Chapter 1

Multistep Methods

Methods using the approximation at more than one previous mesh points to determine the approximation at the next point are called multistep methods.

Definition 1. An $m-$step multistep method for solving the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \tag{1.1}$$

is one whose difference equation for finding the approximation w_{i+1} at mesh point t_{i+1} can be represented by the following equation, where m is an integer grater than 1:

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})] \tag{1.2}$$

for $i = m-1, m, \ldots, N-1$, where $h = (b-a)/N$. The a_0, $a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_m$ are constants and the starting values $w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2, \ldots, w_{m-1} = \alpha_{m-1}$ are specified.

When $b_m = 0$, the method is called explicit or open, since equation (1.2) then gives w_{i+1} explicitly in terms of previously determined values. When $b_m \neq 0$, the method is called implicit or closed, since w_{i+1} occures on both sides of (1.2) and is specified only implicitly.

Adams-Bashforth two-step method:

$$\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1 \\ &w_{i+1} = w_i + \frac{h}{2}[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \quad \text{for } i = 1, 2, 3, \ldots, N-1 \end{aligned} \tag{1.3}$$

Local Truncation error:

$$T_{i+1}(h) = \frac{5}{12} y'''(\mu_i) h^2 \qquad \mu_i \in (t_{i-1}, t_{i+1})$$

Adams-Bashforth three-step method:

$$\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2 \\ &w_{i+1} = w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})] \quad \text{for } i = 2, 3, \ldots, N-1 \end{aligned} \tag{1.4}$$

Local Truncation error:

$$T_{i+1}(h) = \frac{3}{8} y^{(4)}(\mu_i) h^3 \qquad \mu_i \in (t_{i-2}, t_{i+1})$$

Adams-Bashforth four-step method:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3 \tag{1.5}$$

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})] \quad \text{for } i = 3, 4, 5, \ldots, N-1 \tag{1.6}$$

Local Truncation error:

$$T_{i+1}(h) = \frac{251}{720} y^{(5)}(\mu_i) h^4 \qquad \mu_i \in (t_{i-3}, t_{i+1})$$

Adams-Moulton two-step method:

$$\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1 \\ &w_{i+1} = w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})] \quad \text{for } i = 1, 2, 3, \ldots, N-1 \end{aligned} \tag{1.7}$$

Local Truncation error:

$$T_{i+1}(h) = -\frac{1}{24}y^{(4)}(\mu_i)h^3 \qquad \mu_i \in (t_{i-1}, t_{i+1})$$

Adams-Moulton three-step method:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2$$

$$w_{i+1} = w_i + \frac{h}{24}\left[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\right] \quad \text{for } i = 2, 3, \ldots, N-1 \tag{1.8}$$

Local Truncation error:

$$T_{i+1}(h) = -\frac{19}{720}y^{(5)}(\mu_i)h^4 \qquad \mu_i \in (t_{i-2}, t_{i+1})$$

Adams-Moulton four-step method:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + \frac{h}{720}\left[251f(t_{i+1}, w_{i+1}) + 646f(t_i, w_i) - 264f(t_{i-1}, w_{i-1}) + 106f(t_{i-2}, w_{i-2}) - 19f(t_{i-3}, 2_{i-3})\right] \quad \text{for } i = 3, 4, \ldots \tag{1.9}$$

Local Truncation error:

$$T_{i+1}(h) = -\frac{3}{160}y^{(6)}(\mu_i)h^5 \qquad \mu_i \in (t_{i-3}, t_{i+1})$$

Example. Consider the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5$$

and the approximations given by the Adams-Bashforth four-step method and the Adams-Moulton three-step method, both using $h = 0.2$.
The Adams-Bashforth 4th step method has the difference equation

$$w_{i+1} = w_i + \frac{h}{24}\left[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})\right] \quad \text{for } i = 3, 4, 5, \ldots, 9$$

Exact solution: $y'(t) = (t+1)^2 - 0.5e^t$. Also, $f(t, y) = y - t^2 + 1$, $h = 0.2$ and $t_i = 0.2i$.
So,

$$w_{i+1} = \frac{1}{24}\left[35w_i - 11.8w_{i-1} + 7.4w_{i-2} - 1.8w_{i-3} - 0.192i^2 - 0.192i + 4.736\right]$$

Adams-Moulton 3rd step method has the difference equation

$$w_{i+1} = w_i + \frac{h}{24}\left[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\right] \quad \text{for } i = 2, 3, \ldots, 9$$

which reduces to,

$$w_{i+1} = \frac{1}{24}\left[1.8w_{i+1} + 27.88w_i + w_{i-1} + 0.2w_{i-2} - 0.192i^2 - 0.192i + 4.736\right]$$

To use this method explicitly, we solve for w_{i+1} which gives

$$w_{i+1} = \frac{1}{22.2}\left[27.8w_i - w_{i-1} + 0.2w_{i-2} - 0.192i^2 - 0.192i + 4.736\right]$$

The results in the following table were obtained using the exact solutions from $y(t) = (t+1)^2 - 0.5e^t$ for α_0, α_1, α_2 and α_3 for Adams-Bashforth case and α_0, α_1, and α_2 in the Adams-Moulton case.

		Adams- Bashforth		Adams-Moulton	
t_i	Exact	w_i	Error	w_i	Error
0.0	0.5000000 α_0				
0.2	0.8292986 α_1				
0.4	1.2140877 α_2				
0.6	1 .6489406 α_3			1 .6489341	0.0000065
0.8	2.1272295	2.1273124	0.0000828	2.1272136	0.0000160
1.0	2.6408591	2.6410810	0.0002219	2.6408298	0.0000293
1.2	3.17994 15	3.1803480	0.0004065	3.1798937	0.0000478
1.4	3.7324000	3.7330601	0.0006601	3.7323270	0.0000731
1.6	4.2834838	4.2844931	0.0010093	4.2833767	0.0001071
1.8	4.8151763	4.8166575	0.0014812	4.8150236	0.0001527
2.0	5.3054720	5.3075838	0.0021119	5.3052587	0.0002132

Implicit Adams-Moulton method gave better results than the explicit Adams-Bashforth method of the same order.

Note. Implicit method is not always possible to convert an explicit form, for example

$$y' = e^y, \quad 0 \leq t \leq 0.25, \quad y(0) = 1$$

Since $f(t, y) = e^y$ the three-step Adams-Moulton method has the difference equation

$$w_{i+1} = w_i + \frac{h}{24}\left[9e^{w_{i+1}} + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}\right]$$

Which cannot be solved explicitly for w_{i+1}