Fuzzy Topology

MAT514

PROF. DR SHAMSUN NAHER BEGUM

Shahjalal University of Science and Technology

EDITED BY
MEHEDI HASAN

Preface

This is a compilation of lecture notes with some books and my own thoughts. If there are any mistake/typing error or, for any query mail me at mehedi12@student.sust.edu.

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Part I

Sheet

Chapter 1

Fuzzy Sets

Definition 1 (Characteristic function)**.** Let X be a universal set and $A \subseteq X$. Then the function[1]

$$\chi_A(x) = \begin{cases} 1; & x \in A \\ 0; & x \notin A \end{cases}$$

is characteristic function of A in X.

Definition 2 (Fuzzy Set)**.** A fuzzy set[2] $A \subseteq X$ is a mapping $A : X \to [0,1]$, where, $A(x) = y \in [0,1]$ is called the membership function or, grade of membership of x in A. The collection of all fuzzy sets of X is denoted by $\mathcal{F}(X)$.

Definition 3 (Fuzzy subset)**.** A fuzzy set A is called a fuzzy subset of another fuzzy set B if $A(x) \leq B(x)$ $\forall x \in X$. We denote it by $A \leq B$.

Definition 4 (Empty fuzzy set)**.** A fuzzy set A is called empty fuzzy set if $\forall x \in X$ $A(x) = 0$. The empty fuzzy set is denoted by $\underline{0}$. Thus, $\underline{0}(x) = 0$ $\forall x \in X$.

Definition 5 (Total fuzzy set)**.** The total fuzzy set $\underline{1}$ is defined by $\underline{1}(x) = 1$ $\forall x \in X$.

Definition 6 (Equality of two fuzzy sets)**.** Two fuzzy sets A and B of X is said to be equal iff $A \leq B$ and $B \leq A$.

Example (Empty and Total fuzzy set)**.** Suppose, $A : X \to [0,1]$ where $X = [20, 90]$. Then,

$$\underline{0}(x) = \begin{cases} 0 & \text{if } 15 < x < 90 \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \underline{1}(x) = \begin{cases} 1 & \text{if } 20 \leq x < 90 \\ 0 & \text{otherwise} \end{cases}$$

Example (Fuzzy subset)**.** Suppose, $A : X \to [0,1]$ where, $X = [0, 100]$ defined by

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \frac{x}{75}; & \text{if } 40 \leq x < 75 \\ 1; & \text{if } 75 \leq x \leq 100 \end{cases}$$

and $B : X = [0, 100] \to [0,1]$ defined by

$$B(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \frac{x}{95}; & \text{if } 40 \leq x < 95 \\ 1; & \text{if } 95 \leq x \leq 100 \end{cases}$$

Then, $B(x)$ is a subset of $A(x)$. Since, $B(x) \leq A(x)$ $\forall x \in X$.

[1] Some authors use μ as characteristic function.
[2] Sometimes fuzzy set is denoted by $\underset{\sim}{A}$.

1.1 Fuzzy Set Operations

Definition 7 (Union of Fuzzy Sets)**.** Let $A,\ B \in \mathcal{F}(X)$. Then the union of A and B is denoted and defined by, $(A \vee B)(x) = \max\{A(x),\ B(x)\}$, $\forall x \in X$.

Definition 8 (Intersection of Fuzzy Sets)**.** Let $A,\ B \in \mathcal{F}(X)$. Then the intersection of A and B is denoted and defined by, $(A \wedge B)(x) = \min\{A(x),\ B(x)\}$, $\forall x \in X$.

Definition 9 (Complement of Fuzzy Set)**.** Let A be a fuzzy set of X. Then, the complement of A is denoted by A^c and defined by $A^c(x) = 1 - A(x)$, $\forall x \in X$.

Example. Given,

$$A_1 = \begin{cases} 1; & \text{if } 40 \leq x < 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases} \quad \text{and} \quad A_2 = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 0; & \text{if } 70 \leq x \leq 100 \end{cases}$$

1. Find the complement of A_1 and A_2.
2. Find $(A_1 \wedge A_2)(x)$ and $(A_1 \vee A_2)(x)$

<u>Solution:</u>

1. Complement of A_1,

$$A_1{}^c = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 1; & \text{if } 60 \leq x \leq 100 \end{cases}$$

Complement of A_2,

$$A_2{}^c = \begin{cases} 1; & \text{if } 40 \leq x < 50 \\ \dfrac{60-x}{10}; & \text{if } 50 \leq x < 60 \\ \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 1; & \text{if } 70 \leq x \leq 100 \end{cases}$$

2.

$$(A_1 \wedge A_2)(x) = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ 1 - \dfrac{x-50}{10}; & \text{if } 55 \leq x \leq 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases}$$

$$(A_1 \vee A_2)(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ \dfrac{x-50}{10}; & \text{if } 55 \leq x < 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 0; & \text{if } 70 \leq x < 100 \end{cases}$$

Definition 10 (Level Set)**.** Let $A : X \to [0,1]$ be a fuzzy set. The α level set of A is denoted and defined by, A_α or $\alpha_A = \{x \in X \mid A(x) \geq \alpha\}$ where, $0 < \alpha \leq 1$.

Remark. A_α is a classical set not a fuzzy set.

Definition 11 (Core level of a fuzzy set)**.** When $\alpha = 1$, then $A_1 = \{x \in X \mid A(x) = 1\}$ is called the core level of A.

Definition 12 (Support of a fuzzy set)**.** Support of a fuzzy set A is denoted and defined by,$S_A = \{x \in X \mid A(x) > 0\}$.

Example. Given,

$$A = \begin{cases} 0; & \text{if } x \leq 20 \text{ or, } x \geq 60 \\ \dfrac{x-20}{15}; & \text{if } 20 < x < 35 \\ \dfrac{60-x}{15}; & \text{if } 45 < x < 60 \\ 1; & \text{if } 35 \leq x \leq 45 \end{cases} \quad \text{and} \quad B = \begin{cases} 0; & \text{if } x \leq 45 \\ \dfrac{x-45}{15}; & \text{if } 45 < x < 60 \\ 1; & \text{if } x \geq 60 \end{cases}$$

1. (a) Core level of A?
   (b) Support of A?
   (c) Half level of A?
   (d) $\frac{3}{4}$ level of A?
2. (a) Core level of B?
   (b) Support of B?
   (c) Half level of B?

Solution.

1. (a) Core level of A is $A_1 = \{x \in X \mid 35 \leq x \leq 45\}$.
   (b) Support level of A is $S_A = \{x \in X \mid 20 < x < 60\}$.
   (c) Half level of A is $A_{\frac{1}{2}} = \{x \in X \mid 27.5 \leq x \leq 52.5\}$.
   (d) $\frac{3}{4}$ level of A is $A_{\frac{3}{4}} = \{x \in X \mid 31.25 \leq x \leq 48.75\}$.
2. (a) Core level of B is $B_1 = \{x \in X \mid x \geq 60\}$.
   (b) Support level of B is $S_B = \{x \in X \mid x > 45\}$.
   (c) Half level of B is $B_{\frac{1}{2}} = \{x \in X \mid x \geq 52.5\}$.

Example. $A : X \to [0, 1]$ defined by

$$A(x) = \begin{cases} 1; & \text{if } x \leq 20 \\ \dfrac{35-x}{20}; & \text{if } 20 \leq x < 35 \\ 0; & \text{if } x \geq 35 \end{cases}$$

Then find $\frac{1}{2}$ level of A.

Solution.

$$A_{\frac{1}{2}} = \{x \in X \mid x \leq 25\}$$

Problem 1.1. Consider, the two fuzzy sets $A, B : X = [0, 100] \to [0, 1]$ defined by

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \dfrac{x}{75}; & \text{if } 40 \leq x < 75 \\ 1; & \text{if } 75 \leq x < 100 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \dfrac{x}{95}; & \text{if } 40 \leq x < 95 \\ 1; & \text{if } 95 \leq x \leq 100 \end{cases}$$

Then find $(A \wedge B)(x)$ and $(A \vee B)(x)$.

Solution.

$$(A \wedge B)(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \dfrac{x}{95}; & \text{if } 40 \leq x < 95 \\ 1; & \text{if } 95 \leq x \leq 100 \end{cases} \quad \text{and} \quad (A \vee B)(x) = \begin{cases} 0; & \text{if } 0 \leq x \leq 40 \\ \dfrac{x}{75}; & \text{if } 40 \leq x < 75 \\ 1; & \text{if } 75 \leq x \leq 100 \end{cases}$$

Suppose, $X = \mathbb{R}$ and the fuzzy set of real numbers much greater than 5 in X, that could be defined by,

$$A(x) = \begin{cases} 0; & \text{if } x \leq 5 \\ \dfrac{x-5}{50}; & \text{if } 5 < x \leq 55 \\ 1; & \text{if } x \geq 55 \end{cases}$$

Example. Consider, the two fuzzy sets A and B of $\mathcal{F}(X)$, where $X = [0, 100]$

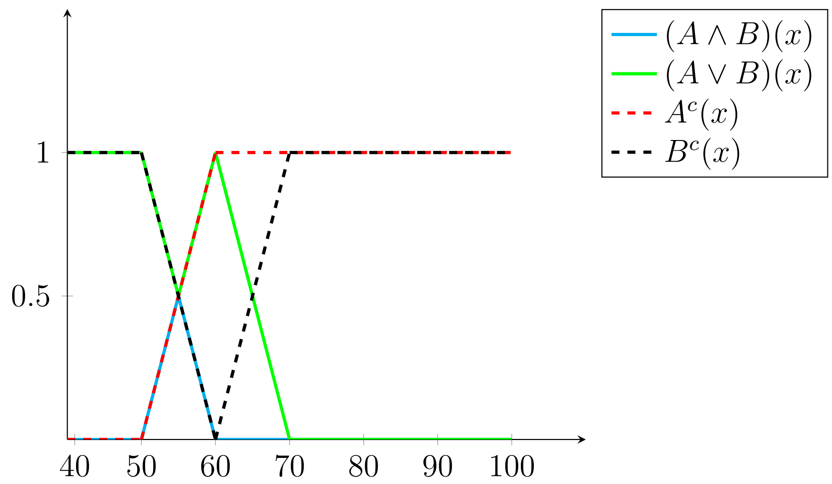
$$A(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0; & \text{if } 40 \leq x \leq 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x \leq 70 \\ 0; & \text{if } 70 \leq x \leq 100 \end{cases}$$

Draw $(A \vee B)(x)$, $(A \wedge B)(x)$, A', B'.

