Chapter 1

Continuous Mappings on Metric Spaces

Definition 1. Let (M, d) and (N, ρ) be two metric spaces, $A \subset M$, and $f : A \to N$ be a mapping. Suppose that x_0 is an accumulation point of A. We say that $b \in N$ is *the limit of f at x_0*, written $\lim_{x \to x_0} f(x) = b$, if given any $\epsilon > 0$ there exists $\delta > 0$ (possibly depending on f, x_0, and ϵ) such that for all $x \in A$ satisfying $x \neq x_0$ and $d(x_0, x) < \delta$, we have $\rho(f(x), b) < \epsilon$.

Intuitively, this says that as x approached x_0, $f(x)$ approaches b. We also write $f(x) \longrightarrow b$ as $x \longrightarrow x_0$.

Definition 2. Let (M, d) and (N, ρ) be two metric spaces and $a \subset M$ and $f : A \to N$ be a mapping. We say that f is continuous at x_0 in its domain if and only if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in A$, $d(x, x_0) < \delta$ implies $\rho(f(x), f(x_0)) < \epsilon$.

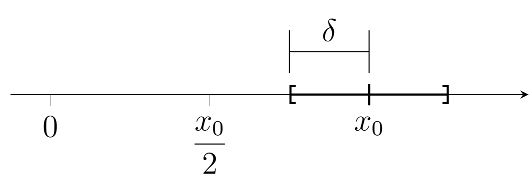
Note. A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x_0 \in A$ if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in A$ with $||x - x_0|| < \delta$, we have $||f(x) - f(x_0)|| < \epsilon$.

Example (HW)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function $x \mapsto x$. Show that f is continuous.

Example. Let $f : (0, \infty) \to \mathbb{R}$, $x \mapsto \frac{1}{x}$, show that f is continuous.

Solution. Fix $x_0 \in (0, \infty)$, that is, fix $x_0 > 0$. To determine how to choose δ, we examine the expression

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x_0 - x|}{|x\,x_0|}$$



If $|x - x_0| < \delta$, then we would get

$$|f(x) - f(x_0)| < \frac{\delta}{|x\,x_0|} = \frac{\delta}{x\,x_0}$$

If $\delta < \frac{x_0}{2}$, then $x > \frac{x_0}{2}$ and so $\frac{\delta}{x\,x_0} < \frac{2\delta}{x_0^2}$. Thus, given $\epsilon > 0$, choose $\delta = \min\left(\frac{x_0}{2}, \frac{\epsilon x_0^2}{2}\right)$. Then $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$, and so f is continuous.

Theorem 1.0.1. Suppose that (M, d) and (N, p) are two metric spaces, $f : M \to N$ is continuous and $K \subset M$ is connected. Then $f(K)$ is connected. Similarly, if K is path-connected, so is $f(K)$.

Proof. Suppose $f(k)$ is not connected. By definition, we can write $f(K) \subset U \cup V$, when $U \cap V \cap f(K) = \varnothing$, $U \cap f(K) \neq \varnothing$, $V \cap f(K) \neq \varnothing$, and U, V are open sets. Now, $f^{-1}(U) = U' \cap K$ for some open set U', and similarly, $f^{-1}(V) = V' \cap K$ for some open set V'. From the conditions on U, V, we see that $U' \cap V' \cap K = \varnothing$, $K \subset U' \cup V'$, $U' \cap K \neq \varnothing$, and $V' \cap K \neq \varnothing$. Thus, K is not connected, which proves the first assertion. □

1.1 The Boundedness of Continuous Functions on Compact Sets

Theorem 1.1.1 (Maximum-Minimum Theorem (Boundedness Theorem)). Let (M, d) be a metric space, let $A \subset M$, and let $f : A \to \mathbb{R}$ be continuous. Let $K \subset A$ be a compact set. Then f is bounded on K; that is, $B = \{f(x) \mid x \in K\} \subset \mathbb{R}$ is a bounded set. Furthermore, there exists points $x_0, x_1 \in K$ such that $f(x_0) = \inf(B)$ and $f(x_1) = \sup(B)$. We call $\sup(B)$ the (absolute) maximum of f on K and $\inf B$ the (absolute) minimum of f on K.

