Chapter 1

Sequence in Metric Space

1.1 Sequence of Real Numbers[2]

A sequence if real numbers in $\mathbb{R}$ is simply a function $f : \mathbb{N} \to \mathbb{R}$ which us usually defined by $f(n) = x_n$ and arranged in a particular order such as $x_1, x_2, x_3, \ldots, x_n, \ldots$.

For example, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be represented as $x_n = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$.

1.2 Convergent Sequence

A sequence x_n in $\mathbb{R}$ is said to converge to a limit $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N such that $|x_n - x| < \epsilon$, whenever $n \geq N$.

In this case we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Note. $N := N(\epsilon)$, often smaller ϵ may require larger N.

1.3 Sequence of points or Vectors in Metric Spaces

A sequence of points in a metric space $M := (M, d)$ is a function $f : \mathbb{N} \to M$, usually defined by $f(n) = x_k$ and arranged in a definite order such as $x_1, x_2, x_3, \ldots, x_n, \ldots$.

1.4 Convergent Sequence in a Metric Space

A sequence x_k in a metric space (M, d) converges to $x \in M$ if for every given $\epsilon > 0$ there is a natural number N such that $n \geq N$ implies $d(x_k)$.

1.5 Convergent Sequence in Normed Space $\mathbb{R}^n$

A sequence v_k of vector converges to the vector $v \in \mathbb{R}^n$ if for every given $\varepsilon > 0$, there exists such that $d(v_k, v) = ||v_k - v|| < \varepsilon$ whenever $k \geq N$.

[2]Marsden. P.36

1.6 Convergent Sequence in Arbitrary Normed Space V

$v_k \in V \to v$, $||v_k - v|| \to 0$ as $k \to \infty$.

If $v, v_k \in \mathbb{R}^n$, we write $v = (v^1, v^2, \ldots, v^n)$, $v_k = (v_k^1, v_k^2, \ldots, v_k^n)$

Theorem 1.6.1. $v_k \to v$ in $\mathbb{R}^n$ if and only if each sequence of coordinates converges to the corresponding coordinate of v as a sequence in $\mathbb{R}$. That is,

$$\lim_{k\to\infty} v_k = v \text{ in } \mathbb{R}^n \text{ if and only if } \lim_{k\to\infty} v^i = v \text{ in } \mathbb{R} \text{ for each } i = 1, 2, \ldots, n$$

or,

$$\lim_{k\to\infty} \left(v_k^1, v_k^2, \ldots, v_k^n\right) = \left(\lim_{k\to\infty} v_k^1, \lim_{k\to\infty} v_k^2, \ldots, \lim_{k\to\infty} v_k^n\right)$$

Problem 1.6.1. Test the convergence of the sequences in $\mathbb{R}^2$

1. $v_k = ({}^1/_2, {}^1/_{k^2})$
2. $v_n = \left({}^{(\sin n)^n}/_n, {}^1/_{n^2}\right)$

Solution.

1. Here the component sequences $\frac{1}{k}$ and $\frac{1}{k^2}$ each converge to 0. Hence, the vector $v_k \to 0,\ 0 = (0,0) \in \mathbb{R}^2$.

2. Use sandwich theorem ($v_n \to (0,0)$)
   Here,
   $$\left|\frac{(\sin n)^n}{n}\right| = \frac{|\sin n|^n}{n} \leq \frac{1}{n} \Rightarrow -\frac{1}{n} \leq \frac{(\sin n)^n}{n} \leq \frac{1}{n}$$
   Hence, by sandwich theorem,
   $$\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$$
   Therefore
   $$\lim_{n\to\infty} \frac{(\sin n)^n}{n} = 0$$
   Again
   $$\lim_{n\to\infty} \frac{1}{n^2} = 0$$
   Therefore, $v_n \to (0,0)$

Theorem 1.6.2. A set $A \subset M$ is closed $\Leftrightarrow$ for every sequence $x_k \in A$ converges to a point $x \in A$.

