# *Fuzzy Topology*

MAT514


Prof. Dr Shamsun Naher Begum

*Shahjalal University of Science and Technology*


Edited by
Mehedi Hasan

# Preface

This is a compilation of lecture notes with some books and my own thoughts. If there are any mistake/typing error or, for any query mail me at mehedi12@student.sust.edu.

# Contents

# Part I

# Sheet

# Chapter 1

# Fuzzy Sets

**Definition 1** (Characteristic function)**.** Let $X$ be a universal set and $A \subseteq X$. Then the function[1]

$$\chi_A(x) = \begin{cases} 1; & x \in A \\ 0; & x \notin A \end{cases}$$

is characteristic function of $A$ in $X$.

**Definition 2** (Fuzzy Set)**.** A fuzzy set[2] $A \subseteq X$ is a mapping $A : X \to [0,1]$, where, $A(x) = y \in [0,1]$ is called the membership function or, grade of membership of $x$ in $A$. The collection of all fuzzy sets of $X$ is denoted by $\mathcal{F}(X)$.

**Definition 3** (Fuzzy subset)**.** A fuzzy set $A$ is called a fuzzy subset of another fuzzy set $B$ if $A(x) \leq B(x)$ $\forall x \in X$. We denote it by $A \leq B$.

**Definition 4** (Empty fuzzy set)**.** A fuzzy set $A$ is called empty fuzzy set if $\forall x \in X$ $A(x) = 0$. The empty fuzzy set is denoted by $\underline{0}$. Thus, $\underline{0}(x) = 0$ $\forall x \in X$.

**Definition 5** (Total fuzzy set)**.** The total fuzzy set $\underline{1}$ is defined by $\underline{1}(x) = 1$ $\forall x \in X$.

**Definition 6** (Equality of two fuzzy sets)**.** Two fuzzy sets $A$ and $B$ of $X$ is said to be equal iff $A \leq B$ and $B \leq A$.

**Example** (Empty and Total fuzzy set)**.** Suppose, $A : X \to [0,1]$ where $X = [20, 80]$. Then,

$$\underline{0}(x) = \begin{cases} 0 & \text{if } 15 < x < 90 \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \underline{1}(x) = \begin{cases} 1 & \text{if } 20 \leq x < 90 \\ 0 & \text{otherwise} \end{cases}$$

**Example** (Fuzzy subset)**.** Suppose, $A : X \to [0,1]$ where, $X = [0, 100]$ defined by

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \frac{x}{75}; & \text{if } 40 \leq x < 75 \\ 1; & \text{if } 75 \leq x \leq 100 \end{cases}$$

and $B : X = [0, 100] \to [0,1]$ defined by

$$B(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \frac{x}{95}; & \text{if } 40 \leq x < 95 \\ 1; & \text{if } 95 \leq x \leq 100 \end{cases}$$

Then, $B(x)$ is a subset of $A(x)$. Since, $B(x) \leq A(x)$ $\forall x \in X$.

[1] Some authors use $\mu$ as characteristic function.

[2] Sometimes fuzzy set is denoted by $\underset{\sim}{A}$.

## 1.1 Fuzzy Set Operations

**Definition 7** (Union of Fuzzy Sets)**.** Let $A,\ B \in \mathcal{F}(X)$. Then the union of $A$ and $B$ is denoted and defined by, $(A \vee B)(x) = \max\{A(x),\ B(x)\},\ \forall x \in X$.

**Definition 8** (Intersection of Fuzzy Sets)**.** Let $A,\ B \in \mathcal{F}(X)$. Then the intersection of $A$ and $B$ is denoted and defined by, $(A \wedge B)(x) = \min\{A(x),\ B(x)\},\ \forall x \in X$.

**Definition 9** (Complement of Fuzzy Set)**.** Let $A$ be a fuzzy set of $X$. Then, the complement of $A$ is denoted by $A^c$ and defined by $A^c(x) = 1 - A(x),\ \forall x \in X$.

