Chapter 1

Legendre Function

1.1 Legendre Function

The differential equation $(1-x)^2\frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} - 2x\frac{\mathrm{d}\, y}{\mathrm{d}\, x} + n(n+1)y = 0$ is known as Legendre's differential equation; where n is a constant (real number). But in most applications only integral values of n are required.

Any solution of the Legendre's equation is called a Legendre function.

1.2 Rodrigues' Formula of Legendre Polynomial

We have obtained the Legendre polynomials as solutions of the Legendre's equation. There is another way of obtaining $P_n(x)$, which may be deduced directly from Legendre's differential equation without solving it. According to this formula $P_n(x)$ is given by,

$$P_n(x) = \frac{1}{2^n\, n!}\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(x^2-1\right)^n$$

This is Rodrigues' formula.

Proof. Let,
$y = (x^2-1)^n$
$\therefore y_1 = 2nx\,(x^2-1)^{n-1}$

$$\begin{aligned}
\therefore & y_1 = 2nx\left(x^2-1\right)^{n-1} \\
\Rightarrow & y_1\left(x^2-1\right) = 2nxy \\
\Rightarrow & y_2\left(x^2-1\right) + 2xy_1 = 2nxy_1 + 2ny \\
\Rightarrow & y_2\left(x^2-1\right) + 2(n-1)xy_1 - 2nx = 0
\end{aligned}$$

Now differentiating n times with respect to x, we get,

$$\Rightarrow y_{n+2}\left(x^2-1\right) + ny_{n+2-1}\cdot 2x + {}^nC_2 y_{n+2-2}\cdot 2 - 2(n-1)xy_{n+1} - 2n(n-1)y_n - 2ny_n = 0$$

$$\text{i.e.,}\qquad y_{n+2}\left(x^2-1\right) + 2xy_{n+1} - n(n+1)y_n = 0 \tag{1.1}$$

Put $\quad y_n = Z$

$$\therefore y_{n+1} = \frac{\mathrm{d}\, Z}{\mathrm{d}\, x}, \qquad y_{n+2} = \frac{\mathrm{d}^2\, Z}{\mathrm{d}\, x^2}$$

Substituting these values in (1.1) we get,

$$\begin{aligned}
&\left(x^2-1\right)\frac{\mathrm{d}^2 Z}{\mathrm{d}\, x^2} + 2x\frac{\mathrm{d}\, Z}{\mathrm{d}\, x} - n(n+1)y_n = 0 \\
\Rightarrow &\left(x^2-1\right)\frac{\mathrm{d}^2 Z}{\mathrm{d}\, x^2} - 2x\frac{\mathrm{d}\, Z}{\mathrm{d}\, x} + n(n+1)y_n = 0
\end{aligned}$$

This is a Legendre's differential equation of order n.
But since $Z = y_n = \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left\{(x^2-1)^n\right\}$, Z is a polynomial of degree n and since Legendre's equation has one and

only one distinct series solution of the form $P_n(x)$, it follows that $P_n(x)$ is a multiple of Z.
Hence,

$$P_n(x) = c \cdot Z = c\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left\{\left(x^2-1\right)^n\right\} \qquad [c \text{ is a constant}]$$

$$\begin{aligned}\text{or,} \sum_{r=0}^{N}(-1)^r\frac{1}{2^n\,r!}\frac{(2n-2r)!}{(n-r)!(n-2r)!}x^{n-2r} &= c\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[x^{2n}-nx^{2(n-1)}+\frac{n(n-1)}{2!}x^{2(n-2)}+\dots\right]\\ &= c\left[\frac{(2n)!}{n!}x^n-\frac{n(2n-2)!}{(n-2)!}x^{n-2}+\dots\right]\end{aligned}$$

Equating the coefficient of x^n on both sides,

$$\frac{2n!}{2^n\,n!\,n!} = c\frac{(2n)!}{n!} \qquad [\text{Putting } r=0]$$

$$\text{ie.,} \qquad c = \frac{1}{2^n\,n!}$$

$$\therefore P_n(x) = c\cdot Z = \frac{2}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n$$

