
FRONT-DOOR REDUCIBILITY: REDUCING ADMGS TO THE STANDARD FRONT-DOOR SETTING VIA A GRAPHICAL CRITERION

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ABSTRACT

Front-door adjustment provides a simple closed-form identification formula under the classical front-door criterion, but its applicability is often viewed as narrow and strict. Although ID algorithm is very useful and is proved effective for causal relation identification in general causal graphs (if it is identifiable), performing ID algorithm does not guarantee to obtain a practical, easy-to-estimate interventional distribution expression. We argue that the applicability of the front-door criterion is not as limited as it seems: many more complicated causal graphs can be reduced to the front-door criterion. In this paper, We introduce front-door reducibility (FDR), a graphical condition on acyclic directed mixed graphs (ADMGs) that extends the applicability of the classic front-door criterion to reduce a large family of complicated causal graphs to a front-door setting by aggregating variables into super-nodes (FDR triple) $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$. After characterizing FDR criterion, we prove a graph-level equivalence between the satisfaction of FDR criterion and the applicability of FDR adjustment. Meanwhile, we then present FDR-TID, an exact algorithm that detects an admissible FDR triple, together with established the algorithm's correctness, completeness, and finite termination. Empirically-motivated examples illustrate that many graphs outside the textbook front-door setting are FDR, yielding simple, estimable adjustments where general ID expressions would be cumbersome. FDR thus complements existing identification method by prioritizing interpretability and computational simplicity without sacrificing generality across mixed graphs.

Keywords Causal inference, Front-door criterion, Interventional distribution

1 Introduction

Front-door confounding is a simple but ubiquitous scenario as shown in Figure 1 (c). The *front-door criterion* is formally defined by Pearl (2009) for its first time:

Definition 1. (Front-door criterion (Pearl, 2009)). Given cause variable X , effect variable Y and a set of variables \mathbf{M} , the causal effect from the cause X to the effect Y is identifiable if:

1. The mediator variable set \mathbf{M} intercepts all directed paths from X to Y ;
2. There is no unblocked backdoor (see Definition 4) path from X to \mathbf{M} ; and

3. All backdoor paths from M to Y are blocked by X .

And for a set of variable M that satisfy this criterion, we can use *do*-calculus to compute the interventional distribution (Pearl, 2010):

$$p(y | do(x)) = \int_{m,x} p(y | x', m) p(m | x) p(x') dm dx \quad (1)$$

Despite its familiarity, simplicity and wide applicability (Mao and Little, 2024), the ordinary front-door setting sometimes is criticized (Robins, 1995; VanderWeele, 2009) as a causal assumption with strict assumptions as depicted in Definition 1. For example, Bellemare et al. (2024) present a front-door case study in economics with robustness checks on real data. And they conclude that the standard front-door adjustment relies on strong structural measurement assumptions and high-quality mediators in empirical study. However, we argue that the applicability of the front-door criterion is not as limited as it seems: many more complicated causal graphs can be reduced to the front-door criterion! For example, Figure 1 (a) is a complicated ADMG, where the joint causal effect of (X, K) on Y is theoretically identifiable and its interventional distribution $p(y | do(x), do(k))$ can be solved by the ID algorithm (Shpitser and Pearl, 2006). Although ID algorithm is very useful and is proved as an effective identification method for an identifiable causal relation in general causal graphs, performing ID algorithm is not necessarily practical in such a complicated graph, considering the complexity of both the recursive algorithm itself and its output expression difficult to estimate or use. In this example, Shpitser and Pearl's ID algorithm gives the interventional distribution shown as:

$$\begin{aligned} p(y | do(x), do(k)) &= \int_{x', m, w, k', u, z, v} p(y | x, u, v, m, w, k', z) p(z | u, v, x', m, w, k) \\ &\quad p(m, u | v, x, k) p(x' | w, v, k) p(v | k) p(w) p(k') dx' dm dw dk' du dz dv \end{aligned} \quad (2)$$

On the contrary, the *front-door adjustment* gives simpler and more practical formula as shown in eq. (1). Meanwhile, the complicated ADGM (Figure 1 (a)) can be reduced to an ADMG (Figure 1 (b)) satisfying front-door criterion with proper operation such as merging and omitting. In this example, the projection can be done by (i) merging X and K as a super-cause node $X^* = \{X, K\}$; (ii) adjusting on $M^* = \{M\}$ as the super-mediator node and let $Y^* = \{Y\}$ be the super-effect node; and (iii) omitting other irrelevant variables. Then the interventional distribution $p(y | do(x^*))$ is reduced to the ordinary front-door adjustment formula as shown in eq. (1). In fact, we show later that a family of ADMGs, e.g., graphs in Figure 2 are all reducible to an ordinary front-door setting.

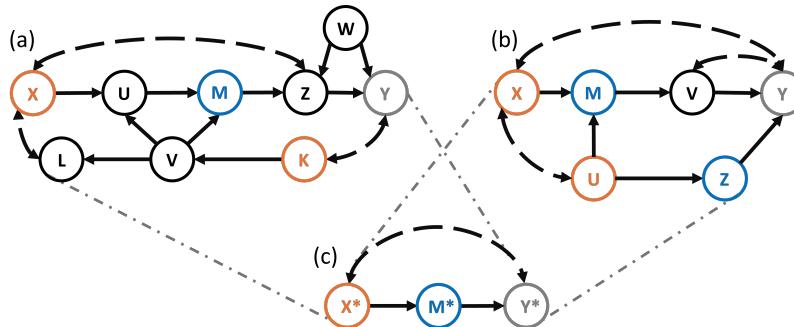


Figure 1: The example of the equivalence projection from two complicated (but reducible) ADMGs (a) and (b) to a simple, well studied ADMG (c) where ordinary front-door adjustment can be used to obtain the interventional distribution $p(Y^* | do(X^*))$. Different colors shows the projection relation of the nodes between (a), (b) and (c), e.g., for (a) $X^* = \{X, K\}$, $M^* = \{M\}$, and for (b) $X^* = \{X, U\}$, $M^* = \{M, Z\}$.

Much recent literature studies and extend the front-door criterion. Fulcher et al. (2020) propose a generalized front-door approach that aims to identify indirect (mediated) effects even when a direct cause-effect path

and unmeasured confounding are present using both parametric and semiparametric methods. However, this method is not applicable to estimate the total effects, and it largely relies on a set of conditions which can be hard to verify in practice. Researchers in the literature (Cui et al., 2024) further extend the framework of proximal causal inference with hidden mediator variable by Ghassami et al. (2021), deriving influence functions and practical estimators for a broad class of proximal identification problems. However, the reported performance is sensitive to proxy quality and modeling choices. Conditional front-door criterion, extension of standard front-door criterion, was proposed by Xu et al. (2024), allows a observable confounder acting as the common cause of the cause, mediator and effect variable, while it introduces even stronger conditioning assumptions for the observable confounder. And CFDiVAE is proposed in this work to provide a variational autoencoder-based estimator for identification, whereas it comes with less interpretability and a heavier computational burden. More recently, Cakiqi and Little (2025) provides an new viewpoint from category theory: it turns identification problem as purely symbolic transformations, offering a formal calculus for implementable identification pipelines. Nevertheless, the empirical performance of the proposed method and its alignment with practical adjustment forms remain to be tested.

The works above collectively extends the existing causal inference and identification techniques, especially attempting to expand the scope of the standard front-door adjustment. However, they often either require strong proximal assumptions or produce algebraically complex identification formula. Our contribution is this work is complementary: we introduce front-door reducibility (FDR), a graphical sufficient and necessary condition that guarantees a simple, closed-form FDR adjustment for causal relation identification in an enlarged family of more general ADMGs. Meanwhile, we characterize when such FDR adjustment is valid, and provide a complete, sound exact algorithm to identify an admissible FDR triple $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$. Our theoretical analysis on the equivalence further shows that whenever an FDR adjustment formula (4) is universally valid, the underlying ADMG necessarily satisfies the FDR criterion.