**Example.** Given,

$$A_1 = \begin{cases} 1; & \text{if } 40 \leq x < 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases} \quad \text{and} \quad A_2 = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 0; & \text{if } 70 \leq x \leq 100 \end{cases}$$

1. Find the complement of $A_1$ and $A_2$.
2. Find $(A_1 \wedge A_2)(x)$ and $(A_1 \vee A_2)(x)$

Solution:

1. Complement of

$$A_1, A_1^c = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 1; & \text{if } 60 \leq x \leq 100 \end{cases}$$

Complement of

$$A_2, A_2^c = \begin{cases} 1; & \text{if } 40 \leq x < 50 \\ \dfrac{60-x}{10}; & \text{if } 50 \leq x < 60 \\ \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 1; & \text{if } 70 \leq x \leq 100 \end{cases}$$

2.

$$(A_1 \wedge A_2)(x) = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ 1 - \dfrac{x-50}{10}; & \text{if } 55 \leq x \leq 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases}$$

$$(A_1 \vee A_2)(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ \dfrac{x-50}{10}; & \text{if } 55 \leq x < 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 0; & \text{if } 70 \leq x < 100 \end{cases}$$

# Chapter 2

# Separation Axioms

**Definition 10** (Quasi $T_0-$space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a quasi $T_0-$space, if for every two distinct fuzzy points $x_a$ and $x_b$ with same support point $x$, there exists $U \in Q_\delta(x_a)$ such that $x_b \not\propto U$ or, there exists $V \in Q_\delta(x_b)$ such that $x_a \not\propto V$.

**Definition 11** (Sub $T_0-$space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a sub $T_0-$space, if for every two distinct $x, y \in X$, there exists $a \in [0,1]$ such that either $\exists U \in Q_\delta(x_a)$ with $y_a \not\propto U$ or, $\exists V \in Q_\delta(y_a)$ with $x_a \not\propto V$.

**Definition 12** ($T_0-$space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a $T_0-$space, if for every two distinct fuzzy points $x_a$ and $y_b$, $\exists U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, $\exists V \in Q_\delta(y_b)$ with $x_a \not\propto V$.

**Definition 13** ($T_1-$space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a $T_1-$space, if for every two distinct fuzzy points $x_a$ and $y_b$ such that $x_a \not\leq y_b$ then there exists $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ and, $\exists V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.

**Definition 14** ($T_2-$space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a $T_2-$space, if for every two distinct fuzzy points $x_a$ and $y_b$ (i.e., $x_a \neq y_b$) then there exists $U \in Q_\delta(x_a)$ and, $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

*Theorem* 2.0.1. Quasi $T_0$ property is heriditary.
or, Every subspace of a Quasi $T_0$ space is Quasi $T_0$ space.

*Proof.* Suppose, $\langle X, \delta \rangle$ be a fuzzy topological space which is Quasi $T_0-$space. Let $\langle Y, \mu \rangle$ be the subspace of $\langle X, \delta \rangle$. We have to prove that, $\langle Y, \mu \rangle$ be a Q-$T_0-$space.

Now, since, $Y \subseteq X$ so every $V \in \mu$, $V = U_{\restriction Y}$ for some $U \in \delta$. Let $y_a$ and $y_b$ be two distinct fuzzy points in $Y$ such that, $y_a \neq y_b$. Then as $Y \subseteq X$, we have $y_a$ and $y_b$ are in $X$ with $y_a \neq y_b$.

Again, since $\langle X, \delta \rangle$ is a Quasi $T_0-$space there exist $U \in Q_\delta(y_a)$ such that $y_b \not\propto U$ or, there exist $V \in Q_\delta(y_b)$ such that $y_a \not\propto V$. This implies, there is $U_{\restriction y} \in Q_{\delta \restriction y}(y_a)$ such that $y_b \not\propto U_{\restriction Y}$ or there is $V_{\restriction Y} \in Q_{\delta \restriction}(y_b)$ such that $y_a \not\propto V_{\restriction Y}$.

