) \right]^\beta \left[\exp \left\{ \overleftarrow{V}_T^{[i+1]}(x_T) \right\} \right]^{1-\beta} dx_T.$$

Similar to the proof in Appendix D, we formulate a quadratic over x_T and $\overleftarrow{m}_T^{[i]}$

$$\begin{aligned}
&- \frac{1}{2} \left[\left[\overleftarrow{m}_T^{[i]} \right]^\top \quad x_T^\top \right] \begin{bmatrix} \overleftarrow{J}_{xx}^{[i+1]} & -\overleftarrow{J}_{xm}^{[i+1]} \\ -\overleftarrow{J}_{mx}^{[i+1]} & \overleftarrow{J}_{mm}^{[i+1]} \end{bmatrix} \begin{bmatrix} \overleftarrow{m}_T^{[i]} \\ x_T \end{bmatrix} + \left[\left[\overleftarrow{m}_T^{[i]} \right]^\top \quad x_T^\top \right] \begin{bmatrix} \overleftarrow{j}_x^{[i+1]} \\ \overleftarrow{j}_m^{[i+1]} \end{bmatrix} + \overleftarrow{\tau}^{[i+1]} \\
&= \beta \log \overleftarrow{q}_T^{[i]}(x_T) + (1 - \beta) \overleftarrow{V}_T^{[i+1]}(x_T),
\end{aligned}$$

where by matching the quadratic factors, we get

$$\overleftarrow{J}_{xx}^{[i+1]} = \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}, \quad \overleftarrow{J}_{xm}^{[i+1]} = \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1},$$

$$\begin{aligned}\overleftarrow{J}_{mm}^{[i+1]} &= (1 - \beta) \overleftarrow{R}_T^{[i+1]} + \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}, & \overleftarrow{J}_{mx}^{[i+1]} &= \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}, \\ \overleftarrow{j}_x^{[i+1]} &= 0, & \overleftarrow{j}_m^{[i+1]} &= (1 - \beta) \overleftarrow{r}_T^{[i+1]}, \\ \overleftarrow{\tau}^{[i+1]} &= (1 - \beta) \overleftarrow{\rho}_T^{[i+1]} - \frac{\beta}{2} \log \left| 2\pi \overleftarrow{P}_T^{[i]} \right|.\end{aligned}$$

Using the identity (43), we get a log-normalizing constant as a quadratic function over $\overleftarrow{m}_T^{[i]}$

$$\log \overleftarrow{\mathcal{Z}}_T^{[i+1]} = -\frac{1}{2} \left[\overleftarrow{m}_T^{[i]} \right]^\top \overleftarrow{U}^{[i+1]} \overleftarrow{m}_T^{[i]} + \left[\overleftarrow{m}_T^{[i]} \right]^\top \overleftarrow{u}^{[i+1]} + \overleftarrow{\eta}^{[i+1]},$$

where

$$\begin{aligned}\overleftarrow{U}^{[i+1]} &= \overleftarrow{J}_{xx}^{[i+1]} - \left[\overleftarrow{J}_{mx}^{[i+1]} \right]^\top \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1} \overleftarrow{J}_{mx}^{[i+1]}, \\ \overleftarrow{u}^{[i+1]} &= \overleftarrow{j}_x^{[i+1]} - \left[\overleftarrow{J}_{mx}^{[i+1]} \right]^\top \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1} \overleftarrow{j}_m^{[i+1]}, \\ \overleftarrow{\eta}^{[i+1]} &= \overleftarrow{\tau}^{[i+1]} - \frac{1}{2} \left| 2\pi \overleftarrow{J}_{mm}^{[i+1]} \right|^{-1} + \frac{1}{2} \left[\overleftarrow{j}_m^{[i+1]} \right]^\top \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1} \left[\overleftarrow{j}_m^{[i+1]} \right].\end{aligned}$$

F Proof of Lemma 1

For a tilted distribution of the form

$$q^{[i+1]}(x) = \left[\mathcal{Z}^{[i+1]} \right]^{-1} \left[q^{[i]}(x) \right]^\beta \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta},$$

with a normalizing constant

$$\mathcal{Z}^{[i+1]} = \int \left[q^{[i]}(x) \right]^\beta \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta} dx,$$

where $q^{[i]}(x) = \mathcal{N}(x \mid m^{[i]}, P^{[i]})$, we compute the moments of $q^{[i+1]}(x)$ by considering the derivatives of $\mathcal{Z}^{[i+1]}$ with respect to $m^{[i]}$. Starting with the first-order derivative

$$\begin{aligned}\frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} &= \beta \left[P^{[i]} \right]^{-1} \int (x - m^{[i]}) \left[q^{[i]}(x) \right]^\beta \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta} dx \\ &= \beta \left[P^{[i]} \right]^{-1} \int x \left[q^{[i]}(x) \right]^\beta \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta} dx \\ &\quad - \beta \left[P^{[i]} \right]^{-1} m^{[i]} \int \left[q^{[i]}(x) \right]^\beta \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta} dx \\ &= \beta \mathcal{Z}^{[i+1]} \left[P^{[i]} \right]^{-1} \left[\mathbb{E}^{[i+1]} [x] - m^{[i]} \right].\end{aligned}$$

After rearranging the terms, we get

$$\begin{aligned}\mathbb{E}^{[i+1]} [x] &= m^{[i]} + \frac{1}{\beta} P^{[i]} \left[\mathcal{Z}^{[i+1]} \right]^{-1} \frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} \\ &= m^{[i]} + \frac{1}{\beta} P^{[i]} \frac{\partial \log \mathcal{Z}^{[i+1]}}{\partial m^{[i]}},\end{aligned}$$

which is an expression for the first moment of $q^{[i+1]}(x)$.

Next, we consider the second-order derivative $\mathcal{Z}^{[i+1]}$ with respect to $m^{[i]}$

$$\begin{aligned}\frac{\partial^2 \mathcal{Z}^{[i+1]}}{\partial m^{[i]} \partial [m^{[i]}]^\top} &= \beta^2 \left[P^{[i]} \right]^{-1} \left[\int (x - m^{[i]})(x - m^{[i]})^\top \left[q^{[i]}(x) \right]^\beta \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta} dx \right] \left[P^{[i]} \right]^{-1} \\ &\quad - \beta \left[P^{[i]} \right]^{-1} \int \left[q^{[i]}(x) \right]^\beta \left[\exp \left\{ V^{[i+1]}(x) \right\} \right]^{1-\beta} dx\end{aligned}$$

