Statics And Dynamics
(Class Note and Sir's PDF)

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Contents

Part I

Statics

Chapter 1

Introduction

1.1 Definitions

Force: Force is anything which changes or tend to change, the state of rest or uniform motion of a body.

Rest: A body is said to be rest when it does not change its position with respect to it's surrounding objects.

Statics: Statics is the science with treats the action of forces on bodies, the forces being so arranged that the bodies are at rest.

Dynamics: The science which treats the action of forces on bodies in motion is called dynamics.

Particle: A particle is a portion of matter which is indefinitely small in size.

Rigid Body: A rigid body is a body whose parts always preserve an invariable position with respect to one another.

Equal Forces: Two forces are said to be equal when if they act on a particle in opposite direction, the particle remains at rest.

Mass: The mass of a body is the quantity of matter in the body.

Weight: The force with which the earth attract any body to itself is called the weight of the body.

Tension: If we tie one end of a string to any point of a body and pull at the other end of the string, we exert a force on that body, such a force exerted by means of string or rod is called tension.

Equilibrium: When two or more forces act upon a body and are so arranged that the body remains at rest, the forces are said to be in equilibrium.

Resultant: If two or more forces $P, Q, S \ldots$ act upon a rigid body and it a single force R can be found whose effect upon the body is the same as that of the forces $P, Q, S \ldots$. This single force R is called the resultant of the other forces and the forces $P, Q, S \ldots$ are called the component of R.

Moments: The moment (or torque) of a force about a turning point is the force multiplied by the perpendicular distance to the force from the turning point. Moments are measured in Newton meters(Nm). Mathematically

$$\text{Moment} = F.d$$

Example 1.1.1. A $10\,N$ force acts at a perpendicular distance of $0.5m$ from the turning point. What is the moment of the force?

Solution.

$$\begin{aligned} M &= F\,.\,d \\ &= 10 \times 0.5 \\ &= 5\,Nm \end{aligned}$$

The principal of momentum: When an object is in equilibrium, the sum of the anti-clockwise moments about a turning point must be equal to the sum of the clockwise moments.

Couple: Two forces of equal magnitudes and opposite direction whose line of action is not same is said to constitute a couple. A couple produces rotational effects in the body on which it acts and this rotational effect is measured by a physical quantity torque. Mathematically, Couple $= F.S$

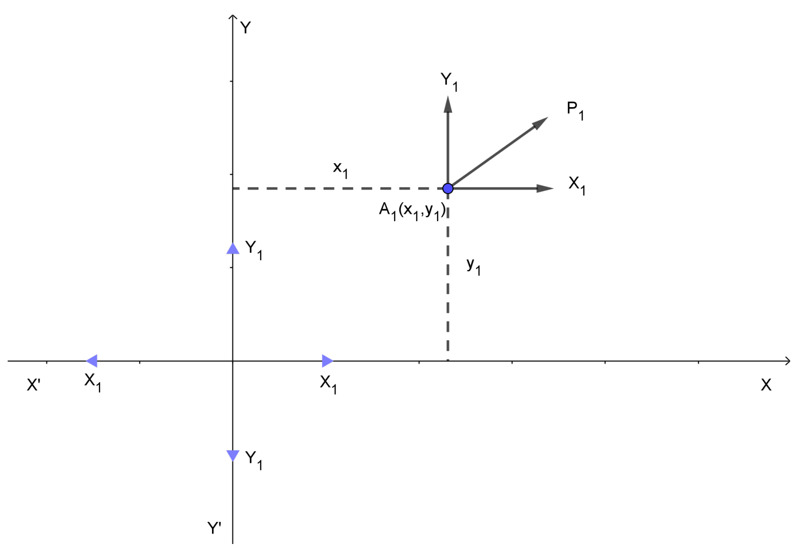
Chapter 2

Reduction of a System of Coplanar Forces

2.1 Reduction of a System of Coplanar Forces

Statement: A system of forces acting in one plane ar different point of a rigid body can be reduces to a single force R through any arbitrary point and a couple, whose moment is equal to the sum of the moments of the given forces about this point.

Proof. Let O be any point in the plane. Through O take any pair of rectangular axes OX and OY . Let the forces $P_1, P_2, P_3, \ldots$ act at points $A_1(x_1, y_1), A_2(x_2, y_2), A_3(x_3, y_3), \ldots$.



Let the resolved parts of these forces in direction parallel to the axes be $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), \ldots$

At O introduce forces X_1 and X_1 along OX and OX' also Y_1 and Y_1 along OY and OY' respectively. So that their actions O are zero. i.e. the body remain unaffected.

Now the force P_1 at A_1 has component X_1 parallel to OX and this component X_1 at O acts along OX. Another two equal forces X_1 at A_1 and X_1 at O along OX' are equal and unlike parallel forces which form a couple of moment $-X_1y_1$ and this couple has a tendency to rotate the body in a clockwise direction.

Similarly, the force Y_1 acting at O along OY' and the force Y_1 as A_1 parallel to OY form a couple of moment x_1Y_1.

Hence the force P_1 at A_1 is equivalent to components X_1, Y_1 along OX, OY and a couple $G_1 = x_1Y_1 - X_1y_1$, which is algebraic sum of the moments of two couples formed above.

Similarly the force P_2 at A_2 is equivalent to components X_2 along OX and Y_2 along OY and a couple of moment $G_2 = x_2Y_2 - X_2y_2$.

Resolving all other forces in similar way, if R is the resultant of forces along OX and OY inclined at angle θ to OX.

