

Exercise session on finite difference approximations of parabolic PDEs, part 3

November 26, 2021

In all exercises, compute the l^2 norm using the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 1

Consider the homogeneous heat equation

$$\frac{\partial c}{\partial t} = \nu \frac{\partial^2 c}{\partial x^2}$$

with **periodic boundary conditions** on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 10$, $T = 5$, $\nu = 0.1$. Assume as initial datum

$$c_0(x) = 10 \exp \left\{ - \left(\frac{x - L/2}{L/10} \right)^2 \right\}.$$

- Compute the exact solution by separation of variables on a uniform mesh of $N = 100$ points.
- Compute on the same mesh a numerical solution by the implicit Euler method using $M = 20$ time steps. Plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.
- Estimate empirically the convergence rate by repeating the computation using $N = 200$ time steps and $M = 40$ time steps.
- Compare the accuracy of the results with those of the explicit Euler method applied with a time step such that $\nu \Delta t / \Delta x^2 = 1/5$.

Exercise 2

Repeat the previous exercise using the Crank-Nicolson method. Compare the accuracy of the results with those of the implicit Euler method.

Exercise 3

Consider the non homogeneous, variable coefficients diffusion equation

$$\begin{cases} \frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left[\frac{x^2}{100} \frac{\partial c}{\partial x} \right] + s(x, t) & x \in (0, 2), t \in (0, 2), \\ y(x, 0) = y_0(x) = x^3 & x \in (0, 2), \end{cases}$$

with $s(x, t) = 2tx^3 - 3x^3(t^2 + 1)/25$. Notice that the exact solution is given by $c(x, t) = x^3(t^2 + 1)$. Solve the equation on a mesh of $N = 200$ nodes and $M = 50$ time steps with the Crank Nicolson method, considering

- Dirichlet boundary conditions $g_0(t) = 0, g_L(t) = 8(t^2 + 1)$;
- Neumann boundary conditions $g_0(t) = 0, g_L(t) = 12(t^2 + 1)$;
- flux boundary conditions $g_0(t) = 0, g_L(t) = 12(t^2 + 1)/25$.

Plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.