Problem 1.6.2. Let $x_n \in \mathbb{R}^m$ be a convergent sequence with $||x_n|| \leq 1$ for all n. Show that x also satisfies $||x|| \leq 1$. If $||x_n|| < 1$, then must we have $||x|| < 1$?

Solution. The unit ball $B = \{y \in \mathbb{R}^m \mid ||y|| \leq 1\}$ is closed. Let $x_n \in B$, and $x_n \to x \Rightarrow x \in B$ as B is closed, by the above theorem. This is not true if $\leq$ is replaced by $<$; for example, on $\mathbb{R}$ consider $x_n = 1 - {}^1/_n$.

1.7 Cauchy Sequence

Let (M, d) be a metric space. A *Cauchy sequence* is a sequence $x_k \in M$ such that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that in $n \geq N$ implies $d(x_m, x_n) < \varepsilon$.

1.8 Complete Metric Space

The metric space M is called *complete* if and only if every Cauchy sequence in M converges to a point in M.

In Normed space, such as $\mathbb{R}^n$, a sequence v_k is Cauchy sequence if for every $\varepsilon > 0$ there us an N such that $||v_k - v_j|| < \varepsilon$ whenever $j, k \geq N$.

1.9 Bounded Sequence

A sequence x_k in a normed space is *bounded* if there is a number $M^{'} > 0$ such that $||x_k|| \leq M$ for every k.

In a metric space we require that there be a point x_c such that $d(x_k, x_c) \leq M^{'}$ for every k.

Theorem 1.9.1. A convergent sequence in a normed or metric space is bounded.

Theorem 1.9.2.

(i) Every convergent sequence in a metric space is a Cauchy sequence.

(ii) A Cauchy sequence in a metric space is bounded.

(iii) If a subsequence of a Cauchy sequence converges to x, then the sequence converges to x.

Theorem 1.9.3. A sequence $x_k \in \mathbb{R}^n$ converges to a point in $\mathbb{R}^n$ if and only if it is a Cauchy sequence.

Problem 1.9.1 (2.8.8 - P.125, Marsden)**.** Let (M, d) be a complete metric space and $B \subset M$ a closed subset. Show that B is complete as well.

Problem 1.9.2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(\sin n)^n}{n^2}, \frac{1}{n^2} \right)$ converges.

Solution. The first component series $\sum_{n=1}^{\infty} \frac{(\sin n)^n}{n^2}$ is absolutely convergent and hence convergent.
For absolutely convergence, $\sum_{n=1}^{\infty} \left| \frac{(\sin n)^n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$, by comparison theorem/test. Since $\sum \frac{1}{n^2}$ is convergent, so $\sum \left| \frac{(\sin n)^n}{n^2} \right|$ is convergent and hence $\sum_{n=1}^{\infty} \frac{(\sin n)^n}{n^2}$ is convergent.
The second component series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, according to p-series test.
Therefore, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(\sin n)^n}{n^2}, \frac{1}{n^2} \right)$ is convergent series in $\mathbb{R}^2$.

1.10 Series of Real Numbers and Vectors

Definition 1. Let V be a normed space. A series $\sum_{k=1}^{\infty} x_k$, where $x_k \in V$, is said to converge to $x \in V$ if the sequence of partial sums $s_k = \sum_{i=1}^{k} x_i$ converges to $x \in V$, and if so we write $\sum_{k=1}^{\infty} x_k = x$ or simply $\sum x_k = x$.

Theorem 1.10.1. $\sum x_k = x$ is equivalent to corresponding component series converging to components of x.

1.11 Cauchy Criterion for Series of Vectors

Let V be a complete normed space (such as $\mathbb{R}^n$). A series $\sum x_k$ in V converges if and only if for every $\varepsilon > 0$, there is an N such that $k \geq N$ implies

$$||x_k + x_{k+1} + \cdots + x_{k+p}|| < \varepsilon \qquad \text{for } p = 0, 1, 2, \ldots$$

1.12 Absolutely Convergent Series

A series $\sum x_k$ is said to be *absolutely convergent* if and only if the real series $\sum ||x_k||$ converges.

