Questions from Previous Years

2014-2015 (2017)

1. Marks : $4 + 4 + 6 = 14$

   (a) Define metric space with example. Let d be a metric on a set M and let ρ be defined by

   $$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

   show that ρ is bounded metric on M.

   (b) Define a normed liner space with example. Prove that an Euclidean space is a normed linear space.

   (c) What is meant by an inner product space? State and prove Cauchy-Schwarz inequality.

2. Marks : $4 + 3 + 7 = 14$

   (a) For a metric space X, define

      i. neighborhood of a point of X,

      ii. limit point of a subset of X,

      iii. closed subset of X,

      iv. interior point of a subset X.

   (b) Prove that every neighborhood is an open set.

   (c) Let X be a metric space. Prove that the intersection of finite collection of open sets is open. Also give an example to show that intersection of any collection of open set may not be open.

3. Marks : $5 + 4 + 5 = 14$

   (a) Define compact set with example. Show that the set of real numbers is not a compact set.

   (b) Explain what is meant by connected set. Is the set of integers a connected set? Support your answer.

   (c) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub-collection of $\{K_\alpha\}$ is non-empty, then prove that $\cap K_\alpha$ is non-empty.

4. Marks : $5 + 4 + 5 = 14$

   (a) Define continuous function in a metric space. Prove that a function f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open X for every open set V in Y.

(b) Prove that every continuous image of a connected subset of a metric space is connected.

(c) Show that the set $\{x \in \mathbb{R}^n : ||x|| = 1\}$ is compact and connected.

5. Marks : $8 + 6 = 14$

   (a) State and prove the Maximum and Minimum theorem. Hence, verify it for

   $$f(x) = \frac{x}{1 + x^2}$$

   on $[0, 1]$.

   (b) State Heine-Borel Theorem. Prove that every compact subset of a metric space is closed.

6. Marks : $5 + 5 + 4 = 14$

   (a) What is the Intermediate Value Theorem? Can you prove it? How?

   (b) Show that there exists a real root of a cube polynomial.

   (c) Let $f : (a, b) \to \mathbb{R}$ be differentiable, and suppose that there is a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that f is uniformly continuous on (a, b).

7. Marks : $5 + 4 + 5 = 14$

   (a) Define pointwise convergent and uniform convergence of a sequence of functions in metric spaces. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

   $$f_n(x) = \frac{\sin x}{n}$$

   Show that $f_n \to f = 0$ uniformly as $n \to \infty$.

   (b) Let $f_n(x) = x^n,\ 0 \leq x \leq 1$. Does f_n converge uniformly? What is the situation for $0 \leq x < 1$?

   (c) Suppose $f_n(x) = \frac{x^n}{1+x^n}$ for $x \in [0, 2]$. Show that $\langle f_n \rangle$ converges pointwise on $[0, 2]$ but that the convergence is not uniformly.

8. Marks : $7 + 7 = 14$

   (a) State the Contraction Mapping Principle. Consider the integral equation $f(x) = a + \int_0^x f(y) x e^{-xy} \, \mathrm{d}\, y$. Check directly on which intervals $[0, r]$ we get a contraction.

   (b) Let Ω be the set of all invertible linear operator on $\mathbb{R}^n$. Then prove that Ω is an open subset of $L(R^n)$, the set of all linear transformation on $\mathbb{R}^n$ and the mapping $A \to A^{-1}$ is continuous on Ω.

2015-2016 (2018)

1. Marks : $5 + 5 + 4 = 14$

   (a) Define Euclidean space. Prove that Euclidean space is a metric space.

   (b) What do you mean by supremum norm? If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space and $||\cdot||$ is defined for $v \in V$ by $||v|| = \sqrt{\langle v, v \rangle}$, then prove that $||\cdot||$ is a norm on V.

(c) Let $C([0,1])$ be the vector space of all continuous functions defined on $[0,1]$. For $f, g \in C([0,1])$ define $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}\,x$. Show that $\langle f, g\rangle$ is an inner product and hence $C([0,1])$ is an inner product space.

2. Marks : $3+4+7=14$

   (a) Define interior, boundary, and closure of a set with examples.

   (b) Let $A \subset \mathbb{R}^n$ be open and $B \subset \mathbb{R}^n$, define $A+B = \{x+y \in \mathbb{R}^n \mid x \in A \text{ and } y \in B\}$, prove that $A+B$ is open.

   (c) In a metric space, prove that the union of finite number of closed subsets is closed, and the intersection of arbitrary collection of closed subsets is closed. What do you say about the union of an infinite collection of closed sets?

3. Marks : $6+8=14$

   (a) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub-collection of $\{K_\alpha\}$ is non-empty, then prove that $\bigcap_\alpha K_\alpha$ is non-empty.

   (b) State and prove the Heine-Borel theorem.

4. Marks : $4+4+6=14$

   (a) Define Cauchy sequence in a metric space. Prove that every convergent sequence in a metric space is a Cauchy sequence.

