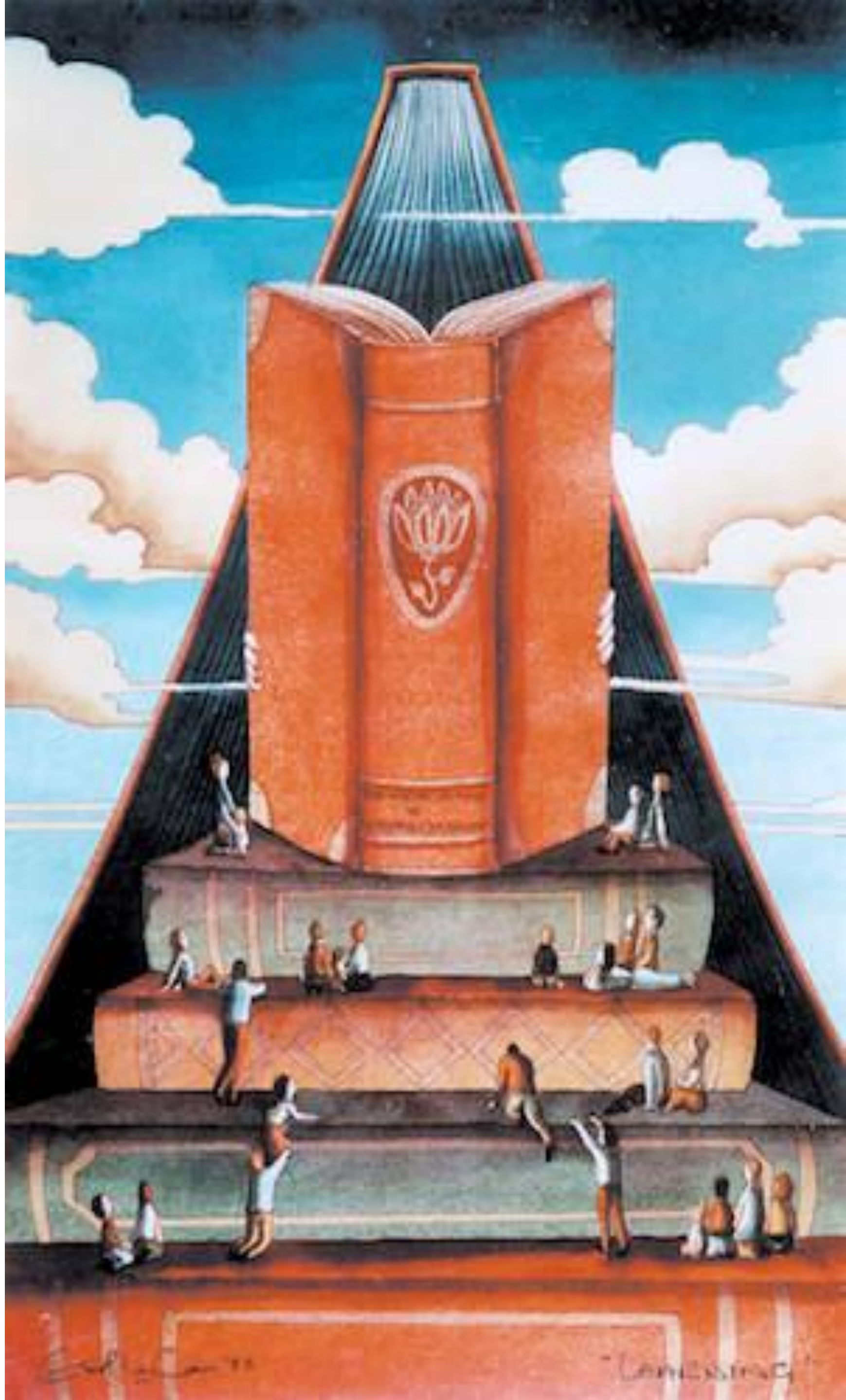
GREEDY ALGORITHM



Outlines

- Introduction
- Knapsack Problem
- Job Sequencing with Deadlines
- Minimum-Cost Spanning Trees
  - Prim’s Algorithm
  - Kruskal’s Algorithm
- Single-Source Shortest Paths
  - Dijkstra’s Algorithm

Introduction

- A greedy algorithm always makes the choice that looks best at the moment.
- That is, it makes a locally optimal choice in the hope that this choice will lead to a globally optimal solution.
- The greedy approach does not always lead to an optimal solution.
- The problems that have a greedy solution are said to posses the greedy-choice property.
- The greedy approach is also used in the context of hard (difficult to solve) problems in order to generate an approximate solution.

Knapsack Problem

- We are given n objects and a knapsack or bag
- Object i has a weight w_i and the knapsack has a capacity W
- If a fraction x_i, $1 \le x_i \le 1$, of object i is placed into the knapsack, the a profit of $p_i x_i$ is earned.
- The objective is to obtain a filling of the knapsack that maximizes the total profit earned. Thus, the problem is as follows:

$$\left.\begin{aligned} &\max \textstyle\sum_{i=1}^{n} p_i x_i \\ &\text{s.t. } \textstyle\sum_{i=1}^{n} w_i x_i \le W \\ &\quad 0 \le x_i \le 1, 1 \le i \le n \end{aligned}\right\}$$

A feasible solution is any set $(x_1,\ldots,x_n)$ satisfying the constraints. An optimal solution is a feasible solution for which the objective is maximized

Knapsack Problem (Cont..)

- Example: Consider the instance of a knapsack problem: n=3, W=20, $(p_1,p_2,p_3)=(25,24,15)$ and $(w_1,w_2,w_3)=(18,15,10)$. Four feasible solutions are:

	(x_1, x_2, x_3)	$\sum w_i x_i$	$\sum p_i x_i$
1.	(1/2, 1/3, 1/4)	16.5	24.25
2.	(1, 2/15, 0)	20	28.2
3.	(0, 2/3, 1)	20	31
4.	(0, 1, 1/2)	20	31.5

- Of these four feasible solutions, solution 4 yields the maximum profit.

Knapsack Problem (Cont..)

```
Greedy_Knapsack(W, n)
    for i = 1 to n do
        x[i] = 0
    end for
    M = W;
    for i = 1 to n do
        if w[i] > M then
            break;
        end if
        x[i] = 1.0; M = M – w[i]
    end for
    if i ≤ n then x[i] = M/w[i] end if
End  Greedy_Knapsack
```

//p[1:n] and w[1:n] contain the profit and weights respectively of the n objects s.t. p[i]/w[i]≥ p[i+1]/w[i+1]. W is the knapsack size and x[1:n] is the solution vector

Job Sequencing with Deadlines

- We are given a set J of n jobs
- Job i is associated with an integer deadline $d_i \geq 0$ and a profit $p_i > 0$
- P_i is earned iff the job is completed by its deadline
- To complete a job, one has to process the job on a machine for one unit of time and one machine is available for processing jobs
- The feasible solution for this problem is $J' \subseteq J$ that can be completed within its deadline
- The value is $\sum_{i \in J'} p_i$
- The optimal solution is a feasible solution with maximum value

Job Sequencing with Deadlines (Cont..)

- Example: let n=4, $(p_1,p_2,p_3,p_4)=(100,10,15,27)$ and $(d_1,d_2,d_3,d_4)=(2,1,2,1)$. The feasible solution s and their values:

	feasible solution	processing sequence	value
1.	(1, 2)	2, 1	110
2.	(1, 3)	1, 3 or 3, 1	115
3.	(1, 4)	4, 1	127
4.	(2, 3)	2, 3	25
5.	(3, 4)	4, 3	42
6.	(1)	1	100
7.	(2)	2	10
8.	(3)	3	15
9.	(4)	4	27

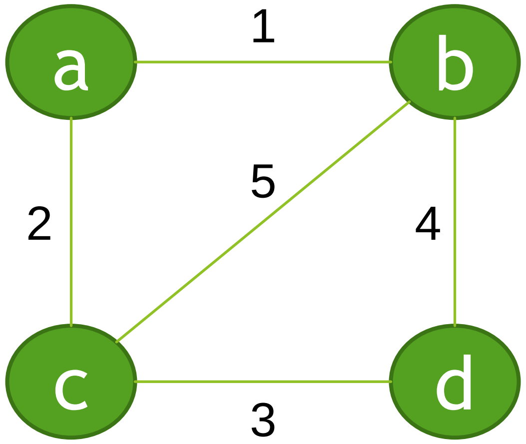
Solution 3 is optimal

Job Sequencing with Deadlines (Cont..)

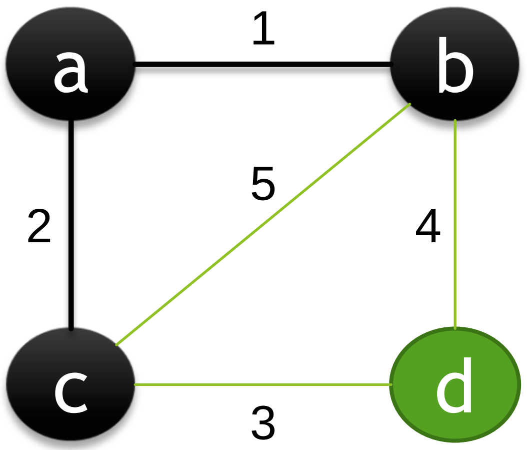
```
Algorithm JS(d, j, n)
// d[i] ≥ 1, 1 ≤ i ≤ n are the deadlines, n ≥ 1. The jobs
// are ordered such that p[1] ≥ p[2] ≥ ··· ≥ p[n]. J[i]
// is the ith job in the optimal solution, 1 ≤ i ≤ k.
// Also, at termination d[J[i]] ≤ d[J[i + 1]], 1 ≤ i < k.
{
    d[0] := J[0] := 0; // Initialize.
    J[1] := 1; // Include job 1.
    k := 1;
    for i := 2 to n do
    {
        // Consider jobs in nonincreasing order of p[i]. Find
        // position for i and check feasibility of insertion.
        r := k;
        while ((d[J[r]] > d[i]) and (d[J[r]] ≠ r)) do r := r − 1;
        if ((d[J[r]] ≤ d[i]) and (d[i] > r)) then
        {
            // Insert i into J[ ].
            for q := k to (r + 1) step −1 do J[q + 1] := J[q];
            J[r + 1] := i; k := k + 1;
        }
    }
    return k;
}
```

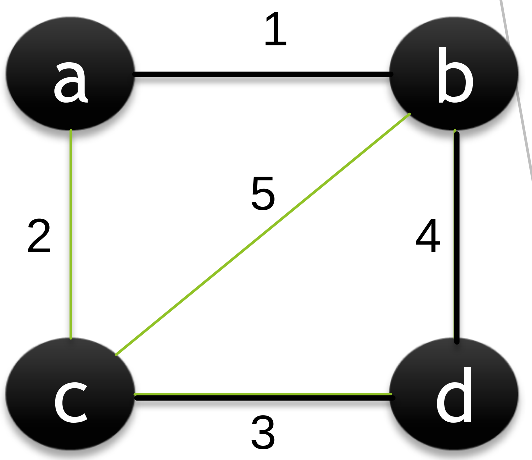
Minimum cost spanning tree (MCST)

- Tree: No cycles; equivalently, for each pair of nodes u and v, there is only one path from u to v
- Spanning: Contains every node in the graph
- Minimum cost: Smallest possible total weight of any spanning tree



MCST (Cont..)



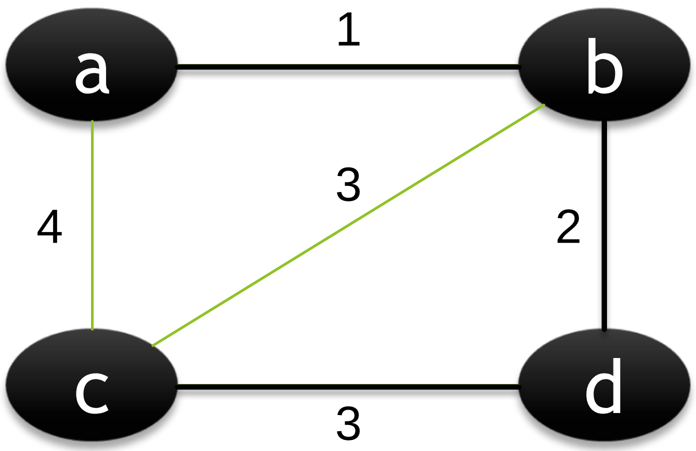


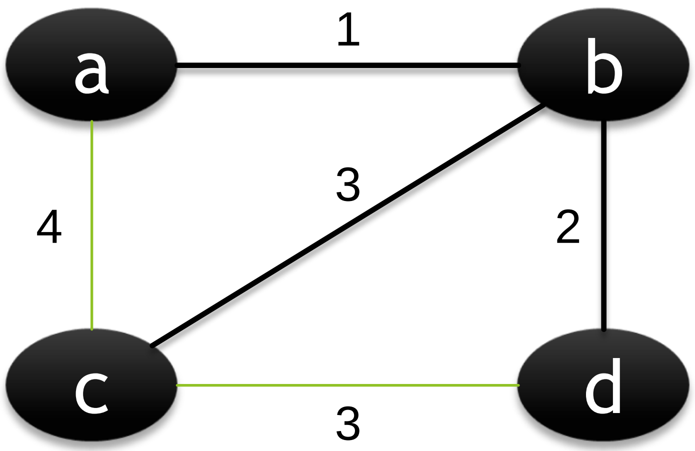
- Black edges and nodes are in T. Is T a MCST?
- Not spanning; d is not in T.

- Black edges and nodes are in T. Is T a MCST?
- Not minimum cost; can swap edges 4 and 2

MCST (Cont..)

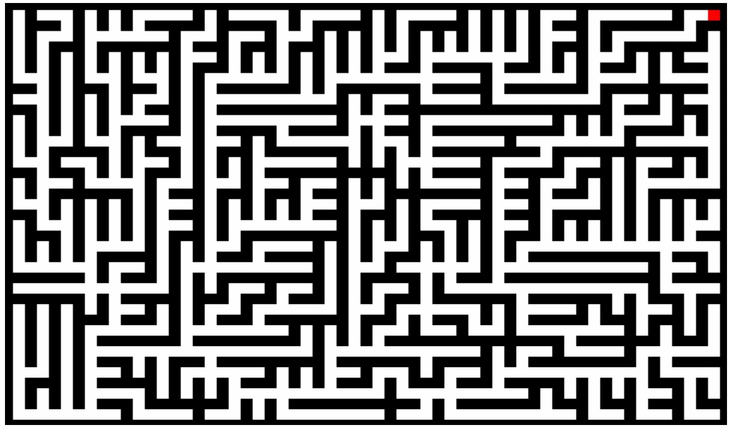
- Which edges form a MCST?





Application of MCST

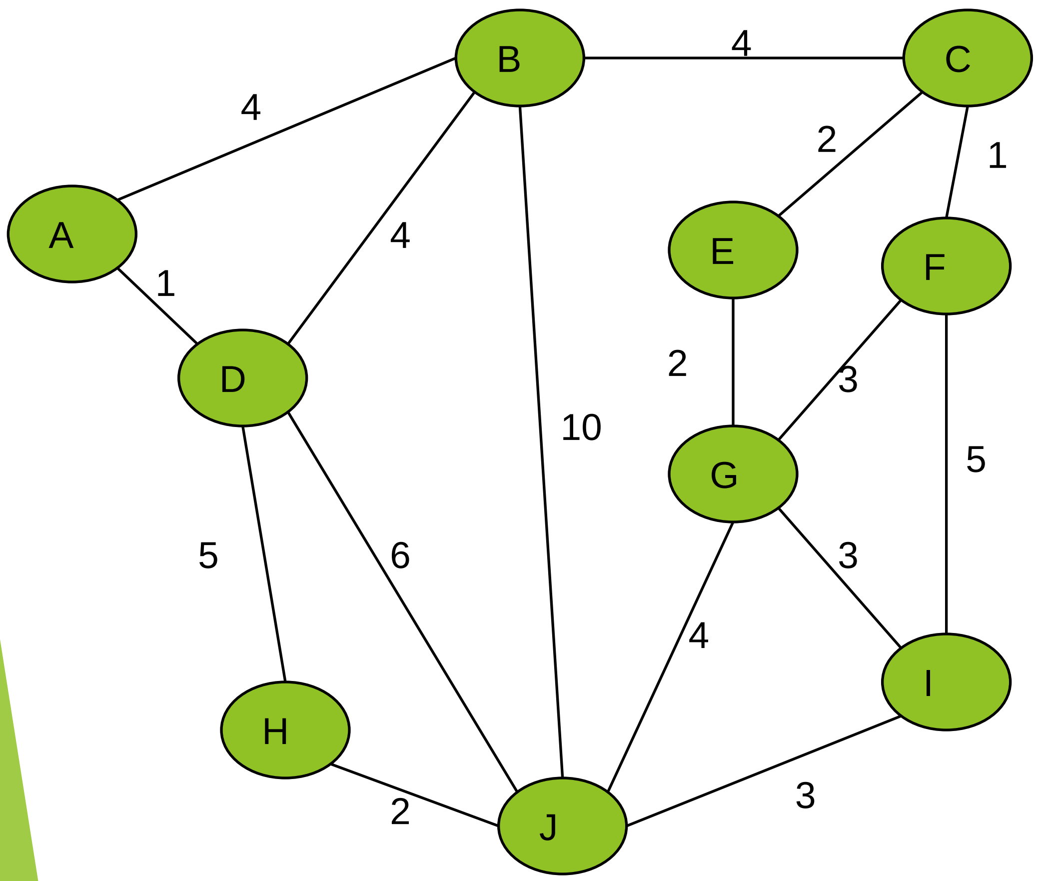
- Electronic circuit designs
- Planning how to lay network cable to connect several locations to the internet
- Planning how to efficiently bounce data from router to router to reach its internet destination
- Creating a 2D maze (to print on cereal boxes, etc.)



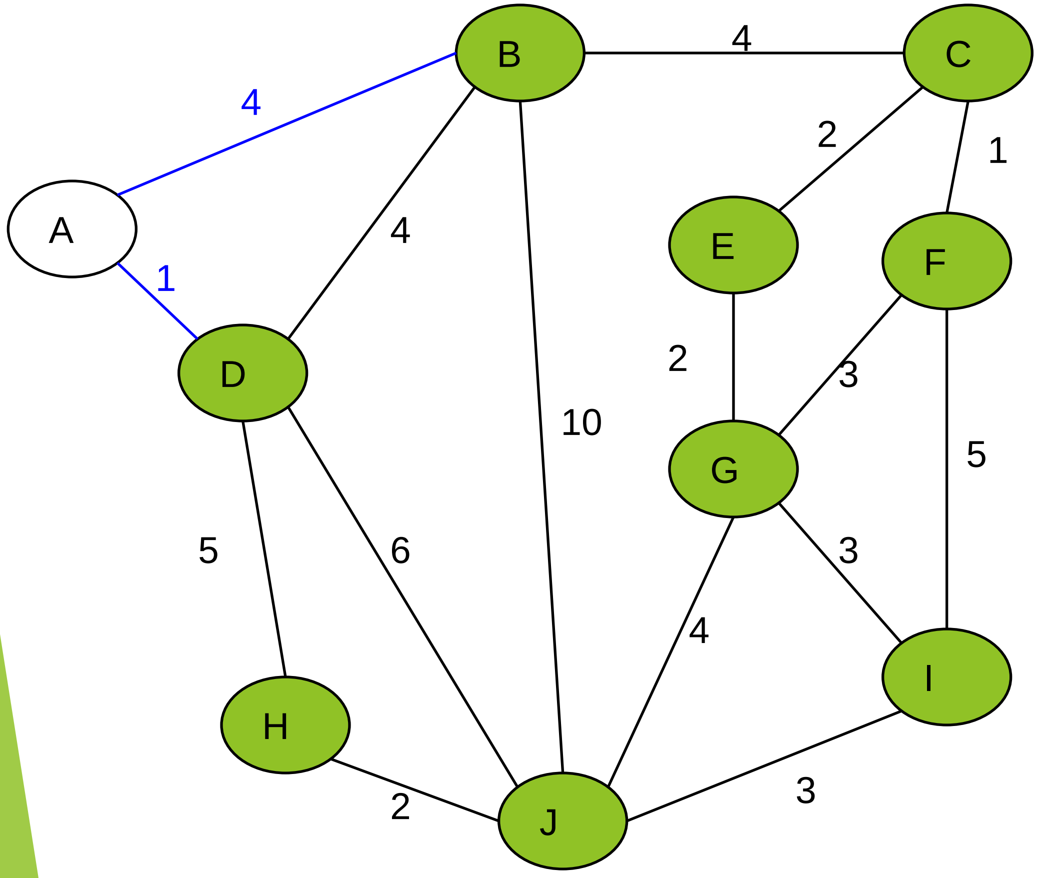
Prim's Algorithm

- Prim’s algorithm takes a graph *G=(V, E)* and builds an MCST *T*
- Major steps of the algorithm:
  - Pick an arbitrary node r from V
  - Add r to T
  - While T contains < |V| nodes
    - Find a minimum weight edge (u, v) where $u \in T$ and $v \notin T$
    - Add node v to T

Complete Graph



Old Graph



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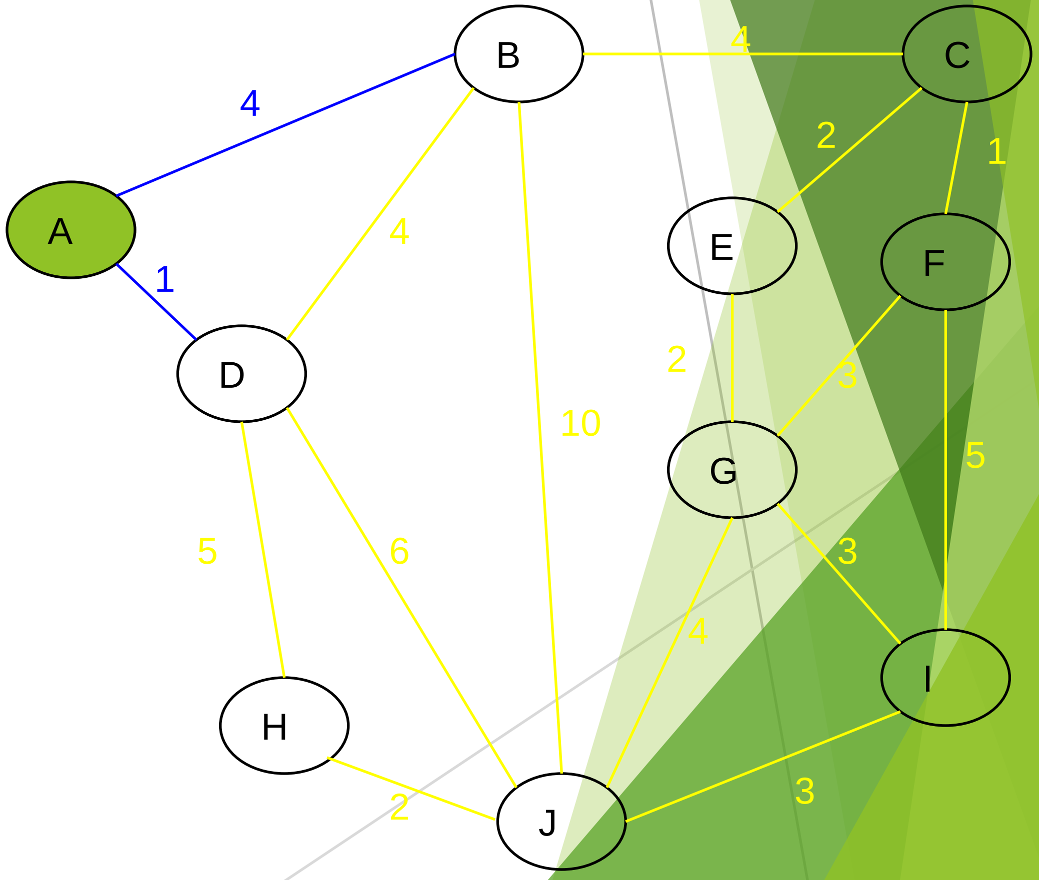

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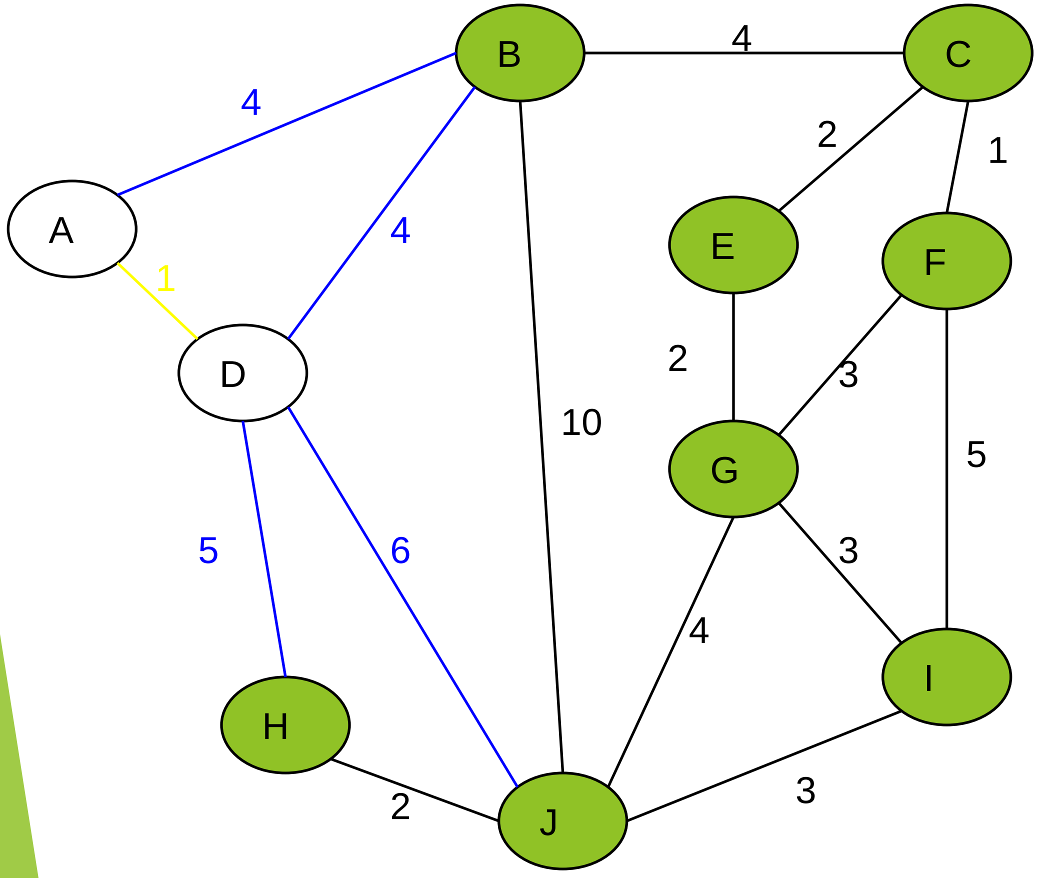

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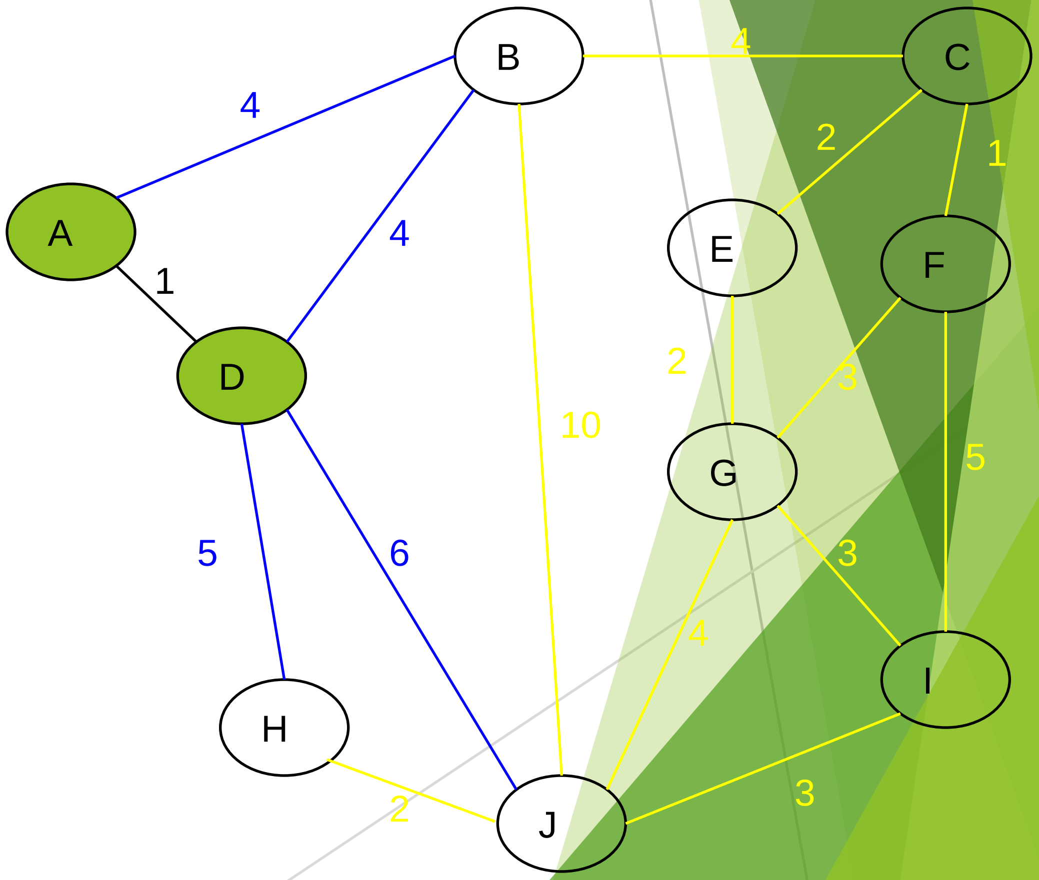