$$\begin{aligned}
&= \beta^2 \mathcal{Z}^{[i+1]} \left[P^{[i]} \right]^{-1} \mathbb{E}^{[i+1]} \left[x x^\top \right] \left[P^{[i]} \right]^{-1} + \beta^2 \mathcal{Z}^{[i+1]} \left[P^{[i]} \right]^{-1} m^{[i]} \left[m^{[i]} \right]^\top \left[P^{[i]} \right]^{-1} \\
&\quad - 2 \beta^2 \mathcal{Z}^{[i+1]} \left[P^{[i]} \right]^{-1} \mathbb{E}^{[i+1]} \left[x \right] \left[m^{[i]} \right]^\top \left[P^{[i]} \right]^{-1} - \beta \mathcal{Z}^{[i+1]} \left[P^{[i]} \right]^{-1},
\end{aligned}$$

which leads to the second *raw* moment of $q^{[i+1]}(x_k)$

$$\begin{aligned}
\mathbb{E}^{[i+1]} \left[x x^\top \right] &= -m^{[i]} \left[m^{[i]} \right]^\top + 2 \mathbb{E}^{[i+1]} \left[x \right] \left[m^{[i]} \right]^\top \\
&\quad + \frac{1}{\beta} P^{[i]} + \frac{1}{\beta^2} \left[\mathcal{Z}^{[i+1]} \right]^{-1} P^{[i]} \frac{\partial^2 \mathcal{Z}^{[i+1]}}{\partial m^{[i]} \partial [m^{[i]}]^\top} P^{[i]}.
\end{aligned}$$

We can now derive the second *central* moment as follows

$$\begin{aligned}
\mathbb{V}^{[i+1]} \left[x \right] &= \mathbb{E}^{[i+1]} \left[x x^\top \right] - \mathbb{E}^{[i+1]} \left[x \right] \mathbb{E}^{[i+1]} \left[x \right]^\top \\
&= m^{[i]} \left[m^{[i]} \right]^\top + 2 \frac{1}{\beta} \left[\mathcal{Z}^{[i+1]} \right]^{-1} P^{[i]} \frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} \left[m^{[i]} \right]^\top + \frac{1}{\beta} P^{[i]} \\
&\quad + \frac{1}{\beta^2} \left[\mathcal{Z}^{[i+1]} \right]^{-1} P^{[i]} \frac{\partial^2 \mathcal{Z}^{[i+1]}}{\partial m^{[i]} \partial [m^{[i]}]^\top} P^{[i]} - m^{[i]} \left[m^{[i]} \right]^\top \\
&\quad - 2 \frac{1}{\beta} \left[\mathcal{Z}^{[i+1]} \right]^{-1} m^{[i]} \left[P^{[i]} \frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} \right]^\top - \frac{1}{\beta^2} \left[\mathcal{Z}^{[i+1]} \right]^{-2} P^{[i]} \frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} \left[\frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} \right]^\top P^{[i]} \\
&= \frac{1}{\beta} P^{[i]} + \frac{1}{\beta^2} \left[\mathcal{Z}^{[i+1]} \right]^{-1} P^{[i]} \frac{\partial^2 \mathcal{Z}^{[i+1]}}{\partial m^{[i]} \partial [m^{[i]}]^\top} P^{[i]} - \frac{1}{\beta^2} \left[\mathcal{Z}^{[i+1]} \right]^{-2} P^{[i]} \frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} \left[\frac{\partial \mathcal{Z}^{[i+1]}}{\partial m^{[i]}} \right]^\top P^{[i]} \\
&= \frac{1}{\beta} P^{[i]} + \frac{1}{\beta^2} P^{[i]} \frac{\partial^2 \log \mathcal{Z}^{[i+1]}}{\partial m^{[i]} \partial [m^{[i]}]^\top} P^{[i]}.
\end{aligned}$$

G Proof of Lemma 2

For a tilted forward-Markov conditional distribution of the form

$$\overrightarrow{q}_k^{[i+1]}(x_{k+1} | x_k) = \left[\overrightarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \left[f_k(x_{k+1} | x_k) \exp \left\{ \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta},$$

with a normalizing function

$$\overrightarrow{\psi}_k^{[i+1]}(x_k) = \int \left[\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \left[f_k(x_{k+1} | x_k) \exp \left\{ \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} dx_{k+1},$$

where $\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) = \mathcal{N}(x_{k+1} | \overrightarrow{F}_k^{[i]} x_k + \overrightarrow{d}_k^{[i]}, \overrightarrow{\Sigma}_k^{[i]})$ and $f_k(x_{k+1} | x_k) \approx \exp \{ \ell_f^{[i]}(x_{k+1}, x_k) \}$

$$\ell_f^{[i]}(x_{k+1}, x_k) \approx -\frac{1}{2} \begin{bmatrix} x_{k+1}^\top & x_k^\top \end{bmatrix} \begin{bmatrix} C_{\bar{x}\bar{x},k}^{[i]} & -C_{\bar{x}x,k}^{[i]} \\ -C_{x\bar{x},k}^{[i]} & C_{xx,k}^{[i]} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} + \begin{bmatrix} x_{k+1}^\top & x_k^\top \end{bmatrix} \begin{bmatrix} c_{\bar{x},k}^{[i]} \\ c_{x,k}^{[i]} \end{bmatrix} + \kappa_k^{[i]}.$$

We compute the *conditional* moments of $q^{[i+1]}(y | x)$ from the derivatives of $\psi^{[i+1]}(x)$ with respect to x . Starting with the first-order derivative

$$\begin{aligned}
\frac{\partial \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k} &= \beta \int \frac{\partial \log \overrightarrow{q}_k^{[i]}(x_{k+1} | x_k)}{\partial x_k} \left[\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} | x_k) \exp \left\{ \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} dx_{k+1} \\
&\quad + (1-\beta) \int \frac{\partial \log f_k(x_{k+1} | x_k)}{\partial x_k} \left[\overrightarrow{q}_k^{[i]}(x_{k+1} | x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} | x_k) \exp \left\{ \overrightarrow{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} dx_{k+1},
\end{aligned}$$

$$\begin{aligned}
&= \beta \int \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \left[x_{k+1} - \vec{F}_k^{[i]} x_k - \vec{d}_k^{[i]} \right] \left[\vec{q}_k^{[i]}(x_{k+1} \mid x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} \mid x_k) \exp \left\{ \vec{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} dx_{k+1} \\
&+ (1-\beta) \int \left[-C_{xx,k}^{[i]} x_k + C_{x\bar{x},k}^{[i]} x_{k+1} + c_{x,k}^{[i]} \right] \left[\vec{q}_k^{[i]}(x_{k+1} \mid x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} \mid x_k) \exp \left\{ \vec{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} dx_{k+1}, \\
&= \vec{\psi}_k^{[i+1]}(x_k) \left[(1-\beta) C_{x\bar{x},k}^{[i]} + \beta \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \right] \mathbb{E}^{[i+1]} [x_{k+1} \mid x_k] \\
&\quad - \vec{\psi}_k^{[i+1]}(x_k) \left[(1-\beta) C_{xx,k}^{[i]} + \beta \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{F}_k^{[i]} \right] x_k \\
&\quad + \vec{\psi}_k^{[i+1]}(x_k) \left[(1-\beta) c_{x,k}^{[i]} - \beta \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{d}_k^{[i]} \right] \\
&= \vec{\psi}_k^{[i+1]}(x_k) \vec{G}_{x\bar{x},k}^{[i+1]} \mathbb{E}^{[i+1]} [x_{k+1} \mid x_k] - \vec{\psi}_k^{[i+1]}(x_k) \vec{G}_{xx,k}^{[i+1]} x + \vec{\psi}_k^{[i+1]}(x_k) \vec{g}_{x,k}^{[i+1]},
\end{aligned}$$

where we use the definition of $\vec{G}_{x\bar{x},k}^{[i+1]}$, $\vec{G}_{xx,k}^{[i+1]}$, and $\vec{g}_{x,k}^{[i+1]}$ from Proposition 4.