2 Preliminaries

We first introduce some necessary definitions and theorems in causal inference. In this report, without special explanations, we use an uppercase letter to denote a variable and a lowercase letter to represent its value. Boldfaced letters are sets of variables and values. Let $\mathcal{G}(\mathbf{V})$ be an acyclic directed mixed graph (ADMG), where \mathbf{V} is the set of nodes indicating observed variables. Denote the ancestor and descendant sets in ADMG \mathcal{G} as $\text{Ang}(\cdot)$ and $\text{Deg}(\cdot)$, for example, $\text{Ang}(V_i)$ represents for the set of observable variables preceding V_i and itself in the topological ordering of \mathcal{G} . We represent m -separation as \perp_m . We use π to represent a path whose nodes are connected by either directed arcs (\rightarrow or \leftarrow) or a bidirected arc (\leftrightarrow), i.e., the directed path from X to Y in Figure 1 (c) is $\pi_{X \rightarrow Y} = (X \rightarrow M \rightarrow Y)$. We use $X \prec_{\mathcal{G}} Y$ to represent that X has smaller topological ordering than Y , and $X \preceq_{\mathcal{G}} Y$ if and only if $X \in \text{Ang}(Y)$. We use $\text{Nb}_{\mathcal{G}}^{\leftrightarrow}(\mathbf{A}) := \{V_i \in \mathbf{V} \setminus \mathbf{A} : \exists A_i \in \mathbf{A} \text{ s.t. } V_i \leftrightarrow A_i \text{ in } \mathcal{G}\}$ to denote the set of first bidirected-arc-connected neighbours of any node in \mathbf{A} in ADMG \mathcal{G} . In the following discussion, we use \mathcal{C} as the shorthand for the C-components (Definition 5) of a given ADMG \mathcal{G} without special mention. In this paper, we largely use the three rules of *do*-calculus (Pearl, 2010, 2009) as shown in 6. Same as other literature for causal inference, we use $\mathcal{G}_{\overline{X}}$ to denote the modified graph obtained by deleting from \mathcal{G} all arcs incoming to X ; and use $\mathcal{G}_{\underline{X}}$ to denote the modified graph obtained by deleting from \mathcal{G} all arcs outgoing from X . For example, $\mathcal{G}_{\overline{X}\underline{Z}}$ represents a modified graph \mathcal{G} after deleting all incoming arcs to X and outgoing arcs from Z .

3 Front-door reducibility

In Figure 2, we demonstrate some example ADMGs, where all of them can be reduced to an ordinary front-door setting with a proper construction (projection) of the super-cause, effect and mediator node. For instance, for Figure 2 (f), we can prove that the interventional distribution $p(Y | do(X))$ has the same expression as the front-door adjustment formula in eq. (1) by considering M as the mediator variable with the fundamental *do*-calculus rules:

$$\begin{aligned}
 p(Y | do(X)) &= \int_m p(y | do(x), m) p(m | do(x)) dm \\
 &= \int_m p(y | do(x), do(m)) p(m | x) dm \\
 &\quad \left(\text{Rule 2: } (M \perp_m Y | X)_{\mathcal{G}_{\overline{X}M}} \text{ and } (M \perp_m X)_{\mathcal{G}_X} \right) \\
 &= \int_m p(m | x) p(y | do(m)) dm \\
 &\quad \left(\text{Rule 3: } (Y \perp_m X | M)_{\mathcal{G}_X} \right) \\
 &= \int_m p(m | x) \int_{x'} p(y | do(m), x') p(x' | do(m)) dx' dm \\
 &= \int_m p(m | x) \int_{x'} p(y | m, x') p(x') dx' dm \\
 &\quad \left(\text{Rule 2: } (Y \perp_m M | X)_{\mathcal{G}_M} ; \text{ and Rule 3: } (X \perp_m M)_{\mathcal{G}_{\overline{M}}} \right)
 \end{aligned} \tag{3}$$

On the other hand, if we test the front-door criterion on the ADMG in Figure 2 (f) for the cause variable X , effect variable Y and mediator variable M , all conditions are satisfied. This example comes with the fact that a wide family of ADMGs such as graphs in Figure 2 can be covered by front-door criterion and hence we could use front-door adjustment to obtain the interventional distribution of interest, rather than applying the over-complex ID algorithm to get an even impractical expression.

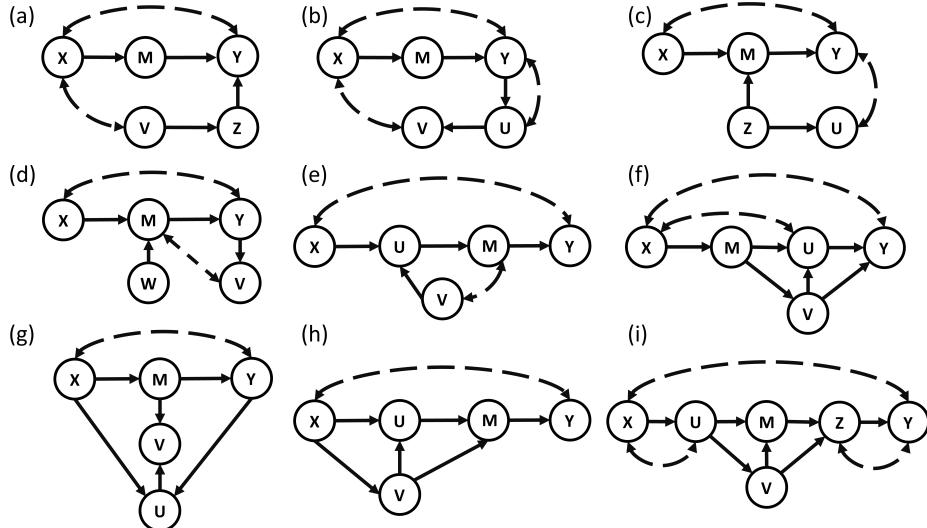


Figure 2: Example ADMGs relative to the cause and effect of interest pair (X, Y) satisfying FDR criterion. In (a-h), an admissible super-cause node is $X^* = \{X\}$, an admissible super-mediator node is $M^* = \{M\}$ and an admissible super-effect node is $Y^* = \{Y\}$; in (i), an admissible super-cause node is $X^* = \{X, U\}$, an admissible super-mediator node is $M^* = \{M, V\}$ and an admissible super-effect node $Y^* = \{Y\}$.

The question that follows is how do we test if a complicated ADMG can be reduced to the ordinary front-door setting? If it is reducible, how can we identify or construct admissible super-cause, effect and mediator node?

3.1 Front-door reducibility (FDR)

First, we formally define the *front-door reducibility* as follows:

Definition 2. (Front-door reducibility criterion, FDR criterion). An ADMG $\mathcal{G}(V)$ relative to cause and effect variables X and Y is said front-door reducible to \mathcal{G}_{X^*, Y^*}^* , iff there exists a triple of disjoint and non-empty supersets $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$, referring to the super-cause, effect and mediator nodes, respectively, such that:

- **FDR1:** For any $X_i \in \mathbf{X}^*$ and any $Y_i \in \mathbf{Y}^*$, all variables on every directed paths from X_i to Y_i intersect with \mathbf{M}^* , which is to say: $\forall X_i \in \mathbf{X}^*, \forall Y_i \in \mathbf{Y}^*, \forall \pi : X_i \rightarrow \dots \rightarrow Y_i, \pi \cap \mathbf{M}^* \neq \emptyset$, equivalently $\mathbf{Y}^* \cap \text{Deg}_{\mathcal{G}(V \setminus M)}(\mathbf{X}^*) = \emptyset$;
- **FDR2:** For any $M_i \in \mathbf{M}^*$, there is no backdoor path between \mathbf{X}^* and M_i , equivalently $(\mathbf{X}^* \perp_m \mathbf{M}^*)_{\mathcal{G}_{\mathbf{X}^*}}$;
- **FDR3:** For any $M_i \in \mathbf{M}^*$, all backdoor paths between \mathbf{Y}^* and \mathbf{M}^* are blocked by $\mathbf{X}^* \cup \mathbf{M}_{-i}^*$, equivalently $\forall M_i \in \mathbf{M}^*, (\mathbf{Y}^* \perp_m M_i | \mathbf{X}^* \cup \mathbf{M}_{-i}^*)_{\mathcal{G}_{\overline{\mathbf{X}^*} \setminus M_i}}$.

The FDR criterion provides a shortcut so that we could use front-door adjustment with trivial substitution for variables to obtain the interventional distribution of interest. The following theorem proves that:

Theorem 1. (Front-door reducible (FDR) adjustment). If an ADMG \mathcal{G} relative to cause and effect variables X and Y is front-door reducible, the causal effect of \mathbf{X}^* on \mathbf{Y}^* is identifiable by adjusting on \mathbf{M}^* . The interventional distribution $p(\mathbf{y}^* | do(\mathbf{x}^*))$ is given by the following FDR adjustment formula:

$$p(\mathbf{y}^* | do(\mathbf{x}^*)) = \int_{\mathbf{m}^*, \mathbf{x}'^*} p(\mathbf{y}^* | \mathbf{x}'^*, \mathbf{m}^*) p(\mathbf{m}^* | \mathbf{x}^*) p(\mathbf{x}'^*) d\mathbf{m}^* d\mathbf{x}^*, \quad (4)$$

where $\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*$ are super-cause, effect and mediator nodes, respectively.

