Exercise session on finite difference approximations

September 23 2021

Exercise 1

Compute analytically the first derivative $f'(x)$ of the function

$$f(x) = \exp(-x^2)\sin(2x+1), \qquad x \in [-3, 3]$$

Write a Matlab script that does the following:

(a) define the function $f(x)$ using a function handle and plot it with a blue line on the interval $x \in [-3, 3]$;

(b) define the first derivative $f'(x)$ and plot it on the interval $x \in [-3, 3]$ with a red line in a new figure window;

(c) compute the approximate first derivative of $f(x)$ at the node $x = 0$ with the forward finite difference method,
$$\delta^+ f = \frac{f(x+h) - f(x)}{h},$$
using $h = 0.01$;

(d) compute the error of the approximation, that can be estimated at a node x_i as
$$E_i = |f'(x_i) - \delta^+ f_i|;$$

(e) repeat the computation at the node $x = 1$.

Exercise 2

Consider the function
$$r(x) = 3x\ln(x^2+1) \qquad x \in [-1, 1]$$
and compute its first derivative $r'(x)$ analytically. Write a MATLAB script that

(a) defines the function $r(x)$ and its derivative and plots them on the same figure using appropriate coloured lines;

(b) defines the vector of $n+1$ equispaced nodes $\mathbf{x} = [x_0, x_1, \ldots, x_n]$, with $n = 50$, on the interval $[-1, 1]$ and evaluates the function r on them

(c) computes the approximate first derivative of r on the vector of nodes $\mathbf{x}_c = [x_1, \ldots, x_{n-1}]$ (that is the vector $\mathbf{x}$ without the first and the last nodes) with the centered finite difference scheme, given by

$$\delta^0 r_i = \frac{r(x_{i+1}) - r(x_{i-1})}{2h}, \qquad i = 1, \ldots, n-1;$$

(d) plot in a new figure the exact derivative $r'(x)$ evaluated in the vector of nodes $\mathbf{x}_c$ with a blue line and the approximate values $\delta^0 r_i$ in the nodes $\mathbf{x}_c$ with red circles;

(e) compute the error using the infinity norm $\|\cdot\|_\infty$, that is

$$E = \max_{i=1,\ldots,n-1} |r'(x_i) - \delta^0 r(x_i)|.$$

Exercise 3

Consider the function $f(x) = \exp(-3x^2)$.

(a) Plot f in the interval $[-1, 2]$ using a mesh of 200 equal subintervals.

(b) Build on the same interval a mesh of 20 equal subintervals with nodes $x_0, \ldots, x_{20}$. Compute the approximations of

$$f'(x) = -6x \exp(-3x^2)$$

given by 1) forward finite difference 2) backward finite difference 3) centered finite difference 4) fourth order finite differences at the internal nodes $x_2, \ldots, x_{18}$. Compute the relative error in norms $||\cdot||_2, ||\cdot||_\infty$ over the whole interval.

(c) Repeat the computation using 40 equal subintervals and check the ratio of the corresponding errors obtained by each method.

(d) Repeat the computation for the function $f(x) = \exp(-30x^2)$, whose first derivative is $f'(x) = -60x\exp(-30x^2)$. Explain the differences between the results in this case and those in the previous case on the basis of the theory of finite difference approximations.

Exercise 4

Consider the function $f(x) = x^{\frac{15}{2}}$ over the interval $[0, 2]$. Note that its second derivative is given by $f''(x) = \frac{195}{4} x^{\frac{11}{2}}$.

(a) Build on the same interval a mesh of 30 equal subintervals with nodes $x_0, \ldots, x_{30}$.

(b) Compute a centered finite difference approximation of the exact second derivative at the internal nodes $x_1, \ldots, x_{29}$. Compute the relative error in the infinity norm. Repeat the computation using 60 subintervals and check the ratio of the corresponding errors obtained by each method.

(c) Repeat the previous computation in the case $f(x) = x^{\frac{5}{2}}$, $f''(x) = \frac{15}{4} x^{\frac{1}{2}}$. Explain the difference between the results in the two cases on the basis of the theory of finite difference approximation.

Exercise 5

Consider function $f(x) = \arctan(2x)$, whose first derivative is given by $f'(x) = 2/(4x^2 + 1)$. Plot f in the interval $[-1, 2]$.

(a) Compute the approximation of with the backward finite difference and with the fourth-order centered finite difference method at the interior nodes of a grid with 25 uniformly spaced nodes over the interval.

(b) Compute the relative errors of point (a) in the $||\cdot||_\infty$ norm over the whole interval. Discuss the differences in the results obtained with the two methods.

(c) Repeat the previous computations with function $f(x) = \arctan(200x)$, whose first derivative is given by $f'(x) = 200/(40000x^2 + 1)$. Explain the differences in the results.

Exercise 6

Consider on the interval $[0, \frac{\pi}{2}]$ the function $f(x) = x \cos x$, whose second derivative is given by $f^{(2)}(x) = -x \cos x - 2 \sin x$.

(a)

(b) Build on the same interval a mesh of 20 equal subintervals with nodes $x_0, \ldots, x_{20}$.

(c) Compute a centered finite difference approximation of the exact second derivative at the internal nodes $x_1, \ldots, x_{19}$. Compute the relative error in the infinity norm.

(d) Repeat the computation using 40 subintervals and check the ratio of the corresponding errors obtained by each method.

(e) Repeat the previous points considering the function $f(x) = x \cos(20x)$, whose second derivative is given by $f^{(2)}(x) = -400x \cos x - 40 \sin(20x)$. Compare the results with those in the previous case and explain the differences on the basis of the theory.

Exercise 7

Write a function that computes a second order approximation of the derivative for the values of a generic function on a uniformly spaced grid, using the centered finite difference approximation at the interior nodes and the second order formulae

$$\delta^{++} f_i = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}, \quad \delta^{--} f_i = \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h}$$

at the first and the last node, respectively. Check that the function implementation is correct using the examples in the previously solved exercises.