Differential Equation (D.E): An equation including derivatives of one or more dependent variables with respect to one or more indepent variables is called a differential equation.

For example ;

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Partial Differential Equation: A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation. (PDE)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 .$$

Ordinary Differential Equation: A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation. (ODE)

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Order: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Degree: The degree of an algebric D.E is the degree of the derivative or differential of the highest order in the equation after the equation is freed from radieal and fractions in the derivatives.

For example ; $\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^5 + 10y = 0$

Here the order of the DE is 2 and degree of the DE is 1

$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{2/3} = \frac{d^2y}{dx^2}$ another DE

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = \left(\frac{d^2y}{dx^2}\right)^3$ here order = 2

degree = 3

Linear ODE: A linear ODE of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in the form,

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x$$

observation : y and its various derivatives occur to the 1st degree only and that no products of y and/or any of its derivatives are present

Non-linear ODE: A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + y = 0$$

[Trigonometric function $\Rightarrow$ maximum non-linear]
(product)

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Variable Separable Technique :

$\frac{dy}{dx} = 1$ or $\frac{ds}{dt} = 1$

$\Rightarrow dy = dx$ $\quad \Rightarrow ds = dt$

$\Rightarrow \int dy = \int dx$ $\quad \Rightarrow \int ds = \int dt$

$\therefore y = x + c$ $\quad \therefore s = t + c$

[The solution of a DE is a function of a set of function]

$\frac{dx}{dt} = 1$

$\Rightarrow dx = dt$

$\Rightarrow \boxed{x(t) = t + c}$ → General ~~equation~~ Solution

$x(t) = t + c$

Given $x(-1) = -1$

$\therefore 1 + c = -1$

$\therefore c = -2$

$x(1) = 1$

$1 + c = 1$

$\therefore c = 0$

Initial value problem (IVP) and Boundary value problems (BVP)

$x(0) = 0$, $t = 0, x = 0$ } initial conditions

IVP ⇒ independent value same

⇒ starts ~~wit~~ from same point

⇒ initial conditions

$\frac{d^2y}{dx^2} + y = 0$

$y(1) = 3$

$y'(1) = -4$

IVP ⇒ unique solution

BVP ⇒ independent value changes

⇒ starts from different point

$\frac{d^2y}{dx^2} + y = 0$

$y(0) = 1$

$y(\frac{\pi}{2}) = 5$

BVP ⇒ Unique solution / No soln / infinitely many soln

Example 1.7

The function $f(x) = 2\sin x + 3\cos x$ defined for all real x is an explicit solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Sol^n: Let $y = f(x) = 2\sin x + 3\cos x$

$$\frac{dy}{dx} = 2\cos x - 3\sin x$$

$$\frac{d^2y}{dx^2} = -2\sin x - 3\cos x$$

Now, the DE ; $\frac{d^2y}{dx^2} + y = 0$

$$\frac{d^2y}{dx^2} + y = -2\sin x - 3\cos x + 2\sin x + 3\cos x$$
$$= 0$$

$\therefore f(x)$ is an explicit solution for all $x \in \mathbb{R}$ of the differential equation.

Example 1.8 The relation $x^2 + y^2 - 25 = 0$ is an implicit solution of the DE $x + y\frac{dy}{dx} = 0$ on the interval I defined by $-5 < x < 5$.

Sol^n:

$$x^2 + y^2 - 25 = 0$$
$$y = \pm\sqrt{25 - x^2}$$

Let $f_1(x) = \sqrt{25 - x^2}$

$f_2(x) = -\sqrt{25 - x^2}$

Now, $f_1'(x) = \frac{1}{2\sqrt{25-x^2}} \cdot -2x$

$$= -\frac{x}{\sqrt{25-x^2}}$$

$$f_2'(x) = \frac{1}{2(-\sqrt{25-x^2})} \cdot (-2x)$$

$$= \frac{x}{\sqrt{25-x^2}}$$

Now,

$$x+y\frac{dy}{dx} = x+y\left(-\frac{x}{\sqrt{25-x^2}}\right)$$

$$= x+\sqrt{25-x^2}\left(-\frac{x}{\sqrt{25-x^2}}\right)$$

$$= x-x$$

$$= 0$$

Similarly,

$$x+y\frac{dy}{dx} = x+y\cdot\frac{x}{\sqrt{25-x^2}}$$

$$= x-\sqrt{25-x^2}\cdot\frac{x}{\sqrt{25-x^2}}$$

$$= x-x$$

$$= 0$$

$\therefore$ y is an implicit solution of the D.E.

Again $x^2+y^2+25=0$ is also ~~an~~ a solution of the DE

$$y^2 = -x^2-25$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow y\frac{dy}{dx} = -x$$

$$\therefore x+y\frac{dy}{dx} = 0$$

hence x^2+y^2+25 is also an implicit ~~fo~~ solution of the DE

Solution of DE: A sol^n of a DE is a relation between the variables (independent and dependent), which is free of derivatives of any order, and which satisfies the DE ~~int~~ identically.

General Solution of DE: The general sol^n to a DE is the most general form that includes all possible solutions and typically includes arbitrary constants (in the case of an ODE) or arbitrary functions (in the case of PDE).

Particular Solution of a DE: A particular sol^n of a DE is a sol^n obtained from the general sol^n by assigning specific values to the arbitrary constants.

IVP: In the field of ~~Di~~ differential equations, an IVP (also called cauchy problem) is an ordinary DE together with a specified value, called the initial condition of the unknown function ~~at~~ a given point in the domain of the solution.

BVP: A BVP is a system of ODE's with solution and derivative values specified at more than one point.

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Exercise

(1) (a) Show that $f(x) = x + 3e^{-x}$ is a solution of the DE $\frac{dy}{dx} + y = x+1$.

Sol^n: Let $y = f(x) = x + 3e^{-x}$

$$\frac{dy}{dx} = 1 - 3e^{-x}$$

Given DE is $\frac{dy}{dx} + y = x + 1$

$\therefore \frac{dy}{dx} + y = 1 - 3e^{-x} + x + 3e^{-x} = 1 + x$

Hence $f(x)$ is a solution of the given DE. showed

(3) (a)

Show that the ~~every~~ function $f(x) = (x^3 + c)e^{-3x}$ is a solution of the DE $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$ where c is an arbitrary constant.

Sol^n: Let $y = f(x) = (x^3+c)e^{-3x}$

$$\frac{dy}{dx} = 3x^2 e^{-3x} + (x^3+c)(-3e^{-3x})$$
$$= 3x^2 e^{-3x} - 3(x^3+c)e^{-3x}$$

Now, $\frac{dy}{dx} + 3y = 3x^2 e^{-3x} - 3(x^3+c)e^{-3x} + 3(x^3+c)e^{-3x}$
$$= 3x^2 e^{-3x}$$

hence y is a solution of the DE $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$.

3(b)

Show that $f(x) = 2 + ce^{-2x^2}$ is a solution of the DE $\frac{dy}{dx} + 4xy = 8x$ where c is an arbitrary constant.

Sol^n: Let $y = f(x) = 2 + ce^{-2x^2}$

$$\frac{dy}{dx} = c\left(-4x e^{-2x^2}\right) = -4cx e^{-2x^2}$$

Now
$$\frac{dy}{dx} + 4xy = -4cx e^{-2x^2} + 4x\left\{2 + ce^{-2x^2}\right\}$$
$$= -4cxe^{-2x^2} + 8x + 4cxe^{-2x^2}$$
$$= 8x$$

$\therefore y$ is a solution of the given DE.

4(a)

Show that $f(x) = c_1 e^{4x} + c_2 e^{-2x}$ where c_1 and c_2 are arbitrary constants, is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0.$$

Sol^n: Let $y = f(x) = c_1 e^{4x} + c_2 e^{-2x}$

$$\frac{dy}{dx} = 4c_1 e^{4x} - 2c_2 e^{-2x}$$

$$\frac{d^2y}{dx^2} = 16c_1 e^{4x} + 4c_2 e^{-2x}$$

Now,
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 16c_1 e^{4x} + 4c_2 e^{-2x} - 2\left\{4c_1 e^{4x} - 2c_2 e^{-2x}\right\} + -8\left\{c_1 e^{4x} + c_2 e^{-2x}\right\}$$
$$= 16c_1 e^{4x} + 4c_2 e^{-2x} - 8c_1 e^{4x} + 4c_2 e^{-2x} - 8c_1 e^{4x} - 8c_2 e^{-2x}$$
$$= 0$$

hence $f(x)$ is a solution of the DE.

4(b)

Show that $g(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x}$, is a solution of the DE $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$, where c_1, c_2 and c_3 are arbitrary constants.

Solⁿ: Let $y = g(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x}$

$$\frac{dy}{dx} = 2c_1 e^{2x} + 2c_2 x e^{2x} + c_2 e^{2x} + -2c_3 e^{-2x}$$

$$\frac{d^2y}{dx^2} = 4c_1 e^{2x} + 4c_2 x e^{2x} + 2c_2 e^{2x} + 2c_2 e^{2x} + 4c_3 e^{-2x}$$

$$= 4c_1 e^{2x} + 4c_2 x e^{2x} + 4c_2 e^{2x} + 4c_3 e^{-2x}$$

$$\frac{d^3y}{dx^3} = 8c_1 e^{2x} + 8c_2 x e^{2x} + 4c_2 e^{2x} + 8c_2 e^{2x} - 8c_3 e^{-2x}$$

$$= 8c_1 e^{2x} + 8c_2 x e^{2x} + 12 c_2 e^{2x} - 8c_3 e^{-2x}$$

Now,

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = \cancel{8c_1 e^{2x}} + \cancel{8c_2 x e^{2x}} + \cancel{12c_2 e^{2x}} - \cancel{8c_3 e^{-2x}} - \cancel{8c_1 e^{2x}}$$

$$- \cancel{8c_2 x e^{2x}} - \cancel{8c_2 e^{2x}} - \cancel{8c_3 e^{-2x}} - \cancel{8c_1 e^{2x}} - \cancel{8c_2 x e^{2x}}$$

$$- \cancel{4c_2 e^{2x}} + \cancel{8c_3 e^{-2x}} + \cancel{8c_1 e^{2x}} + \cancel{8c_2 x e^{2x}} + \cancel{8c_3 e^{-2x}}$$

$$= 0$$

Hence $g(x)$ is a solution of the given DE.