Solution. Here,

$$(A \vee B)(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ \dfrac{x-50}{10}; & \text{if } 55 \leq x \leq 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x \leq 70 \\ 0; & \text{if } 70 \leq x \leq 100 \end{cases} \quad \text{and} \quad (A \wedge B)(x) = \begin{cases} 0; & \text{if } 40 \leq x \leq 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ 1 - \dfrac{x-50}{10}; & \text{if } 55 \leq x \leq 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases}$$

$$A^c(x) = \begin{cases} 0; & \text{if } 40 \leq x \leq 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ 1; & \text{if } 60 \leq x \leq 100 \end{cases} \quad \text{and} \quad B^c(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ \dfrac{x-60}{10}; & \text{if } 60 \leq x \leq 70 \\ 1; & \text{if } 70 \leq x \leq 100 \end{cases}$$



1.2 Fuzzy Relation

Definition 13 (Fuzzy Relation)**.** Let X and Y be two non-empty classical(Fuzzy) sets. Then a fuzzy relation R on $X \times Y$ is a mapping, $R : X \times Y \to [0, 1]$ where, the number $R(x, y) \in [0, 1]$ is called the degree of relationship between x and y.

Example. Let $X = \{a, b, c\}$, $Y = \{c, d\}$. Then $X \times Y = \{(a, c), (a, d), (b, c), (b, d), (c, c), (c, d)\}$ where $R(a, c) = R(a, d) = 0$, $R(b, c) = R(b, d) = R(c, c) = 1$ and $R(c, d) = 0.8$. For the fuzzy relation:

1. Core of R?
2. Support of R?
3. 0.7 of R?

Solution.

1. Core of $R = \{(b, c), (b, d), (c, c)\}$ Since, $R(x, y) = 1$ for $x \in X$ and $y \in Y$.
2. Support of $R = \{(b, c), (b, d), (c, c), (c, d)\}$ Since, $R(x, y) > 0$ for $x \in X$ and $y \in Y$.
3. 0.7 of $R = \{(b, c), (b, d), (c, c), (c, d)\}$ Since, $R(x, y) > 0.7$ for $x \in X$ and $y \in Y$.

Definition 14 (Domain)**.** If $R(x, y)$ is a fuzzy relation, its domain is the fuzzy set *dom* $R(x, y)$ whose membership function is

$$\chi_{dom} R(x) = \max \chi_R(x, y) \forall x \in X$$

Definition 15 (Range)**.** If $R(x, y)$ is a fuzzy relation, its range is the fuzzy set *ran* $R(x, y)$ whose membership function is

$$\chi_{ran} R(y) = \max \chi_R(x, y) \forall y \in y$$

Example. Consider $X = \{x_1, x_2, x_3, x_4\}$ and

$$R(x, x) = \begin{pmatrix} 0.2 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0.0 & 0.4 & 0.0 \\ 0.0 & 0.6 & 0.0 & 0.1 \end{pmatrix}$$

Then the domain is *dom* $R = \{0.5, 0.8, 0.4, 0.6\}$ and the range is *ran* $R = \{0.2, 0.6, 0.7, 0.8\}$.

Definition 16 (Max-min and Min-max Composition)**.** Let R be a fuzzy relation on $X \times Y$ i.e., $R \in \mathcal{F}(X \times Y)$ and S be a fuzzy relation on $Y \times Z$ i.e., $S \in \mathcal{F}(Y \times Z)$. Then $R \circ S \in \mathcal{F}(X \times Z)$ defined by

$$(R \circ S)(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z)$$

is called the Max-Min composition of R and S on $X \times Z$. And

$$(R \circ S)(x, z) = \bigwedge_{y \in Y} R(x, y) \vee S(y, z)$$

is called the Min-Max composition of R and S on $X \times Z$

Problem 1.2. Consider, $X = \{a, b\}$, $Y = \{c, d, e\}$ and $Z = \{u, v\}$ where,

$$R(x, y) = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1.0 & 0.0 & 0.9 \end{pmatrix} \quad \text{and} \quad S(y, z) = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.0 \\ 0.5 & 0.6 \end{pmatrix}$$

then find the max-min and min-max composition of R and S.

Solution. Max-min composition of R and S

$$(R \circ S)(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z) = \begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}$$

Min-max composition of R and S

$$(R \circ S)(x, z) = \bigwedge_{y \in Y} R(x, y) \vee S(y, z) = \begin{pmatrix} 0.5 & 0.3 \\ 0.1 & 0.0 \end{pmatrix}$$

Definition 17 (Reflexive)**.** Let R be a fuzzy relation in $X \times X$. R is called reflexive if

$$\chi_R(x, x) = 1 \quad \forall x \in X$$

Definition 18 (Symmetic)**.** Let R be a fuzzy relation in $X \times X$. R is called symmetric if

$$R(x, y) = R(y, x) \quad \forall x, y \in X$$

Example. The following relation is a symmetric relation:

$$R(x, y) = \begin{pmatrix} 0.0 & 0.1 & 0.0 & 0.1 \\ 0.1 & 1.0 & 0.2 & 0.2 \\ 0.0 & 0.2 & 0.8 & 0.8 \\ 0.1 & 0.3 & 0.8 & 1.0 \end{pmatrix}$$

Definition 19 (Antisymmetic)**.** Let R be a fuzzy relation in $X \times X$. R is called antisymmetric if for

$$x \neq y \qquad \left.\begin{array}{r} \text{either } \chi_R(x, y) \neq \chi_R(y, x) \\ \text{or } \chi_R(x, y) = \chi_R(y, x) = 0 \end{array}\right\} \quad \forall x, y, \in X$$

Example. The following relation is an antisymmetric relation:

$$R(x, y) = \begin{pmatrix} 0.4 & 0.0 & 0.1 & 0.8 \\ 0.8 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.6 & 0.7 & 0.0 \\ 0.0 & 0.2 & 0.0 & 0.0 \end{pmatrix}$$

Definition 20. A fuzzy relation R is called (max-min) transitive if

$$R \circ R \subseteq R$$

Example. The fuzzy relation

$$R = \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}$$

is a transitive relation.
Here using a max-min composition,

$$R \circ R = \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \circ \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ 0.4 & 0.3 \end{pmatrix}$$

$\therefore$ 7; $R \circ R \subseteq R$. Hence, R is a transitive relation.

Definition 21. The max-min composition is called associative if

$$R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3$$

Chapter 2

Fuzzy Topology

Definition 22 (Fuzzy Topology)**.** Let X be a non-empty set. A collection δ of fuzzy sets on X is called the fuzzy topology on X if it satisfies the following conditions:

(i) $\underline{0}$, $\underline{1} \in \delta$.

(ii) If $A, B \in \delta$, then $A \wedge B \in \delta$.

(iii) If $A_i \in \delta$, then $\vee_{i \in I} A_i \in \delta$.

If δ is a topology on X then, $\langle \mathcal{F}(X), \delta \rangle$ is called a fuzzy topological space.

Example. Let $X = \{a, b\}$ and A be a fuzzy set defined by $A(a) = 0.5$ and $A(b) = 0.4$. Then $\delta = \{\underline{0}, \underline{1}, A\}$ be a fuzzy topology and $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space.

Example. Let A, B be a fuzzy sets of $I = [0, 1]$ defined as

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x \leq \dfrac{1}{2} \\ 2x - 1; & \text{if } \dfrac{1}{2} \leq x \leq 1 \end{cases} \qquad \text{and} \qquad B(x) = \begin{cases} 1; & \text{if } 0 \leq x \leq \dfrac{1}{4} \\ -4x + 2; & \text{if } \dfrac{1}{4} \leq x \leq \dfrac{1}{2} \\ 0; & \text{if } \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

Then $\delta = \{\underline{0}, \underline{1}, A, B, A \vee B\}$ is a fuzzy topology on I.

Definition 23 (Open and Closed Fuzzy Sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, the member of δ i.e., each $A \in \delta$ is called the fuzzy open set. A fuzzy set B is called a fuzzy closed set if $B^c \in \delta$.

