DNF

HOW TO CREATE IT?

Logical Operators

$\vee$ - Disjunction

$\wedge$ - Conjunction

$\neg$ - Negation

$\rightarrow$ - Implication

$\oplus$ - Exclusive or

$\leftrightarrow$ - Biconditional

Do we need all these?

$p \rightarrow q \Leftrightarrow \neg p \vee q$

$(p \wedge \neg q) \vee (\neg p \wedge q)$

$p \leftrightarrow q \Leftrightarrow$

$(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow$

$(\neg p \vee q) \wedge (\neg q \vee p)$

Functionally Complete

- A set of logical operators is called functionally complete if every compound proposition is logically equivalent to a compound proposition involving only this set of logical operators.
- $\wedge$, $\vee$, and $\neg$ form a functionally complete set of operators.

Are $\neg(p \lor (\neg p \land q))$ and $(\neg p \land \neg q)$ equivalent?

$\neg(p \lor (\neg p \land q))$	
$\Leftrightarrow \neg p \land \neg(\neg p \land q)$	DeMorgan
$\Leftrightarrow \neg p \land (\neg\neg p \lor \neg q)$	DeMorgan
$\Leftrightarrow \neg p \land (p \lor \neg q)$	Double Negation
$\Leftrightarrow (\neg p \land p) \lor (\neg p \land \neg q)$	Distribution
$\Leftrightarrow (p \land \neg p) \lor (\neg p \land \neg q)$	Commutative
$\Leftrightarrow F \lor (\neg p \land \neg q)$	And Contradiction
$\Leftrightarrow (\neg p \land \neg q) \lor F$	Commutative
$\Leftrightarrow (\neg p \land \neg q)$	Identity

Are $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ equivalent?

- Even though both are expressed with only $\wedge$, $\vee$, and $\neg$, it is still hard to tell without doing a proof.
- What we need is a unique representation of a compound proposition that uses $\wedge$, $\vee$, and $\neg$.
- This unique representation is called the Disjunctive Normal Form.

Disjunctive Normal Form

- A disjunction of conjunctions where every variable or its negation is represented once in each conjunction (a *minterm)*
  - each minterm appears only once

Example: DNF of $p \oplus q$ is

$(p \wedge \neg q) \vee (\neg p \wedge q)$.

Truth Table

p	q	$p \oplus q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	F
T	*F*	*T*	*T*
F	*T*	*T*	*T*
F	F	F	F

Method to construct DNF

- Construct a truth table for the proposition.
- Use the rows of the truth table where the proposition is True to construct minterms
  - If the variable is true, use the propositional variable in the minterm
  - If a variable is false, use the negation of the variable in the minterm
- Connect the minterms with $\vee$'s.

How to find the DNF of $(p \vee q) \rightarrow \neg r$

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \rightarrow \neg r$
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

There are five sets of input that make the statement true. Therefore there are five minterms.

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \rightarrow \neg r$
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

From the truth table we can set up the DNF

$$(p \vee q) \rightarrow \neg r \Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

Can we show that just $\neg$ and $\wedge$ form a set of functionally complete operands?

It is sufficient to show that $p \vee q$ can be written in terms of $\neg$ and $\wedge$. Then using DNF, we can write every compound proposition in terms of $\neg$ and $\wedge$.

$(p \vee q)$

$\Leftrightarrow (\neg\neg p \vee \neg\neg q)$ Double negation (2)

$\Leftrightarrow \neg(\neg p \wedge \neg q)$ DeMorgan

Find an expression equivalent to $p \to q$ that uses only conjunctions and negations.

p	q	$p \to q$
T	*T*	*T*
T	F	F
F	*T*	*T*
F	*F*	*T*

How many minterms in the DNF?

The DNF of $p \to q$ is $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$.

Then, applying DeMorgan's Law, we get that this is equivalent to

$\neg[\neg(p \wedge q) \wedge \neg(\neg p \wedge q) \wedge \neg(\neg p \wedge \neg q)]$.

Now can we write an equivalent statement to $p \rightarrow q$ that uses only disjunctions and negations?

$p \rightarrow q$	
$\Leftrightarrow \neg[\neg(p \wedge q) \wedge \neg(\neg p \wedge q) \wedge \neg(\neg p \wedge \neg q)]$	From Before
$\Leftrightarrow \neg[(\neg p \vee \neg q) \wedge (\neg\neg p \vee \neg q) \wedge (\neg\neg p \vee \neg\neg q)]$	DeMorgan
$\Leftrightarrow \neg[(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge (p \vee q)]$	Doub. Neg.
$\Leftrightarrow [\neg(\neg p \vee \neg q) \vee \neg(p \vee \neg q) \vee \neg(p \vee q)]$	DeMorgan