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Ex-1.12

(IVP)

$$\frac{d^2y}{dx^2} + y = 0$$

$y(1) = 3$

$y'(1) = -4$

Solⁿ: Auxilary equation

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$y(x) = c_1 e^{ix} + c_2 e^{-ix}$

$= c_1(\cos x + i\sin x) + c_2(\cos x - i\sin x)$

$= c_1 \cos x + c_2 \sin x$ —— (1)

$y'(x) = -c_1 \sin x + c_2 \cos x$

$\therefore y(1) = c_1 \cos(1) + c_2 \sin(1) = 3$

$y'(1) = -c_1 \sin(1) + c_2 \cancel{\sin}\cos(1) = -4$

$\Rightarrow c_2 - c_1 \sin(1) = -4$

Solving c_1 and c_2 we have to put the values at (1) which will be the solution of the given DE.

Ex-1.13

(BVP) $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 1$

$y(\frac{\pi}{2}) = 5$

Solⁿ: Auxilary equation,

$r^2 + 1 = 0$

$r = \pm i$

$\therefore y(x) = c_1 \cos x + c_2 \sin x$ —— (1)

$y(0) = c_1 \cos(0) + c_2 \sin(0)$

$\Rightarrow 1 = c_1 + 0$

$\therefore c_1 = 1$

$y(\frac{\pi}{2}) = c_1 \cos(\frac{\pi}{2}) + c_2 \sin(\frac{\pi}{2})$

$\Rightarrow 5 = 0 + c_2$

$\therefore c_2 = 5$

putting the value of c_1, c_2 at ① we get

$y(x) = \cos x + 5\sin x$, which is the solution of the

~~BVP~~ given DE.

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Ex-1 State the order and degree of the following D.E.

(i) $\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^5 + 10y = 0$

order $\rightarrow$ 2

degree $\rightarrow$ 1

(ii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}} = \frac{d^2y}{dx^2}$

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = \left(\frac{d^2y}{dx^2}\right)^3$

order $\rightarrow$ 2

degree $\rightarrow$ 3

(iii) $y\left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} - y = 0$

order $\rightarrow$ 1

degree $\rightarrow$ 2

(iv) $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 1+x$

$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = (1+x)^2$

order $\rightarrow$ 1

degree $\rightarrow$ 2

Ex-2 find the DE of which each of the following functions is the general solution (A, B being arbitrary constants):

(i) $y = A\sin x + B\cos x$

$\frac{dy}{dx} = A\cos x - B\sin x$

$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$

$= -(A\sin x + B\cos x)$

$= -y$

$\therefore \frac{d^2y}{dx^2} + y = 0$

which is the required DE.

(ii) $y = e^{-x}(A\cos x + B\sin x)$

$\Rightarrow y e^{x} = A\cos x + B\sin x$

$\Rightarrow y e^{x} + e^{x}\frac{dy}{dx} = -A\sin x + B\cos x$

$\cancel{\Rightarrow e^{x}\frac{dy}{dx} = -A\sin x + B}$

$y e^{x} + \frac{dy}{dx}e^{x} + e^{x}\frac{d^2y}{dx^2} + e^{x}\frac{dy}{dx} = -A\cos x - B\sin x$

$= -(A\cos x + B\sin x)$

$= -y e^{x}$

$\therefore \frac{d^2y}{dx^2}e^{x} + \cancel{2e^{x}}\,2\frac{dy}{dx}e^{x} + 2ye^{x} = 0$

$\therefore \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

which is the required D.E.

(iii) $y = A\cos(px - B)$; p being a parameter.

$\frac{dy}{dx} = -Ap\sin(px - B)$

$\frac{d^2y}{dx^2} = -Ap^2\cos(px - B)$

$= -p^2 y$

$\therefore \frac{d^2y}{dx^2} + p^2 y = 0$

which is the D.E.

(iv) $y = kx + k - k^3$, k being a parameter

$\frac{dy}{dx} = k$

Now,

$y = x\frac{dy}{dx} + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$

$\therefore y = (1+x)\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$

is the required DE.

(v) $y = A + \frac{B}{x}$

$$\frac{dy}{dx} = -\frac{B}{x^2}$$

$\Rightarrow x^2 \frac{dy}{dx} = -B$

$\Rightarrow 2x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} = 0$

$\therefore \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$

is the required DE.

(vi) $y = Ae^{2x} + Be^{-2x}$

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$$

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$$
$$= 4(Ae^{2x} + Be^{-2x})$$
$$= 4y$$

$\therefore \frac{d^2y}{dx^2} - 4y = 0$, is the required DE.

Ex-3 Find the differential equation of

(i) All parabolas having their axes parallel to the y axis.

Sol^n: Equation of any parabola whose axes is parallel to the ~~x~~ y axis is

$$y = Ax^2 + Bx + C$$

$\Rightarrow \frac{dy}{dx} = 2Ax + B$

$\Rightarrow \frac{d^2y}{dx^2} = 2A$

$\therefore \frac{d^3y}{dx^3} = 0$ which is the required DE.

(ii) All circles passing through the origin and having their centres ~~at~~ on the x-axis.

Sol^n: Let the centre be $(a, 0)$

$\therefore (x-a)^2 + (y-0)^2 = r^2$ which passes through origin.

$\therefore a^2 + 0^2 = r^2 \Rightarrow r = a$

The equation of the circle is: $(x-a)^2+y^2=a^2$

$\Rightarrow x^2-2ax+a^2+y^2=a^2$

$\Rightarrow x^2+y^2-2ax=0$

where a is a parameter.

Now, $x^2+y^2-2ax=0$ —— (I)

$2x+2y\frac{dy}{dx}-2a=0$

$2a=2x+2y\frac{dy}{dx}$ —— (II)

From (I) & (II) $x^2-y^2-\left(2x+2y\frac{dy}{dx}\right)x=0$

$\Rightarrow x^2-y^2-2x^2-2xy\frac{dy}{dx}=0$

$\Rightarrow y^2-x^2-2xy\frac{dy}{dx}=0$ is the required DE

(III) All the circles having a constant radius.

Sol^n: Let, the centre be (a,b) & radius r

$\therefore (x-a)^2+(y-b)^2=r^2$ —— (I)

$\Rightarrow 2(x-a)+2(y-b)\frac{dy}{dx}=0$

$\Rightarrow (y-b)\frac{dy}{dx}=-(x-a)$

$\therefore \frac{dy}{dx}=-\frac{(x-a)}{(y-b)}$ —— (II)

$$\frac{d^2y}{dx^2}=-\frac{(y-b)-(x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$y_2=\frac{(y-b)-(x-a)\cdot\left(-\frac{(x-a)}{(y-b)}\right)}{(y-b)^2}$$

$$=-\frac{(y-b)+\frac{(x-a)^2}{(y-b)}}{(y-b)^2}$$

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$$= -\frac{(y-b)^2 + (x-a)^2}{(y-b)^3}$$

$$= -\frac{r^2}{(y-b)^3}$$

$$\Rightarrow (y-b)^3 = -\frac{r^2}{y_2} \quad \text{--- (III)}$$

from (I)

$$\frac{(x-a)^2}{(y-b)^2} + 1 = \frac{r^2}{(y-b)^2}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{r^2}{(y-b)^2} \Rightarrow (y-b)^2 = \frac{r^2}{1+y_1^2} \quad \text{--- (IV)}$$

from (III) & (IV) $\qquad [(III)^2 = (IV)^3]$

$$\left(\frac{r^2}{y_2}\right)^2 = \left(\frac{r^2}{1+y_1^2}\right)^3$$

$$\Rightarrow \frac{r^4}{y_2^2} = \frac{r^6}{(1+y_1^2)^3}$$

$$\Rightarrow r^2 y_2^2 = (1+y_1^2)^3$$

$$\therefore \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$ which is the required DE.

(b) The system of circles having a constant radius a and having their centres on the x axis.

Sol^n: Let the centre be $(\alpha, 0)$

$\therefore$ The equation of circle is ~~(x-a)~~ $(x-\alpha)^2 + y^2 = a^2$ --- (I) where a is the ~~differentiating~~ constant radius.

Differentiating both sides with respect to x in (I)

$$2(x-\alpha) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2(x-\alpha)$$

$$\left\{y\frac{dy}{dx}\right\}^2 = \left\{-(x-\alpha)\right\}^2$$

$$\Rightarrow y^2\left(\frac{dy}{dx}\right)^2 = (x-\alpha)^2 \quad \text{--- (II)}$$

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From (I) & (II) ;

$$y^2 \left(\frac{dy}{dx}\right)^2 + y^2 = a^2$$

$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} y^2 = a^2$ is the required DE.

Chapter TWO

Exact Differential Equation

$$df(x,y) = \frac{\partial F(x,y)}{\partial x} dx + \frac{\partial F(x,y)}{\partial y} dy$$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Exact DE: The expression $M(x,y)dx + N(x,y)dy$ — (I) is called an exact differential equation in a domain if there exists a function F of two real variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$. That is the expression (I) is an exact DE in differential in D if there exists a function F such that $\frac{\partial F(x,y)}{\partial x} = M(x,y)$ and

$\frac{\partial F(x,y)}{\partial y} = N(x,y)$ for all $(x,y) \in D$.

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the DE $M(x,y)dx + N(x,y)dy = 0$ is called an exact DE.

Theorem 2.1

Consider the differential equation $M(x,y)dx + N(x,y)dy = 0$ — ①, where M and N have continuous first partial derivatives at all the points (x,y) in a rectangular domain D

If the DE ① is exact in D, then $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ for all $(x,y) \in D$.

Conversely, if $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ for all $(x,y) \in D$ then the DE ① is exact in D.

