Chapter 1

Continuity

The central concept in topology is continuity, defined for functions between sets equipped with a notion of nearness (topological spaces) which is preserved by a continuous function. Topology is one kind of geometry in which the important properties of a figure are those are preserved under continuous motions.

Definition 1.1. Let X and Y be two topological spaces and $f : X \to Y$ be a mapping. Then f is said to be continuous at p in X if given any open set V containing $f(p)$ there exist an open set U containing p such that $f(U) \subseteq V$.

If f is continuous for each $p \in X$, then f is said to be continuous on X.

Example 1.1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$ be a topology on X. Define $f : X \to Y$ by $f(a) = b$, $f(b) = d$, $f(c) = b$, $f(d) = c$. Discuss/examine/check the continuity at c and d.

Solution. Continuity of f at c:
We see that at c, $f(c) = b$ and the open sets containing $f(c)$ are $X, \{b\}, \{a, b\}, \{b, c, d\}$. If we take $V = \{a, b\}$, then

$$f^{-1}(V) = f^{-1}(\{a, b\}) = \{a, c\}$$

The open sets containing c are X and $\{b, c, d\}$. Now, $f(X) = \{b, c, d\}$, and $f(\{b, c, d\}) = \{b, c, d\}$. But none of them contained in $V = \{a, b\}$. Hence, f is not continuous at c. **Continuity of f at d:** Here $f(d) = c$ and the open sets containing $f(d)$ are X and $\{b, c, d\}$. Also, the open sets containing d are X and $\{b, c, d\}$.

If we take $V = X$, then we get an open set $\{b, c, d\}$ containing d with $f(\{b, c, d\}) = \{b, c, d\} \subseteq V$.
And if we take $V = \{b, c, d\}$, we get the open set $\{b, c, d\}$ containing d with $f(\{b, c, d\}) = \{b, c, d\} \subseteq V$. Therefore, f is continuous at d.

Example 1.2. If a singleton set $\{p\}$ is an open in a topological space (X, τ) then any function $f : X \to Y$, is continuous at $p \in X$.

Proof. Suppose Hbe a open set containing $f(p)$. But

$$f(p) \in H \quad \text{implies} \quad p \in f^{\leftarrow}(H) \quad \text{implies} \quad \{p\} \subseteq f^{\leftarrow}(H)$$

This implies $f(\{P\}) \subseteq H$. Hence, f is continuous at p. □

From this example we can say that any function defined on a discrete space is continuous.

Theorem 1.1. A function $f : X \to Y$ is continuous iff for each open subset V in Y, $fa(V)$ is open in X.

Proof. First suppose f is continuous on X and let V be any open subset of Y. Let $U = f^{\leftarrow}(V)$. Choose any point $p \in U$. Then $f(p) \in V$. Since f is continuous at p, there exist an open set W_p containing p such that $f(W_p) \subseteq V$. Then $p \in W_p \subseteq f^{\leftarrow}(V) = U$. Hence, U is a neighborhood of p. Since p is arbitrary, so U is a neighborhood of each point of U. Therefore, $U = f^{\leftarrow}(V)$ is open.

Conversely, let for each open subset V of Y, $f^{\leftarrow}(V)$ is open in X. Let $U = f^{\leftarrow}(V)$. Then $f(U) = f(f^{\leftarrow}(V)) \subseteq V$.
Hence, by definition, f is continuous. □

Example 1.3. Let $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ be given by $f(x) = x$ for all $x \in \mathbb{R}$; that is, f is an identity function. Then for any open set V in $\mathbb{R}$, $f^{\leftarrow}(V) = V$ and so $f^{\leftarrow}(V)$ is open. Hence, f is continuous.

Example 1.4. Let $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ be given by $f(x) = c$ for all $x \in \mathbb{R}$; that is, f is a constant function. Then for any open set V in $\mathbb{R}$, clearly $f^{\leftarrow}(V) = \mathbb{R}$ if $c \in V$ and $f^{\leftarrow}(V) = \varnothing$ if $c \notin V$. In both cases $f^{\leftarrow}(V)$ is open. Hence, f is continuous.

