Exercise session on matrix manipulation and numerical solution of linear systems

November 5, 2021

Exercise 1

Build the following matrices

$$\mathbf{A} = \begin{bmatrix} 50 & 1 & 3 \\ 1 & 6 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 50 & 1 \\ 3 & 20 \\ 10 & 4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}.$$

(a) Compute $\mathbf{D} = \mathbf{I} + \mathbf{BC}$.

(b) Check that $\mathbf{A}$ is different from $\mathbf{A}^T$. Compute

$$\mathbf{A}_s = \frac{\mathbf{A} + \mathbf{A}^T}{2} \qquad \mathbf{A}_{as} = \frac{\mathbf{A} - \mathbf{A}^T}{2}$$

and chech that $\mathbf{A}_s = \mathbf{A}_s^T$ and $\mathbf{A}_{as} = -\mathbf{A}_{as}^T$.

(c) Check that $\mathbf{AD}$ is different from $\mathbf{DA}$. and compute the commutator

$$[\mathbf{A}, \mathbf{D}] = \mathbf{AD} - \mathbf{DA}.$$

(d) Compute $\mathbf{E} = \mathbf{I} + 2\mathbf{A}^T\mathbf{A} + 3\mathbf{A}^3$. Check that $\mathbf{E}$ is invertible by computing its determinant with the MATLAB function `det` and compute its inverse with the MATLAB function `inv` .

(e) Check that $\mathbf{E}^{-1}$ and `inv`($\mathbf{E}$) coincide up to roundoff errors.

Exercise 2

Build the matrices

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{D} & \mathbf{E} \\ -\mathbf{E}^T & \mathbf{D}^{-1} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{A}_1 \\ 2\mathbf{I} & -\mathbf{A}_1 & \mathbf{I} \\ \mathbf{A}_1^T & \mathbf{0} & 3\mathbf{I} \end{bmatrix},$$

where $\mathbf{I}, \mathbf{0}$ are the identity and the zero matrix of the appropriate dimension. Compute the dimensions of $\mathbf{A}_1, \mathbf{A}_2$ using the MATLAB function `size`.

Exercise 3

(a) Build the Toeplitz symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

a) using the MATLAB command `toeplitz` b) applying repeatedly command `diag` c) using repeated for cycles.

(b) Extract the upper and lower triangular matrices contained in $\mathbf{A}$, using the commands `triu` and `tril`. Check what happens with the commands `tril(A,2)` o `triu(A,-2)`.

(c) Compute the determinants of the upper and lower triangular matrices contained in $\mathbf{A}$. Compute the eigenvalues of the upper and lower triangular matrices contained in $\mathbf{A}$ and check that the determinants are a) the product of the eigenvalues b) the product of the terms on the main diagonal.

Exercise 4

Build a vector $\mathbf{v} = [2, 4, 2, 4, \ldots, 2, 4]^T$, $\mathbf{v} \in \mathbf{R}^{100}$.

(a) Build a matrix $\mathbf{A}$ that has $\mathbf{v}$ on the main diagonal, all components equal to -1 on the first superdiagonal and all components equal to 1 on the first subdiagonal.

(b) Compute

$$\mathbf{B} = \frac{-3\mathbf{A} + 2\mathbf{A}^2}{\mathbf{I} + 4\mathbf{A} - \mathbf{A}^4}.$$

(c) Check if $\mathbf{B}$ is invertible by computing its determinant with the MATLAB function `det` and compute its inverse with the MATLAB function `inv`.

(d) Compute the vector $\mathbf{d} = \mathbf{B}\mathbf{x}_{ex}$, where $\mathbf{x}_{ex} = [-1, -1, \ldots, -1, -1]^T$, $\mathbf{x}_{ex} \in \mathbf{R}^{100}$.

(e) Solve the system $\mathbf{B}\mathbf{x} = \mathbf{d}$ using a) the inverse of $\mathbf{B}$ computed by `inv` b) the inverse of $\mathbf{B}$ computed as $\mathbf{B}^{-1}$ c) the command \ (backslash).

Exercise 5

Build the Hilbert matrix of dimension n, given by

$$\mathbf{A} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+1) \\ 1/3 & 1/4 & 1/5 & \cdots & 1/(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/(n+1) & 1/(n+2) & \cdots & 1/(2n-1) \end{pmatrix}$$

using command `hilb`. Consider then the previous matrix $\mathbf{A}$ in the case $n = 7$.

(a) Extract the third column of the matrix; substitute it with a column containing values equal to one. Repeat the procedure with the first three components of the fifth row of the matrix, using in each case a single MATLAB command.

(b) Extract all the diagonals of $\mathbf{A}$ using the command `diag`.

(c) Compute the determinant of matrix $\mathbf{A}$.

(d) Compute the eigenvalues of matrix $\mathbf{A}$.

(e) Compute the vector $\mathbf{b} = \mathbf{A}\mathbf{x}_{ex}$, where $\mathbf{x}_{ex} = [1, 1, \dots, 1, 1]^T$, $\mathbf{x}_{ex} \in \mathbf{R}^7$.

(f) Solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ using a) the inverse of $\mathbf{A}$ computed by `inv` b) the inverse of $\mathbf{A}$ computed as $\mathbf{A}^{-1}$ c) the command \ (backslash). In each case, compute the absolute and relative error of the computed solution with respect to $\mathbf{x}_{ex}$ in the l^2 and l^∞ norm.

Exercise 6

Build a matrix $\mathbf{A}$ of dimension 20×20 such that:

- it has all the integers from 11 to 30 on the main diagonal
- it has all values equal to π on the second upper diagonal
- it has all values equal to 2 on the first lower diagonal
- it has all values equal to 5 on the tenth column (for the values that have not been defined yet)
- it has all values equal to zero elsewhere.

Compute the determinant of matrix $\mathbf{A}$. Compute the eigenvalues of matrix $\mathbf{A}$. Compute the vector $\mathbf{b} = \mathbf{A}\mathbf{x}_{ex}$, where $\mathbf{x}_{ex} = [1, -1, \dots, 1, -1]^T$, $\mathbf{x}_{ex} \in \mathbf{R}^{20}$. Solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ using a) the inverse of $\mathbf{A}$ computed by `inv` b) the inverse of $\mathbf{A}$ computed as $\mathbf{A}^{-1}$ c) the command \ (backslash). In each case, compute the absolute and relative error of the computed solution with respect to $\mathbf{x}_{ex}$ in the l^2 and l^∞ norm.