Example 2.5

Solve the equation $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$

Sol^n: Lets determine whethere the equation is exact or not.

$$M(x,y)dx + N(x,y)dy = 0$$

$\therefore$ $M(x,y) = 3x^2 + 4xy$, $N(x,y) = 2x^2 + 2y$

$$\frac{\partial M(x,y)}{\partial y} = 0 + 4x = 4x \qquad \frac{\partial N(x,y)}{\partial x} = 4x$$

Here $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$ Hence the equation is exact.

For all $(x,y) \in D$ we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 3x^2 + 4xy$$ and

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^2 + 2y$$

$$\therefore F(x,y) = \int M(x,y)\,\partial x + \varphi(y)$$

$$= \int (3x^2+4xy)\,\partial x + \varphi(y)$$

$$= x^3 + 2x^2y + \varphi(y)$$

$$\frac{\partial F(x,y)}{\partial y} = 2x^2 + \varphi'(y)$$

$$\cancel{F(x,y) = \int N(x,y)\,\partial y + \varphi}$$

$$\Rightarrow 2x^2+2y = 2x^2 + \varphi'(y) \qquad \left[\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^2+2y\right]$$

$$\Rightarrow \varphi'(y) = 2y$$

$$\Rightarrow \varphi(y) = \int 2y\,dy = y^2 + c_0$$

$$\therefore F(x,y) = x^3 + 2x^2y + y^2 + c_0 = c_1$$ is the required solution.

$$\Rightarrow \boxed{x^3 + 2x^2y + y^2 = c}$$

Example 2.6

Solve the ~~in~~ IVP $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0$, $y(0) = 2$.

Sol^n:

$$M(x,y) = 2x\cos y + 3x^2y$$

$$N(x,y) = x^3 - x^2\sin y - y$$

$$\frac{\partial M(x,y)}{\partial y} = -2x\sin y + 3x^2, \quad \frac{\partial N(x,y)}{\partial x} = 3x^2 - 2x\sin y$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

hence the equation is exact.

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Now, for all $(x,y) \in D$, there must be F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 2x\cos y + 3x^2 y$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = x^3 - x^2\sin y - y$$

$$F(x,y) = \int M(x,y)\,dx + \varphi(y) = \int (2x\cos y + 3x^2 y)\,dx + \varphi(y)$$

$$= \cancel{x^2\cos y + \frac{3}{2}x^3}$$

$$= x^2\cos y + x^3 y + \varphi(y)$$

$$\frac{\partial F(x,y)}{\partial y} = -x^2\sin y + x^3 + \varphi'(y)$$

$$\Rightarrow x^3 - x^2\sin y - y = -x^2\sin y + x^3 + \varphi'(y)$$

$$\Rightarrow \varphi'(y) = -y$$

$$\varphi(y) = -\int y\,dy = -\frac{y^2}{2} + c_0$$

$$\therefore F(x,y) = x^2\cos y + x^3 y - \frac{y^2}{2} + c_0 = c_1$$

$$\therefore x^2\cos y + x^3 y - \frac{y^2}{2} = c$$

when $y(0) = 2$, $c = -2$

$\therefore x^2\cos y + x^3 y - \frac{y^2}{2} = -2$ is the required solution.

Exercise

P-37

(11) Solve the IVP, $(2xy-3)dx+(x^2+4y)dy=0$, $y(1)=2$.

Sol^n: Here,

$$M(x,y)=2xy-3$$
$$N(x,y)=x^2+4y$$

Now,

$$\frac{\partial M(x,y)}{\partial y}=2x \quad , \quad \frac{\partial N(x,y)}{\partial x}=2x$$

$\therefore$ The equation is exact.

Now, $\frac{\partial f(x,y)}{\partial x}=M(x,y)=2xy-3$ & $\frac{\partial f(x,y)}{\partial y}=N(x,y)=x^2+4y$

$$f(x,y)=\int M(x,y)dx+\varphi(y)$$
$$=\int(2xy-3)dx+\varphi(y)$$
$$=2y\cdot\frac{x^2}{2}-3x+\varphi(y)$$
$$=x^2y-3x+\varphi(y)$$

$$\frac{\partial}{\partial y}f(x,y)=x^2+\varphi'(y)$$

$$\Rightarrow x^2+4y=x^2+\varphi'(y)$$

$$\Rightarrow \varphi'(y)=4y$$

$$\varphi(y)=4\int y\,dy=\cancel{4}2y^2+c_0$$

$$\therefore f(x,y)=x^2y-3x+2y^2+c_0=c_1$$

$$\Rightarrow x^2y-3x+2y^2=c$$

applying the initial condition we get $\cancel{\$}$

$c=7$

$$\left[\begin{array}{l} y(1)=2 \\ (1)^2(2)-3(1)+2(2)^2=c \\ 2-3+8=c \\ c=7 \end{array}\right]$$

$$\therefore x^2y-3x+2y^2=7$$

which is the required solution.

12. Solve the IVP : $(3x^2y^2 - y^3 + 2x)dx + (2x^3y - 3xy^2 + 1)dy = 0$; $y(-2) = 1$.

Sol^n: Here $M(x,y) = 3x^2y^2 - y^3 + 2x$

$N(x,y) = 2x^3y - 3xy^2 + 1$

Now, $\frac{\partial M(x,y)}{\partial y} = 6x^2y - 3y^2$, $\frac{\partial N(x,y)}{\partial x} = 6x^2y - 3y^2$

$\therefore$ the equation is exact

$\frac{\partial f(x,y)}{\partial x} = M(x,y) = 3x^2y^2 - y^3 + 2x$ and $\frac{\partial f(x,y)}{\partial y} = N(x,y) = 2x^3y - 3xy^2 + 1$

$f(x,y) = \int M(x,y)dx + \varphi(y)$

$= \int (3x^2y^2 - y^3 + 2x)dx + \varphi(y)$

$= \cancel{x^3y^2 + x^2 + \varphi(y)} = x^3y^2 - xy^3 + x^2 + \varphi(y)$

$\frac{\partial f(x,y)}{\partial y} = 2x^3y \cancel{+ \varphi'(y)} - 3xy^2 + \varphi'(y)$

$\Rightarrow 2x^3y - 3xy^2 + 1 = 2x^3y \cancel{+ \varphi'(y)} - 3xy^2 + \varphi'(y)$

$\Rightarrow \varphi'(y) = \cancel{-3xy^2} + 1$

$\cancel{\varphi(y) = \int(-3xy^2+1)dy}$ $\varphi(y) = \int 1 dy = y + C_0$

$= \cancel{-xy^3}$

$\therefore f(x,y) = x^3y^2 - xy^3 + x^2 + y + C_0 = C_1$

$\Rightarrow x^3y^2 - xy^3 + x^2 + y = C$

applying the initial condition $y(-2) = 1$, we get

$(-2)^3(1)^2 - (-2)1^3 + (-2)^2 + 1 = C$

$-8 + 2 + 4 + 1 = C$

$C = -1$

$\therefore x^3y^2 - xy^3 + x^2 + y = -1$

which is the required solution.

19. Solve the IVP : $(2y\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0$, $y(0)=3$

Sol^n: Here $M(x,y) = 2y\sin x\cos x + y^2\sin x$

$N(x,y) = \sin^2 x - 2y\cos x$

$\frac{\partial M(x,y)}{\partial y} = 2\sin x\cos x + 2y\sin x$, $\frac{\partial N(x,y)}{\partial x} = 2\sin x\cos x + 2y\sin x$

hence the equation is exact.

$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 2y\sin x\cos x + y^2\sin x$

$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \sin^2 x - 2y\cos x$

$F(x,y) = \int M(x,y)dx$

$= \int (2y\sin x\cos x + y^2\sin x)dx$

$= \cancel{-2y\sin^2 x + 2y\cos^2 x}$

$= -2y\sin x\cos x - y^2 \cancel{\sin x}\cos x + \varphi(y)$

$\frac{\partial F(x,y)}{\partial y} = -2\sin x\cos x - 2y\cos x + \varphi'(y)$

$\Rightarrow 2\sin^2 x - 2y\cos x = -2\sin x\cos x - 2y\cos x + \varphi'(y)$

$\varphi'(y) = \sin^2 x + \sin 2x$

$\varphi(y) = y\sin^2 x + y\sin 2x + c_0$

$\therefore F(x,y) = -2y\sin x\cos x - y^2\cos x + y\sin^2 x + y\sin 2x + c_0$

$\therefore \cancel{2\sin x} F(x,y) = -y^2\cos x + y^2\sin^2 x + c_0$

$\frac{dF(x,y)}{dx} = -2y\cos x\frac{dy}{dx} + 2y\sin x\cos x + \sin^2 x\frac{dy}{dx} + y^2\sin x$

$= (2y\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy$

$= 0$

$\therefore f(x,y) = c_1$

$y \sin^2 x - y^2 \cos x + c_0 = c_2$

$y \sin^2 x - y^2 \cos x = c$

Applying the initial condition we get $c = -9$

$\left[9 \cdot 0 - 9\,(3)^2 \cdot 1 = c \quad c = -9 \right]$

$\therefore y \sin^2 x - y^2 \cos x = -9$ is the required solution.

14. Solve the IVP : $(ye^x + 2e^x + y^2)\,dx + (e^x + 2xy)\,dy = 0$, $y(0) = 6$.

Sol^n: $(ye^x + 2e^x + y^2) + (e^x + 2xy)\dfrac{dy}{dx} = 0$ ——— (1)

$\Rightarrow M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$

$\dfrac{\partial M(x,y)}{\partial y} = e^x + 2y$ and $\dfrac{\partial N(x,y)}{\partial x} = e^x + 2y$

$\therefore$ the equation is exact.

Now, $f_x = M(x,y) = ye^x + 2e^x + y^2$

$f_y = N(x,y) = e^x + 2xy$

$f(x,y) = \int M(x,y)\,\partial x$

$= \int (ye^x + 2e^x + y^2)\,\partial x$

$= ye^x + 2e^x + y^2 x + \varphi(y)$

$\Rightarrow f_y(x,y) = e^x + 2xy + \varphi'(y)$

$\Rightarrow e^x + 2xy = e^x + 2xy + \varphi'(y)$

$\Rightarrow \varphi'(y) = 0$

$\varphi(y) = c_0$

$\therefore f(x,y) = ye^x + 2e^x + xy^2 + c_0$

Again $f(x,y) = ye^x + 2e^x + xy^2 + c_0 = c_1$

$\Rightarrow \quad ye^x + 2e^x + xy^2 = c$

applying the initial condition we get $c = 8$

$\left[\begin{array}{l} y(0) = 6 \\ 6.1 + 2 + 0.6^2 = c \\ c = 8 \end{array}\right]$

$\therefore \quad ye^x + 2e^x + xy^2 = 8$

which is the required solution.