□

1.3 Generating Function for $P_n(x)$

Legendre polynomial $P_n(x)$ is the coefficient if h^n in $(1-2xh+h^2)^{-\frac{1}{2}}$ that is

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)h^n$$

Proof. The function $(1-2xh+h^2)^{-\frac{1}{2}}$ can be replaced by using binomial theorem as

$$\begin{aligned}\left(1-2xh+h^2\right)^{-\frac{1}{2}} &= \{1-h(2x-h)\}^{-\frac{1}{2}}\\ &= 1+\frac{1}{2}h(2x-h)+\frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{2!}h^2(2x-h)^2+\frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}h^3(2x-h)^3+\dots\\ &= 1+\frac{1}{2}h(2x-h)+\frac{3}{4\cdot 2}h^2\left(4x^2-4xh+h^2\right)+\frac{15}{8\cdot 6}h^3\left(8x^3-12x^2h+6xh^2-h^3\right)+\dots\\ &= 1+xh-\frac{h^2}{2}+\frac{3}{2}x^2h^2-\frac{3}{2}xh^3+\frac{3}{8}h^4+\frac{5}{2}x^3h^3+\dots\\ &= 1+xh+\left(\frac{3}{2}x^2-\frac{1}{2}\right)h^2+\left(\frac{5}{2}x^3-\frac{3}{2}x\right)h^3+\dots \qquad (1.2)\end{aligned}$$

Again, we have,

$$P_n(x) = \frac{1}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n$$

Another form to find the value of c - Rajput. P-654
Integral form - Rajput. P-654

Putting $n = 0, 1, 2, 3, \ldots$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2} 2x = x$$

$$\begin{aligned}
P_2(x) &= \frac{1}{4 \cdot 2} \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} \left(x^2 - 1\right)^2 \\
&= \frac{1}{8} \frac{\mathrm{d}\,\{2\,(x^2 - 1)\,2x\}}{\mathrm{d}\,x} \\
&= \frac{1}{2} \frac{\mathrm{d}\,(x^3 - x)}{\mathrm{d}\,x} \\
&= \frac{1}{2} \left(3x^2 - 1\right)
\end{aligned}$$

$$\begin{aligned}
P_3(x) &= \frac{1}{8 \cdot 6} \frac{\mathrm{d}^3}{\mathrm{d}\,x^3} \left(x^2 - 1\right)^3 \\
&= \frac{1}{8 \cdot 6} \frac{\mathrm{d}^3}{\mathrm{d}\,x^3} [x^6 - 3x^4 + 3x^2 - 1] \\
&= \frac{1}{8 \cdot 6} \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} \left(6x^5 - 12x^3 + 6x\right) \\
&= \frac{1}{8 \cdot 6} \left(6 \cdot 5 \cdot 4x^3 - 12 \cdot 3 \cdot 2x\right) \\
&= \left(\frac{5}{2}x^3 - \frac{3}{2}x\right)
\end{aligned}$$

So from (1.2) we get,

$$\begin{aligned}
\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} &= P_0(x) + P_1(x)h + P_2(x)h^2 + P_3(x)h^3 + \ldots \\
&= \sum_{n=0}^{\infty} P_n(x)h^n
\end{aligned}$$

Thus by expanding $(1 - 2xh + h^2)^{-\frac{1}{2}}$, we can obtain the Legendre's polynomials of different order as the coefficient of corresponding power of h.
This is why $(1 - 2xh + h^2)^{-\frac{1}{2}}$ is known as the generating function if $P_n(x)$. □

1.4 Recurrence Relation for $P_n(x)$

1.4.1 First Relation $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$

We know,

$$\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)h^n$$

Now differentiating with respect to h we have

$$\begin{aligned}
&(x - h)\left(1 - 2xh + h^2\right)^{-\frac{3}{2}} = h \sum_{n=0}^{\infty} P_n(x)h^{n-1} \\
\Rightarrow &(x - h)\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = h\left(1 - 2xh + h^2\right) \sum_{n=0}^{\infty} P_n(x)h^{n-1} \\
\Rightarrow &(x - h) \sum P_n(x)h^n = \sum \left[nP_n(x)h^{n-1} - 2hxP_n(x)h^n + nP_n h^{n+1}\right] \\
\Rightarrow &x \sum P_n(x)h^n - \sum P_n(x)h^{n+1} = \sum \left[nP_n(x)h^{n-1} \ldots\right]
\end{aligned}$$