Proof. Since every FDR criterion is just the extension of the standard front-door criterion in terms of merged super-node, the reduced ADMG \mathcal{G}_{X^*, Y^*}^* has the FDR adjustment formula shown in (4) by simple substitution the variables with the super-node sets in eq. (1). In other words, we show the derivation as below:

$$p(\mathbf{y}^* | do(\mathbf{x}^*)) = \int_{\mathbf{m}^*} p(\mathbf{y}^* | do(\mathbf{x}^*), \mathbf{m}^*) p(\mathbf{m}^* | do(\mathbf{x}^*)) d\mathbf{m}^* \quad (5)$$

$$= \int_{\mathbf{m}^*} p(\mathbf{y}^* | do(\mathbf{x}^*), do(\mathbf{m}^*)) p(\mathbf{m}^* | do(\mathbf{x}^*)) d\mathbf{m}^* \quad (6)$$

(Rule 2 iteratively $\forall M_i \in \mathbf{M}^*, (\mathbf{Y}^* \perp_m M_i | \mathbf{X}^* \cup \mathbf{M}_{-i}^*)_{\mathcal{G}_{\overline{\mathbf{X}^*} \setminus M_i}}$, FDR 3)

$$= \int_{\mathbf{m}^*} p(\mathbf{y}^* | do(\mathbf{m}^*)) p(\mathbf{m}^* | \mathbf{x}^*) d\mathbf{m}^* \quad (7)$$

(Rule 3 : $(\mathbf{Y}^* \perp_m \mathbf{X}^* | \mathbf{M}^*)_{\mathcal{G}_{\overline{\mathbf{X}^*}}}$, FDR criterion 1; and Rule 2: $(\mathbf{M}^* \perp_m \mathbf{X}^*)_{\mathcal{G}_{\overline{\mathbf{X}^*}}}$, FDR 2)

$$= \int_{\mathbf{m}^*} p(\mathbf{m}^* | \mathbf{x}^*) \int_{\mathbf{x}'^*} p(\mathbf{y}^* | do(\mathbf{m}^*), \mathbf{x}'^*) p(\mathbf{x}'^* | do(\mathbf{m}^*)) d\mathbf{x}'^* d\mathbf{m}^* \quad (8)$$

$$= \int_{\mathbf{m}^*} p(\mathbf{m}^* | \mathbf{x}^*) \int_{\mathbf{x}'^*} p(\mathbf{y}^* | \mathbf{m}^*, \mathbf{x}'^*) p(\mathbf{x}'^*) d\mathbf{x}'^* d\mathbf{m}^* \quad (9)$$

(Rule 2 iteratively $\forall M_i \in \mathbf{M}^*, (\mathbf{Y}^* \perp_m M_i | \mathbf{X}^* \cup \mathbf{M}_{-i}^*)_{\mathcal{G}_{\overline{\mathbf{X}^*} \setminus M_i}}$, FDR 3;

and Rule3: $(\mathbf{X}^* \perp_m \mathbf{M}^*)_{\mathcal{G}_{\overline{\mathbf{M}^*}}}$, truncated factorization semantics)

□

Theorem 1 shows that, for all possible ADMGs that satisfy FDR criterion, FDR adjustment formula exists, and its expression is given by eq. (4). Next, we prove that if there exists an interventional distribution in

the form of FDR adjustment formula, the corresponding ADMG must satisfy FDR criterion. In other words, satisfaction of FDR criterion is equivalent to applicability FDR adjustment.

Theorem 2. (Equivalence between FDR adjustment and FDR criterion satisfaction). Let $\mathcal{G}(V)$ be an ADMG, X, Y be the cause and effect variables of interest. The following statements are equivalent:

1. **FDR adjustment:** There exists a triple of disjoint and non-empty supersets $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$, such that the interventional distribution $p(\mathbf{y}^* | do(\mathbf{x}^*))$ equals the FDR adjustment formula eq. (4) in \mathcal{G} .
2. **FDR criterion:** There exists a triple of disjoint and non-empty supersets $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$, such that $\mathcal{G}(V)$ is front-door reducible to $\mathcal{G}_{\mathbf{X}^*, \mathbf{Y}^*}^*$.

Proof. To show the equivalence (Statement 1 \iff Statement 2), we prove both directions:

1. Statement 1 is sufficient for Statement 2: Every steps in the derivation eq. (5)-(9) that deleting or replacing a $do(\cdot)$ operator must satisfy the corresponding graphical precondition of the do-calculus rules, e.g., if and only if $(\mathbf{M}^* \perp_m \mathbf{X}^*)_{\mathcal{G}_{\mathbf{X}^*}}$ to replace $p(\mathbf{m}^* | do(\mathbf{x}^*))$ with $p(\mathbf{m}^* | \mathbf{x}^*)$; otherwise, there exists a model compatible with \mathcal{G} under which that equality fails, proved by Lemma 4. Note that the necessity and sufficiency of the manipulation of expressions with $do(\cdot)$ operators and corresponding graphical preconditions in do-calculus rules have been shown in existing literature (Huang and Valtorta, 2012; Pearl, 2009). Let's now consider the key steps in eq. (5)-(9):
 - (a) In eq. (7), we use Rule 2 to replace $p(\mathbf{m}^* | do(\mathbf{x}^*))$ with $p(\mathbf{m}^* | \mathbf{x}^*)$. Then the graph \mathcal{G} must satisfy the graphical precondition $(\mathbf{M}^* \perp_m \mathbf{X}^*)_{\mathcal{G}_{\mathbf{X}^*}}$, which corresponds to FDR 2.
 - (b) In eq. (6) and eq. (9), we use Rule 2 to replace $p(\mathbf{y}^* | do(\mathbf{x}^*), \mathbf{m}^*)$ with $p(\mathbf{y}^* | do(\mathbf{x}^*), do(\mathbf{m}^*))$, and replace $p(\mathbf{y}^* | do(\mathbf{m}^*), \mathbf{x}'^*)$ with $p(\mathbf{y}^* | \mathbf{m}^*, \mathbf{x}'^*)$, respectively. Then the graph \mathcal{G} must satisfy the graphical precondition $\forall M_i \in \mathbf{M}^*, (\mathbf{Y}^* \perp_m M_i | \mathbf{X}^* \cup \mathbf{M}_i^*)_{\mathcal{G}_{\overline{\mathbf{X}^* \cup M_i}}}$, which corresponds to FDR 3.
 - (c) In eq. (7), we use Rule 3 to replace $p(\mathbf{y}^* | do(\mathbf{x}^*), do(\mathbf{m}^*))$ with $p(\mathbf{y}^* | do(\mathbf{m}^*))$. Then the graph \mathcal{G} must satisfy the graphical precondition $(\mathbf{Y}^* \perp_m \mathbf{X}^* | \mathbf{M}^*)_{\mathcal{G}_{\overline{\mathbf{X}^*}}}$, which is equivalent to the FDR condition 1, i.e., $\mathbf{Y}^* \cap \text{De}_{\mathcal{G}(V \setminus \mathbf{M})}(\mathbf{X}^*) = \emptyset$.
 - (d) Additionally, the step of deleting $do(\mathbf{m}^*)$ from $p(\mathbf{x}'^* | do(\mathbf{m}^*))$ as $p(\mathbf{x}'^*)$ in eq. (9) follows directed from truncated factorization. Since the mediator super node must be the descendants of \mathbf{X}^* , intervening on \mathbf{M}^* does not affect \mathbf{X}^* . That means no FDR condition is needed for this deleting operation.

The above proves that if the interventional distribution $p(\mathbf{y}^* | do(\mathbf{x}^*))$ can be written as eq. (4) for \mathcal{G} , then \mathcal{G} must be front-door reducible to $\mathcal{G}_{\mathbf{X}^*, \mathbf{Y}^*}^*$.

2. Statement 1 is necessary for Statement 2: This has been proved in the proof of Theorem 1. □

With the proved necessity and sufficiency, we prove that the two statement are equivalent.

