Exercise session on numerical solution of ODEs, 3

October 29, 2021

Exercise 1

Consider the initial value problem

$$\begin{cases} y' = -\dfrac{1}{2}y^2 + 5y\sin(t) & t \in (0, T), \\ y(0) = y_0, \end{cases}$$

where $T = 3$, $y_0 = 7$.

(a) Build a reference solution using the `ode45` solver enforcing a maximum time step of $T/4000$ and returning the result at a of equally spaced time levels with $N = 400$ time levels.

(b) Compute the an approximation of the solution using the Heun method and the two step Adams Bashforth method, using $N = 400$ time levels. For the Adams Bashforth method, use the reference solution to provide the supplementary initial conditions. Compute the relative errors in the infinity norm for both methods and say which of the two is more accurate for this problem.

Exercise 2

(a) Repeat exercise 1 using the BDF2 two step method, using the Crank-Nicolson method to provide the supplementary initial conditions. Use $N = 40$ time steps and a tolerance value of 10^{-8} for the nonlinear solver required by the implicit methods.

(b) Repeat exercise 1 using the BDF3 three step method, use $N = 40$ time steps and a tolerance value of 10^{-8} for the nonlinear solver required by the implicit methods.

(c) Repeat exercise 1 using $N = 50000$ and $N = 10^5$ steps. Compute the empirical estimate of the convergence order for the two methods, checking if the result is coherent with the theory and, if not, explaining why.

Exercise 3

Consider the Cauchy problem

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t) \qquad \mathbf{y}(0) = [200, 100]^T \qquad t \in [0, T].$$

where $T = 2$ and

$$\mathbf{A} = \begin{bmatrix} -1 & 20 \\ -30 & -\frac{1}{1000} \end{bmatrix} \qquad \mathbf{g}(t) = \begin{bmatrix} t\sin(t) \\ 0 \end{bmatrix}.$$

- Compute a reference solution by the `ode23tb` solver with relative error tolerance 10^{-11}.
- Compute a numerical solution using the Heun method and the Crank Nicolson method with an appropriate value of the parameter θ, using $N = 200$ time steps for both methods and a tolerance value of 10^{-8} for the nonlinear solver required by the Crank Nicolson method and using the exact solution to provide supplementary initial conditions. Compute the relative errors in the infinity norm (separately for the first and second component) over the whole time interval. Say which method is more accurate in this case.
- Repeat the computation using $N = 400$ time steps and compute the empirical convergence rate for both methods, saying if the results are coherent with the theory and why.