Consequently, the first conditional moment of $\vec{q}_k^{[i+1]}(x_{k+1} \mid x_k)$ is

$$\mathbb{E}^{[i+1]} [x_{k+1} \mid x_k] = \left[\vec{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \vec{G}_{xx,k}^{[i+1]} x_k - \left[\vec{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \vec{g}_{x,k}^{[i+1]} + \left[\vec{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial \log \vec{\psi}_k^{[i+1]}(x_k)}{\partial x_k}.$$

Now we consider the second-order derivatives of $\vec{\psi}_k^{[i+1]}(x_k)(x)$ with respect to x_k

$$\begin{aligned}
\frac{\partial^2 \vec{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} &= \beta^2 \int \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \left[x_{k+1} - \vec{F}_k^{[i]} x_k - \vec{d}_k^{[i]} \right] \left[\vec{q}_k^{[i]}(x_{k+1} \mid x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} \mid x_k) \exp \left\{ \vec{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} \\
&\quad \times \left[x_{k+1} - \vec{F}_k^{[i]} x_k - \vec{d}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{F}_k^{[i]} dx_{k+1} \\
&+ \beta(1-\beta) \int \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \left[x_{k+1} - \vec{F}_k^{[i]} x_k - \vec{d}_k^{[i]} \right] \left[\vec{q}_k^{[i]}(x_{k+1} \mid x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} \mid x_k) \exp \left\{ \vec{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} \\
&\quad \times \left[-C_{xx,k}^{[i]} x + C_{x\bar{x},k}^{[i]} y + c_{x,k}^{[i]} \right]^\top dx_{k+1} \\
&+ (1-\beta)^2 \int \left[-C_{xx,k}^{[i]} x + C_{x\bar{x},k}^{[i]} y + c_{x,k}^{[i]} \right] \left[\vec{q}_k^{[i]}(x_{k+1} \mid x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} \mid x_k) \exp \left\{ \vec{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} \\
&\quad \times \left[-C_{xx,k}^{[i]} x + C_{x\bar{x},k}^{[i]} y + c_{x,k}^{[i]} \right]^\top dx_{k+1} \\
&+ \beta(1-\beta) \int \left[-C_{xx,k}^{[i]} x + C_{x\bar{x},k}^{[i]} y + c_{x,k}^{[i]} \right] \left[\vec{q}_k^{[i]}(x_{k+1} \mid x_k) \right]^\beta \\
&\quad \times \left[f_k(x_{k+1} \mid x_k) \exp \left\{ \vec{V}_{k+1}^{[i+1]}(x_{k+1}) \right\} \right]^{1-\beta} \\
&\quad \times \left[x_{k+1} - \vec{F}_k^{[i]} x_k - \vec{d}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{F}_k^{[i]} dx_{k+1} \\
&- \beta \vec{\psi}_k^{[i+1]}(x_k) \left[\vec{F}_k^{[i]} \right]^\top \left[\vec{\Sigma}_k^{[i]} \right]^{-1} \vec{F}_k^{[i]} - (1-\beta) \vec{\psi}_k^{[i+1]}(x_k) C_{xx,k}^{[i]}
\end{aligned}$$

$$\begin{aligned}
&= \overrightarrow{\psi}_k^{[i+1]}(x_k) \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right] \mathbb{E}^{[i+1]} \left[x_{k+1} x_{k+1}^\top \mid x_k \right] \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right] \\
&\quad + \overrightarrow{\psi}_k^{[i+1]}(x_k) \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right] \left[x_k x_k^\top \right] \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^\top \\
&\quad + \overrightarrow{\psi}_k^{[i+1]}(x_k) \left[\overrightarrow{g}_{x,k}^{[i+1]} \right] \left[\overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \\
&\quad - 2 \overrightarrow{\psi}_k^{[i+1]}(x_k) \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right] \mathbb{E}^{[i+1]} \left[x_{k+1} \mid x_k \right] x_k^\top \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^\top \\
&\quad + 2 \overrightarrow{\psi}_k^{[i+1]}(x_k) \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right] \mathbb{E}^{[i+1]} \left[x_{k+1} \mid x_k \right] \left[\overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \\
&\quad - 2 \overrightarrow{\psi}_k^{[i+1]}(x_k) \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right] x_k \left[\overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \\
&\quad - \overrightarrow{\psi}_k^{[i+1]}(x_k) \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right].
\end{aligned}$$

This gives us an expression for the second *raw* conditional moment

$$\begin{aligned}
\mathbb{E}^{[i+1]} \left[x_{k+1} x_{k+1}^\top \mid x_k \right] &= \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right] \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad + \left[\overrightarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial^2 \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad - \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right] \left[x_k x_k^\top \right] \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad - \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{g}_{x,k}^{[i+1]} \right] \left[\overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad + 2 \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right] \mathbb{E}^{[i+1]} \left[x_{k+1} \mid x_k \right] x_k^\top \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad - 2 \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right] \mathbb{E}^{[i+1]} \left[x_{k+1} \mid x_k \right] \left[\overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad + 2 \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right] x_k \left[\overrightarrow{g}_{x,k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^{-1}.
\end{aligned}$$

Finally, we can derive the second *central* moment according to

$$\begin{aligned}
\mathbb{V}^{[i+1]} \left[x_{k+1} \mid x_k \right] &= \mathbb{E}^{[i+1]} \left[x_{k+1} x_{k+1}^\top \mid x_k \right] - \mathbb{E}^{[i+1]} \left[x_{k+1} \mid x_k \right] \mathbb{E}^{[i+1]} \left[x_{k+1} \mid x_k \right]^\top \\
&= \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right] \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad + \left[\overrightarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial^2 \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&\quad - \left[\overrightarrow{\psi}_k^{[i+1]}(x_k) \right]^{-2} \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k} \left[\frac{\partial \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k} \right]^\top \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \\
&= \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \left[\overrightarrow{G}_{xx,k}^{[i+1]} \right] \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} + \left[\overrightarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial^2 \log \overrightarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} \left[\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1}.
\end{aligned}$$

