Taylor Series' Pi approximation

potiron-lotte, mehdiben7

May 2022

We want to estimate the digits of pi using solely using trigonometric definitions of the constant.

1 Trigonometric definition of π

Let T be a right triangle of unknown hypotenuse where both cathetus are of size a, and where both of the non-right angles are $\frac{\pi}{4}$ radians.

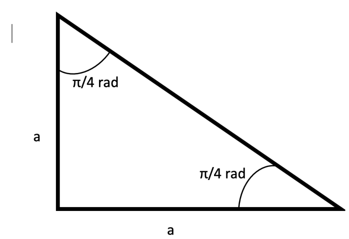

Figure 1: Right triangle T

$$tan(\theta) = \frac{a}{a} \tag{1}$$

It is therefore easy to derived a definition for π

$$tan(\theta) = 1$$

$$arctan(1) = \theta = \frac{\pi}{4}$$

$$4arctan(1) = \pi \tag{2}$$

QED

2 Taylor Series

2.1 Definition

Provided a power series approximation for the function $f(x)$ about $x = a$ exists, the **Taylor Series** of this function is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

2.2 Taylor series approximation of the $arctan(x)$ function

Since we want to approximate $arctan(1)$, our approximation will be around $x = 1$, and so will be the Taylor series approximation. However, finding the Taylor series expansion for the function $arctan(x)$ is a bit tricky. Will use a trick to do so:

$$\frac{d}{dx}(arctan(x)) = \frac{1}{1+x^2}$$

The Taylor series of the derivative of the $arctan(x)$ function can therefore be defined as:

$$1 - x^2 + x^4 - x^6 + ... + (-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

And using integration, we can determine the Taylor series approximation for the $arctan(x)$ function

$$\int \frac{dx}{1+x^2} = arctan(x)$$

$$\sum_{n=0}^{\infty} \int (-1)^n x^{2n} dn = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = arctan(x)$$

2.3 Using Taylor series approximation to find π

Earlier, we found a trigonometric definition for π on equation 2. It the becomes easy to use the Taylor series we found in the previous section to estimate π.

$$\pi = 4arctan(1) = 4\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$4 \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(-1)^n}{2n+1} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \pi$$

Hence, the larger k gets, the more accurate the estimation π gets.

$$4\sum_{n=0}^{m} \frac{(-1)^n}{2n+1} \approx \pi \mid m \in N, > 0 \qquad (3)$$