Example 1.5. Let (X, τ) and (Y, τ^*) be two topologies defined by $X = \{a, b, c\}$ $\tau = \{X, \varnothing, \{a\}, \{b, c\}\}$ and $Y = \{p, q, r\}$ $\tau^* = \{Y, \varnothing, \{r\}, \{p, q\}\}$. Define $f : X \to Y$ by $f(a) = p$, $f(b) = q$, $f(c) = r$. The f is not continuous, because if we take the open set $V = \{r\}$ in Y, the $f^{\leftarrow}(V) = \{c\}$ which is not open in X.

Theorem 1.2. A function $f : ((X, \tau)) \to (Y, \tau^*)$ is continuous iff for each member of a base $\mathcal{B}$ for Y, $f^{\leftarrow} = (B)$ is open in X.

Proof. Let f be continuous and $B \in \mathcal{B}$. Then B is open in Y since it is a member of base $\mathcal{B}$, and hence $f^{\leftarrow}(B)$ is open in X.

Conversely, let V be any open set in Y. We show that $f^{\leftarrow}(V)$ is open in X. Since $\mathcal{B}$ is a base for Y, every open set in Y is the union of members of $\mathcal{B}$ and so $V = \cup\{B : B \in \mathcal{B}\}$. Then

$$f^{\leftarrow}(V) = f^{\leftarrow}(\cup\{B : B \in \mathcal{B}\}) = \cup\{f^{\leftarrow}(B) : B \in \mathcal{B}\}$$

But, by the hypothesis $f^{\leftarrow}(B)$ is open in X and their union is also open in X. Hence, $f^{\leftarrow}(V)$ is open. Thus, f is continuous. □

Theorem 1.3. Let $f : ((X, \tau)) \to (Y, \tau^*)$ and $\mathcal{A}$ be a subbase for the topology τ^* on Y. Then f is continuous iff the inverse of each member of the subbase $\mathcal{A}$ is an open subset of X.

Proof. Let $f : ((X, \tau)) \to (Y, \tau^*)$ be continuous and $\mathcal{A}$ be a subbase to τ^*. Then each element $\mathcal{S}$ of $\mathcal{A}$ is open in Y and so $f^{\leftarrow}(\mathcal{S})$ is open in X, f being continuous.

Conversely, suppose for any $\mathcal{S} \in \mathcal{A}$, $f^{\leftarrow}(\mathcal{S})$ is open in X. We show that f is continuous; i.e., $G \in \tau^*$ implies $f^{\leftarrow}(G) \in \tau$. Let $G \in \tau^*$. Then by definition of subbase,

$$G = \cup(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n), \quad \text{where } \mathcal{S}_i \in \mathcal{A}$$

Hence

$$\begin{aligned} f^{\leftarrow}(G) &= f^{\leftarrow}(\cup(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n)) \\ &= \cup f^{\leftarrow}(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n) \\ &= \cup[f^{\leftarrow}(\mathcal{S}_1) \cap f^{\leftarrow}(\mathcal{S}_2) \cap \cdots \cap f^{\leftarrow}(\mathcal{S}_n)] \end{aligned}$$

But, by hypothesis, $\mathcal{S}_i \in \mathcal{A}$ implies $f^{\leftarrow}(\mathcal{S}_i)$ is open in X and hence $f^{\leftarrow}(G)$ is open in X since the union of finite intersection of open sets is open. Therefore, f is continuous. □

Theorem 1.4. A function $f : X \to Y$ is continuous iff for any subset of Y, $f^{\leftarrow}(B^\circ) \subseteq f^{\leftarrow}(B)^\circ$.

Proof. Suppose f is continuous on X and let B be any subset of Y. Then B° is open in Y and so $f^{\leftarrow}(B^\circ)$ is open in X. Now, we have, $B^\circ \subseteq B$ and so $f^{\leftarrow}(B^\circ) \subseteq f^{\leftarrow}(B)$, and then $[f^{\leftarrow}(B^\circ)]^\circ \subseteq [f^{\leftarrow}(B^\circ)]^\circ = f^{\leftarrow}(B)^\circ$.
But $f^{\leftarrow}(B^\circ)$ is open, so $[f^{\leftarrow}(B^\circ)]^\circ = f^{\leftarrow}(B^\circ)$ and hence $f^{\leftarrow}(B^\circ) \subseteq f^{\leftarrow}(B)^\circ$.