Thus, by definition of a Q-$T_0-$space $\langle Y, \mu \rangle$ is a Q-$T_0-$ space. □

*Theorem* 2.0.2. Every subspace of a $T_0-$space is $T_0-$space.

*Proof.* Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $\langle Y, \mu \rangle$ be a subspace of $\langle X, \delta \rangle$. Let $x_a$ and $y_b$ be two distinct points in $Y$. Then since, $Y \subseteq X$, we have, $x_a$ and $y_b$ in $X$ with $x_a \neq y_b$. Now since $\langle X, \delta \rangle$ is a fuzzy $T_0-$space. We have either there is $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, there is $V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.
Now, $U_{\restriction Y} \in Q_{\delta \restriction Y}(x_a)$ such that $y_b \not\propto U_{\restriction Y}$ as $x_a$, $y_b \in Y$ and $V_{\restriction Y} \in Q_{\delta \restriction Y}(y_b)$ such that $x_a \not\propto V_{\restriction Y}$.
Thus, $\langle Y, \mu \rangle$ is a $T_0-$space. □

*Theorem* 2.0.3. A fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ is a quasi-$T_0-$space iff for every $x \in X$ and $a \in [0,1]$ there exists $B \in \delta$ such that $B(x) = a$.

*Proof.* Suppose, $\langle \mathcal{F}(X), \delta \rangle$ be a quasi $T_0-$space. If $a = 0$, then it suffices to take $B = \underline{0}$. If $0 < a < 1$, we take a strictly monotonic increasing sequence of positive real numbers converging to $a$. Let $\Delta_n = (a_n, a_{n+1}]$, $n = 1, 2, 3, \ldots$.

Since $\langle \mathcal{F}(X), \delta \rangle$ be a quasi $T_0-$space, then for any $x \in X$ and $\Delta = (a_1, a_2)$ with $0 \leq a_1 < a_2 < 1$, there exists $B \in \delta$ such that $B(x) \in \Delta$.
From this property, we can say that, $\exists B_n \in \delta$ such that $B_n(x) \in \Delta_n$, for each $n$

$$\therefore\ B = \bigvee_{n=1}^{\infty} B_n \in \delta \qquad \text{and} \qquad B(x) = a.$$

Conversely, suppose $x_a$ and $x_b$ are two fuzzy points with $b < a$ where $a, b \in [0, 1]$. Then by hypothesis, there is an open set $B$ such that $B(x) = 1 - b > 1 - a$.
This implies, $B$ is an open Q-nbd of $x_a$ but not quasi-conincident with $x_b$ [since, $B$ is a nbd of $x_{1-a}$]. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a quasi $T_0-$space. □

*Theorem* 2.0.4. A fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ is $T_1-$space iff for every $x \in X$ and each $a \in [0, 1]$ there exists $B \in \delta$ such that $B(x) = 1 - a$ and $B(y) = 1$ for $y \neq x$.
Or, $\langle \mathcal{F}(X), \delta \rangle$ is a $T_1-$space $\Leftrightarrow$ every fuzzy point in $\langle X, \delta \rangle$ is closed.

*Proof.* Suppose $\langle \mathcal{F}(X), \delta \rangle$ be a $T_1-$space. If $a = 0$ then it suffices to take $B = \underline{1}$.
Suppose, $a > 0$ and $x_a$ is a fuzzy point. Since, every fuzzy point in a $T_1-$space is closed, so, $x_a$ is a closed set.
$\therefore$ We have, $B = 1 - x_a \in \delta$ and hence $B(x) = 1 - a$ and $B(y) = 1$. if $y \neq x$.
Conversely, let $x_a$ be a fuzzy point. Then by hypothesis there exists $B \in \delta$ such that $B(x) = 1 - a$ and $B(y) = 1$ with $y \neq x$. This implies, $B = 1 - x_a$ and hence $B^c = x_a$ which is closed. Thus, $B \in \delta$. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a $T_1-$space. □

**Definition 15** (Purely $T_2-$space)**.** $\langle \mathcal{F}(X), \delta \rangle$ is called purely $T_2-$space if for every two zero-meet fuzzy points $x_a$ and $y_b$, $\exists U \in Q_\delta(x_a)$ and $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

**Definition 16.** A fuzzy topological space $\langle X, \delta \rangle$ is said to be fuzzy regular iff for each $x \in X$ and each closed fuzzy set $U$ with $U(x) = 0$ there exists $V, W \in \delta$ such that $V(x) = 1$ and $V \subseteq 1 - W$.