H Proof of Lemma 3

For a tilted reverse-Markov conditional distribution of the form

$$\overleftarrow{q}_k^{[i+1]}(x_{k-1} \mid x_k) = \left[\overleftarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overleftarrow{q}_k^{[i]}(x_{k-1} \mid x_k) \right]^\beta \left[f_{k-1}(x_k \mid x_{k-1}) \exp \left\{ \overleftarrow{V}_{k-1}^{[i+1]}(x_{k-1}) \right\} \right]^{1-\beta},$$

with a normalizing function

$$\overleftarrow{\psi}_k^{[i+1]}(x_k) = \int \left[\overleftarrow{q}_k^{[i]}(x_{k-1} \mid x_k) \right]^\beta \left[f_{k-1}(x_k \mid x_{k-1}) \exp \left\{ \overleftarrow{V}_{k-1}^{[i+1]}(x_{k-1}) \right\} \right]^{1-\beta} dx_{k-1},$$

where $\overleftarrow{q}_k^{[i]}(x_{k-1} \mid x_k) = \mathcal{N}(x_{k-1} \mid \overleftarrow{F}_k^{[i]} x_k + \overleftarrow{d}_k^{[i]}, \overleftarrow{\Sigma}_k^{[i]})$ and $f_{k-1}(x_k \mid x_{k-1}) \approx \exp \{ \ell_f^{[i]}(x_k, x_{k-1}) \}$

$$\ell_f^{[i]}(x_k, x_{k-1}) \approx -\frac{1}{2} \begin{bmatrix} x_k^\top & x_{k-1}^\top \end{bmatrix} \begin{bmatrix} C_{\bar{x}\bar{x},k}^{[i]} & -C_{\bar{x}x,k}^{[i]} \\ -C_{x\bar{x},k}^{[i]} & C_{xx,k}^{[i]} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} + \begin{bmatrix} x_k^\top & x_{k-1}^\top \end{bmatrix} \begin{bmatrix} c_{\bar{x},k}^{[i]} \\ c_{x,k}^{[i]} \end{bmatrix} + \kappa_k^{[i]}.$$

The following steps are analogous to those described in Appendix G, thus we will omit redundant steps to avoid repetition. Starting with the first-order derivative of $\overleftarrow{\psi}_k^{[i+1]}(x_k)$

$$\begin{aligned} \frac{\partial \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k} &= \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[(1-\beta) C_{\bar{x}x,k}^{[i]} + \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \right] \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] \\ &\quad - \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[(1-\beta) C_{\bar{x}\bar{x},k}^{[i]} + \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{F}_k^{[i]} \right] x_k \\ &\quad + \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[(1-\beta) c_{\bar{x},k}^{[i]} - \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]} \right] \\ &= \overleftarrow{\psi}_k^{[i+1]}(x_k) \overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] - \overleftarrow{\psi}_k^{[i+1]}(x_k) \overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} x + \overleftarrow{\psi}_k^{[i+1]}(x_k) \overleftarrow{g}_{\bar{x},k}^{[i+1]}, \end{aligned}$$

where we use the definition of $\overleftarrow{G}_{\bar{x}x,k}^{[i+1]}$, $\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]}$, and $\overleftarrow{g}_{\bar{x},k}^{[i+1]}$ from Proposition 5. Consequently, the first conditional moment of $\overleftarrow{q}_k^{[i+1]}(x_{k-1} \mid x_k)$ is

$$\mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] = \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} x_k - \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \overleftarrow{g}_{\bar{x},k}^{[i+1]} + \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \frac{\partial \log \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k}.$$

Now we consider the second-order derivatives of $\overleftarrow{\psi}_k^{[i+1]}(x_k)(x)$ with respect to x_k

$$\begin{aligned} \frac{\partial^2 \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} &= \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right] \mathbb{E}^{[i+1]} [x_{k-1} x_{k-1}^\top \mid x_k] \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \\ &\quad + \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \left[x_k x_k^\top \right] \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \\ &\quad + \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right] \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \\ &\quad - 2 \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right] \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] x_k^\top \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \\ &\quad + 2 \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right] \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \\ &\quad - 2 \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] x_k \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \\ &\quad - \overleftarrow{\psi}_k^{[i+1]}(x_k) \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]. \end{aligned}$$

This gives us an expression for the second *raw* conditional moment

$$\begin{aligned}
\mathbb{E}^{[i+1]} [x_{k-1} x_{k-1}^\top \mid x_k] &= \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad + \left[\overleftarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial^2 \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad - \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \left[x_k x_k^\top \right] \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad - \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right] \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad + 2 \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] x_k^\top \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad - 2 \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad + 2 \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] x_k \left[\overleftarrow{g}_{\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1}.
\end{aligned}$$

Finally, we can derive the second *central* moment according to

$$\begin{aligned}
\mathbb{V}^{[i+1]} [x_{k-1} \mid x_k] &= \mathbb{E}^{[i+1]} [x_{k-1} x_{k-1}^\top \mid x_k] - \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k] \mathbb{E}^{[i+1]} [x_{k-1} \mid x_k]^\top \\
&= \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad + \left[\overleftarrow{\psi}_k^{[i+1]}(x_k) \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial^2 \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&\quad - \left[\overleftarrow{\psi}_k^{[i+1]}(x_k) \right]^{-2} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \frac{\partial \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k} \left[\frac{\partial \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k} \right]^\top \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} \\
&= \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \left[\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right] \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1} + \left[\overleftarrow{G}_{\bar{x}x,k}^{[i+1]} \right]^{-1} \frac{\partial^2 \log \overleftarrow{\psi}_k^{[i+1]}(x_k)}{\partial x_k \partial x_k^\top} \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^{-1}.
\end{aligned}$$

I Recursive Bayesian Inference Algorithms

Algorithm 1 Backward Recursion of the Forward Proximal Variational Smoother

Require: Damping: β , first Gaussian marginal: $\overrightarrow{m}_0^{[i]}, \overrightarrow{P}_0^{[i]}$,
forward affine-Gaussian conditionals: $\overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]}$,
log prior ℓ_p : $L_0^{[i]}, l_0^{[i]}$, log measurement ℓ_h : $L_{1:T}^{[i]}, l_{1:T}^{[i]}$,
log dynamics ℓ_f : $C_{\bar{x}\bar{x},0:T-1}^{[i]}, C_{\bar{x}x,0:T-1}^{[i]}, C_{xx,0:T-1}^{[i]}, c_{\bar{x},0:T-1}^{[i]}, c_{x,0:T-1}^{[i]}$.