Example. Let $X = \{a, b\}$, $B : X \to [0, 1]$ such that $B(a) = 0.5$, $B(b) = 0.6$. Then, $B^c(a) = 0.5$, $B^c(b) = 0.4$, $\delta = \{\underline{0}, \underline{1}, A\}$, $A(a) = 0.5$, $A(b) = 0.4$.
$\therefore$ B is closed under $\delta/\delta-$closed. i.e., B^c is open.

Difference between classical and fuzzy sets: Classical set contains elements that satisfy precise properties of membership while fuzzy set contains elements that satisfy imprecise properties of membership.

Definition 24 (Interior and Closure of fuzzy sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and A be a non-empty subset of X.

The interior of A is denoted by A° and defined as the union of all open sets contained in A. i.e., $A^\circ = \cup\{G \in \delta \mid G \leq A\}$. (Largest open set contained in A).

The closure of A is denoted by $\bar{A}$ and defined as the intersection of all closed sets containing A. i.e., $\bar{A} = \cap\{F \mid F^c \in \delta \text{ and } A \leq F\}$. (Smallest closed set containing A).

Example. Consider, $X = \{a, b, c\}$ and

$$\begin{aligned} A &: \ a \mapsto 0.2, \ b \mapsto 0.4, \ c \mapsto 0.8 \\ B &: \ a \mapsto 0.4, \ b \mapsto 0.6, \ c \mapsto 0.8 \\ C &: \ a \mapsto 0.6, \ b \mapsto 0.8, \ c \mapsto 1.0 \end{aligned}$$

Then, $\delta = \{\underline{0}, \underline{1}, A, B, C\}$ be a fuzzy topology on X. Here $U : X \to [0, 1]$ and $U : a \mapsto 0.8$, $b \mapsto 0.7$, $c \mapsto 0.8$. Find U° and $\bar{U}$.

Solution. 1. We know that, $U^\circ = \cup\{G \in \delta : g \leq U\} = \cup\{\underline{0}, A, B\} = B$. Since, $\underline{0} \leq A \leq B$.

2. At first, $A^c : a \mapsto 0.8,\ b \mapsto 0.6,\ c \mapsto 0.2$
$B^c : a \mapsto 0.6,\ b \mapsto 0.4,\ c \mapsto 0.2$
$C^c : a \mapsto 0.4,\ b \mapsto 0.2,\ c \mapsto 0.0$
$\underline{0}^c = \underline{1}$ and $\underline{1}^c = \underline{0}$
We know that $\bar{U} = \cap\{F \mid F^c \in \delta \text{ and } U \leq F\} = \underline{1}$.

Theorem 2.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, the following conditions hold:

(i) $\underline{0}^\circ = \underline{0}$ and $\underline{1}^\circ = \underline{1}$

(ii) $\forall A \in \mathcal{F}(X),\ A^\circ \leq A$

(iii) $\forall A \in \mathcal{F}(X),\ A^{\circ\circ} = A^\circ$

(iv) for $A, B \in \mathcal{F}(X)$ with $A \leq B$ implies $A^\circ \leq B^\circ$

(v) for $A, B \in \mathcal{F}(X)$, $(A \wedge B)^\circ = A^\circ \wedge B^\circ$

Proof.

(i) By definition, $\underline{0}^\circ = \cup\{G \in \delta \mid G \leq \underline{0}\} = \underline{0}$ and $\underline{1}^\circ = \cup\{G \in \delta \mid G \leq \underline{1}\} = \underline{1}$

(ii) By definition, $A^\circ = \cup\{G \in \delta \mid G \leq A\}$. Since, the arbitrary union of open sets is open, A° is the open set of $\mathcal{F}(X)$ and also, A° is the largest open set contained in A. $\therefore\ A^\circ \leq A$.

(iii) From (ii), $A^\circ \leq A \ \Rightarrow A^{\circ\circ} \leq A^\circ$. But A° is the largest open set contained in A. So, $A^\circ \leq A^{\circ\circ}$. Hence, $A^{\circ\circ} = A^\circ$.

(iv) Let $A, B \in \mathcal{F}(X)$ such that $A \leq B$. Now, since $A^\circ \leq A$, hence $A^\circ \leq B$. But B° is the set of all open sets contained in B. So, $B^\circ \leq B$. Therefore, $A^\circ \leq B^\circ$.

(v) Let $A, B \in \mathcal{F}(X)$. Then,

$$\begin{aligned} & A^\circ \leq A,\ B^\circ \leq B \\ \Rightarrow\ & A^\circ \wedge B^\circ \leq\ A \wedge B \\ \Rightarrow\ & (A^\circ \wedge B^\circ)^\circ \leq\ (A \wedge B)^\circ \end{aligned} \tag{2.1}$$

Here, A° is the largest open set contained in A and B° is the largest open set contained in B. Hence, $A^\circ \wedge B^\circ$ is also an open set of X. So, $(A^\circ \wedge B^\circ)^\circ \leq\ (A \wedge B)^\circ$.
From (2.1),

$$A^\circ \wedge B^\circ \leq\ (A \wedge B)^\circ \tag{2.2}$$

Again, Since,

$$\begin{aligned} & A \wedge B \leq A,\ B \\ \Rightarrow\ & (A \wedge B)^\circ \leq A^\circ,\ B^\circ \\ \Rightarrow\ & (A \wedge B)^\circ \leq A^\circ \wedge B^\circ \end{aligned} \tag{2.3}$$

From, (2.2) and (2.3), $A^\circ \wedge B^\circ = (A \wedge B)^\circ$.

□

Note. If A be a fuzzy open set of the topological space $\langle X, \delta \rangle$, then $A^\circ = A$, $\bar{A} = A$ iff A is closed.

Definition 25 (Fuzzy Point)**.** A fuzzy set x_a on X is called a fuzzy point on X if $\forall y \in X$,

$$x_a(y) = \begin{cases} a; & \text{if } x = y \\ 0; & \text{if } x \neq y \end{cases} \quad ; \qquad \text{where, } 0 < a \leq 1$$

The set of all fuzzy points on X is denoted by $P(X)$. The fuzzy points x_{1-a} is called the dual point of the fuzzy points x_a.

Example. $X = [0,1]$, where $X = \{x, y, z\}$. We need to find $x_a(y)$ where $y \in X$.

$$\begin{array}{llll} x_a : x \to a & y_a : x \to 0 & z_a : x \to 0 & \text{dual of } x_a,\ x_{1-a} : x \to (1-a) \\ \quad y \to 0 & \quad y \to a & \quad y \to 0 & \qquad\qquad\qquad y \to 0 \\ \quad z \to 0 & \quad z \to 0 & \quad z \to a & \qquad\qquad\qquad z \to 0 \end{array}$$

Definition 26 (Neighborhood of a fuzzy point)**.** Let $\langle X, \delta \rangle$ be a fuzzy topological space and $x_a \in P(X)$. Then $U \in \delta$ is called a fuzzy neighborhood of x_a if $x_a \in U$.
The set of all fuzzy neighborhood of x_a is denoted by $\mathcal{N}_\delta(x_a)$.

Example. $X = \{a, b, c\}$, $\delta = \{\underline{0}, \underline{1}, A, B\}$, $A : a \to 0.0$, $B : a \to 0.2$. Find the neighborhood of $a_{0.4}$, $b_{0.7}$, $c_{0.8}$.

$$\begin{array}{ll} b \to 0.2 & b \to 0.4 \\ c \to 0.7 & c \to 0.8 \end{array}$$

Solution.

1. $a_{0.4} : a \to 0.4$; Fuzzy neighborhood of $a_{0.4} : \{\underline{1}\}$.
   $b \to 0.0$
   $c \to 0.0$

2. $b_{0.7} : a \to 0.0$; Fuzzy neighborhood of $b_{0.7} : \{\underline{1}\}$.
   $b \to 0.7$
   $c \to 0.0$

3. $c_{0.8} : a \to 0.0$; Fuzzy neighborhood of $c_{0.8} : \{B, \underline{1}\}$.
   $b \to 0.0$
   $c \to 0.8$

Theorem 2.0.2. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $A \subseteq X$. Then a fuzzy point $x_a \in A^\circ \Leftrightarrow x_a$ has a neighborhood U such that $U \subseteq A$.

Proof. Suppose, $x_a \in A^\circ$. By the definition of A°, $A^\circ = \cup\{G \in \delta \mid G \subseteq A\}$. $\therefore\ x_a \in \cup\{G \in \delta \mid G \subseteq A\}$. Thus we have $x_a \in U$ for some $U \in \delta \ni U \subseteq A$. $\therefore$ There exists a neighborhood U of x_a such that $U \subseteq A$.

Conversely, suppose, U be a neighborhood of a fuzzy point $x_a \ni U \subseteq A$. This implies, $x_a \in U \subseteq A$. Now, since A° is the largest open set contained in A, we have $U \subseteq A^\circ$. Thus, $x_a \in A^\circ$. ☐

Definition 27 (Quasi-Coincident of a fuzzy point)**.** Let $\langle X, \delta \rangle$ be a fuzzy topological space. A fuzzy point x_a is called quasi-coincident of a fuzzy set A denoted by $x_a \propto A$ iff $x_a \not\leq A^c$ i.e., $a > A^c(x) \Rightarrow a + A(x) > 1$,

Definition 28 (Quasi-Coincident of a fuzzy set)**.** A fuzzy set A is said to be quasi-coincident with a fuzzy set B iff there exists an $x \in X$ such that $A(x) > B^c(x)$ i.e., $A(x) + B(x) > 1$ for some $x \in X$.