<u>Homogeneous equation</u>: The first-order differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be homogeneous if, when written in the derivative form $dy/dx = f(x,y)$, there exists a function g such that $f(x,y)$ can be expressed in the form $g(y/x)$.

<u>Theorem 2.3</u>

If $M(x,y)dx + N(x,y)dy = 0$ —— (1) is a homogeneous equation, then the change of variables $y = vx$ transforms (1) into a separable equation in the variables v and x.

<u>Proof</u>: Since $M(x,y)dx + N(x,y) = 0$ is homogeneous, it may be written in the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

Let $y = vx$ then, $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}, \quad g\left(\dfrac{y}{x}\right) = g(v)$

$\therefore \quad v + x\frac{dv}{dx} = g(v)$

$\Rightarrow [v - g(v)]\,dx + \cancel{x\,dy} = x\,dv = 0$ which is a variable seperable equation.

$$\frac{dx}{x} + \frac{dv}{v-g(v)} = 0 \quad \text{[separating the variables]}$$

Then $\displaystyle\int \frac{dx}{x} + \int \frac{dv}{v-g(v)} = c$

$\left[\text{Let } f(v) = \displaystyle\int \frac{dv}{v-g(v)}\right]$

$\therefore \quad \ln|x| + f\left(\frac{y}{x}\right) = c$

$\therefore \quad F\left(\frac{y}{x}\right) + \ln|x| = c.$

$\boxed{}$ $\displaystyle\frac{dy}{dx} = \frac{x+y}{x}$

$\displaystyle = 1 + \frac{y}{x} = f\left(\frac{y}{x}\right)$

$\therefore$ this is homogeneous

Now, $\displaystyle\frac{y}{x} = v$

$\Rightarrow y = vx$

$\displaystyle\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \displaystyle 1 + \frac{y}{x} = v + x\frac{dv}{dx}$

$\Rightarrow \displaystyle 1 = x\frac{dv}{dx}$

$\Rightarrow \displaystyle \frac{dv}{dx} = \frac{1}{x}$

$\Rightarrow \displaystyle \int dv = \int \frac{1}{x}\,dx$

$v = \ln|x| + c$

$\therefore \quad y = x\ln|x| + cx.$

Example 2.9

Solve the IVP $x \sin y\, dx + (x^2+1)\cos y\, dy = 0$; $y(1) = \frac{\pi}{2}$

Sol^n: $\frac{x}{x^2+1} dx = -\frac{\cos y}{\sin y} dy$

$\Rightarrow \int \frac{x}{x^2+1} dx = -\int \frac{\cos y}{\sin y} dy$

$\Rightarrow \frac{1}{2}\int \frac{2x}{x^2+1} dx = -\cancel{\ln|\sin y| + c_1} \int \frac{\cos y}{\sin y} dy$

$\Rightarrow \frac{1}{2} \ln|x^2+1| = -\ln|\sin y| + c_1$

$\Rightarrow \ln|x^2+1| + 2\ln|\sin y| = 2c_1 = c$

$\Rightarrow \ln|x^2+1| + \ln|\sin^2 y| = \ln c$

$\Rightarrow \ln\{(x^2+1)\sin^2 y\} = \ln c$

$\Rightarrow \therefore (x^2+1)\sin^2 y = c$

applying the initial condition, we get $c = 2$

$\therefore (x^2+1)\sin^2 y = 2$; which is the required equation.

Example 2.12:

Solve the equation ; $(x^2-3y^2)dx + 2xy\,dy = 0$.

Sol^n: $(x^2-3y^2)dx + 2xy\,dy = 0$

$\Rightarrow (x^2-3y^2) + 2xy\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{3y^2-x^2}{2xy} = \frac{3y}{2x} - \frac{1}{2}\frac{1}{\left(\frac{y}{x}\right)}$

$= \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\frac{1}{\left(\frac{y}{x}\right)}$

$= g\left(\frac{y}{x}\right)$

$\therefore$ The equation is homogeneous

Now, let $\frac{y}{x} = v \Rightarrow y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{3}{2}\frac{y}{x} - \frac{x}{2y} = v + x\frac{dv}{dx}$$

$$\Rightarrow \frac{3v}{2} - \frac{1}{2v} = v + x\frac{dv}{dx}$$

$$\Rightarrow \frac{3v^2 - 1 - 2v^2}{2v} = x\frac{dv}{dx}$$

$$\Rightarrow 2vx\,dv = (v^2-1)\,dx$$

$$\Rightarrow \int \frac{2v}{v^2-1}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \ln|v^2-1| = \ln|x| + c_1$$

$$\Rightarrow \ln\left|\frac{v^2-1}{x}\right| = \ln c_1$$

$$\Rightarrow v^2 - 1 = |cx|$$

$$\Rightarrow \left|\frac{y^2}{x^2} - 1\right| = |cx|$$

$\Rightarrow y^2 - x^2 = cx^3$; which is the required ~~equation~~ solution.

Example - 2.13

Solve the IVP $(y + \sqrt{x^2+y^2})\,dx - x\,dy = 0$; $y(1) = 0$.

Sol^n^: $(y + \sqrt{x^2+y^2})\,dx = x\,dy$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$
$$= g\left(\frac{y}{x}\right)$$

$\therefore$ the equation is homogeneous.

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$$\therefore \frac{dy}{dx} = v + \sqrt{1+v^2}$$

Let $\frac{y}{x} = v$

$\therefore y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \cancel{v} + \sqrt{1+v^2} = \cancel{v} + x\frac{dv}{dx}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dv}{\sqrt{1+v^2}}$$

$$\Rightarrow \ln|x| = \ln\left|v + \sqrt{1+v^2}\right| + c_1$$

$$\Rightarrow \ln\left|\frac{v+\sqrt{1+v^2}}{x}\right| = -c_1 = c = \ln c$$

$$\Rightarrow \frac{v+\sqrt{1+v^2}}{x} = c$$

$$\therefore v + \sqrt{1+v^2} = cx$$

$$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = cx$$

$$\Rightarrow \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = cx$$

$$\Rightarrow y + \sqrt{x^2+y^2} = cx^2$$

Applying the initial condition we ge $c = 1$

$\therefore y + \sqrt{x^2+y^2} = x^2$; which is the required solution.

Exercise

11. Solve the D.E: $\left(x\tan\frac{y}{x} + y\right)dx - xdy = 0$

Sol^n: $\left(x\tan\frac{y}{x} + y\right)dx = xdy$

$$\frac{x\tan\frac{y}{x} + y}{x} = \frac{dy}{dx}$$

$$\tan\frac{y}{x} + \frac{y}{x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = g(x)$$

$\therefore$ it is homogeneous equation

Let $\frac{y}{x} = v$

$y = vx$

$$\frac{dy}{dx} = \tan v + v \quad \text{—} (1)$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{—} (11)$$

From (1) & (11)

$$\tan v + v = v + x\frac{dv}{dx}$$

$$\frac{dv}{\tan v} = \frac{dx}{x}$$

$$\Rightarrow \frac{\cos v}{\sin v} dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{\cos v}{\sin v} dv = \int \frac{dx}{x}$$

$$\Rightarrow \ln|\sin v| = \ln|x| + \ln c$$

$$\Rightarrow \ln\left|\frac{\sin v}{x}\right| = \ln c$$

$$\Rightarrow \sin v = cx$$

$$\sin\frac{y}{x} = cx$$

Which is the required solution.

15. Solve the I.V.P $(y+2)dx + y(x+4)dy = 0$

Linear Equations

A first order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be written in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

For example.
$$x\frac{dy}{dx} + (x+1)y = x^3$$

$$\Rightarrow \frac{dy}{dx} + \left(1+\frac{1}{x}\right)y = x^2 \qquad P(x) = 1+\frac{1}{x}$$
$$Q(x) = x^2$$

Bernoulli Equations

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is called a Bernoulli differential equation.

Integrating factor

1. $\frac{dy}{dx} + P(x)y = Q(x)$ 1. $IF = e^{\int P(y)dy}\ e^{\int P(x)\,dx}$

2. $\frac{dx}{dy} + P(y)x = Q(y)$ 2. $IF = e^{\int P(y)dy}$

3. $\frac{ds}{dt} + P(t)s = Q(t)\int P(x)dx$ 3. $IF = e^{\int P(t)dt}$

Theorem 2.5

Suppose $n \neq 0$ or 1. Then the transformation $v = y^{1-n}$ reduceds the Bernoulli equation is $\frac{dy}{dx} + P(x)y = Q(x)y^n$ to a linear equation in v.

Proof: $\frac{dy}{dx} + P(x)y = Q(x)y^n$ —— (I)

Multiply (I) by y^{-n} and we get

$$\frac{dy}{dx} y^{-n} + P(x) y^{1-n} = Q(x) \text{ —— (II)}$$

If we let $v = y^{1-n}$, then $\frac{dv}{dx} = (1-n) y^{-n} \frac{dy}{dx}$

and (II) transforms into $\frac{1}{1-n} \frac{dv}{dx} + P(x) v = Q(x)$ —— (III)

Let $(1-n)P(x) = P_1(x)$ and $(1-n)Q(x) = Q_1(x)$

then from (III) we get

$\frac{dv}{dx} + P_1(x) v = Q_1(x)$ which is linear in v.

p-51

Example 2.14 $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}$

Solⁿ: Here $P(x) = \frac{2x+1}{x}$

$Q(x) = e^{-2x}$

$$I.F = e^{\int P(x)dx} = e^{\int \frac{2x+1}{x} dx} = e^{\int (2+\frac{1}{x})dx} = e^{2x+\ln x}$$

$$= e^{2x} \cdot e^{\ln x}$$

$$= x e^{2x}$$

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Now, multiplying the equation by this integrating factor, we get

$$x e^{2x} \frac{dy}{dx} + x e^{2x}\left(\frac{2x+1}{x}\right) y = x e^{-2x} \cdot e^{-2x}$$

$$\Rightarrow x e^{2x} \frac{dy}{dx} + e^{2x}(2x+1) y = x$$

$$\Rightarrow \frac{d}{dx}\left(x e^{2x} \cdot y\right) = x \qquad \left[\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$x e^{2x} y = \frac{x^2}{2} + c$$

$$\therefore y = \frac{1}{2} x e^{-2x} + \frac{c}{x} e^{-2x}$$; where c is an arbitrary constant.