Conversely, let for any subset B of Y, $f^{\leftarrow}(B^\circ) \subseteq [f^{\leftarrow}(B)]^\circ$. Let V be any open set in Y. Then $f^{\leftarrow}(V^\circ) \subseteq [f^{\leftarrow}(B)]^\circ$. But V is open, so $V^\circ = V$. Hence $f^{\leftarrow}(V^\circ) = f^{\leftarrow}(V) \subseteq [f^{\leftarrow}(V)]^\circ$.
Bur it is always the case that $[f^{\leftarrow}(V)]^\circ \subseteq f^{\leftarrow}(V)$. Hence $f^{\leftarrow}(V) = [f^{\leftarrow}(V)]^\circ$ which is open in X. Therefore, f is continuous on X. □

Theorem 1.5. A function $f : X \to Y$ is continuous iff for each closed subset F of Y, $f^{\leftarrow}(F)$ is a closed subset in X.

Proof. Suppose f is continuous on X and let F be any closed subset of Y. Let $V = Y - F$. Then V is open in Y and hence $f^{\leftarrow}(V)$ is open in X. Now,

$$X - f^{\leftarrow}(F) = f^{\leftarrow}(Y - F) = f^{\leftarrow}(V)$$

which is open in X. Hence, $f^{\leftarrow}(F)$ is closed in X.

Conversely, let V be any open set in Y. Then $Y - V$ is closed and hence $f^{\leftarrow}(Y - V)$ is closed ain X. But,

$$f^{\leftarrow}(Y - V) = X - f^{\leftarrow}(V)$$

which is closed in X. Hence, $f^{\leftarrow}(V)$ is open in X. Thus, f is continuous on X. □

Theorem 1.6. A function $f : X \to Y$ is continuous iff for any subset A of X, $f(\overline{A}) \subseteq \bar{(}f(A))$.

Proof. Suppose f is continuous on X and let A be any subset of X. Then $f(A)$ is a subset of Y and $\overline{f(A)}$ is closed in Y; hence $f^{\leftarrow}(\overline{f(A)})$ is closed in X. Now, we have

$$f(A) \subseteq \overline{f(A)}$$

and so

$$f^{\leftarrow}(f(A)) \subseteq f^{\leftarrow}(\overline{f(A)})$$

But $A \subseteq f^{\leftarrow}(f(A))$; hence $A \subseteq f^{\leftarrow}(\overline{f(A)})$. Since $f^{\leftarrow}(\overline{f(A)})$ is closed and $\overline{A}$ is the smallest closed set containing, it follows that

$$\overline{A} \subseteq f^{\leftarrow}(\overline{f(A)})$$

and so

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Conversely, let for any subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$. Let F be any closed set in Y. Then $f^{\leftarrow}(F)$ is subset of X. We claim that $f^{\leftarrow}(F)$ is closed in X. Since $fa(F)$ is subset of X, so

$$f\overline{(f^{\leftarrow}(F))} \subseteq \overline{f(f^{\leftarrow}(F))} = \overline{F} = F$$

$\therefore \overline{(f^{\leftarrow}(F))} \subseteq f^{\leftarrow}(F)$.
But it is always the case that $f^{\leftarrow}(F) \subseteq \overline{(f^{\leftarrow}(F))}$. Hence $f^{\leftarrow}(F) = \overline{(f^{\leftarrow}(F))}$ i.e., $f^{\leftarrow}(F)$ is closed and therefore f is continuous on X. □

1.1 Sequential Continuity

Definition 1.2. A function $f : X \to Y$ is said to be sequentially continuous at a point $p \in X$ iff for every sequence $\langle a_n \rangle$ converging to p, the sequence $f(a_n)$ converges to $f(p)$; i.e., iff $a_n \to p$ implies $f(a_n) \to f(p)$.

