Exercise session on numerical solution of ODEs, 4

November 4, 2021

Exercise 1

Consider the Cauchy problem

$$\begin{cases} \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad t \in [0, T], \\ \mathbf{y}(0) = [1, 2, -10]^T, \end{cases}$$

where $\mathbf{g} = [2, 2, 2]^T$ and

$$\mathbf{A} = \begin{pmatrix} -30 & 0 & -28 \\ -29 & -1 & -29 \\ 0 & 0 & -2 \end{pmatrix},$$

where $T = 4$.

- Compute a reference solution using the MATLAB solver `ode45` with a relative error tolerance of 10^{-9} and a maximum time step $h = T/N, N = 200$.
- Compute a numerical solution using the three step BDF3 method with $N = 200$ time steps. Use the reference solution to provide the numerical initial condition and a tolerance value of 10^{-8}. Estimate the required computational time using the `tic` and `toc` commands.
- Compute for each component the relative errors over the whole time interval in the infinity norm. Say which component has the largest relative error.
- Repeat the previous points with the three stage Runge Kutta method. Say which of the two methods is more efficient in this case.
- Repeat the previous points using $N = 20$. Say which of the two methods is more efficient and accurate in this case.

Exercise 2

Consider the second order equation

$$y'' = -y - y' \quad y(0) = 1 \;\; y'(0) = 0 \quad t \in [0, 4],$$

whose exact solution is

$$y_{ex}(t) = \exp(-\frac{t}{2}) \cos(\frac{\sqrt{3}}{2} t) + \frac{1}{\sqrt{3}} \exp(-\frac{t}{2}) \sin(\frac{\sqrt{3}}{2} t).$$

- Transform the equation in a first order system.
- Solve the resulting system using the Heun method and the two step Adams Bashforth method, using $N = 400$ time steps. Compute the relative errors in the infinity norm over the whole time interval with respect to the exact solution.
- Repeat the computation using the $\theta-$ method and the BDF2 method. Use the $\theta-$ method to provide the supplementary initial condition for BDF2. Use $\theta = 0.53$ and use the value 10^{-9} for the tolerance of the nonlinear solver employed by the implicit methods.
- Compare the errors obtained and the times required by these methods to obtain the solutions with those required by the Heun method and the two step Adams Bashforth method.
- Compare the errors obtained and the times required by these methods to obtain the solutions with that required by the leapfrog (Verlet) method.

Exercise 3

Consider the second order Van der Pol equation

$$y'' = -y + \mu(1 - y^2) y' \quad y(0) = 1 \;\; y'(0) = -1 \quad t \in [0, 20].$$

- Transform the equation in a first order system.
- In the case $\mu = 2$, compute a reference solution using the MATLAB solver `ode15s` with a relative error tolerance of 10^{-10} and a maximum time step $h = T/N, N = 500$.
- Solve the resulting system using the third order Runge Kutta method and the three step Adams Bashforth method, using $N = 500$ time steps. Use the third order Runge Kutta method to provide the supplementary initial condition for the three step Adams Bashforth method. Compute the relative errors in the infinity norm over the whole time interval with respect to the exact solution.
- Repeat the computation for the Crank Nicolson method and the BDF2 method using $N = 100$ time steps in the case $\mu = 20$. Use the Crank Nicolson method to provide the supplementary initial condition for the BDF2 method. Use the value 10^{-7} for the tolerance of the nonlinear solver employed by the implicit methods.
- Compare the errors obtained and the times required by these methods to obtain the solutions with that required by the leapfrog (Verlet) method.