   (b) Define diameter of a subset of a metric space. Prove that diam$\bar{E}$ =diam$E$ where $\bar{E}$ is the closure of a subset E of a metric space.

   (c) Define complete metric space. Prove that $\mathbb{R}^k$ is complete.

5. Marks : $7+7=14$

   (a) Define pointwise convergence and uniform convergence of sequence of functions in a metric space. Distinguish between these two sorts of convergences. Show that the sequence of functions

$$f_k(x) = \begin{cases} 0 & x \geq {}^1/_k \\ -kx+1 & 0 \leq x < {}^1/_k \end{cases}$$

   converges pointwise on $[0,1]$.

   (b) Can we differentiate and integrate an infinite series of function? When and how? Examine the uniform convergence of the series

$$\sum_{n=0}^{\infty} \frac{(\sin nx)^2}{\pi^2}$$

   on $\mathbb{R}$.

6. Marks : $7+7=14$

   (a) Prove that a mapping $f : X \to Y$ of metric spaces is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

(b) Define uniformly continuous mapping on a metric space. Prove that every continuous mapping on compact metric spaces is uniformly continuous.

7. Marks : $7 + 4 + 3 = 14$

   (a) State and prove the Maximum-Minimum Theorem. Hence, justify it for $f(x) = x^3 - x$ on $[-1, 1]$.

   (b) Let $f : [1, 2] \to [0, 3]$ be continuous such that $f(1) = 0$ and $f(2) = 3$, show that f has a fixed point.

   (c) Let $f : (0, 1] \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $[a, 1]$ for $a > 0$.

8. Marks : $7 + 7 = 14$

   (a) State and prove the Contraction Mapping Principle.

   (b) Find a function f such that $f'(x) = xf(x)$ for x near 0 and $f(0) = 3$.

2016-2017 (2019)

1. Marks : $4 + 4 + 6 = 14$

   (a) Define metric space with an example. Let (M, d) be a metric space. Construct a function $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Then show that (M, ρ) is a metric space and $\rho(x, y)$ is bounded by 1.

   (b) What is meant by a normed space? If $(V, ||\cdot||)$ is a normed space, then prove that it is a metric space defined by the function $d(x, y) = ||x - y||$.

   (c) State and prove Cauchy Schwarz Inequality. Put the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}\,x$ on the space of all continuous functions $C([0, 1])$. Then verify the stated inequality with $f(x) = x$ and $g(x) = x^2$ on $[0, 1]$.

2. Marks : $5 + 6 + 3 = 14$

   (a) Define neighbourhood of a point in a metric space with example. Prove that neighbourhood of a point in a metric space is an open set.

   (b) Prove that intersection of a finite number of open subsets of a metric space is open. How is about the intersection of an arbitrary family of open sets? Justify your answer.

   (c) Find the interior, exterior, and boundary of $S = (x, y) : x^2 + y^2 \leq 1$.

3. Marks : $5 + 4 + 5 = 14$

   (a) Let $K \subset Y \subset X$. Then prove that K is compact relative to X if and only if K is compact relative to Y.

   (b) Prove that every infinite subset E of a compact space K has a limit point in K.

   (c) Prove that every $k-$cell is compact.

4. Marks : $7 + 7 = 14$

(a) Define compact set, connected set, and path-connected set with example. Show that $A = x \in \mathbb{R}^n : ||x|| \leq 2$ is compact and connected.

(b) State and prove the Heine-Borel Theorem.

5. Marks : $7 + 7 = 14$

(a) Define continuous function on a metric space. Prove that a mapping $f : X \to Y$, where X and Y are metric space, is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

(b) Define uniformly continuous function on a metric space. Prove that every continuous mapping from a compact metric space into a metric space is uniformly continuous.

6. Marks : $5 + 4 + 5 = 14$

(a) State and prove the Intermediate-Value Theorem. Show that f is uniformly continuous on (a, b).

(b) Show that the equation $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$ has at least a real solution.

(c) Let $f : (a, b) \to \mathbb{R}$ be differentiable, and suppose that there is a constant $M > 0$ such that $||f'(x)|| \leq M$ for all $x \in (a, b)$. Here, a or b may be $\pm\infty$, and f' stands for the derivative of f.

7. Marks : $8 + 6 = 14$

(a) Distinguish between pointwise convergence and uniform convergence for sequence of functions. Show that the sequence $f_n(x) = \frac{x^n}{(1+x^n)}$, $x \in [0, 2]$ converges pointwise on $[0, 2]$ but that the convergence is not uniform.

(b) Prove that every Cauchy sequence in a compact metric space is convergent.

8. Marks : $7 + 7 = 14$

(a) State and prove the Inverse Function Theorem.

(b) Define contraction of a metric space. If X is a complete metric space, and φ is a contraction on X, then prove that there is a unique $x \in X$ such that $\varphi(x) = x$.