Chapter 1

Hermite Polynomial

The differential equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} - 2x\dfrac{\mathrm{d}\, y}{\mathrm{d}\, x} + 2ny = 0$ when n is a constant, is called Hermite's differential equation. The solutions of Hermite's equation is called the Hermite polynomial. Hermite polynomial of order n is denoted and defined by

$$H_n(x) = \sum_{r=0}^{N} (-1)^r \frac{n!}{r!\,(n-2r)!}(2x)^{n-2r},\ N = \begin{cases} \dfrac{n}{2}, & n \text{ is even} \\ \dfrac{n-1}{2}, & n \text{ is odd} \end{cases} \tag{1.1}$$

1.1 Relation Between Legendre and Hermite Polynomial

Problem 1.1.1. Prove that

$$P_n(x) = \frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2} H_n(xt)\,\mathrm{d}\, t$$

Proof. We have

$$H_n(x) = \sum_{r=0}^{N} (-1)^r \frac{n!}{r!\,(n-2r)!}(2x)^{n-2r}$$

$$\Rightarrow H_n(xt) = \sum_{r=0}^{N} (-1)^r \frac{n!}{r!\,(n-2r)!}(2xt)^{n-2r}$$

Now,

$$\begin{aligned} &\frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2} H_n(xt)\,\mathrm{d}\, t \\ =&\frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2}\left[\sum_{r=0}^{N}(-1)^r \frac{n!}{r!\,(n-2r)!}(2xt)^{n-2r}\right]\mathrm{d}\, t \\ =&\sum_{r=0}^{N}\frac{(-1)^r 2^{n-2r+1} x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!}\int_0^\infty e^{-t^2} t^{2n-2r}\,\mathrm{d}\, t \end{aligned} \tag{1.2}$$

Now,

$$\begin{aligned} &\int_0^\infty e^{-t^2} t^{2n-2r-1+1}\,\mathrm{d}\, t \\ =&\int_0^\infty e^{-t^2} t^{2\left(n-r+\frac{1}{2}\right)-1}\,\mathrm{d}\, t \\ =&\frac{1}{2}\Gamma\left(n-r+\frac{1}{2}\right) && \left|\, 2\int_0^\infty e^{-t^2} t^{2n-1}\,\mathrm{d}\, t = \Gamma(n)\right. \\ =&\frac{1}{2}\cdot\frac{(2n-2r)!}{2^{2n-2r}(n-r)!}\sqrt{\pi} && \left|\, \Gamma\left(x+\frac{1}{2}\right) = \frac{(2x)!}{2^{2x}\,(x)!}\sqrt{\pi}\right. \end{aligned}$$

From (1.2),

$$
\begin{aligned}
&\sum_{r=0}^{N} \frac{(-1)^r 2^{n-2r+1} x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!} \frac{1}{2} \cdot \frac{(2n-2r)!}{2^{2n-2r}(n-r)!} \sqrt{\pi} \\
=&\sum_{r=0}^{N} (-1)^r \frac{(2n-2r)}{2^n\, r!\,(n-r)!\,(n-2r)!} x^{n-2r} \\
=&P_n(x)
\end{aligned}
$$

$$
\therefore P_n(x) = \frac{2}{\sqrt{\pi}\, n!} \int_0^{\infty} t^n e^{-t^2} H_n(xt)\, \mathrm{d}\, t
$$

□

1.2 Generating Function of Hermite Polynomial

Problem 1.2.1. Prove that

$$
e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)
$$

Proof.

$$
\begin{aligned}
e^{2tx-t^2} &= e^{2tx} \cdot e^{-t^2} \\
&= \left\{1 + \frac{2tx}{1!} + \frac{2^2t^2x^2}{2!} + \cdots + \frac{(2tx)^r}{r!} + \ldots\right\} \times \left\{1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} \cdots + \frac{(t^2)^s}{s!} + \ldots\right\} \\
&= \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \sum_{s=0}^{\infty} \frac{(-1)^s\,(t^2)^s}{s!} \\
&= \sum_{r,s=0}^{\infty} (-1)^s \frac{(2x)^r \cdot t^{r+2s}}{r!\, s!}
\end{aligned}
$$

Let $r + 2s = n$ so that $r = n - 2s$
so for a fixed value of s, the coefficient of t^n is given by

$$
(-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\, s!}
$$

The total value of t^n is obtained by summing over all allowed values of s and since $r = n - 2s$.
$\therefore\ n - 2s \geq 0$ or $s \leq \dfrac{n}{2}$
Thus if n is even, s goes from 0 to $\dfrac{n}{2}$ and if n is odd, s goes from 0 to $\dfrac{n-1}{2}$.
So coefficient of t^n

$$
\begin{aligned}
t^n &= \sum_{s=0}^{\frac{n}{2}} \frac{(-1)^s (2x)^{n-2s}}{(n-2s)!\, s!} \\
&= \sum_{s=0}^{\frac{n}{2}} \frac{(-1)^s n!}{(n-2s)!\, s!} \cdot (2x)^{n-2s} \cdot \frac{1}{n!} \\
&= \frac{H_n(x)}{n!}
\end{aligned}
$$