Note that Theorem 2 does not discuss ad-hoc numeric counterexamples, while it rests entirely on the do-calculus rules. Consequently, if any such precondition fails, there exists a model compatible with \mathcal{G} under which that equality breaks; hence eq. (4) cannot serve as a universally valid formula in \mathcal{G} . More importantly, observing eq. (4) which matches $p(\mathbf{y}^* | do(\mathbf{x}^*))$ for a single empirical distribution does not imply that \mathcal{G} relative to cause and effect variables X and Y is front-door reducible to $\mathcal{G}_{\mathbf{X}^*, \mathbf{Y}^*}^*$. The equivalence we establish is graph-level, which means eq. (4) is required to hold for all semi-Markovian causal models with respect to \mathcal{G} , not merely for a specific dataset or parameterization.

3.2 FDR triple

Next, we construct the triple of super-nodes that satisfies the FDR criterion (so we call *FDR triple*) and present its characteristics and corresponding proof. The following definition gives a formal construction for the super-nodes \mathbf{X}^* , \mathbf{Y}^* and \mathbf{M}^* mentioned above searching through a constructive space whose admissible sets are exactly the FDR triples satisfying FDR criterion.

Definition 3. (FDR triple). Let $\mathcal{G}(\mathbf{V})$ be an ADMG, $X, Y \in \mathbf{V}$ be the cause and effect of interest. The super-effect node \mathbf{Y}^* is simply:

$$\mathbf{Y}^* = \{Y\} \quad (10)$$

Define the potential super-cause family \mathcal{X} as a set of candidate super-cause set by:

$$\mathcal{X} = \{\mathbf{S} \subseteq \{X\} \cup (\text{An}_{\mathcal{G}}(\mathbf{Y}^*) \setminus \mathbf{Y}^*) : X \in \mathbf{S}\} \quad (11)$$

For $\mathbf{S}' \in \mathcal{X}$, define the candidate mediator region relative to \mathbf{S}' as $\mathbf{Z}(\mathbf{S}')$ by:

$$\mathbf{Z}(\mathbf{S}') = (\text{An}_{\mathcal{G}}(\mathbf{Y}^*) \cap \text{De}_{\mathcal{G}}(\mathbf{S}')) \setminus (\text{Nb}_{\mathcal{G}}^{\leftrightarrow}(\mathbf{S}') \cup \text{Nb}_{\mathcal{G}}^{\leftrightarrow}(\mathbf{Y}^*) \cup \mathbf{S}' \cup \mathbf{Y}^*) \quad (12)$$

define the candidate super-mediator family relative to \mathbf{S}' as $\mathcal{M}(\mathbf{S}')$ by:

$$\begin{aligned} \mathcal{M}(\mathbf{S}') = & \left\{ \mathbf{M} \subseteq \mathbf{Z}(\mathbf{S}') : \underbrace{\mathbf{Y}^* \cap \text{De}_{\mathcal{G}(\mathbf{V} \setminus \mathbf{M})}(\mathbf{S}') = \emptyset}_{\text{Condition 1}} \wedge \underbrace{(\mathbf{S}' \perp_m \mathbf{M})_{\mathcal{G}_{\underline{\mathbf{S}'}}} \wedge}_{\text{Condition 2}} \right. \\ & \left. \underbrace{\forall M_i \in \mathbf{M} : (\mathbf{Y}^* \perp_m M_i | \mathbf{S}' \cup \mathbf{M}_{-i})_{\mathcal{G}_{\overline{\mathbf{S}' \setminus M_i}}}}_{\text{Condition 3}} \right\} \end{aligned} \quad (13)$$

If there exists a \mathbf{S}' such that it induces a non-empty set $\mathbf{M}' \in \mathcal{M}(\mathbf{S}')$, then we define the super-cause node \mathbf{X}^* and the super-mediator node as \mathbf{M}^* by:

$$\mathbf{X}^* = \mathbf{S}' \quad (14)$$

$$\mathbf{M}^* = \mathbf{M}' \quad (15)$$

We call $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ as the FDR triple of \mathcal{G} relative to (X, Y) . Meanwhile, by construction, $\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*$ are pairwise disjoint.

In the above construction, we fix $\mathbf{Y}^* = \{Y\}$ without any enlargement, though intuitively in some scenarios we may consider to enlarge the super-effect node. For example, in Figure 2 (i), due to the bidirected edge connecting Z and Y , we may consider an enlarged super-effect node $\mathbf{Y}^+ = \{Z, Y\}$ to reduce the ADMG with $\mathbf{M}^* = \{M, V\}$ and $\mathbf{X}^* = \{X, U\}$. In fact, the triple $(\mathbf{X}^* = \{X, U\}, \mathbf{Y}^+ = \{Z, Y\}, \mathbf{M}^* = \{M, V\})$ also hold FDR criterion in Definition 2. Nevertheless, in this example, the triple $(\{X, U\}, \{Y\}, \{M, V\})$ is still sufficient to hold FDR criterion, where (i) \mathbf{M}^* intercepts all directed paths from \mathbf{X}^* to Y ; (ii) there is no backdoor between \mathbf{X}^* and \mathbf{M}^* ; (iii) backdoor paths between Y and M are blocked by conditioning on $\mathbf{X}^* \cup \{V\}$ and backdoor paths between Y and V are blocked by conditioning on $\mathbf{X}^* \cup \{M\}$. Without loss of generality, we have the proposition of effect minimality:

Proposition 1. (Effect minimality). Let $\mathcal{G}(\mathbf{V})$ be an ADMG and (X, Y) be the cause-effect pair of interest. If there exists a super effect node $Y \in \mathbf{Y}^+$ and corresponding triple $(\mathbf{X}^*, \mathbf{Y}^+, \mathbf{M}^*)$ satisfies FDR criterion, then the triple $(\mathbf{X}^*, \mathbf{Y}^* = \{Y\}, \mathbf{M}^*)$ also satisfies FDR criterion. Hence, w.l.o.g., we fix $\mathbf{Y}^* = \{Y\}$.

Proof. This proposition can be proved by reviewing the criterion shown in 2: \square

1. The interception of \mathbf{M}^* is monotone under shrinking the super-effect node, because if \mathbf{M}^* intercepts all directed paths from \mathbf{X}^* to \mathbf{Y}^+ , it also intercepts those from \mathbf{X}^* to \mathbf{Y}^* .
2. If FDR 2 holds for triple $(\mathbf{X}^*, \mathbf{Y}^+, \mathbf{M}^*)$, then it must hold for $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$, because it is independent of the super-effect \mathbf{Y}^* .
3. For all $M_i \in \mathbf{M}^*$, if $(\mathbf{Y}^+ \perp_m M_i | \mathbf{X}^* \cup \mathbf{M}_{-i})_{\mathcal{G}_{\overline{\mathbf{X}^* \cup \mathbf{M}_i}}}$, then $(Y \perp_m M_i | \mathbf{X}^* \cup \mathbf{M}_{-i})_{\mathcal{G}_{\overline{\mathbf{X}^* \cup \mathbf{M}_i}}}$ since $Y \in \mathbf{Y}^+$.

Next, we show the correctness of the construction of the FDR triple $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ in 3 must satisfy FDR criterion.

Theorem 3. (Correctness of the FDR triple construction). If there exist a triple of non-empty supersets $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ constructed as Definition 3 for an ADMG $\mathcal{G}(\mathbf{V})$ with cause and effect of interest X and Y , then $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ must satisfy the FDR criterion.

Proof. Three criteria holds for the FDR triple constructed in Definition 3 since the construction of the candidate super-mediator family relative to \mathbf{S}' as $\mathcal{M}(\mathbf{S}')$ is formulated with the same conditions as criterion 1-3 in Definition 2. Therefore, given an ADMG $\mathcal{G}(\mathbf{V})$, if there exists a FDR triple of non-empty supersets $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ constructed as shown in Definition 3, then the ADMG $\mathcal{G}(\mathbf{V})$ relative to cause and effect variables of interest X and Y is said front-door reducible to $\mathcal{G}_{\mathbf{X}^*, \mathbf{Y}^*}^*$. \square

With the FDR triple correctness, we now have an important corollary to show the applicability of FDR adjustment shown in eq. (4) to an ADMG which is FDR.

Corollary 1. (Applicability of FDR adjustment on the FDR triple). Given an ADMG $\mathcal{G}(\mathbf{V})$ with the cause and effect of interest X and Y , if there exist a triple of non-empty supersets $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ constructed as Definition 3, the interventional distribution $p(\mathbf{Y}^* | do(\mathbf{X}^*))$ is given by eq. (4).