1.13 Conditionally Convergent Series

A series that is converged but not absolute convergent is said to be conditionally convergent.

Example.

1. If a series of non-negative real numbers is convergent, then it is obviously absolutely convergent.
2. The series $\sum \frac{(-1)^n}{n^3}$ is *absolutely convergent* because $\sum \left|\frac{(-1)^n}{n^3}\right| = \sum \frac{1}{n^3}$ is convergent.
3. The series $\sum \frac{(-1)^{n-1}}{n}$ is convergent (by Leibniz alternating test) but not absolutely convergent because the harmonic series $\sum \left|\frac{(-1)^{n-1}}{n}\right| = \sum \frac{1}{n}$ is divergent. So, $\sum \frac{(-1)^{n-1}}{n}$ is *conditionally convergent.*

Theorem 1.13.1. In a complete normed space, if $\sum x_k$ converges absolutely, then $\sum x_k$ converges.

1.13.1 P-series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

1.14 Geometric Series

The series $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Problem 1.14.1. Let $x_n = \left(\frac{1}{n^2}, \frac{1}{n}\right)$. Does $\sum x_n$ converge?

Solution. No, because the harmonic series $\sum \frac{1}{n}$ diverges even though the $p = 2$ series $\sum \frac{1}{n^2}$ converges.

Problem 1.14.2. Let $||x_n|| \leq \frac{1}{2^n}$; prove that $\sum x_n$ converges and $||\sum_{n=0}^{\infty} x_n|| \leq 2$.

Solution.

$$\sum_{n=0}^{\infty} ||x_n|| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - {}^1/_2} = 2 \qquad \text{(Geometric series } \sum \frac{1}{2^n} \text{ is convergent)}$$

By comparison theorem with the convergent geometric series $\sum {}^1/_{2^n}$, the series $\sum x_n$ is absolutely convergent and hence is convergent.
Again the partial sums satisfy

$$||s_n|| = \left|\left|\sum_{k=0}^{n} x_k\right|\right| \leq \sum_{k=0}^{n} ||x_k|| \leq \sum_{k=0}^{n} \frac{1}{2^n} = 2$$

Let $B = \{y \in \mathbb{R}^n \mid ||y|| \le 2\}$. Clearly B is closed. If $s_n \in B$ and $s_n \to s$, then $s \in B$ as B is closed. Hence, $||s|| \le 2$.

Problem 1.14.3. Test for convergence: $\sum_{n=1}^{\infty} \frac{n}{3^n}$

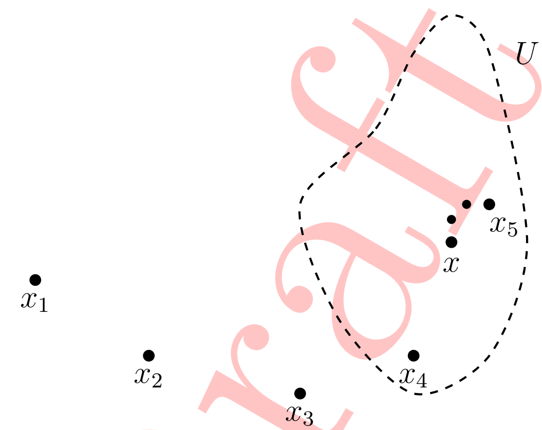
Solution. The ratio test is applicable: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{3 \cdot 3^n} \cdot \frac{3^n}{n} = \frac{1}{3} \cdot \frac{n+1}{n} \to \frac{1}{3}$ and so the series converges.

Problem 1.14.4. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges.

Solution. Observe that $\frac{n}{n^2+1} \ge \frac{n}{n^2+n^2} = \frac{1}{2^n}$, and so by comparison with divergent series $\frac{1}{2}\sum \frac{1}{n}$, we get divergence.