Proof. First, B is bounded, for $B = f(K)$ is compact, since the continuous image of a compact set is compact. Therefore, it is closed and bounded, by the definition of compactness. Second, we want to produce an x_1 such that $x_1 \in K$ and $f(x_1) = \sup B$. Now, since B is closed, $\sup B \in B = f(k)$. Thus, $\sup B = f(x_1)$ for some $x_1 \in K$. The case of $\inf B$ is similar. □

To appreciate the result, let us consider what can happen on a non-compact set:

Note. First, *a continuous function need not be bounded.* Consider the function $f(x) = \frac{1}{x}$ on $(0, 1)$. As x gets closer to 0, the function becomes arbitrarily large, but f is nevertheless continuous.

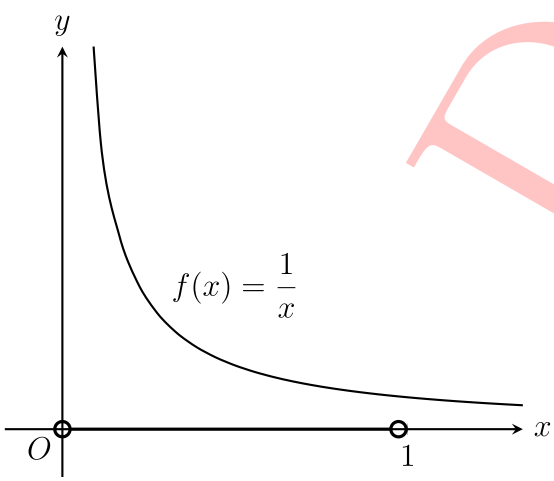


Figure 1.1: An unbounded continuous function

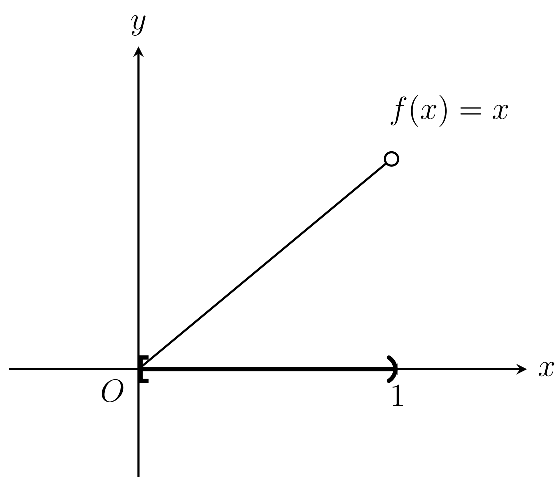


Figure 1.2: A function with no maximum

Note. Second, *if a function is bounded and continuous, it might not assume its maximum at any points of its domain.*
Let $f(x) = x$ on $[0, 1)$. This function never attain a maximum value, because even though there are an infinite number of points as near to 1 as we please, there is no point x for which $f(x) = 1$.

Problem 1.1.1. Give an example of an unbounded discontinuous function on a compact set.

Solution. Let $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ if $x > 0$ and $f(0) = 0$. Clearly, this function exhibits the same unboundedness property as does $\frac{1}{x}$ on $(0, 1]$

Problem 1.1.2. Verify the Maximum-Minimum theorem for $f(x) = \frac{x}{x^2+1}$ on $[0, 1]$

Solution. $f(0) = 0$, $f(1) = \frac{1}{2}$. We shall verify explicitly that, the maximum is at $x = 1$, and the minimum is at $x = 0$. First, as $0 \leq x \leq 1$, so $\frac{x}{x^2+1} \geq 0$, since $x \geq$, $x^2 + 1 \geq 1$, so that $f(x) \geq f(0)$ for $0 \leq x \leq 1$. Thus, 0 is the minimum. Next, note that $0 \leq (x - 1)^2 = x^2 - 2x + 1$, so that $x^2 + 1 \geq 2x$, and hence for $x \neq 0$, $\frac{x}{x^2+1} \leq \frac{x}{2x} = \frac{1}{2}$ so that $f(x) \leq f(1) = \frac{1}{2}$ and thus $x = 1$ is the maximum point.