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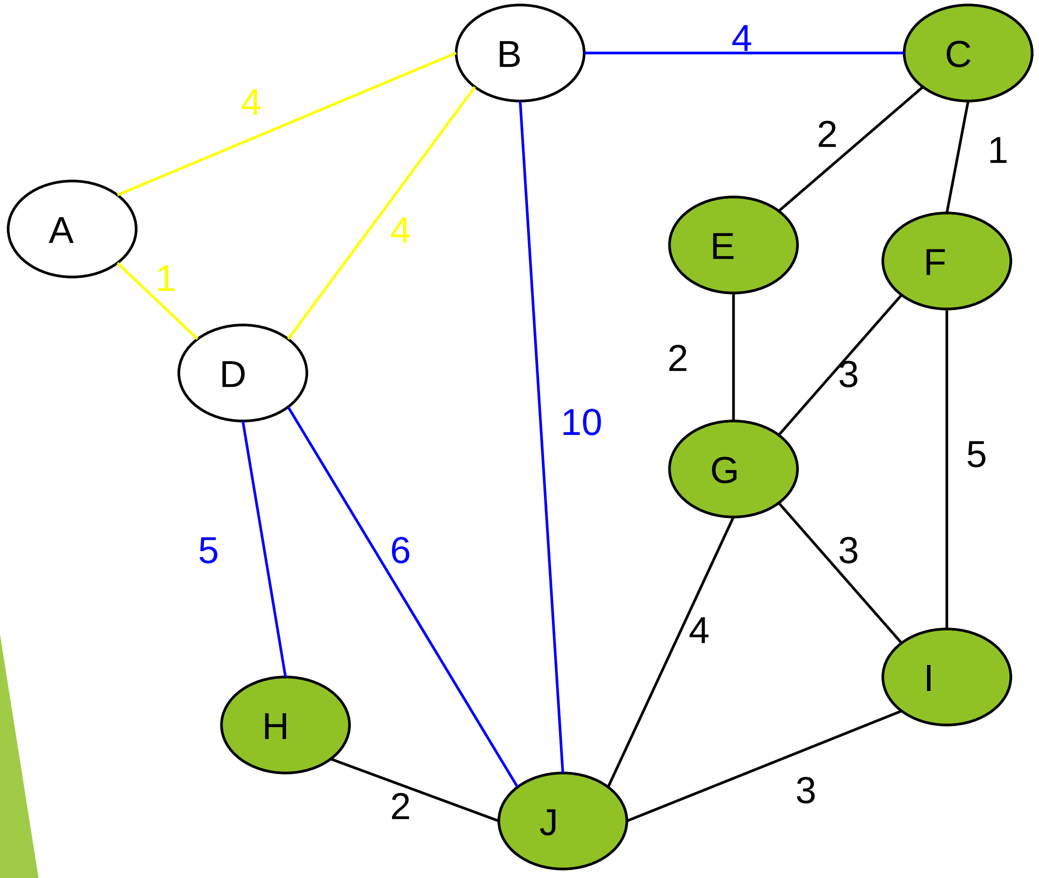

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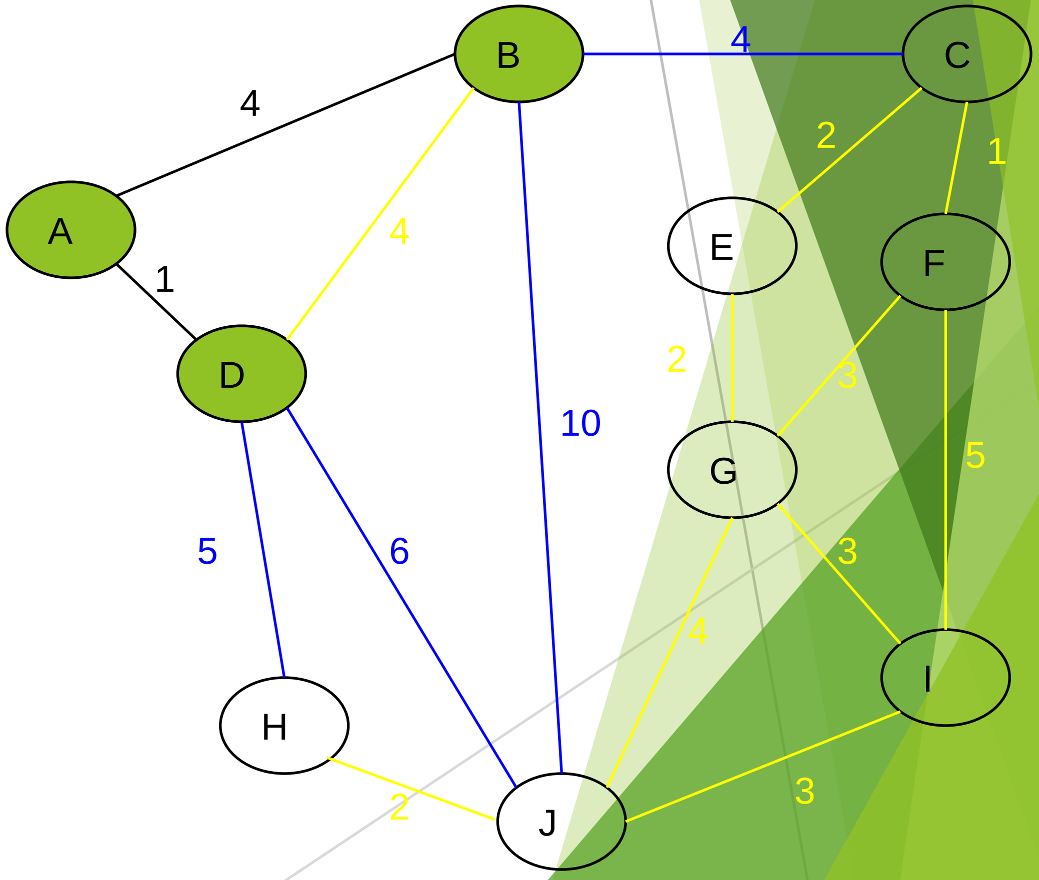

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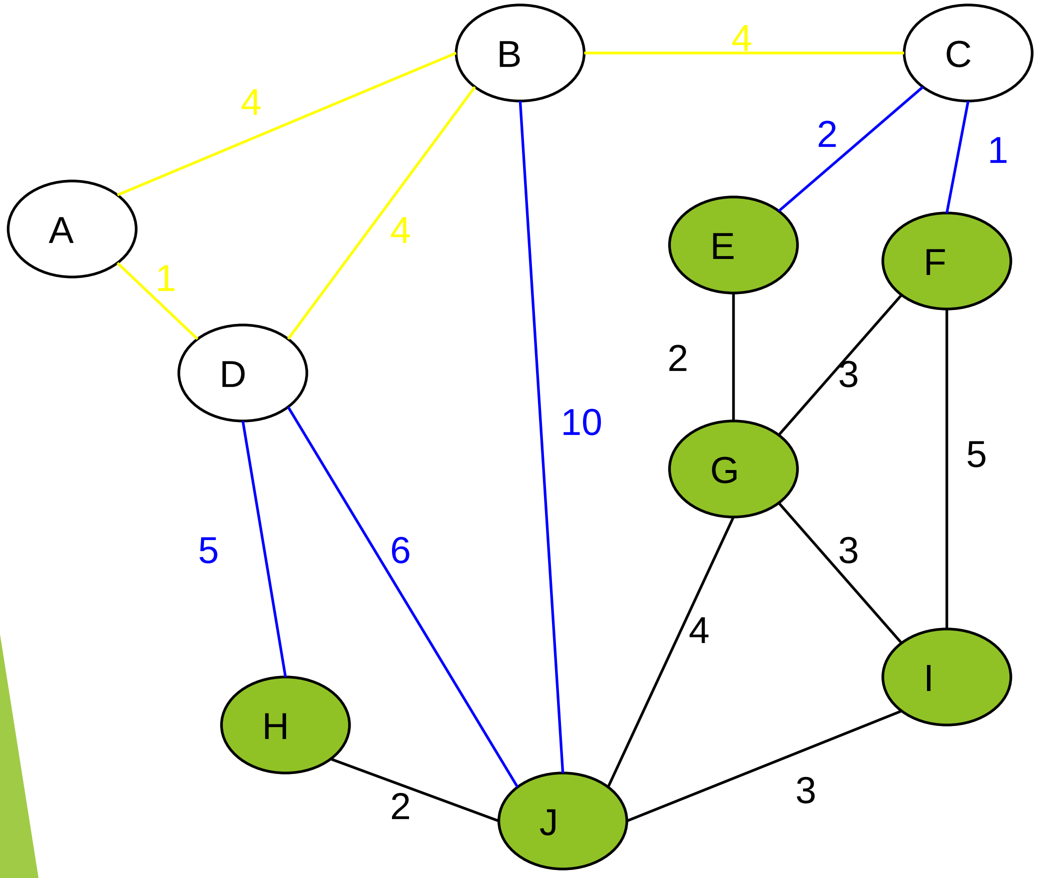

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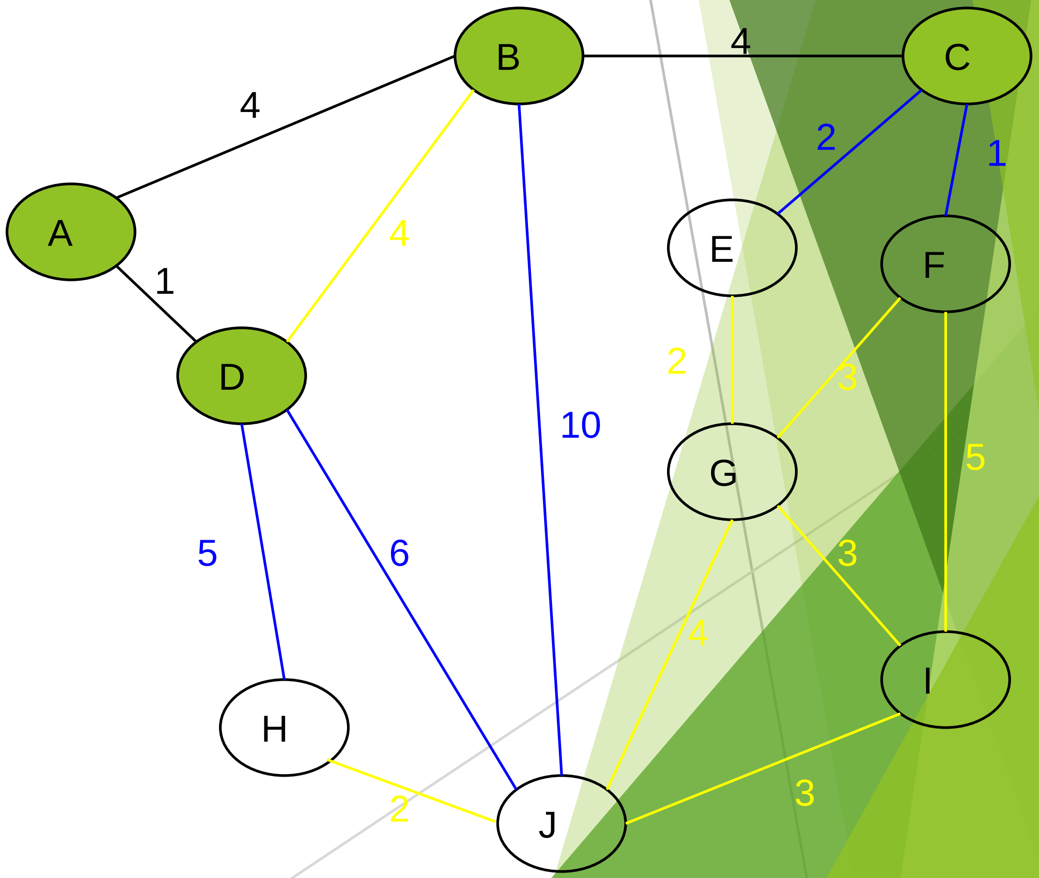

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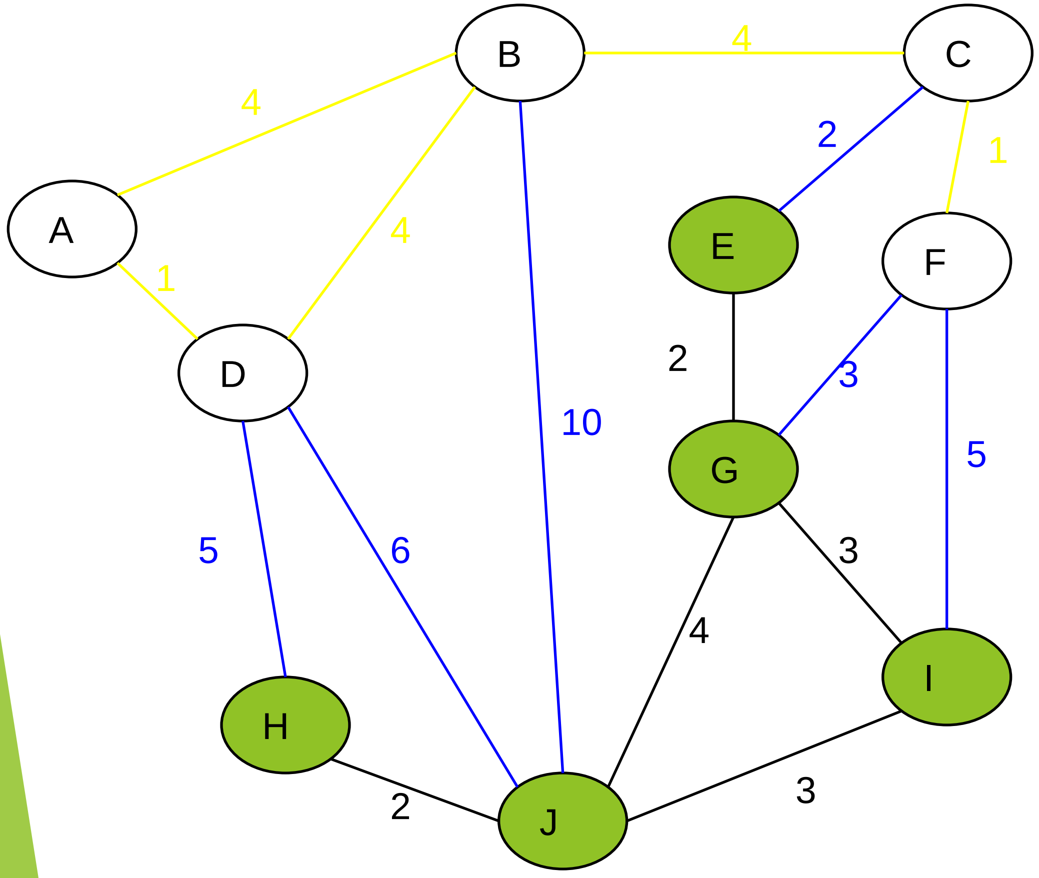

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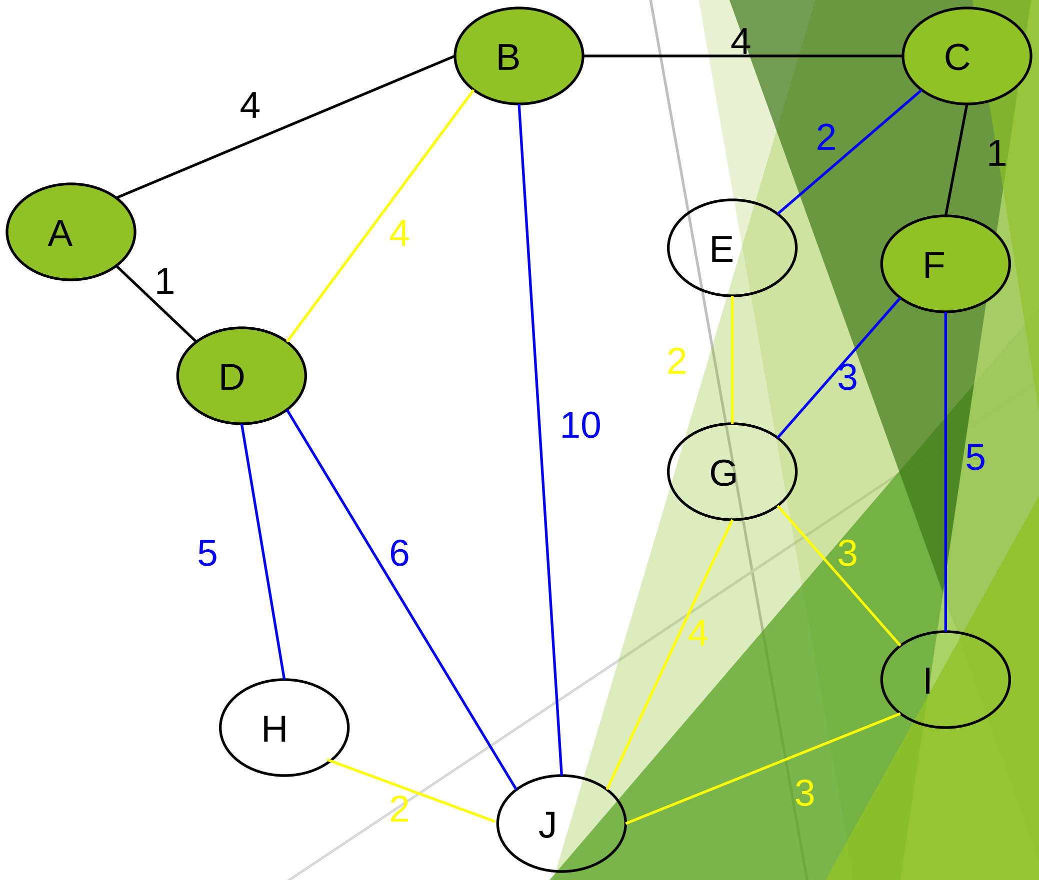

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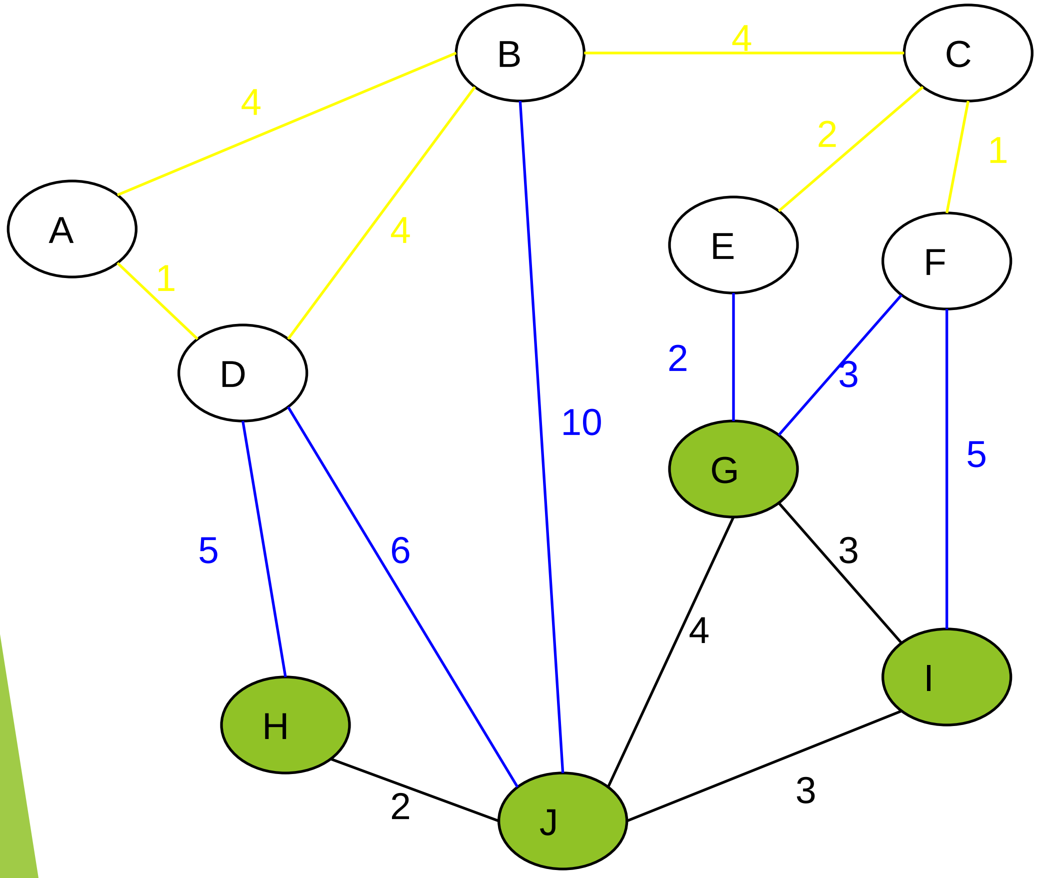

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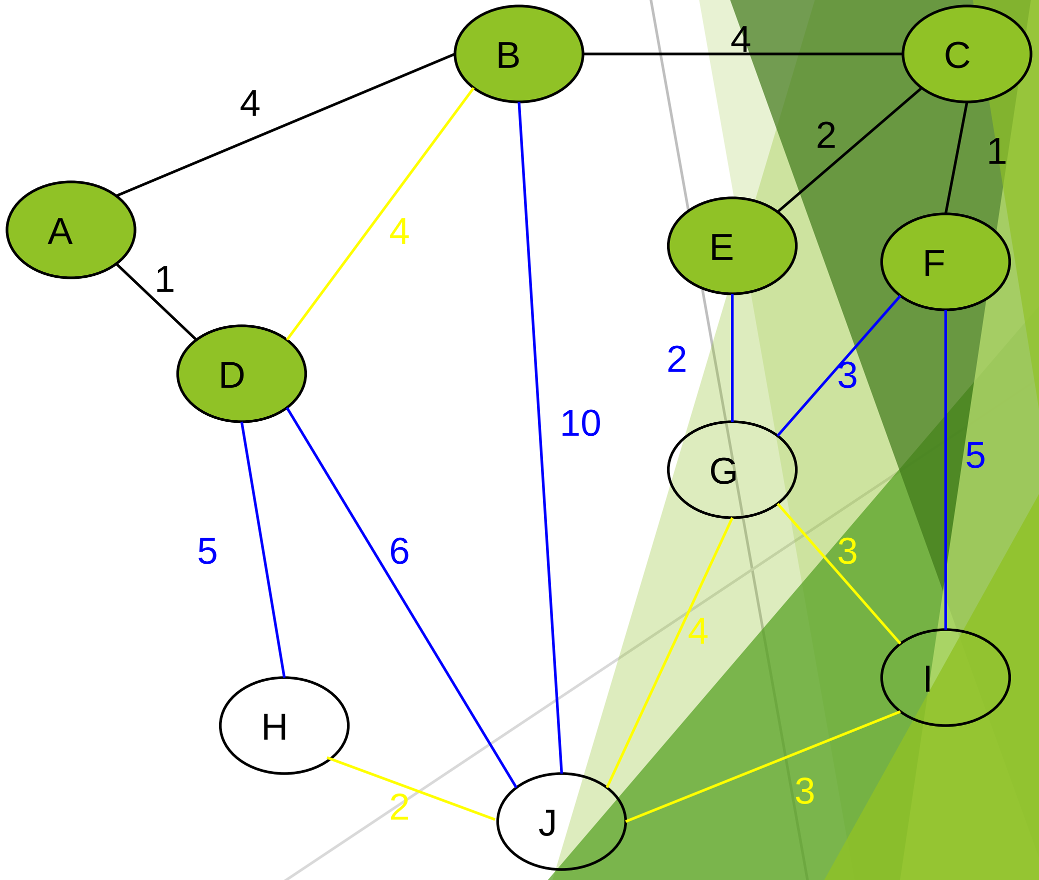

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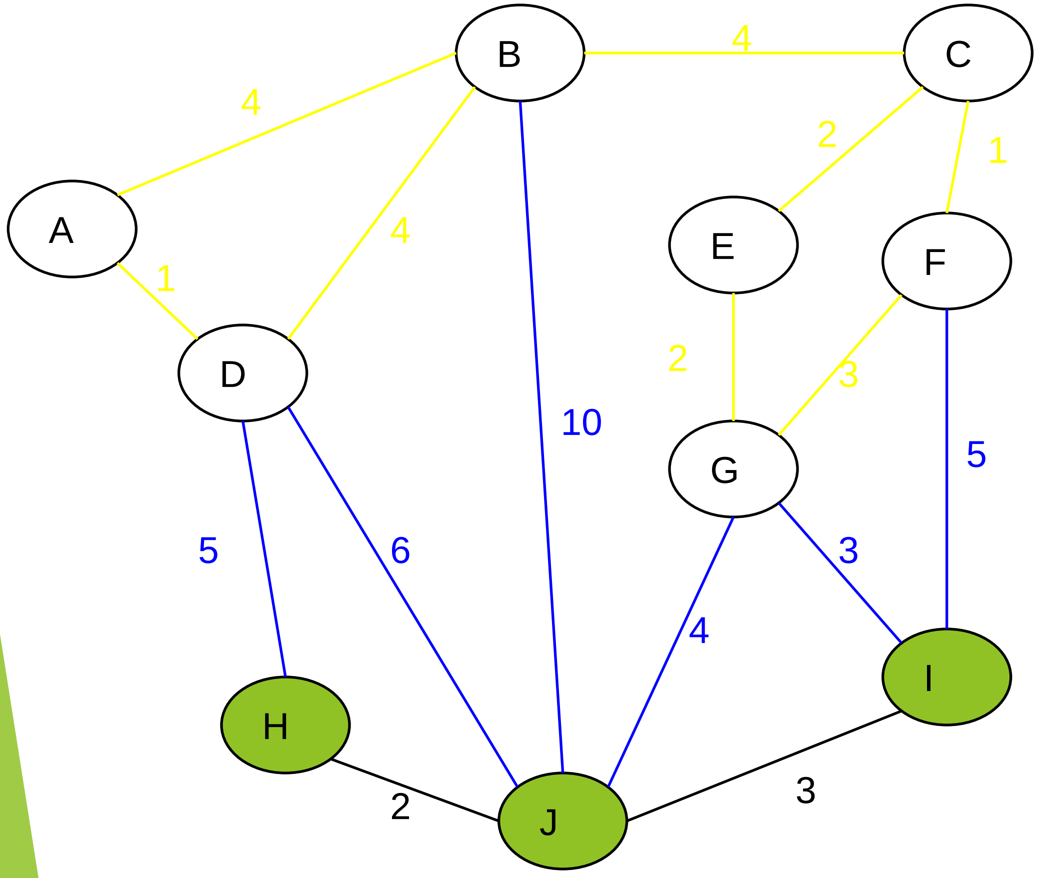

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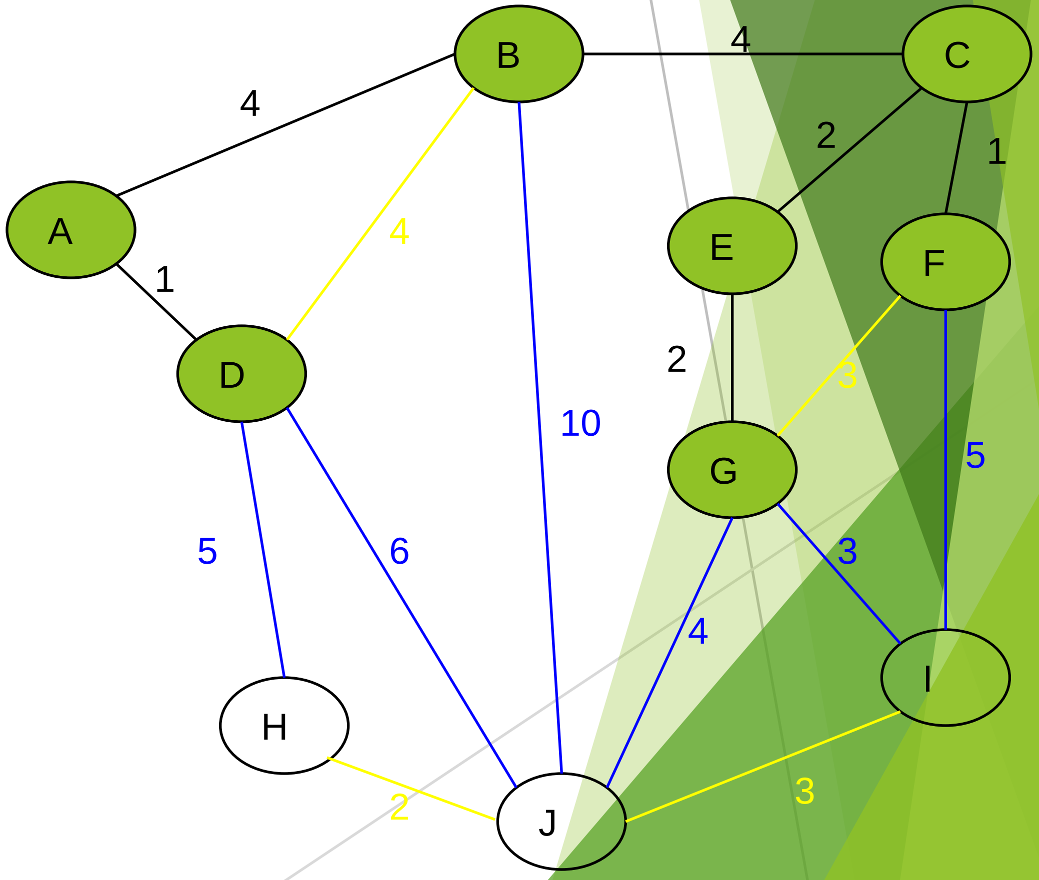