1.15 Sequence in Metric Space

Definition 2. Let (M, d) be a metric space, and $\langle x_n \rangle$ a sequence of points in M. We say that $\langle x_n \rangle$ converges to a point $x \in M$, written $\lim_{k\to\infty} x_k = x$ or $x_k \to x$ as $k \to \infty$.



Provided that for every open set U containing x, there us an integer N such that $x_k \in U$ whenever $k \ge N$.
This definition coincides with the usual $\varepsilon - \delta$ definition as the next theorem shows.

Proposition 1.15.1. A sequence $\langle x_k \rangle$ in M converges to $x \in M$ if and on;y if for every $\varepsilon > 0$ there is an N such that $k \ge N$ implies $d(x, x_k) < \varepsilon$.

Thus, a sequence $\langle v_k \rangle$ of points in $\mathbb{R}^n$ converges to $v \in \mathbb{R}^n$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(v, v_k) = ||v_k - v|| < \varepsilon$ whenever $k \ge N$.

Definition 3. Let (M, d) be a metric space. A Cauchy sequence is a sequence $\langle x_k \rangle$ in M such that for all $\varepsilon > 0$, there is an N such that $k, l \ge N$ implies $d(x_k, x_l) < \varepsilon$. The space M is called *complete* if and only if every Cauchy sequence in M converges to a point in M.

In a normed space, such as $\mathbb{R}^n$, a sequence v_k is a Cauchy sequence if for every $\varepsilon > 0$ there is an N such that $||v_k - v_j|| < \varepsilon$ whenever $k, j \ge N$.

Definition 4. A sequence $\langle x_k \rangle$ in a normed space is bounded if there is a number M such that $||x_k|| \le M \forall k$. In a metric space, we require that there be a point x_0 such that $d(x_k, x_0) \le M$ for all k.

Theorem 1.15.2. (i) Every convergent sequence in a metric space is a Cauchy sequence.

(ii) A Cauchy sequence in a metric space is bounded.

x If a subsequence of a Cauchy sequence converges to x then the sequence converges to x.

Proof. *H.W.* $\square$

Example. $\mathbb{R}$ is a complete metric space. An example of an incomplete metric space is the set of rational numbers with $d(x, y) = |x - y|$.
Another example is $\mathbb{R} \mid \{0\}$ with the same metric.

Theorem 1.15.3 (Completeness of the metric space $\mathbb{R}^n$). A sequence $\langle x_k \rangle$ in $\mathbb{R}^n$ converges to a point in $\mathbb{R}^n$ if and only if it is a Cauchy sequence.

Proof. If x_k converges to x, then for $\varepsilon > 0$, choose N so that $k \geq N$ implies $||x_k - x|| < {}^{\varepsilon}/_2$. Then, for $k, l \geq N$, $||x_k - x_l|| = ||(x_k - x) + (x - x_l)|| \leq ||x_k - x|| + ||x - x_l|| < {}^{\varepsilon}/_2 + {}^{\varepsilon}/_2 = \varepsilon$, by the triangle inequality. Thus, $\langle x_k \rangle$ is a Cauchy sequence.

Conversely, suppose $\langle x_k \rangle$ is a Cauchy sequence. Since $|x_k^i - x_l^i| \leq ||x_k - x_l||$, the components are also Cauchy sequence on the real line. By the completeness of $\mathbb{R}$, x_k^i converges to, say, x^i.
Therefore, $\langle x_k \rangle$ converges to $x = (x^1, x^2, \ldots, x^n)$. $\square$

1.16 Contraction Mapping

A function $\varphi : (M, d) \to (M, d)$ is called a contraction mapping if there exists a number $k (0 < k < 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq k d(x, y) \qquad \text{for all } x, y \in M$$

A point x_k is said to be a fixed point of φ if $\varphi(x_k) = x_k$.