Proof. The construction shown in Definition 3 mainly based on the FDR criterion, i.e., the three conditions are explicitly included when constructing the candidate super-mediator family $\mathcal{M}(\mathbf{S}')$. \square

Meanwhile, the construction of FDR triple shown in Definition 3 also includes all possible triples that makes an ADMG front-door reducible. Before we provide the completeness of such a construction, we show the completeness for the potential super-cause family \mathcal{X} and the candidate mediator region $\mathbf{Z}(\mathbf{S}')$ first.

Lemma 1. (Completeness of the potential super-cause family \mathcal{X}). Given an ADMG $\mathcal{G}(\mathbf{V})$, suppose there exists an triple $(\mathbf{X}^+, \mathbf{Y}^*, \mathbf{M}^+)$ satisfying FDR criterion. Then there exists:

$$\mathbf{X}' = \{X\} \cup (\mathbf{X}^+ \cap (\text{An}(\mathbf{Y}^*) \setminus \mathbf{Y}^*)) \quad (16)$$

Then, $\mathbf{X}' \in \mathcal{X}$, and there exists some $\mathbf{M}' \subseteq \mathbf{M}^+$ such that the triple $(\mathbf{X}', \mathbf{Y}^*, \mathbf{M}')$ is a FDR triple. That is to say, the potential super-cause family \mathcal{X} is complete.

Proof. Since $\mathbf{X}' \subseteq \{X\} \cup (\text{An}(\mathbf{Y}^*) \setminus \mathbf{Y}^*)$, matching the construction of \mathcal{X} in eq. (11), i.e., $\mathbf{X}' \in \mathcal{X}$. Let the shrinking set $\mathbf{S}_0 = \mathbf{X}^+ \setminus \mathbf{X}'$, so $\mathbf{S}_0 \cap \text{An}(\mathbf{Y}^*) = \emptyset$. We next prove shrinking \mathbf{X}^+ to \mathbf{X}' does not break the FDR criterion with possible shrunk \mathbf{M}^+ to a subset \mathbf{M}' :

1. If there exists a directed path $\pi_{X' \rightarrow Y^*}$ that is not intercepted by M^+ , then it is also a directed path $\pi_{X^+ \rightarrow Y^*}$ without interception by M^+ , contradicting the FDR criterion for (X^+, Y^*, M^+) . In other words, there is no directed path $\pi_{S_0 \rightarrow Y^*}$, which means removing S_0 cannot create a new directed path connecting X' with Y^* . This proves that shrinking X^+ to X' does not break FDR 1.
2. If there is a m-connected path in $\mathcal{G}_{\underline{X'}}$ with outgoing arcs from S_0 , the path must contain $S'_0 \in S_0$ as a non-collider with an outgoing arc toward M^+ . That means $S'_0 \in \text{An}(M') \subseteq \text{An}(Y^*)$, contradicting $S_0 \cap \text{An}(Y^*) = \emptyset$. This proves that shrinking X^+ to X' does not break FDR 2.
3. Shrinking X^+ to X' only removes incoming edges to the super-cause node fewer in the ADMG $\mathcal{G}_{\overline{X^+} \underline{M_i^+}}$ for all $M_i^+ \in M^+$, which does not create a new backdoor path between $\forall M_i^+ \in M^+$ and Y^* . This proves that shrinking X^+ to X' does not break FDR 3.

□

Lemma 1 comes with the conclusion that only the nodes that can be potentially merged into the super-cause node in a given ADMG $\mathcal{G}(V)$ will be included in \mathcal{X} . For example, in Figure 2 (b), $\mathcal{X} = \{\{X\}, \{X, M\}\}$. In other words, $\nexists S \in \mathcal{X}$ such that $Y \in S$ or $U \in S$ or $V \in S$. Intuitively, we don't consider V and U potentially in the super-cause node because the path $(X \leftrightarrow V \leftarrow U \leftarrow Y)$ is naturally blocked by the collider V , and including any of U or V (or both) in the super-cause node will break the FDR criterion.

Lemma 2. (Completeness of the candidate mediator region $Z(S')$). Given an ADMG $\mathcal{G}(V)$, let $S' \in \mathcal{X}$ as Definition 3. Suppose there exists $M^+ \subseteq V$ such that the triple (S', Y^*, M^+) satisfies FDR criterion. Then there exists:

$$M' \subseteq Z(S') = (\text{An}_{\mathcal{G}}(Y^*) \cap \text{Deg}(S')) \setminus (\text{Nb}_{\mathcal{G}}^{\leftrightarrow}(S') \cup \text{Nb}_{\mathcal{G}}^{\leftrightarrow}(Y^*) \cup S' \cup Y^*), \quad (17)$$

such that an FDR triple (S', Y^*, M') also satisfies FDR criterion. That is to say, the candidate mediator region $Z(S')$ is **complete**.

Proof. Start from any feasible M^+ , the pruning involves three aspects without breaking the FDR criterion:

1. Restriction to the interior region $I = \text{An}_{\mathcal{G}}(Y^*) \cap \text{Deg}(S') \setminus (S' \cup Y^*)$: I is then the set of all interior nodes on directed paths $\pi_{S' \rightarrow Y^*}$. If M^+ is a feasible super-mediator that satisfies FDR criterion, then for any $M_i^+ \in M^+$ such that $M_i^+ \notin I$, M_i^+ cannot be on any directed path of $\pi_{S' \rightarrow Y^*}$. So, any node $V_i \notin I$ can be safely excluded.
2. Removal of the first bidirected-arc-connected neighbours of S' ($\text{Nb}_{\mathcal{G}}^{\leftrightarrow}(S')$): If there exists $M_i^+ \in M^+ \cap \text{Nb}_{\mathcal{G}}^{\leftrightarrow}(S')$, then there must exist a m-connected path $(M_i^+ \leftrightarrow S')$ in the modified ADMG $\mathcal{G}_{\underline{S'}}$, contradicting FDR 2. So, removing $\text{Nb}_{\mathcal{G}}^{\leftrightarrow}(S')$ is safe.
3. Removal of the first bidirected-arc-connected neighbours of Y^* ($\text{Nb}_{\mathcal{G}}^{\leftrightarrow}(Y^*)$): If there exists $M_i^+ \in M^+ \cap \text{Nb}_{\mathcal{G}}^{\leftrightarrow}(Y^*)$, then there must exist a m-connected path $(M_i^+ \leftrightarrow Y^*)$ in the modified ADMG $\mathcal{G}_{\overline{X^+} \underline{M_i^+}}$ regardless of conditioning on $S' \cup M_{-i}$, contradicting FDR 3. So, removing $\text{Nb}_{\mathcal{G}}^{\leftrightarrow}(Y^*)$ is safe.

□

Lemma 2 comes with the conclusion that only the nodes that can be potentially merged into the super-mediator node in a given ADMG $\mathcal{G}(V)$ will be included in $Z(S')$. For example, in Figure 2 (g), given $S' = \{X\}$, the candidate mediator region relative to S' is $Z(S') = \{M\}$. In other words, $V \notin Z(S')$ or $U \notin Z(S')$. Intuitively, adjusting on a super-mediator node with any (or both) of them open the backdoor path between Y

and the actual mediator M when conditioning on X in $\mathcal{G}_{\overline{X}M}$, and also open the backdoor path between X and M in \mathcal{G}_X .