Definition 29 (Quasi-neighborhood)**.** An open set $U \in \delta$ is called a quasi-neighborhood of a fuzzy point x_a if x_a is a quasi-coincident of U. The set of all quasi-coincident of x_a is denoted by $\mathcal{Q}_\delta(x_a)$.

Example. Consider, $X = \{a, b, c\}$, $\delta = \{\underline{0}, \underline{1}, A, B\}$,

$$\begin{aligned} A :&\ a \mapsto 0.0,\ b \mapsto 0.2,\ c \mapsto 0.7 \\ B :&\ a \mapsto 0.6,\ b \mapsto 0.4,\ c \mapsto 0.8 \\ \text{Given, } P :&\ a \mapsto 0.0,\ b \mapsto 0.4,\ c \mapsto 0.9 \end{aligned}$$

Find the quasi-neighborhood of x_a at $a = 0.4$.

Solution. Here, $B^c :\ a \mapsto 0.4,\ b \mapsto 0.6,\ c \mapsto 0.2$. Since, $a = 0.4 \geq B^c(a) = 0.4$ so, $x_{0.4}$ is a quasi-coincident of B and $\mathcal{Q}_\delta(x_a) = \{\underline{1}, B\}$.

Theorem 2.0.3. A quasi-neighborhood of x_a is exactly a neighborhood of x_{1-a}.

Proof. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $U \in \delta$ be a quasi-neighborhood of x_a. By the definition of quasi-neighborhood of x_a,

$$\begin{aligned} & a > U^c(x), \quad \text{for some } x \in X, \\ \Leftrightarrow\ & a > 1 - U(x), \quad \text{for some } x \in X, \\ \Leftrightarrow\ & a + U(x) > 1, \quad \text{for some } x \in X, \\ \Leftrightarrow\ & 1 - a < U(x), \quad \text{for some } x \in X, \\ \Leftrightarrow\ & x_{1-a} \in U, \\ \Leftrightarrow\ & U \text{ is a neighborhood of } x_{1-a} \end{aligned}$$

□

Proposition 1. Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $A, B \subseteq X$. Then $A \leq B$ iff A and B^c are not quasi-coincident.

Proof. Suppose, $A \leq B$, then, $A(x) \leq B(x)$, for all $x \in X$. Now, $A(x) + B^c(x) = A(x) + 1 - B(x) \leq 1$, for all $x \in X$ [Since, $A(x) \leq B(x)$]
Hence, A and B^c are not quasi-coincident.

Conversely, suppose $A(x)$ and $B^c(x)$ are not quasi-coincident. Then,

$$\begin{aligned} & A(x) + B^c(x) \leq 1 \\ \Rightarrow\ & A(x) + 1 - B(x) \leq 1 \\ \Rightarrow\ & A(x) - B(x) \leq 0 \\ \Rightarrow\ & A(x) \leq B(x) \end{aligned}$$

□

Theorem 2.0.4. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then, the following conditions hold:

1. $x_a \in A^\circ$ iff $x_{1-a} \notin \bar{A}^c$.
2. $x_a \in \bar{A}$ iff each neighborhood of its dual point x_{1-a} is quasi-coincident with A.

Proof.

1. Let $x_a \in A^\circ$. Then by definition of A°, there exists $B \in \delta$ such that $x_a \in B \subseteq A$ i.e., B is a neighborhood of x_a and hence B is a quasi-neighborhood of x_{1-a}. Hence $x_{1-a} \not\leq B^c$ i.e., $x_{1-a} \notin B^c$. Since, $B \subseteq A$ and $\bar{A}$ is the smallest closed set containing A, we have, $B \subseteq A \subseteq \bar{A}$ implies $\bar{A}^c \subseteq B^c$. Hence we can show that $x_{1-a} \notin \bar{A}^c$.

   Conversely, suppose $x_{1-a} \notin \bar{A}^c$. Then there is a neighborhood B of x_a which is not quasi-coincident with A^c. Thus,

   $$\begin{aligned} & B(x) + A^c(x) \leq 1 \quad \forall x \in X \\ \Rightarrow\ & B(x) \leq A(x) \quad \forall x \in X \end{aligned}$$

   $\therefore$ $B^c \subseteq A$ and so $x_a \in B \subseteq A$ i.e., $x_a \in A^\circ$.

2. Let N be the neighborhood of x_{1-a}. Now, N is a quasi-coincident with A implies

   $$\begin{aligned} & N(x) + A(x) > 1, \quad \forall x \in X \\ \Rightarrow\ & N \text{ and } A \text{ intersect at } x \\ \Rightarrow\ & x_a \in \bar{A} \end{aligned}$$

   Conversely, suppose $x_a \in \bar{A}$. Then, N and A intersect at x. This implies,

   $$\begin{aligned} & N(x) + A(x) > 1, \quad \forall x \in X \\ \Rightarrow\ & N \text{ is a quasi-coincident with } A \text{ at } x \\ \Rightarrow\ & \text{each neighborhood } N \text{ of } x_{1-a} \text{ is quasi-coincident with } A \end{aligned}$$

□

Definition 30 (Subspace)**.** Let $\langle X, \delta \rangle$ be a fuzzy topological space and let Y be a nonempty set ($Y \neq \varnothing$) such that $Y \subseteq X$. Define $\delta_{\restriction_Y} = \{U_{\restriction_Y} \mid U \in \delta\}$ [where, $U_{\restriction_Y} = U \cap Y = U$ restricted in Y]. Then $\delta_{\restriction_Y}$ is a fuzzy topology on Y. The fuzzy topological space $\langle Y, \delta_{\restriction_Y} \rangle$ is called a subspace of $\langle X, \delta \rangle$.

Example. Let, $X = \{a, b, c\}$ and $Y = \{b, c\}$. Let $\delta = \{\underline{0}, \underline{1}, A, B\}$ where

$$\begin{aligned} A &: \ a \mapsto 0.2, \ b \mapsto 0.4, \ c \mapsto 1.0 \\ B &: \ a \mapsto 0.1, \ b \mapsto 0.4, \ c \mapsto 0.8 \end{aligned}$$

Then $\delta_{\restriction_Y} = \{\underline{0}_{\restriction_Y}, \underline{1}_{\restriction_Y}, A_{\restriction_Y}, B_{\restriction_Y}\}$ where,

$$\begin{aligned} A_{\restriction_Y} &: \ b \mapsto 0.4, \ c \mapsto 1.0 \\ B_{\restriction_Y} &: \ b \mapsto 0.4, \ c \mapsto 0.8 \end{aligned}$$

is a fuzzy topology on Y and hence $\langle Y, \delta_{\restriction_Y} \rangle$ is a fuzzy subspace of $\langle X, \delta \rangle$.

Example. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 3, 4\}$. Find a non-trivial fuzzy topology on X and hence, find a fuzzy subspace of $\langle X, \delta \rangle$.

Solution. Let $\delta = \{\underline{0}, \underline{1}, A, B\}$ be a fuzzy topology on X where,

$$\begin{aligned} A &: \ 1 \mapsto 0.3, \ 2 \mapsto 0.1, \ 3 \mapsto 0.6, \ 4 \mapsto 0.2 \\ B &: \ 1 \mapsto 0.7, \ 2 \mapsto 0.4, \ 3 \mapsto 0.1, \ 4 \mapsto 0.2 \end{aligned}$$

Then $\delta_{\restriction_Y} = \{\underline{0}_{\restriction_Y}, \underline{1}_{\restriction_Y}, A_{\restriction_Y}, B_{\restriction_Y}\}$ where,

$$\begin{aligned} A_{\restriction_Y} &: \ 1 \mapsto 0.3, \ 3 \mapsto 0.6, \ 4 \mapsto 0.2 \\ B_{\restriction_Y} &: \ 1 \mapsto 0.7, \ 3 \mapsto 0.1, \ 4 \mapsto 0.2 \end{aligned}$$

is a fuzzy topology on Y and hence $\langle Y, \delta_{\restriction_Y} \rangle$ is a fuzzy subspace of $\langle X, \delta \rangle$.

Remark. Let $\langle X, \tau \rangle$ be a fuzzy topological space. The two fuzzy sets A and B in X are said to be intersecting $\Leftrightarrow$ there exists a point $x \in X$ such that $(A \wedge B)(x) \neq 0$.
For such a case, we say that, A and B intersect at x.

Again, if A and B are quasi-coincident at x, then, $A(x) + B(x) > 1$ i.e., both $A(x)$ and $B(x)$ are not zero and here A and B intersect at x.

- $x_a \to$ quasi-coincident of A if $a > A^c(y)$ for some $y \in X$.
- $U \in \delta \to$ quasi-neighborhood if x_a is a quasi-coincident of U.

Definition 31 (Adherent point)**.** A fuzzy point x_a is called an adherent point of a fuzzy set A iff every quasi-neighborhood of x_a is a quasi-coincident with A.

Problem 2.1. Give an example of an adherent point.

Definition 32 (Accumulation Point)**.** A fuzzy point x_a is called an accumulation point of a fuzzy set A iff x_a is an adherent point of A and every quasi-neighborhood of x_a and A are quasi-coincident at some point different from sup x_a, whenever, $x_a \in A$.

Definition 33 (Base)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $\mathcal{B}$ is a base for δ iff every open set $G \in \delta$ is the union of members of $\mathcal{B}$ i.e., $G = \cup B_i$, $\forall\, B_i \in \mathcal{B}$.