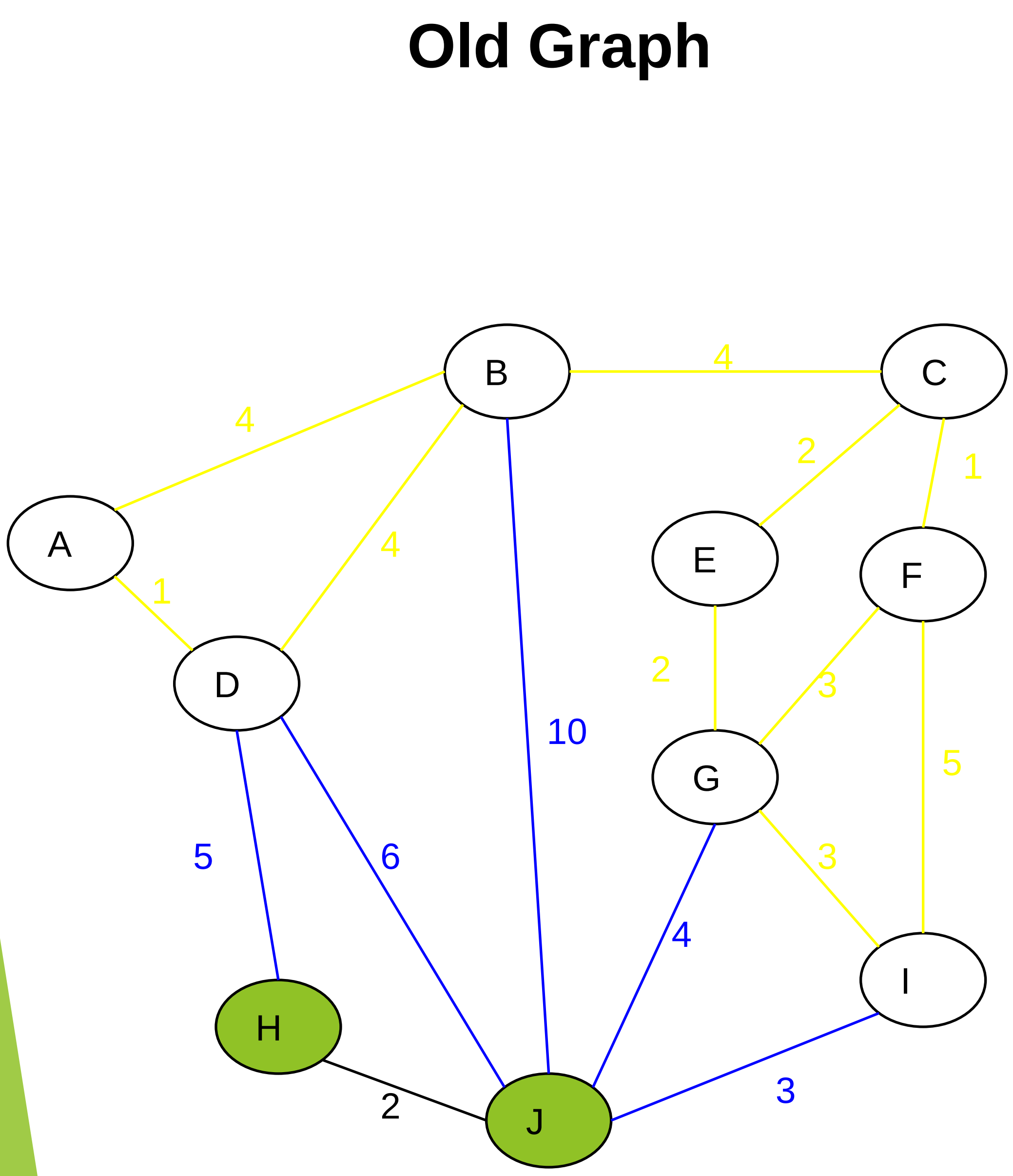
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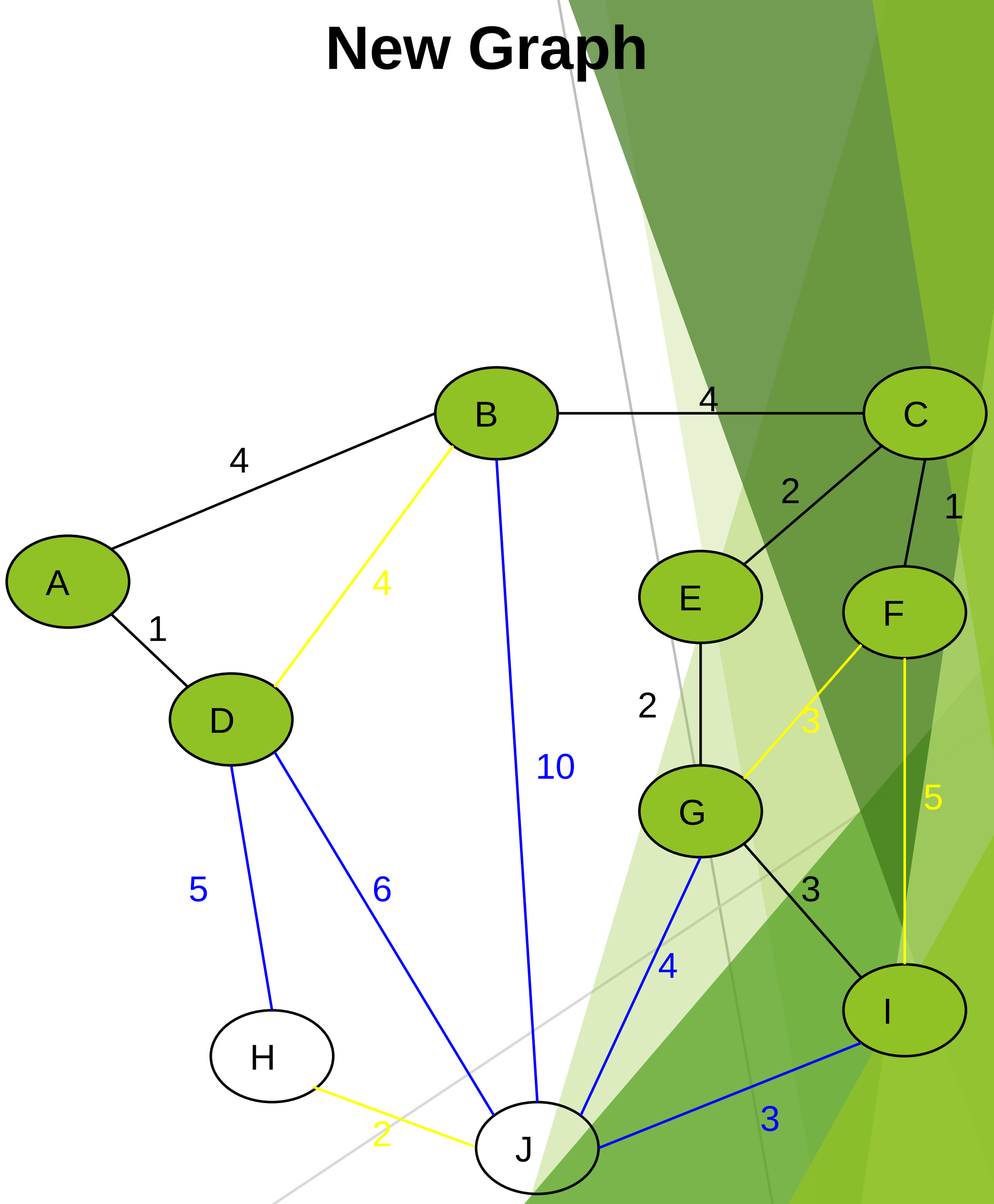
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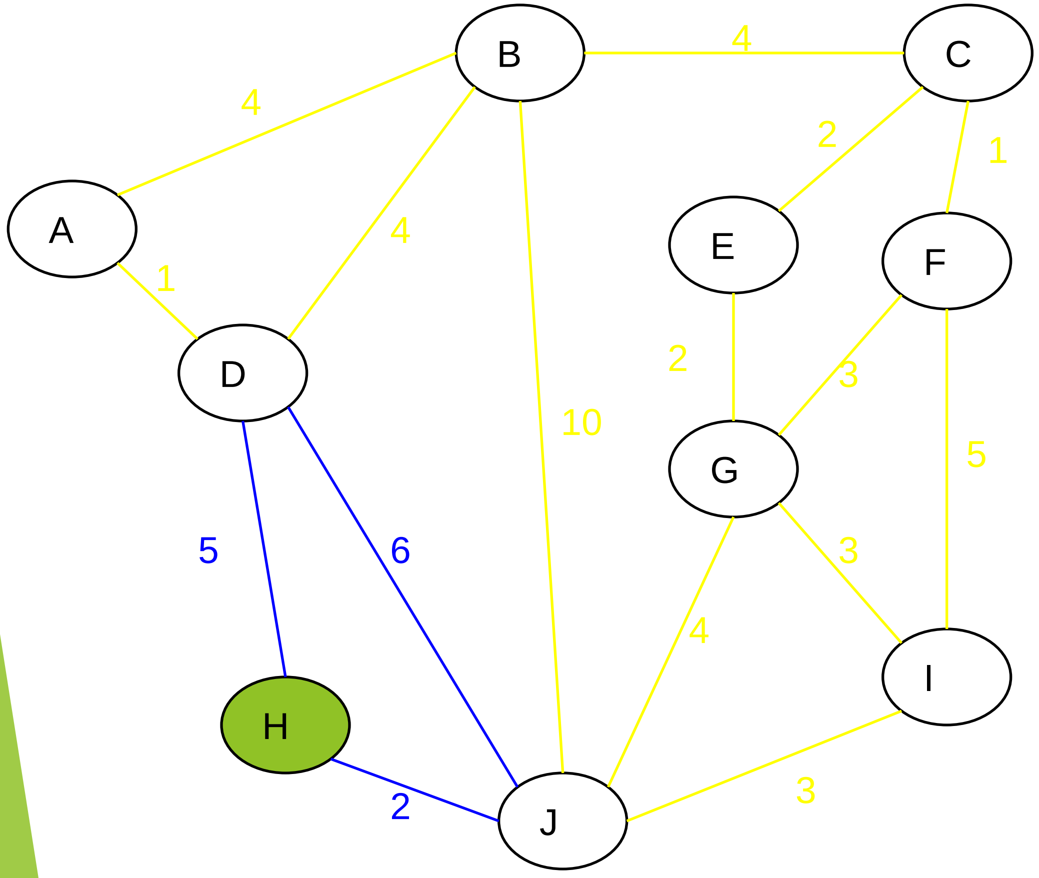

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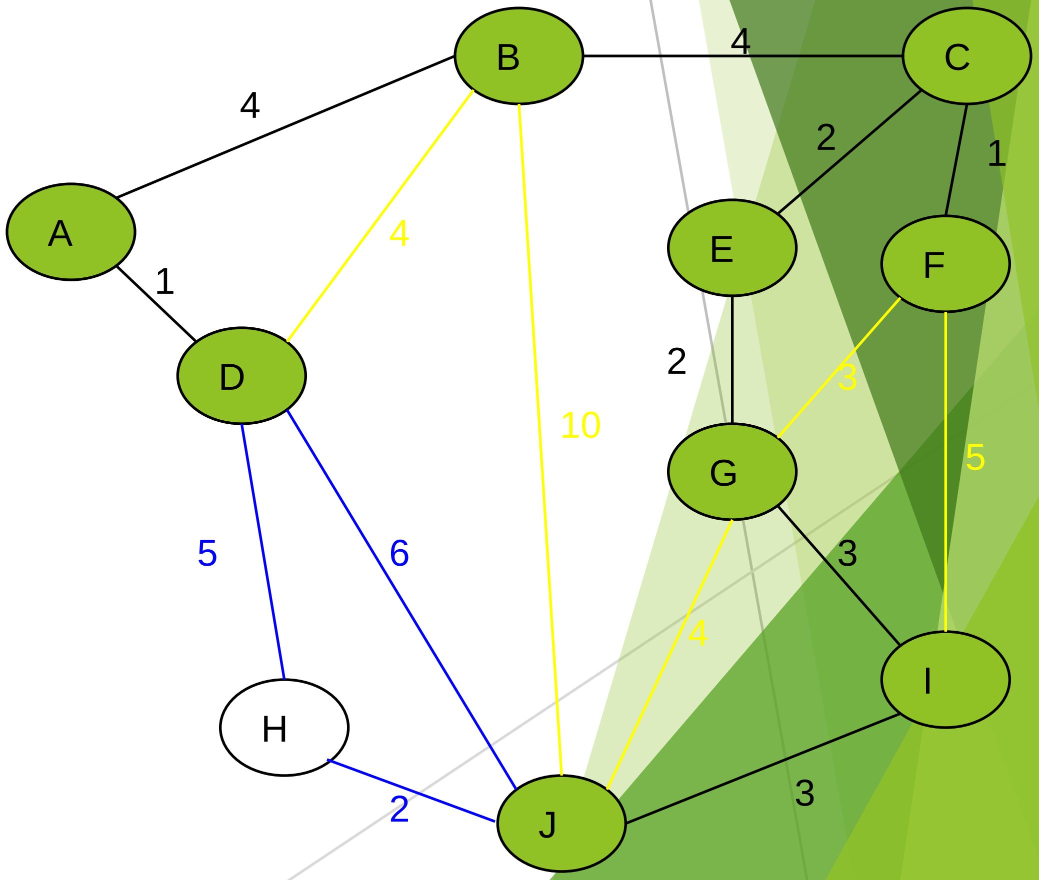

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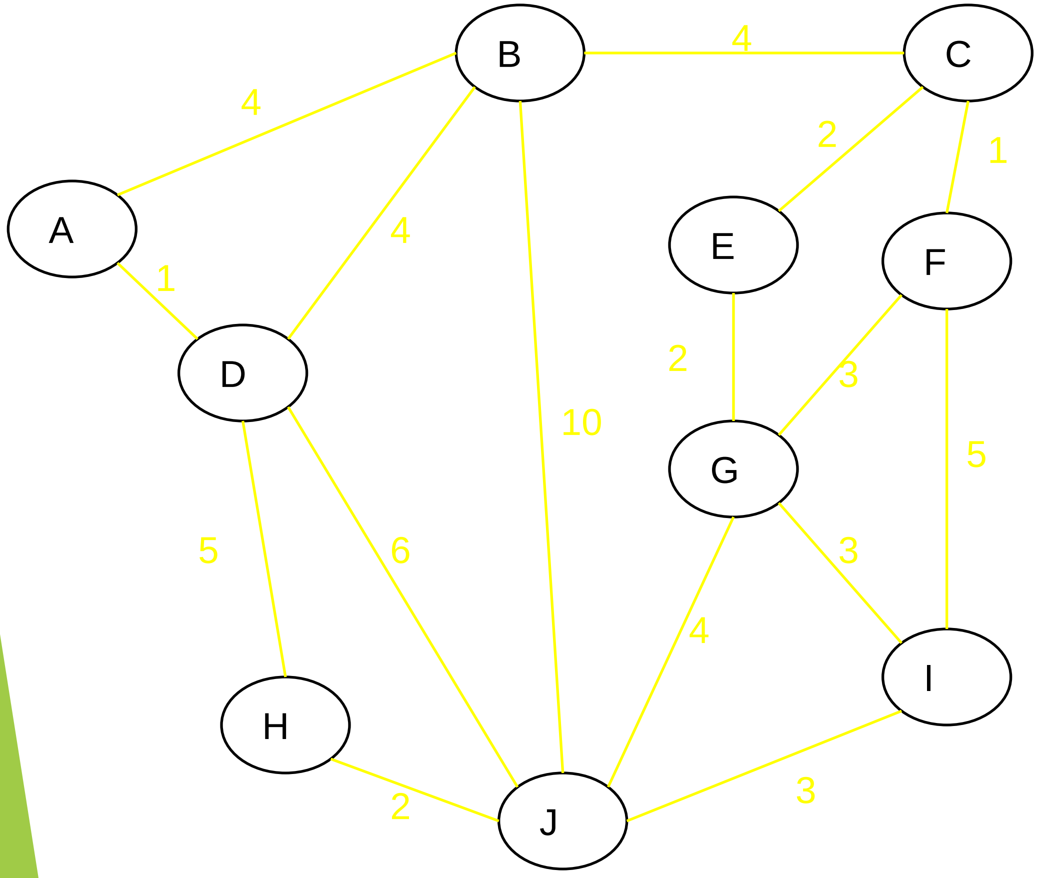

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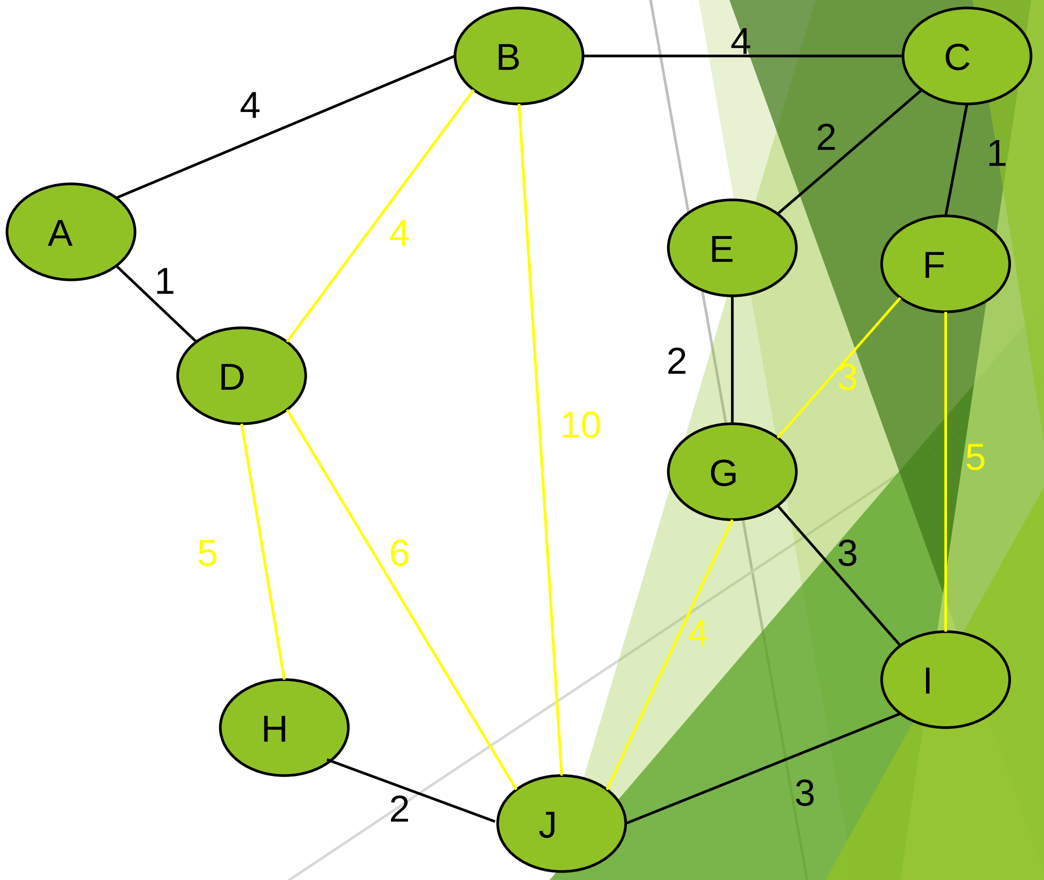
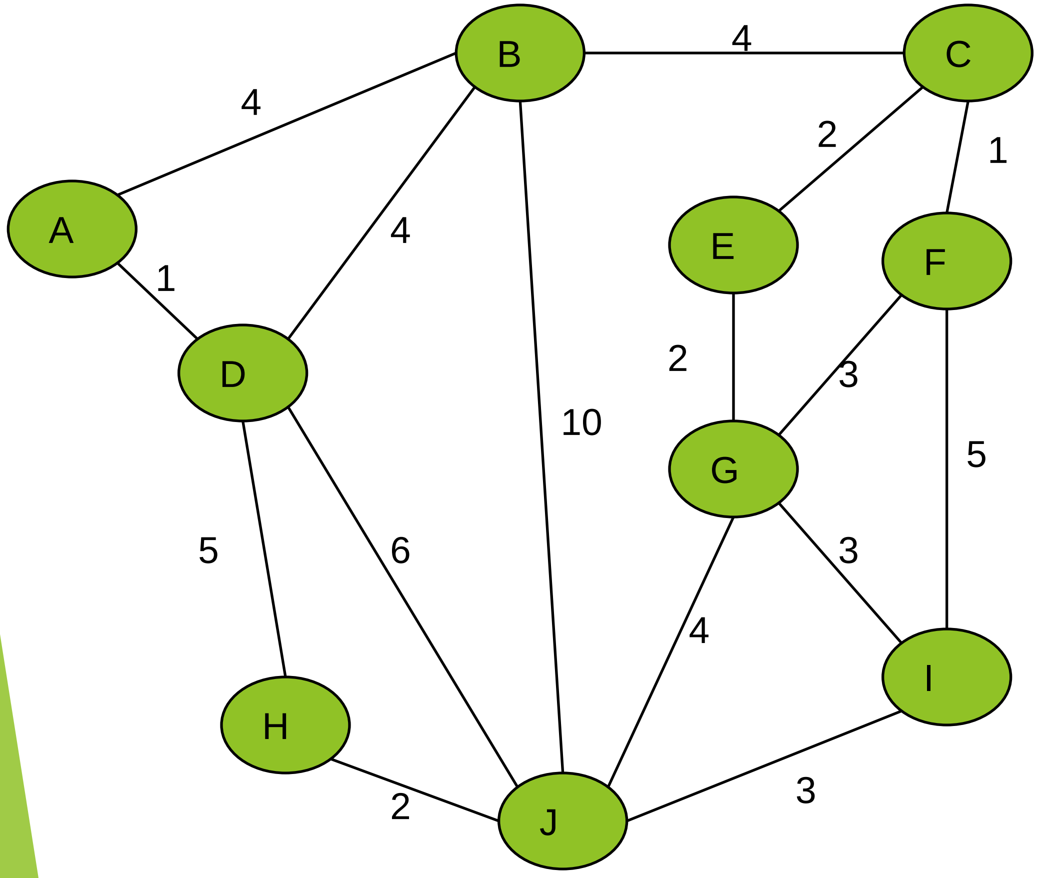
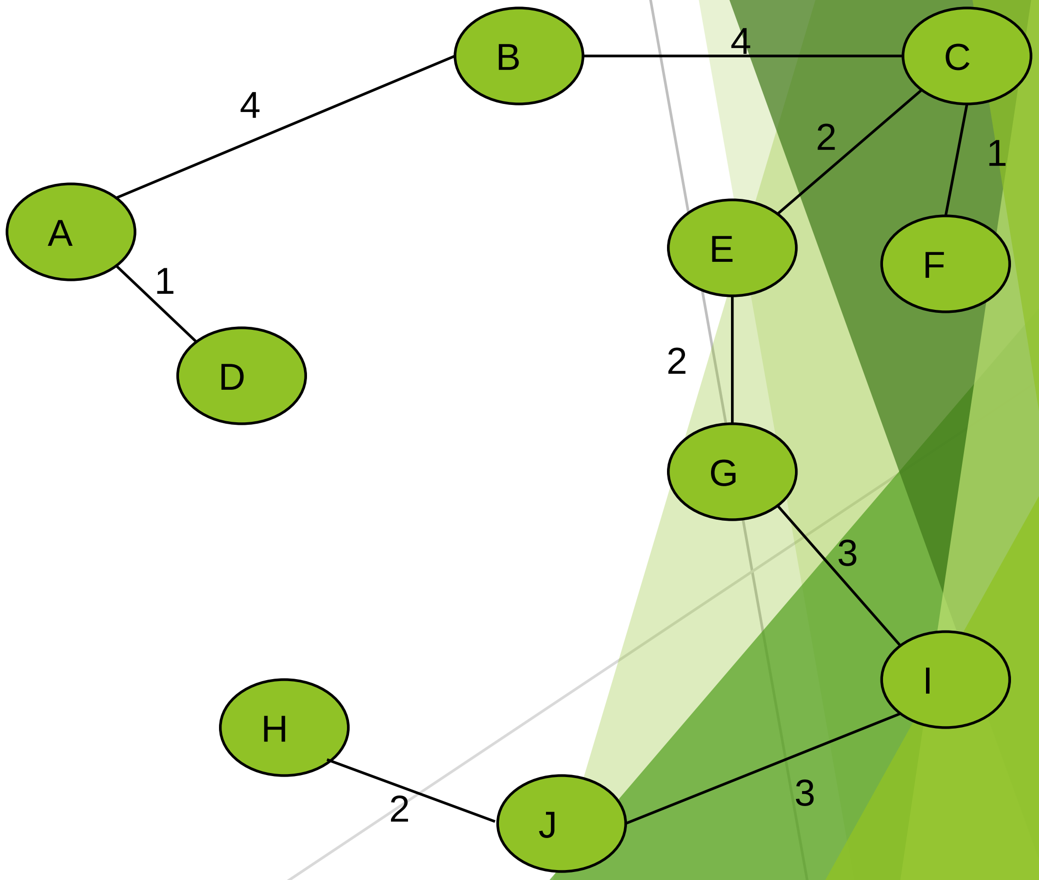


Complete Graph



Minimum Spanning Tree



```
Algorithm Prim(E, cost, n, t)
// E is the set of edges in G. cost[1 : n, 1 : n] is the cost
// adjacency matrix of an n vertex graph such that cost[i, j] is
// either a positive real number or ∞ if no edge (i, j) exists.
// A minimum spanning tree is computed and stored as a set of
// edges in the array t[1 : n − 1, 1 : 2]. (t[i, 1], t[i, 2]) is an edge in
// the minimum-cost spanning tree. The final cost is returned.
{
    Let (k, l) be an edge of minimum cost in E;
    mincost := cost[k, l];
    t[1, 1] := k; t[1, 2] := l;
    for i := 1 to n do  // Initialize near.
        if (cost[i, l] < cost[i, k]) then near[i] := l;
        else near[i] := k;
    near[k] := near[l] := 0;
    for i := 2 to n − 1 do
    { // Find n − 2 additional edges for t.
        Let j be an index such that near[j] ≠ 0 and
        cost[j, near[j]] is minimum;
        t[i, 1] := j; t[i, 2] := near[j];
        mincost := mincost + cost[j, near[j]];
        near[j] := 0;
        for k := 1 to n do // Update near[ ].
            if ((near[k] ≠ 0) and (cost[k, near[k]] > cost[k, j]))
                then near[k] := j;
    }
    return mincost;
}
```

Analysis of Prim's Algorithm

- Running Time = O(m + n log n) (m = edges, n = nodes)
- If a heap is not used, the run time will be O(n^2) instead of O(m + n log n). However, using a heap complicates the code since you're complicating the data structure. A Fibonacci heap is the best kind of heap to use, but again, it complicates the code.
- Unlike Kruskal's, it doesn't need to see all of the graph at once. It can deal with it one piece at a time. It also doesn't need to worry if adding an edge will create a cycle since this algorithm deals primarily with the nodes, and not the edges.
- For this algorithm the number of nodes needs to be kept to a minimum in addition to the number of edges. For small graphs, the edges matter more, while for large graphs the number of nodes matters more

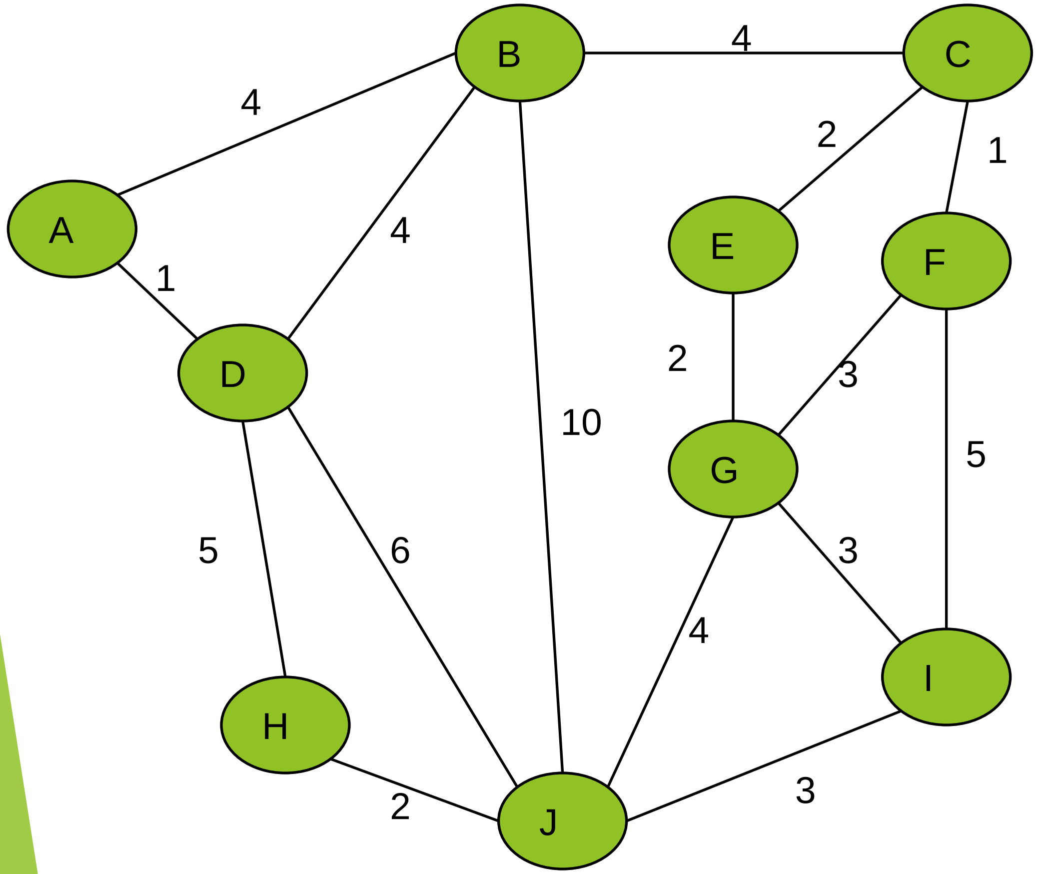
Kruskal's Algorithm

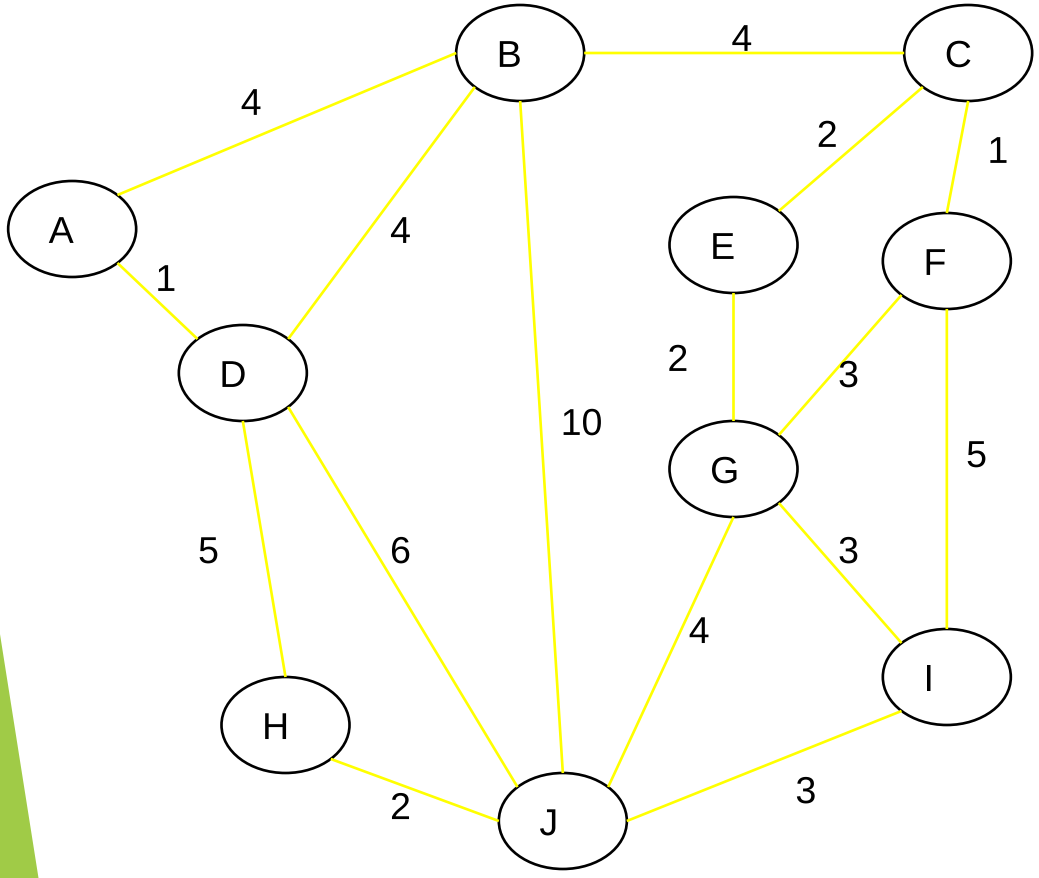
- This algorithm creates a forest of trees.
- Initially the forest consists of n single node trees (and no edges).
- At each step, we add one edge (the cheapest one) so that it joins two trees together.
- If it were to form a cycle, it would simply link two nodes that were already part of a single connected tree, so that this edge would not be needed.