Problem 1.1.3. Verify the Maximum-Minimum theorem for $f(x) = x^3 - x$ on $[-1, 1]$

1.2 The Intermediate Value Theorem (IMVT)

From the context of elementary calculus, it states that a continuous function on an interval assumes all values between any two given elements of its range (Fig 1.3 below). This theorem is not true for the case of a discontinuous function (Fig 1.4 below). Also, this is not true when the function is continuous the domain is not connected (Fig 1.5 below). Therefore, the crucial assumptions for this theorem to hold for a function f be *continuous* and the domain of definition be *connected.*

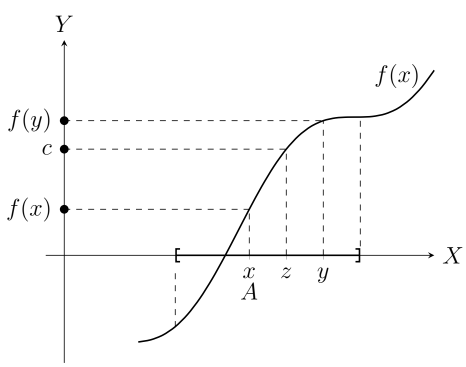


Figure 1.3: IMVT

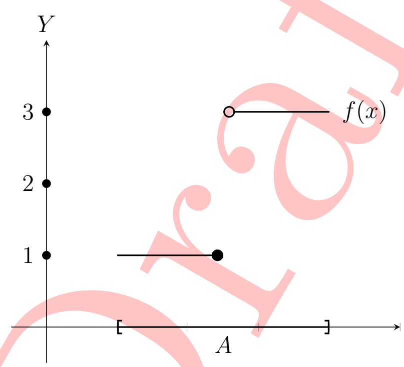


Figure 1.4: IMVT

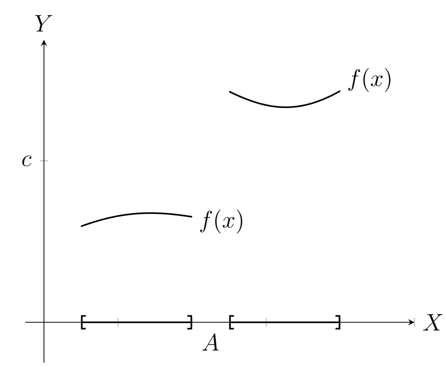


Figure 1.5: Continuous function with disconnected domain

1.2.1 Intermediate Value Theorem

Suppose M is a metric space, $A \subset M$, and $f : A \to \mathbb{R}$ is continuous. Suppose that $K \subset A$ is connected and $x, y \in K$. For every number $c \in \mathbb{R}$ such that $f(x) < c < f(y)$, there exists a point $z \in K$ such that $f(z) = c$.

Proof. Suppose no such z exists. Let $U = (-\infty, c)$ and $V = (c, \infty)$. Clearly, U and V are open sets. Since f is continuous, we have $f^{-1}(U) = U_0 \cap K$ an open set U_0, and similarly, $f^{-1}(V) = V_0 \cap K$ an open set V_0 (by the following theorem). By the definition if U and V, we have $U_0 \cap V_0 \cap K = \varnothing$, and by the assumption that $\{z \in K \mid f(z) = c\} = \varnothing$. We have $U_0 \cap V_0 \supset K$. Also, $U_0 \cap K \neq \varnothing$, since $x \in U$; and $V_0 \cap K \neq \varnothing$, since $y \in V$. Hence, K is not connected, a contradiction. $\square$

Theorem 1.2.1. Let $f : A \subset M \to N$ be a mapping. Then the following assumptions are equivalent:

(i) f is continuous on A

(ii) For each convergent sequence $x_k \to x_0$ in A, we have $f(x_k) \to f(x_0)$

(iii) For each open set U in N, $f^{-1}(U) \subset A$ is open relative to A; i.e., $f^{-1}(U) = U_0 \cap A$ for some open set U_0

(iv) For each closed set $F \subset N$, $f^{-1}(F) \subset A$ is closed relative to A; i.e., $f^{-1}(F) = F_0 \cap A$ for some closed set F_0

Problem 1.2.1. Let $f(x)$ be a cubic polynomial. Show that f has a (real) root x_0 (i.e., $f(x_0) = 0$).