Theorem 4. (*Completeness of FDR triple for FDR criterion*). *Given an ADMG $\mathcal{G}(V)$ with at least one directed path $\pi_{X \rightarrow Y}$ and the cause and effect of interest $X, Y \in V$, if there does **not** exist a FDR triple of non-empty sets $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ constructed as stated in the Definition 3, such that:*

$$\begin{aligned} \emptyset \neq \mathbf{X}^* &\in \mathcal{X}, \text{ and} \\ \mathbf{Y}^* &= \{Y\}, \text{ and} \\ \emptyset \neq \mathbf{M}^* &\subseteq \mathbf{Z}(\mathbf{X}^*), \end{aligned} \tag{18}$$

*then the $\mathcal{G}(V)$ is **not** front-door reducible. Equivalently, the construction of Definition 3 is complete for the FDR criterion.*

Proof. By Lemma 1, we obtain $\mathbf{X}^* \in \mathcal{X}$ with necessary and harmless shrinking. By Proposition 1, we proved the minimal super-effect is $\mathbf{Y}^* = \{Y\}$. By Lemma 2, we can prune mediators to some $\mathbf{M}^* \subseteq \mathbf{Z}(\mathbf{X}^*)$ while preserving FDR criterion. \square

Corollary 2. (*The irrelevance of FDR for strict descendants of Y*). *Let $\mathcal{G}(V)$ be an ADMG with cause and effect variables of interest X, Y . Let \mathcal{G}^- be the induced subgraph on $\{X\} \cup (\text{Ang}_{\mathcal{G}}(\mathbf{Y}^*))$. We use term **strict descendants** of \mathbf{Y}^* here to refer $V_i \in \text{Deg}_{\mathcal{G}}(\mathbf{Y}^*) \setminus \mathbf{Y}^*$. Then:*

1. \mathcal{G} relative to cause and effect variables X and Y is front-door reducible iff. \mathcal{G}^- relative to cause and effect variables X and Y is front-door reducible.
2. All admissible FDR triples as constructed in Definition 3 $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ in \mathcal{G} are also all admissible FDR triples in \mathcal{G}^- .

In other words, $\text{Deg}_{\mathcal{G}}(Y)$ is irrelevant to that whether an ADMG satisfies FDR or not, and deleting or adding descendant nodes of Y does not change the admissible set of FDR triples.

Proof. By Definition 3, the potential super-cause family \mathcal{X} only contains the region $\mathbf{R} = \{X\} \cup (\text{Ang}_{\mathcal{G}}(\mathbf{Y}^*) \setminus \mathbf{Y}^*)$ as shown in eq. (11), and the candidate mediator region \mathbf{Z} also contains the region at most $\text{Ang}_{\mathcal{G}}(\mathbf{Y}^*)$. That means the construction of an admissible FDR triple is solely on the region $\mathbf{R} \cup \mathbf{Y}^*$. Meanwhile, the FDR criterion stated as Definition 2 shows that all m-separation tests are for nodes in the region $\mathbf{R} \cup \mathbf{Y}^*$. If there exists a strict descendant V_i such that attempts to traverse to the region \mathbf{R} from \mathbf{Y}^* , it either breaks the fundamental acyclicity for ADMG or hits a collider on the path, which is naturally blocked without conditioning on the collider or its strict descendants. That proves \mathcal{G} and \mathcal{G}^- is equivalent for FDR criterion and FDR triple. \square

Corollary 2 is very useful to simplify the ADMG, for example, the descendants of Y in Figure 2 (b), (d) and (g) can be removed harmlessly for FDR criterion test and FDR triple construction. In Figure 2 (d), V is the only strict descendant of Y , and it is obvious that the path $(Y \rightarrow V \leftrightarrow M)$ is blocked by the collider V itself. Removing it from the ADMG does not affect FDR criterion test and also the construction of an admissible FDR triple, i.e., the only admissible FDR triple is given by $(\mathbf{X}^* = \{X\}, \mathbf{Y}^* = \{Y\}, \mathbf{M}^* = \{M\})$.

3.3 Non-FDR examples

Figure 3 demonstrates some examples which do not satisfy FDR criterion. Let's check one of the failure example as shown in Figure 3 (b). Using the construction in Definition 3, the super-effect node $\mathbf{Y}^* = \{Y\}$. In the potential super-cause family \mathcal{X} has 8 elements (sets), it is easy to figure out none of them $\mathbf{S}' \in \mathcal{X} = \{\mathbf{S} \subseteq \{X, U, M, V\} : X \in \mathbf{S}\}$ can lead to a non-empty $\mathbf{M}^* \in \mathbf{M}(\mathbf{S}')$. Let's explore the two candidate super-cause nodes that seems most likely to satisfy FDR, i.e., (i) $\mathbf{S}'_1 = \{X\}$; and (ii) $\mathbf{S}'_2 = \{X, V\}$:

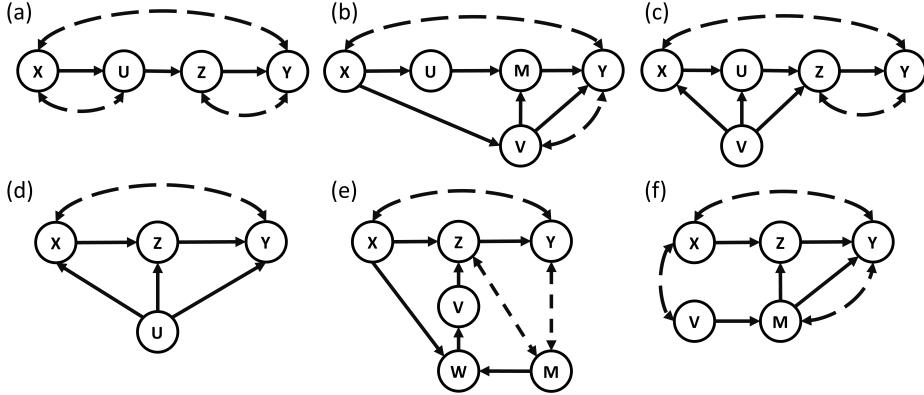


Figure 3: Example ADMGs relative to the cause and effect of interest X, Y not satisfying FDR criterion.

1. $S'_1 = \{X\}$: Using eq. (12), we have $Z(S'_1) = \{U, M\}$. Then, it is obvious that adjusting either solely on U, M or on their joint, there exists a directed path $(X \rightarrow V \rightarrow Y)$. This breaks the condition 1 in 13, which it leads to a candidate super-mediator family with only an empty set, i.e., $\mathcal{M}(S'_1) = \{\emptyset\}$.
2. $S'_2 = \{X, V\}$: Using eq. (12), we have $Z(S'_2) = \{M\}$. There is a directed path $V \rightarrow Y$ which cannot be intercepted by M . That also breaks the condition 1, so $\mathcal{M}(S'_2) = \{\emptyset\}$.

However, non-FDR does not imply an causal relation of interest is not identifiable. For example, $p(y | do(x))$ is still identifiable using ID algorithm (Shpitser and Pearl, 2006) for Figure 3 (c), (d) and (f), but they are non-FDR. As a simple example, the interventional distribution $p(y | do(x))$ for the ADMG shown in Figure 3 (c) can be written as:

$$p(y | do(x)) = \int_{u, v, x'} p(y | u, v, x') p(u | v, x) p(v, x'), \quad (19)$$

which cannot be reduce to a FDR adjustment formula by neither trivial simplification nor grouping a few variables as a super-node.

4 Admissible FDR triple identification algorithm

Given the above discussion and formal definition with proved properties of FDR and FDR triple, we now attempt to solve the FDR triple identification problem if an ADMG is FDR. Based on aforementioned properties and FDR triple definition, we propose admissible FDR triple identification (FDR-TID) algorithm as shown in Algorithm 1.

Before we show the correctness and completeness of the proposed FDR-TID algorithm, we give the following example to show the operation steps of the algorithm. Consider the ADMG $\mathcal{G}(\mathbf{V})$ in Figure 2 (h), where we want to identify a FDR triple for the cause and effect variable pair (X, Y) .

Line 1 fixes $\mathbf{Y}^* = \{Y\}$, and $\mathcal{G}(\mathbf{V})$ cannot be shrunk in line 2 since $\mathbf{V} = \{X\} \cup \text{Ang}_{\mathcal{G}}(\mathbf{Y}^*)$, hence $\mathcal{G}^- = \mathcal{G}$. When the algorithm invokes the outer loop for \mathbf{S}' in line 3, let's start with $\mathbf{S}' = \{X\} \subseteq \{X\} \cup \text{Ang}_{\mathcal{G}}(\mathbf{Y}^*)$. Line 5 gives the candidate mediator region relative to $\mathbf{S}' = \{X\}$: $Z_{\mathbf{S}'} = \{U, M, V\}$. Again in the inner loop for \mathbf{M}' , let's start from $\mathbf{M}' = \{U\} \subseteq Z_{\mathbf{S}'}$ without loss of generality. Here, $\text{De}_{\mathcal{G}^-}(\mathbf{V}_{\mathcal{G}^- \setminus \mathbf{M}'})(\mathbf{S}') = \{M, X, Y, V\}$, so that $\mathbf{Y}^* \cap \text{De}_{\mathcal{G}^-}(\mathbf{V}_{\mathcal{G}^- \setminus \{U\}})(X) = \{Y\} \neq \emptyset$. Condition 1 check does not pass, which means U fails to intercept all directed path from \mathbf{S}' to \mathbf{Y}^* . Then the algorithm goes to the next iteration by selecting $\mathbf{M}' = \{M\} \subseteq Z_{\mathbf{S}'}$. In this case, $\text{De}_{\mathcal{G}^-}(\mathbf{V}_{\mathcal{G}^- \setminus \{M\}})(X) = \{X, V, U\}$, and $\mathbf{Y}^* \cap \text{De}_{\mathcal{G}^-}(\mathbf{V}_{\mathcal{G}^- \setminus \{M\}})(X) = \emptyset$. The algorithm then goes to check condition 2 after condition 1 check passed. After deleting all outgoing arcs from X , it is clear that there is no backdoor paths between \mathbf{S}' and \mathbf{M}' , such that $(X \perp_m M)_{\mathcal{G}_X^-}$, passing