Kruskal's Algorithm (Steps)

- The forest is constructed - with each node in a separate tree.
- The edges are placed in a priority queue.
- Until we've added n-1 edges,
  - Extract the cheapest edge from the queue,
  - If it forms a cycle, reject it,
  - Else add it to the forest. Adding it to the forest will join two trees together.
- Every step will have joined two trees in the forest together, so that at the end, there will only be one tree in T

Complete Graph





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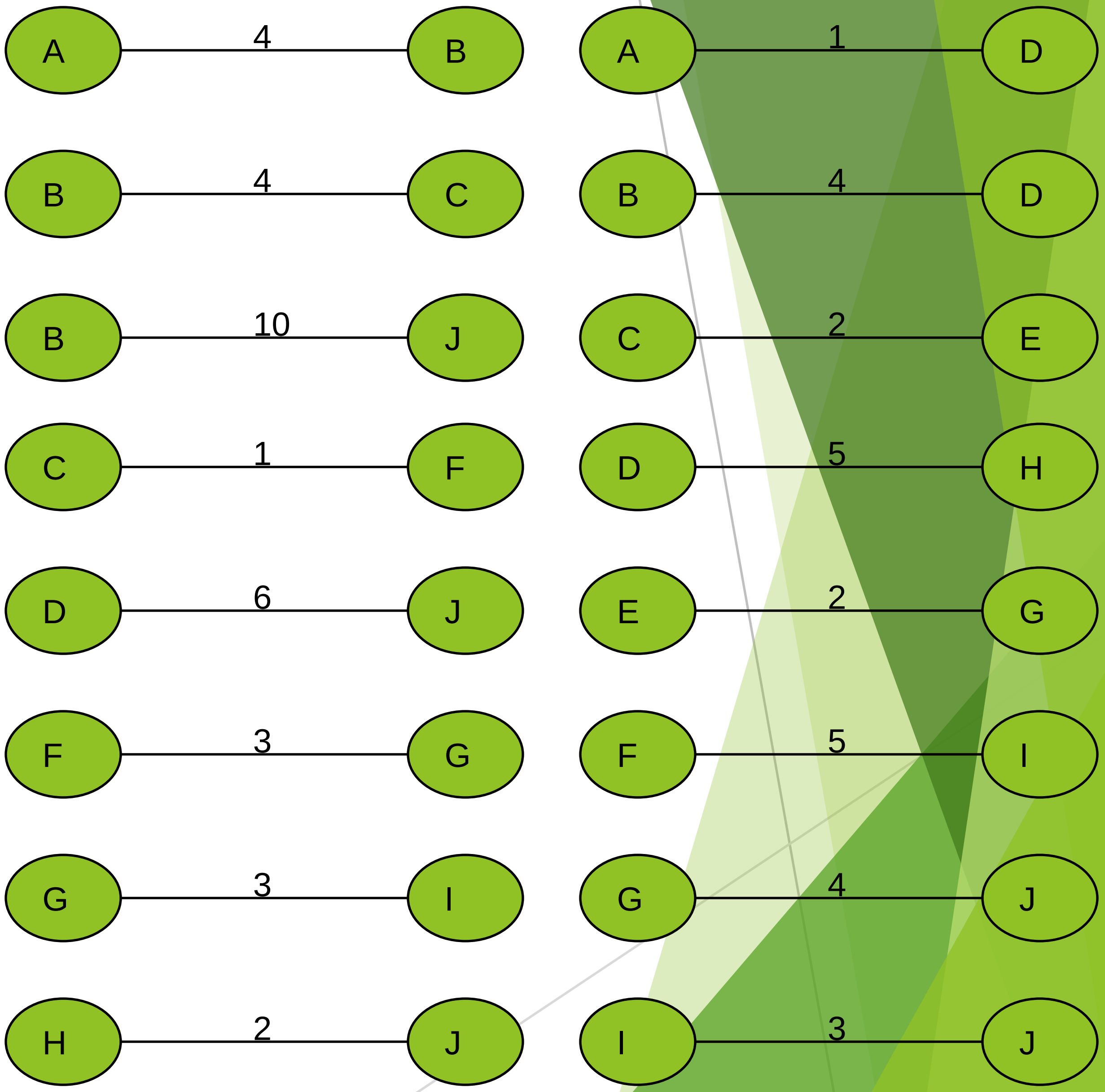

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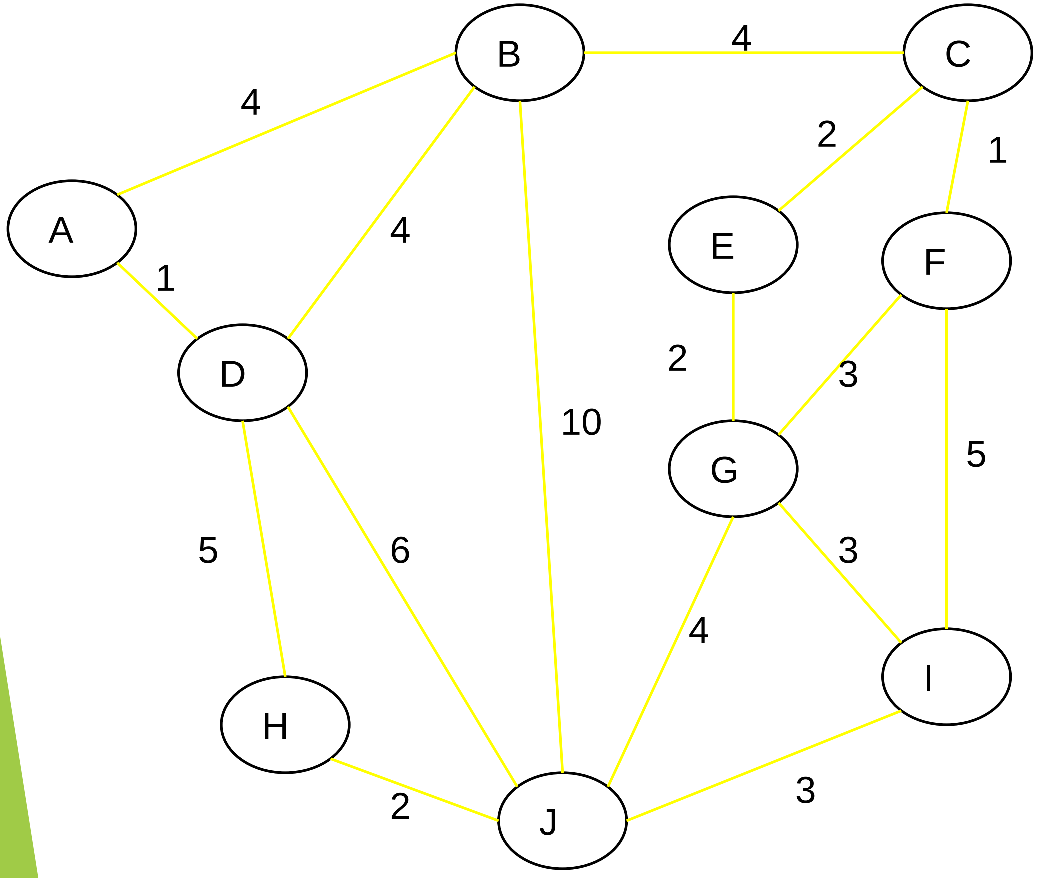
Sort Edges

(in reality they are placed in a priority queue - not sorted - but sorting them makes the algorithm easier to visualize)



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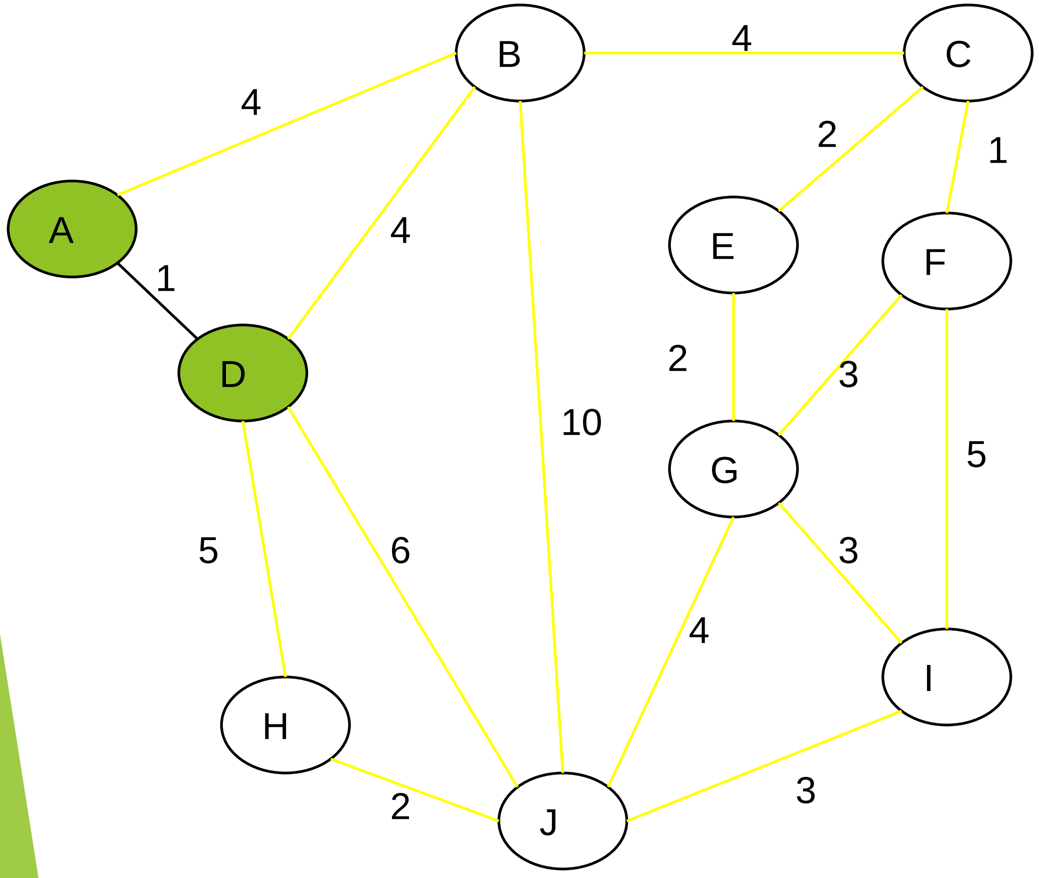
Add Edge



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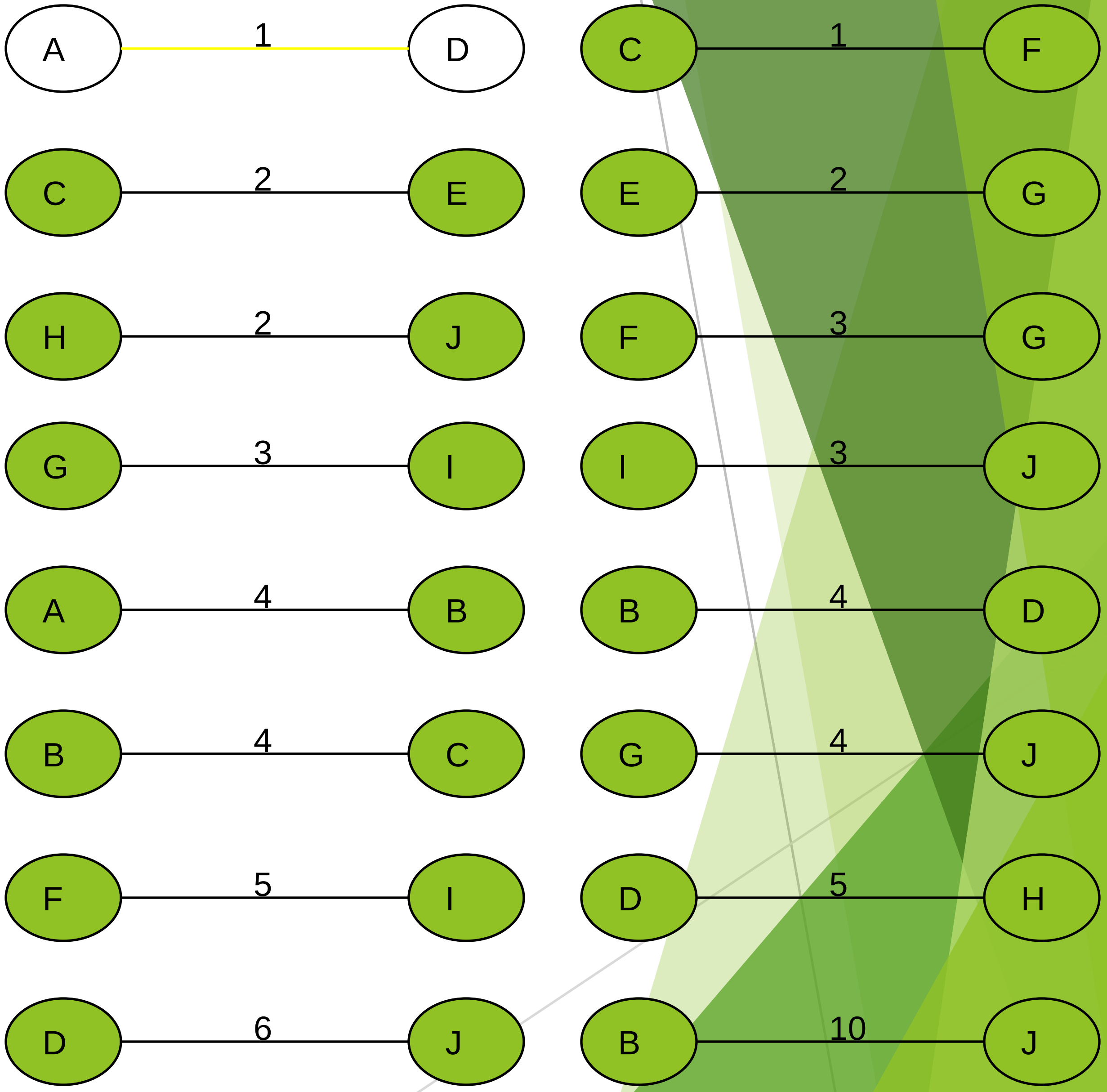

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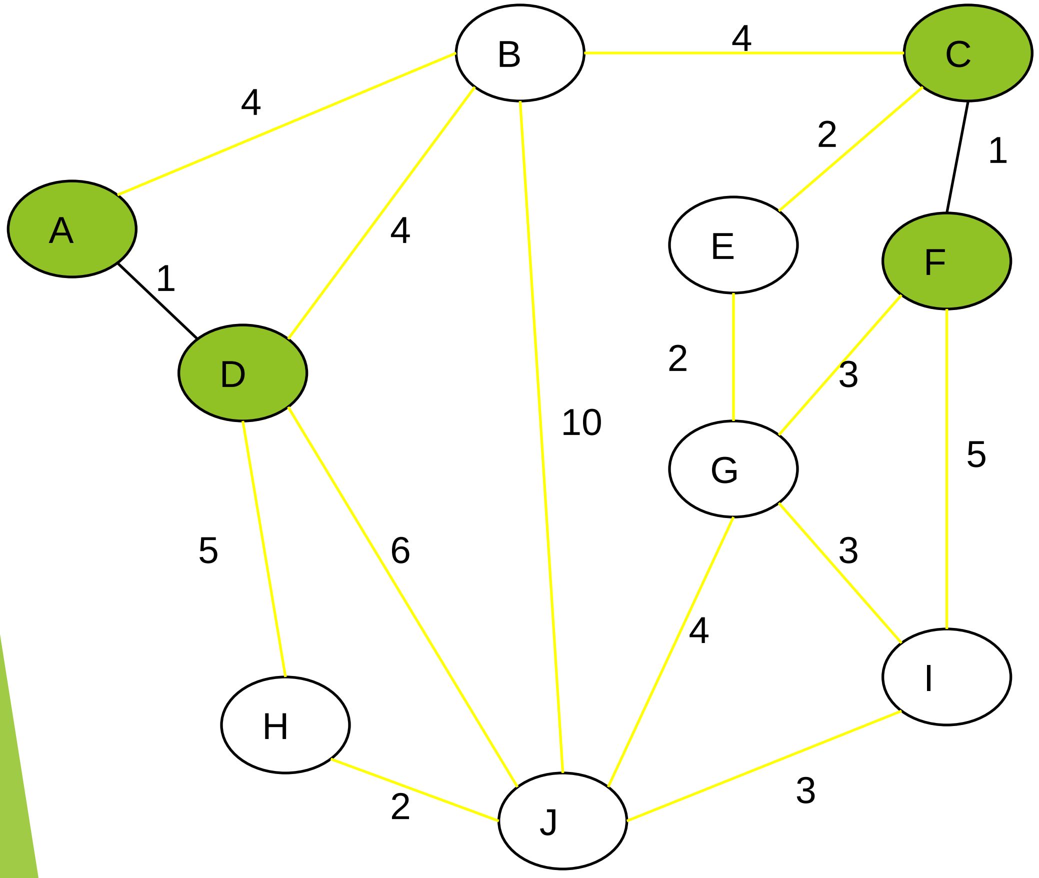
D 5 H
D 6 J
B 10 J

Add Edge



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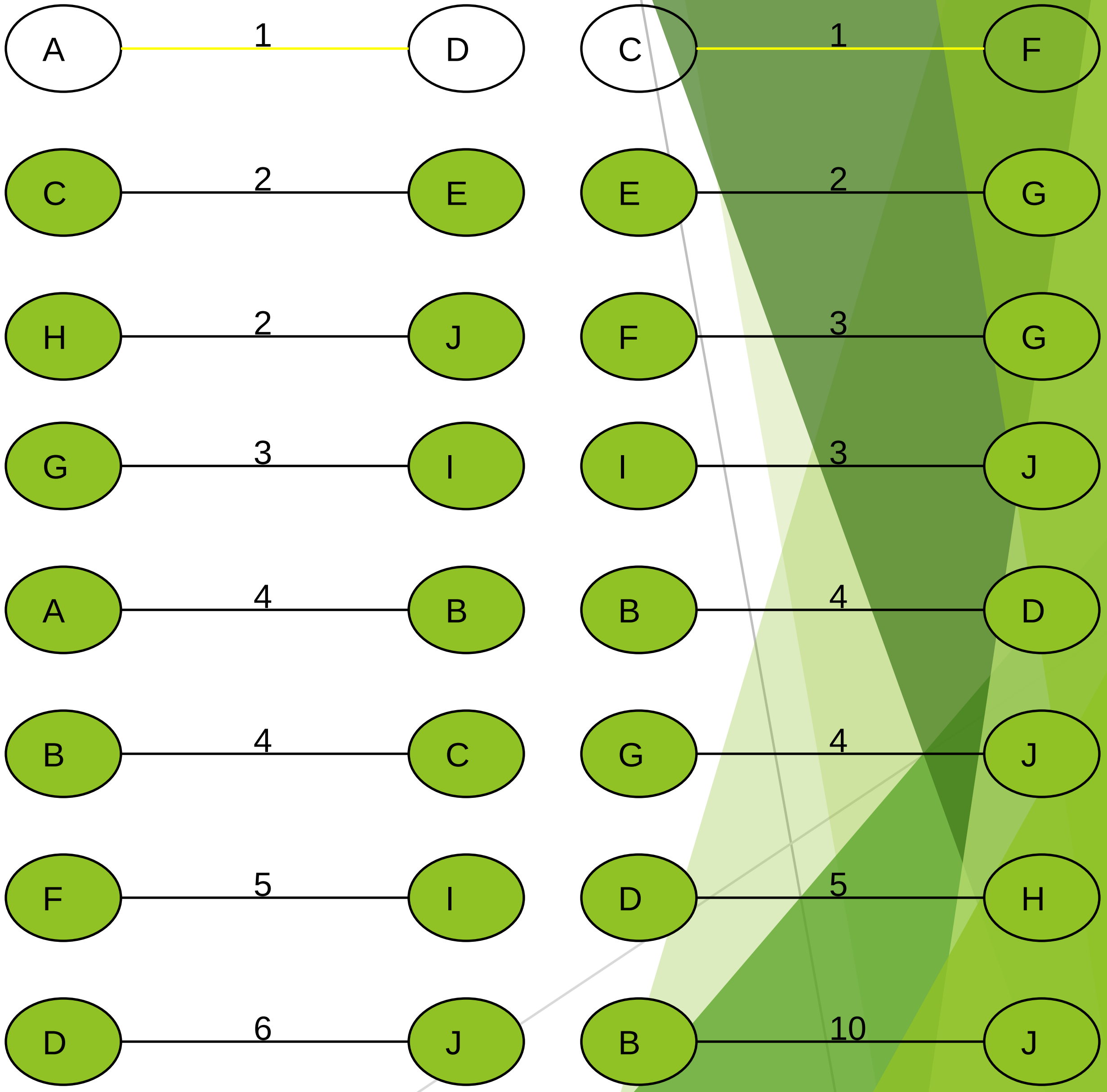

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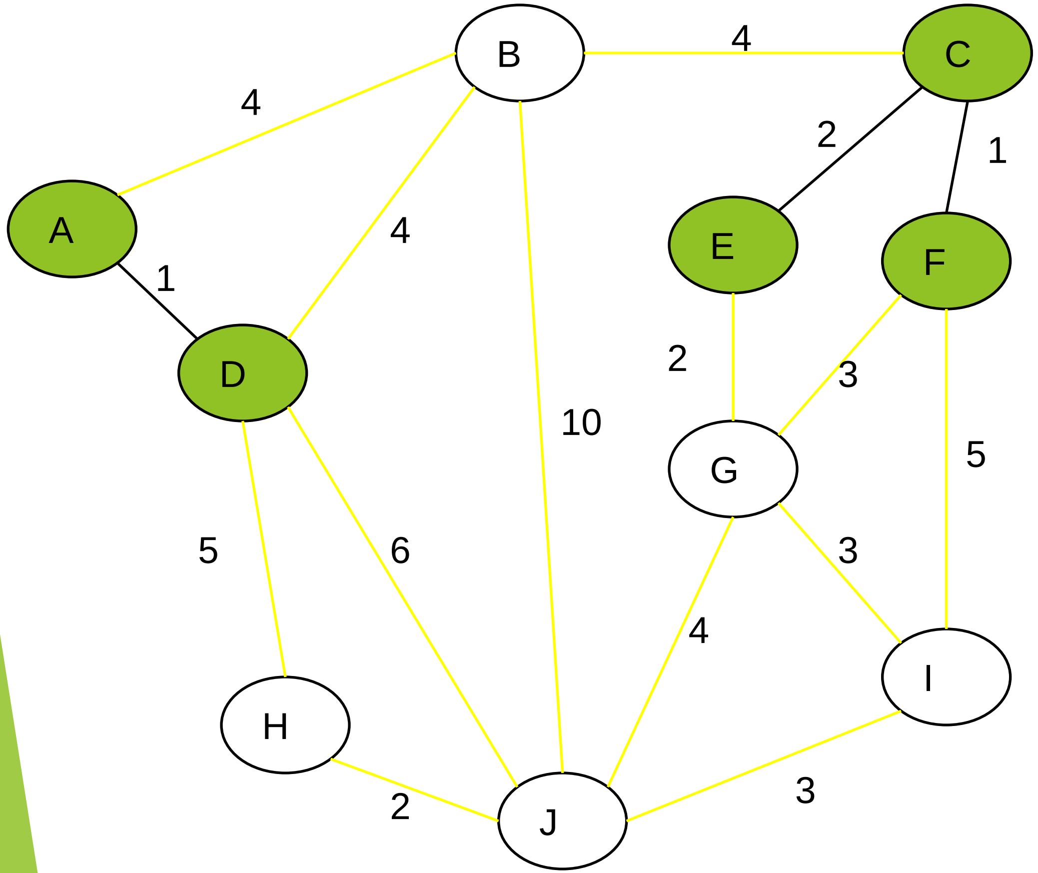
Add Edge



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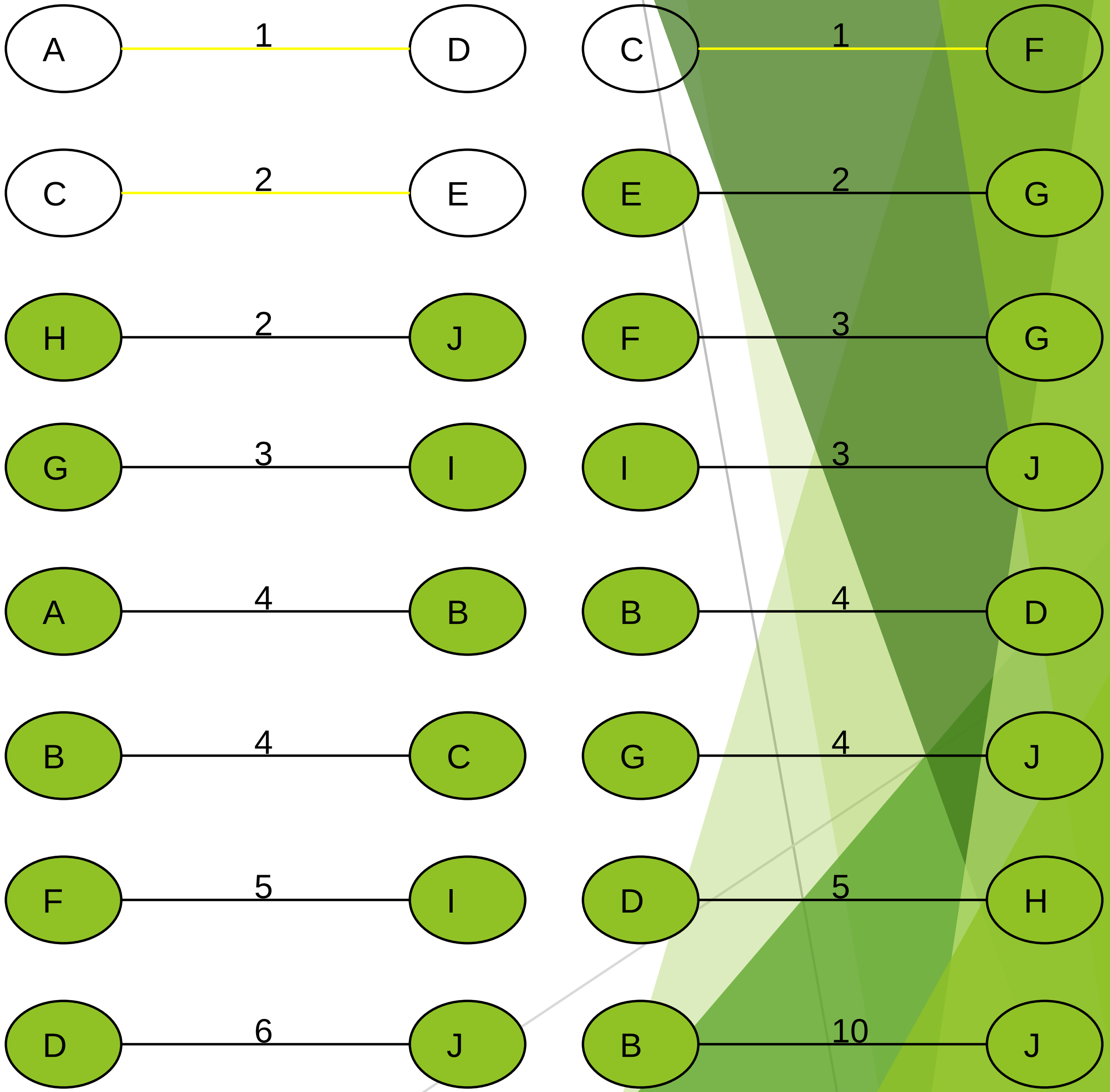

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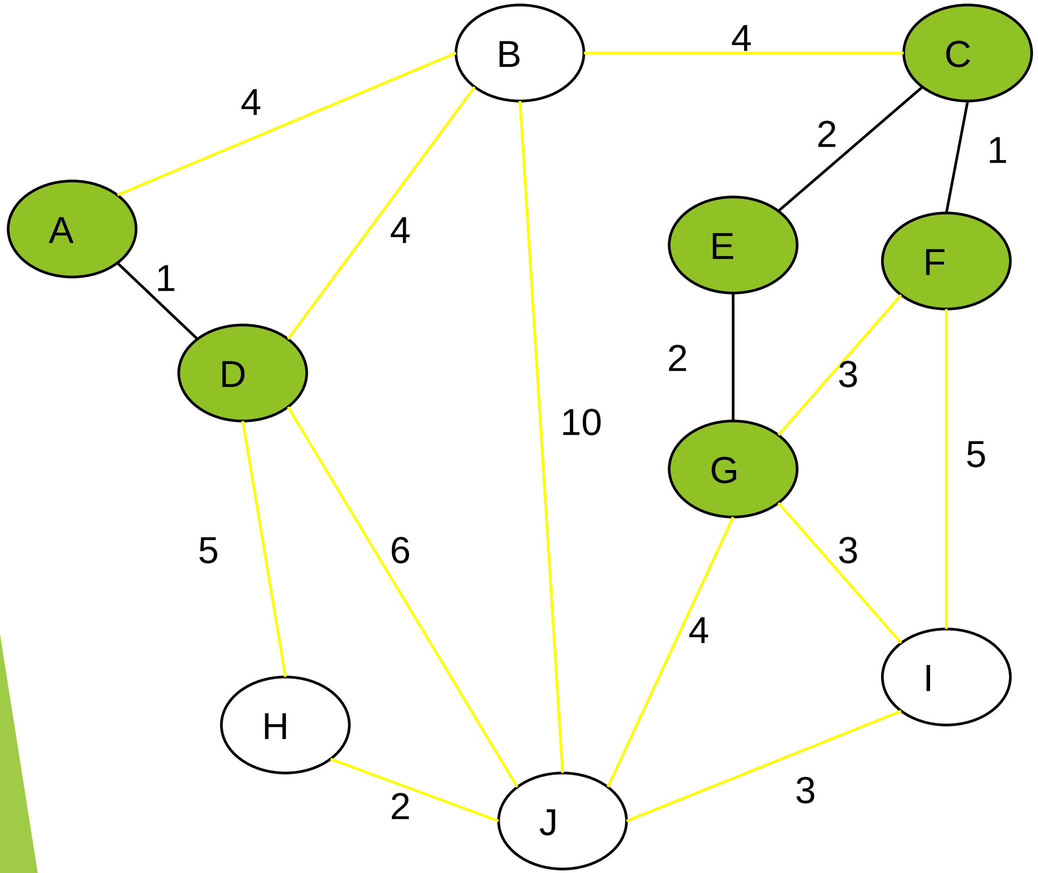
B 10 J