1: $\overrightarrow{R}_T^{[i+1]} = L_T^{[i]}, \quad \overrightarrow{r}_T^{[i+1]} = l_T^{[i]}$ ▷ initialize backward recursion

2: **for** $k \leftarrow T-1, \dots, 0$ **do**

3: $\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} = (1-\beta) \left[C_{\bar{x}\bar{x},k}^{[i]} + \overrightarrow{R}_{k+1}^{[i+1]} \right] + \beta \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1}$

4: $\overrightarrow{G}_{xx,k}^{[i+1]} = (1-\beta) C_{xx,k}^{[i]} + \beta \left[\overrightarrow{F}_k^{[i]} \right]^\top \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{F}_k^{[i]}$

5: $\overrightarrow{G}_{\bar{x}x,k}^{[i+1]} = (1-\beta) C_{\bar{x}x,k}^{[i]} + \beta \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{F}_k^{[i]}$

6: $\overrightarrow{g}_{\bar{x},k}^{[i+1]} = (1-\beta) \left[c_{\bar{x},k}^{[i]} + \overrightarrow{r}_{k+1}^{[i+1]} \right] + \beta \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{d}_k^{[i]}$

7: $\overrightarrow{g}_{x,k}^{[i+1]} = (1-\beta) c_{x,k}^{[i]} - \beta \left[\overrightarrow{F}_k^{[i]} \right]^\top \left[\overrightarrow{\Sigma}_k^{[i]} \right]^{-1} \overrightarrow{d}_k^{[i]}$

8: $\overrightarrow{S}_k^{[i+1]} = \overrightarrow{G}_{xx,k}^{[i]} - \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]}$ ▷ update log-normalizers

9: $\overrightarrow{s}_k^{[i+1]} = \overrightarrow{g}_{x,k}^{[i]} + \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^\top \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \overrightarrow{g}_{\bar{x},k}^{[i+1]}$

10: $\overrightarrow{R}_k^{[i+1]} = L_k^{[i]} + 1/(1-\beta) \overrightarrow{S}_k^{[i+1]}$ ▷ update potential functions

11: $\overrightarrow{r}_k^{[i+1]} = l_k^{[i]} + 1/(1-\beta) \overrightarrow{s}_k^{[i+1]}$

12: $\overrightarrow{F}_k^{[i+1]} = \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \overrightarrow{G}_{\bar{x}x,k}^{[i+1]}$ ▷ update conditional posteriors

13: $\overrightarrow{d}_k^{[i+1]} = \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1} \overrightarrow{g}_{\bar{x},k}^{[i+1]}$

14: $\overrightarrow{\Sigma}_k^{[i+1]} = \left[\overrightarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} \right]^{-1}$

15: **end for**

16: $\overrightarrow{J}_{xx}^{[i+1]} = (1-\beta) \overrightarrow{R}_0^{[i+1]} + \beta \left[\overrightarrow{P}_0^{[i]} \right]^{-1}$

17: $\overrightarrow{J}_{xm}^{[i+1]} = \beta \left[\overrightarrow{P}_0^{[i]} \right]^{-1}$

18: $\overrightarrow{j}_x^{[i+1]} = (1-\beta) \overrightarrow{r}_0^{[i+1]}$

19: $\overrightarrow{m}_0^{[i+1]} = \left[\overrightarrow{J}_{xx}^{[i+1]} \right]^{-1} \left[\overrightarrow{j}_x^{[i+1]} + \overrightarrow{J}_{xm}^{[i+1]} \overrightarrow{m}_0^{[i]} \right]$ ▷ update boundary marginal

20: $\overrightarrow{P}_0^{[i+1]} = \left[\overrightarrow{J}_{xx}^{[i+1]} \right]^{-1}$

21: **return** $\overrightarrow{m}_0^{[i+1]}, \overrightarrow{P}_0^{[i+1]}, \overrightarrow{F}_{0:T-1}^{[i+1]}, \overrightarrow{d}_{0:T-1}^{[i+1]}, \overrightarrow{\Sigma}_{0:T-1}^{[i+1]}, \overrightarrow{R}_{0:T}^{[i+1]}, \overrightarrow{r}_{0:T}^{[i+1]}, \overrightarrow{S}_{0:T-1}^{[i+1]}, \overrightarrow{s}_{0:T-1}^{[i+1]}$.

Algorithm 2 Forward Recursion of the Forward Proximal Variational Smoother

Require: First Gaussian marginal: $\overrightarrow{m}_0^{[i+1]}, \overrightarrow{P}_0^{[i+1]}$,
forward affine-Gaussian conditionals: $\overrightarrow{F}_{0:T-1}^{[i+1]}, \overrightarrow{d}_{0:T-1}^{[i+1]}, \overrightarrow{\Sigma}_{0:T-1}^{[i+1]}$.

- 1: **for** $k \leftarrow 0, \dots, T-1$ **do**
- 2: $\overrightarrow{m}_{k+1}^{[i+1]} = \overrightarrow{F}_k^{[i+1]} \overrightarrow{m}_k^{[i+1]} + \overrightarrow{d}_k^{[i+1]}$ ▷ update marginals
- 3: $\overrightarrow{P}_{k+1}^{[i+1]} = \overrightarrow{F}_k^{[i+1]} \overrightarrow{P}_k^{[i+1]} \left[\overrightarrow{F}_k^{[i+1]} \right]^\top + \overrightarrow{\Sigma}_k^{[i+1]}$
- 4: **end for**
- 5: **return** $\overrightarrow{m}_{0:T}^{[i+1]}, \overrightarrow{P}_{0:T}^{[i+1]}$.

Algorithm 3 Optimal Damping for Forward Proximal Variational Smoother

Require: Maximum multiplier: α_{\max} , minimum multiplier: α_{\min} ,
initial multiplier: α_0 , first Gaussian marginal: $\overrightarrow{m}_0^{[i]}, \overrightarrow{P}_0^{[i]}$,
forward affine-Gaussian conditionals: $\overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]}$,
log prior ℓ_p , log measurement ℓ_h , log dynamics ℓ_f .

- 1: $\alpha \leftarrow \alpha_0$
- 2: **repeat**
- 3: $\beta \leftarrow \alpha/(1+\alpha)$
- 4: $\overrightarrow{m}_0^{[i+1]}, \overrightarrow{P}_0^{[i+1]}, \overrightarrow{F}_{0:T-1}^{[i+1]}, \overrightarrow{d}_{0:T-1}^{[i+1]}, \overrightarrow{\Sigma}_{0:T-1}^{[i+1]} \leftarrow$
FORWARD CONDITIONALS $(\beta, \ell_p, \ell_h, \ell_f, \overrightarrow{m}_0^{[i]}, \overrightarrow{P}_0^{[i]}, \overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]})$ ▷ Algorithm 1
- 5: $\overrightarrow{m}_{0:T}^{[i+1]}, \overrightarrow{P}_{0:T}^{[i+1]} \leftarrow$
FORWARD MARGINALS $(\overrightarrow{m}_0^{[i+1]}, \overrightarrow{P}_0^{[i+1]}, \overrightarrow{F}_{0:T-1}^{[i+1]}, \overrightarrow{d}_{0:T-1}^{[i+1]}, \overrightarrow{\Sigma}_{0:T-1}^{[i+1]})$ ▷ Algorithm 2
- 6: **if** $\varepsilon - \mathbb{D}_{\text{KL}} \left[\overrightarrow{q}^{[i+1]} \parallel \overrightarrow{q}^{[i]} \right] > 0$ **then**
- 7: $\alpha_{\max} \leftarrow \alpha$
- 8: $\alpha \leftarrow \sqrt{\alpha \cdot \alpha_{\min}}$ ▷ reduce multiplier
- 9: **else if** $\varepsilon - \mathbb{D}_{\text{KL}} \left[\overrightarrow{q}^{[i+1]} \parallel \overrightarrow{q}^{[i]} \right] < 0$ **then**
- 10: $\alpha_{\min} \leftarrow \alpha$
- 11: $\alpha \leftarrow \sqrt{\alpha \cdot \alpha_{\max}}$ ▷ increase multiplier
- 12: **end if**
- 13: **until** $\varepsilon - \mathbb{D}_{\text{KL}} \left[\overrightarrow{q}^{[i+1]} \parallel \overrightarrow{q}^{[i]} \right] \approx 0$ ▷ gradient vanishes
- 14: $\beta \leftarrow \alpha/(1+\alpha)$
- 15: **return** β