$R\cos\theta = X = X_1 + X_2 + X_3 + \cdots = \sum X_i$
$R\sin\theta = Y = Y_1 + Y_2 + Y_3 + \cdots = \sum Y_i$
If G represents the algebraic sum of the moments of the couple, then

$$G = G_1 + G_2 + G_3 + \cdots = \sum (x_iY_i - X_iy_i)$$

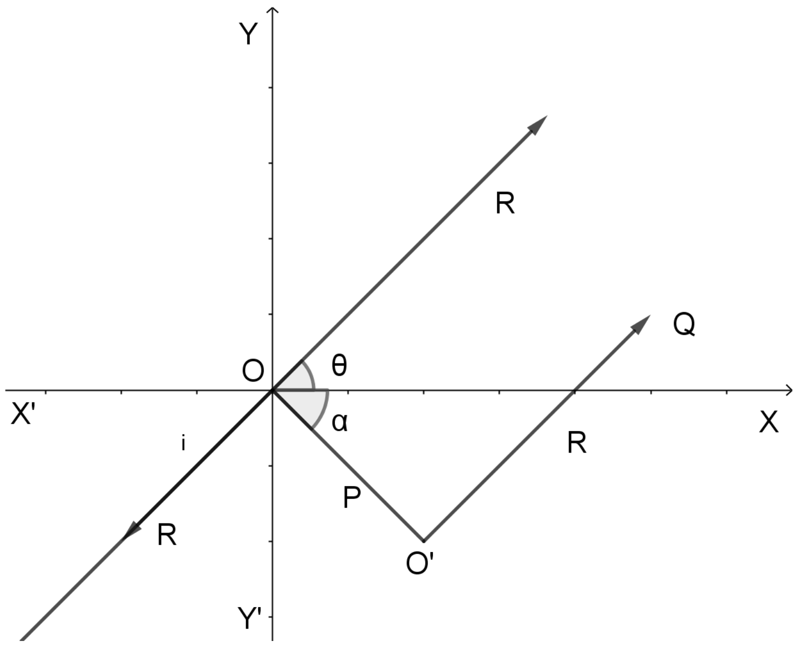
Thus,

$$R = \sqrt{X^2 + Y^2} \qquad \text{Where, } X = \sum X_i \text{ and } Y = \sum Y_i$$
$$\theta = \tan^{-1}\frac{Y}{X}$$
$$G = \sum (x_iY_i - X_iy_i)$$

□

2.2 Equation to the resultant of a system of forces in one plane

We have studied that the system of coplanar forces can be reduced to a single force R through O inclined at an angle θ with OX and a couple of moments G about O.



Now replace the couple G by two equal and unlike parallel forces one at O opposite to the direction of R and other at O' perpendicular to OO' so that,

$$R.OO' = G \Rightarrow OO' = \frac{G}{R} \Rightarrow P = \frac{G}{R}$$

Two forces at O are equal and opposite and they cancel each other.

Another force R at O' along $O'Q$ where $OO' = P$ is left whose equation is to be determined.

Let $x\cos\alpha + y\sin\alpha = p$ be the equation of the line OQ', where OO' makes angle α with OX and $OO' = p = \frac{G}{R}$.
Now, $\alpha = -\angle O'OX = -(\frac{\pi}{2} - \theta)$

Hence the equation of $O'Q$ is,

$$
\begin{aligned}
& x\cos\left\{-\left(\frac{\pi}{2}\right)-\theta\right\}+y\sin\left\{-\left(\frac{\pi}{2}\right)-\theta\right\}=\frac{G}{R}\\
\Rightarrow\ & x\sin\theta-y\cos\theta=\frac{G}{R}\\
\Rightarrow\ & x\,R\,\sin\theta-y\cos\theta=G\\
\Rightarrow\ & xY-yX=G \qquad\qquad [\because X=R\cos\theta \text{ and } Y=R\sin\theta]\\
\therefore\ & G-xY+yX=0
\end{aligned}
$$

2.3 Condition of equilibrium of a system of coplanar forces

We have,

$$R=\sqrt{X^2+Y^2} \text{ and } G=\sum(x_iY_i-y_iX_i)$$

For equilibrium,

$$R=0 \quad \Rightarrow X=0, Y=0 \text{ and } G=0$$

The conditions of equilibrium can be stated as:

(i) The algebraic sum of the resolved parts of all the forces along OX should be zero.

(ii) The algebraic sum of the resolved parts of all the forces along OY should be zero.

(iii) The algebraic sum of the moments of all the forces about O should be zero.

Hence the necessary and sufficient conditions for equilibrium are $X=0$, $Y=0$, $G=0$. These are known as the general conditions of equilibrium of forces.

2.4 A static equilibrium

A body is acted on by a system of coplanar forces, which keep it at rest. If all the forces are turned through the same angle about their points of application and if the equilibrium is not disturbed by such a rotation, then the equilibrium is said to be static.

2.5 Theorems

Theorem 2.5.1. *If all the forces in a coplanar system are rotated about their angle in their own plane, their resultant passes through an fixed point in the body.*

Proof. Let $P_1, P_2, P_3, \ldots$ be the system of forces acting on the body at the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ and make angle $\theta_1, \theta_2, \theta_3, \ldots$ with x axis. Let X, Y be the components of the system along

OX, OY and G be the moment of the forces about the origin.
Now we have,

$$\left.\begin{aligned} X &= \sum P_1 \cos \theta_1 \\ Y &= \sum P_1 \sin \theta_1 \\ G &= \sum P_1(x_1 \sin \theta_1 - y_1 \cos \theta_1) \end{aligned}\right\} \tag{2.1}$$