Add Edge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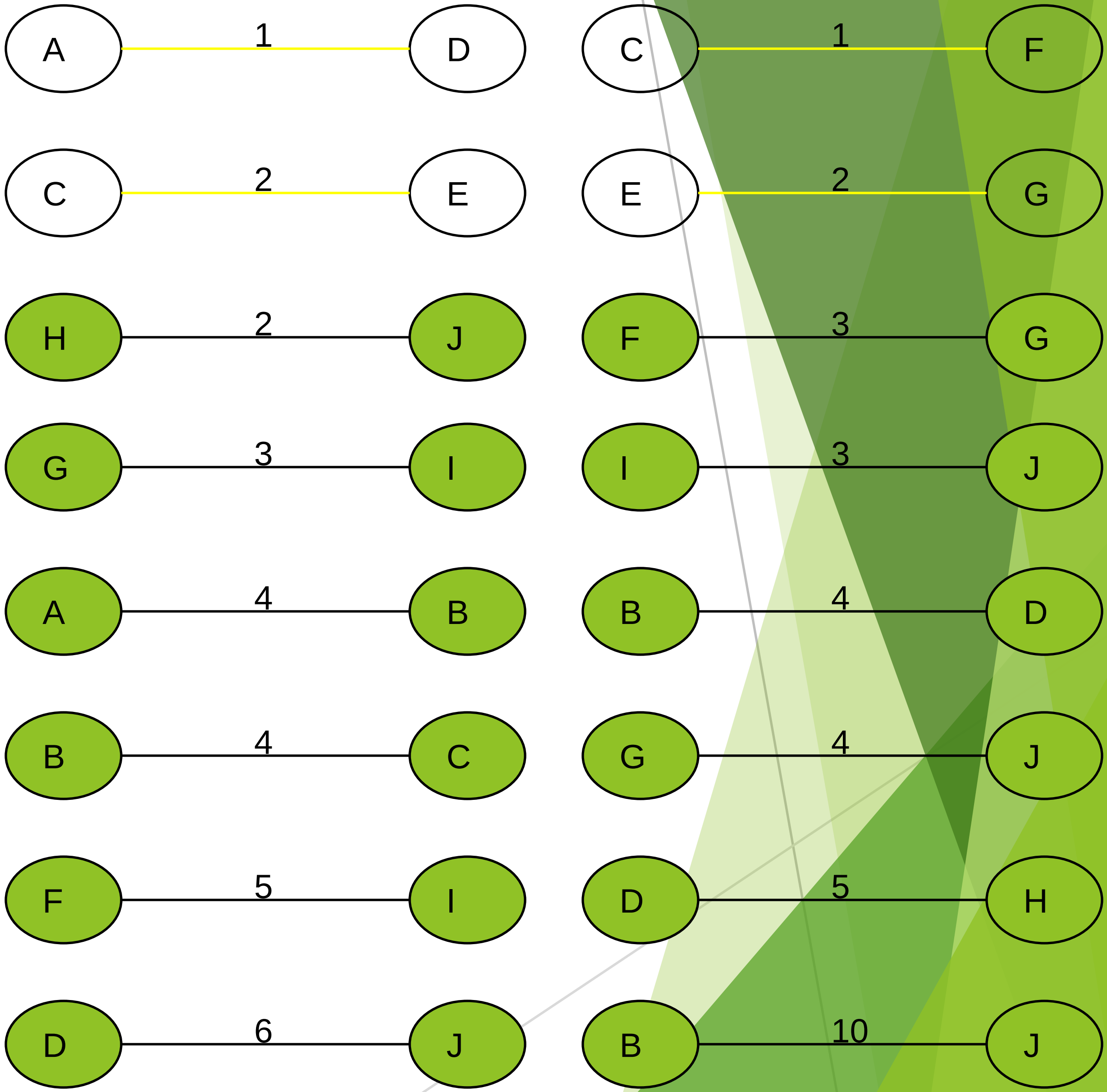
A 1 D
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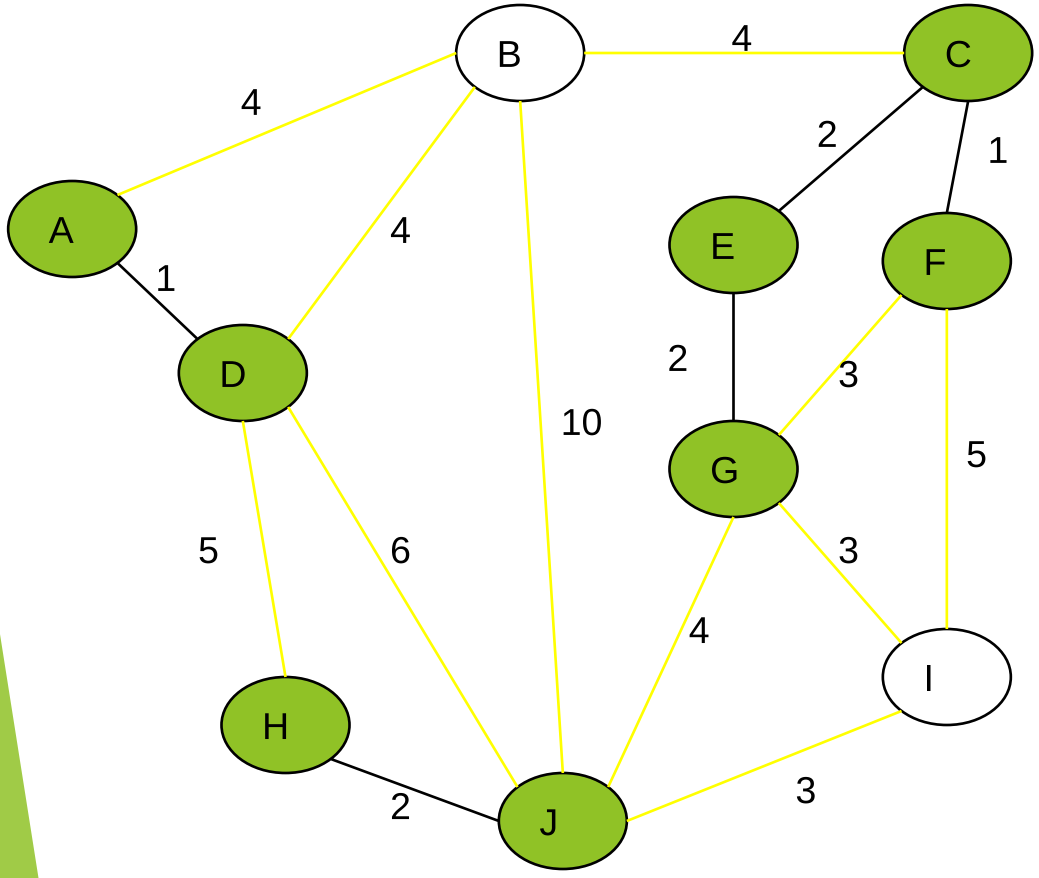
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge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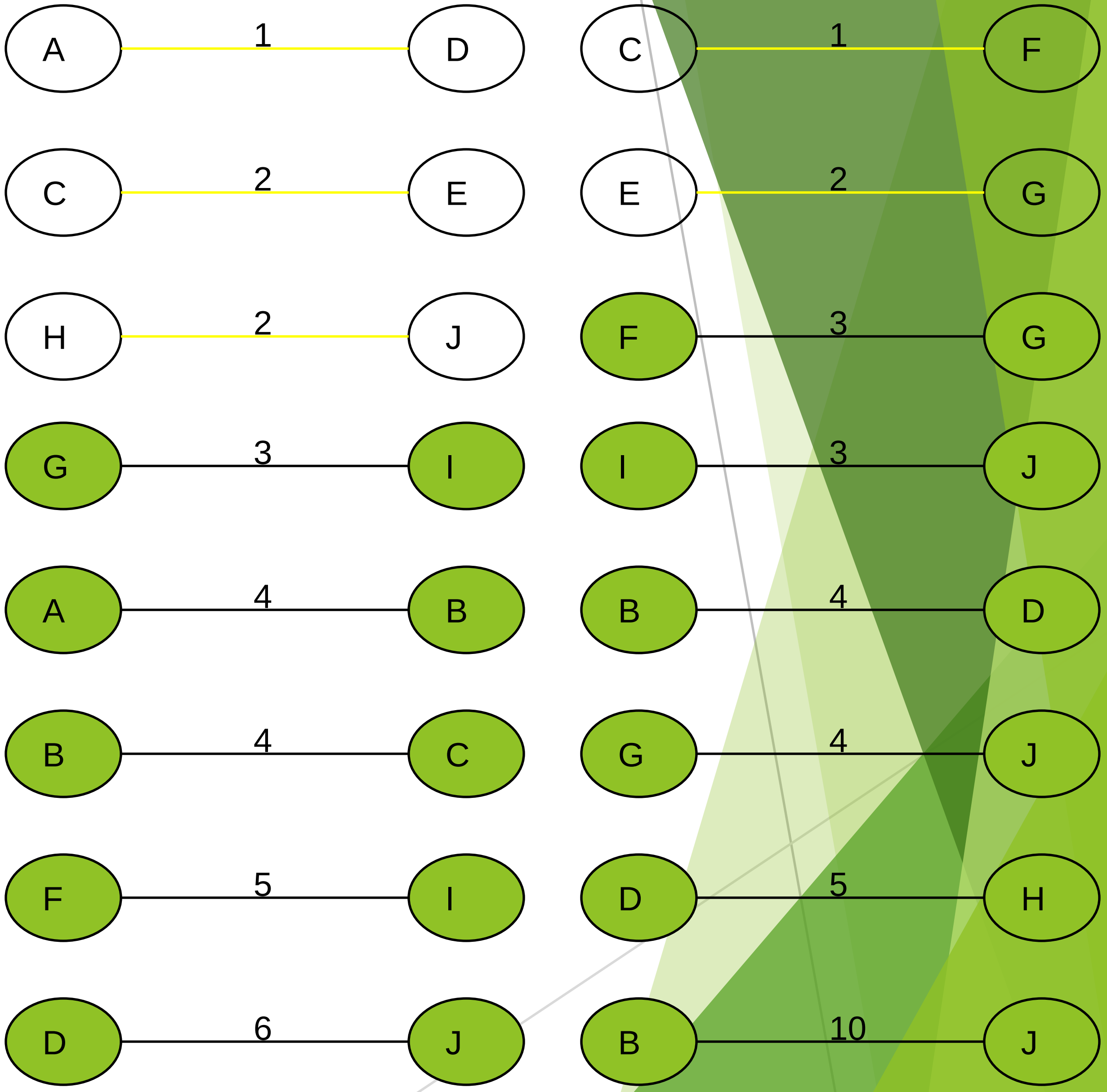
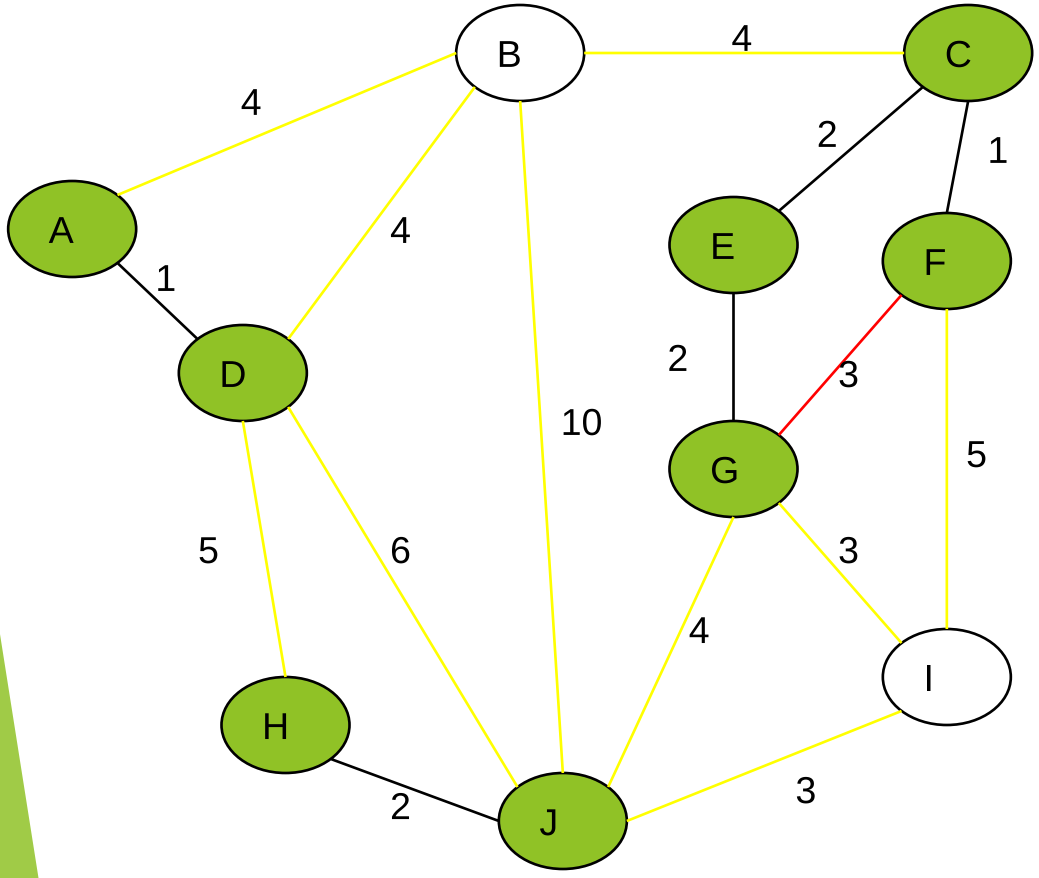

A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

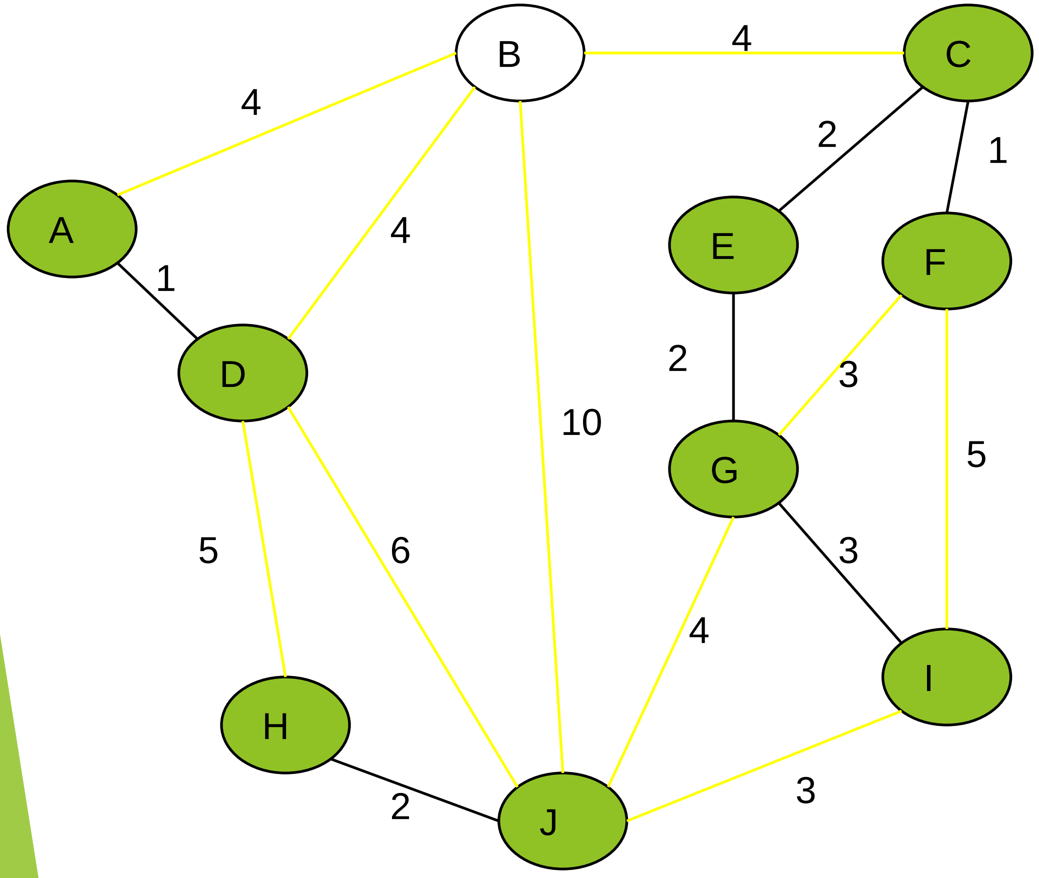
Cycle

Don’t Add Edge



A	1	D
C	2	E
H	2	J
G	3	I
A	4	B
B	4	C
F	5	I
D	6	J
C	1	F
E	2	G
F	3	G
I	3	J
B	4	D
G	4	J
D	5	H
B	10	J

Add Edge



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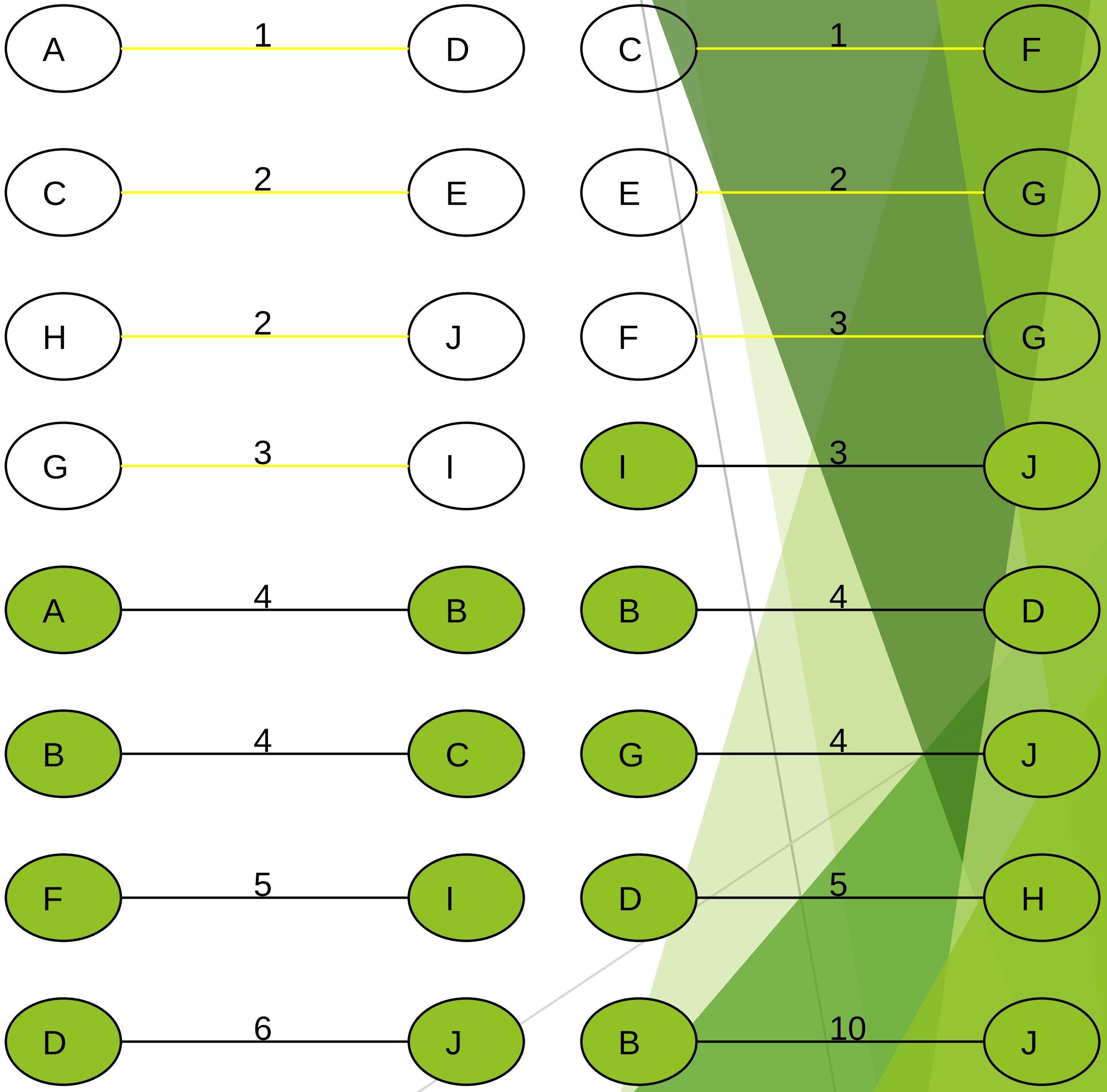
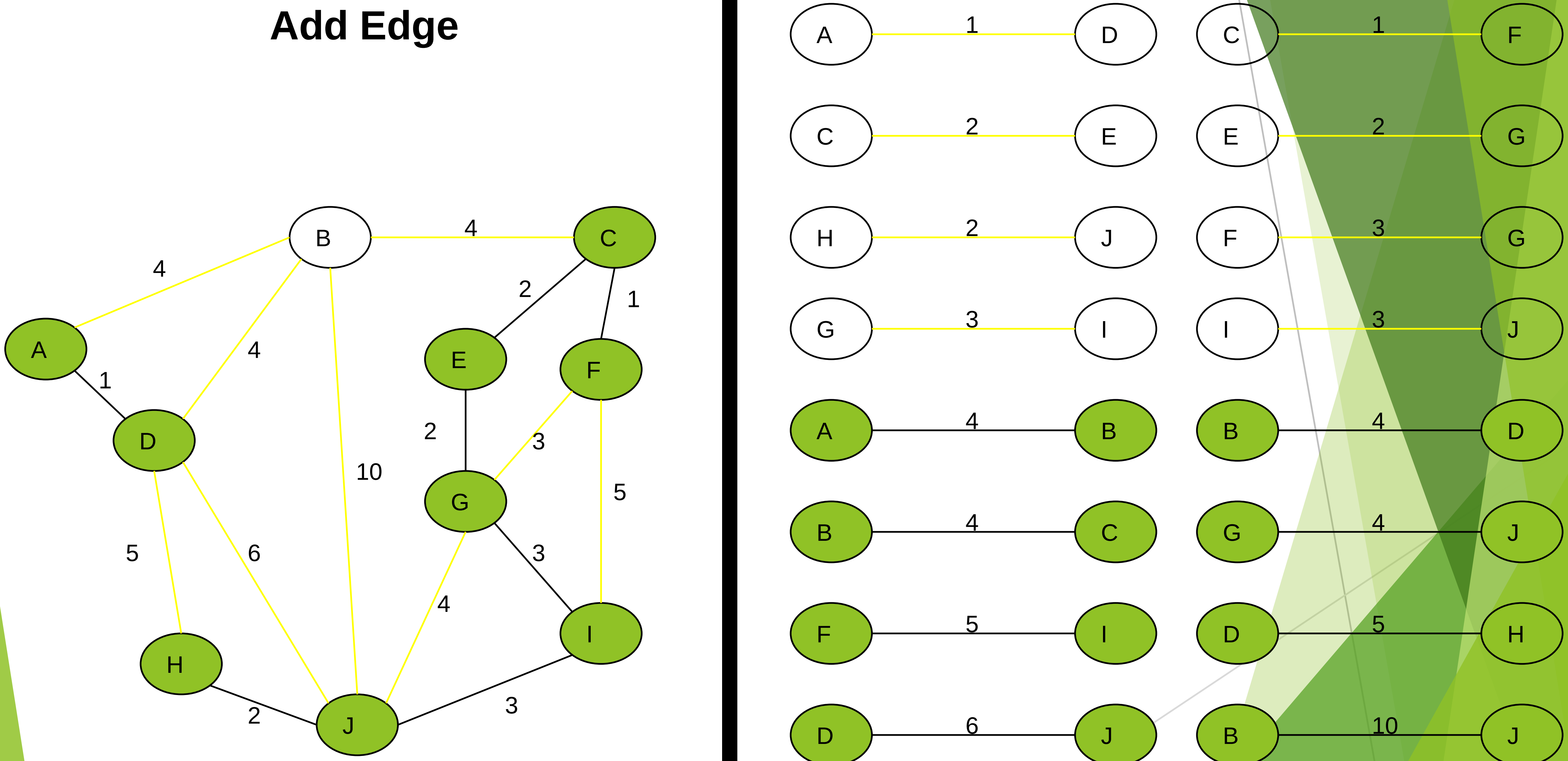


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge



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C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
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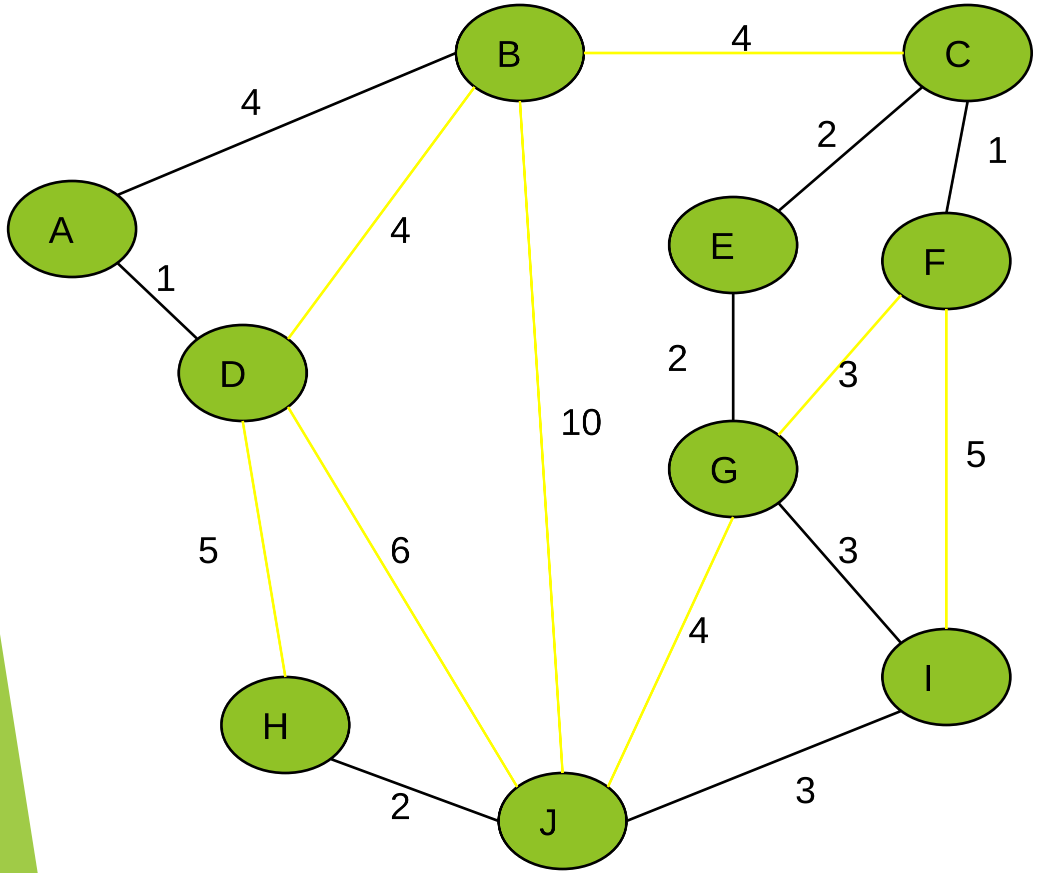
D 6 J
B 10 J