Definition 34 (Subbase)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $\mathcal{S} \in X$ is called a subbase iff finite intersection of member of $\mathcal{S}$ form a base for δ.

Chapter 3

Separation Axioms

Definition 35 (Quasi T_0-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a quasi T_0-space, if for every two distinct fuzzy points x_a and x_b with same support point x, there exists $U \in Q_\delta(x_a)$ such that $x_b \not\propto U$ or, there exists $V \in Q_\delta(x_b)$ such that $x_a \not\propto V$.

Definition 36 (Sub T_0-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a sub T_0-space, if for every two distinct $x, y \in X$, there exists $a \in [0, 1]$ such that either $\exists U \in Q_\delta(x_a)$ with $y_a \not\propto U$ or, $\exists V \in Q_\delta(y_a)$ with $x_a \not\propto V$.

Definition 37 (T_0-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_0-space, if for every two distinct fuzzy points x_a and y_b, $\exists U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, $\exists V \in Q_\delta(y_b)$ with $x_a \not\propto V$.

Definition 38 (T_1-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_1-space, if for every two distinct fuzzy points x_a and y_b such that $x_a \not\leq y_b$ then there exists $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ and, $\exists V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.

Definition 39 (T_2-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_2-space, if for every two distinct fuzzy points x_a and y_b (i.e., $x_a \neq y_b$) then there exists $U \in Q_\delta(x_a)$ and, $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

Theorem 3.0.1. Quasi T_0 property is heriditary.
or, Every subspace of a Quasi T_0 space is Quasi T_0 space.

Proof. Suppose, $\langle X, \delta \rangle$ be a fuzzy topological space which is Quasi T_0-space. Let $\langle Y, \mu \rangle$ be the subspace of $\langle X, \delta \rangle$. We have to prove that, $\langle Y, \mu \rangle$ be a Q-T_0-space.

Now, since, $Y \subseteq X$ so every $V \in \mu$, $V = U_{\restriction Y}$ for some $U \in \delta$. Let y_a and y_b be two distinct fuzzy points in Y such that, $y_a \neq y_b$. Then as $Y \subseteq X$, we have y_a and y_b are in X with $y_a \neq y_b$.

Again, since $\langle X, \delta \rangle$ is a Quasi T_0-space there exist $U \in Q_\delta(y_a)$ such that $y_b \not\propto U$ or, there exist $V \in Q_\delta(y_b)$ such that $y_a \not\propto V$. This implies, there is $U_{\restriction y} \in Q_{\delta \restriction y}(y_a)$ such that $y_b \not\propto U_{\restriction Y}$ or there is $V_{\restriction Y} \in Q_{\delta \restriction}(y_b)$ such that $y_a \not\propto V_{\restriction Y}$.

Thus, by definition of a Q-T_0-space $\langle Y, \mu \rangle$ is a Q-T_0- space. □

Theorem 3.0.2. Every subspace of a T_0-space is T_0-space.

Proof. Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $\langle Y, \mu \rangle$ be a subspace of $\langle X, \delta \rangle$. Let x_a and y_b be two distinct points in Y. Then since, $Y \subseteq X$, we have, x_a and y_b in X with $x_a \neq y_b$. Now since $\langle X, \delta \rangle$ is a fuzzy T_0-space. We have either there is $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, there is $V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.
Now, $U_{\restriction Y} \in Q_{\delta \restriction Y}(x_a)$ such that $y_b \not\propto U_{\restriction Y}$ as x_a, $y_b \in Y$ and $V_{\restriction Y} \in Q_{\delta \restriction Y}(y_b)$ such that $x_a \not\propto V_{\restriction Y}$.
Thus, $\langle Y, \mu \rangle$ is a T_0-space. □

Theorem 3.0.3. A fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ is a quasi-T_0-space iff for every $x \in X$ and $a \in [0, 1]$ there exists $B \in \delta$ such that $B(x) = a$.

Proof. Suppose, $\langle \mathcal{F}(X), \delta \rangle$ be a quasi T_0-space. If $a = 0$, then it suffices to take $B = \underline{0}$. If $0 < a < 1$, we take a strictly monotonic increasing sequence of positive real numbers converging to a. Let $\Delta_n = (a_n, a_{n+1}]$, $n = 1, 2, 3, \ldots$.

Since $\langle \mathcal{F}(X), \delta \rangle$ be a quasi T_0-space, then for any $x \in X$ and $\Delta = (a_1, a_2)$ with $0 \leq a_1 < a_2 < 1$, there exists $B \in \delta$ such that $B(x) \in \Delta$.
From this property, we can say that, $\exists B_n \in \delta$ such that $B_n(x) \in \Delta_n$, for each n

$$\therefore B = \bigvee_{n=1}^{\infty} B_n \in \delta \qquad \text{and} \qquad B(x) = a.$$

Conversely, suppose x_a and x_b are two fuzzy points with $b < a$ where $a, b \in [0, 1]$. Then by hypothesis, there is an open set B such that $B(x) = 1 - b > 1 - a$.
This implies, B is an open Q-nbd of x_a but not quasi-conincident with x_b [since, B is a nbd of x_{1-a}]. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a quasi T_0-space. □

Theorem 3.0.4. A fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ is T_1-space iff for every $x \in X$ and each $a \in [0, 1]$ there exists $B \in \delta$ such that $B(x) = 1 - a$ and $B(y) = 1$ for $y \neq x$.
Or, $\langle \mathcal{F}(X), \delta \rangle$ is a T_1-space $\Leftrightarrow$ every fuzzy point in $\langle X, \delta \rangle$ is closed.

Proof. Suppose $\langle \mathcal{F}(X), \delta \rangle$ be a T_1-space. If $a = 0$ then it suffices to take $B = \underline{1}$.
Suppose, $a > 0$ and x_a is a fuzzy point. Since, every fuzzy point in a T_1-space is closed, so, x_a is a closed set.
$\therefore$ We have, $B = 1 - x_a \in \delta$ and hence $B(x) = 1 - a$ and $B(y) = 1$. if $y \neq x$.
Conversely, let x_a be a fuzzy point. Then by hypothesis there exists $B \in \delta$ such that $B(x) = 1 - a$ and $B(y) = 1$ with $y \neq x$. This implies, $B = 1 - x_a$ and hence $B^c = x_a$ which is closed. Thus, $B \in \delta$. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a T_1-space. □

Definition 40 (Purely T_2-space)**.** $\langle \mathcal{F}(X), \delta \rangle$ is called purely T_2-space if for every two zero-meet fuzzy points x_a and y_b, $\exists U \in Q_\delta(x_a)$ and $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

Theorem 3.0.5. For a fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ the following statements are equivalent

1. $\langle X, \delta \rangle$ is a fuzzy T_0-space.
2. For $x, y \in X$, $x \neq y$, $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.

Proof. (1) $\Rightarrow$ (2), Suppose $\langle X, \delta \rangle$ is a fuzzy T_0-space. Thus, we have $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$. □

Theorem 3.0.6 (X). A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called a fuzzy T_0 space iff

(i) $\forall x, y \in X$, $\exists U \in \delta$ such that $U(x) = 1$, $U(y) = 0$ or $U(y) = 1$, $U(x) = 0$.

(ii) For all $\forall x, y \in X$, $x \neq y$, $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$

(iii) $\forall x, y \in X$, $x \neq y$, such that $U(x) < U(y)$ or $U(y) < U(x)$

Theorem 3.0.7. For a fuzzy topological space $\langle X, \delta \rangle$ the following statements are equivalent

(i) For all $x,\ y \in X$, $x \neq y$, $\bar{x}_1(y) \cap \bar{y}_1(x) < 1$,

(ii) For all $x,\ y \in X$, $x \neq y$, $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.

Proof. (i) $\Rightarrow$ (ii)
Suppose $\langle X, \delta \rangle$ be a fuzzy T_0 space. Thus, we have, $\bar{x}_1(y) \cap \bar{y}_1(x) < 1$. This implies that either $\bar{x}_1(y) < 1$ or $\bar{y}_1(x) < 1$. Consider $\bar{x}_1(y) < 1$. Hence, $1 - \bar{x}_1(y) > 0$ and $1 - \bar{x}_1(x) = 0$.
Let, $1 - \bar{x}_1 = U$, then $U \in \delta$ and $U(x) = 0$, $U(y) > 0$.
Similarly, from $\bar{y}_1(x) < 1$ we can show that there exists $V \in \delta$ such that $V(y) = 0$, $V(x) > 0$.