Example 2.15

Solve the IVP $(x^2+1)\frac{dy}{dx} + 4xy = x$; $y(2) = 1$

Sol^n:

$$\frac{dy}{dx} + \frac{4x}{x^2+1} y = \frac{x}{x^2+1} \text{ --- (1)}$$

$$\therefore P(x) = \frac{4x}{x^2+1}$$

$$Q(x) = \frac{x}{x^2+1}$$

$$I.f = e^{\int \frac{4x}{x^2+1} dx} = e^{2\int \frac{2x}{x^2+1} dx} = e^{2\ln|x^2+1|}$$

$$= e^{\ln(x^2+1)^2}$$

$$= (x^2+1)^2$$

$u = x^2+1$

$du = 2x\, dx$

$\therefore \int \frac{4x}{x^2+1} dx = 2\int \frac{du}{u}$

$= 2\ln u = 2\ln(x^2+1)$

Multiply (1) by the integrating factor and we get

$$(x^2+1)^2 \frac{dy}{dx} + (x^2+1)^2 \frac{4x}{(x^2+1)} y = x(x^2+1)$$

$$\Rightarrow (x^2+1)^2 \frac{dy}{dx} + 4x(x^2+1) y = x(x^2+1)$$

$$\frac{d}{dx}\left\{(x^2+1)^2 y\right\} = x^3+x$$

$$\Rightarrow (x^2+1)^2 y = \frac{x^4}{4}+\frac{x^2}{2}+c \quad \text{(ii)}$$

Applying the initial condition we get:

$$25 = 4+2+c \quad \therefore c = 19$$

And (ii) becomes $(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19$.

P-50 $\frac{\partial}{\partial y}\left[\mu(x) P(x) y - \mu(x) Q(x)\right] = \frac{\partial}{\partial x}\left[\mu(x)\right]$ [μ is an IF].

Example 2.17 $\frac{dy}{dx} + y = xy^3$.

Sol^n: $\frac{dy}{dx} + y = xy^3$ ——— (i)

The equation is a Bernoulli equation where $P(x) = 1 = x^0$

$Q(x) = x$, $n = 3$

Multiply the equation with y^{-3}, we get

$$y^{-3}\frac{dy}{dx} + y^{-2} = x \quad \text{(ii)}$$

Let $v = y^{1-n} = y^{-2}$

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\therefore y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

Now from (ii) $-\frac{1}{2}\frac{dv}{dx} + v = x$

$$\Rightarrow \frac{dv}{dx} - 2v = -2x \quad \text{(iii)}$$

: which is a linear equation where $P(x) = -2$

$$\therefore e^{\int P(x)dx} = e^{\int -2dx} = e^{-2x}$$

Now multiplying (III) with e^{-2x} we get,

$$e^{-2x}\frac{dv}{dx} - 2v.e^{-2x} = -2xe^{-2x}$$

$$\Rightarrow \frac{d}{dx}(e^{-2x}.v) = -2xe^{-2x}$$

$$\Rightarrow e^{-2x} v = -2\int xe^{-2x} = -2\left[x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}dx\right]$$

$$= xe^{-2x} - \int e^{-2x}dx = xe^{-2x} - \frac{e^{-2x}}{-2} + c$$

$$= xe^{-2x} + \frac{1}{2}e^{-2x} + c$$

$$\therefore v = x + \frac{1}{2} + ce^{2x}$$

$$\Rightarrow y^{-2} = x + \frac{1}{2} + ce^{2x}$$

$$\therefore \frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$$

Exercise

P-56 (i) Solve the D.E : $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$

Sol^n: Here $P(x) = \frac{3}{x}$, $Q(x) = 6x^2$

$$IF = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Multiply the equation with the IF & we get

$$x^3\frac{dy}{dx} + x^2 3y = 6x^5$$

$$\Rightarrow \frac{d}{dx}(x^3.y) = 6x^5$$

$$x^3 y = 6\cdot\frac{x^6}{6} + c$$

$$\Rightarrow x^3 y = x^6 + c$$

$$\Rightarrow \therefore y = x^3 + cx^{-3}$$

$\therefore\ y = x^3 + cx^{-3}$ is the required solution.

(3) Solve the DE : $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$

Sol^n:

here $P(x) = 3$

$IF = e^{\int P(x)dx} = e^{3\int dx} = e^{3x}$

Now multiply the equation with this IF & we get

$$e^{3x}\frac{dy}{dx} + 3ye^{3x} = 3x^2$$

$$\Rightarrow \frac{d}{dx}(e^{3x}\cdot y) = 3x^2$$

$$\Rightarrow e^{3x} y = x^3 + c$$

$$\Rightarrow y = x^3 e^{-3x} + c e^{-3x} = (x^3 + c) e^{-3x}$$

which is the required solution.

11. Solve: $\frac{dr}{d\theta} + r\tan\theta = \cos\theta$

Sol^n: here $P(\theta) = \tan\theta$

$IF = e^{\int P(\theta)d\theta} = e^{\int \tan\theta\, d\theta} = e^{\ln|\sec\theta|} = |\sec\theta|$

Now multiply the equation with this IF and we get

$$\sec\theta\frac{dr}{d\theta} + \sec\theta\tan\theta\, r = 1$$

$$\Rightarrow \frac{d}{d\theta}(\sec\theta\cdot r) = 1$$

$$\Rightarrow \sec\theta\cdot r = \int d\theta = x + c$$

$$\Rightarrow r = (\theta + c)\cos\theta$$

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15. $\dfrac{dy}{dx} - \dfrac{y}{x} = -\dfrac{y^2}{x}$

Sol^n: general form of the equation is ; $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

where $P(x) = -\dfrac{1}{x}$, $Q(x) = -\dfrac{1}{x}$, $n = 2$

Multiplying the equation with y^{-2} we get

$$y^{-2}\dfrac{dy}{dx} - \dfrac{1}{xy} = -\dfrac{1}{x} \quad \text{——— (I)}$$

Let $v = y^{-1}$

$$\dfrac{dv}{dx} = -y^{-2}\dfrac{dy}{dx}$$

From (I) $-\dfrac{dv}{dx} - \dfrac{1}{xy}v = -\dfrac{1}{x}$ ——— (II)

$$\Rightarrow \dfrac{dv}{dx} + \dfrac{v}{x} = \dfrac{1}{x}$$

Now IF $= e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

multiplying (II) with x

$$x\dfrac{dv}{dx} + v = 1$$

$$\Rightarrow \dfrac{d}{dx}(xv) = 1$$

$$\Rightarrow xv = x + C$$

$$v = 1 + Cx^{-1}$$

$$\therefore y = (1 + Cx^{-1})^{-1}$$

19. Solve the IVP $x\frac{dy}{dx} - 2y = 2x^4$, $y(2)=8$

Sol^n: $\frac{dy}{dx} - \frac{2}{x}y = 2x^3$ —— (1) which is a linear equation.

here $P(x) = -\frac{2}{x}$

$IF = e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}}$
$= x^{-2} = \frac{1}{x^2}$

Multiplying (1) with IF we get

$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = 2x$

$\Rightarrow \frac{d}{dx}(x^{-2}.y) = 2x$

$\Rightarrow x^{-2}y = x^2 + c$

$\therefore y = x^4 + cx^2$

Applying the initial condition we get $8 = 2^4 + c\,2^2$
$c = -2$

$\therefore y = x^4 - 2x^2$ which is the solution.

23. Solve the IVP : $\frac{dr}{d\theta} + r\tan\theta = \cos^2\theta$, $r(\frac{\pi}{4}) = 1$

Sol^n Here $P(\theta) = \tan\theta$

$IF = e^{\int \tan\theta\, d\theta} = e^{\ln(\sec\theta)} = |\sec\theta|$

multiplying the equation with IF we get

$$\frac{dr}{d\theta}\sec\theta + r\sec\theta\tan\theta = \sec\theta\cos^2\theta$$

$$\Rightarrow \frac{d}{d\theta}(r.\sec\theta) = \cos\theta$$

$$\Rightarrow r\sec\theta = \sin\theta + c$$

$$\Rightarrow r = (\sin\theta + c)\cos\theta$$

Applying the initial condition we get $c = \frac{1}{\sqrt{2}}$ and the equation become $r = \sin\theta\cos\theta + \frac{1}{\sqrt{2}}\cos\theta$

$\therefore 2r = \sin 2\theta + \sqrt{2}\cos\theta$; which is the required solution.