Add Edge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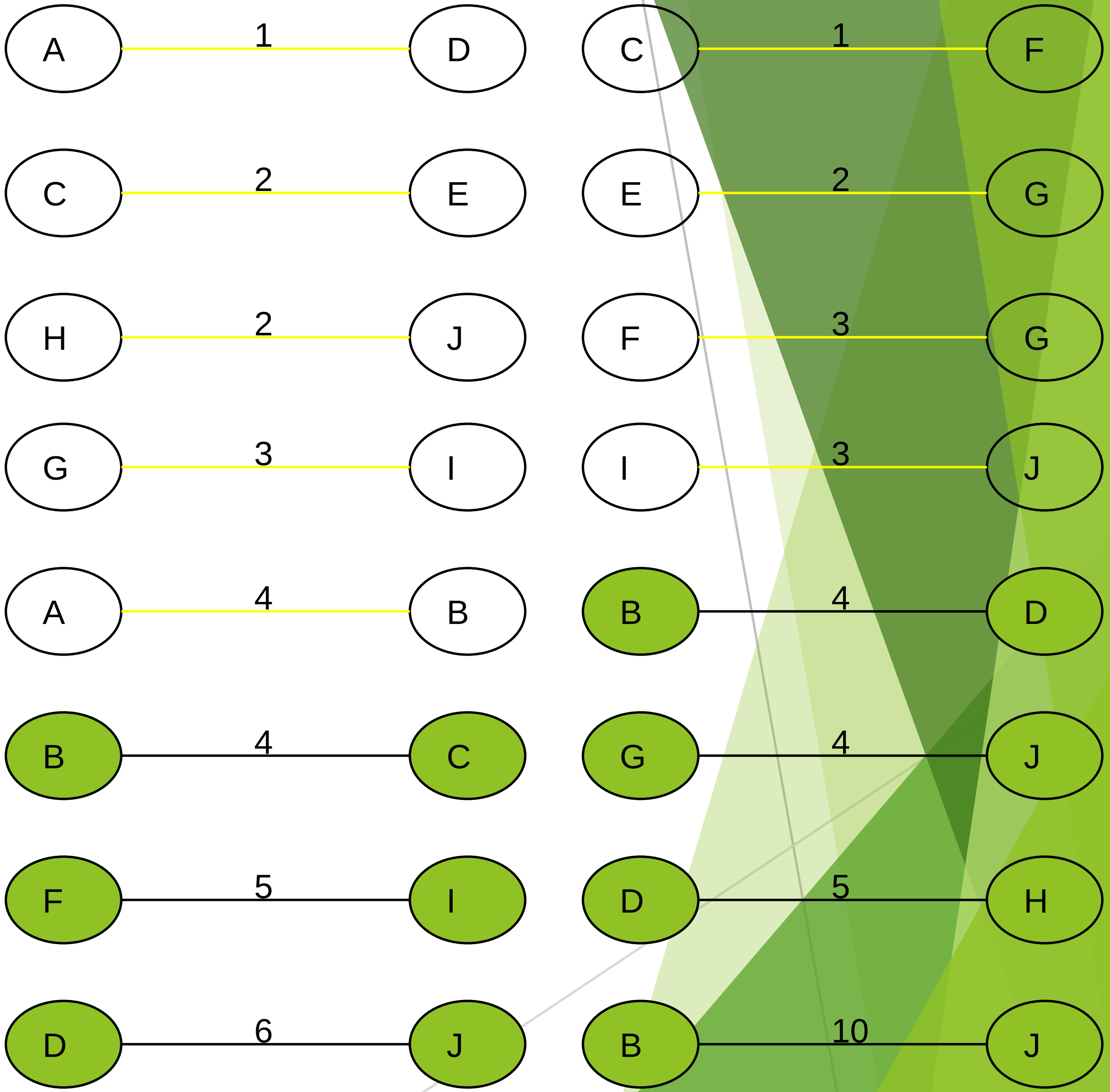
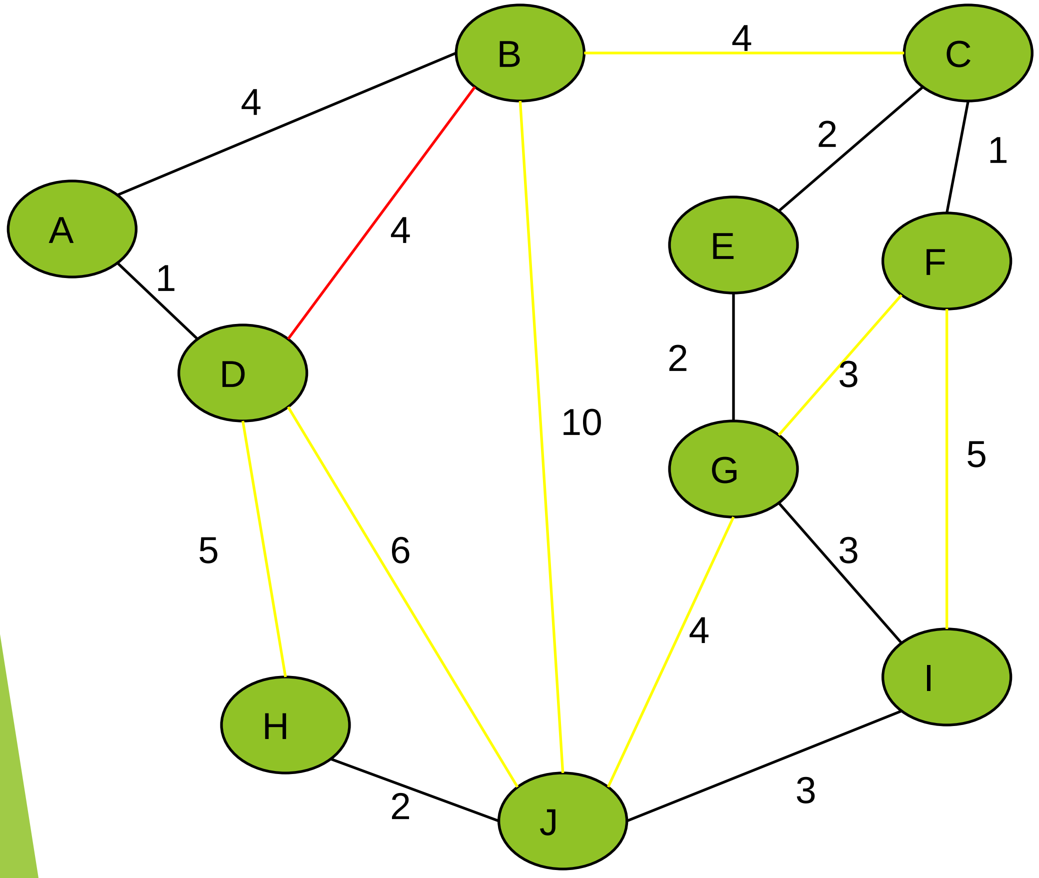
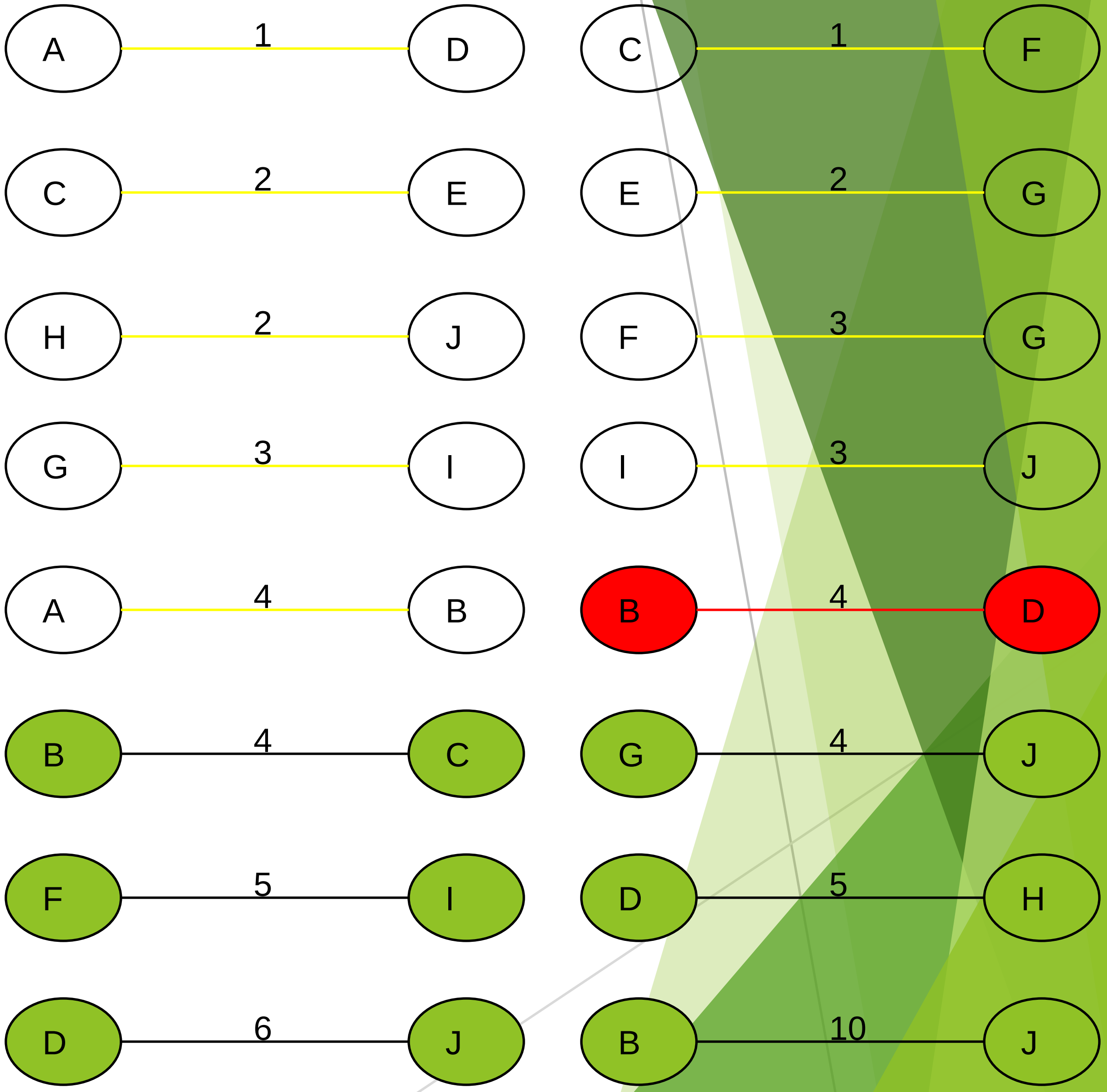

A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

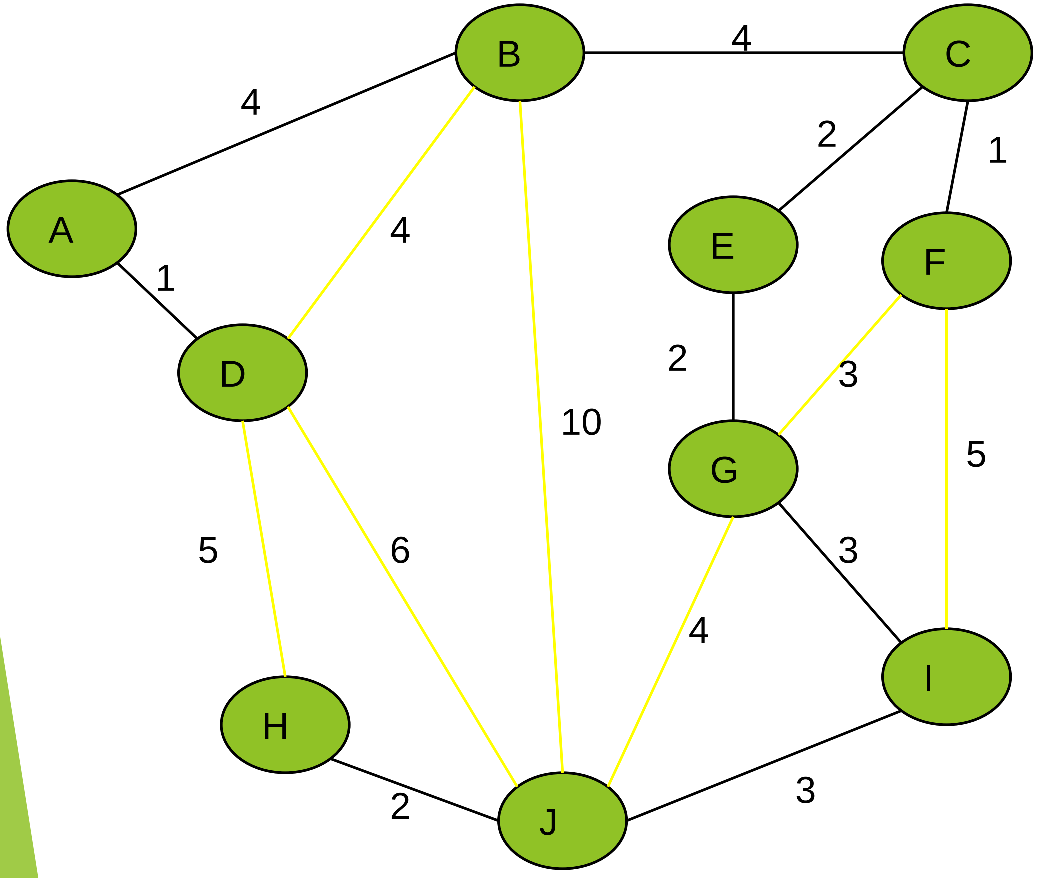
Cycle

Don't Add Edge





Add Edge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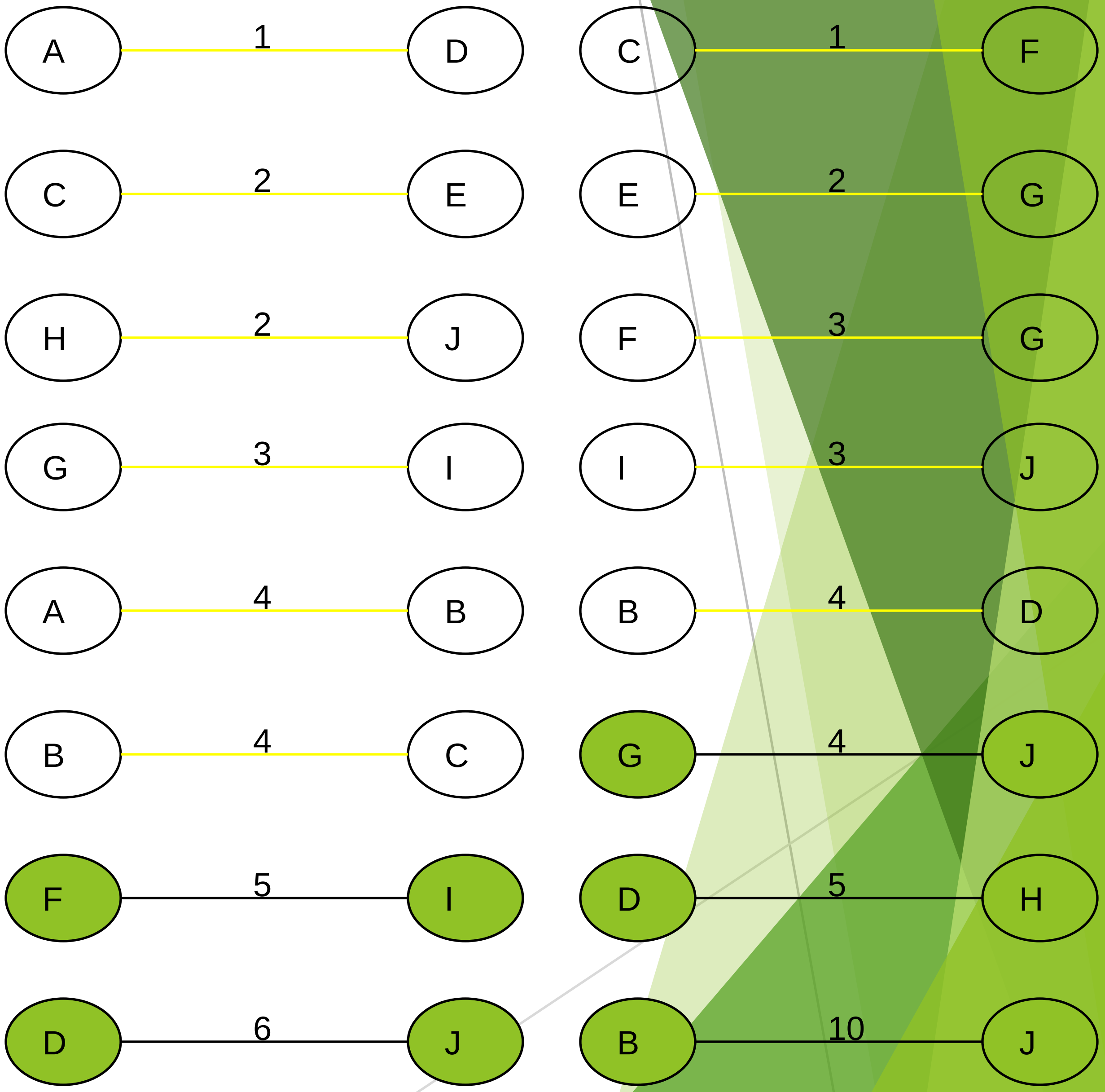
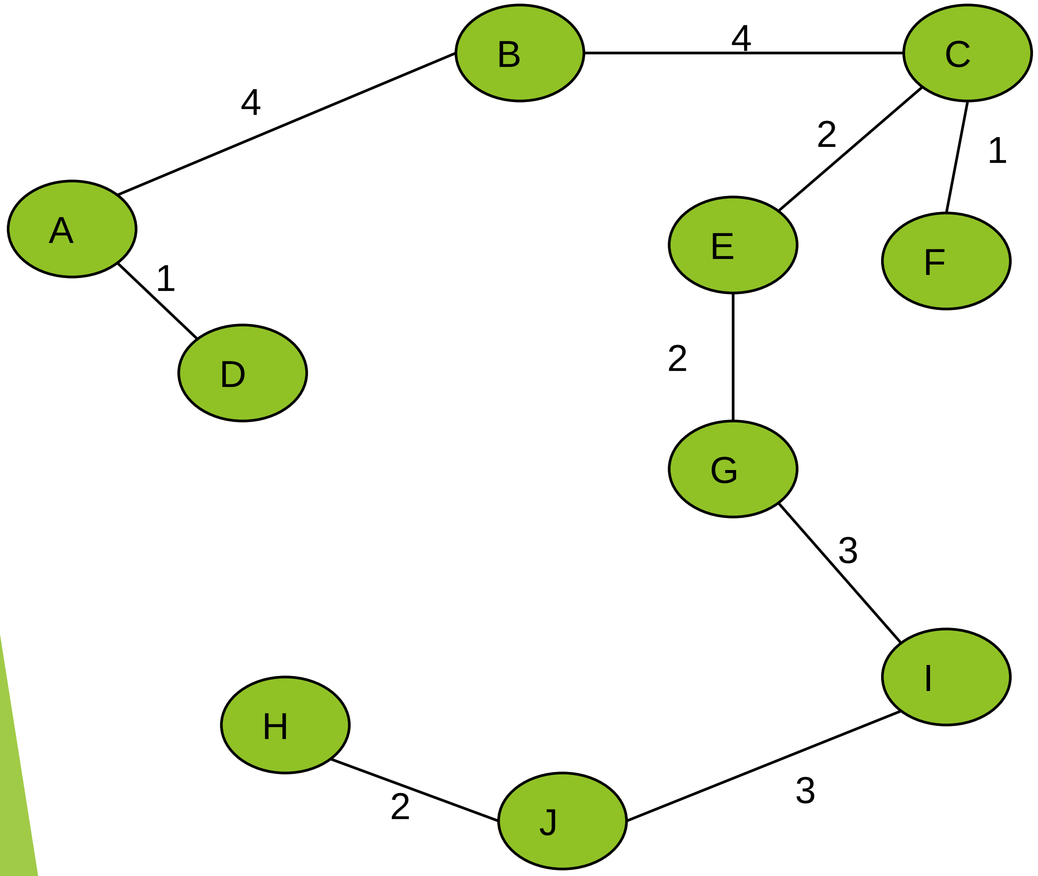
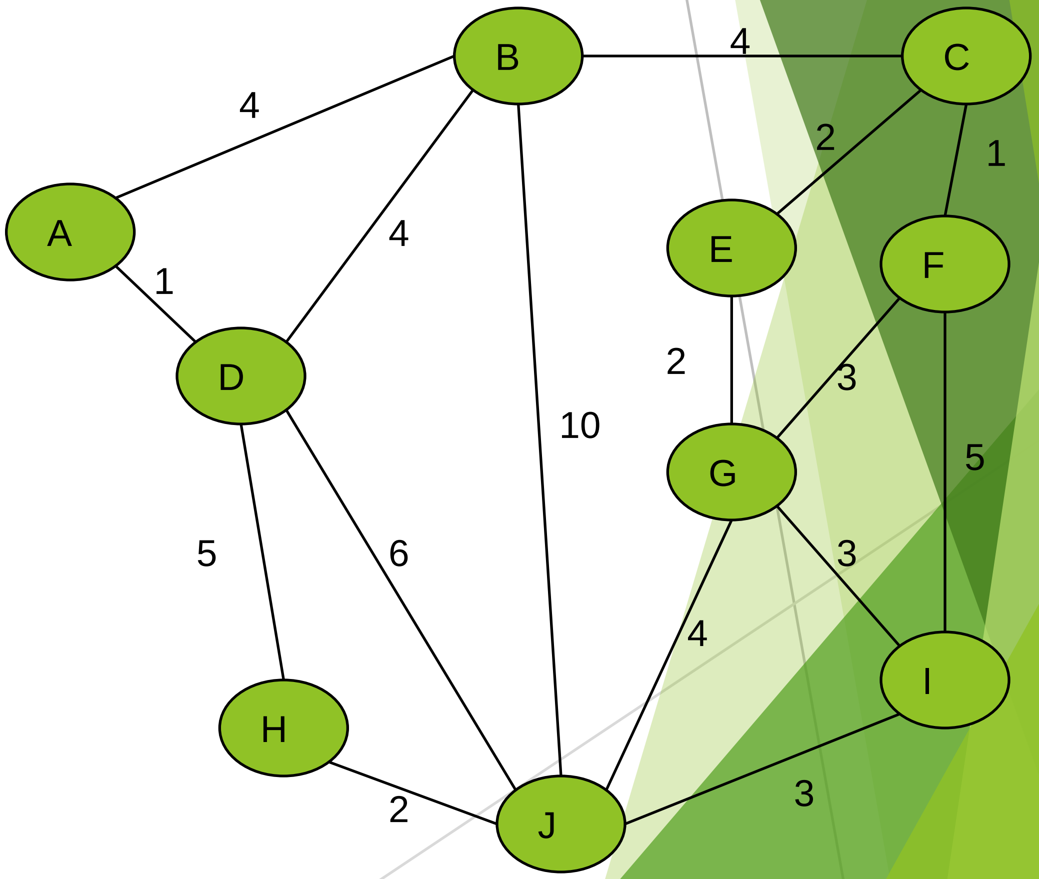

A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Minimum Spanning Tree



Complete Graph



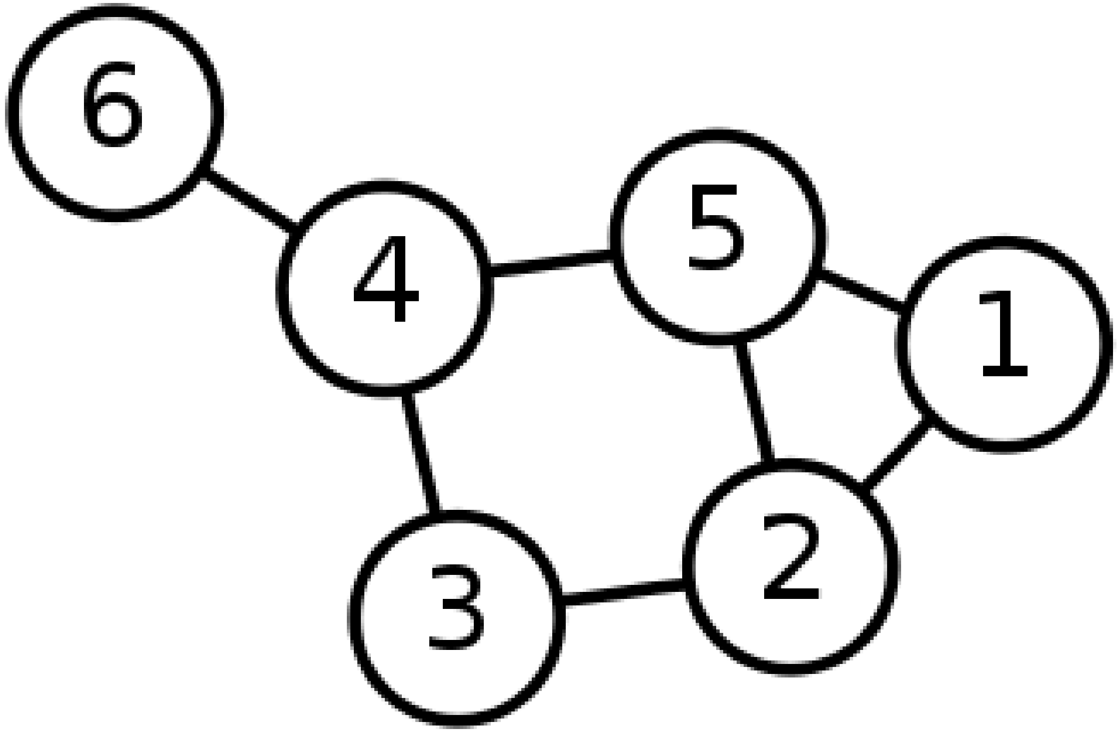
```
Algorithm Kruskal(E, cost, n, t)
// E is the set of edges in G. G has n vertices. cost[u, v] is the
// cost of edge (u, v). t is the set of edges in the minimum-cost
// spanning tree. The final cost is returned.
{
    Construct a heap out of the edge costs using Heapify;
    for i := 1 to n do parent[i] := -1;
    // Each vertex is in a different set.
    i := 0; mincost := 0.0;
    while ((i < n - 1) and (heap not empty)) do
    {
        Delete a minimum cost edge (u, v) from the heap
        and reheapify using Adjust;
        j := Find(u); k := Find(v);
        if (j ≠ k) then
        {
            i := i + 1;
            t[i, 1] := u; t[i, 2] := v;
            mincost := mincost + cost[u, v];
            Union(j, k);
        }
    }
    if (i ≠ n - 1) then write ("No spanning tree");
    else return mincost;
}
```

Analysis of Kruskal's Algorithm

- Running Time = O(m log n) (m = edges, n = nodes)
- Testing if an edge creates a cycle can be slow unless a complicated data structure called a “union-find” structure is used.
- It usually only has to check a small fraction of the edges, but in some cases (like if there was a vertex connected to the graph by only one edge and it was the longest edge) it would have to check all the edges.
- This algorithm works best, of course, if the number of edges is kept to a minimum

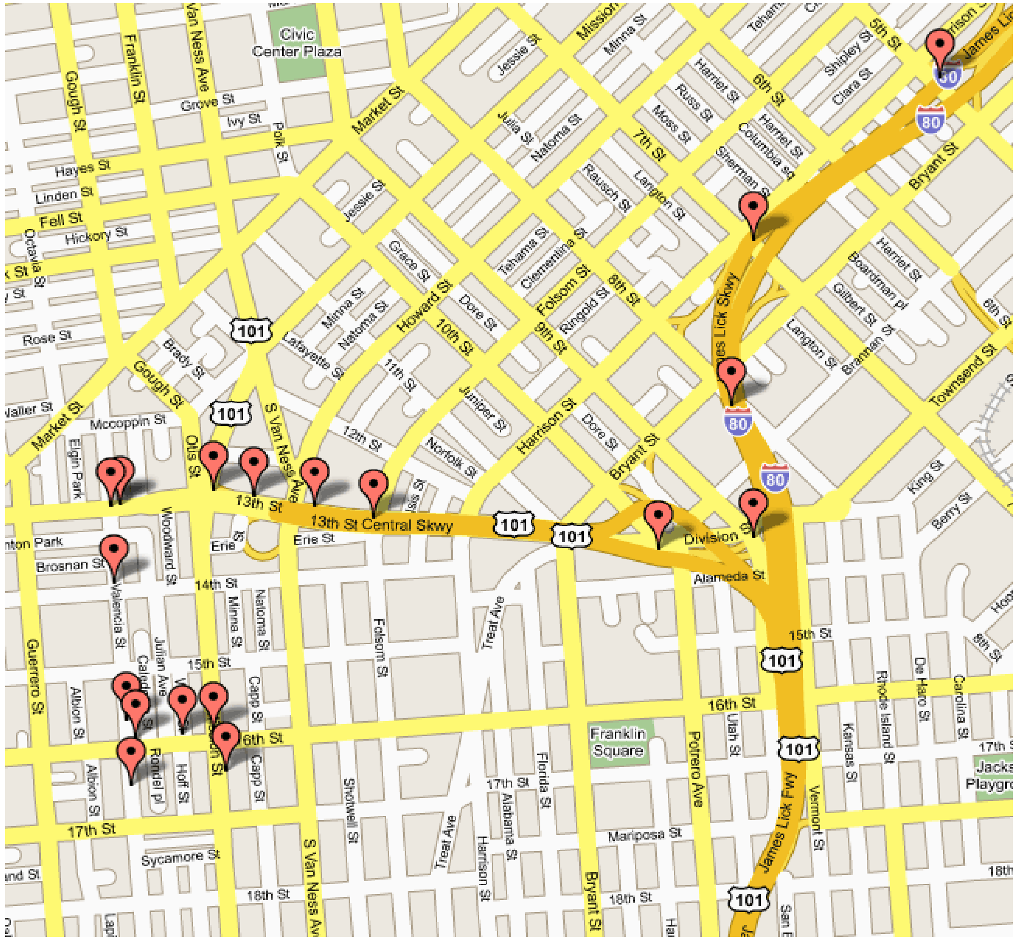
Single-Source Shortest Paths

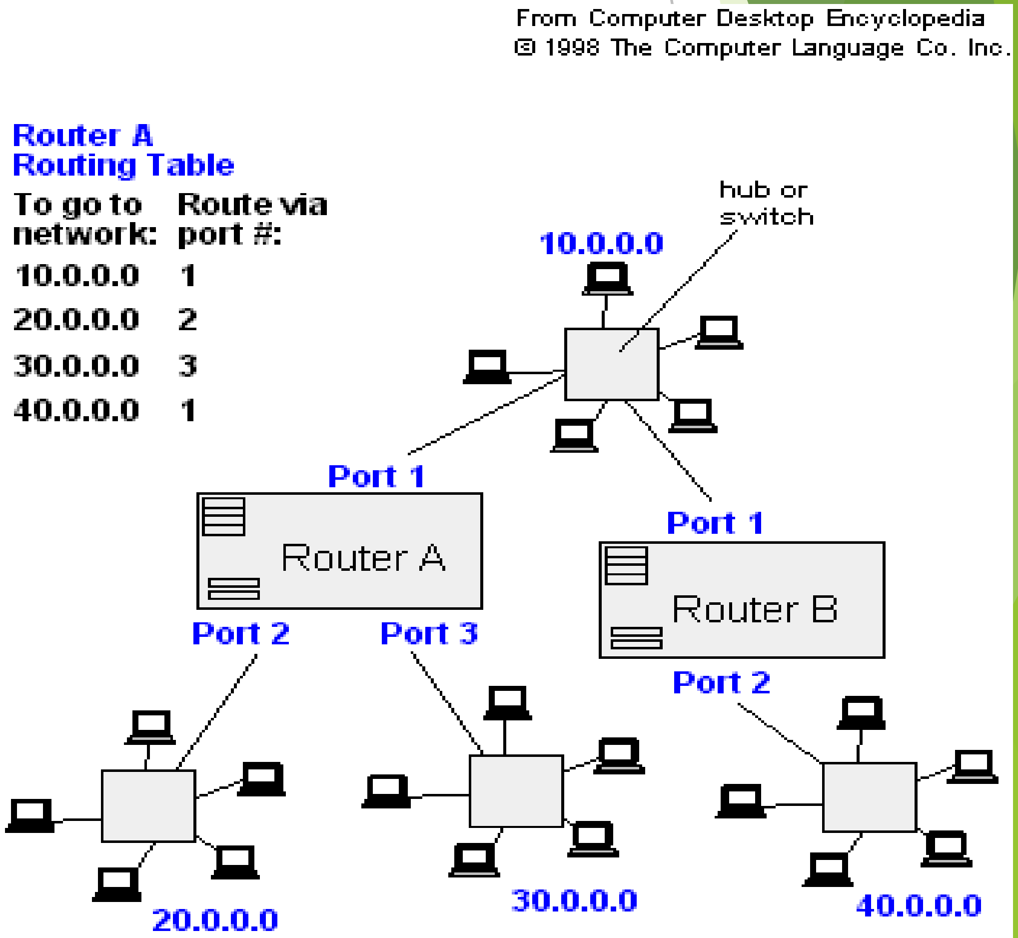
- The problem of finding shortest paths from a source vertex v to all other vertices in the graph.



Application of SSSP

- Maps (Map Quest, Google Maps)
- - Routing Systems





Dijkstra's Algorithm

- This is a solution to the single-source shortest path problem in graph theory.

- Works on both directed and undirected graphs. However, all edges must have nonnegative weights.

- Input: Weighted graph $G=\{E,V\}$ and source vertex $v \in V$, such that all edge weights are nonnegative

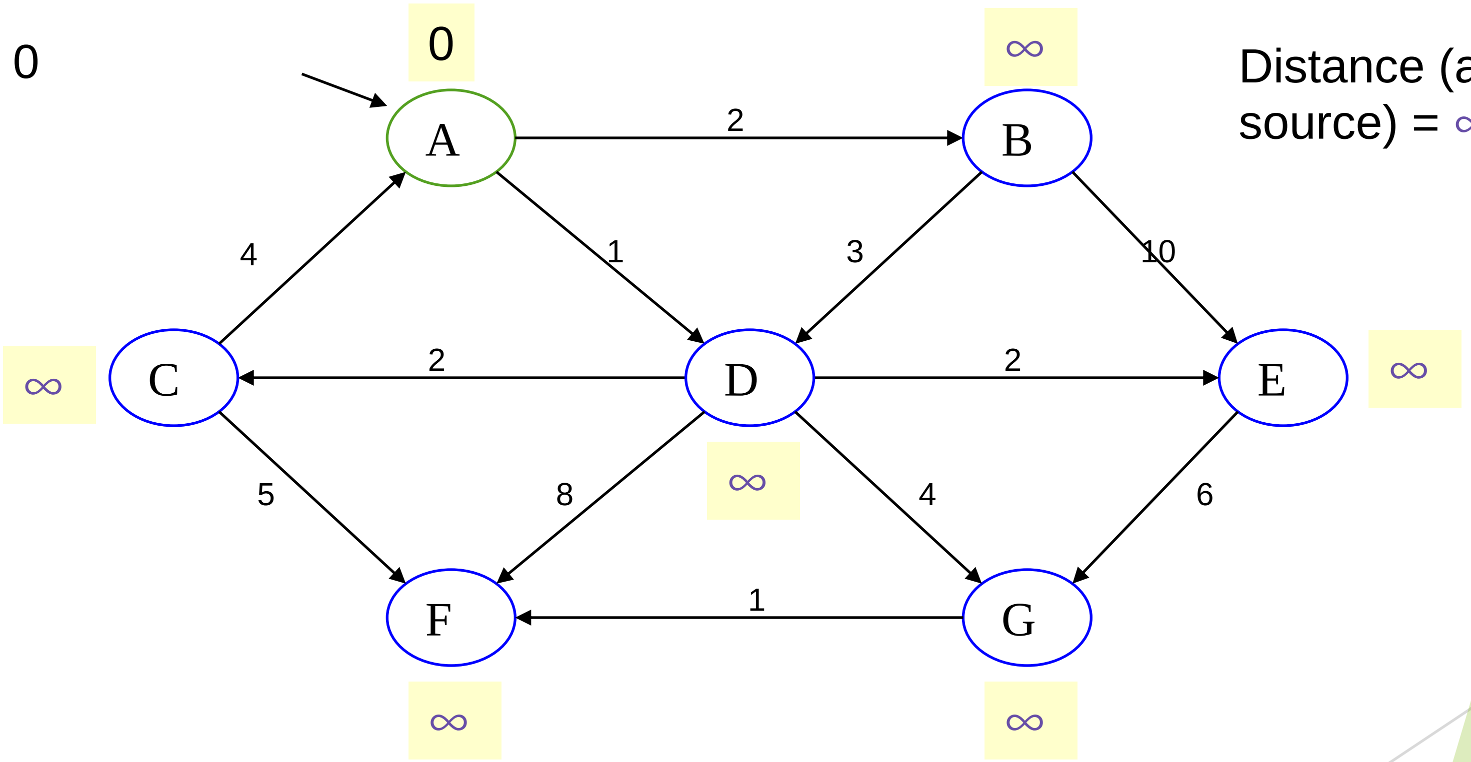
- Output: Lengths of shortest paths (or the shortest paths themselves) from a given source vertex $v \in V$ to all other vertices

Approach

- The algorithm computes for each vertex u the distance to u from the start vertex v, that is, the weight of a shortest path between v and u.
- The algorithm keeps track of the set of vertices for which the distance has been computed, called the cloud C
- Every vertex has a label D associated with it. For any vertex u, D[u] stores an approximation of the distance between v and u. The algorithm will update a D[u] value when it finds a shorter path from v to u.
- When a vertex u is added to the cloud, its label D[u] is equal to the actual (final) distance between the starting vertex v and vertex u.