The equation of the resultant is

$$G - xY + Xy = 0 \tag{2.2}$$

Let the forces be rotated through angle α, about their points of applications, so that, $\theta_1, \theta_2, \ldots$ becomes $\theta_1 + \alpha, \theta_2 + \alpha, \ldots$. Let X', Y' and G' be the new system at the origin. Then,

$$\begin{aligned} X' &= \sum P_1 \cos(\theta_1 + \alpha) \\ &= \sum (P_1 \cos \theta_1 \cos \alpha - P_1 \sin \theta_1 \sin \alpha) \\ &= \sum (P_1 \cos \theta_1) \cos \alpha - \sum (P_1 \sin \theta_1) \sin \alpha \\ &= X \cos \alpha - Y \sin \alpha \qquad [\text{by } 2.1] \end{aligned}$$

and,

$$\begin{aligned} Y' &= \sum P_1 \sin(\theta_1 + \alpha) \\ &= \sum (P_1 \sin \theta_1 \cos \alpha + P_1 \cos \theta_1 \sin \alpha) \\ &= \left(\sum P_1 \sin \theta_1\right) \cos \alpha + \left(\sum P_1 \cos \theta_1\right) \sin \alpha \\ &= Y \cos \alpha + X \sin \alpha \qquad [\text{by } 2.1] \end{aligned}$$

and,

$$\begin{aligned} G' &= \sum \left\{P_1 x_1 \sin(\theta_1 + \alpha) - P_1 y_1 \cos(\theta_1 + \alpha)\right\} \\ &= \cos \alpha \sum P_1(x_1 \sin \theta_1 - y_1 \cos \theta_1) + \sin \alpha \sum P_1(x_1 \cos \theta_1 + y_1 \sin \theta_1) \\ &= G \cos \alpha + V \sin \alpha \qquad [\text{by } 2.1] \end{aligned}$$

Where, $V = \sum P_1(x_1 \cos \theta_1 + y_1 \sin \theta_1)$ which is called *viral of the system.*

The equation of the new resultant of the system is

$G' - xY' + X'y = 0$
$\Rightarrow G \cos \alpha + V \sin \alpha - x(Y \cos \alpha + X \sin \alpha) + y(X \cos \alpha - Y \sin \alpha) = 0$ [Putting the value of G', X', Y']
$\Rightarrow \sin \alpha(xX + yY - V) + \cos \alpha(xY - yX - G) = 0$ (2.3)

For all values of α 2.3 passes through the intersection of $xX+yY-V=0$ and $xY-yX-G=0$.i.e. through the point

$$\left(\frac{GY-VX}{X^2+Y^2}, \frac{VY-GX}{X^2+Y^2}\right) \tag{2.4}$$

Note: $x\sin\alpha + y\cos\alpha = 0$ passes through the origin.

Since the coordinates are independent of α, hence it is a fixed point in the body. The point is called *Astatic centre.*

Suppose the forces are in equilibrium before displacement, then $X=0$, $Y=0$ and $G=0$.

Hence after displacement, the system will be in equilibrium if,

$$\begin{aligned} X' &= X\cos\alpha - Y\sin\alpha = 0 \\ Y' &= Y\sin\alpha - x\cos\alpha = 0 \\ G' &= G\cos\alpha + V\sin\alpha = 0 \end{aligned}$$

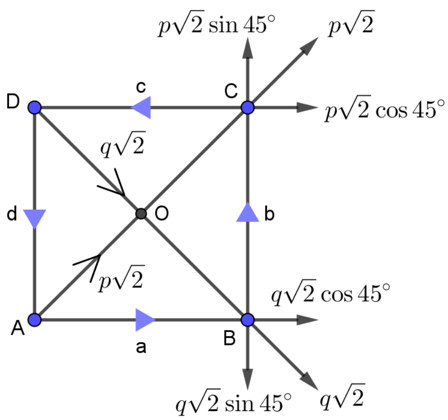
The equilibrium will be static if $V=0$.
Hence the equation for static equilibrium is $V=0$.

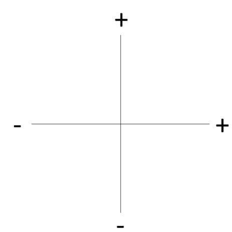
From the above discussion, we came to the conclusion that if the original system of forces are in equilibrium and each is turned through an angle α, they are equivalent to a couple whose moment is $V\sin\alpha$. Where V is a constant independent of α □

2.6 Problems

Problem 2.6.1. $ABCD$ is a square whose side is 2 units in length. Forces a, b, c, d act along the sides AB, BC, CD, DA taken in order and $p\sqrt{2}$, $q\sqrt{2}$ act along AC and DB respectively. Show that if $p+q=c-a$ and $p-q=d-b$, the forces are equivalent to a couple of moment $a+b+c+d$.

Solution. Resolve the forces along AB and AD,





$$
\begin{aligned}
X &= a - c + p\sqrt{2}\cos 45° + q\sqrt{2}\cos 45° \\
&= a - c + p + q \\
&= 0 \qquad [\therefore p+q=c-a] \\
Y &= b - d + p\sqrt{2}\sin 45° - q\sqrt{2}\sin 45° \\
&= b - d + p - q \\
&= 0 \qquad [\therefore p-q=d-b]
\end{aligned}
$$

Moment = Force×perp. dist. If the force passes through the point then no moment.