24 $\frac{dy}{dt} = \frac{dx}{dt} - x = \sin 2t$; $x(0) = 0$

Sol^n:

$P(t) = -1$, $Q(t) = \sin 2t$

$IF = e^{\int -1\, dt} = e^{-t}$

Multiply the equation with IF and we get

$$e^{-t}\frac{dx}{dt} - xe^{-t} = e^{-t}\sin 2t$$

$$\frac{d}{dt}(e^{-t}.x) = e^{-t}\sin 2t \qquad \left[\int e^{-t}\sin 2t\, dt = \int uv\right]$$

$$\Rightarrow e^{-t}x = -\frac{1}{5}e^{-t}(\sin 2t + 2\cos 2t) + c$$

$$\therefore x = -\frac{1}{5}(\sin 2t + 2\cos 2t) + ce^{t}$$

Applying the initial condition $c = \frac{2}{5}$

$\therefore x = -\frac{1}{5}(\sin 2t + 2\cos 2t) + \frac{2}{5}e^{-t}$

$\Rightarrow 5x = 2e^{-t} - \sin 2t - 2\cos 2t$; which is the required solution.

$$u = \int e^{-t} \sin 2t \, dt$$

$$\int u = -\sin 2t \, e^{-t} + \int 2\cos 2t \, e^{-t} dt$$

$$= -\sin 2t \, e^{-t} + 2\left[-\cos 2t \, e^{-t} - 2\int \sin 2t \, e^{-t} dt\right]$$

$$= -\sin 2t \, e^{-t} - 2\cos 2t \, e^{-t} - 4 \int \sin 2t \, e^{-t} dt$$

$$[4u]$$

$$5u = -e^{-t}(\sin 2t + 2\cos 2t)$$

$$u = -\frac{1}{5} e^{-t} (\sin 2t + 2\cos 2t) + C$$

25. $$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}, \quad y(1) = 2$$

Sol^n: The equation is a Bernoulli equation where

$P(x) = \frac{1}{2x}$, $Q(x) = x$, $y^n = y^{-3}$, $n = -3$

Multiply the equation with y^{-3}

$$y^{-3} \frac{dy}{dx} + \frac{y^{-2}}{2x} = x$$

$$\Rightarrow \cancel{\frac{d}{dx} \quad y^{-3} \, y}$$

$$\Rightarrow y^3 \frac{dy}{dx} + \frac{y^4}{2x} = x \quad \text{—— (1)}$$

Let $v = y^{1-n} = y^4$

$$\Rightarrow \frac{dv}{dx} = 4y^3 \frac{dy}{dx} \quad \therefore y^3 \frac{dy}{dx} = \frac{1}{4} \frac{dv}{dx}.$$

$\Rightarrow \frac{d}{x}$

from (I) $\frac{1}{4} \frac{dv}{dx} + \frac{v}{2x} = x$

$\Rightarrow \frac{dv}{dx} + \frac{2v}{x} = 4x$ — (II) which is a

linear DE on v

where $P(x) = \frac{2}{x}$, $Q(x) = 4x$

IF $= e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

Now multiplying the equation (II) with this IF

$x^2 \frac{dv}{dx} + 2vx = 4x^3$

$\frac{d}{dx}(x^2 . v) = 4x^3$

$\Rightarrow x^2 v = x^4 + c$

$\Rightarrow v = x^2 + cx^{-2}$

$y^4 = x^2 + cx^{-2}$

$\Rightarrow$ Or, $x^2 y^4 = x^4 + c$

Now applying initial conditions $c = 15$

$\therefore x^2 y^4 - x^4 = 15$ is the required solution.

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Chapter Three

$x(t)$ the amount / volume of p human population at any time t.

$$\frac{d\,x(t)}{dt} \propto x(t)$$

$$\frac{dx(t)}{dt} = k\,x(t); \quad k>0 \quad [\text{grow/ increase}]$$

$$\frac{dx(t)}{dt} = -k\,x(t); \quad k \quad [\text{decline/decrease}]$$

P-89
① Example 3.8

Let $x(t)$ be the amount of radioactive nuclei present after t years.

$\frac{dx}{dt}$ represents the rate of nuclei decay.

$$\frac{dx(t)}{dt} = -k\,x(t)$$

[amount of x is clearly positive, but as it is decreasing so $k<0$]

Let x_0 denotes the amount initially presents

Here, the initial conditions are $x(0) = x_0$

$x(1500) = \frac{1}{2}x_0$

Now, $\frac{dx}{dt} = -kx$

$$\Rightarrow \int \frac{1}{x}\,dx = -k\int dt$$

$$\Rightarrow \ln|x(t)| = -kt + c_1$$

$$x(t) = e^{-kt}\cdot \underbrace{e^{c_1}}_{c}$$

$$x(t) = c\,e^{-kt} \quad \text{——— (1)}$$

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Applying the initial condition that is $x = x_0$ whe $t = 0$,

we get $x(0) = c \cdot e^{-k \cdot 0} = c$

$c = x_0$

$\therefore$ From (1) we get $x = x_0 e^{-kt}$ —— (11)

Again for the 2nd condition when $t = 1500$, $x = \frac{1}{2} x_0$

$$\therefore \frac{1}{2} x_0 = x_0 e^{-1500 \cdot k}$$

$$\Rightarrow (e^{-k})^{1500} = \frac{1}{2}$$

$$\Rightarrow e^{-k} = \left(\frac{1}{2}\right)^{\frac{1}{1500}}$$

From (11)

$$x = x_0 \cdot e^{-kt}$$

$$= x_0 \left\{ \left(\frac{1}{2}\right)^{\frac{1}{1500}} \right\}^t$$

$$= x_0 \left(\frac{1}{2}\right)^{\frac{t}{1500}}$$

when $t = 4500$, $x = x_0 \left(\frac{1}{2}\right)^{\frac{4500}{1500}} = \frac{1}{8} x_0$

2. we have $x = x_0 \left(\frac{1}{2}\right)^{\frac{t}{1500}}$

when $x = \frac{1}{10} x_0$ then

$$\frac{1}{10} x_0 = x_0 \left(\frac{1}{2}\right)^{\frac{t}{1500}}$$

$$\Rightarrow \frac{1}{10} = \left(\frac{1}{2}\right)^{\frac{t}{1500}}$$

$$\Rightarrow \ln(1/10) = \frac{t}{1500} \ln(1/2)$$

$$\Rightarrow \frac{t}{1500} = \frac{\ln(1/10)}{\ln(1/2)} = \frac{\ln 10}{\ln 2}$$

$$\therefore t = \frac{1500 \ln 10}{\ln 2} \approx 4983 \text{ (years)}$$

Example 3.9

The population x of a certain city satisfies the logistic law $\frac{dx}{dt} = \frac{1}{100}x - \frac{1}{(10)^8}x^2$ where time t is measured in years.
Given that the population of the city is 100,000 in 1980.
determine the population as a function of time for $t > 1980$.

(a) What will be the population in 2020?

(b) In what year does the 1980 population double?

(c) Assuming the given DE applies for all $t > 1980$, how large will the population ultimately be?

Sol^n: initial condition $x(1980) = 100000$

From the D.E, $\frac{dx}{10^{-2}x - 10^{-8}x^2} = dt$

$$\Rightarrow \frac{dx}{10^{-2}x\{1-10^{-6}x\}} = dt$$

$$\Rightarrow 100\left\{\frac{1}{x} + \frac{10^{-6}}{(1-10^{-6}x)}\right\}dx = dt$$

$$\Rightarrow 100\left\{\ln|x| - \ln|1-10^{-6}x|\right\} = t + c$$

$$\Rightarrow 100\ln\frac{x}{1-10^{-6}x} = t + c$$

$$\Rightarrow \frac{x}{1-10^{-6}x} = ce^{\frac{t}{100}}$$

$$\Rightarrow x + ce^{\frac{t}{100}}(10)^{-6}x = ce^{\frac{t}{100}}$$

$$\therefore x = \frac{ce^{\frac{t}{100}}}{1+ce^{t/100}\cdot(10^{-6})}$$

$$\Rightarrow 10^5 = \frac{ce^{\frac{1980}{100}}}{1+ce^{\frac{1980}{100}}\cdot(10^{-6})}$$

$$\Rightarrow \quad 10^5 + c e^{19.8} 10^{-1} = c e^{19.8}$$

$$\Rightarrow \quad c = \frac{10^5}{e^{19.8}(1 - 10^{-1})}$$

$$\therefore \quad c = \frac{10^6}{9 e^{19.8}}$$

The final equation : $x = \dfrac{\frac{10^6}{9e^{19.8}} e^{t/100}}{1 + \frac{10^6}{9e^{19.8}} \cdot e^{t/100} . 10^{-6}}$

$$= \frac{10^6}{\frac{9e^{19.8}}{e^{t/100}} + \frac{9e^{19.8}}{e^{t/100}} \cdot \frac{10^6}{9e^{19.8}} e^{t/100} . 10^{-6}}$$

$$x = \frac{10^6}{1 + 9e^{19.8} . e^{-t/100}}$$

(a) In 2000, the population will be

$$x = \frac{10^6}{1 + 9e^{19.8} e^{-20}} \approx 119405$$

(b) The population will be double in

$$2 \times 10^5 = \frac{10^6}{1 + 9e^{19.8} e^{-t/100}}$$

$$\Rightarrow \quad 1 + 9e^{19.8} e^{-t/100} = 5$$

$$\Rightarrow \quad e^{-t/100} = \frac{4}{9} e^{-19.8}$$

$$\Rightarrow \quad -t/100 = \ln\left(\tfrac{4}{9}\right)(-19.8)$$

$$\Rightarrow \quad -t = \left(\ln \tfrac{4}{9} - 19.8\right) 100$$

$$\therefore \quad t = \left(19.8 - \ln \tfrac{4}{9}\right) 100$$

$$\approx 2061$$

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(C) $\lim\limits_{t \to \infty} x = \lim\limits_{t \to \infty} \dfrac{10^6}{1 + 9e^{19.8} \cdot e^{-t/100}}$

$= 10^6$.

Mixture Problem

Example 3.10

A tank initially contains 50 gal of pure water. Starting at time $t = 0$ a brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate.

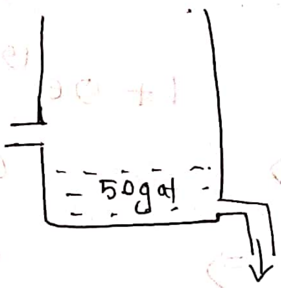
1. How much salt is in the tank at any time $t > 0$?
2. " " " " present at the end of 25 min?
3. " " " " " after a long time?

Sol^n. Let $x(t)$ denote the amount of salt present at time t.