Since

$$
\sum_{r=0}^{\frac{n}{2}} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}
$$

Hence

$$
e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)
$$

$$
\text{Or, } e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)
$$

□

1.3 Hermite Polynomials of Different Forms

Theorem 1.3.1. Prove that

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(e^{-x^2}\right)$$

Proof. Using the generating function, we have

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \tag{1.3}$$

Or,

$$f(x,t) = e^{x^2-(t-x)^2} = \frac{H_0(x)}{0!}t^0 + \frac{H_1(x)}{1!}t^1 + \frac{H_2(x)}{2!}t^2 + \cdots + \frac{H_n(x)}{n!}t^n + \frac{H_{n+1}(x)}{(n+1)!}t^{n+1} + \ldots \tag{1.4}$$

Differentiating both sides of (1.4) partially with respect to t n times

$$e^{x^2} \cdot \frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\} = 0 + \frac{H_n(x)}{n!}n! + \frac{H_{n+1}(x)}{(n+1)!}(n+1)n(n-1)\ldots 2t \cdots + \ldots \tag{1.5}$$

Putting $t = 0$ in (1.5), we get

$$\begin{aligned}
e^{x^2}\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} &= \frac{H_n(x)\, n!}{n!} + 0 \\
\Rightarrow e^{x^2}\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} &= H_n(x) \\
\Rightarrow H_n(x) = e^{x^2}\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} & \qquad (1.6)
\end{aligned}$$

Putting $t - x = u$ so that $\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial u}$
But at $t = 0$ $-x = u$ i.e., $x = -u$
Therefore,

$$\begin{aligned}
\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} &= \frac{\partial^n}{\partial u^n}\left(e^{-u^2}\right) \\
&= (-1)^n \frac{\partial^n}{\partial\, x^n}\left(e^{-x^2}\right) \\
&= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(e^{-x^2}\right)
\end{aligned}$$

Thus from (1.6), we get

$$\begin{aligned}
H_n(x) &= e^{x^2} \cdot (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(e^{-x^2}\right) \\
\Rightarrow H_n(x) &= (-1)^n \cdot e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(e^{-x^2}\right)
\end{aligned}$$

Which is also known as the *Rodrigue's formula for $H_n(x)$.* □

1.4 Orthogonality Properties of Hermite Polynomials

Problem 1.4.1. Prove that

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,\mathrm{d}\, x = \begin{cases} 0 & \text{if } m \neq n \\ 2^n\sqrt{\pi}\, n! & \text{if } m = n \end{cases}$$

Proof. From generating function of Hermite polynomial, we have

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{1.7}$$

and

$$e^{-s^2+2sx} = \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!} \tag{1.8}$$

Multiplying the corresponding sides of (1.7) and (1.8), we can write

$$\begin{aligned} e^{-t^2+2tx} \cdot e^{-s^2+2sx} &= \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \cdot \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!} \\ \Rightarrow e^{2tx-t^2+2sx-s^2} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{H_n(x)\, H_m(x)\, t^n\, s^m}{n!\, m!} \end{aligned} \tag{1.9}$$

Multiplying both sides of (1.9) by e^{-x^2} and then integrating both sides with respect to x from $-\infty$ to ∞, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \,\mathrm{d}\, x \right] \frac{t^n\, s^m}{n!\, m!} \\ =& \int_{-\infty}^{\infty} e^{-x^2+2(t+s)x\left(t^2+s^2\right)} \,\mathrm{d}\, x \\ =& \int_{-\infty}^{\infty} e^{-x^2+2(t+s)x\left(t^2+s^2\right)} \times e^{(t+s)^2-\left(t^2+s^2\right)} \,\mathrm{d}\, x \\ =& e^{2st} \int_{-\infty}^{\infty} e^{-(x-(t+s))^2} \,\mathrm{d}\, x \end{aligned}$$

Putting $x-(t+s)=y$ so that $\mathrm{d}\, x = \mathrm{d}\, y$

$$\begin{aligned} &\text{Limits} \quad \left.\begin{matrix} x=\infty \\ y=\infty \end{matrix}\right\} \qquad \left.\begin{matrix} x=-\infty \\ y=-\infty \end{matrix}\right\} \\ =& e^{2st} \cdot \int_{-\infty}^{\infty} e^{-y^2} \,\mathrm{d}\, y \\ =& e^{2st} \cdot 2 \int_{0}^{\infty} e^{-y^2} \,\mathrm{d}\, y \\ =& 2e^{2st} \cdot \frac{\sqrt{pi}}{2} \qquad \text{since } \int_0^{\infty} e^{-y^2} \,\mathrm{d}\, y = \frac{\sqrt{\pi}}{2} \\ =& \sqrt{\pi} e^{2st} \\ =& \sqrt{\pi} \left[1 + \frac{2st}{1!} + \frac{(2st)^2}{2!} + \frac{(2st)^3}{3!} + \ldots \right] \\ =& \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2st)^n}{n!} \\ =& \sqrt{\pi} \sum_{n=0}^{\infty} 2^n \frac{s^n t^n}{n!} \end{aligned}$$