Example: Initialization

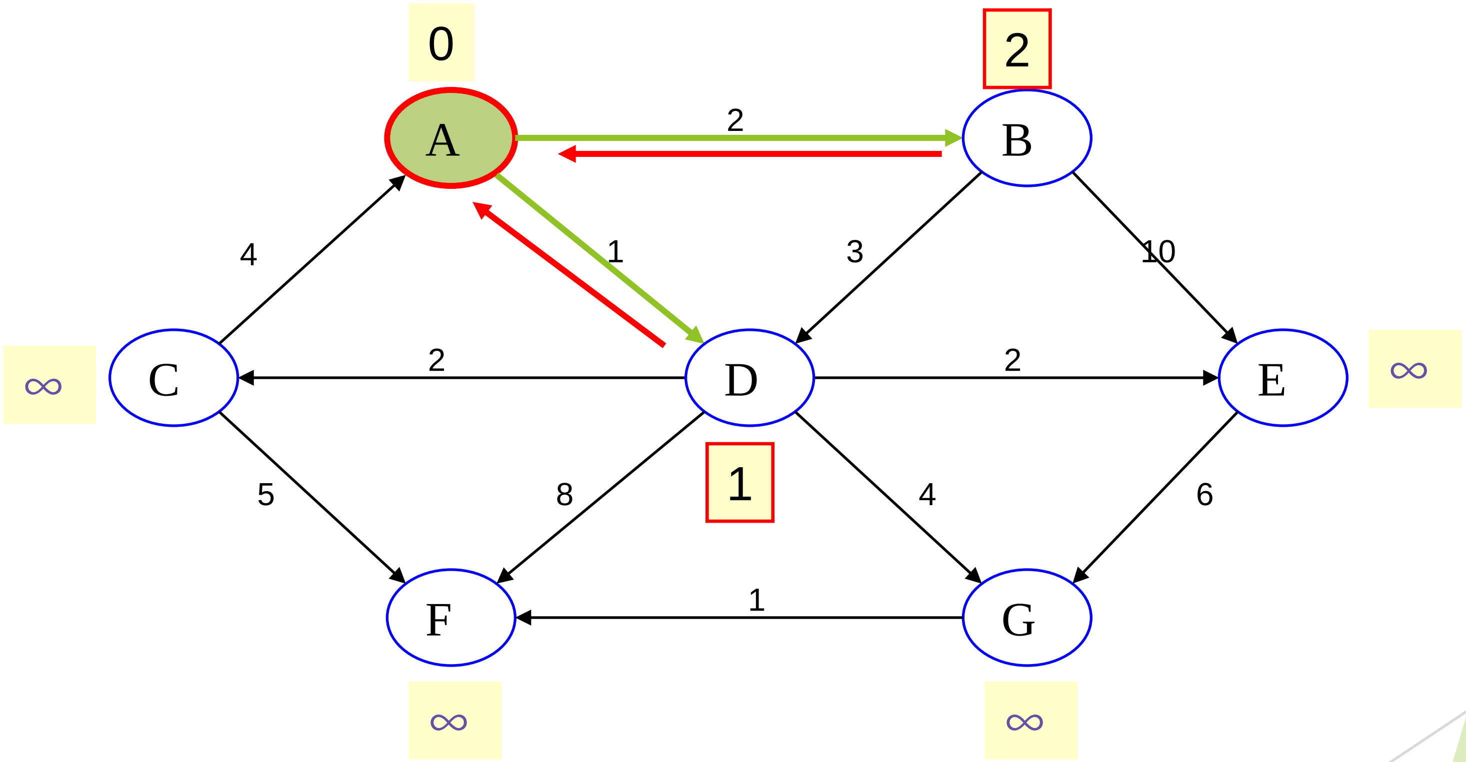
Distance(source) = 0



Distance (all vertices but source) = ∞

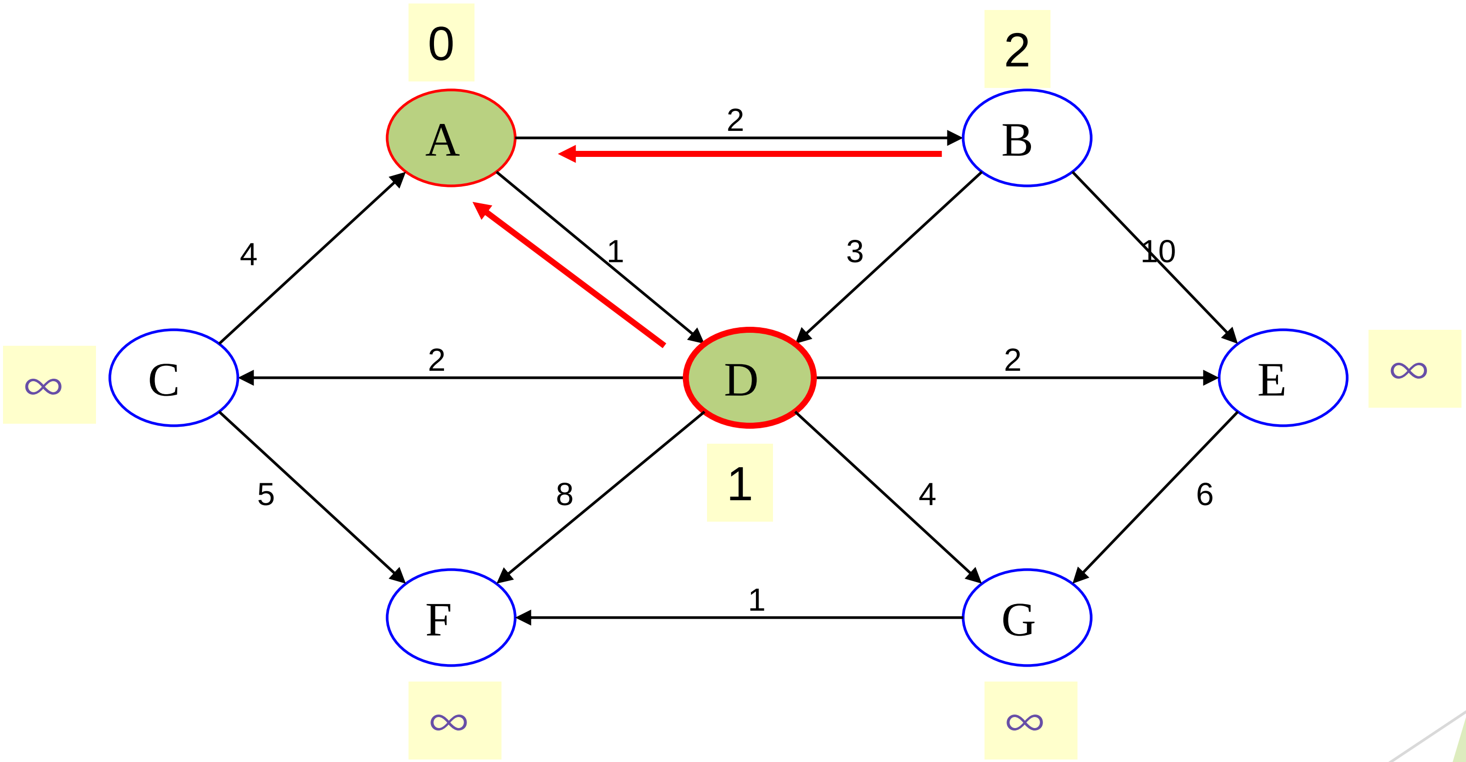
Pick vertex in List with minimum distance.

Example: Update neighbors' distance



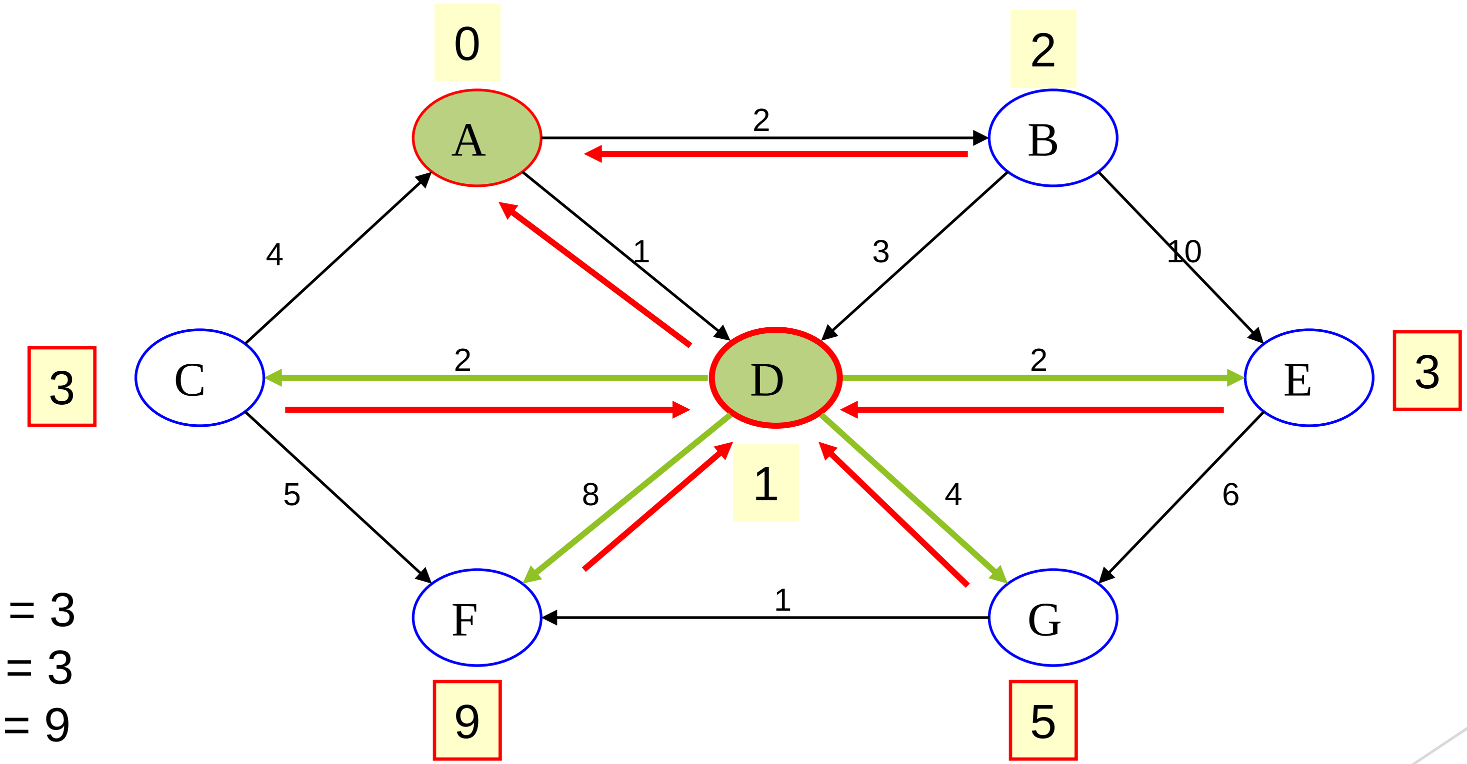
Distance(B) = 2
Distance(D) = 1

Example: Remove vertex with minimum distance



Pick vertex in List with minimum distance, i.e., D

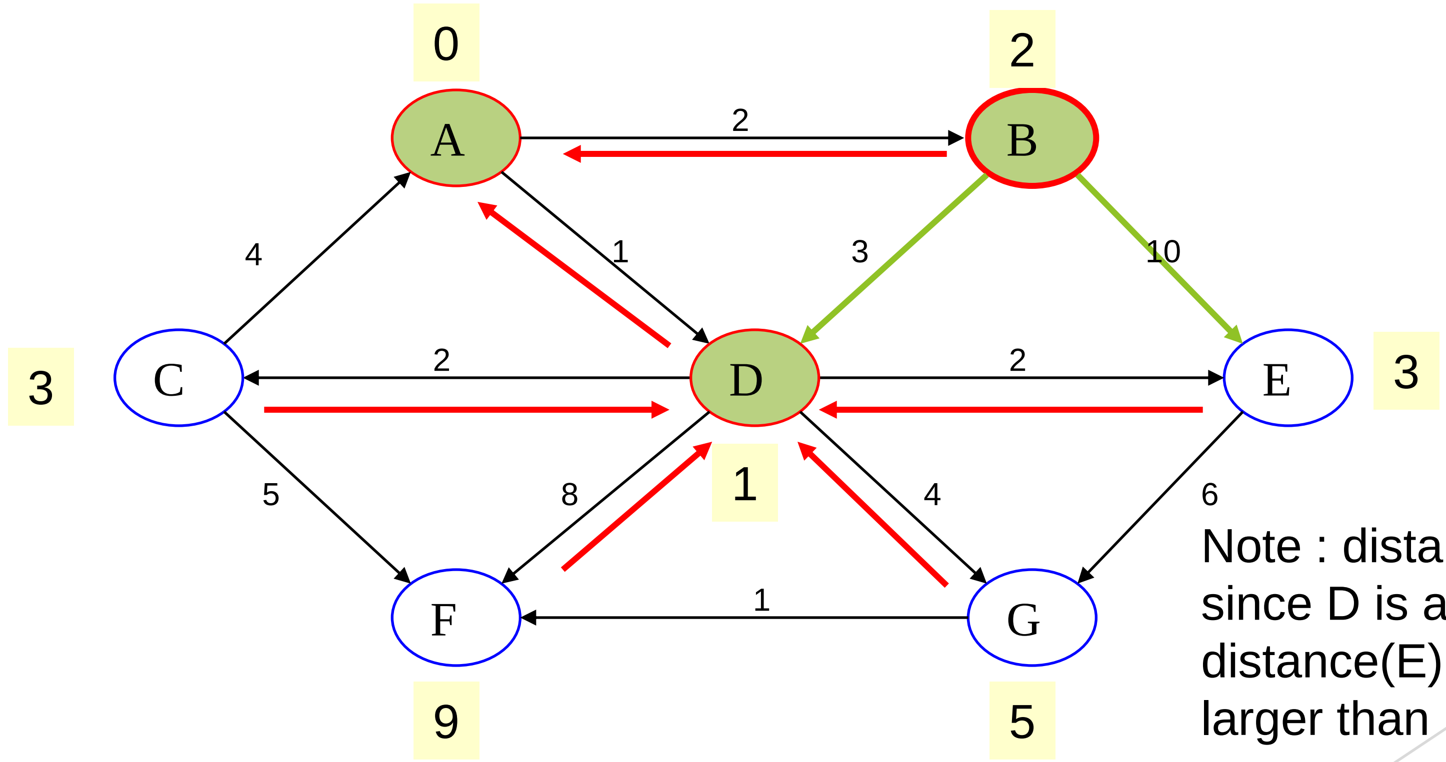
Example: Update neighbors



Distance(C) = 1 + 2 = 3
Distance(E) = 1 + 2 = 3
Distance(F) = 1 + 8 = 9
Distance(G) = 1 + 4 = 5

Example: Continued...

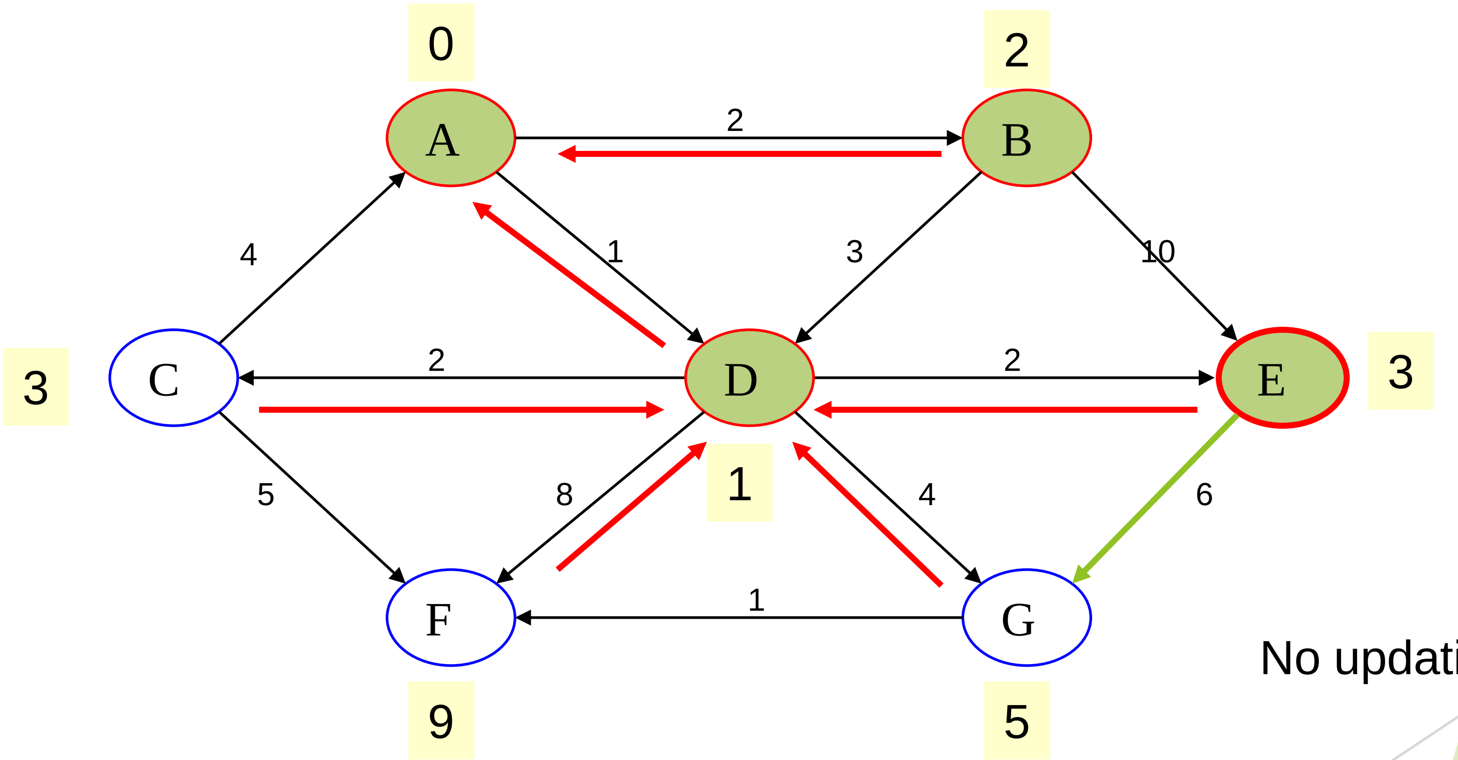
Pick vertex in List with minimum distance (B) and update neighbors



Note : distance(D) not updated since D is already known and distance(E) not updated since it is larger than previously computed

Example: Continued...

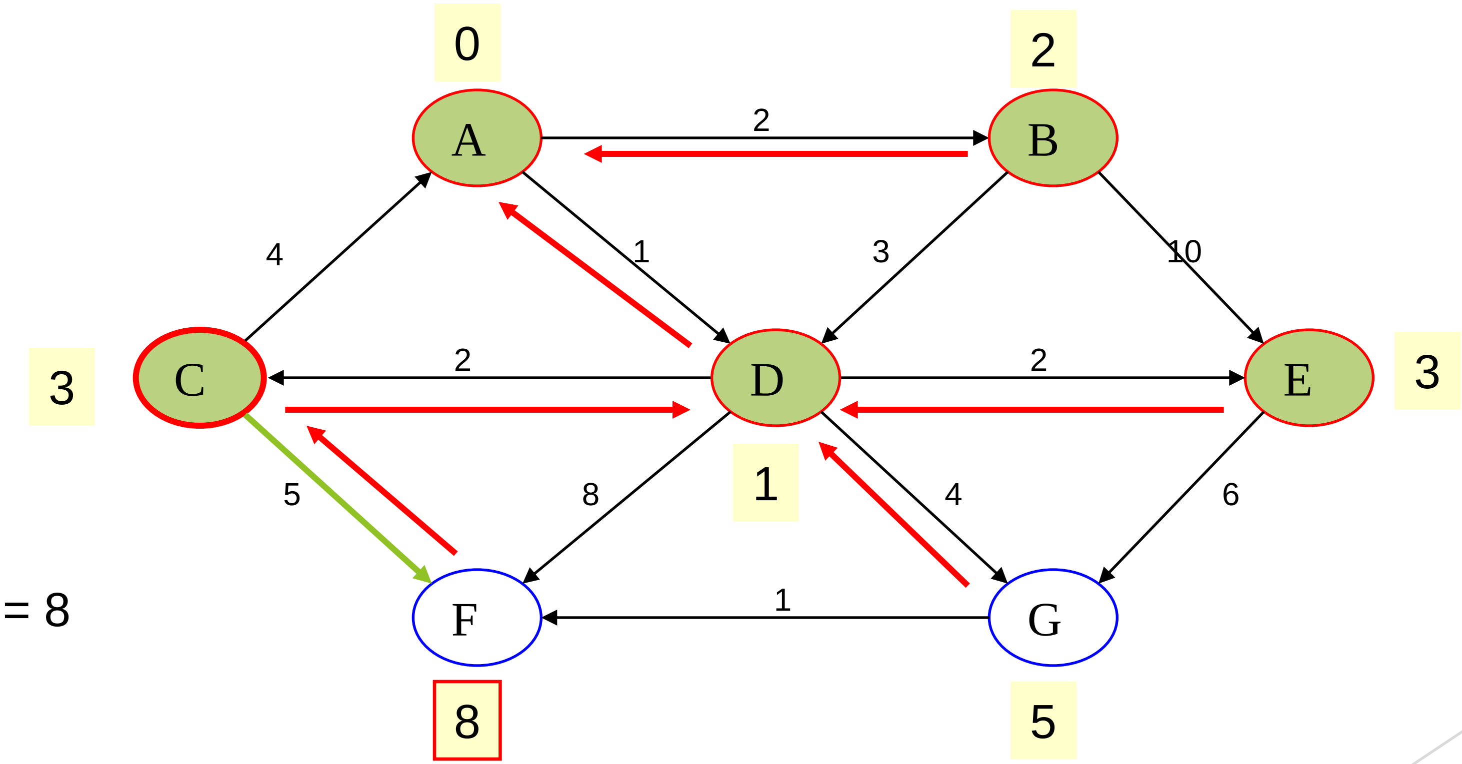
Pick vertex List with minimum distance (E) and update neighbors



No updating

Example: Continued...

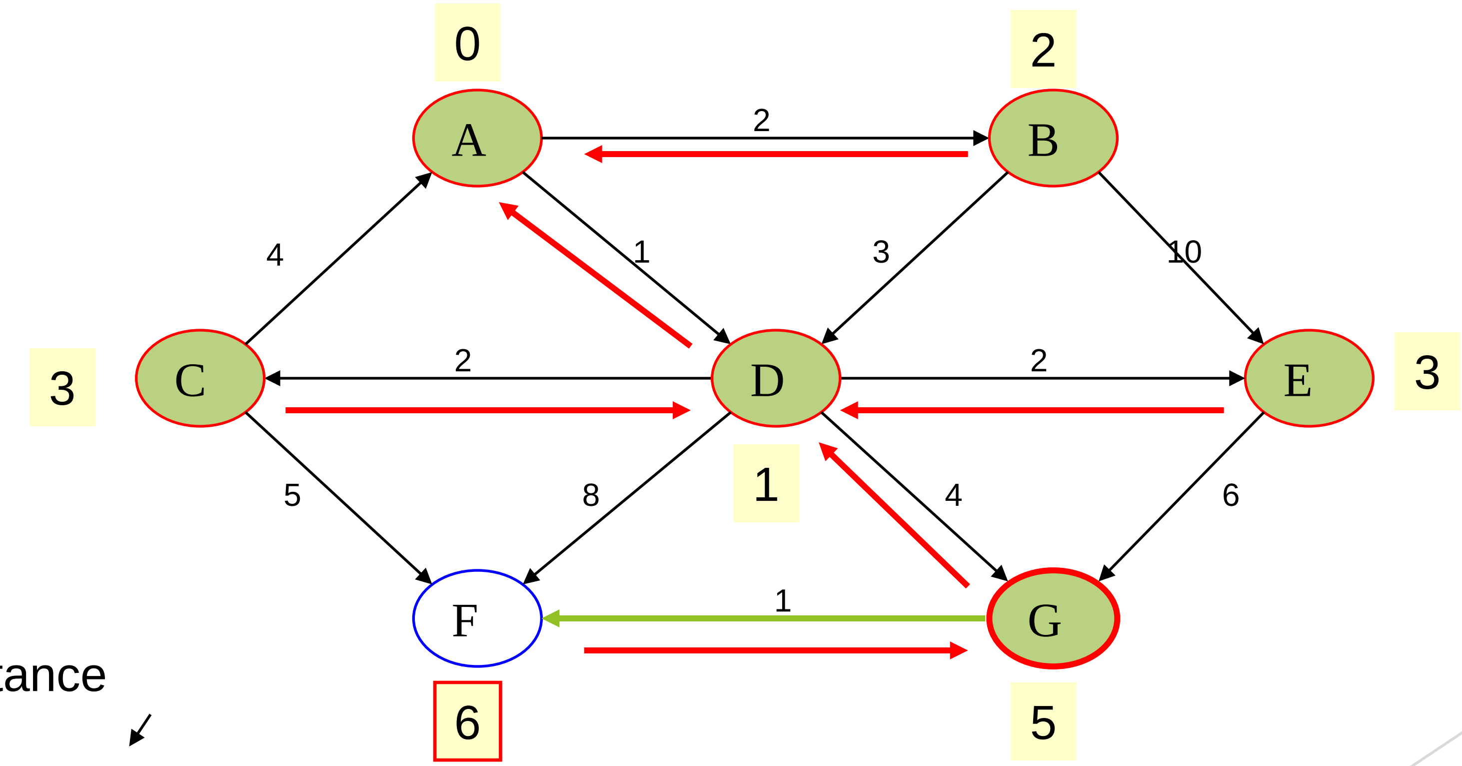
Pick vertex List with minimum distance (C) and update neighbors



Distance(F) = 3 + 5 = 8

Example: Continued...

