Chapter 1

Liapunov's Direct Method

Definition 1. Let $E(x, y)$ have continuous first partial derivatives at all points (x, y) in a domain D containing the origin $(0, 0)$.

1. The function E is called positive defined in D if $E(0, 0) = 0$ and $E(x, y) > 0$ for all other points in (x, y) in D.

2. The function E is called positive semi-defined in D if $E(0, 0) = 0$ and $E(x, y) \geq 0$ for all other points in (x, y) in D.

3. The function E is called negative defined in D if $E(0, 0) = 0$ and $E(x, y) < 0$ for all other points in (x, y) in D.

4. The function E is called negative semi-defined in D if $E(0, 0) = 0$ and $E(x, y) \leq 0$ for all other points in (x, y) in D.

1.1 Liapunov Function

Consider the non-linear autonomous system,

$$\left.\begin{aligned} \tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= P(x, y) \\ \tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Q(x, y) \end{aligned}\right\} \tag{1.1}$$

has isolated critical point at $(0, 0)$ and P and Q have continuous first partial derivatives for all (x, y). If there exists a differentiable function $E(x, y)$ such that,

(i) $E(x, y)$ is positive defined and

(ii) $\dot{E}(x, y)$ is negative semi defined.

Then $E(x, y)$ is called Liapunov function for the system (1.1) in D.

1.2 Three theorem on Liapunov

Theorem 1.2.1. Consider the system,

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x, y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x, y)$$

if

(i) $(0,0)$ is an isolated critical point

(ii) $E(x,y)$ is a Liapunov function

Then $(0,0)$ is called stable critical point.

Theorem 1.2.2. Consider the system,

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x,y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x,y)$$

if

(i) $(0,0)$ is an isolated critical point

(ii) $E(x,y)$ is a Liapunov function

(iii) $\dot{E}(x,y)$ is a negative defined

Then $(0,0)$ is asymptotically stable.

Theorem 1.2.3. Consider the system,

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x,y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x,y)$$

if there exists a function $E(x,y)$ such that

$$E(0,0) = 0$$
$$E(x,y) > 0 \qquad \text{for } x \neq 0,\ y \neq 0$$

Then $(0,0)$ is unstable.

Problem 1.1. For what value of A, $V(x,y) = x^2 + y^2$ is a Liapunov function for the system.

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = Ax + xy^2$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Ay - yx^2$$

and discuss the stability of the critical point $(0,0)$.

Solution. We have,

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = Ax + xy^2$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Ay - yx^2$$

Again, we have,

$$\begin{aligned} V(x,y) &= x^2 + y^2 \\ \dot{V}(x,y) &= 2x\dot{x} + 2y\dot{y} \\ &= 2x(Ax + xy^2) + 2y(Ay - yx^2) \\ &= 2Ax^2 - 2x^2y^2 + 2Ay^2 - 2x^2y \\ &= 2A(x^2 + y^2) - 4x^2y^2 \end{aligned}$$

If $V(x, y)$ is a Liapunov function then,

$$\begin{aligned} &\dot{V}(x, y) \leq 0 \\ \Rightarrow\ &2A(x^2 + y^2) - 4x^2y^2 \leq 0 \\ \Rightarrow\ &A \leq \frac{4x^2y^2}{2(x^2 + y^2)} \end{aligned}$$

Now, we observe that

(i) V is a differentiable function of x and y

(ii) V is positive defined

(iii) $\dot{V}(x, y) \leq 0$ if $A \leq \frac{4x^2y^2}{2(x^2+y^2)}$

The critical point $(0, 0)$ is stable for the given system for $A \leq \frac{4x^2y^2}{2(x^2+y^2)}$.

Problem 1.2. For the autonomous system

$$\begin{aligned} \frac{\mathrm{d}\, x}{\mathrm{d}\, t} &= -x - y - x^3 \\ \frac{\mathrm{d}\, y}{\mathrm{d}\, t} &= x - y - y^3 \end{aligned}$$

Construct a Liapunov function of the form $Ax^2 + By^2$ where A and B are the constant and use the function to determine the stability of the trivial solution of the system.