(ii) $\Rightarrow$ (i)
From (ii), we have for any two points $x, y \in X$ with $x \neq y$ $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.
Then

$$\begin{aligned} 1 - U(x) &< 1 \qquad \text{and} \qquad 1 - U(y) = 1 \\ \text{or, } 1 - U(x) &= 1 \qquad \text{and} \qquad 1 - U(y) < 1 \end{aligned}$$

Since the complement of U is closed, we see that $\bar{x}_1(y) < 1$ or $\bar{y}_1(x) < 1$. This implies $\bar{x}_1(y) \cap \bar{y}_1(x) < 1$. □

Theorem 3.0.8. For any fuzzy topological space $\langle X, \delta \rangle$ the following are equivalent:

(i) For all $x,\ y \in X,\ x \neq y,\ \exists U, V \in \delta$ such that $U(x) = 1 = V(y)$ and $U \subset V^c$

(ii) If $x \in X$, then for each $y \in X,\ y \neq x,\ \exists U \in \delta$ such that $U(x) = 1$ and $\bar{U}(y) = 0$

Proof. Let $\langle X, \delta \rangle$ be a fuzzy topological space. Let $x, y \in X$ with $x \neq y$. Then by (i) there exist $U, V \in \delta$ such that

$$\begin{aligned} &U(x) = 1 = V(y) \qquad \text{and} \qquad U \subset V^c = 1 - V \\ \Rightarrow &\bar{U} \subset \overline{1 - V} \subset 1 - V \end{aligned}$$

So,

$$\bar{U}(y) \subset (1 - V)(y) = 1 - V(y) = 0 \quad \text{i.e., } \bar{U}(y) = 0$$

(ii) $\Rightarrow$ (i)
Let $\langle X, \delta \rangle$ be a fuzzy topological space. Let $x, y \in X$ with $x \neq y$. By (ii) there exist $U \in \delta$ with $U(x) = 1$ and $\bar{U}(y) = 0$ or, there exist $V \in \delta$ with $V(y) = 1$ and $\bar{V}(x) = 0$.
Let $V = 1 - \bar{U}$. Then,

$$\begin{aligned} V(y) &= (1 - \bar{U})(y) \\ &= 1 - \bar{U}(y) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Again, $U \subset \bar{U} = 1 - V = V^c$. Hence $U \subset V^c$, $U(x) = V(y) = 1$ and it is also clear that $U,\ V \in \delta$. □

Definition 41 (Regular Space)**.** A fuzzy topological space $\langle X, \delta \rangle$ is said to be fuzzy regular iff for each $x \in X$ and each closed fuzzy set U with $U(x) = 0$ there exists $V, W \in \delta$ such that $V(x) = 1$, $U \subset W$ and $V \subseteq 1 - W$.

Theorem 3.0.9. Let $\langle X, \delta \rangle$ be a fuzzy topological space. Then the following are equivalent:

(i) $\langle X, \delta \rangle$ is a fuzzy regular space.

(ii) For each $x \in X$, $U \in \delta$ with $U(x) = 1$, $\exists V \in \delta$ with $V(x) = 1$ and $V \subset \bar{V} \subset U$.

(iii) For each $x \in X$, $U \in \delta^c$ with $U(x) = 0$, $\exists V \in \delta$ with $V(x) = 1$ such that $U \subseteq 1 - \bar{V}$ or $\bar{V} \subseteq 1 - U$.

Proof. (i) $\Rightarrow$ (ii)
Let $\langle X, \delta \rangle$ be a fuzzy topological space. Let $x \in X$ and $U \in \delta$ with $U(x) = 1$. Then $1 - U \in \delta^c$. This implies

$$(1 - U)(x) = 1 - U(x) = 1 - 1 = 0 \quad \text{i.e., } U^c(x) = 0$$

Then be definition of fuzzy regular space, there exists $V, W \in \delta$ such that $1 - U \subset W$ and $V \subseteq 1 - W$. Now,

$$\begin{aligned} &V \subseteq 1 - W \\ \Rightarrow &\bar{V} \subseteq \overline{1 - W} \\ \Rightarrow &\bar{V} \subseteq 1 - W \\ \Rightarrow &\bar{V} \subset U \end{aligned}$$

Hence, $V \subset \bar{V} \subset U$.

(ii) $\Rightarrow$ (iii)
Let $x \in X$ and $U \in \delta^c$ with $U(x) = 0$. Now $1 - U \in \delta$ and $(1 - U)(X) = 1 - U(x) = 1$. Hence, from (ii) there exists $V \in \delta$ with $V(x) = 1$ such that $V \subset \bar{V} \subset 1 - U$ or $U \subset 1 - \bar{V}$.

Lastly, (iii) $\Rightarrow$ (i)
Let $x \in X$ and $U \in \delta^c$ with $U(x) = 0$. From (iii), there exists $V \in \delta$ with $V(x) = 1$ and $\bar{V} \subset 1 - U$. This implies $U \subset 1 - \bar{V}$. Consider $1 - \bar{V} = W$. Therefore, $W \in \delta$ and also $V \subset \bar{V} = 1 - W$. Hence, $U \subset W$ and $V \subseteq 1 - W$. Therefore, by the definition of regular space $\langle X, \delta \rangle$ is a regular space. □

Chapter 4

Connected Fuzzy Topological Space

Definition 42 (Separated Fuzzy Sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $A, B \in \mathcal{F}(X)$ are called separated sets if $\bar{A} \wedge B = \underline{0} = A \wedge \bar{B}$.

Lemma. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and A, B, $C \in \mathcal{F}(X)$. If B and C are separated sets then $A \wedge B$ and $A \wedge C$ are separated.

Proof. Since B and C are separated, we have, $\bar{B} \wedge C = \underline{0} = B \wedge \bar{C}$.
We have, $A \wedge B < B \;\; \Rightarrow \overline{A \wedge B} < \overline{B}$ and $A \wedge C < C$.
This implies, $\overline{(A \wedge B)} \wedge (A \wedge C) \leq \bar{B} \wedge C = \underline{0}$. Similarly, $(A \wedge B) \wedge \overline{(A \wedge C)} \leq B \wedge \bar{C} = \underline{0}$.
Hence, $A \wedge B$ and $A \wedge C$ are separated. □

Definition 43 (Connected Fuzzy Sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. A fuzzy set A on X is called connected if there do not exist $C, D \in \mathcal{F}(X) \setminus \{\underline{0}\}$ such that $A = C \vee D$.
Or, A set A is connected if $A = B \vee C$ then either $B = \underline{0}$ or, $C = \underline{0}$.

Theorem 4.0.1. Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then the following are equivalent:

1. A is connected.
2. B, $C \in \mathcal{F}(X)$ are separated, $A \leq B \vee C$ implies $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$.
3. B, $C \in \mathcal{F}(X)$ are separated, $A \leq B \vee C$ implies $A \leq B$ or, $A \leq C$.

Proof. (1) $\Rightarrow$ (2), Since, B and C are separated set. By the above lemma, we have $(A \wedge B)$ and $(A \wedge C)$ are separated. Since, A is connected and $A \leq B \vee C$ implies

$$\begin{aligned} A &= A \wedge (B \vee C) \\ &= (A \wedge B) \vee (A \wedge C) \end{aligned}$$

then by definition of connectedness, either $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$. Hence, (2) holds.

(2) $\Rightarrow$ (3), Suppose, $A \wedge B = \underline{0}$, then,

$$\begin{aligned} A &= (A \wedge B) \vee (A \wedge C) \\ &= \underline{0} \vee (A \wedge C) \\ &= (A \wedge C). \end{aligned}$$

So, $A \leq C$. Similarly, if $A \wedge C = \underline{0}$, then we can prove that $A \leq B$. Thus, (3) holds.

Finally, (3) $\Rightarrow$ (1), Suppose, (3) holds, we need to show that, A is connected. Let $B, C \in \mathcal{F}(X)$ are two separated fuzzy sets such that $A \leq B \vee C$. We need to prove that, either, $B = \underline{0}$ or, $C = \underline{0}$.
By (3), we have either $A \leq B$ or, $A \leq C$. Now if $A \leq B$ then $C \wedge A \leq C \wedge B \leq C \wedge \bar{B}$. But since, B, C are separated sets so, $C \wedge \bar{B} = \underline{0}$. $\therefore$ $C \wedge A = \underline{0}$.
Again, $C \wedge A = C \wedge (B \vee C) = C$. So $C = \underline{0}$. Now if $A \leq C$, we can similarly prove that $B = \underline{0}$.
Thus, A is connected. □

Theorem 4.0.2. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$ is connected such that $A \leq B \leq \bar{A}$. Show that B is connected.

Proof. Suppose, C and D are two separated fuzzy sets such that, $B = C \vee D$. To show that, B is connected we need only to show either, $C = \underline{0}$ or, $D = \underline{0}$.
By the lemma, we have $(A \wedge C)$ and $(A \wedge D)$ are separated sets. Let $F = A \wedge C$, $G = A \wedge D$. Now

$$\begin{aligned} F \vee G &= (A \wedge C) \vee (A \wedge D) \\ &= A \wedge (C \vee D) \\ &= A \wedge B \\ &= A \end{aligned}$$

Since, A is connected, we have either $F = \underline{0}$ or, $G = \underline{0}$.
Suppose, $F = \underline{0}$. Then, $A = F \vee G = G = A \wedge D$. This implies, $A \leq D$. Thus, $\bar{A} \leq \bar{D}$. i.e., $B \leq \bar{A} \leq \bar{D}$.
Now, $C \wedge B \leq C \wedge \bar{A} \leq C \wedge \bar{D} = \underline{0}$. i.e.,

$$\begin{aligned} & C \wedge B \leq \underline{0} \\ \Rightarrow & C \wedge (C \vee D) \leq \underline{0} \\ \Rightarrow & C = \underline{0} \end{aligned}$$

Similarly, if $G = \underline{0}$, then we can show that $D = \underline{0}$. Hence, B is connected. □

Definition 44 (Connected Fuzzy Topological Space)**.** If the fuzzy set $\underline{1}$ is connected i.e., there does not exist separated sets $C, D \in \mathcal{F}(X) \setminus \{\underline{0}\}$ such that $\underline{1} = C \vee D$, then the fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ is called a connected fuzzy topological space.

