

Solving Quadratic Equations

Feel free to skip this part if you feel confident in your knowledge regarding quadratic equations.

Definition

By definition, a quadratic equation is an equation of second order with one unknown variable. “Order”, in this case, refers to the highest power of the unknown variable in the equation. In our case, that’s two.

Here are three quadratic equations. Notice that we have x to the power of **two** in all of them.

$$x^2 + x + 1 = 0$$

$$x^2 - 4 = 0$$

$$x^2 + 10x = 0$$

And here we have three equations which are of the first order. As you can see, the highest power is **one**:

$$x - 1 = 0$$

$$x - 2 = 4$$

$$2x + 3 - 5 = 0$$

Here is a quadratic equation in its general form:

$$ax^2 + bx + c = 0$$

The letters **a**, **b** and **c** are constant coefficients, while **x** is the unknown variable.

There are *three* case scenarios regarding the number of possible solutions:

- the equation can have *two distinct solutions*, also called *roots* of the equation
- the equation can have a *single solution* – a *double root*
- the equation has *no existing solutions* at all

The formula for the discriminant of this equation looks like this.

$$D = b^2 - 4ac$$

The discriminant is just a number computed by subtracting four times **a** times **c** from the square of the coefficient **b**. Based on it, we decide on *the number of solutions* the equation has.

- if the discriminant is a positive number, we have *two distinct solutions*
- if **D** equals zero, the quadratic equation has *a single solution*
- if the discriminant is a negative number, then there are *no solutions* to the quadratic equation

When the discriminant is a positive number, the following formula gives the two solutions, x_1 and x_2 , of the equation.

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

But what happens when the discriminant equals zero?

Well, setting D to zero here, the solution to our equation is unique and is given by:

$$x = \frac{-b}{2a}$$

Let's see an example with a positive discriminant and two distinct solutions first.

Solving the Quadratic Equation – First Approach

Here is an example equation with integer coefficients.

$$x^2 - 7x + 6 = 0$$

It's generally good practice to calculate the discriminant first, before deciding on the number of solutions there may be. In our example this looks as follows:

$$\begin{aligned} D &= b^2 - 4ac \\ &= 49 - 24 \\ &= 25 > 0 \end{aligned}$$

Finally, apply the formula for the solutions. So, first, we'll calculate D which is 25 – a positive number. Therefore, there are two distinct solutions to the equation.

$$\begin{aligned} x_1 &= \frac{-b + \sqrt{D}}{2a} \\ &= \frac{7 + \sqrt{25}}{2 \cdot 1} \\ &= \frac{7 + 5}{2} \\ &= 6 \\ x_2 &= \frac{-b - \sqrt{D}}{2a} \\ &= \frac{7 - \sqrt{25}}{2} \\ &= \frac{7 - 5}{2} \\ &= 1 \end{aligned}$$

By applying the formula, we find that our two distinct solutions are:

$$x_1 = 6$$

$$x_2 = 1$$

Let's cover cases where we have no solution and one solution possible, respectively.

Here is our second quadratic equation.

$$3x^2 - 6x + 3 = 0$$

When we calculate its discriminant, we see that it equals zero.

$$D = 36 - 36 = 0$$

Thus, there is a single solution to this equation. By applying the formula with D equals zero, we get:

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{-6}{2 * 3} = \frac{6}{6} = 1 \end{aligned}$$

Finally, here is our third equation:

$$x^2 - x + 3 = 0$$

We calculate the discriminant, which equals a negative number:

$$\begin{aligned} D &= b^2 - 4ac \\ &= 1 - 4 * 3 = -11 \end{aligned}$$

Here, we conclude that there are no solutions to this equation for real numbers.

Solving the Quadratic Equation – Second Approach

After we've seen the three types of quadratic equations, let's briefly discuss an alternative approach to solving those, in which the discriminant is either zero or a positive number.

There is another way we can solve this equation,

$$ax^2 + bx + c = 0$$

which is: writing it as a product of expressions comprising the unknown variable. Let's demonstrate that with the same equation. We start with the equation with the general form again,

$$ax^2 + bx + c = 0$$

but this time the goal is to rewrite it in the following form.

$$(x - \alpha)(x - \beta) = 0$$

Here, the letters **alpha** and **beta** stand for expressions comprising all the coefficients a, b, and c in the equation.

Consider our example from earlier:

$$x^2 - 7x + 6 = 0$$

We'll re-write the following way:

$$x^2 - x - 6x + 6 = 0$$

$$x(x - 1) - 6(x - 1) = 0$$

Extracting x minus one as a factor, the expression is equivalent to:

$$(x - 1)(x - 6) = 0$$

Finally, we find the same roots as with the previous approach by equating each of these factors to zero:

$$x_1 - 1 = 0$$

$$x_1 = 1$$

$$x_2 - 6 = 0$$

$$x_2 = 6$$

Note that sometimes one way is much faster than the other. Take this equation as an example.

$$x^2 - 4 = 0$$

Here we use the following formula:

$$a^2 - b^2 = (a - b)(a + b)$$

This way, we rewrite the last equation as:

$$(x - 2)(x + 2) = 0$$

We obtain the solutions by setting x minus two to equal zero, giving the first solution as two and, similarly, x plus two equals zero gives the second solution as minus two.

$$x_1 - 2 = 0$$

$$x_2 + 2 = 0$$

This has been our preliminary lecture on solving quadratic equations.

Thanks for sticking around!