Solution. The given autonomous system,

$$\begin{aligned} \frac{\mathrm{d}\, x}{\mathrm{d}\, t} &= -x - y - x^3 \\ \frac{\mathrm{d}\, y}{\mathrm{d}\, t} &= x - y - y^3 \end{aligned}$$

Let us consider a Liapunov function is,

$$E(x, y) = Ax^2 + By^2$$

which is differentiable function of x and y.
Now,

$$\begin{aligned} \dot{E}(x, y) &= 2Ax\dot{x} + 2By\dot{y} \\ &= 2Ax(-x - y - x^3) + 2By(x - y - y^3) \\ &= -2Ax^2 - 2Axy - 2Ax^4 + 2Bxy - 2By^2 - 2By^4 \\ &= 2(Bxy - Axy) - 2\left\{A(x^2 + x^4) + B(y^2 + y^4)\right\} \end{aligned}$$

For Liapunov function

$$\begin{aligned} &\dot{E}(x, y) \leq 0 \\ \Rightarrow\ &2(Bxy - Axy) = 0 \\ \Rightarrow\ &Bxy = Axy \\ \Rightarrow\ &\frac{A}{B} = \frac{1}{1} \\ \therefore\ &A = 1 \\ \therefore\ &B = 1 \end{aligned}$$

Hence, $E(x, y) = x^2 + y^2$
The function E is defined by $E(x, y) = x^2 + y^2$ is positive defined in every domain D containing $(0, 0)$.
Clearly, $E(0, 0) = 0$
Also, $\dot{E}(x, y) < 0$ for all (x, y)
Hence, $(0, 0)$ is asymptotically stable point.

Problem 1.3. For the system

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -x + 2x^2 + y^2$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = -y + xy$$

Construct a Liapunov function of the form $Ax^2 + By^2$ where A and B are the constant and use the function to determine whether the critical point $(0, 0)$ of the system is asymptotically stable or at least stable.

Solution. The given system is,

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -x + 2x^2 + y^2$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = -y + xy$$

Let us consider the Liapunov function,

$$E(x, y) = Ax^2 + By^2$$

Now,

$$\begin{aligned} \dot{E}(x, y) &= 2Ax\dot{x} + 2By\dot{y} \\ &= 2Ax(-x + 2x^2 + y^2) + 2By(-y + xy) \\ &= -2Ax^2 + 4Ax^3 + 2Axy^2 - 2By^2 + 2Bxy^2 \\ &= x^2(-2A + 4Ax) + y^2(2Ax - 2B + 2Bx) \end{aligned}$$

For Liapunov function,

$$\begin{aligned} & \dot{E}(x, y) \leq 0 \\ \Rightarrow \ & -2A + 4Ax = 0 \end{aligned} \tag{1.2}$$

and

$$\Rightarrow \ 2Ax - 2B + 2Bx = 0 \tag{1.3}$$

From (1.2) we get,

$$\begin{aligned} & -2A = -4Ax \\ \Rightarrow \ & 1 = 2x \\ \Rightarrow \ & x = \frac{1}{2} \end{aligned}$$

From (1.3) we get,

$$\begin{aligned} & 2A\frac{1}{2} - 2B + 2B\frac{1}{2} = 0 \\ \Rightarrow \ & A - 2B + B = 0 \\ \Rightarrow \ & A - B = 0 \\ \Rightarrow \ & \frac{A}{B} = \frac{1}{1} \end{aligned}$$

Now,

$$\begin{aligned}
\dot{E}(x,y) &= 2x\dot{x} + 2y\dot{y} \\
&= 2x(-x + 2x^2 + y^2) + 2y(-y + xy) \\
&= -2x^2 + 4x^3 + 2xy^2 - 2y^2 - 2y^2 + 2xy^2 \\
&= -2(x^2 + y^2) + 4x^3 + 4xy^2 \\
&= -2(x^2 + y^2) + 4x(x^2 + y^2)
\end{aligned}$$

Here, $E(0,0) = 0$ and $\dot{E}(x,y) < 0$
Hence, $(0,0)$ is an asymptotically stable point.

Problem 1.4. Find the Liapunov function of the dynamical system

$$\begin{aligned}
\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= -y - \frac{x}{2} - \frac{x^3}{4} \\
\frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= x - \frac{y}{2} - \frac{y^3}{4}
\end{aligned}$$

and examine the stability of $(0,0)$.

Solution. The given system is,

$$\left.\begin{aligned}
\tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= -y - \tfrac{x}{2} - \tfrac{x^3}{4} \\
\tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= x - \tfrac{y}{2} - \tfrac{y^3}{4}
\end{aligned}\right\} \tag{1.4}$$