Continuity and sequential continuity at a point are related as follows:

Theorem 1.7. If a function $f : X \to Y$ is continuous at $p \in X$, then it is sequentially continuous at p.

Proof. Let the sequence $\langle a_n \rangle$ in X converges to p. Let M be the neighborhood of $f(p)$. Then f being continuous at p implies $f^{\leftarrow}(M)$ is open in X containing p. Let $N = f^{\leftarrow}(M)$. Then, since $\langle a_n \rangle$ converges to p, so $a_n \in N$ for almost all $n \in \mathbb{N}$. This implies $f(a_n) \in f(N) = f(f^{\leftarrow}(M)) = M$ for almost all $n \in \mathbb{N}$. So, the sequence $\langle f(a_n) \rangle$ converges to $f(p)$. Hence, f is sequentially continuous at p. □

1.2 Open and Closed functions

A function $f : X \to Y$ is called an **open function** if the image of every open set is open.
Similarly, a function $f : X \to Y$ is called a **closed function** if the image of every closed set is closed.
In general, functions which are not open need not be closed and vice versa.

Example 1.6. Let $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ be given by $f(x) = c$ for all $x \in \mathbb{R}$. Then f is continuous (see ex 1.3). Let V be a open set and H be a closed set in R. Then,

$$f(v) = \{c\} \text{ and } f(H) = \{c\} \text{ for all } x \in V \text{ and for all } x \in H$$

Since $\{c\}$ is finite, it is closed but not open. Therefore f is a closed map and continuous but it is not open.

Example 1.7. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$, $Y = \{p, q, r\}$ and $\tau^* = \{\varnothing, \{p\}, \{p, r\}, Y\}$.

1. Define $f : X \to Y$ by $f(a) = p$, $f(b) = q$, $f(c) = r$. Then f is an open map but it is not continuous.
2. Define $g : X \to Y$ by $g(x) = q$ for all $x \in X$. Then g is a closed map and it is continuous but not open.
3. Define $h : X \to Y$ by $h(x) = p$ for all $x \in X$. Then h is an open map and it is not continuous and not open.

1.3 Homeomorphism

Between any two topological spaces (X, τ) and (Y, τ^*), there are many functions $f : X \to Y$. We choose to discuss continuous, or open or closed functions rather than arbitrary functions since these functions preserves some aspects of the structure of the spaces (X, τ) and (Y, τ^*).
If the function $f : X \to Y$ defines a one to one correspondence between the open sets in X and the open sets in Y, then the spaces (X, τ) and (Y, τ^*) are identical from the topological point of view.

Definition 1.3. Let X and Y be topological spaces. A bijective function $f : X \to Y$ is said to be a homeoporphism if f is open and continuous, or equivalently, both f and $f^{\leftarrow}$ are continuous.
If there exists a homeoporphism between X and Y, we say that X and Y are **homeomorphic** spaces, or that they are topologically equivalent, and write $X \cong Y$.

Lemma 1.1. If $f : X \to Y$ is a homeoporphism, then so is the inverse map $f^{\leftarrow} : Y \to X$.

Lemma 1.2. If $f : X \to Y$ and $g : Y \to Z$ are homeoporphisms, then so is the composite map $gf : X \to Z$.

Example 1.8. For each space X the identity function $i_d : X \to X$, with $i_d(x) = x$ for all $x \in X$, is a homeoporphism.

Example 1.9. Any two open intervals of the real line are homeomorphic. For example, if $S = (-1, 1)$ and $T = (0, 5)$, then define $f : S \to T$ and $g : T \to S$ by $f(x) = \frac{5}{2}(x+1)$, $g(x) = \frac{2}{5}(x-1)$. These maps are continuous, being composites of addition and multiplication, and it is easy to verify that they are inverse to each other. So f and g are homeomorphisms, and $(-1, 1)$ and $(0, 5)$ are homeomorphic.