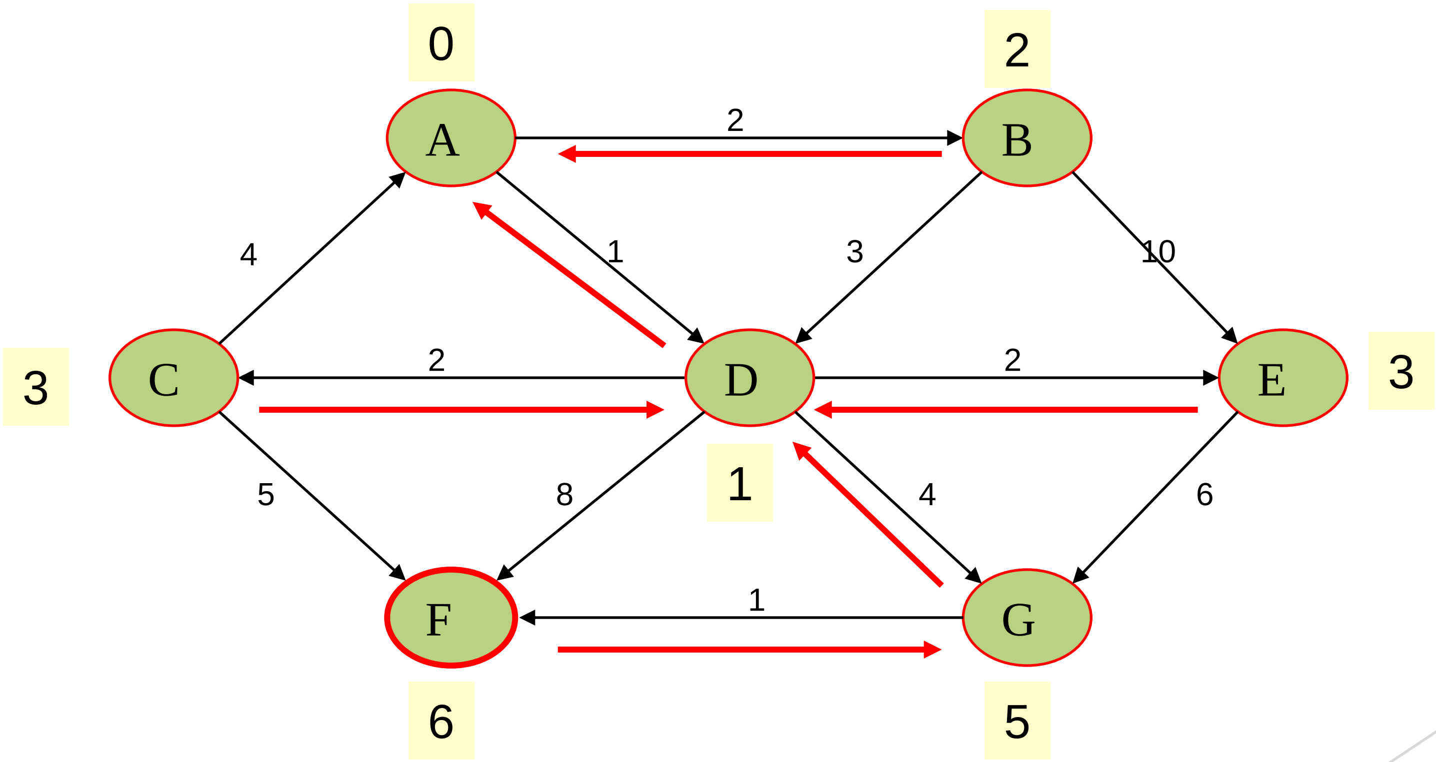
Pick vertex List with minimum distance (G) and update neighbors



Previous distance

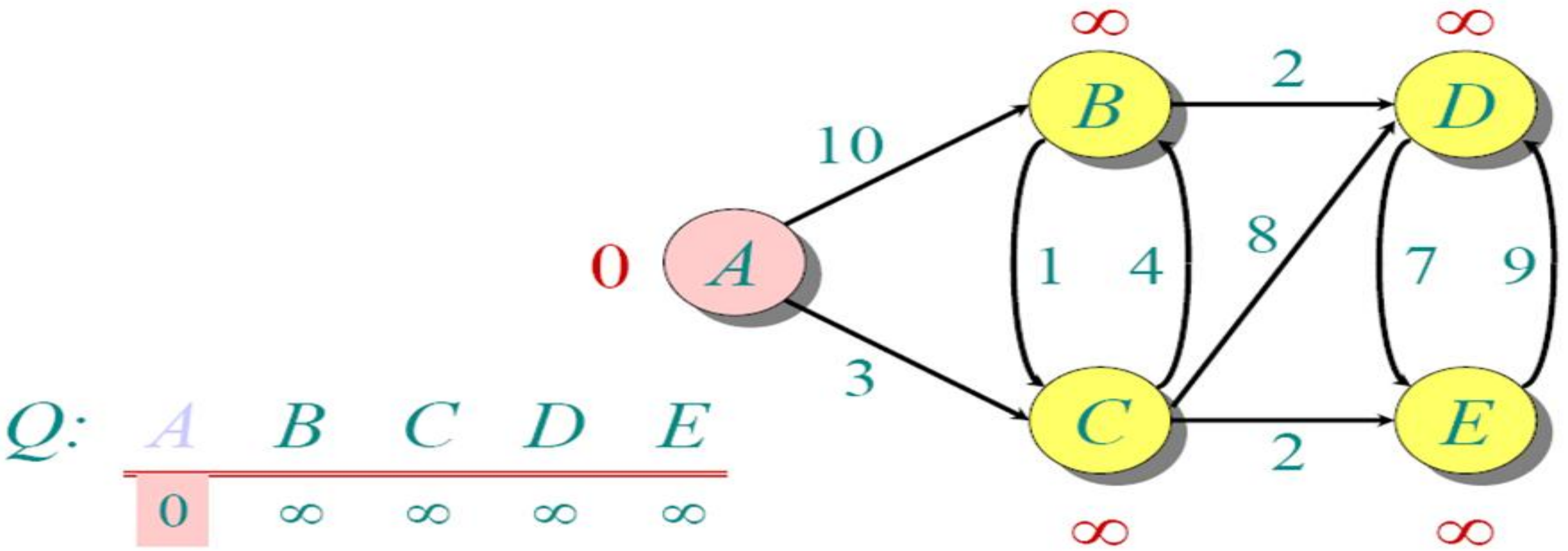
Distance(F) = min (8, 5+1) = 6

Example (end)

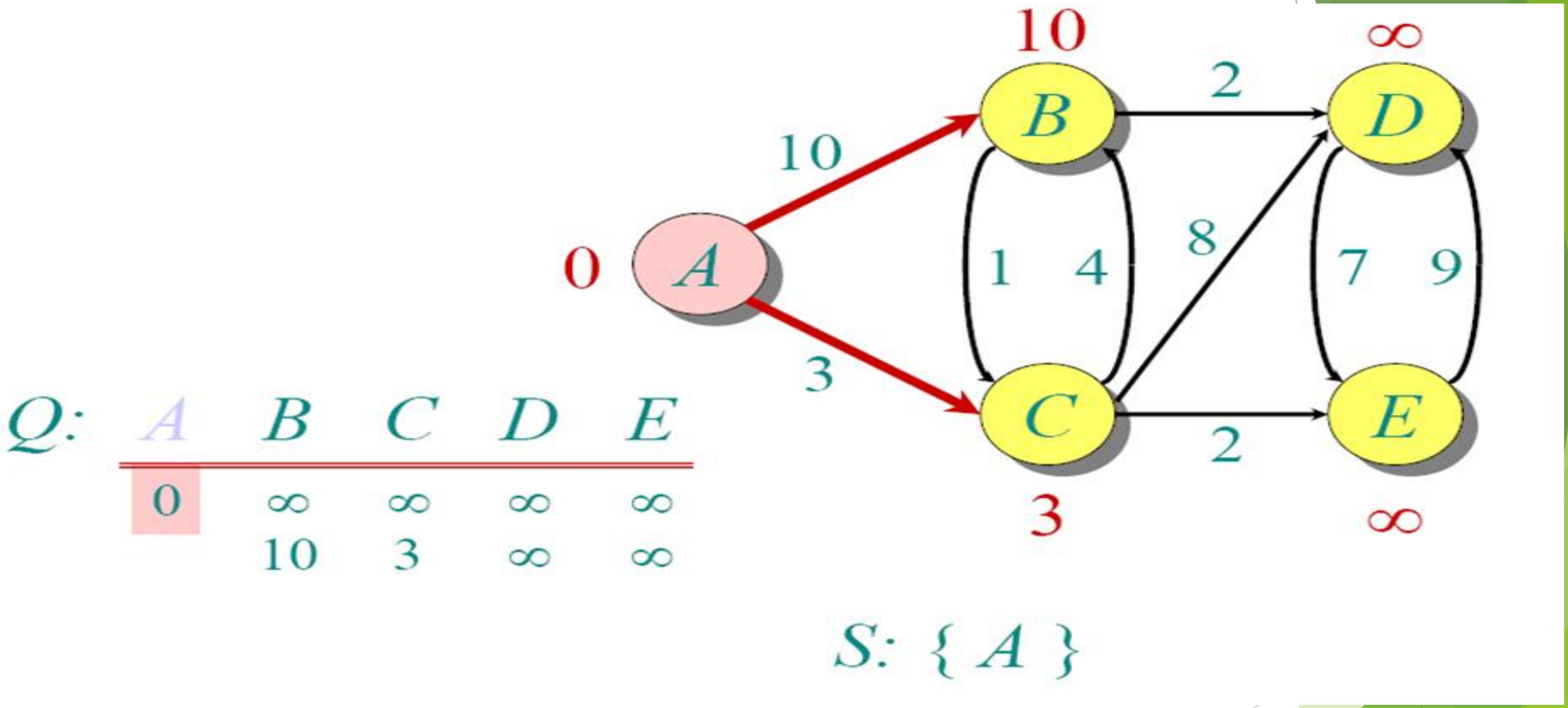


Pick vertex not in S with lowest cost (F) and update neighbors

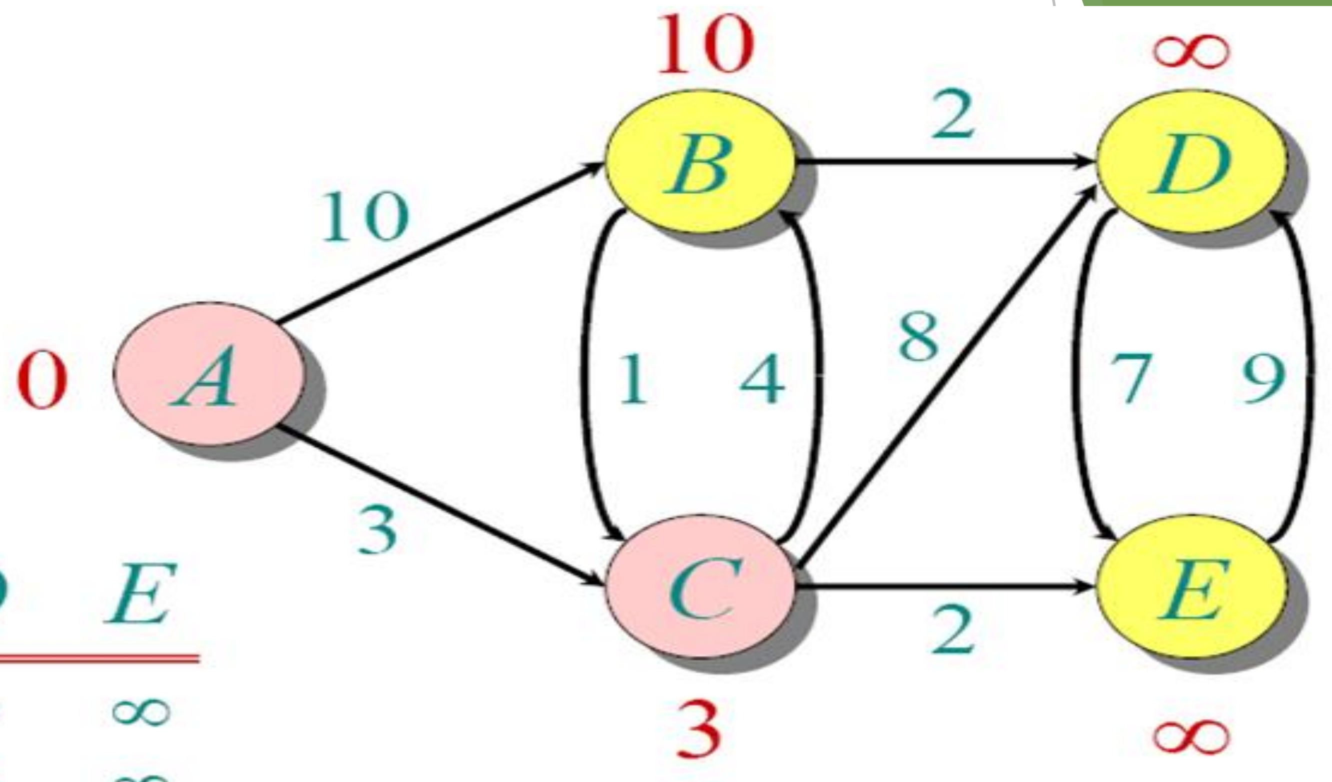
Another Example



Another Example



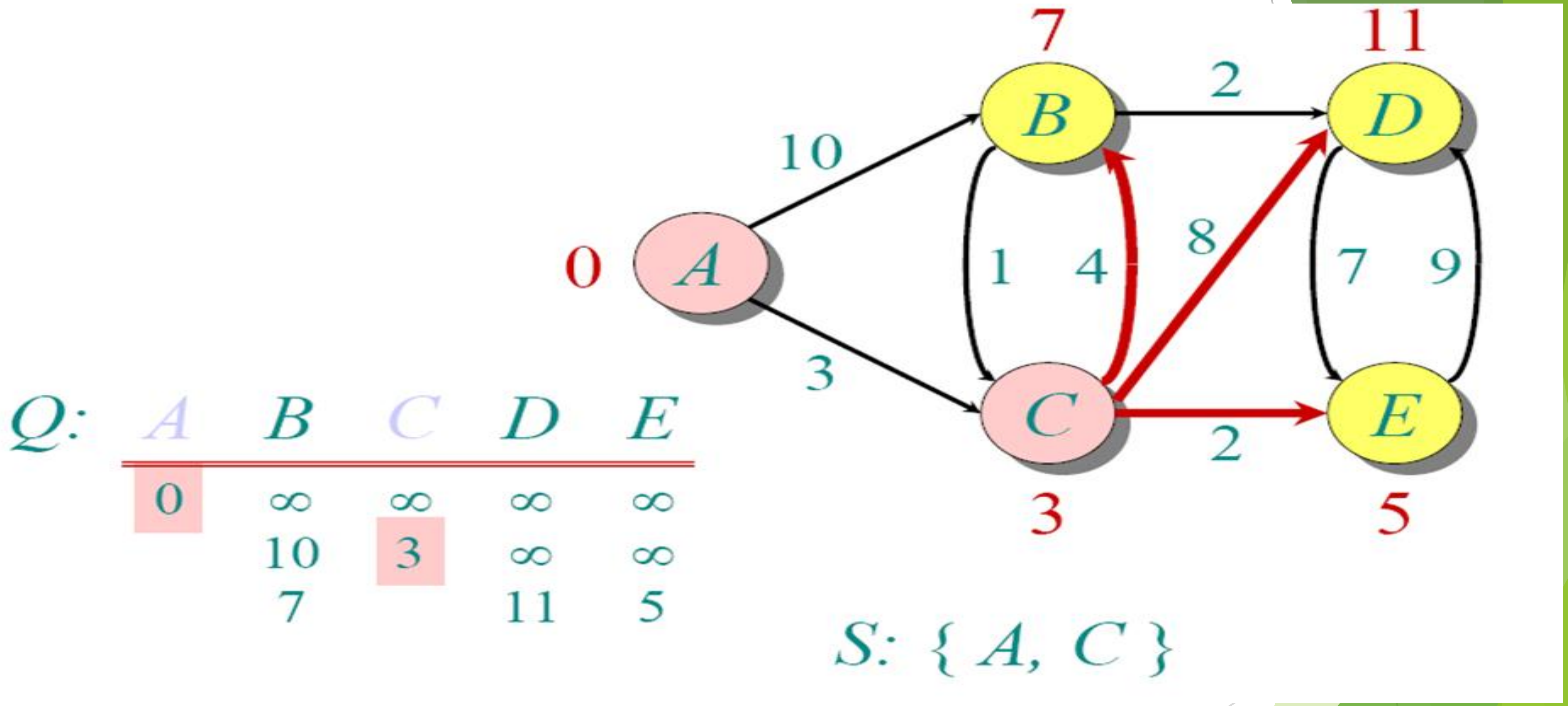
Another Example



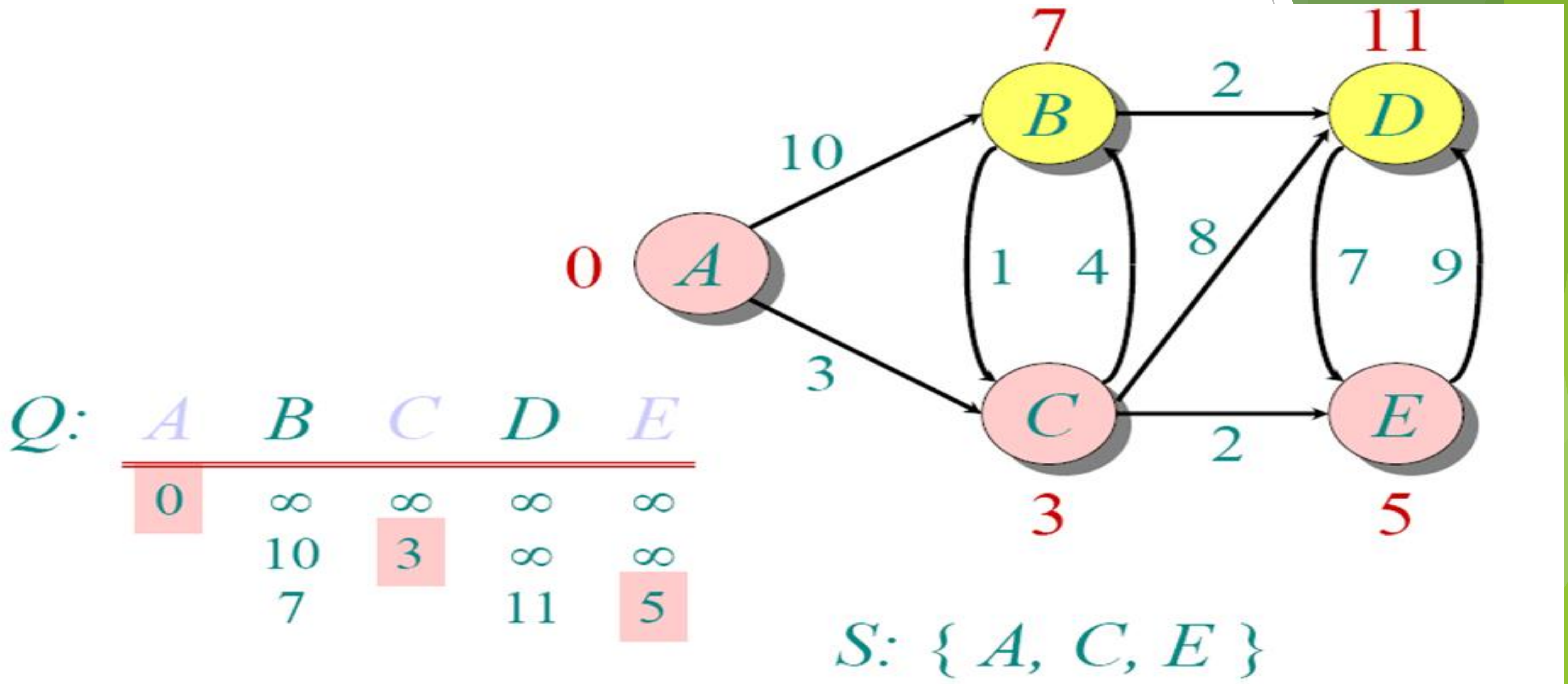
Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A, C }

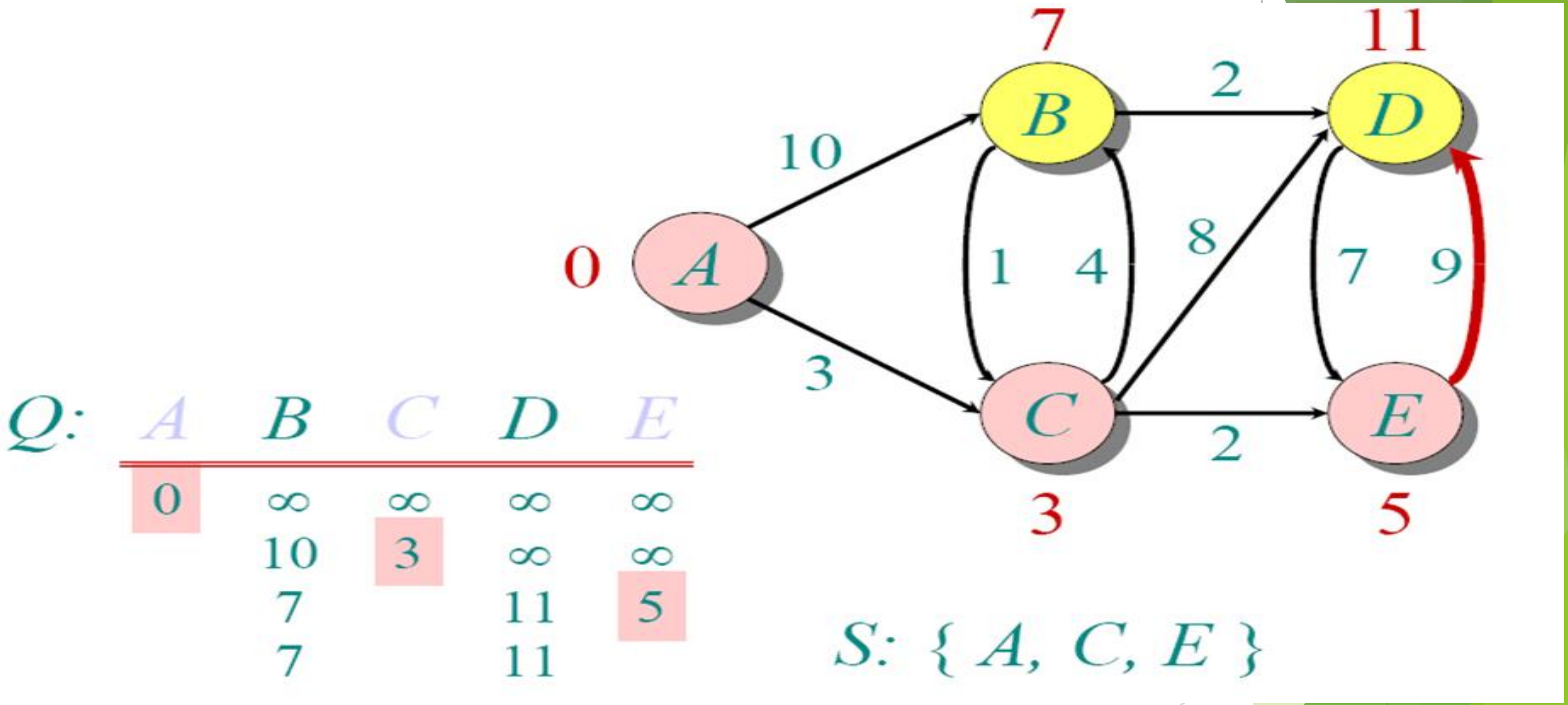
Another Example



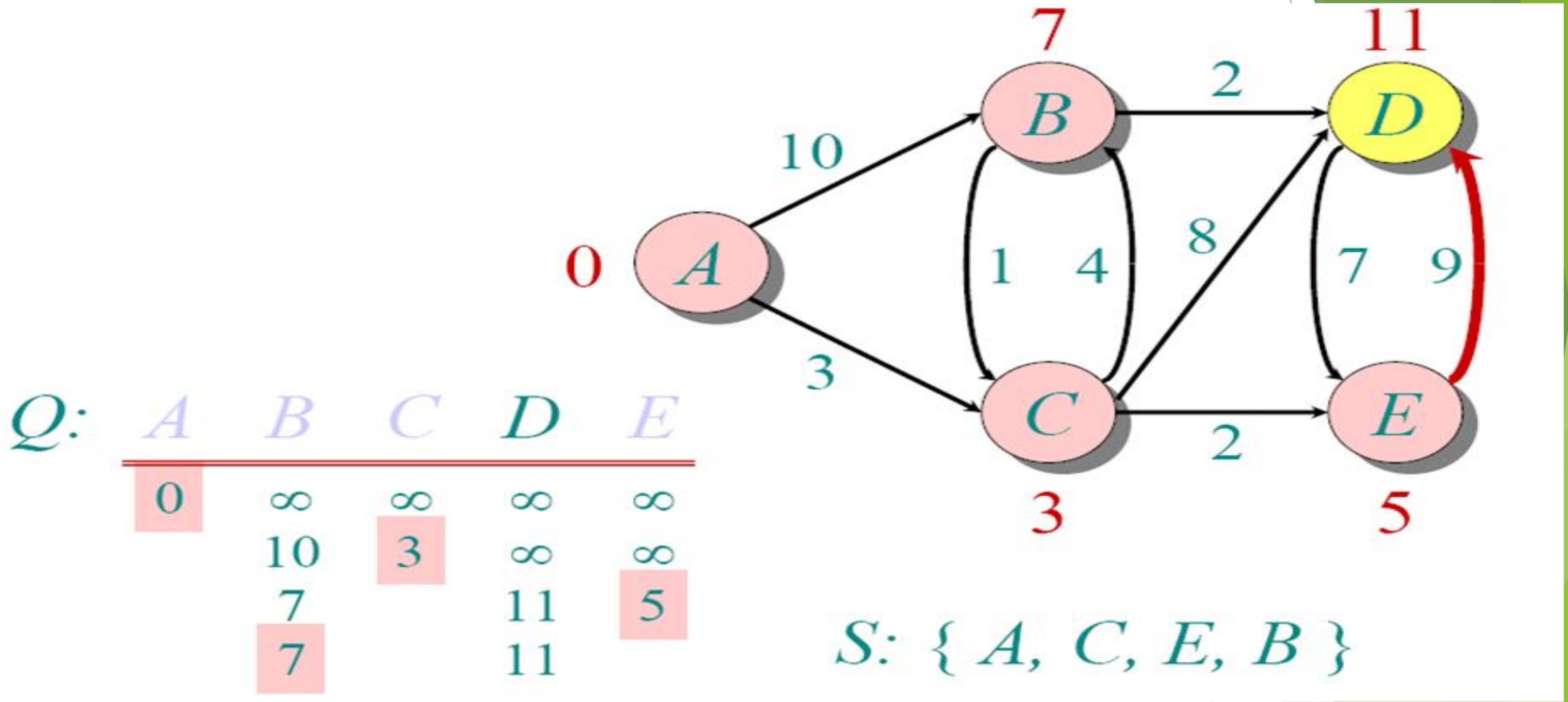
Another Example



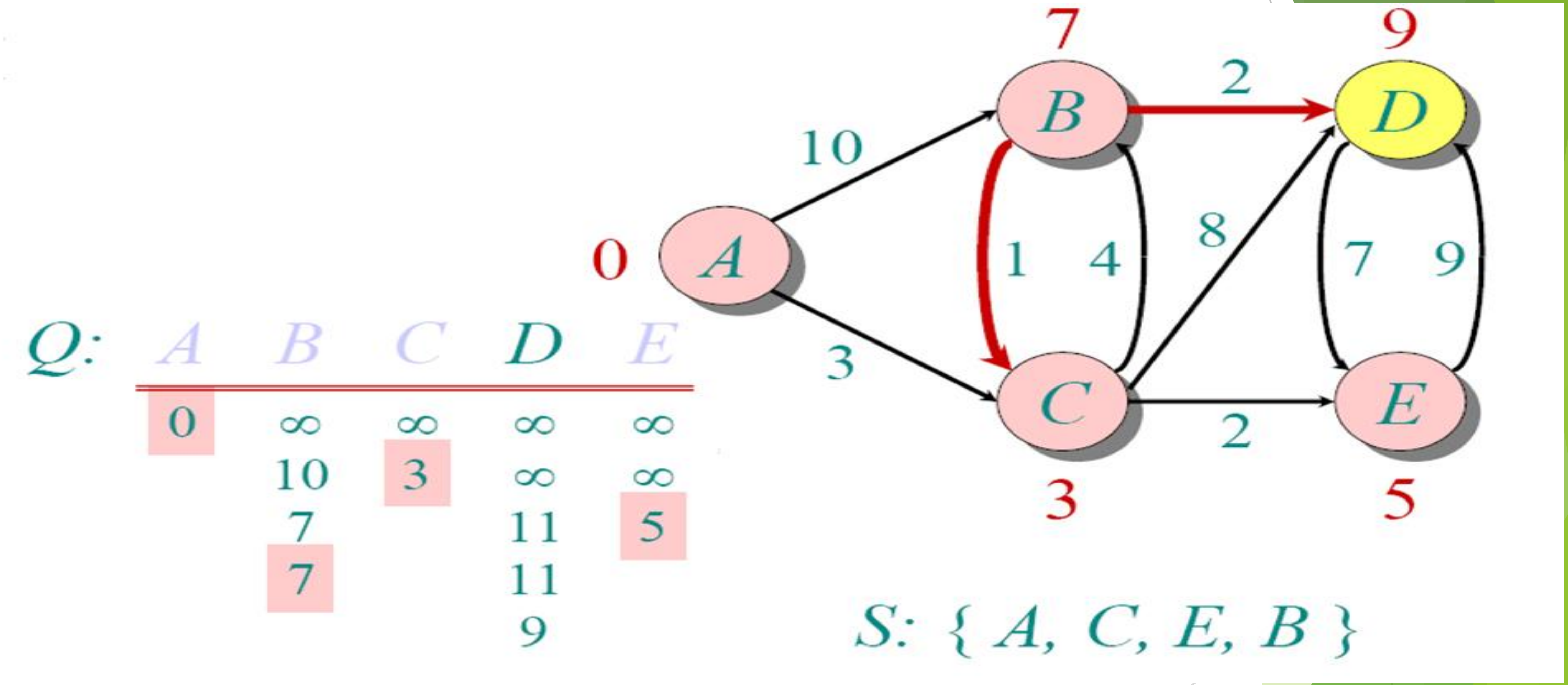
Another Example



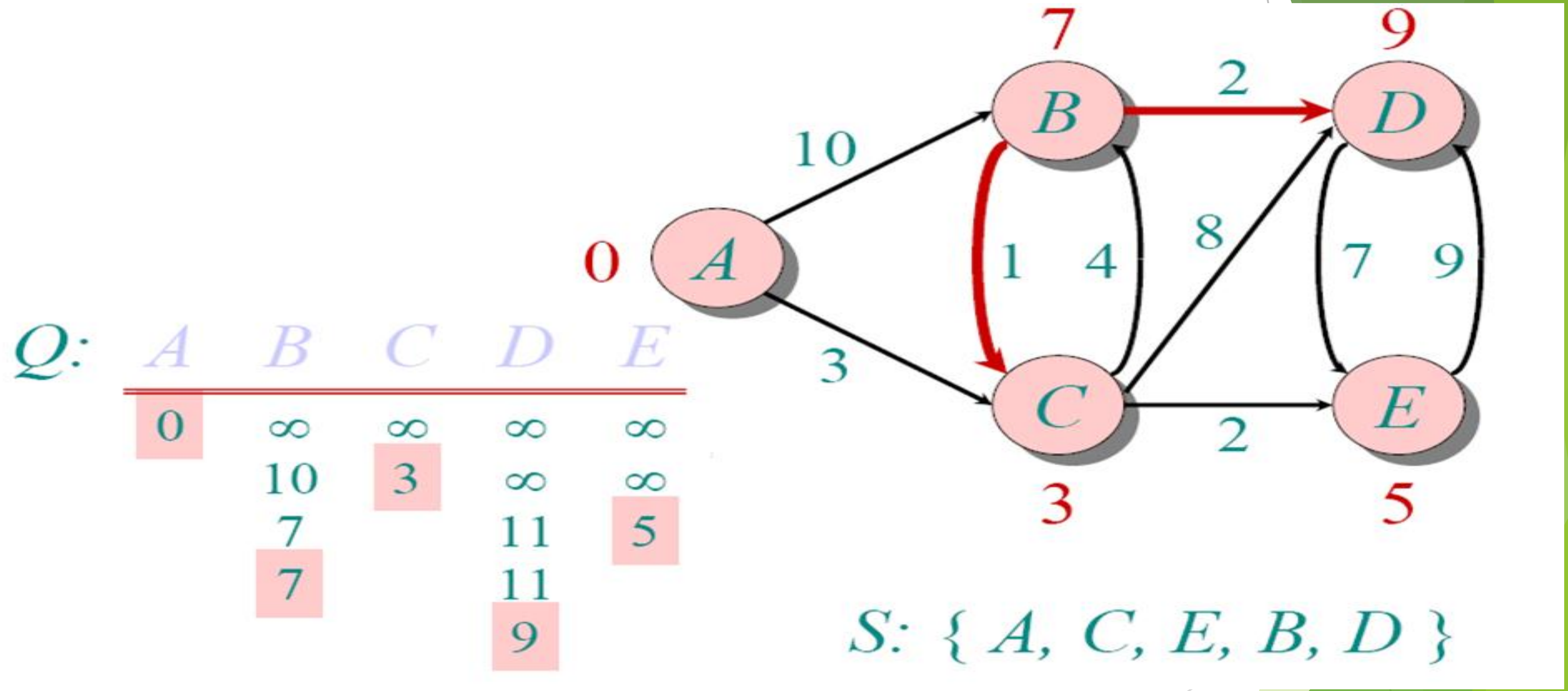
Another Example



Another Example



Another Example



Dijkstra's Pseudo Code

- Graph *G*, weight function *w*, root *s*

DIJKSTRA(G, w, s)

for each $v \in V$
    do $d[v] \leftarrow \infty$
$d[s] \leftarrow 0$
$S \leftarrow \emptyset$ $\triangleright$ Set of discovered nodes
$Q \leftarrow V$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        $S \leftarrow S \cup \{u\}$
        for each $v \in Adj[u]$
            do if $d[v] > d[u] + w(u, v)$
                then $d[v] \leftarrow d[u] + w(u, v)$

relaxing edges

Thanks for your Attention

Q&A