$\dfrac{dx}{dt} = \text{IN} - \text{OUT}$

initial condition

$x(0) =$



IN = (2 lb/gal) (3 gal/min)

$=$ 6 lb / min

OUT $= \left(\dfrac{x}{50} \text{ lb/gal}\right)$ (3 gal/min)

$= \dfrac{3x}{50}$ lb/min

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$$\frac{dx}{dt} = 6 - \frac{3x}{50} = \frac{3(100-x)}{50}$$

$$\Rightarrow \int \frac{dx}{100-x} = \frac{3}{50} \int dt$$

$$\Rightarrow -\ln|100-x| = \frac{3}{50} t + c_1$$

$$\Rightarrow -(100-x) = c e^{\frac{3}{50} t}$$

$$\Rightarrow x = c e^{-\frac{3}{50} t} + 100$$

At initial condition $t=0, x=0$

$$\therefore 0 = c e^{-\frac{3}{50} \cdot 0} + 100$$

$$c = -100$$

$$\therefore x = 100 \left(1 - e^{-\frac{3}{50} t}\right)$$

(1) $x = 100\left(1 - e^{-\frac{3t}{50}}\right)$

(2) $x(25) = 100\left(1 - e^{-\frac{3}{50} \times 25}\right)$

$$\approx 78 \text{ lb}$$

(3) $t \to \infty$

$$x = 100\left(1 - e^{-\infty}\right)$$

$$= 100\left(1 - \frac{1}{e^{\infty}}\right)$$

$$= 100(1-0)$$

$$= 100 \text{ lb.}$$

Example : 3.11 A large tank initially contains 50 gal of brine in which there is dissolved 10 lb of salt. Brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring, and the stirred mixture simultaneously flows out at the slower rate of 3 gal/min. How much salt is in the tank at any time $t > 0$?

Sol^n: Let x = the amount of salt at time t.

$$\frac{dx}{dt} = In - Out$$

In $= (2\,lb/gal) \times (5\,gal/min) = 10\,lb/min$

Out $= \left(\frac{x}{50+2t}\,lb/gal\right)(3\,gal/min) = \frac{3x}{50+2t}\,lb/min$

[In 5 gal/min, out 3 gal/min
Net gal gain $= (5-3) = 2$ gal/min of brine
At the end of t min, the amount of brine in the tank is $50+2t$ gal.]

$$\Rightarrow \frac{dx}{dt} = 10 - \frac{3x}{50+2t}$$

$$\Rightarrow \frac{dx}{dt} + \frac{3}{50+2t}x = 10$$

[linear D.E $\frac{dx}{dt} + P(t)x = Q(t)$]

$$\Rightarrow e^{\int \frac{3}{50+2t}dt} = e^{\frac{3}{2}\int \frac{2}{50+2t}dt}$$

$$= e^{\frac{3}{2}\ln(50+2t)} = (50+2t)^{\frac{3}{2}}$$

Now, $\frac{dx}{dt}(2t+50)^{\frac{3}{2}} + 3(50+2t)^{\frac{1}{2}}x = 10(2t+50)^{\frac{3}{2}}$

$$\Rightarrow \frac{d}{dt}\left[(2t+50)^{\frac{3}{2}}x\right] = 10(2t+50)^{3/2}$$

$$\Rightarrow (2t+50)^{3/2}x = \cancel{10}^{2}\,\frac{(2t+50)^{5/2}}{\frac{\cancel{5}}{2}\times \cancel{2}} + c$$

$$\Rightarrow x = \frac{2(2t+50)^{5/2}}{(2t+50)^{3/2}} + \frac{c}{(2t+50)^{3/2}}$$

$$= 2(2t+50) + \frac{c}{(2t+50)^{3/2}}$$

Initial condition $x(0) = 10$

Applying the condition,

$$10 = 100 + \frac{c}{(50)^{3/2}}$$

$$\therefore c = -(90)(50)^{3/2} = -(90)(50)(50)^{\frac{1}{2}}$$

$$= -(90\times 50\times 25)\sqrt{2} = -22500\sqrt{2}$$

$\therefore$ The amount of salt at any time $t > 0$ is

$$x = 4t + 100 - \frac{22500\sqrt{2}}{(2t+50)^{3/2}}$$

Chapter : 3.1

* Orthogonal trajectories :

Let $f(x, y, c) = 0$ ——— ① be a given one parameter family of curves in the xy plane. A curve that intersects the curves of the family ① at right angles is called an orthogonal trajectory of the given family.

Ex : 3.1 : Consider the family of circles $x^2+y^2=c^2$ ——— ⊛ is straight line through the origin $y = kx$ is an orthogonal trajectory of the family of circles ⊛

orthogonal trajectories

Step 1 : From the eqn $f(x, y, c) = 0$ of the given family of curves find the D.E's [we have to eliminate the parameter there is any.]

$\frac{dy}{dx} = f(x, y)$ of this family.

Step 2 : Replace $f(x, y)$ by it's negative reciprocal. $\frac{-1}{f(x,y)}$

i.e $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

Step : 3 Obtain a one-parameter family $G(x, y, c) = 0$ or $y = f(x, c)$.

Solⁿ: $x^2+y^2=c^2$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow x + y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$-\frac{dx}{dy} = -\frac{x}{y}$

$\Rightarrow \int \frac{1}{x} dx = \int \frac{1}{y} dy$

$\Rightarrow \ln|x| = \ln|y| + \ln|k_1|$

$\Rightarrow \ln|x| = \ln|yk_1|$

$\Rightarrow x = ky$

$\therefore y = kx$

Ex. 3.3 * H.W

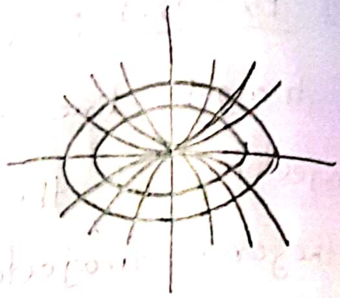
Find the orthogonal trajectories of the family of parabola $y = cx^2$.

Soln: $y = cx^2 \Rightarrow c = \frac{y}{x^2}$ ——— (i)

$\Rightarrow \frac{dy}{dx} = 2cx$ ——— (ii)

From (i) & (ii) $\frac{dy}{dx} = 2\frac{y}{x^2}x = \frac{2y}{x}$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and we get



$-\frac{dx}{dy} = \frac{2y}{x}$

$\Rightarrow \int x\,dx = \int -2y\,dy$

$\Rightarrow \frac{x^2}{2} = -y^2 + k$

$\therefore x^2 + 2y^2 = k^2$; which is the required solution where k is an arbitrary constant.

The required eqn is the orthogonal trajectories of given family of family of curves of parabola and the soln is the family of ellipse.

***Exercise:**

(1) Find the O.T of the family of curves $y = cx^3$ & sketch.

Soln: $y = cx^3 \Rightarrow c = \frac{y}{x^3}$

$\Rightarrow \frac{dy}{dx} = 3cx^2 = 3\frac{y}{x^3}x^2 = \frac{3y}{x}$

replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$,

$-\frac{dx}{dy} = \frac{3y}{x}$

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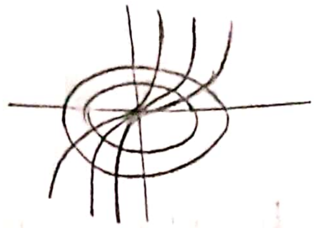
$\Rightarrow \int x\,dx = \int -3y\,dy$

$\Rightarrow \frac{x^2}{2} = -\frac{3}{2}y^2 + k$

$\therefore x^2 + 3y^2 = k^2$; which is the required O.T.



Extra

(2) Find the O.T of the family of curves $y^2 = cx$ and sketch.

Soln: $y^2 = cx$ $\Rightarrow c = \frac{y^2}{x}$

$2y\frac{dy}{dx} = c = \frac{y^2}{x}$

$\Rightarrow 2x\cancel{y}\,dy = y\cancel{y^2}\,dx$

$\Rightarrow \frac{2x}{dx} = -\frac{y}{dy}$ $\Rightarrow \int \frac{1}{2x}\,dx = \int \frac{1}{y}\,dy$

$\Rightarrow \frac{1}{2}\ln|x| = \ln|y| + \ln|k|$

$\therefore x^2 = ky$

(3) Find the O.T of the family of curves $cx^2 + y^2 = 1$ and sketch.

Soln: $cx^2 + y^2 = 1$

Chapter-4

Definition of Linear Differential Equations:

A linear ordinary differential equation of order n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in, the form

$$\underbrace{a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y}_{\text{Homogeneous}} = \underbrace{f(x)}_{\text{Non-homogeneous}}$$

25.02.20

Theorem-4.11:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \quad (1)$$

Consider the nth-order homogeneous linear differential equation (1) with constant co-efficient. If the auxillary equation

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \quad (2)$$

has the n distinct real roots $m_1, m_2, \ldots m_n$. then the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Theorem-4.12:

① Consider the nth-order homogeneous linear differential equation ① with constant coefficients. If the auxillary equation ② has the real root m occuring k times, then the part of the general solution of ① corresponding to this k-fold repeated root is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{mx}$$

② If, further, the remaining roots of the auxillary equation ② are the distinct real numbers $m_{k+1}, \ldots, m_n$, then the general solution of ① is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{mx} + c_{k+1} e^{m_{k+1} x} + \cdots + c_n e^{m_n x}$$

③ If, however, any of the remaing roots are also repeated, then the parts of the general solution of ① corresponding to each of these other repeated roots are expressions similar to that corresponding to m in part 1.

