Exercise session on finite difference approximations of parabolic PDEs, part 1

November 22, 2021

In all exercises, compute the l^2 norm using the formula

$$\|g\|_2 = \sqrt{\sum_{i=0}^{N_x} |g_i|^2 \Delta x}.$$

Exercise 1

Consider the homogeneous heat equation

$$\frac{\partial c}{\partial t} = \nu \frac{\partial^2 c}{\partial x^2}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 10$, $T = 5$, $\nu = 0.05$. Assume as initial datum

$$c_0(x) = 10 \exp\Big\{ - \Big(\frac{x - L/2}{L/10} \Big)^2 \Big\}.$$

- Compute the exact solution by separation of variables on a uniform mesh of $N = 100$ points.
- Compute on the same mesh a numerical solution by the explicit Euler method using $M = 200$ time steps.
- Estimate empirically the convergence rate by repeating the computation using $N = 200$ time steps and $M = 400$ time steps.

Plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.

Exercise 2

Consider the non homogeneous heat equation

$$\frac{\partial c}{\partial t} = \nu \frac{\partial^2 c}{\partial x^2} + \exp\Big\{ - \Big(\frac{x - L/3}{L/20} \Big)^2 \Big\}$$

with periodic boundary conditions on the spatial domain $[0, L]$ and on the time interval $[0, T]$. Assume $L = 4$, $T = 5$, $\nu = 0.1$. Assume as initial datum $c_0(x) = 0$.

- Compute the exact solution by separation of variables on a uniform mesh of $N = 50$ points.
- Compute on the same mesh a numerical solution by the explicit Euler method using $M = 500$ time steps.
- Estimate empirically the convergence rate by repeating the computations on a mesh of $N = 100$ points and using $M = 1000$ time steps.

In all cases, plot the absolute value of the difference between the numerical and the exact solution, and compute the l^2 and l^∞ relative errors.

Exercise 3

Consider the non homogeneous, variable coefficients diffusion equation with Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(\dfrac{1+x}{1+t}\right)\dfrac{\partial y}{\partial x}\right] + s(x,t) & x \in (0, 2\pi), t \in (0,1), \\ y(x,0) = y_0(x) = \sin(8x) & x \in (0,2\pi), \\ y(0,t) = 0 \quad y(2\pi, t) = 0 & t \in (0,1) \end{cases}$$

with

$$s(x,t) = -\sin(8x)\exp(-t) - 8\frac{\cos(8x)\exp(-t)}{t+1} + 64\frac{\sin(8x)\exp(-t)(x+1)}{t+1}.$$

Notice that the exact solution is given by $y(x,t) = \exp(-t)\sin(8x)$.

- Compute a numerical solution on a uniform mesh of $N = 50$ intervals by the explicit Euler method using $M = 4000$ time steps.
- Estimate empirically the convergence rate by repeating the computations on a mesh of $N = 100$ points and using $M = 8000$ time steps.