Algorithm 1 Admissible FDR triple identification (FDR-TID)

Input: An ADMG $\mathcal{G}(\mathbf{V})$, the cause and effect variable pair of interest (X, Y) .
Output: FDR triple $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ or FAIL

```

1: Fix:  $\mathbf{Y}^* = \{Y\}$                                 ▷ Proposition 1
2: Shrink to subgraph:  $\mathcal{G}^- = \mathcal{G}(\{X\} \cup \text{Ang}_{\mathcal{G}}(\mathbf{Y}^*))$           ▷ Corollary 2
3: for  $S' \subseteq \{X\} \cup (\text{Ang}_{\mathcal{G}^-}(\mathbf{Y}^*) \setminus \mathbf{Y}^*)$  s.t.  $X \in S'$  do          ▷ Lemma 1
4:   Identify interior nodes:  $I = (\text{Ang}_{\mathcal{G}^-}(\mathbf{Y}^*) \cap \text{De}_{\mathcal{G}^-}(S')) \setminus (S' \cup \mathbf{Y}^*)$ 
5:   Identify the candidate mediator region relative to  $S'$ :
        $Z(S') = I \setminus (\text{Nb}_{\mathcal{G}^-}^\leftrightarrow(S') \cup \text{Nb}_{\mathcal{G}^-}^\leftrightarrow(\mathbf{Y}^*))$           ▷ Lemma 2
6:   for  $M' \subseteq Z(S')$  do
7:     if  $\mathbf{Y}^* \cap \text{De}_{\mathcal{G}^-}(V_{\mathcal{G}^- \setminus M'}) (S') \neq \emptyset$  then
8:       continue                                ▷ If FDR 1 fails to hold, skip to the next  $M'$  search.
9:     end if
10:    if  $(S' \not\perp_m M')_{\mathcal{G}_{S'}^-}$  then
11:      continue                                ▷ If FDR 2 fails to hold, skip to the next  $M'$  search.
12:    end if
13:    if  $\exists M_i \in M' : (\mathbf{Y}^* \not\perp_m M_i | S' \cup M_{-i})_{\mathcal{G}_{S' \setminus M_i}^-}$  then
14:      continue                                ▷ If FDR 3 fails to hold, skip to the next  $M'$  search.
15:    end if
16:    return  $(X^* = S', Y^* = \{Y\}, M^* = M')$ 
17:  end for
end for
return FAIL

```

condition 2 check. Line 13 for condition 3 check is not triggered, since there is no backdoor path between Y and M after conditioning on X in the modified graph $\mathcal{G}_{\overline{X}\underline{M}}^-$, hence $(Y \perp_m M | X)_{\mathcal{G}_{\overline{X}\underline{M}}^-}$. Finally, the FDR-TID returns an admissible FDR triple $(\{X\}, \{Y\}, \{M\})$.

It is worth mentioning that triple $(\{X\}, \{Y\}, \{M\})$ is not the only admissible FDR triple. In fact, triples such as $(\{X\}, \{Y\}, \{M, V\})$, $(\{X, V\}, \{Y\}, \{M, U\})$, $(\{X, V, U\}, \{Y\}, \{M\})$, etc., are also admissible FDR triples (also as shown in Table 1). The above demonstration just considers the simplest construction leading to FDR for the given ADMG, whereas this does not affect the correctness and completeness of the proposed FDR-TID. We next give formal discussion about the algorithm's correctness and completeness.

Theorem 5. (Correctness of FDR-TID). *If FDR-TID returns a triple $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$, then it must be a FDR triple as defined in Definition 3.*

Proof. FDR-TID follows the constructions in Definition 3 whose correctness has been proved in Theorem 3. The algorithm enumerates every pairs (S', M') (line 3 and 6), and check the three conditions in eq. (13) (line 7-15), equivalent to the defined construction of the FDR triple. \square

Next, we provide proof of FDR-TID termination and analyse its time complexity.

Lemma 3. (FDR-TID finite search space and termination). *Let $\mathcal{G}(\mathbf{V})$ be a finite ADMG, X, Y be the cause and effect variables of interest. Denote the subgraph $\mathcal{G}^- = \mathcal{G}(\{X\} \cup \text{Ang}_{\mathcal{G}}(\mathbf{Y}^*))$. The algorithm FDR-TID always has finite search space and terminates.*

Proof. The search space for \mathbf{X}^* and \mathbf{M}^* are finite, and each condition check in the FDR-TID algorithm must terminate in finite round:

1. Finite \mathbf{X}^* -search. Note that \mathbf{X}^* takes the first admissible supersets $S' \ni X$ if \mathcal{G} is FDR. FDR-TID enumerates S' inside $\text{Ang}_{\mathcal{G}}(\mathbf{Y}^*) \setminus \mathbf{Y}^*$, where the size of search space is $2^{|\text{Ang}_{\mathcal{G}}(\mathbf{Y}^*) \setminus (\mathbf{Y}^* \cup \{X\})|}$, hence finite.
2. Finite \mathbf{M}^* -search per S' . For each S' , the candidate mediator region $Z_{S'}$ is a finite set as shown in line 5 (also in (12)). FDR-TID enumerates nonempty $M' \subseteq Z_{S'}$, whose size of search space is $2^{|Z_{S'}|} - 1 < \infty$.
3. Each condition check must terminate. For every pair of (S', M') , the three condition checks are done by a finite number of graph operations, e.g., edge deletions, m-separation queries on an ADMG. Each such graph operation run in time polynomial in $|V| + |E|$, where $|E|$ is the total number of arcs in an ADMG.

Therefore, FDR-TID always terminates no later than the first admissible FDR triple is found. \square

With Lemma 3, we can conclude that a crude worst-case bound for the time complexity of FDR-TID is that:

$$T(n) \leq \sum_{S' \subseteq \{X\} \cup (\text{Ang}_{\mathcal{G}} - (\mathbf{Y}^*) \setminus \mathbf{Y}^*)} (2^{|Z_{S'}|} - 1) \kappa T_{\text{op}}(|V|, |E|), \quad (20)$$

where $\kappa > 0$ is a constant bounding the number of primitive calls per condition check, and $T_{\text{op}}(|V|, |E|)$ is a uniform upper bound on the running time of any single primitive graph operation on an ADMG with $|V|$ nodes and $|E|$ arcs. Now we can show the completeness of FDR-TID algorithm.

Theorem 6. (Completeness of FDR-TID). *If there exists an FDR triple for a given ADMG $\mathcal{G}(V)$ and the cause and effect variable pair of interest (X, Y) , then FDR-TID returns an admissible FDR triple $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$ within finite time.*

Proof. Lemma 3 has shown its termination property under finite search steps. The algorithm incrementally enumerates all possible pair (S', M') and returns when the first admissible FDR triple rather than gives all possible FDR triple for the sake of time complexity, which does not affect its completeness. \square

Table 1 shows all admissible FDR triples for ADMGs shown in Figure 2 with respect to the cause and effect variable pair of interest (X, Y) . Every admissible FDR triple guarantees that the interventional distribution $p(\mathbf{y}^* | do(\mathbf{x}^*))$ in the reduced ADMG \mathcal{G}^* can be expressed by eq. (4). This shows that the FDR criterion together with the FDR-TID algorithm provide a more efficient way to obtain an interpretable, easy-to-estimate and practical interventional distribution expression under a few well-defined criteria, rather than conducting a sophisticated and computationally costly ID algorithm to obtain an equivalent but impractical and unestimatable expression such as shown in eq. (2).