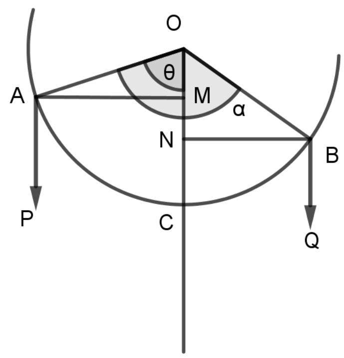
Thus, there is no single resultant force.
Take momentum about A,

$$
\begin{aligned}
&= b\,.\,2 + c\,.\,2 - q\sqrt{2}\,.\,\mathrm{OA} \\
&= 2b + 2c - q\sqrt{2}\,.\,2\cos 45° \\
&= 2(b+c) - 2q \qquad [\therefore p+q=c-a \text{ and } p-q=d-b] \\
&= a+b+c+d
\end{aligned}
$$

'+ve' in Anticlock wise and '-ve'in clock wise

Problem 2.6.2. A rigid wire without weight in the form of the arc of a circle subtending an angle α at its centre and having two weights P and Q its extremties rests with its convexity downwards upon a horizontal plane. Show that if θ be the inclination to the vertical of the radius to the end at which P is suspended then $\tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha}$

Solution. Let ABC be the wire subtend an angle $\alpha = \angle AOB$ at the centre O. OC is the vertical axis. OA makes an angle θ with the vertical OC. AM and BN are the perpendicular drawn from A and B with respect to OC. Weights P & Q are respectively acting at A & B.



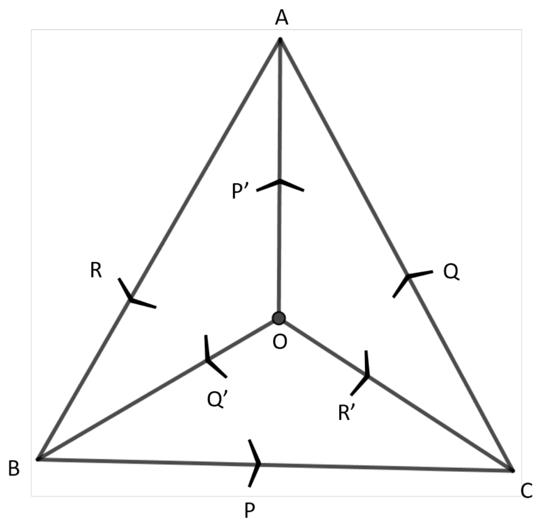
Taking moment about O,

$$
\begin{aligned}
& P.AM - Q.BN = 0 \\
\Rightarrow\, & P.OA\sin\theta - Q.OB\sin(\alpha-\theta) = 0 \\
\Rightarrow\, & P\sin\theta = Q\sin\alpha\cos\theta - Q\cos\alpha\sin\theta \quad [\because OA = OB] \\
\Rightarrow\, & P\tan\theta = Q\sin\alpha - Q\cos\alpha\tan\theta \\
\Rightarrow\, & \tan\theta = \frac{Q\sin\alpha}{P+Q\cos\alpha}
\end{aligned}
$$

Problem 2.6.3. Forces P, Q, R act along the sides BC, CA, AB of the triangle ABC and forces P', Q', R' act along OA, OB, OC where O is the circumcenter in the sense indicated by the order of the letters. If the six forces are in equilibrium, show that

$$\frac{PP'}{a} + \frac{QQ'}{b} + \frac{RR'}{c} = 0$$

Solution. Let the area of $\Delta ABC = \Delta$



Consider d_1 = length of the perpendicular A to BC, then,

$$\Delta = \frac{1}{2} BC.d_1$$
$$\Rightarrow d_1 = \frac{2\Delta}{BC}$$
$$\Rightarrow d_1 = \frac{2\Delta}{a}$$

Similarly, $d_2 = \frac{2\Delta}{b}$, $d_3 = \frac{2\Delta}{c}$ where d_2 and d_3 are perpendicular from B and C to AC and AB respectively.
Again, length of perpendicular from A to CO is $P_1 = \frac{2\Delta AOC}{OC}$
Similarly, length of perpendicular from A to BO is $P_2 = \frac{2\Delta BOA}{OB}$
Similarly, length of perpendicular from B to OA and B to OC are

$$P_3 = \frac{2\Delta AOB}{OA} \quad \text{and } P_4 = \frac{2\Delta BOC}{OC}$$

and, length of perpendicular from C to OB and C to AO are

$$P_5 = \frac{2\Delta BOC}{OB} \quad \text{and } P_6 = \frac{2\Delta AOC}{OA}$$

Now, taking the moments of the forces about A,

$$\begin{aligned} &P.\frac{2\Delta}{a} + R'.P_1 - Q'.P_2 = 0 \\ \Rightarrow &P.\frac{2\Delta}{a} + R'.\frac{2\Delta AOC}{OC} - Q'.\frac{2\Delta BOA}{OB} = 0 \end{aligned}$$

Again, taking moment about B and C separately,

$$Q.\frac{2\Delta}{b} + P'.\frac{2\Delta AOB}{OA} - R'\frac{2\Delta BOC}{OC} = 0$$

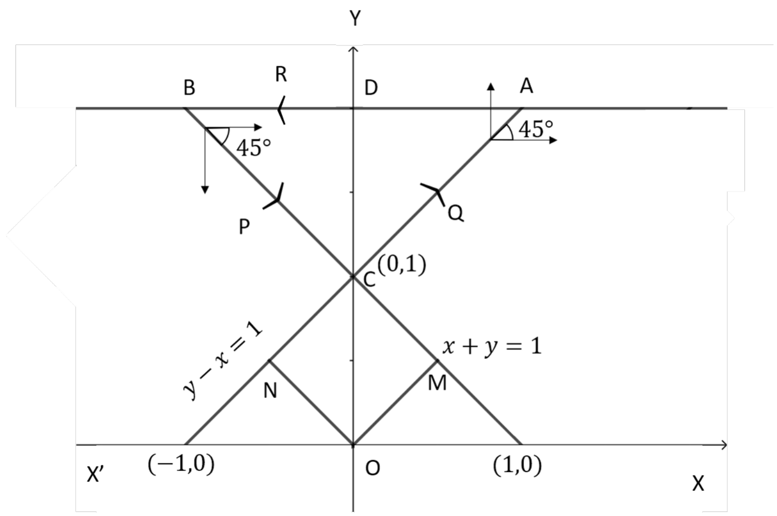
$$\text{and } R.\frac{2\Delta}{c} + Q'\frac{2\Delta BOC}{OB} - P.\frac{2\Delta AOC}{OA} = 0$$

