Chapter 1

Ordered Sets

1.1 Order/Partial Order

Definition 1. Let P be a set. An *order* (or, *partial order*) on P is a binary relation ($f : A \times A \to A$) $\leq$ on P such that $\forall x, y, z \in P$

(i) $x \leq x$, (reflexivity)

(ii) $x \leq y$ and $y \leq x$ imply $x = y$, (antisymmetry)

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$. (transitivity)

A set P equipped with an order relation $\leq$ is said to be an ordered set (or, partially ordered set) or, poset. The order relation overtly we write $\langle P; \leq \rangle$.

On any set $=$ is an order, called *discrete order*. A relation $\leq$ on a set P which is reflexive and transitive but not necessarily antisymmetric is called a *quasi-order/pre-order*.

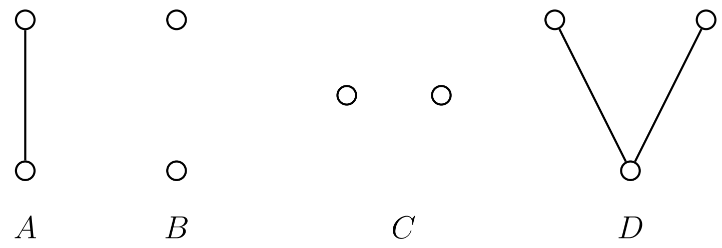
1.2 Chain, Antichain

Definition 2. Let P be an ordered set. Then P is a *chain* if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of P are comparable). Alternative names for chains are *linearly ordered set* and *totally ordered set*.

Definition 3. The ordered set P is an *antichain* if $x \leq y$ in P only if $x = y$.

Note. With the induced order, any subset of a chain (an antichain) is a chain (antichain).

Let P be the $n-$element set $\{0, 1, \ldots, n-1\}$. We write $\mathbf{n}$ to denote the chain obtained by giving P the order in which $0 < 1 < \cdots < n-1$ and $\bar{\mathbf{n}}$ for P regarded as an antichain. Any set S may be converted into antichain $\bar{S}$ by giving S the discrete order.



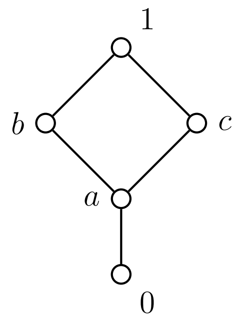
Here A, D are chains B, C and are antichains.

1.3 Cover

Definition 4. Let P be an ordered set and let $x, y \in P$. We say x is *covered by* y (or y covers x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there be no element z of P with $x < z < y$.

Examples

- In the chain $\mathbb{N}$, we have $m \prec n$ if and only if $n = m + 1$.
- In $\mathbb{R}$, there are no pairs x, y such that $x \prec y$.
- In $\mathcal{P}(X)$, we have $A \prec B$ if and only if $B = A \cup \{b\}$, for some $b \in X \setminus A$.



Here, 1 covers b and c, b covers a, c covers a and a covers 0.

1.4 Diagrams

Let P be a finite ordered set. We can represent P be a configuration of circles (representing the elements of P) and interconnecting lines (indicating the covering relation). The construction goes as follows

1. To each point $x \in P$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with center at $p(x)$.
2. For each covering pair $x \prec y$ in P, take a line segment $\ell(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.
3. Carry out (1) and (2) in such a way that
   (a) if $x \prec y$, then $p(x)$ is 'lower' than $p(y)$ (that is, in standard Cartesian coordinates, has a strictly smaller second coordinate),
   (b) the circle at $p(z)$ does not intersect the line segment $\ell(x, y)$ if $z \neq x$ and $z \neq y$.

A configuration satisfying these conditions is called a *diagram* (or *Hasse diagram*) of P.

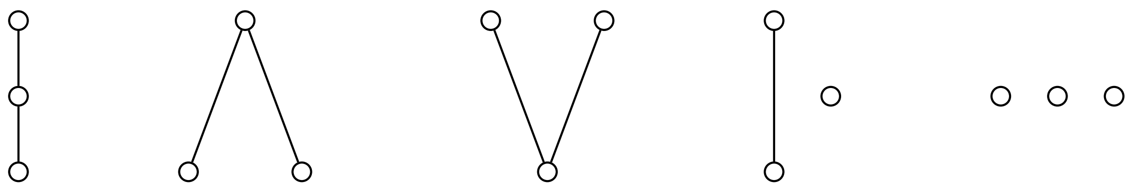


Figure 1.1: All possible sets with three elements.

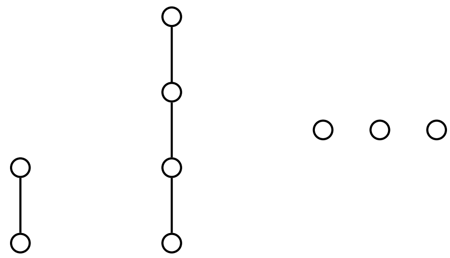


Figure 1.2: Diagrams of $\mathbf{2}$, $\mathbf{4}$ and $\bar{\mathbf{3}}$

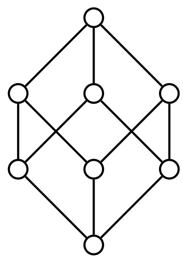


Figure 1.3: $\mathcal{P}(\{1,2,3\})$. Also known as the *cube*.

