Chapter 1

System of IVPs

1.1 Improved Euler/Modified Euler

$$
\begin{aligned}
y_1 &= y_0 + h\left[\frac{f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))}{2}\right] \\
y_{n+1} &= y_n + h\left[\frac{f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))}{2}\right]
\end{aligned}
$$

1.2 Runge-Kutta Method Of Order 4

IVP: $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y)$; $y(x_0) = y_0$

$$
y_{i+1} = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); \quad i = 0, 1, 2, 3, \dots
$$

where,

$$
\begin{aligned}
k_1 &= hf(x_i, y_i) \\
k_2 &= hf(x_i + h/2, y_i + k_1/2) \\
k_3 &= hf(x_i + h/2, y_i + k_2/2) \\
k_4 &= hf(x_i + h, y_i + k_3)
\end{aligned}
$$

Example. Solve $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 3x + y^2$; $y(1) = 1.2$ with RK4 at $x = 1.1$.

Here $f(x, y) = 3x + y^2$, $x_0 = 1$, $y_0 = 1.2$ and $x_1 = x_0 + h \Rightarrow 1.1 = 1.h \Rightarrow h = 0.1$.
Note. We cane choose $h = 0.05$ then $x_1 = x_0 + h = 1 + 0.05 = 1.05$, $x_2 = x_1 + h = 1.05 + 0.05 = 1.1$
$x_n = x_0 + nh$; $n = 0, 1, 2, \dots$

$$
\begin{aligned}
k_1 &= hf(x_0, y_0) \\
&= h(3x_0 + y_0^2) \\
&= 0.1(3.1 + 1.2^2) \\
&= 0.444
\end{aligned}
$$

or

$$
\begin{aligned}
k_1 &= hf(x_0, y_0) = hf(1, 1.2) \\
&= 0.1 \times (3 \times 1 + 1 \cdot 2^2) = 0.444
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= hf(x_0 + h/2, y_0 + k_1/2) = hf(1 + 0.1/2, 1.2 + 0.4444) \\
&= 0.1 \times (3 \times (1 + 0.1/2) + (1.2 + 0.444/2)^2) = 0.5172 \\
k_3 &= hf(x_0 + h/2, y_0 + k_2/2) = hf(1 + 0.1/2, 1.2 + 0.5172/2) \\
&= 0.1 \times (3 \times (1 + 0.1/2) + (1.2 + 0.5172/2)^2) = 0.5278 \\
k_4 &= hf(x_0 + h, y_0 + k_3) = hf(1 + 0.1,\ 1.2 + 0.5278) \\
&= 0.1 \times (3 \times (1 + 0.1) + (1.2 + 0.5278)^2) = 0.6285
\end{aligned}
$$

$$\begin{aligned} \therefore y_1 = y(1.1) &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ &= 1.2 + .5271 = 1.7271 \end{aligned}$$

1.3 Modified Euler Method

Example.

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y) = x + y, \qquad y(0) = 1$$

with $h = 0.2$ carry out 2

Solution. Given, $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = x + y$ with $x_0 = 0$, $y_0 = 1$, $h = 0.2$
Predictor formula:

$$y_n^p = y_{n-1} h[f(x_{n-1}, y_{n-1})] \tag{1.1}$$

$x_n = x_0 + nh$; $h =$ step length
Corrector formula:

$$y_n^c = y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1} + f(x_n, y_n^p))] \tag{1.2}$$

Form (1.1), $n = 1$

$$\begin{aligned} y_1^p &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.2f(0, 1) \\ &= 1.2 \end{aligned}$$

Form (1.2), $n = 1$

$$\begin{aligned} y_1^c &= y_0 + \frac{h}{2}[f(x_0, y_0 + f(x_1, y_1^p))] \\ &= 1 + \frac{0.2}{2}[f(0, 1) + f(0.2, 1.2)] \\ &= 1.24 \end{aligned}$$

Now,

$$\begin{aligned} y_1^{c_1} &= 1 + \frac{0.2}{2}\left[f(0, 1) + f(0.2, 1.24)\right] \\ &= 1.244 \\ y_1^{c_2} &= 1 + \frac{0.2}{2}\left[f(0, 1) + f(0.2, 1.244)\right] \\ &= 1.244 \end{aligned}$$

So $y_1^{c_2} = y(0.2) = 1.244$

1.4 Adam-Bashforth Predictor and Corrector Method

$$y_4^p = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right] \tag{1.3}$$

$$y_4^c = y_3 + \frac{h}{24}\left[9f_4^p + 19f_3 - 5f_2 + f_1\right] \tag{1.4}$$

Example. Given $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = x^2(1 + y)$ by Adam-Bashforth method using $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, find $y(1.4)$.

Solution. Given $\frac{dy}{dx} = x^2(1+y)$

x	y	$f = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$f_0 = x_0^2(1+y_0) = 1^2(1+1) = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$f_1 = x_1^2(1+y_1) = 1.1^2(1+1.233) = 2.70193$
$x_2 = 1.2$	$y_2 = 1.548$	$f_2 = x_2^2(1+y_2) = 1.2^2(1+1.548) = 3.69912$
$x_3 = 1.3$	$y_3 = 1.979$	$f_3 = x_3^2(1+y_3) = 1.3^2(1+1.979) = 5.0345$
$x_4 = 1.4$	$y_4^p = 2.5273$	$f_4^p = x_4^2(1+y_4^p) = 1.4^2(1+2.5273) = 7.0017$
	$y_c = 2.5749$	

From (1.3),

$$\begin{aligned} y_4^p &= y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0] \\ &= 1.979 + \frac{0.1}{24}[55(5.03451) - 59(3.69912) + 37(2.70193) - 9(2)] \\ &= 2.5723 \end{aligned}$$

From (1.4),

$$\begin{aligned} y_4^c &= y_3 + \frac{h}{24}[9f_4^p + 19f_3 - 5f_2 + f_1] \\ &= 1.979 + \frac{0.1}{24}[9(7.0017) + 19(5.03451) - 5(3.69912) + 2(2.70193)] \\ &= 2.5749 \end{aligned}$$

Note. If the value of y_4^c need to correct up to required decimal point, then we need to follow the following steps:

$$f_4^p = x_4^2(1+y_4^c) = (1.4)^2(1+2.5749) = ...$$

Then

$$y_4^c = y_3 + \frac{h}{24}[9f_4^p + 19f_3 - 5f_2 + f_1] = \dots$$

using y_4^c
Continue this process until get required accuracy.

1.5 Adams-Moulton or Modified Adams Methods

- $y' = \frac{dy}{dx} = f(x,y)$ given
- Three starting values of y (i.e., y_1, y_2, y_3) will be given, if not then use Picard, Taylor, Euler or Runge-Kutta method to find these values.
- Find the corresponding values of $y' = f(x,y)$
- Step size h will also be given