# Chapter 1

# System of IVPs

## 1.1 Runge-Kutta Order 4

**Problem 1.1.** Consider the 2nd order IVP

$$y'' - 2y' + 2y = e^{2t}\sin t; \qquad 0 \le t \le 1$$

with $y(0) = -0.4$, $y'(0) = -0.6$.

**Solution.** Let

$$\begin{aligned} & u_1(t) = y(t) \\ \text{and } & y'(t) = u_1^{'}(t) = u_2(t) \\ \therefore\ & y^{''}(t) = u_1^{''}(t) = u_2^{'}(t) \end{aligned}$$

So, $u_1^{'}(t) = \dfrac{\mathrm{d}\,u_1}{\mathrm{d}\,t} = u_2(t)$

$$\begin{aligned} & u_2^{'}(t) - 2u_2(t) + 2u_1(t) = e^{2t}\sin t \\ \Rightarrow\ & u_2^{'}(t) = 2u_2(t) - 2u_1(t) + e^{2t}\sin t \end{aligned}$$

So,

$$\begin{aligned} & f_1 = u_1^{'} = \frac{\mathrm{d}\,u_1}{\mathrm{d}\,t} = u_2 \\ \Rightarrow\ & u_1^{'} = f_1(t, u_1, u_2) \\ & u_2^{'} = \frac{\mathrm{d}\,u_2}{\mathrm{d}\,t} = e^{2t}\sin t - 2u_1 + 2u_2 \\ \Rightarrow\ & u_2^{'} = f_2(t, u_1, u_2) \end{aligned}$$

with $u_1(0) = -0.4 \Rightarrow \quad u_1^0 = -0.4$ and $u_2(0) = -0.6 \Rightarrow \quad u_2^0 = -0.6$
Using 4th order Runge-Kutta with $h = 0.1$,

$$\begin{aligned} k_1 &= hf_1(t_0,\, u_1^0,\, u_2^0) \\ &= 0.1f_1(0,\, -0.4,\, -0.6) \\ &= 0.1 \times [(-0.6)] \\ &= -0.06 \\ l_1 &= hf_2(t_0,\, u_1^0,\, u_2^0) \\ &= 0.1f_1(0,\, -0.4,\, -0.6) \\ &= 0.1 \times [e^{2\times 0}\sin\, 0 - 2 \times (-0.4) + 2 \times (-0.6)] \\ &= -0.04 \end{aligned}$$

$$\begin{aligned}
k_2 &= hf_1\left(t_0+\frac{h}{2},\, u_1^0+\frac{k_1}{2},\, u_2^0+\frac{l_1}{2}\right)\\
&= hf_1\left(0+\frac{0.1}{2},\, -0.4+\frac{-0.06}{2},\, -0.6+\frac{-0.04}{2}\right)\\
&= hf_1\left(\frac{0.1}{2},\, -0.4-\frac{0.06}{2},\, -0.6-\frac{0.04}{2}\right)\\
&= h\left(-0.6-\frac{0.04}{2}\right)\\
&= -0.062\\
l_2 &= hf_2\left(t_0+\frac{h}{2},\, u_1^0+\frac{k_1}{2},\, u_2^0+\frac{l_1}{2}\right)\\
&= hf_2\left(0.05,\, -0.4+\frac{-0.06}{2},\, -0.6+\frac{-0.04}{2}\right)\\
&= hf_2\left(0.05,\, -0.43,\, -0.62\right)\\
&= 0.1\times\left[e^{2\times 0.05}\sin(0.05)-2\times(-0.43)+2\times(-0.62)\right]\\
&= -0.032476
\end{aligned}$$

$$\begin{aligned}
k_3 &= hf_1\left(t_0+\frac{h}{2},\, u_1^0+\frac{k_2}{2},\, u_2^0+\frac{l_2}{2}\right)\\
&= hf_1\left(0.05,\, -0.4+\frac{-0.062}{2},\, -0.6+\frac{-0.032476}{2}\right)\\
&= h\left(-0.6-\frac{0.032476}{2}\right)\\
&= -0.06162381\\
l_3 &= hf_2\left(t_0+\frac{h}{2},\, u_1^0+\frac{k_2}{2},\, u_2^0+\frac{l_2}{2}\right)\\
&= hf_2\left(0.05,\, -0.4+\frac{-0.062}{2},\, -0.6+\frac{-0.032476}{2}\right)\\
&= hf_2\left(0.05,\, -0.431,\, -0.616238\right)\\
&= 0.1\times\left[e^{2\times 0.05}\sin(0.05)-2\times(-0.431)+2\times(-0.616238)\right]\\
&= -0.03152409
\end{aligned}$$

$$\begin{aligned}
k_4 &= hf_1\left(t_0+h,\, u_1^0+k_3,\, u_2^0+l_3\right)\\
&= hf_1\left(0+0.1,\, -0.4-0.06162381,\, -0.6-0.03152409\right)\\
&= h\left(-0.6-0.03152409\right)\\
&= -0.063152409\\
l_4 &= hf_2\left(t_0+h,\, u_1^0+k_3,\, u_2^0+l_3\right)\\
&= hf_2\left(0.1,\, -0.46162381,\, -0.63152409\right)\\
&= 0.1\times\left[e^{2\times 0.1}\sin(0.1)-2\times(-0.46162381)+2\times(-0.63152409)\right]\\
&= -0.02178637
\end{aligned}$$