Since, $OA = OB = OC =$ radius of the circumcircle. Multiply the equations by P', Q', R' respectively and adding we get,

$$\begin{aligned} &PP'.\frac{2\Delta}{a} + QQ'\frac{2\Delta}{b} + RR'\frac{2\Delta}{c} = 0 \\ \Rightarrow &\frac{PP'}{a} + \frac{QQ'}{b} + \frac{RR'}{c} = 0 \end{aligned}$$

Problem 2.6.4. Three forces P, Q, R act along the sides of a triangle formed by the lines $x + y = 1$, $y - x = 1$ and $y = 2$. Find the equation of the line of the action of the resultant.

Solution. The line of action of forces are shown in the figure.



Resolve the forces along and perpendicular to OX

$$
\begin{aligned}
X &= P.\cos 45^\circ + Q\cos 45^\circ + R\cos 180^\circ \\
&= (P + Q - R\sqrt{2})/\sqrt{2} \\
Y &= -P\sin 45^\circ + Q\sin 45^\circ + R\sin 180^\circ \\
&= -(P - Q)/\sqrt{2}
\end{aligned}
$$

Moments about O,

$$
\begin{aligned}
G &= -P.OM - Q.ON + R.OD \\
&= -P(1\sin 45^\circ) - Q(1\sin 45^\circ) + R(2) \\
&= -(P + Q - 2\sqrt{2}R)/\sqrt{2}
\end{aligned}
$$

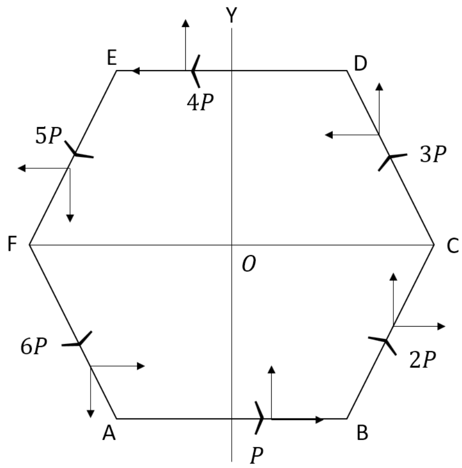
The equation of the line action of the resultant is,

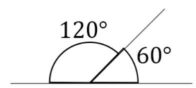
$$
\begin{aligned}
&G + yX - xY = 0 \\
\Rightarrow &\frac{(-P - Q + 2\sqrt{2}R)}{\sqrt{2}} + \frac{x(P - Q)}{\sqrt{2}} + \frac{y(P - Q - R\sqrt{2})}{\sqrt{2}} = 0 \\
\Rightarrow &P(x + y - 1) + Q(y - x - 1) - R(y - 2)\sqrt{2} = 0
\end{aligned}
$$

This is the required equation.

Problem 2.6.5. If six forces of relative magnitudes $1, 2, 3, 4, 5$ and 6 act along the side of a regular hexagon taken in order. Show that the single equivalent force is of relative magnitude 6 and that is acts along a line parallel to the force 5 at a distance from the centre of the hexagon $3\frac{1}{2}$ times the distance of side from the centre.

Solution. Let the horizontal and vertical through O be taken as the axes of coordinates and bc be the perpendicular distance of O from each side.





Resolve the forces along and perpendicular to OC

$$
\begin{aligned}
X &= P + 2P\cos 60^\circ - 3P\cos 60^\circ - 4P - 5P\cos 60^\circ + 6P\cos 60^\circ \\
&= P + P - \frac{3P}{2} - 4P - \frac{5P}{2} + 3P \\
&= -3P \\
Y &= P\sin 60^\circ + 2P\sin 60^\circ + 3P\sin 60^\circ + 4P\sin 180^\circ - 5P\sin 60^\circ - 6P\sin 60^\circ \\
&= \sqrt{3}P + \frac{3\sqrt{3}P}{2} - \frac{5\sqrt{3}P}{2} - 3\sqrt{3}P \\
&= -3\sqrt{3}P
\end{aligned}
$$

Now,

$$
\begin{aligned}
R &= \sqrt{X^2 + Y^2} \\
&= \sqrt{9P^2 + 21P^2} \\
&= 6p
\end{aligned}
$$

and,

$$\tan\theta = \frac{Y}{X}$$
$$\Rightarrow \tan\theta = \frac{-3\sqrt{3}P}{-3P}$$
$$\Rightarrow \tan\theta = \sqrt{3}$$
$$\Rightarrow \tan\theta = \tan 60^\circ$$
$$\Rightarrow \theta = 60^\circ$$

i.e., the resultant is parallel to the force $5P$.
Now the sum of the moments of the forces about O,

$$\begin{aligned} G &= P.d + 2P.d + 3P.d + 4P.d + 5P.d + 6P.d \\ &= 21P.d \end{aligned}$$

Now, the equation of resultant is,

$$\begin{aligned} &G + yX - xY = 0 \\ \Rightarrow &21Pd - 3Py + 3\sqrt{3}Px = 0 \\ \Rightarrow &\sqrt{3}x - y + 7d = 0 \end{aligned}$$

Distance of the line acting of the resultant from $O(0,0)$ is

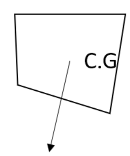
$$P = \frac{0 - 0 + 7d}{\sqrt{(\sqrt{3})^2 + 1^2}} = \frac{7}{2}d = 3\frac{1}{2}d$$

H.W. P.K Bhattacharjee, Ch-1: ex:1,2,3,4,6,8,9,11,15,17.