Thus coefficient of $t^n s^m$ in the expansion of $\int_{-\infty}^{\infty} e^{-x^2+2(t+s)x\left(t^2+s^2\right)} \,\mathrm{d}\, x$ is $\begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2^n \sqrt{\pi}}{n!} & \text{if } m = n \end{cases}$

Hence

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \,\mathrm{d}\, x = \begin{cases} 0 & \text{if } m \neq n \\ 2^n \sqrt{\pi}\, n! & \text{if } m = n \end{cases}$$

□

Making use of the *kronecker delta* we have

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \,\mathrm{d}\, x = \sqrt{\pi} 2^n\, n!\, \delta_{mn} \qquad \text{since } \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \\ \Rightarrow &\int_{-\infty}^{\infty} e^{-x^2} (H_n(x))^2 \,\mathrm{d}\, x = \sqrt{\pi} 2^n\, n! \end{aligned}$$

1.5 Recurrence Relation of Hermite Polynomial

(i) $H_n^{'}(x) = 2nH_{n-1}(x)$, $n \geq 1$; $H_0^{'}(x) = 0$.

(ii) $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$, $n \geq 1$; $H_1(x) = 2xH_0(x)$.

(iii) $H_n^{'}(x) = 2xH_n(x) - H_{n+1}(x)$.

(iv) $H_n^{''}(x) - 2xH_n^{'}(x) + 2nH_n(x) = 0$.

1.6 Integral Formula for Hermite Polynomial

Let us assume

$$y_n = \frac{1}{2\pi\, i} \oint \rho^{-n-1} e^{\left\{x^2-(\rho-x)^2\right\}} \,\mathrm{d}\,\rho \tag{1.10}$$

where the contour is taken around a circle having centre at origin.
If we differentiate (1.10) with respect to x we get,

$$\begin{aligned}\frac{\mathrm{d}\,y_n}{\mathrm{d}\,x} &= \frac{1}{2\pi i} \oint 2\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}} \,\mathrm{d}\,\rho \\ \Rightarrow \frac{\mathrm{d}^2\,y_n}{\mathrm{d}\,x^2} &= \frac{1}{2\pi i} \oint 4\rho^{-n+1} e^{\left\{x^2-(\rho-x)^2\right\}} \,\mathrm{d}\,\rho\end{aligned}$$

Thus

$$\begin{aligned}& y_n^{''} - 2xy_n^{'} - 2ny_n \\ =& \frac{1}{2\pi i} \oint 4\rho^{-n+1} e^{\left\{x^2-(\rho-x)^2\right\}} \,\mathrm{d}\,\rho - \frac{2x}{2\pi i} \oint 2\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}} \,\mathrm{d}\,\rho + \frac{2n}{2\pi\, i} \oint \rho^{-n-1} e^{\left\{x^2-(\rho-x)^2\right\}} \,\mathrm{d}\,\rho \\ =& \frac{1}{2\pi i} \oint \left(4\rho^2 - 4x\rho + 2n\right) e^{\left\{x^2-(\rho-x)^2\right\}} \rho^{-n-1} \,\mathrm{d}\,\rho \\ =& \frac{-2}{2\pi i} \oint \frac{\mathrm{d}}{\mathrm{d}\,\rho} \left[\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}}\right] \,\mathrm{d}\,\rho\end{aligned}$$

But

$$\oint \frac{\mathrm{d}}{\mathrm{d}\,\rho} \left[\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}}\right] \,\mathrm{d}\,\rho = 0$$

$$\therefore y_n^{''} - 2xy_n^{'} - 2ny_n = 0$$

Which is Hermite equation and hence y_n given by (1.10) is also a solution of the Hermite equation.
So we may have $H_n(x) = cy_n(x)$, c is a constant but if we put $x = 0$ in (1.1) we have,

$$H_n(0) = (-1)^{\frac{n}{2}} \frac{n!}{\frac{n}{2}}$$

and form (1.10),

$$y_n(0) = \frac{1}{2\pi\, i} \oint \rho^{-n-1} e^{-(\rho-x)^2} \,\mathrm{d}\,\rho$$

But by contour integration we have,

$$\oint \rho^{-n-1} e^{-(\rho-x)^2} \,\mathrm{d}\,\rho = \frac{2\pi i(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}$$

$$\therefore y_n(0) = \frac{(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}$$

Thus

$$\begin{aligned}& H_n(0) = cy_n(0) \\ \Rightarrow & (-1)^{\frac{n}{2}} \frac{n!}{\frac{n}{2}} = \frac{c(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \Rightarrow \qquad c = n!\end{aligned}$$

$$\begin{aligned}& \therefore H_n(x) = n!\, y_n(x) \\ \Rightarrow & H_n(x) = \frac{n!}{2\pi\, i} \oint \rho^{-n-1} e^{\left\{x^2-(\rho-x)^2\right\}} \,\mathrm{d}\,\rho\end{aligned}$$

Which is the integral form of Hermite polynomial.