Exercise session on the numerical solution of linear systems, 2

November 11, 2021

Exercise 1

Build a matrix $\mathbf{A}$ of dimension 400×400 that has all elements equal to -6 on the main diagonal, all elements equal to -1 on the first subdiagonal and superdiagonal and all elements equal to 1 on the fourth subdiagonal and superdiagonal (use the command `diag`).

(a) Check that the matrix is diagonally dominant by rows.

(b) Check that the matrix is symmetric and positive definite.

(c) Compute the vector $\mathbf{b} = \mathbf{A}\mathbf{x}_{ex}$, where $\mathbf{x}_{ex} = [1, -1, \ldots, 1, -1]^T$, $\mathbf{x}_{ex} \in \mathbf{R}^{400}$.

(d) Solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ using by the Cholesky factorization method.

(e) Compute the absolute and relative error of the computed solution with respect to $\mathbf{x}_{ex}$ in the l^2 and l^∞ norm and compare them with the *a priori* and *a posteriori* error estimates.

Exercise 2

Consider the $n \times n$ matrix $\mathbf{H}_n$, where $\mathbf{H}_n$ the Hilbert matrix of dimension n, built using the command `hilb`. For $n = 1, \ldots, 20$, check if the matrix is diagonally dominant by rows and if the matrix is symmetric and positive definite; then

(a) compute the vector $\mathbf{b} = \mathbf{A}\mathbf{x}_{ex}$, where $\mathbf{x}_{ex} = [2, 2, \ldots, 2, 2]^T$, $\mathbf{x}_{ex} \in \mathbf{R}^n$;

(b) solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ using by the Cholesky factorization method;

(c) compute the absolute and relative error of the computed solution with respect to $\mathbf{x}_{ex}$ in the l^2 and l^∞ norm and compare them with the *a priori* and *a posteriori* error estimates.

(d) report the errors and the *a priori* error estimates on a semilogarithmic plot as a function of $n = 1, \ldots, 20$.

Exercise 3

Build a matrix $\mathbf{B}$ of dimension 300×300 that has all elements equal to 4 on the main diagonal, all elements equal to -1 on the first subdiagonal, all elements equal to 1 on the first superdiagonal, all elements equal to 0.5 on the tenth subdiagonal and all elements equal to -0.3 on the ninth superdiagonal (use the command `diag`). Consider the 300×100 matrix $\mathbf{C}$ that consists of the first 100 columns of $\mathbf{B}$ and build the 100×100 matrix $\mathbf{A} = \mathbf{C}^T\mathbf{C}$.

(a) check that the matrix $\mathbf{A}$ is symmetric and positive definite

(b) compute the vector $\mathbf{b} = \mathbf{A}\mathbf{x}_{ex}$, where $\mathbf{x}_{ex} = [1, 2, \ldots, 99, 100]^T$, $\mathbf{x}_{ex} \in \mathbf{R}^n$;

(c) solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ using by the Cholesky factorization method;

(d) compute the absolute and relative error of the computed solution with respect to $\mathbf{x}_{ex}$ in the l^2 and l^∞ norm and compare them with the *a priori* and *a posteriori* error estimates.

Exercise 4

Let $\mathbf{A} = 10^{-8}\mathbf{I} + \mathbf{H}_{20}$ and consider the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ so that $\mathbf{x}_{ex} = [1, 1, \ldots, 1, 1]^T$. Consider then the perturbed system $\mathbf{A}\tilde{\mathbf{x}} = \mathbf{b}+\mathbf{e}$, where $\mathbf{e}$ is a vector of independent, identically distributed Gaussian variables of zero mean and variance σ built using the command `randn`. Say which is the maximum value of σ for which an *a priori* error bound of 10^{-2} on the computation of $\mathbf{x}$ can be guaranteed.