Now equating the coefficient of h^n from both sides,

$$\begin{aligned} xP_n(x) - P_{n-1}(x) &= (n+1)P_{n+1}(X) - 2xnP_n(x) + (n-1)P_{n-1}(x) \\ \Rightarrow &(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \end{aligned}$$

Replacing n by $n-1$, we get

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0$$

The other relations are

- $P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x) = P_{n-1}(x)$
- $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$
- $P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x)$
- $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$
- $(x^2-1)\,P_n'(x) = n\left\{xP_n(x) - P_{n-1}(x)\right\}$
- $(x^2-1)\,P_n'(x) = (n+1)\left\{P_{n+1}(x) - xP_n(x)\right\}$

1.5 Orthogonal Properties of Legendre Polynomial

Problem 1.5.1. Prove that

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\,x = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

Proof. Since $P_n(x)$ is a solution of the Legendre's differential equation, we have

$$\begin{aligned} &(1-x)^2 \frac{\mathrm{d}^2}{\mathrm{d}\,x^2}(P_n(x)) - 2x\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x)) + n(n+1)P_n(x) = 0 \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\,x}\left\{\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x))\right\} + n(n+1)P_n(x) = 0 \\ &\Rightarrow \int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}\,x}\left\{\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x))\right\}P_m(x)\,\mathrm{d}\,x + n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \\ &\Rightarrow \left[P_m(x)\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x))\right]_{-1}^{1} - \int_{-1}^{1}P_m(x)\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x))\,\mathrm{d}\,x + n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \\ &\Rightarrow -\int_{-1}^{1}\left(1-x^2\right)P_m'(x)P_n'(x)\,\mathrm{d}\,x + n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \qquad (1.3) \end{aligned}$$

Interchanging m and n in (1.3), we get

$$\Rightarrow -\int_{-1}^{1}\left(1-x^2\right)P_n'(x)P_m'(x)\,\mathrm{d}\,x + m(m+1)\int_{-1}^{1}P_n(x)P_m(x)\,\mathrm{d}\,x = 0 \qquad (1.4)$$

Subtracting (1.4) from (1.3),

$$\begin{aligned} &\Rightarrow (n-m)(m+n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \\ &\Rightarrow (n-m)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \quad \Big|\, m+n+1 \neq 0 \\ &\Rightarrow \int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0, \text{ if } m \neq n \end{aligned}$$

Again, we have

$$P_n(x) = \frac{1}{2^n\, n!}\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(x^2-1\right)^n$$
$$P_m(x) = \frac{1}{2^m\, m!}\frac{\mathrm{d}^m}{\mathrm{d}\, x^m}\left(x^2-1\right)^m$$

$$\begin{aligned}
&\therefore \int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\, x\\
&= \frac{1}{2^{m+n}\, m!\, n!}\int_{-1}^{1}\frac{\mathrm{d}^m}{\mathrm{d}\, x^m}\left(x^2-1\right)^m\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(x^2-1\right)^n\,\mathrm{d}\, x\\
&= \frac{1}{2^{m+n}\, m!\, n!}\left\{\left[\frac{\mathrm{d}^m}{\mathrm{d}\, x^m}\left(x^2-1\right)^m\frac{\mathrm{d}^{n-1}}{\mathrm{d}\, x^{n-1}}\left(x^2-1\right)^n\right]_{-1}^{1} - \int_{-1}^{1}\frac{\mathrm{d}^{m+1}}{\mathrm{d}\, x^{m+1}}(x^2-1)^m\frac{\mathrm{d}^{n-1}}{\mathrm{d}\, x^{n-1}}\left(x^2-1\right)^n\,\mathrm{d}\, x\right\}\\
&= -\frac{1}{2^{m+n}\, m!\, n!}\int_{-1}^{1}\frac{\mathrm{d}^{m+1}}{\mathrm{d}\, x^{m+1}}\left(x^2-1\right)^m\frac{\mathrm{d}^{n-1}}{\mathrm{d}\, x^{n-1}}\left(x^2-1\right)^n\,\mathrm{d}\, x
\end{aligned}$$