Theorem 4.0.3 (Characterization Theorem). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then the followings are equivalent

1. $\langle \mathcal{F}(X), \delta \rangle$ is connected.
2. $A, B \in \delta$, $A \vee B = \underline{1}$, $A \wedge B = \underline{0}$, implies $\underline{0} \in \{A, B\}$.
3. $A, B \in \delta'$, $A \vee B = \underline{1}$, $A \wedge B = \underline{0}$, implies $\underline{0} \in \{A, B\}$.

Proof. (1) $\Rightarrow$ (2), Suppose, (2) is false. Then there are $A, B \in \delta \setminus \{\underline{0}\}$ such that

$$\begin{aligned} & A \vee B = \underline{1}, \qquad A \wedge B = \underline{0} \\ \Rightarrow & A^c \wedge B^c = \underline{0}, \text{ and } \ A^c \vee B^c = \underline{1} \quad \text{[By De Morgan's Law]} \\ \Rightarrow & \bar{A}^c \wedge B^c = \underline{0}, \text{ and } \ A^c \wedge \bar{B}^c = \underline{0} \quad \text{[Since, } A^c,\ B^c \text{ are closed.]} \end{aligned}$$

$\therefore$ We have by definition, A^c and B^c are two separated sets. Therefore, we have $A^c \vee B^c = \underline{1}$ and A^c, B^c are two separated sets. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is disconnected. Hence, (2) is true.

(2) $\Rightarrow$ (3), Let $A, B \in \delta'$ such that $A \vee B = \underline{1}$ and $A \wedge B = \underline{0}$. Then by De Morgan's Laws, $A^c \wedge B^c = \underline{0}$ and $A^c \vee B^c = \underline{1}$. By (2), $\underline{0} \in \{A^c, B^c\}$. Hence, $\underline{0} \in \{A, B\}$.

(3) $\Rightarrow$ (1), If $\langle \mathcal{F}(X), \delta \rangle$ is not connected, then there exists non-zero separated sets $A, B \in \delta' \setminus \{\underline{0}\}$ such that $A \vee B = \underline{1}$, which contradicts (3).
Hence, $\langle \mathcal{F}(X), \delta \rangle$ is connected. □

Chapter 5

Compactness

Definition 45 (Cover and $C-$compactness)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then, $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a cover of A if $A \subseteq \vee \mathcal{A}$.

- $\langle \mathcal{F}(X), \delta \rangle$ is called $C-$compact if every open cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite subcover.
- $\mathcal{A}$ is called an open cover of A, if $\mathcal{A} \subseteq \delta$ and if $\mathcal{A}$ is a cover of A.
- $\mathcal{B} \subseteq \mathcal{A}$ is called a subcover if $\mathcal{B}$ is still a cover of A.

In particularly, $\mathcal{A}$ is a cover of $\langle \mathcal{F}(X), \delta \rangle$ if $\mathcal{A}$ is a cover of $\underline{1}$.

Definition 46 ($\alpha-$cover and $\alpha-$compactness)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an $\alpha-$cover, if for very $x \in X \exists A \in \mathcal{A} \ni A(x) > \alpha$.

- $\langle \mathcal{F}(X), \delta \rangle$ is called an $\alpha-$compact, if for every open $\alpha-$cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite sub$-\alpha-$cover where $\alpha \in [0, 1)$.

Definition 47 (Strong Compact)**.** A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called strongly compact if it is $\alpha-$compact for every $\alpha \in [0, 1)$.

Definition 48 (α^*-cover and α^*-compactness)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an α^*-cover, if for every $x \in X$, there exists $A \in \mathcal{A}$ such that, $A(x) \geq \alpha$.

- For $\alpha \in [0, 1)$, $\langle \mathcal{F}(X), \delta \rangle$ is called an α^*-compact, if for every open α^*-cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite sub α^*-cover.

Example. Given $X = \{a, b, c\}$, $\mathcal{A} = \{A, B, C\}$, $\alpha \in [0, 1)$, $\delta = \{\underline{0}, \underline{1}, A, B, C\}$ where,

$$\begin{aligned} A &: \ a \mapsto 0.2, \ b \mapsto 0.4, \ c \mapsto 0.6; \\ B &: \ a \mapsto 0.4, \ b \mapsto 0.6, \ c \mapsto 0.8; \\ C &: \ a \mapsto 0.6, \ b \mapsto 0.8, \ c \mapsto 0.9; \end{aligned}$$

Check whether $\mathcal{A}$ is $\alpha-$compact or, α^*-compact corresponding to the given value of α.

Solution.

1. Let $\alpha = 0.7$
   $a \in X : \alpha = 0.7 > A(a), \ B(a), C(a)$.
   Hence, for $\alpha = 0.7$, $\mathcal{A}$ is not an $\alpha-$cover.

2. Let $\alpha = 0.3$
   $a \in X : \alpha = 0.3 < C(a) = 0.6, \ B(a) = 0.4$
   $b \in X : \alpha = 0.3 < A(b) = 0.4, \ B(b) = 0.6, \ C(b) = 0.8$
   $c \in X : \alpha = 0.3 < A(c) = 0.6, \ B(c) = 0.8, \ C(c) = 0.9$
   $\therefore \mathcal{A}$ is an $\alpha-$compact space for $a = 0.3$.

3. Let $\alpha = 0.6$
   For, $a \in X : \alpha = 0.6 = C(a)$
   For, $b \in X : \alpha = 0.6 = B(b), \ \alpha = 0.6 < C(b) = 0.8$
   For, $c \in X : \alpha = 0.6 = A(c) = 0.6, \ \alpha = 0.6 < B(c) = 0.8, \ C(c) = 0.9$
   $\therefore \mathcal{A}$ is an α^*-compact space for $a = 0.6$.

Definition 49 ($Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a $Q-$cover of A if for every $x \in Supp(A)$, there exists $U \in \mathcal{A}$ such that $x_{A(x)} \propto U$.

Definition 50 ($Q-$compact)**.** A fuzzy set A is called $Q-$compact if every open $Q-$cover of A has a finite sub $Q-$cover. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called $Q-$compact if $\underline{1}$ is $Q-$compact.

Example. Consider, $X = \{a, b, c\}$, $\delta = \{\underline{0}, \underline{1}, U, V, W\}$ where

$$\begin{aligned} U &: a \mapsto 0.3,\ b \mapsto 0.5,\ c \mapsto 0.7; \\ V &: a \mapsto 0.4,\ b \mapsto 0.6,\ c \mapsto 0.8; \\ W &: a \mapsto 0.6,\ b \mapsto 0.8,\ c \mapsto 0.9; \end{aligned}$$

Consider $\mathcal{A} = \{U, V\} \subseteq \delta$ and let, $A : a \mapsto 0.1,\ b \mapsto 0.2,\ c \mapsto 0.3$. Then, find the $Q-$cover of A.

Solution. Here, $Supp(A) = \{a, b, c\}$
For, $x = a$, $a_{A(a)} = a_{0.1} = 0.1$
For, $x = b$, $b_{A(b)} = b_{0.2} = 0.2$
For, $x = c$, $c_{A(c)} = c_{0.3} = 0.3$
For $x = a$, we have $U_a, : 0.3 + 0.1 < 1$, $V_a = 0.4 + 0.1 < 1$. Hence $\mathcal{A}$ is not a $Q-$cover of A.

\# If $A : a \mapsto 0.7,\ b \mapsto 0.6,\ c \mapsto 0.5$.
Then, For $x = a$, $a_{A(a)} = a_{0.7} = 0.7$
For, $x = b$, $b_{A(b)} = b_{0.6} = 0.6$
For, $x = c$, $c_{A(c)} = c_{0.5} = 0.5$
For, $x = a, 0.3 + 0.7 \geq 1,\ 0.4 + 0.7 > 1$
For, $x = b, 0.5 + 0.6 > 1,\ 0.6 + 0.6 > 1$
For, $x = c, 0.7 + 0.5 > 1,\ 0.8 + 0.5 > 1$
Hence, for every $x \in Supp(A)$, $x_{A(x)} \propto U$.
$\therefore \mathcal{A}$ is a $Q-$cover of A.

Definition 51 ($\alpha - Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\varphi \subseteq \mathcal{F}(X)$ is called an $\alpha - Q-$cover of A, if for every $x_a \subseteq A$, there exists $U \in \varphi$ such that $x_a \propto U$. It is denoted by $\vee \varphi \hat{q} A(\alpha)$.

Definition 52 ($\bar{\alpha} - Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\varphi \subseteq \mathcal{F}(X)$ is called an $\bar{\alpha} - Q-$cover of A, if there exists $\gamma \in B^*(\alpha)$ such that γ is a $\gamma - Q-$cover of A.

- $B(b) = \{a \in L : a \propto b\}$, where the binary relation $\propto$ is defined as, for $a, b \in L$, $a \propto b \Leftrightarrow$ for every subset $D \subseteq L$, $b \leq Sup\, D$ implies the existence of $d \in D$ with $a \leq$
- $B^*(b) = B(b) \cap M(L)$, where, $M(L) = (0, 1]$.

Definition 53. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. A is called $N-$compact if for every $\alpha \in (0, 1] - M([0, 1])$, every open $\alpha - Q-$cover of A has a finite subfamily which is an $\bar{\alpha} - Q-$cover of A. $\langle \mathcal{F}(X), \delta \rangle$ is called $N-$compact, if $\underline{1}$ is compact.