Chapter 3

Centre of Gravity

Definition 3.1 (center of Gravity)**.** The center of gravity of a body or system of particles rigidly connected together is that point through which the line of action of the weight of the body always passes.



3.1 General Formula for the determination of the center of gravity(2D)

3.1.1 For a scattered system of particle

Let $w_1, w_2, \ldots, w_n$ be the weights of a system of particles scattered on a plane situated at points $A_1, A_2, \ldots, A_n$ respectively with coordinates $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Let (x', y') be the coordinates of their center of gravity G.

Now the weights of all particles are parallel to forces acting vertically downwards, hence their resultant force is a single force of magnitude $W = w_1 + w_2 + \cdots + w_n$ acting at G also vertically downwards.

Taking moment of forces about OY axis then,

$$\begin{aligned} x'W &= w_1x_1 + w_2x_2 + \cdots + w_nx_n \\ \Rightarrow x'W &= \sum w_ix_i; \quad i = 1, 2, \ldots, n \\ \Rightarrow x' &= \frac{\sum w_ix_i}{W} \\ \therefore x' &= \frac{\sum w_ix_i}{\sum w_i}; \quad i = 1, 2, \ldots, n \end{aligned}$$

Similarly by taking the moment of the forces about OX axis, we have,

$$\begin{aligned} y'W &= w_1y_1 + w_2y_2 + \cdots + w_ny_n \\ \therefore y' &= \frac{\sum w_iy_i}{\sum w_i}; \quad i = 1, 2, \ldots, n \end{aligned}$$

Remark. For a scattered system of masses $m_1, m_2, \ldots, m_n$ at $A_1, A_2, \ldots, A_n$. Since the weights of the particles are proportional to their corresponding masses then $G(x', y')$ is,

$$x' = \frac{\sum m_ix_i}{\sum m_i} \quad y' = \frac{\sum m_iy_i}{\sum m_i}$$

If we consider a continuous distribution of matter in the form of a rigid lamina of any shape, the coordinate center of gravity is ,

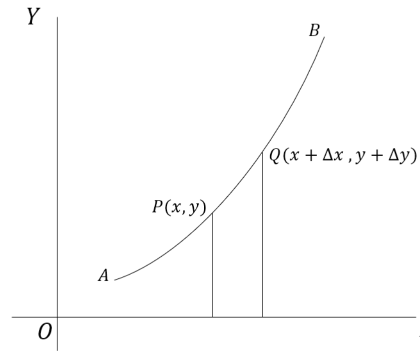
$$x' = \frac{\sum x\delta w}{\delta w} \quad y' = \frac{\sum y\delta w}{\delta w}$$

When δw is the weight of a element of lamina with coordinates at (x, y). If we subdivide the lamina into indefinitely large number of such ...., then δw becomes an indefinitesimal and in the limit we have,

$$\left.\begin{aligned} x' &= \frac{\int x\,\mathrm{d}w}{\int \mathrm{d}w}, \quad y' = \frac{\int y\,\mathrm{d}w}{\int \mathrm{d}w} \\ \text{or, } x' &= \frac{\int x\,\mathrm{d}m}{\int \mathrm{d}m}, \quad y' = \frac{\int y\,\mathrm{d}m}{\int \mathrm{d}m} \end{aligned}\right\} \tag{3.1}$$

where the integration is such as to include the whole body.

3.1.2 Center Of Gravity of an Arc



Let, $AP = s$, $AQ = s + \Delta s$, $PQ = \Delta s$, and $\rho =$ density of the arc at P. Then $\mathrm{d}m = \rho\,\mathrm{d}s$.
$P(x, y)$, $Q(x + \Delta x, y + \Delta y)$ and center of gravity $G(x', y')$ then,

$$x' = \frac{\int x\rho\,\mathrm{d}s}{\int \rho\,\mathrm{d}s}, \quad y' = \frac{\int y\rho\,\mathrm{d}s}{\int \rho\,\mathrm{d}s} \tag{3.2}$$

The limit of integration extending from one end to the other of the arc considered.
If the density is uniform then,

$$\left.\begin{aligned} x' &= \frac{\int x\,\mathrm{d}s}{\int \mathrm{d}s} = \frac{\int x\frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x}{\int \frac{\mathrm{d}s}{\mathrm{d}x}\,\mathrm{d}x} \\ y' &= \frac{\int y\,\mathrm{d}s}{\int \mathrm{d}s} = \frac{\int y\frac{\mathrm{d}s}{\mathrm{d}y}\,\mathrm{d}y}{\int \frac{\mathrm{d}s}{\mathrm{d}y}\,\mathrm{d}y} \end{aligned}\right\} \tag{3.3}$$

Here, $\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$, $\frac{\mathrm{d}s}{\mathrm{d}y} = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}$

In polar coordinates, say, $r = f(\theta)$
then, $\Delta s = \sqrt{\Delta r^2 + r^2(\Delta\theta)^2}$ and $x = r\cos\theta$, $y = r\sin\theta$.
Then we have,

$$\left.\begin{aligned} x' &= \frac{\int r\cos\theta.\rho\,\mathrm{d}s}{\int \mathrm{d}s} \\ y' &= \frac{\int r\sin\theta.\rho\,\mathrm{d}s}{\int \mathrm{d}s} \end{aligned}\right\} \tag{3.4}$$

3.1.3 Center of Gravity of a Plane

If $\mathrm{d}A$ be an elementary area whose center of gravity is at $P(x, y)$ and ρ is the mass per unit area at P_1 then,

$$x' = \frac{\int \rho x\,\mathrm{d}A}{\int \rho\,\mathrm{d}A}, \quad y' = \frac{\int \rho y\,\mathrm{d}A}{\int \rho\,\mathrm{d}A}$$