Continuing this process m times, we get

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\, x = \frac{(-1)^m}{2^{m+n}\, m!\, n!}\int_{-1}^{1}\frac{\mathrm{d}^{m+m}}{\mathrm{d}\, x^{m+m}}\left(x^2-1\right)^m\frac{\mathrm{d}^{n-m}}{\mathrm{d}\, x^{n-m}}\left(x^2-1\right)^n\,\mathrm{d}\, x$$

If $m = n$,

$$\begin{aligned}
\int_{-1}^{1}\{P_n(x)\}^2\,\mathrm{d}\, x &= \frac{(-1)^n}{2^{2n}\,(n!)^2}\int_{-1}^{1}\left\{\frac{\mathrm{d}^{2n}}{\mathrm{d}\, x^{2n}}\left(x^2-1\right)^n\right\}(x^2-1)^n\,\mathrm{d}\, x\\
&= (-1)^n\frac{1}{2^{2n}\,(n!)^2}\int_{-1}^{1}(2n)!\left(x^2-1\right)^n\left(x^2-1\right)^n\,\mathrm{d}\, x\\
&= \frac{2(-1)^{2n}(2n)!}{2^{2n}\,(n!)^2}\int_{0}^{1}\left(1-x^2\right)^n\,\mathrm{d}\, x\\
&= \frac{2(2n)!}{2^{2n}\,(n!)^2}\int_{0}^{\frac{\pi}{2}}\cos^{2n+1}\theta\,\mathrm{d}\,\theta\\
&= \frac{2(2n)!}{2^{2n}\,(n!)^2}\frac{2n(2n-2)(2n-4)\ldots 6\cdot 4\cdot 2}{(2n+1)(2n-1)(2n-3)\ldots 5\cdot 3\cdot 1}\\
&= \frac{2(2n)!}{2^{2n}\,(n!)^2}\frac{\{2^n\, n!\}^2}{(2n+1)!}\\
&= \frac{2}{2n+1}
\end{aligned}$$

□

Problem 1.5.2. Show that $P_n(-x) = (-1)^n P_n(x)$

Solution. we have,

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)h^n \tag{1.5}$$

Now replacing x by $-x$ and h by $-h$ in (1.5) we get,

$$\left(1+2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(-x)h^n \tag{1.6}$$

$$\left(1+2xh+h^2\right)^{-\frac{1}{2}} = (-1)^n\sum_{n=0}^{\infty} P_n(x)h^n \tag{1.7}$$

$\frac{\mathrm{d}^{2n}}{\mathrm{d}\, x^{2n}}\left(x^2-1\right)^n = (2n)!$
$(2n+1)! = (2n+1)(2n)(2n-1)(2n-2)\ldots$
$\{2n(2n-2)(2n-4)\ldots\}^2$
$= [2\,\{n(n-1)(n-2)\ldots\}]^2$
$= \{2^n n!\}^2$
If $I_n = \int_0^{\pi/2}\cos^n\,\mathrm{d}\, x$, then $I_n = \frac{n-1}{n} I_{n-2}$

From (1.6) and (1.7) we get

$$\sum P_n(-x)h^n = (-1)^n \sum P_n(x)h^n$$

Equating the coefficients of h^n we get,

$$P_N = n(x) = (-1)^n P_n(x)$$

Problem 1.5.3. Prove that

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x = \frac{2n}{4n^2-1}$$

Proof. We have the recurrence relation

$$\begin{aligned} nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \\ \Rightarrow (2n-1)xP_{n-1}(x) &= nP_n(x) + (n-1)P_{n-2}(x) \end{aligned}$$

Multiplying both sides of the above equation by $P_n(x)$ and then integrating from -1 to 1 we get,

$$(2n-1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x = n\int_{-1}^{1} [P_n(x)]^2\,\mathrm{d}\,x + (n-1)\int_{-1}^{1} P_n(x)P_{n-2}(x)\,\mathrm{d}\,x \tag{1.8}$$