*Theorem* 2.0.5. For a fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ the following statements are equivalent

1. $\langle X, \delta \rangle$ is a fuzzy $T_0$-space.

2. For $x, y \in X$, $x \neq y$, $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.

*Proof.* (1) $\Rightarrow$ (2), Suppose $\langle X, \delta \rangle$ is a fuzzy $T_0-$space. Thus, we have $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$. □

# Chapter 3

# Connected Fuzzy Topological Space

**Definition 17** (Separated Fuzzy Sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $A, B \in \mathcal{F}(X)$ are called separated sets if $\bar{A} \wedge B = \underline{0} = A \wedge \bar{B}$.

*Lemma.* Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological spaceand $A$, $B$, $C \in \mathcal{F}(X)$. If $B$ and $C$ are separated sets then $A \wedge B$ and $A \wedge C$ are separated.

*Proof.* Since $B$ and $C$ are separated, we have, $\bar{B} \wedge C = \underline{0} = B \wedge \bar{C}$.
We have, $A \wedge B < B \;\; \Rightarrow \overline{A \wedge B} < \overline{B}$ and $A \wedge C < C$.
This implies, $\overline{(A \wedge B)} \wedge (A \wedge C) \leq \bar{B} \wedge C = \underline{0}$. Similarly, $(A \wedge B) \wedge \overline{(A \wedge C)} \leq B \wedge \bar{C} = \underline{0}$.
Hence, $A \wedge B$ and $A \wedge C$ are separated. □

**Definition 18** (Connected Fuzzy Sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. A fuzzy set $A$ on $X$ is called connected if there do not exist $C, D \in \mathcal{F}(X) \setminus \{\underline{0}\}$ such that $A = C \vee D$.
Or, A set $A$ is connected if $A = B \vee C$ then either $B = \underline{0}$ or, $C = \underline{0}$.

*Theorem* 3.0.1. Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then the following are equivalent:

1. $A$ is connected.
2. $B$, $C \in \mathcal{F}(X)$ are separated, $A \leq B \vee C$ implies $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$.
3. $B$, $C \in \mathcal{F}(X)$ are separated, $A \leq B \vee C$ implies $A \leq B$ or, $A \leq C$.

*Proof.* (1) $\Rightarrow$ (2), Since, $B$ and $C$ are separated set. By the above lemma, we have $(A \wedge B)$ and $(A \wedge C)$ are separated. Since, $A$ is connected and $A \leq B \vee C$ implies

$$\begin{aligned} A &= A \wedge (B \vee C) \\ &= (A \wedge B) \vee (A \wedge C) \end{aligned}$$

then by definition of connectedness, either $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$. Hence, (2) holds.

(2) $\Rightarrow$ (3), Suppose, $A \wedge B = \underline{0}$, then,

$$\begin{aligned} A &= (A \wedge B) \vee (A \wedge C) \\ &= \underline{0} \vee (A \wedge C) \\ &= (A \wedge C). \end{aligned}$$

So, $A \leq C$. Similarly, if $A \wedge C = \underline{0}$, then we can prove that $A \leq B$. Thus, (3) holds.

