Chapter 1

Sequences in Metric spaces

1.1 Sequences of real numbers

A sequence of real numbers in $\mathbb{R}$ is simply a function $f : \mathbb{N} \to \mathbb{R}$ which is usually defined by $f(n) = x_n$ and arranged in a particular order such as $x_1, x_2, \ldots, x_n, \ldots$.

For example, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be represented as $x_n = \frac{1}{n}$ for $n = 1,\ 2,\ 3, \ldots$

1.2 Convergent Sequence

A sequence x_n in $\mathbb{R}$ is said to be *converged* to a *limit* $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N such that $|x_n - x| < \epsilon$ whenever $n \geq N$. In this case we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$

Note. $N := N(\epsilon)$, often smaller ϵ may require larger N.

1.3 Sequences of points or vectors in Metric Spaces

A sequence of points in a metric space $M := (M, d)$ is a function $f : \mathbb{N} \to M$, usually defined by $f(n) = x_k$ and arranged in a definite order such as $x_1, x_2, \ldots, x_n, \ldots$.

1.4 Convergent sequence in a Metric Space

A sequence x_k in a metric space (M, d) converges to $x \in M$ if for every given $\epsilon > 0$ there is a natural number N such as $n \geq N$ implies $d(x_k, x_n) < \epsilon$

1.5 Convergent sequence in normed space $\mathbb{R}^n$

A sequence v_k of vectors in $\mathbb{R}$ converges to the vector $v \subset \mathbb{R}^n$ if for every given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(v_k, v) = ||v_k - v|| < \epsilon$ whenever $k \geq N$.

1.6 Convergent sequence in arbitrary normed space V

$v_k \in V \to v \in V$ if $||v_k - v|| \to 0$ as $k \to \infty$.
If $v, v_k \in \mathbb{R}^n$, we write $v = (v^1, v^2, \ldots, v^n)$, $v_k = (v_k^1, v_k^2, \ldots, v_k^n)$

Theorem 1.6.1. $v_k \to v$ in $\mathbb{R}^n$ if and only if each sequence of coordinates converges to the corresponding coordinate of v as a sequence in $\mathbb{R}$. That is,
$\lim_{k\to\infty} v_k = v$ in $\mathbb{R}^n$ if and only if $\lim_{k\to\infty} v_k^i = v$ in $\mathbb{R}$ for each $i = 1, 2, \ldots, n$
or,

$$\lim_{k\to\infty} (v_k^1, \ldots, v_k^n) = \left(\lim_{k\to\infty} v_k^1, \ldots, \lim_{k\to\infty} v_k^n\right)$$

Example. Test the convergence of the sequences in $\mathbb{R}^2$:

(i) $v_k = (1/k,\ 1/k^2)$

(ii) $v_k = ((\sin n)^n/n,\ 1/n^2)$

Solution.

(i) Here the component sequences $1/k$ and $1/k^2$ each converges to 0. Hence, the vectors $v_k \to 0$, $0 = (0, 0) \in \mathbb{R}^2$

(ii) Use Sandwich theorem ($v_n \to (0, 0)$).
Here, $\left|\frac{(\sin n)^n}{n}\right| = \frac{|\sin n|^n}{n} \leq \frac{1}{n} \Rightarrow -\frac{1}{n} \leq \frac{(\sin n)^n}{n} \leq \frac{1}{n}$
Hence by sandwich theorem, $\lim_{n\to\infty} -\frac{(\sin n)^n}{n} = 0 = \lim_{n\to\infty} \frac{(\sin n)^n}{n}$, therefore, $\lim_{n\to\infty} \frac{(\sin n)^n}{n} = 0$.
Again $\lim_{n\to\infty} \frac{1}{n^2} = 0$
Therefore, $v_n \to (0, 0)$

Theorem 1.6.2. A set $A \subset M$ is closed $\Leftrightarrow$ for every sequence $x_k \in A$ converges to a point $x \in A$.

Example. Let $x_n \in \mathbb{R}^m$ be a convergent sequence with $||x_n|| \leq 1$ for all n. Show that the limit x also satisfies $||x|| \leq 1$. If $||x_n|| < 1$, then must we have $||x|| < 1$?

Solution. The unit ball $B = \{y \in \mathbb{R}^m \mid ||y|| \leq 1\}$ is closed. Let $x_n \in B$ and $x_n \to x \Rightarrow x \in B$ as B is closed, by theorem 1.6.2. This is not true if $\leq$ is replaced by $<$; for example, on $\mathbb{R}$, consider $x_n = 1 - \frac{1}{n}$.

1.7 Cauchy sequence

Let (M, d) be a metric space. A *Cauchy sequence* is a sequence $x_k \in M$ such that for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon$.

1.8 Complete Metric Space

The metric space M is called *complete* if and only if every Cauchy sequence in M converges to a point in M.

In normed space, such as $\mathbb{R}^n$ a sequence v_k is Cauchy sequence if for every $\epsilon > 0$ there is an N such that $||v_k - v_j|| < \epsilon$ whenever $j, k \geq N$.

1.9 Bounded Sequence

A sequence x_k in a normed space is bounded if there is a number $M' > 0$ such that $||x_k|| \leq M$ for every k.

In a metric space we require that there be a point x_0 such that $d(x_k, x_0) \leq M'$ for all k.