From the orthogonal property, we have,

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\,x = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

$$\begin{aligned} \therefore \int_{-1}^{1} P_n(x)P_{n-2}(x)\,\mathrm{d}\,x &= 0 \quad \text{since} n \neq n-2 \\ \text{And } \int_{-1}^{1} [P_n(x)]^2\,\mathrm{d}\,x &= \frac{2}{2n+1} \end{aligned}$$

So from (1.8),

$$\begin{aligned} (2n-1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x &= \frac{2n}{2n+1} \\ \therefore \int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x &= \frac{2n}{4n^2-1} \end{aligned}$$

□

Problem 1.5.4. Prove that

$$P_n(1) = 1$$

Proof. If $x = 1$ then,

$$\begin{aligned} \left(1-2xh+h^2\right)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} P_n(x)h^n \\ \Rightarrow (1-h)^{-1} &= \sum_{n=0}^{\infty} h_n P_n(1) \\ \Rightarrow (1-h)^{-1} &= 1 + hP_1(x) + h^2P_2(1) + \cdots + h^nP_n(1) + \ldots \\ \Rightarrow 1 + h + h^2 + h^3 + \cdots + h^n + \cdots &= 1 + hP_1(x) + h^2P_2(1) + \cdots + h^nP_n(1) + \ldots \end{aligned}$$

Equating the coefficients of h^n from both sides,

$$P_n(1) = 1$$

□

Problem 1.5.5. Show that

$$\int_{-1}^{1} P_0(x)P_1(x)\,\mathrm{d}\,x = 0$$

Solution.

$$P_n(x) = \frac{1}{2^n\,n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n$$

$$\therefore P_0(x) = 1, \quad P_1(x) = \frac{1}{2\cdot 1}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left(x^2-1\right) = x$$

$$\int_{-1}^{1} P_0(x)P_1(x)\,\mathrm{d}\,x = \int_{-1}^{1} x\,\mathrm{d}\,x = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

Problem 1.5.6. Compute

$$\int_{-1}^{1} [P_2(x)]^{\frac{1}{2}}\,\mathrm{d}\,x$$

Solution.

$$P_2(x) = \frac{1}{2^2\,2!}\,\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}\left(x^2-1\right) = \frac{1}{2}\left(3x^2-1\right)$$

$$\begin{aligned}
\therefore \int_{-1}^{1}\left\{\frac{1}{2}\left(3x^2-1\right)\right\}^{\frac{1}{2}}\mathrm{d}\,x &= \frac{1}{\sqrt{2}}\int_{-1}^{1}\sqrt{\left\{(\sqrt{3}x)^2-1\right\}}\,\mathrm{d}\,x \\
&= \frac{1}{\sqrt{2}\sqrt{3}}\left[\frac{\sqrt{3}x\sqrt{3x^2-1}}{2} + \frac{1}{2}\log\left(3x^2+\sqrt{3x^2-1}\right)\right]_{-1}^{1} \\
&= \frac{1}{\sqrt{6}}\frac{\sqrt{3}\sqrt{3-1}}{2} + \frac{1}{2}\log\left(\sqrt{3}+\sqrt{3-1}\right) + \frac{\sqrt{3}\sqrt{3-1}}{2} - \frac{1}{2}\log\left(-\sqrt{3}+\sqrt{3-1}\right) \\
&= \frac{1}{\sqrt{6}}\left\{\frac{\sqrt{6}}{2} + \frac{1}{2}\log\ldots\right\}
\end{aligned}$$

Remark.

$$\begin{aligned}
&\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum h^n P_n(x) \\
\Rightarrow &\left\{1-h\left(2x-h\right)\right\}^{-\frac{1}{2}} = \sum h^n P_n(x) \\
\Rightarrow &1 + hx + h^2\frac{3x^2-1}{2} + h^3\frac{5x^3-3x}{2} + \cdots = P_0(x) + hP_1(x) + h^2P_2(x) + \ldots
\end{aligned}$$

Equating,

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2-1}{2} \text{ and so on}$$