So,

$$\begin{aligned}
u_1^1 &= u_1^0+\frac{1}{6}(k_1+2k_2+2k_3+k_4) = -0.4617333\\
u_2^1 &= u_2^0+\frac{1}{6}(l_1+2l_2+2l_3+l_4) = -0.63163124
\end{aligned}$$

Hence,

$$u_1^1 = y(0.1) = -0.4617333$$

and

$$u_2^1 = y'(0.1) = -0.63163124$$

Actual Solution: $y(t) = 0.2e^{2t}(\sin t - 2\cos t)$

\# $\dfrac{\mathrm{d}\,y}{\mathrm{d}\,t} = f(t, y); \qquad y(t_0) = y_0$

RK4: $k_1 = hf(t_i, y_i), \qquad k_2 = hf\left(t_i + \frac{h}{2},\, y_i + \frac{k_1}{2}\right), \qquad k_3 = hf\left(t_i + \frac{h}{2},\, y_i + \frac{k_2}{2}\right), \qquad k_4 = hf(t_{i+1}, y_i + k_3)$

$$\therefore \quad y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

\#

$$\begin{aligned}
\frac{\mathrm{d}\,u_1}{\mathrm{d}\,t} &= f(t, u_1, u_2, \ldots, u_m)\\
\frac{\mathrm{d}\,u_2}{\mathrm{d}\,t} &= f(t, u_1, u_2, \ldots, u_m)\\
&\vdots\\
\frac{\mathrm{d}\,u_m}{\mathrm{d}\,t} &= f(t, u_1, u_2, \ldots, u_m)
\end{aligned}$$

for $a \le t \le b$ with initial condition [Here $a = t_0$].

$$\begin{aligned}
&u_1(a) = \alpha_1, \quad u_2(a) = \alpha_2, \quad \ldots, \quad u_m(a) = \alpha_m\\
\Rightarrow\ &u_1^0 = \alpha_1, \quad u_2^0 = \alpha_2, \quad \ldots, \quad u_m^0 = \alpha_m \text{ [1]}
\end{aligned}$$

Consider, $u_1^j,\, u_2^j, \ldots, u_m^j$ have been computed. We obtain, $u_1^{j+1},\, u_2^{j+1}, \ldots, u_m^{j+1}$ by first calculating

$$\begin{aligned}
k_{1,i} &= hf_i(t_i, u_1^j, u_2^j, \ldots, u_m^j) && \text{for each } i = 1, 2, \ldots, m\\
k_{2,i} &= hf_i\left(t_i + \frac{h}{2},\, u_1^j + \frac{k_{1,1}}{2},\, u_2^j + \frac{k_{1,2}}{2}, \ldots, u_m^j + \frac{k_{1,m}}{2}\right) && \text{for each } i = 1, 2, \ldots, m\\
k_{3,i} &= hf_i\left(t_i + \frac{h}{2},\, u_1^j + \frac{k_{2,1}}{2},\, u_2^j + \frac{k_{2,2}}{2}, \ldots, u_m^j + \frac{k_{2,m}}{2}\right) && \text{for each } i = 1, 2, \ldots, m\\
k_{4,i} &= hf_i\left(t_i + h,\, u_1^j + k_{3,1},\, u_2^j + k_{3,2}, \ldots, u_m^j + k_{3,m}\right) && \text{for each } i = 1, 2, \ldots, m
\end{aligned}$$

and then

$$u_i^{j+1} = u_i^j + \frac{1}{6}\left(k_{1,i} + 2k_{2,i} + 2k_{3,i} + k_{4,i}\right) \qquad \text{for each } i = 1,\, 2,\, \ldots, m \text{ and } j = 0,\, 1,\, 2,\, \ldots$$

*Note.* $k_{1,1},\, k_{1,2},\, k_{1,3}, \ldots, k_{1,m}$ must be computed before any of the terms of the form $k_{2,1},\, k_{2,2},\, k_{2,3}, \ldots, k_{2,m}$ i.e., $k_{2,i}$. In general, each $k_{l,1},\, k_{l,2},\, k_{l,3}, \ldots, k_{l,m}$ must be computed before any of the expressions $k_{l+1,i}$.

[1]
$$\begin{aligned}
&u_1(a) = \alpha_1\\
\Rightarrow\ &u_1(t_0) = \alpha_1\\
\Rightarrow\ &u_1^0 = \alpha_1
\end{aligned}$$