Exercise session on finite difference approximations of parabolic PDEs, part 2

November 25, 2021

In all exercises, compute the l^2 norm using the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 1

Consider the non homogeneous, variable coefficients diffusion equation

$$\begin{cases} \dfrac{\partial c}{\partial t} = \dfrac{\partial}{\partial x}\left[\nu(x)\dfrac{\partial c}{\partial x}\right] + s(x,t) & x \in (0,\pi), t \in (0,1), \\ c(x,0) = c_0(x) = 1 + \dfrac{x^3}{10} & x \in (0,\pi) \end{cases}$$

with $\nu(x) = \frac{1}{10} + \frac{1}{100}\cos(x)$ and

$$s(x,t) = -\sin(t)\left(1 + \frac{x^3}{10}\right) + \frac{3x^2}{1000}\sin(x)\cos(t) - \left(\frac{1}{10} + \frac{1}{100}\cos(x)\right)\cos(t)\frac{6x}{10}.$$

Notice that the exact solution is given by

$$c(x,t) = \cos(t)\left(1 + \frac{x^3}{10}\right).$$

Compute a numerical solution by the explicit Euler method using a uniform mesh of $N = 100$ intervals, a number of time steps such that the stability condition is satisfied and applying either

- Dirichlet boundary conditions with $g_0(t) = \cos(t),\ g_\pi(t) = \cos(t)(1 + \pi^3/10)$;
- Neumann boundary conditions with $g_0(t) = 0,\ g_\pi(t) = \cos(t)3\pi^2/10)$;
- flux boundary conditions with $g_0(t) = 0,\ g_\pi(t) = \cos(t)(27\pi^2/1000)$.