In cartesian coordinate, $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, then,

$$x' = \frac{\iint \rho x\,\mathrm{d}x\,\mathrm{d}y}{\iint \rho\,\mathrm{d}x\,\mathrm{d}y}, \quad y' = \frac{\iint \rho y\,\mathrm{d}x\,\mathrm{d}y}{\iint \rho\,\mathrm{d}x\,\mathrm{d}y} \tag{3.5}$$

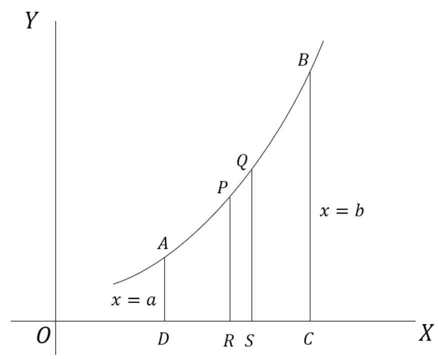
In polar coordinate,

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

then,

$$\left.\begin{aligned} x' &= \frac{\iint \rho\, r\cos\theta r\,\mathrm{d}r\,\mathrm{d}\theta}{\iint \rho r\,\mathrm{d}r\,\mathrm{d}\theta} = \frac{\iint \rho\, r^2\ \cos\theta\,\mathrm{d}r\,\mathrm{d}\theta}{\iint \rho r\,\mathrm{d}r\,\mathrm{d}\theta} \\ y' &= \frac{\iint \rho\, r\sin\theta r\,\mathrm{d}r\,\mathrm{d}\theta}{\iint \rho r\,\mathrm{d}r\,\mathrm{d}\theta} = \frac{\iint \rho\, r^2\ \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta}{\iint \rho r\,\mathrm{d}r\,\mathrm{d}\theta} \end{aligned}\right\} \tag{3.6}$$

3.1.4 Center of Gravity of area bounded by the curve $y = f(x)$, $x-$axis and coordinates $x = a$ and $x = b$, when density is uniform



$$x' = \frac{\int_a^b xy\,\mathrm{d}x}{\int_a^b y\,\mathrm{d}x}, \quad y' = \frac{1}{2}\frac{\int_a^b y^2\,\mathrm{d}x}{\int_a^b y\,\mathrm{d}x} \tag{3.7}$$

Remark. If area is symmetric with respect to x-axis, the center of gravity is at $(x', 0)$, and for symmetric about $y-$axis center of gravity is at $(0, y')$.

3.1.5 Center of Gravity of the area bounded between the curve $r = f(\theta)$ and two radii vector $\theta = \alpha$, $\theta = \beta$, ρ is the density of the uniform curve

Chapter 4

Moments of Inertia

Part II

Dynamics

Chapter 5

Kinematics In Two Dimensions

5.1 Displacement

When a point changes its position and occupies different positions at different times relative to an object which we consider to be fixed, then the point is said to be in motion and is displaced. The curve drawn through the successive positions of the particle is called its *path.*

Definition 5.1 (Displacement)**.** Let us take a particle at a point P and later on it shifts to another point Q. We say that P is displaced from P to Q and PQ is called displacement.

We should note here that PQ has a magnitude as well as direction $P \to Q$ and hence it is a vector quantity. The particle in reaching Q may travel along a straight line or a curved path. We shall first confine our attention to displacement in a straight line.

5.2 Velocity

Consider a point O taken to be fixed and relative to which we shall consider the motion of a particle which moves along the straight line OX. The position P of the particle at any instant is the defined by $\overrightarrow{OP}$. At any subsequent instant let it be at Q; then PQ is the displacement during that interval of time. If the time required to move a distance PQ along the line be t then $\frac{PQ}{t}$ measures the average velocity of the particle during the interval t. We thus have *average velocity* $= \frac{displacement}{time}$ or we may define velocity as the rate of displacement. If the particle moves at a uniform rate (i.e. it covers equal distance in equal times) the velocity is uniform or it is said to be moving with uniform velocity. Here again we should note that it is a vector quantity because the rate of displacement in a given direction is termed as velocity in that direction.

Two velocities may have same magnitude but different directions and hence they will not be called equal. Whenever we may say that a velocity is changing it may mean that there is only change in its magnitude or only in its direction or change in both.

Let again the particle be at P at a distance x from the fixed point O and let it be displaced to a point P' such that $PP' = \delta x$ during an interval of time δt when δt is small, then $\delta x/\delta t$ is the average velocity during this interval. The particle may not be moving at a uniform rate during that interval and hence we have called it average velocity. As we decrease the interval δt and make it smaller and smaller, velocity of the particle during the interval differs from its velocity at P by a smaller and smaller amount. Ultimately if we think that $\delta t \to 0$ so also δx (because δx depends on δt whatever the velocity may be), the velocity during the infinitesimal interval δt will be the same as that of the particle when at P.

Hence $v = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{\mathrm{d}x}{\mathrm{d}t}$ along x increasing.[1]
Speed is a scaler quantity or it is only the magnitude of velocity while velocity has a direction too.

5.3 Acceleration

When the velocity is not uniform it may increase or decrease.

Definition 5.2 (Acceleration)**.** The rate of change of velocity os defined as acceleration.