Algorithm 4 Forward Recursion of the Reverse Proximal Variational Smoother

Require: Damping: β , last Gaussian marginal: $\overleftarrow{m}_T^{[i]}, \overleftarrow{P}_T^{[i]}$,
 reverse affine-Gaussian conditionals: $\overleftarrow{F}_{0:T-1}^{[i]}, \overleftarrow{d}_{0:T-1}^{[i]}, \overleftarrow{\Sigma}_{0:T-1}^{[i]}$,
 log prior $\ell_p(x_0)$: $L_0^{[i]}, l_0^{[i]}$, log measurement $\ell_h(x_k)$: $L_{1:T}^{[i]}, l_{1:T}^{[i]}$,
 log dynamics ℓ_f : $C_{\bar{x}\bar{x},0:T-1}^{[i]}, C_{\bar{x}x,0:T-1}^{[i]}, C_{xx,0:T-1}^{[i]}, c_{\bar{x},0:T-1}^{[i]}, c_{x,0:T-1}^{[i]}$.

1: $\overleftarrow{R}_0^{[i+1]} = L_0^{[i]}, \quad \overleftarrow{r}_0^{[i+1]} = l_0^{[i]}$ ▷ initialize forward recursion

2: **for** $k \leftarrow 1, \dots, T$ **do**

3: $\overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} = (1 - \beta) C_{\bar{x}\bar{x},k-1}^{[i]} + \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{F}_k^{[i]}$

4: $\overleftarrow{G}_{xx,k}^{[i+1]} = (1 - \beta) \left[C_{xx,k-1}^{[i]} + \overleftarrow{R}_{k-1}^{[i+1]} \right] + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1}$

5: $\overleftarrow{G}_{x\bar{x},k}^{[i+1]} = (1 - \beta) C_{x\bar{x},k-1}^{[i]} + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{F}_k^{[i]}$

6: $\overleftarrow{g}_{\bar{x},k}^{[i+1]} = (1 - \beta) c_{\bar{x},k-1}^{[i]} - \beta \left[\overleftarrow{F}_k^{[i]} \right]^\top \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]}$

7: $\overleftarrow{g}_{x,k}^{[i+1]} = (1 - \beta) \left[c_{x,k-1}^{[i]} + \overleftarrow{r}_{k-1}^{[i+1]} \right] + \beta \left[\overleftarrow{\Sigma}_k^{[i]} \right]^{-1} \overleftarrow{d}_k^{[i]}$

8: $\overleftarrow{S}_k^{[i+1]} = \overleftarrow{G}_{\bar{x}\bar{x},k}^{[i+1]} - \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overleftarrow{G}_{x\bar{x},k}^{[i+1]}$ ▷ update log-normalizers

9: $\overleftarrow{s}_k^{[i+1]} = \overleftarrow{g}_{\bar{x},k}^{[i+1]} + \left[\overleftarrow{G}_{x\bar{x},k}^{[i+1]} \right]^\top \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overleftarrow{g}_{x,k}^{[i+1]}$

10: $\overleftarrow{R}_k^{[i+1]} = L_k^{[i]} + 1/(1 - \beta) \overleftarrow{S}_k^{[i+1]}$ ▷ update potential functions

11: $\overleftarrow{r}_k^{[i+1]} = l_k^{[i]} + 1/(1 - \beta) \overleftarrow{s}_k^{[i+1]}$

12: $\overleftarrow{F}_k^{[i+1]} = \left[\overleftarrow{G}_{x2,k}^{[i+1]} \right]^{-1} \overleftarrow{G}_{x\bar{x},k}^{[i+1]}$ ▷ update conditional posteriors

13: $\overleftarrow{d}_k^{[i+1]} = \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1} \overleftarrow{g}_{x,k}^{[i+1]}$

14: $\overleftarrow{\Sigma}_k^{[i+1]} = \left[\overleftarrow{G}_{xx,k}^{[i+1]} \right]^{-1}$

15: **end for**

16: $\overleftarrow{J}_{mm}^{[i+1]} = (1 - \beta) \overleftarrow{R}_T^{[i+1]} + \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}$

17: $\overleftarrow{J}_{xm}^{[i+1]} = \beta \left[\overleftarrow{P}_T^{[i]} \right]^{-1}$

18: $\overleftarrow{j}_m^{[i+1]} = (1 - \beta) \overleftarrow{r}_T^{[i+1]}$

19: $\overleftarrow{m}_T^{[i+1]} = \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1} \left[\overleftarrow{j}_m^{[i+1]} + \overleftarrow{J}_{xm}^{[i+1]} \overleftarrow{m}_T^{[i]} \right]$ ▷ update boundary marginal

20: $\overleftarrow{P}_T^{[i+1]} = \left[\overleftarrow{J}_{mm}^{[i+1]} \right]^{-1}$

21: **return** $\overleftarrow{m}_T^{[i+1]}, \overleftarrow{P}_T^{[i+1]}, \overleftarrow{F}_{1:T}^{[i+1]}, \overleftarrow{d}_{1:T}^{[i+1]}, \overleftarrow{\Sigma}_{1:T}^{[i+1]}, \overleftarrow{R}_{0:T}^{[i+1]}, \overleftarrow{r}_{0:T}^{[i+1]}, \overleftarrow{S}_{1:T}^{[i+1]}, \overleftarrow{s}_{1:T}^{[i+1]}$.

Algorithm 5 Backward Recursion of the Reverse Proximal Variational Smoother

Require: Last Gaussian marginal: $\overleftarrow{m}_T^{[i+1]}, \overleftarrow{P}_T^{[i+1]}$,
 reverse affine-Gaussian conditionals: $\overleftarrow{F}_{1:T}^{[i+1]}, \overleftarrow{d}_{1:T}^{[i+1]}, \overleftarrow{\Sigma}_{1:T}^{[i+1]}$.