1.17 Contraction Mapping Principle (Banach Fixed Point Theorem)

Let φ be a contraction mapping on a complete metric space M. Then there is a unique fixed point for φ. In fact, if x_0 is any point in M, and we define $x_1 = \varphi(x_0)$, $x_2 = \varphi(x_2), \ldots, x_{n+1} = \varphi(x_n), \ldots$, then $\lim_{n\to\infty} x_n = x_*$.

Intuitively, φ is shrinking distances, and so as φ iterates, points bunch up.

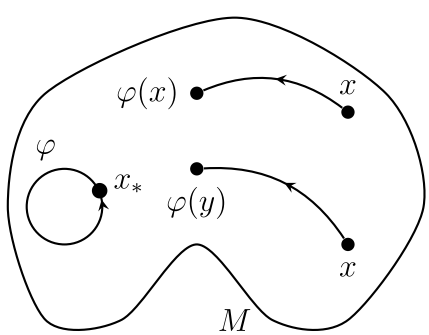


Figure 1.1: A contraction shrinks distances between points

Proof. First we show the existence of a fixed point, then its uniqueness. Let $x_0 \in M$ and $x_1, x_2, x_3, \ldots$ be as in the theorem. If $x_1 = x_0$, $\varphi(x_0) = x_0$ and so x_0 is fixed. If not, then $d(x_1, x_0)$ is not 0, and we start by showing that the points $\{x_n\}$ form a Cauchy sequence in M.
To show this, we write

$$d(x_2, x_1) = d(\varphi(x_1), \varphi(x_0)) \leq k\, d(x_1, x_0)$$
$$d(x_3, x_2) = d(\varphi(x_2), \varphi(x_1)) \leq k\, d(x_2, x_1) \leq k^2\, d(x_1, x_0);$$

inductively, $d(x_{n+1}, x_n) \leq k^n\, d(x_1, x_0)$.
Also,

$$d(x_{n+p}, x_n) \leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n)$$

by the triangle inequality, and so

$$d(x_{n+p}, x_n) \leq \left(k^{n+p-1} + k^{n+p-2} + \cdots + k^n\right) D(x_1, x_0)$$

But the geometric series $\sum_{i=0}^{\infty} k^i$ converges, since $0 \leq k < 1$, and so it satisfies the Cauchy criterion for the series: given $\varepsilon > 0$, there is an N such that $k^{n+p-1} + \cdots + k^n < \frac{\varepsilon}{d(x_1, x_0)}$ if $n \geq N$ and p is arbitrary. Hence, $d(x_{n+p}, x_n) < \varepsilon$ if $n \geq N$ with p arbitrary, and so $\{x_n\}$ is a Cauchy sequence.

By completeness of M, $\lim_{n\to\infty} x_n$ exists in M. Call this limit x_*; i.e., $x_* = \lim_{n\to\infty} x_n$.
We now show that φ is (uniformly) continuous. Given $\varepsilon > 0$, let $\delta = {}^{\epsilon}/_{k}$. Then $d(x, y) < \delta \Rightarrow d(\varphi(x), \varphi(y)) \leq k\, d(x, y) < k\,\delta = \varepsilon$.
Consider, $x_{n+1} = \varphi(x_n)$; $x_{n+1} \to x_*$, and by the continuity of φ, $\varphi(x_n) \to \varphi(x_*)$. Thus, $x_* = \varphi(x_*)$, so x_* is fixed.

Finally, we prove the uniqueness of the fixed point x_*. Let y_* be another point, i.e., $\varphi(y_*) = y_*$. Then

$$d(x_*, y_*) = d(\varphi(x_*), \varphi(y_*)) \leq k\, d(x_*, y_*) \qquad \text{i.e., } (1-k)\,d(x_*, y_*) \leq 0$$

By $k < 1$, and so $(1-k) > 0$, implying $d(x_*, y_*) = 0$, i.e., $x_* = y_*$, and thus the fixed point is unique. □