Theory of Groups

MAT511

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Preface

This is a compilation of lecture notes with some books and my own thoughts. If there are any mistake/typing error or, for any query mail me at mehedi12@student.sust.edu.

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Part 1

Sheet

CHAPTER 1

Automorphism

Definition 1 (Automorphism). An automorphism of a group G is an isomorphism[1] of G onto itself.

Theorem 0.1. The set $Aut(G)$ of all automorphisms of a group G is a group under the operation of composition of mappings.

Proof. Here $Aut(G)$ is the set of all automorphisms of a group G and the operation is the composition of mappings.
Let $f, g \in Aut(G)$.
Then the composite map $g \circ f$ is bijective, because f and g are bijective.
Using the hypotheses that f and g are group homomorphisms, we can conclude that $g \circ f$ is also a group homomorphism, because

$$\begin{aligned}(g \circ f)(ab) &= g(f(ab)) \\ &= g(f(a)f(b)) \\ &= g(f(a))g(f(b)) \\ &= (g \circ f)(a)(g \circ f)(b)\end{aligned}$$

So, $g \circ f \in Aut(G)$.
This is the closure property.

The associative law holds for $Map(G)$, the set of all mappings of G into itself; so it holds in $Aut(G)$, because $Aut(G)$ is closed under composition of mappings.

Clearly, 1_G is the identity element of $Aut(G)$.

If $f \in Aut(G)$, the inverse mapping $f^{-1} : G \to G$ exists and is likewise bijective.
Let $f \in Aut(G)$ and $a, b, x, y \in G$ such that $f(a) = x$ and $f(b) - y$. Then we have $a = f^{-1}(x)$ and $b = f^{-1}(y)$.
Since f is a group homomorphism, we have $f(ab) = f(a)f(b) = xy$.
It gives, $f^{-1}(xy) = ab = f^{-1}(x)f^{-1}(y)$.
This implies that f^{-1} is also a group homomorphism.
Hence, $f^-1 \in Aut(G)$.
Therefore, $Aut(G)$ is a group under composition of mappings. □

[1]Homomorphism: Suppose G, G' are multiplicative groups. A mapping $f : G \to G'$ is called a group homomorphism iff $f(ab) = f(a)f(b)$ holds for all $a, b \in G$.
Isomorphism: A bijective group homomorphism is called an isomorphism.

1. Inner Automorphisms

For any fixed $a \in G$, we define a mapping $f_a : G \to G$ by setting $f_a(x) = axa^{-1}$, $f_a \in Aut(G)$ for every $a \in G$.

PROOF. f_a is injective (by the cancellation law), for

$$f_a(x) = f_a(y) \;\; \Rightarrow \;\; axa^{-1} = aya^{-1} \;\; \Rightarrow \;\; x = y.$$

f_a is surjective, because

$$f_a(a^{-1}xa) = a(a^{-1}xa)a^{-1} = x.$$

f_a is group homomorphism, because for all $x, y \in G$, we have

$$f_a(xy) = a(xy)a^{-1} = (axa^{-1})(aya^{-1}) = f_a(x)f_a(y).$$

□

DEFINITION 2 (Inner Automorphism). For any fixed $a \in G$ the mapping $f_a : G \to G$ defined by $f_a(x) = axa^{-1}$ is called the inner automorphism determined by a.

CHAPTER 2

Conjugacy and Class Equation

Definition 3. Let G be a group. The *normalizer* of a non-empty subset $S \subseteq G$ is defined by $N_S = \{x \in G : xS = Sx\}$.

Definition 4. Let G be a group and $a \in G$. Then the set $N_a = \{x \in G : ax = xa\}$ is called the *normalizer* of $a \in G$ in G.

Thus, N_a is the set of those elements of G which commute with a.

Example. N_a is a subgroup of G.

Proof. We know that $N_a = \{x \in G : ax = xa\}$ when $a \in G$. Let $x, y \in N_a$. Then $ax = xa$ and $ay = ya$.
Hence, we have

$$a(xy) = (ax)y = (xa)y = x(ay) = x(ya) = (xy)a.$$

Besides, we also get

$$x^{-1}(ax)x^{-1} = x^{-1}(xa)^{-1} \text{ which implies that } \quad x^{-1}a = ax^{-1}.$$

Thus, it follows that $xy \in N_a$ and $x^{-1} \in N_a$ for all $x, y \in N_a$.
So, N_a is a subgroup of G. □

Note. $N_a = G \;\Leftrightarrow\; a = Z$.

Example. N_a need not be a normal subgroup[1] of G.

Proof. In order to show that N_a need not be normal in G, consider an element (23) of the symmetric group S_3.
It is easy to verify that $N_{(23)} = \{(1), (23)\}$ is a subgroup of S_3.
But $(12) \circ N_{(23)} = \{(12), (123)\}$ and $N_{(23)} \circ (12) = \{(12), (132)\}$
Thus $(12) \circ N_{(23)} \neq N_{(23)} \circ (12)$.
This shows that $N_{(23)}$ is not a normal subgroup of S_3. □

[1]Normal subgroup: A subgroup N of a group G is called normal subgroup iff $aN = Na$ holds $\forall a \in G$. Denoted by $N \lhd G$.