Theorem – 4.13

① Consider the n-th homogeneous linear differential equation ① with constant co-efficient. If the auxillary equation ② has the conjugate complex roots $a \pm ib$, neither repeated then the corresponding part of the general solution of ① may be written.

$$y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$$

② If, $a \pm ib$ are each k-fold roots of the auxillary equation ① then the the corresponding part of the general solution of ① may be written.

$$y = e^{ax}\left[(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) \sin bx + (c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots + c_{2k} x^{k-1}) \cos bx\right]$$

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Example - 4.28

$(D^2 - 6D + 25)y = 0 \quad , \quad y(0) = -3 \, , \quad y'(0) = 1$

Auxillary equation is,

$m^2 - 6m + 25 = 0$

$\Rightarrow m = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

$\therefore y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x)$ —— (1)

$y' = e^{3x}(4c_1 \cos 4x - 4c_2 \sin 4x) + 3e^{3x}(c_1 \sin 4x + c_2 \cos 4x)$

$= e^{3x}(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x$ —— (2)

When $x = 0, \; y = -3$

(1) $\Rightarrow \; -3 = e^0(c_1 \sin 0 + c_2 \cos 0)$

$\therefore c_2 = -3$

When $x = 0, \; y' = 1$

(2) $\Rightarrow \quad 1 = e^0(3c_1 - 4c_2)\sin 0 + (4c_1 + 3c_2)\cos 0$

$\Rightarrow 4c_1 + 3c_2 = 1$

$\Rightarrow 4c_1 - 9 = -1 \;\; \Rightarrow 4c_1 = 8 \;\; \therefore c_1 = 2$

$\therefore$ (1) $\Rightarrow \; y = e^{3x}(2 \sin 4x - 3\cos 4x)$

Ans

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① Given that,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Auxillary equation is,

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow m^2 - 3m - 2m + 6 = 0$$

$$\Rightarrow (m-3)(m-2) = 0$$

$$\therefore m = 2, 3$$

$\therefore$ The solution is, $y = c_1 e^{2x} + c_2 e^{3x}$ Ans.

$y = e^{mx}$

$\therefore \frac{dy}{dx} = me^{mx}$

$\frac{d^2y}{dx^2} =$

③ Given that,

$$4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 5y = 0$$

Auxillary equation is,

$$4m^2 - 12m + 5 = 0$$

$$\Rightarrow 4m^2 - 10m - 2m + 5 = 0$$

$$\Rightarrow 2m(2m-5) - 1(2m-5) = 0$$

$$\Rightarrow (2m-5)(2m-1) = 0$$

$$\therefore m = \frac{5}{2}, \frac{1}{2}$$

$\therefore$ The solution is

$$y = c_1 e^{\frac{5}{2}x} + c_2 e^{\frac{1}{2}x}$$

Ans.

(5) Given that,

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = 0$$

$y = e^{mx}$

The Auxillary equation is,

$$m^3 - 3m^2 - m + 3 = 0$$

$$\Rightarrow m^2(m-3) - 1(m-3) = 0$$

$$\Rightarrow (m-3)(m^2-1) = 0$$

$$\Rightarrow (m-3)(m+1)(m-1) = 0$$

$$\therefore m = -1, 1, 3$$

$\therefore$ The solution is,

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}$$

Ans.

(7) Given that,

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

The Auxillary equation is,

$$m^2 - 8m + 16 = 0$$

$$\Rightarrow \cancel{m^2}\ (m-4)^2 = 0$$

$$\boxed{y = c_1 e^{mx} + c_2 x e^{mx}}$$

$$\therefore m = 4, 4$$

$\therefore$ The solution ~~eq~~ is,

$$y = c_1 e^{4x} + c_2 x e^{4x}$$

Ans.

(25)

Given that,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0 \quad , \; y(0) = 3 \; , \; y'(0) = 5$$

The Auxillary equation is,

$$m^2 - m - 12 = 0$$

$$\Rightarrow m^2 - 4m + 3m - 12 = 0$$

$$\Rightarrow m(m-4) + 3(m-4) = 0$$

$$\Rightarrow (m-4)(m+3) = 0$$

$$\therefore m = -3, 4$$

The solution is,

$$y = c_1 e^{-3x} + c_2 e^{4x} \quad \text{——— (1)}$$

When $x = 0$, $y = 3$

$$3 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 = 3$$

$$\Rightarrow c_1 = 3 - c_2$$

$$(1) \Rightarrow \cancel{\frac{dy}{dx}} \; y' = -3 c_1 e^{-3x} + 4 c_2 e^{4x}$$

When $x = 0$, $y'(0) = 5$

$$5 = -3 c_1 e^0 + 4 c_2 e^0$$

$$\Rightarrow -3c_1 + 4c_2 = 5$$

$$\Rightarrow -3(3 - c_2) + 4c_2 = 5 \Rightarrow -9 + 3c_2 + 4c_2 = 5$$

$$\Rightarrow 7c_2 = 14 \quad \therefore c_2 = 2$$

$$\therefore c_1 = 3 - 2 = 1$$

$$\therefore \cancel{y = 7e^{3x} - 4e^{4x}} \quad \therefore y = e^{-3x} + 2e^{4x}$$

(29) Given that,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0, \quad y(0) = 2, \quad y'(0) = -3$$

$\therefore$ The Auxillary equation is

$$m^2 + 6m + 9 = 0$$

$$\Rightarrow (m+3)^2 = 0$$

$$\therefore m = -3, -3$$

The solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \quad \text{——— (1)}$$

Whegn $x = 0$, $y = 2$

$$2 = c_1 e^0 + 0 \quad \therefore c_1 = 2$$

Differenciating (1),

$$y' = -3c_1 e^{-3x} - 3c_2 x e^{-3x} + c_2 e^{-3x}$$

When $x = 0$, $y'(0) = -3$

$$-3 = -3c_1 - 0 + c_2$$

$$\Rightarrow c_2 - 3c_1 = -3$$

$$\Rightarrow c_2 - 6 = -3$$

$$\therefore c_2 = 3$$

$$\therefore y = 2e^{-3x} + 3xe^{-3x}$$

$$= (3x+2)e^{-3x} \quad \text{Ans.}$$

(35) Given that,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0 \quad , \; y(0) = 3, \; y'(0) = -1$$

The Auxillary equation is;

$$m^2 + 6m + 13 = 0$$

$$\therefore m = \frac{-6 \pm \sqrt{36-52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

The solution is,

$$y = e^{-3x}(c_1 \sin 2x + c_2 \cancel{\sin x} \cos 2x) \quad \text{--- (1)}$$

When, $x = 0$, $y = 3$

$$3 = e^0(c_1 \sin 0 + c_2 \cos 0)$$

$$\therefore c_2 = 3$$

Differentiating (1)

$$y' = -e^{-3x}(c_1 \sin 2x + c_2 \cos 2x) + e^{-3x}(2c_1 \cos 2x - 2c_2 \sin 2x)$$

$$= -e^{-3x}(c_1 + 2c_2) \sin 2x - e^{-3x}(c_2 - 2c_1) \cos 2x$$

When $x = 0$, $y' = -1$

$$-1 = -e^0(c_1 + 2c_2) \sin 0 - e^0 (c_2 - 2c_1) \cos 0$$

$$\Rightarrow -c_2 + 2c_1 = -1$$

$$\Rightarrow -3 + 2c_1 = -1 \Rightarrow 2c_1 = 2 \; \therefore c_1 = 1$$

$$\therefore y = e^{-3x}(\sin 2x + 3 \cos 2x)$$

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(41) Given that,

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0 \quad ; y(0)=1, \; y'(0)=-8, \; y''(0)=-4$$

$m=-1$
$m+1=0$

The Auxillary equation is,

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow \cancel{m^3 - 4} \Rightarrow m^3 + m^2 - 4m^2 - 4m + 4m + 4 = 0$$

$$\Rightarrow m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$

$$\Rightarrow (m+1)(m^2 - 4m + 4) = 0$$

$$\Rightarrow (m+1)(m-2)^2 = 0$$

$$\therefore m = -1, 2, 2$$

The solution is,

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x} \quad \text{--- (i)}$$

When $x=0, y=1$

$$\therefore 1 = c_1 + c_2 \quad \text{--- (ii)}$$

$$y' = -c_1 e^{-x} + 2c_2 e^{2x} + c_3 e^{2x} + 2c_3 x e^{2x}$$

When $x=0$, $y'=-8$

$$-8 = -c_1 + 2c_2 + c_3 \quad \text{--- (iii)}$$

$$y'' = c_1 e^{-x} + 4c_2 e^{2x} + 2c_3 e^{2x} + 2c_3 e^{2x} + 4c_3 x e^{2x}$$

$$= c_1 e^{-x} + 4c_2 e^{2x} + 4c_3 e^{2x} + 4c_3 x e^{2x}$$

When $x=0$, $y''=-4$

$$-4 = c_1 + 4c_2 + 4c_3 \quad \text{--- (iv)}$$

$4\times(iii) - (iv) \Rightarrow$

$\cancel{4 - 6c_2 + 5c_3}$

$12 = -2c_1 - 2c_2$

$\Rightarrow c_1 + c_2 = -6$ --- (v)

(ii) $-$ (v) $\Rightarrow$

(ii) $\Rightarrow$ $c_1 + c_2 = 1$

(iv) $- 4\times(iii) \Rightarrow$

$28 = 5c_1 - 4c_2$ --- (v)

$(v) + 4\times(ii) \Rightarrow$

$32 = 9c_1$

$\therefore c_1 = \frac{32}{9}$

(ii) $\Rightarrow \frac{32}{9} + c_2 = 1$

$\Rightarrow c_2 = \frac{-23}{9}$

(iv) $\Rightarrow$

$-4 = \frac{32}{9} + \left(\frac{-92}{9}\right) + 4c_3$

$4c_3 = \frac{92}{9} - \frac{32}{9} - 4$

$= \frac{92 - 32 - 36}{9}$

$= \frac{24}{9}$

$c_3 = \frac{24}{9\times 4} = \frac{2}{3}$

$$\therefore y = \frac{32}{9}e^{-x} - \frac{23}{9}e^{2x} + \frac{2}{3}e^{2x}$$

Ans

$$\boxed{y = y_c + y_p}$$

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(1) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$

Auxillary equation,

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\therefore m = 1, 2$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$

$y_p = A + Bx + Cx^2$

$y'p = B + 2Cx$

$y''p = 2C$

$\therefore 2C - 3B - 6Cx + 2A + 2Bx + 2Cx^2 = 4x^2$

$2C = 4$ | $(2B - 6C) = 0$ | $2C - 3B + 2A = 0$

$\therefore C = 2$ | $\Rightarrow 2B = 6C$ | $\Rightarrow 4 - 18 + 2A = 0$

| $\Rightarrow B = 6$ | $\Rightarrow 2A = 14$

| | $\therefore A = 7$

$y^p = 7 + 6x + 2x^2$

$\therefore y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} + 7 + 6x + 2x^2$