5 Conclusion

This paper proposes and systematically studies the graphical criterion of *front-door reducibility* (FDR) to obtain a closed-form FDR adjustment formula where the standard front-door adjustment makes unclear conclusion. Our main conclusion are as follows:

1. We formalized the FDR criterion by three FDR conditions (FDR 1-3) on super-cause, effect and mediator nodes $(\mathbf{X}^*, \mathbf{Y}^*, \mathbf{M}^*)$.
2. We proved that FDR is both sufficient and necessary for the applicability of an FDR adjustment functional, establishing a graph-level equivalence.
3. We proposed FDR-TID, an exact, terminating algorithm that finds an admissible triple when it exists.

These results show that a surprisingly broad family of ADMGs can have simple, easy-to-estimate adjustment formula, offering a practical alternative to general causal relation identification methods, e.g., ID algorithm, often produce algebraically complex identification formula.

ADMG	\mathbf{X}^*	\mathbf{Y}^*	\mathbf{M}^*
(a)	$\{X\}$ $\{X, V\}$	$\{Y\}$	$\{M\}$ $\{M, Z\}$
(b)	$\{X\}$	$\{Y\}$	$\{M\}$
(c)	$\{X\}$ $\{X, Z\}$	$\{Y\}$	$\{M\}$
(d)	$\{X\}$ $\{X, V\}$	$\{Y\}$	$\{M\}$
(e)	$\{X\}$ $\{X, V\}$	$\{Y\}$	$\{M\}, \{U, M\}$ $\{U\}$
(f)	$\{X\}$	$\{Y\}$	$\{M\}, \{V, M\}$
(g)	$\{X\}$	$\{Y\}$	$\{M\}$
(h)	$\{X\}$ $\{X, V\}$ $\{X, V, U\}$	$\{Y\}$	$\{M\}, \{U, V\}, \{V, M\}, \{U, M\}, \{V, U, M\}$ $\{M\}, \{U, M\}$ $\{M\}$
(i)	$\{X, U\}$	$\{Y\}$	$\{M\}$

Table 1: All admissible FDR triples for ADMGs shown in Figure 2 with respect to the cause and effect variable pair of interest (X, Y) , where the interventional distribution $p(\mathbf{y}^* | do(\mathbf{x}^*))$ can be expressed by eq. (4).

Overall, FDR criterion and adjustment together with the FDR-TID algorithm provides an implementable identification method for a board family of ADMGs which are originally out of the scope of standard front-door adjustment. The future work can proceed in two folds: (i) develop more efficient and robust estimators (e.g., doubly robust estimators) for ADMGs satisfying FDR criterion. (ii) integrate FDR-TID into general identification methods, such that returning FDR adjustment formula if possible and falling back to ID expression only when the reduction is impossible.

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A Appendix of fundamental causal inference concepts

In this section, we present some of the most important concepts and definitions commonly used for causal inference. For example, as a fundamental confounding scenario, we give the definition of backdoor criterion as follow:

Definition 4. (Backdoor criterion (Pearl, 2009)). Given cause variable X , effect variable Y and a set of variables U , the causal effect from the cause X to the effect Y is identifiable if:

1. No variable in U is a descendant of X ; and
2. U blocks every path between X and Y that contains an arrow into X .

And another important concept is the so-called C-component (a.k.a. district), as defined as follows:

Definition 5. (C-component (Pearl, 2009)). Let \mathcal{G} be a ADMG such that a subset of its bidirected edges froms a spanning tree over all nodes in \mathcal{G} . Then the nodes V in \mathcal{G} form a *C-component* (a.k.a. *district*). If $\mathcal{G}(V)$ is not a C-component, it can be uniquely partitioned into a set $\mathcal{C}_{\mathcal{G}}$ of subgraphs, each a maximal C-component.

B Appendix of *do*-calculus rules

The *do*-calculus (Pearl, 1995) consists of three rules that can be used to transform expresssions involving $do(\cdot)$ operators into other expression of this type, whenever certain graphical preconditions hold in the causal diagram (ADMG) \mathcal{G} . We give the definition of the three rules of *do*-calculus as follow:

Definition 6. (Three rule of *do*-calculus (Pearl, 1995)). The following three rules are valid for every interventional distribution compatible with any ADMG \mathcal{G} :

- Rule 1 (Insertion/deletion of observations):

$$p(y | do(x), z, w) = p(y | do(x), w) \text{ if } (Y \perp_m Z | X, W)_{\mathcal{G}_{\overline{X}}} \quad (21)$$

- Rule 2 (Intervention/observation exchange):

$$p(y | do(x), do(z), w) = p(y | do(x), z, w) \text{ if } (Y \perp_m Z | X, W)_{\mathcal{G}_{\overline{X}Z}} \quad (22)$$

- Rule 3 (Insertion/deletion of Interventions):

$$p(y | do(x), do(z), w) = p(y | do(x), w) \text{ if } (Y \perp_m Z | X, W)_{\mathcal{G}_{\overline{XZ(W)}}}, \quad (23)$$

where $Z(W)$ is the set of Z -nodes that are not ancestors of any W -node in $\mathcal{G}_{\overline{X}}$.

As a preparation to prove the equivalence between FDR adjustment and FDR criterion satisfaction so as in Theorem 2, we need to show a necessity lemma for these *do*-calculus rules asserting that the corresponding graphical preconditions are required for the equalities to hold for all models compatible with \mathcal{G} .

Lemma 4. (*The necessity of do-calculus rules*). *For disjoint Y, Z, X, W if:*

1. $p(y|do(x), z, w) = p(y|do(x), w)$ holds for all models compatible with \mathcal{G} , then $(Y \perp_m Z | X, W)_{\mathcal{G}_{\bar{X}}}$.
2. $p(y|do(x), do(z), w) = p(y|do(x), z, w)$ holds for all models compatible with \mathcal{G} , then $(Y \perp_m Z | X, W)_{\mathcal{G}_{\bar{X}Z}}$.
3. $p(y|do(x), do(z), w) = p(y|do(x), w)$ holds for all models compatible with \mathcal{G} , then $(Y \perp_m Z | X, W)_{\mathcal{G}_{\bar{X}Z(W)}}$.

Proof. The proof set up with two standard facts:

1. The existence of faithful distributions for ADMGs: if Z and W are not m-separated by U in a modified ADMG \mathcal{G}' (e.g. $\mathcal{G}_{\bar{X}}$, $\mathcal{G}_{\bar{X}Z}$ and $\mathcal{G}_{\bar{X}Z(W)}$), there exists a distribution Q compatible with \mathcal{G}' such that $Z \not\perp_m W | U$ under Q .
2. The truncated factorization: for any Q compatible with a modified ADMG $\mathcal{G}'(V)$, there exists a structural causal model (SCM) \mathcal{M} compatible with \mathcal{G} whose post-intervention distribution under $do(S)$ is $Q = p_{\mathcal{M}}(\mathbf{V} | do(S))$.

And so we are able to argue by contraposition using the two facts above for the three rules:

1. Rule 1: If $(Y \not\perp_m Z | X, W)_{\mathcal{G}_{\bar{X}}}$, pick Q compatible with $\mathcal{G}_{\bar{X}}$ where $(Y \not\perp_m Z | X, W)_{\mathcal{G}_{\bar{X}}}$, and lift to a SCM \mathcal{M} compatible with \mathcal{G}' , then $p_{\mathcal{M}}(y|do(x), z, w) \neq p_{\mathcal{M}}(y|do(x), w)$. This contradicts the universal validity, which implies $(Y \perp_m Z | X, W)_{\mathcal{G}_{\bar{X}}}$.
2. Rule 2: If $(Y \not\perp_m Z | X, W)_{\mathcal{G}_{\bar{X}Z}}$, pick Q compatible with $\mathcal{G}_{\bar{X}Z}$, and lift to a SCM \mathcal{M} compatible with \mathcal{G}' , then $p(y|do(x), do(z), w) \neq p(y|do(x), z, w)$. That also contradicts the universal validity, which implies $(Y \perp_m Z | X, W)_{\mathcal{G}_{\bar{X}Z}}$.
3. Rule 3: If $(Y \not\perp_m Z | X, W)_{\mathcal{G}_{\bar{X}Z(W)}}$, pick Q compatible with $\mathcal{G}_{\bar{X}Z(W)}$, and lift to a SCM \mathcal{M} , then $p(y|do(x), do(z), w) \neq p(y|do(x), w)$. That again contradicts the universal validity, which implies $(Y \perp_m Z | X, W)_{\mathcal{G}_{\bar{X}Z(W)}}$.

□