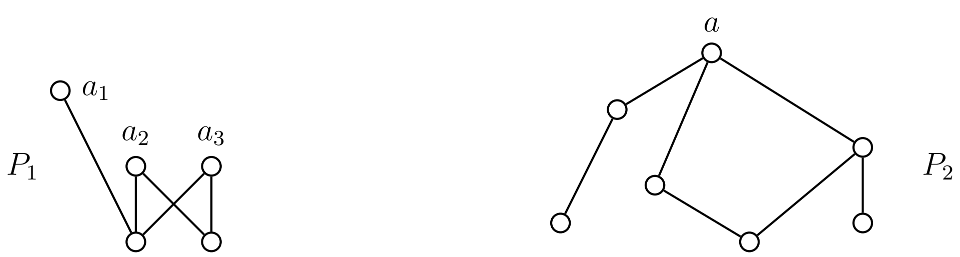
1.5 Bottom and Top

Let P be an ordered set. We say P has a bottom element if there exists $\bot \in P$ (called *bottom*) with the property that $\bot \leq x$ for all $x \in P$. Dually, P has a top element if there exists $\top \in P$ such that $x \leq \top$ for all $x \in P$.

1.6 Maximal and Minimal Element

Let P be an ordered set and let $Q \subseteq P$. Then $a \in Q$ is a *maximal* element of Q if $a \leq x$ and $x \in Q$ imply $a = x$. We denote the set of maximal elements of Q by $\max Q$. If Q (with the order inherited from P) has a top element, $\top_Q$, then $\max Q = \{\top_Q\}$; in this case $\top_Q$ is called the *greatest* (or *maximum*) element of Q, and we write $\top_Q = \max Q$.

A *minimal* element of $Q \subseteq P$ and $\min Q$, the *least* (or *minimum*) element of Q (when these exist) are defined dually, that is by reversing the order.



In the above figure P_1 has maximal elements a_1, a_2, a_3, but no greatest element; a is the greatest element of P_2.

Let P be a finite ordered set. Then any non-empty subset of P has at least one maximal element and, for each $x \in P$, there exists $y \in \max P$ with $x \leq y$. In general a subset Q of an ordered set P may have many maximal elements, just one, or none. A subset of the chain $\mathbb{N}$ has a maximal element if and only if it is finite and non-empty.

1.7 Sums of Ordered Sets

1.7.1 Disjoint Union

Suppose that P and Q are (disjoint) ordered sets. The disjoint union $P \dot\cup Q$ of P and Q is the ordered set formed by defining $x \le y$ in $P \dot\cup Q$ if and only if either $x, y \in P$ and $x \le y$ in P or $x, y \in Q$ and $x \le y$ in Q. A diagram for $P \dot\cup Q$ is formed by placing side by side diagrams for P and Q.

1.7.2 Linear Sum

Let P and Q be (disjoint) ordered sets. The linear sum $P \oplus Q$ is defined by taking the following order relation on $P \cup Q$: $x \le y$ if and only if

$$\begin{aligned} & x, y \in P \text{ and } x \le y \text{ in } P, \\ \text{or} \quad & x, y \in Q \text{ and } x \le y \text{ in } Q, \\ \text{or} \quad & x \in P \text{ and } y \in Q. \end{aligned}$$

A diagram for $P \oplus Q$ (when P and Q are finite) is obtained by placing a diagram for P directly below a diagram for Q and then adding a line segment from each maximal element of P to each minimal element of Q.

Note. Each of the operations $\dot\cup$ and $\oplus$ is associative; for (pairwise disjoint) ordered sets P, Q and R,

$$P \dot\cup (Q \dot\cup R) = (P \dot\cup Q) \dot\cup R \qquad \text{and} \qquad P \oplus (Q \oplus R) = (P \oplus Q) \oplus R$$

1.7.3 Examples

1.

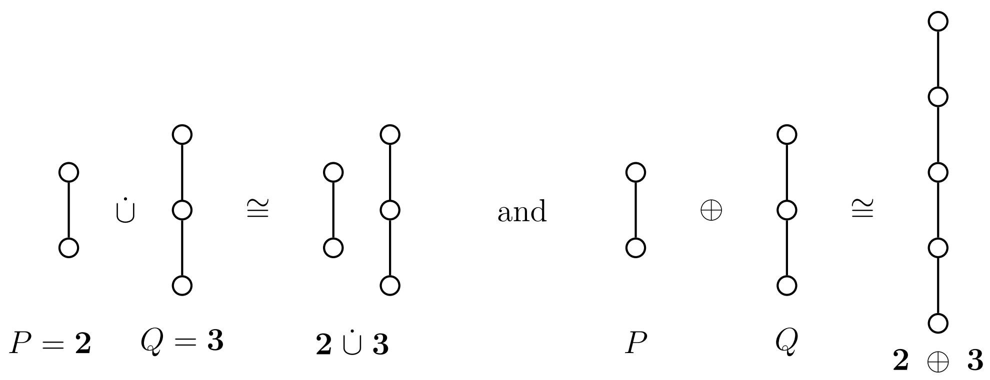


Figure 1.4: $P = \mathbf{2}$, $Q = \mathbf{3}$, $P \dot\cup Q = \mathbf{2} \dot\cup \mathbf{3}$ and $P \oplus Q = \mathbf{2} \oplus \mathbf{3} \cong \mathbf{5}$

2.