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1: for  $k \leftarrow T, \dots, 1$  do
2:    $\overleftarrow{m}_{k-1}^{[i+1]} = \overleftarrow{F}_k^{[i+1]} \overleftarrow{m}_k^{[i+1]} + \overleftarrow{d}_k^{[i+1]}$                                  $\triangleright$  update marginals
3:    $\overleftarrow{P}_{k-1}^{[i+1]} = \overleftarrow{F}_k^{[i+1]} \overleftarrow{P}_k^{[i+1]} \begin{bmatrix} \overleftarrow{F}_k^{[i+1]} \\ \overleftarrow{\Sigma}_k^{[i+1]} \end{bmatrix}^\top + \overleftarrow{\Sigma}_k^{[i+1]}$ 
4: end for
5: return  $\overleftarrow{m}_{0:T}^{[i+1]}, \overleftarrow{P}_{0:T}^{[i+1]}$ .

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Algorithm 6 Optimal Damping for the Reverse Proximal Variational Smoother

Require: Maximum multiplier: α_{\max} , minimum multiplier: α_{\min} ,
 initial multiplier: α_0 , last Gaussian marginal: $\overleftarrow{m}_T^{[i]}, \overleftarrow{P}_T^{[i]}$,
 reverse affine-Gaussian conditionals: $\overleftarrow{F}_{1:T}^{[i]}, \overleftarrow{d}_{1:T}^{[i]}, \overleftarrow{\Sigma}_{1:T}^{[i]}$,
 log prior ℓ_p , log measurement ℓ_h , log dynamics ℓ_f .

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1:  $\alpha \leftarrow \alpha_0$ 
2: repeat
3:    $\beta \leftarrow \alpha/(1+\alpha)$ 
4:    $\overleftarrow{m}_T^{[i+1]}, \overleftarrow{P}_T^{[i+1]}, \overleftarrow{F}_{1:T}^{[i+1]}, \overleftarrow{d}_{1:T}^{[i+1]}, \overleftarrow{\Sigma}_{1:T}^{[i+1]} \leftarrow$  REVERSE CONDITIONALS  $(\beta, \ell_p, \ell_h, \ell_f, \overleftarrow{m}_T^{[i]}, \overleftarrow{P}_T^{[i]}, \overleftarrow{F}_{1:T}^{[i]}, \overleftarrow{d}_{1:T}^{[i]}, \overleftarrow{\Sigma}_{1:T}^{[i]})$   $\triangleright$  Algorithm 4
5:    $\overleftarrow{m}_{0:T}^{[i+1]}, \overleftarrow{P}_{0:T}^{[i+1]} \leftarrow$  REVERSE MARGINALS  $(\overleftarrow{m}_T^{[i+1]}, \overleftarrow{P}_T^{[i+1]}, \overleftarrow{F}_{1:T}^{[i+1]}, \overleftarrow{d}_{1:T}^{[i+1]}, \overleftarrow{\Sigma}_{1:T}^{[i+1]})$   $\triangleright$  Algorithm 5
6:   if  $\varepsilon - \mathbb{D}_{\text{KL}}\left[\overleftarrow{q}^{[i+1]} \parallel \overleftarrow{q}^{[i]}\right] > 0$  then
7:      $\alpha_{\max} \leftarrow \alpha$ 
8:      $\alpha \leftarrow \sqrt{\alpha \cdot \alpha_{\min}}$   $\triangleright$  reduce multiplier
9:   else if  $\varepsilon - \mathbb{D}_{\text{KL}}\left[\overleftarrow{q}^{[i+1]} \parallel \overleftarrow{q}^{[i]}\right] < 0$  then
10:     $\alpha_{\min} \leftarrow \alpha$ 
11:     $\alpha \leftarrow \sqrt{\alpha \cdot \alpha_{\max}}$   $\triangleright$  increase multiplier
12:   end if
13: until  $\varepsilon - \mathbb{D}_{\text{KL}}\left[\overleftarrow{q}^{[i+1]} \parallel \overleftarrow{q}^{[i]}\right] \approx 0$   $\triangleright$  gradient vanishes
14:  $\beta \leftarrow \alpha/(1+\alpha)$ 
15: return  $\beta$ 

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Algorithm 7 Forward Proximal Variational Smoother

Require: First Gaussian marginal: $\overrightarrow{m}_0^{[0]}, \overrightarrow{P}_0^{[0]}$,
forward affine-Gaussian conditionals: $\overrightarrow{F}_{0:T-1}^{[0]}, \overrightarrow{d}_{0:T-1}^{[0]}, \overrightarrow{\Sigma}_{0:T-1}^{[0]}$.

- 1: $i \leftarrow 0$
- 2: **while** not converged **do**
- 3: $\overrightarrow{m}_{0:T}^{[i]}, \overrightarrow{P}_{0:T}^{[i]} \leftarrow \text{FORWARD MARGINALS}(\overrightarrow{m}_0^{[i]}, \overrightarrow{P}_0^{[i]}, \overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]})$ ▷ Algorithm 2
- 4: $\ell_p, \ell_h, \ell_f \leftarrow \text{APPROXIMATE MODEL}(p, f, h, \overrightarrow{m}_{0:T}^{[i]}, \overrightarrow{P}_{0:T}^{[i]})$ ▷ Definition 2/3
- 5: $\beta \leftarrow \text{OPTIMAL DAMPING}(\ell_p, \ell_h, \ell_f, \overrightarrow{m}_0^{[i]}, \overrightarrow{P}_0^{[i]}, \overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]})$ ▷ Algorithm 3
- 6: $\overrightarrow{m}_0^{[i+1]}, \overrightarrow{P}_0^{[i+1]}, \overrightarrow{F}_{0:T-1}^{[i+1]}, \overrightarrow{d}_{0:T-1}^{[i+1]}, \overrightarrow{\Sigma}_{0:T-1}^{[i+1]} \leftarrow$
 $\text{FORWARD CONDITIONALS}(\beta, \ell_p, \ell_h, \ell_f, \overrightarrow{m}_0^{[i]}, \overrightarrow{P}_0^{[i]}, \overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]})$ ▷ Algorithm 1
- 7: $i \leftarrow i + 1$
- 8: **end while**
- 9: **return** $\overrightarrow{m}_{0:T}^{[\infty]}, \overrightarrow{P}_{0:T}^{[\infty]}$.

Algorithm 8 Reverse Proximal Variational Smoother

Require: Last Gaussian marginal: $\overleftarrow{m}_T^{[0]}, \overleftarrow{P}_T^{[0]}$,
reverse affine-Gaussian conditionals: $\overleftarrow{F}_{1:T}^{[0]}, \overleftarrow{d}_{1:T}^{[0]}, \overleftarrow{\Sigma}_{1:T}^{[0]}$.