Solution. We can write $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Suppose that $a > 0$. For x large and positive, ax^3 is large (and positive) and will be bigger than the other terms, so that $f(x) > 0$ if x is large. To see it exactly, note that $ax^3 + bx^2 + cx + d = ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right)$ and the factor in parentheses tends to 1 as $x \to \infty$. Similarly, $f(x) < 0$ if x is large and negative. Hence, we can apply the Intermediate value with $K = \mathbb{R}$ to conclude the existence of a point x_0 where $f(x_0) = 0$.

Problem 1.2.2. Let $f : [1, 2] \to [0, 3]$ be a continuous function satisfying $f(1) = 0$ and $f(2) = 3$. Show that f has a fixed point. That is, show that there us a point $x_0 \in [1, 2]$ such that $f(x_0) = x_0$.

Solution. Let $g(x) = f(x) - x$. Then g is continuous, $g(1) = f(1) - 1 = -1$ and $g(2) = f(2) - 2 = 3 - 2 = 1$. Hence, by the Intermediate value theorem g must vanish at some $x_0 \in [1, 2]$, and this x_0 is a fixed point for $f(x)$.

1.3 Uniform Continuity

Definition 3. Let (M, d) and (N, ρ) be metric spaces, $A \subset M$, $f : A \to N$, and $B \subset A$. We say that f is *uniformly continuous on the set B* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $x, y \in B$ and $d(x, y) < \delta$ imply $\rho(f(x), f(y)) < \epsilon$.

The definition is similar to that of continuity, except that here we are required to chose δ to work for all x, y once ϵ is given. For continuity, we were required only to choose a δ once we were given $\epsilon > 0$ and a particular x_0. Clearly, if f is uniformly continuous, then f is continuous.

For example, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then f is certainly continuous, but it is not uniformly continuous. Indeed, for $\epsilon > 0$ and $x_0 > 0$ given, the $\delta > 0$ we need is at least as small as $\epsilon/(x_0)$ (WHY?), and so if we choose x_0 large, δ must get smaller; i.e., no single δ will do for all $x > 0$. The phenomenon cannot happen on compact sets, as the next theorem shows:

Theorem 1.3.1 (Uniform Continuity Theorem). Let $f : A \subset M \to N$ be continuous and let $K \subset A$ be compact set. Then f is uniformly continuous.

Proof. Given $\epsilon > 0$ and $x \in K$, choose δ_x such that $d(x, y) < \delta_x$ implies $\rho(f(x), f(y)) < \frac{\epsilon}{2}$. The sets $D(x, \frac{\delta_x}{2})$ cover K and are open. Therefore, there is a fine covering, say, $D(x_1, \frac{\delta_{x_1}}{2}), D(x_2, \frac{\delta_{x_2}}{2}), \ldots, D(x_N, \frac{\delta_{x_N}}{2})$. Let $\delta =$ minimum $\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots, \frac{\delta_{x_N}}{2}$. If $d(x, y) < \delta$, then there is an x_i such that $d(x_i, y) \leq d(x, x_i) + d(x_i, y)$. Thus, by the choice of δ, $\rho(f(x), f(y)) \leq \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Problem 1.3.1. Let $f : (0, 1] \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $[a, 1]$ for $a > 0$.

Solution. Since $[a, 1]$ is a compact subset of $(0, 1]$ where $a > 0$ and f is continuous on $(0, 1]$, and therefore, by the uniform continuity theorem, we conclude that f is uniformly continuous in $[a, 1]$.

Problem 1.3.2. Let $f : (a, b) \to \mathbb{R}$ be differentiable and suppose that there is a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Here a and b may be $\pm\infty$. Show that f is uniformly continuous on (a, b).

Solution. The definition of uniform continuity asks us to estimate the difference $|f(x) - f(y)|$ in terms of $|x - y|$. This suggests using the mean value theorem.

Indeed, $f(x) - f(y) = f'(x_0)(x - y)$ for some x_0 between x and y. Hence, $|f(x) - f(y)| \leq M\,|x - y|$; a mapping with this property is called *Lipschitz*.

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$. Then $|x - y|$ implies $|f(x) - f(y)| < M \cdot \frac{\epsilon}{M} = \epsilon$. Hence, f is uniformly continuous on (a, b).

Problem 1.3.3. Show that $\sin x : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.