If v be the velocity of the particle at certain instant t and $v + \delta v$ be its velocity after small subsequent interval δt i.e. at time $t + \delta t$, then the acetylation f is given by

$$\begin{aligned} f &= \lim_{\delta t \to 0} \frac{\text{change in velocity duting the small interval } \delta t}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{(v + \delta v) - v}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} \\ &= \frac{\mathrm{d}v}{\mathrm{d}t} \end{aligned}$$

Thus acceleration is given by $\mathrm{d}v/\mathrm{d}t$; but we have, $v = \frac{\mathrm{d}x}{\mathrm{d}t}$;

$$\therefore f = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}$$

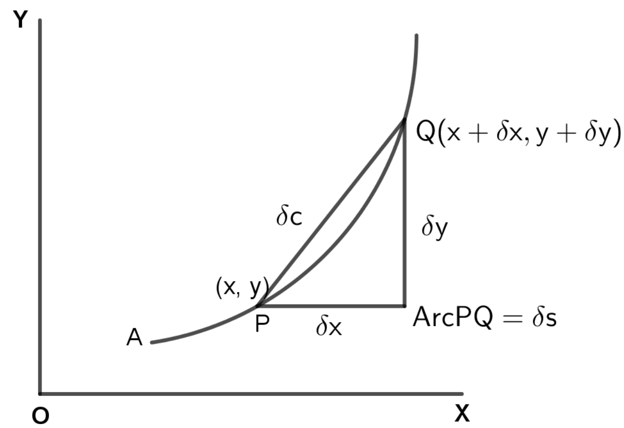
Again

$$f = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

[1] Usually differential coefficient of variables with regard to time are denoted by putting dots above them, e.g. $\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$, $\frac{\mathrm{d}u}{\mathrm{d}t} = \dot{u}$, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}$ etc.

5.4 Motion in a Plane Curve

5.4.1 Cartesian System



Taking the first case, let O be the origin (a fixed point), OX and OY be two axes perpendicular to one another.Let the particle be at a point $P\,(x, y)$ at a given instant. Let it then move to a point $Q\,(x + \delta x, y + \delta y)$ during a subsequent interval δt or at time $t + \delta t$. This suggests that during this interval the particle has gone to Q along a curved path and the displacements in the x and y directions respectively are δx and δy.

Let the path of the particle be APQ where A be a point from where we measure the distance along the arc. During the interval δt the displacement of the particle is given by the chord PQ and hence the velocity is given by $\frac{\text{chord } PQ}{\delta t}$ provided $\delta t \to 0$.

$$\therefore \text{ Velocity at } P \text{ at time } t = \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta t} = \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t}$$

Where δs represents the arcual distance PQ along the curved path, s being measured from A. As $\delta t \to 0$ we have δs also small and in limit chord PQ and δs become equal.

$$\therefore \text{ Velocity } = \lim_{\delta t \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta t} = 1 \times \frac{\delta s}{\delta t} = \frac{\mathrm{d}s}{\mathrm{d}t}$$

As Q approaches P the chord PQ becomes the tangent to the curve at P and also in the direction in which s is increasing.
Hence velocity at P is given by $\frac{\mathrm{d}s}{\mathrm{d}t}$ and its direction is along the tangent to the curve at P int the sense of s increasing.
As given above the displacement $\overrightarrow{PQ}$ is made up of two components, one δx along the direction of x-axis and the other δy along the direction of y-axis.
The component of velocity along the x-axis,

$$\lim_{\delta t \to 0} \frac{\text{displacement in direction of } x\text{-axis}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$$

Similarly the component of velocity along the y-axis= $\frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y}$

Hence if the co-ordinates of a particle be $(x,\, y)$ then components of its velocity parallel to x and y axes are given by $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ respectively.

Since the two components of velocity are along perpendicular direction, therefore resultant velocity v is given by,

$$v^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \text{ or } v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

and the direction is given by $\tan\psi = \frac{\dot{y}}{\dot{x}}$ where ψ is the angle which the direction of resultant velocity makes with x-axis.

5.5 Components of Acceleration of a Particle in a Plane

Let u and v be the components of velocity of the particle parallel to the axes when the particle is at P at time t and $u + \delta u$, $v + \delta v$ be those at time $t + \delta t$ when the particle is at Q. The changes in velocity during the interval δt in the direction of axes are then δu and δv. Hence acceleration along $x-$axis,

$$\lim_{\delta t \to 0} \frac{\text{change of velocity in direction of } x\text{-axis}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \ddot{x} \text{ or } u\frac{\mathrm{d}u}{\mathrm{d}x}$$

Similarly component of acceleration along the $y-$ axis

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \ddot{y} \text{ or } v\frac{\mathrm{d}v}{\mathrm{d}y}$$

and the resultant acceleration $f = \sqrt{\ddot{x}^2 + \ddot{y}^2}$
and the resultant will make an angle $\tan^{-1}\frac{\ddot{y}}{\ddot{x}}$, with the $x-$axis.

5.6 Problems

Problem 5.6.1. A particle moves along a straight line such that its displacement x from a fixed point on the line at time t, is given by $x = t^3 - 9t^2 + 24t + 6$. determine the following.

(i) The instant when the acceleration becomes zero.

(ii) The position of the particle at that instant.

(iii) The velocity of the particle at that instant.

Solution. Given,

$$x = t^3 - 9t^2 + 24t + 6$$

Now,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 18t + 24$$
$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 6t - 18$$

so,

$$
\begin{aligned}
6t - 18 &= 0 \\
\Rightarrow t &= 3
\end{aligned}
$$

$\therefore$ the acceleration becomes zero when t is 3.
At $t = 3$ the position of the particle is,

$$
\begin{aligned}
x &= (3)^3 - 9(3)^2 + 24(3) + 6 \\
\Rightarrow x &= 24
\end{aligned}
$$

At $t = 3$ the velocity of the particle is,

$$
\begin{aligned}
v &= 3(3)^2 - 18(3) + 24 \\
\Rightarrow v &= -3
\end{aligned}
$$