- 1: $i \leftarrow 0$
- 2: **while** not converged **do**
- 3: $\overleftarrow{m}_{0:T}^{[i]}, \overleftarrow{P}_{0:T}^{[i]} \leftarrow \text{REVERSE MARGINALS}(\overleftarrow{m}_T^{[i]}, \overleftarrow{P}_T^{[i]}, \overleftarrow{F}_{1:T}^{[i]}, \overleftarrow{d}_{1:T}^{[i]}, \overleftarrow{\Sigma}_{1:T}^{[i]})$ ▷ Algorithm 5
- 4: $\ell_p, \ell_h, \ell_f \leftarrow \text{APPROXIMATE MODEL}(p, f, h, \overleftarrow{m}_{0:T}^{[i]}, \overleftarrow{P}_{0:T}^{[i]})$ ▷ Definition 2/3
- 5: $\beta \leftarrow \text{OPTIMAL DAMPING}(\ell_p, \ell_h, \ell_f, \overleftarrow{m}_T^{[i]}, \overleftarrow{P}_T^{[i]}, \overleftarrow{F}_{1:T}^{[i]}, \overleftarrow{d}_{1:T}^{[i]}, \overleftarrow{\Sigma}_{1:T}^{[i]})$ ▷ Algorithm 6
- 6: $\overleftarrow{m}_T^{[i+1]}, \overleftarrow{P}_T^{[i+1]}, \overleftarrow{F}_{1:T}^{[i+1]}, \overleftarrow{d}_{1:T}^{[i+1]}, \overleftarrow{\Sigma}_{1:T}^{[i+1]} \leftarrow$
 $\text{REVERSE CONDITIONALS}(\beta, \ell_p, \ell_h, \ell_f, \overleftarrow{m}_T^{[i]}, \overleftarrow{P}_T^{[i]}, \overleftarrow{F}_{1:T}^{[i]}, \overleftarrow{d}_{1:T}^{[i]}, \overleftarrow{\Sigma}_{1:T}^{[i]})$ ▷ Algorithm 4
- 7: $i \leftarrow i + 1$
- 8: **end while**
- 9: **return** $\overleftarrow{m}_{0:T}^{[\infty]}, \overleftarrow{P}_{0:T}^{[\infty]}$.

Algorithm 9 Marginals of the Hybrid Proximal Variational Smoother

Require: Damping: β , prior Gaussian marginal: $m_{1:T-1}^{[i]}, P_{1:T-1}^{[i]}$,
forward log-normalizing functions: $\overrightarrow{S}_{1:T-1}^{[i+1]}, \overrightarrow{s}_{1:T-1}^{[i+1]}$,
reverse potential functions: $\overleftarrow{R}_{1:T-1}^{[i+1]}, \overleftarrow{r}_{1:T-1}^{[i+1]}$.

- 1: **for** $k \leftarrow 1, \dots, T-1$ **do**
- 2: $P_k^{[i+1]} = \left[(1-\beta) \left[\overleftarrow{R}_k^{[i+1]} + \overrightarrow{S}_k^{[i+1]} \right] + \beta \left[P_k^{[i]} \right]^{-1} \right]^{-1}$ ▷ update marginals
- 3: $m_k^{[i+1]} = P_k^{[i+1]} \left[(1-\beta) \left[\overleftarrow{r}_k^{[i+1]} + \overrightarrow{s}_k^{[i+1]} \right] + \beta \left[P_k^{[i]} \right]^{-1} m_k^{[i]} \right]$
- 4: **end for**
- 5: **return** $m_{1:T-1}^{[i+1]}, P_{1:T-1}^{[i+1]}$.

Algorithm 10 Hybrid Proximal Variational Smoother

Require: Initial Gaussian marginals: $m_{0:T}^{[0]}, P_{0:T}^{[0]}$,
forward affine-Gaussian conditionals: $\overrightarrow{F}_{0:T-1}^{[0]}, \overrightarrow{d}_{0:T-1}^{[0]}, \overrightarrow{\Sigma}_{0:T-1}^{[0]}$,
reverse affine-Gaussian conditionals: $\overleftarrow{F}_{1:T}^{[0]}, \overleftarrow{d}_{1:T}^{[0]}, \overleftarrow{\Sigma}_{1:T}^{[0]}$.

- 1: $i \leftarrow 0$
- 2: **while** not converged **do**
- 3: $\ell_p, \ell_h, \ell_f \leftarrow \text{APPROXIMATE MODEL } (p, f, h, m_{0:T}^{[i]}, P_{0:T}^{[i]})$ ▷ Definition 2/3
- 4: $\beta \leftarrow \text{OPTIMAL DAMPING } (\ell_p, \ell_h, \ell_f, m_0^{[i]}, P_0^{[i]}, \overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]})$ ▷ Algorithm 3
- 5: $m_0^{[i+1]}, P_0^{[i+1]}, \overrightarrow{F}_{0:T-1}^{[i+1]}, \overrightarrow{d}_{0:T-1}^{[i+1]}, \overrightarrow{\Sigma}_{0:T-1}^{[i+1]}, \overrightarrow{S}_{1:T-1}^{[i+1]}, \overrightarrow{s}_{1:T-1}^{[i+1]} \leftarrow$
FORWARD NORMALIZERS $(\beta, \ell_p, \ell_h, \ell_f, m_0^{[i]}, P_0^{[i]}, \overrightarrow{F}_{0:T-1}^{[i]}, \overrightarrow{d}_{0:T-1}^{[i]}, \overrightarrow{\Sigma}_{0:T-1}^{[i]})$ ▷ Algorithm 1
- 6: $m_T^{[i+1]}, P_T^{[i+1]}, \overleftarrow{F}_{1:T}^{[i+1]}, \overleftarrow{d}_{1:T}^{[i+1]}, \overleftarrow{\Sigma}_{1:T}^{[i+1]}, \overleftarrow{R}_{1:T-1}^{[i+1]}, \overleftarrow{r}_{1:T-1}^{[i+1]} \leftarrow$
REVERSE POTENTIALS $(\beta, \ell_p, \ell_h, \ell_f, m_T^{[i]}, P_T^{[i]}, \overleftarrow{F}_{1:T}^{[i]}, \overleftarrow{d}_{1:T}^{[i]}, \overleftarrow{\Sigma}_{1:T}^{[i]})$ ▷ Algorithm 4
- 7: $m_{1:T-1}^{[i+1]}, P_{1:T-1}^{[i+1]} \leftarrow$
HYBRID MARGINALS $(\beta, m_{1:T-1}^{[i]}, P_{1:T-1}^{[i]}, \overrightarrow{S}_{1:T-1}^{[i+1]}, \overrightarrow{s}_{1:T-1}^{[i+1]}, \overleftarrow{R}_{1:T-1}^{[i+1]}, \overleftarrow{r}_{1:T-1}^{[i+1]})$ ▷ Algorithm 9
- 8: $i \leftarrow i + 1$
- 9: **end while**
- 10: **return** $m_{0:T}^{[\infty]}, P_{0:T}^{[\infty]}$.