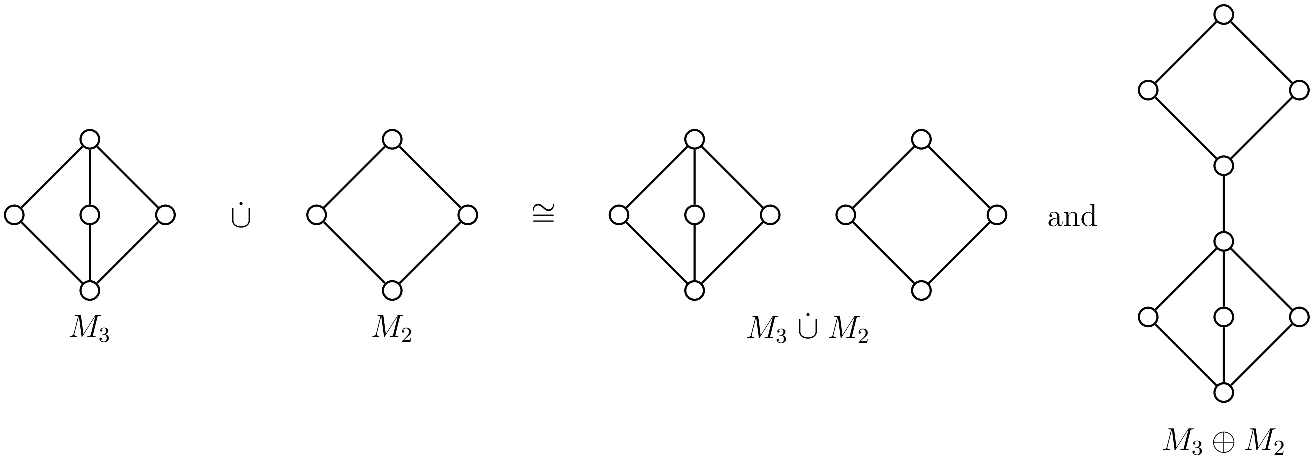
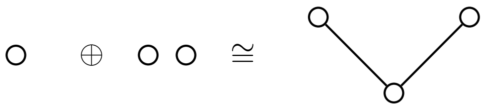
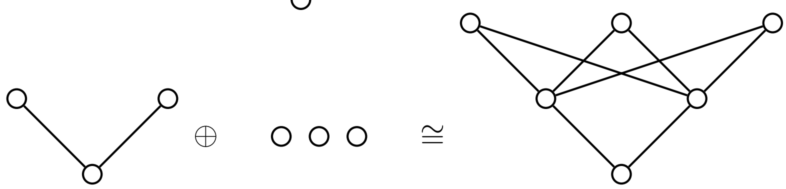
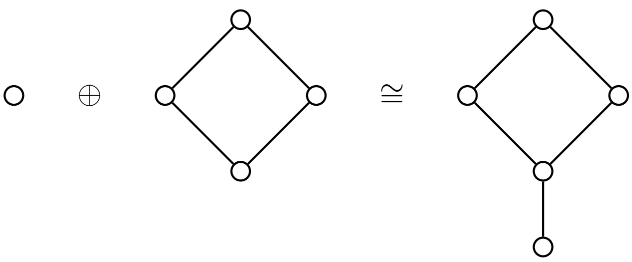


Figure 1.5: $P = M_3$, $Q = M_2$, $P \dot\cup Q = M_3 \dot\cup M_2$ and $P \oplus Q = M_3 \oplus M_2$

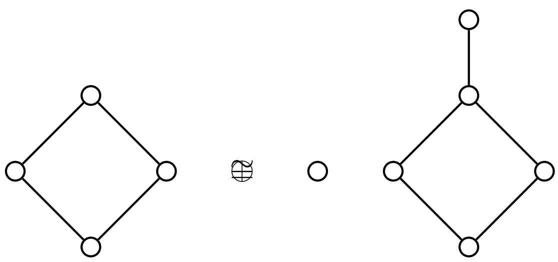
3. For $P \oplus Q$, we consider $P = \bar{\mathbf{1}}$, $Q = \bar{\mathbf{2}}$, $R = \bar{\mathbf{3}}$.

$\therefore P \oplus Q =$ 

Also, $P \oplus Q \oplus R =$ 

$1 \oplus 2^2 =$ 

4.

and $2^2 \oplus 1 =$ 

5. a