Chapter 1

Newton's Method For Non-Linear Systems of Equations

1.1 Newton's Method

$$\mathbf{x}^{(k)} = \mathbf{G}\left(\mathbf{x}^{(k-1)}\right) = \mathbf{x}^{(k-1)} - J\left(\mathbf{x}^{(k-1)}\right)^{-1}\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$$

where

$$J(\mathbf{x}^{(k)}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - J(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$$

Example.

$$\begin{aligned} &3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0 \\ &{x_1}^2 - 81(x_2+0.1)^2 + \sin x_3 + 1.06 = 0 \\ &e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0 \end{aligned}$$

So,

$$J(x_1,x_2,x_3) = \begin{bmatrix} 3 & x_3\sin x_2x_3 & x_2\sin x_2x_3 \\ 2x_1 & -162(x_2+0.1) & \cos x_3 \\ -x_2e^{-x_1x_2} & -x_1e^{-x_1x_2} & 20 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix}$$

where

$$\begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix} = -\left(J\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)\right)^{-1}\mathbf{F}\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)$$

Thus at the kth step, the linear system $J\left(\mathbf{x}^{(k-1)}\right)\mathbf{y}^{(k-1)} = -\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$ must be solved, where,

$$J(\mathbf{x}^{(k-1)}) = \begin{bmatrix} 3 & x_3^{(k-1)}\sin x_2^{(k-1)}x_3^{(k-1)} & x_2^{(k-1)}\sin x_2^{(k-1)}x_3^{(k-1)} \\ 2x_1^{(k-1)} & -162(x_2^{(k-1)}+0.1) & \cos x_3^{(k-1)} \\ -x_2^{(k-1)}e^{-x_1^{(k-1)}x_2^{(k-1)}} & -x_1^{(k-1)}e^{-x_1^{(k-1)}x_2^{(k-1)}} & 20 \end{bmatrix}, \qquad \mathbf{y}^{(k-1)} = \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix}$$

and

$$\mathbf{F}\left(x^{(k-1)}\right) = \begin{bmatrix} 3x_1^{(k-1)} - \cos\left(x_2^{(k-1)}x_3^{(k-1)}\right) - \frac{1}{2} \\ \left(x_1^{(k-1)}\right)^2 - 81\left(x_2^{(k-1)}+0.1\right)^2 + \sin x_3^{(k-1)} + 1.06 \\ e^{-x_1^{(k-1)}x_2^{(k-1)}} + 20x_3^{(k-1)} + \frac{10\pi-3}{3} \end{bmatrix}$$

Let the initial approximation be $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$. The results using this iterative procedure are shown in the table below.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\Vert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\Vert_\infty$
0	0.10000000	0.10000000	-0.10000000	
1	0.50003702	0.01946686	-0.52152047	0.422
2	0.50004593	0.00158859	-0.52355711	1.79×10^{-2}
3	0.50000034	0.00001244	-0.52359845	1.58×10^{-3}
4	0.50000000	0.00000000	-0.52359877	1.24×10^{-5}
5	0.50000000	0.00000000	-0.52359877	0

Newton's method can converges very rapidly with comparison of fixed point iteration.

Problem 1.1. Use Newton's and Quasi-Newton's method to find a solution to the following non-linear systems with the given initial approximation. Iterate until $\left\Vert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\Vert_\infty < 10^{-6}$

1.

$$\begin{aligned}
&3x_1^2 - x_2^2 = 0 \\
&3x_1x_2^2 - x_1^3 - 1 = 0 \\
&\mathbf{x}^{(0)} = (1,1)^t
\end{aligned}$$

2.

$$\begin{aligned}
&\log\left(x_1^2 + x_2^2\right) - \sin(x_1x_2) = \log 2 + \log \pi \\
&e^{x_1 - x_2} + \cos(x_1x_2) = 0 \\
&\mathbf{x}^{(0)} = (2,2)^t
\end{aligned}$$

3.

$$\begin{aligned}
&x_1^2 + x_2 - 37 = 0 \\
&x_1 - x_2^2 - 5 = 0 \\
&x_1 + x_2 + x_3 - 3 = 0 \\
&\mathbf{x}^{(0)} = (0,0,0)^t
\end{aligned}$$

4.

$$\begin{aligned}
&x^2 - y^2 = 4 \\
&x^2 + y^2 = 16 \\
&\mathbf{x}^{(0)} = (2\sqrt{2}, 2\sqrt{2})^t
\end{aligned}$$

See book of Sastry for problem 4.