Example 1.10. The function $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \dfrac{x}{1-x^2}$ is homeoporphism. To find the inverse of f, we rewrite the equation $\dfrac{x}{1-x^2} = y$ as $yx^2 + x - y = 0$ and solve for x as a function of $y \in \mathbb{R}$, namely

$$f^{-1}(y) = \frac{-1 + \sqrt{1 + 4y^2}}{2y} = \frac{2y}{1 + \sqrt{1 + 4y^2}}$$

It is well known that both f and f^{-1} are continuous, hence $\mathbb{R}$ is homeomorphic to any open interval (a, b).

If we define a continuous map $f : (-1, 1) \to \mathbb{R}$ by

$$f(x) = \tan(\frac{\pi}{2}x)$$

This is a bijection and has a continuous invgrse $g : \mathbb{R} \to (-1, 1)$ given by

$$g(x) = \frac{2}{\pi}\tan^{-1}(x)$$

Example 1.11. A solid square is homeomorphic to a solid disc.
We will illustrate this with the square $Q = \left\{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\right\}$ and disc $D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq\right.$
Define $f : D \to Q$ by

$$f(x, y) = \frac{\sqrt{x^2 + y^2}}{\max(|x|, |y|)}(x, y)$$

if $(x, y) \neq (0, 0)$ and $f(0, 0) = (0, 0)$. Its inverse $g : Q \to D$ is given by

$$g(x, y) = \frac{\max(|x|, |y|)}{\sqrt{x^2 + y^2}}(x, y)$$

if $(x, y) \neq (0, 0)$ and $g(0, 0) = (0, 0)$.

The idea of these maps is that f pushes the disc out radially to form a square, and g contracts the square radially to form a disc. Using this idea, you can see that the preimage of an open subset of Q under f will be

Insert fig

open in D and similarly for g. So, they are continuous maps.

1.4 Topological Properties

A property P is said to be a topological property or a topological invariant if, whenever a topological space (X, τ) has the property P, then every space homeomorphic to (X, τ) also has the property P.
Briefly, a property, that is preserved under a homeomorphism, is called a topological property or topological invariant.

Example 1.12. Let $X = (0, \infty)$. Define a function $f : X \to X$ by $f(x) = \dfrac{1}{x}$. Then f is a homeomorphism. Observe that the sequence

$$\langle a_n \rangle = 1, \frac{1}{2}, \frac{1}{3}, ...$$

correspond, under homeoporphism, to the sequence

$$\langle f(a_n) \rangle = 1, 2, 3, ...$$

We see that the sequence $\langle a_n \rangle$ is a Cauchy sequence but the sequence $\langle f(a_n) \rangle$ is not. Hence the property of being a Cauchy sequence is not topological.

Example 1.13. Being a finite topological space, having the discrete, trivial or cofinte topology, or being a Hausdorff space, are all examples of topological properties. So, if X is a Hausdorff space and $X \cong Y$ then Y is a Hausdorff space. Compactness and connectedness are also topological properties.

Problem 1.1. Show that an identity map on a topological space is continuous but the identity map in different topological spaces may not be continuous.

Solution. Let $f : (X, \tau) \to (X, \tau)$ defined by $f(x) = x$ for all $x \in X$; that is, f is an identity map. Then for any open set V in X, $fa(V) = V$ and so $f^{\leftarrow}(V)$ is open. Hence f is continuous.
To prove the 2nd part, let $\tau =$ co-finite topology on $\mathbb{R}$ and $\tau_u =$ usual topology on $\mathbb{R}$.
Let $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ be an identity map. Let $V = (0, 1) \in \tau_u$. Then $i^{\leftarrow}(0, 1) = (0, 1) \notin \tau$ because $\mathbb{R} - (0, 1)$ is not finite. Thus we can see that, though $V = (0, 1)$ is open in $(\mathbb{R}, \tau_u)$, $i^{\leftarrow}(0, 1)$ is not open in $(\mathbb{R}, \tau)$. Hence the identity map $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ is not continuous.

Again, let $i : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau)$ be an identity map. Let $G \in \tau$. Then $\mathbb{R} - G$ is finite. Hence $i^{\leftarrow}(\mathbb{R} - G) = \mathbb{R} - G$ is closed in $\mathbb{R}, \tau_u$. Hence G is open in $\mathbb{R}$. Thus i is continuous.
Therefore, the identity map on different topological spaces may not be continuous.