Theorem 5.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. Then A is $N-$compact iff the following conditionds hold:

(a) For every $\alpha \in (0, 1]$, every open $\alpha - Q-$cover of A has a finite sub $\alpha - Q-$cover.

(b) For every $\alpha \in (0, 1]$, every open $\alpha - Q-$cover of A which consists of just one subset is an $\bar{\alpha} - Q-$cover of A.

Proof. (a) Let, A be $N-$compact, $\alpha \in (0, 1]$ and φ is an open $\alpha - Q-$cover of A. By the definition of $N-$compact, φ has a finite subfamily ψ such that, ψ is an $\bar{\alpha} - Q-$cover of A. Hence, $\vee \psi \hat{q} A(\alpha)$ i.e., ψ is an $\alpha - Q-$cover of A.

(b) Suppose, $U \in \delta$ and $\varphi = \{U\}$ is an open $\alpha - Q-$cover of A. Then, by the $N-$compactness of A, φ has a subfamily ψ such that ψ is an $\bar{\alpha} - Q-$cover of A. But, clearly, $\varphi = \psi$. Hence, ψ is an open $\alpha - Q-$cover of A.

Conversely, suppose (a) and (b) holds.
Let $\alpha \in (0,1]$ and φ is an open $\alpha - Q-$cover of A.
By (a), φ has a finite sub $\alpha - Q-$cover ψ of A. Take $U = \vee\psi$. Then $\{U\}$ is an $\alpha - Q-$cover of A.
By (b), $\{U\}$ is also an $\bar{\alpha} - Q-$cover of A. By the definition of $\bar{\alpha} - Q-$cover, there exists $\gamma \in B^*(\alpha)$ such that x_γ is a quasi-coincident with U for every $x_\gamma \subseteq A$. Hence, $\gamma + U(x) > 1 \Rightarrow \gamma > 1 - U(x)$
i.e., $\gamma \leq (U(x))' \ \Rightarrow \gamma \not\leq (U\psi(x))' = \wedge\{(W(x))'|W \in \psi\}$
i.e., $W \in Q_\gamma(x_\gamma)$. So, ψ is an $\bar{\alpha} - Q-$cover of A. Hence, A is $N-$compact. □

Theorem 5.0.2. Continuous image of an $N-$compact space is $N-$compact.

Proof. Let $f^{\rightarrow} : \langle \mathcal{F}(X), \delta\rangle \rightarrow \langle \mathcal{F}(Y), \mu\rangle$ be a continuous fuzzy mapping and A be a $N-$compact fuzzy set in $\mathcal{F}(X)$. For $\alpha \in (0,1]$, let $\mathcal{A}$ be an open $\alpha - Q-$cover of $f^{\rightarrow}(A)$. Then for every $x_\alpha \leq A$, $f^{\rightarrow}(x_\alpha) = f(x)_\alpha \leq f^{\rightarrow}(A)$, there exists $U \in \mathcal{A}$ such that $f(x)_\alpha \propto U \ \Rightarrow f(x)_\alpha \not\propto U^c \ \Rightarrow \alpha \not\leq U^c(f(x)) \ \Rightarrow \alpha \not\leq f^{\leftarrow}(U^c)(x) = f^{\leftarrow}(U)^c(x)$. That is $x_a \propto f^{\leftarrow}(U)$. Since, $f^{\rightarrow}$ is continuous, $f^{\leftarrow}(U) \in \delta$ and hence $f^{\leftarrow}(U) \in Q(x_\alpha)$. Thus, $f^{\leftarrow}(A)$ is an open $\alpha - Q-$cover of A.

Since A is $N-$compact, $\mathcal{A}$ has a finite subfamily $\mathcal{A}_n = \{U_i : 1 \leq i \leq n\}$ such that $f^{\leftarrow}(\mathcal{A}_n)$ is an $\bar{\alpha} - Q-$cover of A.

Now, we show that, $\mathcal{A}_n$ is an $\bar{\alpha} - Q-$cover of $f^{\rightarrow}(A)$. Since, $f^{\leftarrow}(\mathcal{A})_n$ is an open $\bar{\alpha} - Q-$cover of A, there exists $\gamma \in \mathcal{B}(\alpha)$ such that $f^{\leftarrow}(\mathcal{A}_\backslash)$ is $\gamma - Q-$cover of A. This implies, $\gamma \sqsubseteq a$ and hence $\exists\lambda \in (0,1]$ such that $\gamma \sqsubseteq \lambda \sqsubseteq \alpha$. So, $\lambda \in \mathcal{B}(\alpha)$ and hence we have, $\lambda \leq f^{\leftarrow}(A)(y) = \vee\{A(x) : x \in X, f(x) = y\}$. Now, $\gamma \sqsubseteq \lambda$ implies, $\gamma \not\leq (f^{\leftarrow}(U_i))^c(x) = f^{\leftarrow}(U_i^c)(x) = U_i^x(f(x)) = U_i^c(y)$, for some $1 \leq i \leq n$ such that $x_\gamma \propto f^{\leftarrow}(U_i)$.
By $\gamma \sqsubseteq \lambda$ and hence $\gamma \leq \lambda$, we have $\lambda \not\leq U_i^c(y)$. Thus $y_\lambda \propto U_i$ for some $1 \leq i \leq n$. So, $\mathcal{A}_n$ is an open $\lambda - Q-$cover of $f^{\rightarrow}(A)$ and hence $\mathcal{A}_n$is an $\bar{\alpha} - Q-$cover of $f^{\leftarrow}(A)$.
Therefore, $f^{\rightarrow}(A)$ is an $N-$compact. □

Definition 54 (Net in X)**.** Let X be a non-empty ordinary set and D be a directed set then every mapping $S : D \rightarrow X$ is called a net in X and D is called the index set of S.

Theorem 5.0.3. Let $\langle \mathcal{F}(X), \delta\rangle$ be a fuzzy topological space. Let $A, B, C \in \mathcal{F}(X)$ such that A be a $N-$compact and B be closed. Then $A \wedge B$ is $N-$compact.

Proof. Let S be an $\alpha-$net in $A \wedge B$. Then S is also an $\alpha-$net in A. Since, A is $N-$compact, S has a cluster point x_α in A such that $ht(\alpha) = \alpha$. But, S is also a net in closed subset B, we have $x_\alpha \leq B$.
So, $x_a \leq A \wedge B$, i.e., x_a is a cluster point of δ in $A \wedge B$ such that $ht(\alpha) = \alpha$. Hence, $A \wedge B$ is $N-$compact. □

Chapter 6

Fuzzy Mapping

Definition 55 (Fuzzy Mapping)**.** Let X and Y be two non-empty set and let $f : X \to Y$ be an ordinary mapping. A fuzzy mapping $f^{\to} : \langle \mathcal{F}(X), \delta\rangle \to \langle \mathcal{F}(Y), \mu\rangle$ is defined by $f^{\to}(A)(y) = \bigvee\{A(x)|x \in X, f(x) = y\}\forall y \in Y$, and a fuzzy reverse mapping $f^{\leftarrow} : \langle \mathcal{F}(Y), \mu\rangle \to \langle \mathcal{F}(X), \delta\rangle$ is defined by $f^{\leftarrow}(B)(x) = B(f(x))\,\forall x \in X$.

Definition 56 (Continuous Fuzzy Mapping)**.** Let $\langle \mathcal{F}(X), \delta\rangle$ and $\langle \mathcal{F}(Y), \mu\rangle$ be two fuzzy topological space. A fuzzy mapping $f^{\to} : \langle \mathcal{F}(X), \delta\rangle \to \langle \mathcal{F}(Y), \mu\rangle$ is called continuous if for each $v \in \mu$, $f^{\to}(v) \in \delta$.

Definition 57 (Open Fuzzy Mapping)**.** Let $\langle \mathcal{F}(X), \delta\rangle$ and $\langle \mathcal{F}(Y), \mu\rangle$ be two fuzzy topological space. A fuzzy mapping $f^{\to}$ is called open if for each $u \in \delta$, $f^{\to}(u) \in \mu$.

Definition 58 (Closed Fuzzy Mapping)**.** Let $\langle \mathcal{F}(X), \delta\rangle$ and $\langle \mathcal{F}(Y), \mu\rangle$ be two fuzzy topological space. A fuzzy mapping $f^{\to}$ is called closed if for each closed set $F \in \delta$, $f^{\to}(F)$ is closed in μ.

Theorem 6.0.1. Let $\langle \mathcal{F}(X), \delta\rangle$ and $\langle \mathcal{F}(Y), \mu\rangle$ be two fuzzy topological spaces and $f : X \to Y$ be an ordinary mapping. Then for each $a \in [0, 1]$ and every $A \in \mathcal{F}(X)$, $f^{\to}(aA) = af^{\to}(A)$.

Proof. For all $a \in [0, 1]$, $\forall A \in \mathcal{F}(X)$ and $\forall y \in Y$ we have,

$$\begin{aligned} f^{\to}(aA)(y) &= \vee\{(aA)(x)|x \in X,\ f(x) = y\} \\ &= \vee\{a \wedge (A)(x)|x \in X,\ f(x) = y\} \\ &= a \wedge (\vee\{(A)(x)|x \in X,\ f(x) = y\}) \\ &= a \wedge f^{\to}(A)(y) \\ &= (af^{\to}(A))(y) \end{aligned}$$

Thus, $f^{\to}(aA) = af^{\to}(A)$. $\square$