Finally, (3) $\Rightarrow$ (1), Suppose, (3) holds, we need to show that, $A$ is connected. Let $B, C \in \mathcal{F}(X)$ are two separated fuzzy sets such that $A = B \vee C$.
We need to prove that, either, $B = \underline{0}$ or, $C = \underline{0}$. By (3), we have either $A \leq B$ or, $A \leq C$. Now if $A \leq B$ then $C \wedge A \leq C \wedge B \leq C \wedge \bar{B}$. But since, $B$, $C$ are separated sets so, $C \wedge \bar{B} = \underline{0}$. $\therefore$ $C \wedge A = \underline{0}$. Again, $C \wedge A = C \wedge (B \vee C) = C$. So $C = \underline{0}$.
Now if $A \leq C$, we can similarly prove that $B = \underline{0}$. Thus, $A$ is connected. □

*Theorem* 3.0.2. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$ is connected such that $A \leq B \leq \bar{A}$. Show that $B$ is connected.

*Proof.* Suppose, $C$ and $D$ are two separated fuzzy sets such that, $B = C \vee D$. To show that, $B$ is connected we need only to show either, $C = \underline{0}$ or, $D = \underline{0}$.
By the lemma, we have $(A \wedge C)$ and $(A \wedge D)$ are separated sets. Let $F = A \wedge C$, $G = A \wedge D$. Now

$$\begin{aligned} F \vee G &= (A \wedge C) \vee (A \wedge D) \\ &= A \vee (C \wedge D) \\ &= A \vee B \\ &= A \end{aligned}$$

Since, $A$ is connected, we have either $F = \underline{0}$ or, $G = \underline{0}$.
Suppose, $F = \underline{0}$. Then, $A = F \vee G = G = A \wedge D$. This implies, $A \leq D$. Thus, $\bar{A} \leq \bar{D}$. i.e., $B \leq \bar{A} \leq \bar{D}$.
Now, $C \wedge B \leq C \wedge \bar{A} \leq C \wedge \bar{D} = \underline{0}$. i.e.,

$$\begin{aligned} & C \wedge B \leq \underline{0} \\ \Rightarrow & C \wedge (C \vee D) \leq \underline{0} \\ \Rightarrow & C = \underline{0} \end{aligned}$$

Similarly, if $G = \underline{0}$, then we can show that $D = \underline{0}$. Hence, $B$ is connected. □

**Definition 19** (Connected Fuzzy Topological Space)**.** If the fuzzy set $\underline{1}$ is connected i.e., there does not exist separated sets $C, D \in \mathcal{F}(X) \setminus \{\underline{0}\}$ such that $\underline{1} = C \vee D$, then the fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ is called a connected fuzzy topological space.

*Theorem* 3.0.3 (Characterization Theorem). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then the followings are equivalent

1. $\langle \mathcal{F}(X), \delta \rangle$ is connected.
2. $A, B \in \delta$, $A \vee B = \underline{1}$, $A \wedge B = \underline{0}$, implies $\underline{0} \in \{A, B\}$.
3. $A, B \in \delta'$, $A \vee B = \underline{1}$, $A \wedge B = \underline{0}$, implies $\underline{0} \in \{A, B\}$.

*Proof.* (1) $\Rightarrow$ (2), Suppose, (2) is false. Then there are $A, B \in \delta \setminus \{\underline{0}\}$ such that

$$\begin{aligned} & A \vee B = \underline{1}, \qquad A \wedge B = \underline{0} \\ \Rightarrow & A^c \wedge B^c = \underline{0}, \text{ and } A^c \vee B^c = \underline{1} \quad \text{[By De Morgan's Law]} \\ \Rightarrow & \bar{A}^c \wedge B^c = \underline{0}, \text{ and } A^c \wedge \bar{B}^c = \underline{0} \quad \text{[Since, } A^c,\ B^c \text{ are closed.]} \end{aligned}$$

$\therefore$ We have by definition, $A^c$ and $B^c$ are two separated sets. Therefore, we have $A^c \vee B^c = \underline{1}$ and $A^c$, $B^c$ are two separated sets. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is disconnected. Hence, (2) is true.

(2) $\Rightarrow$ (3), Let $A, B \in \delta'$ such that $A \vee B = \underline{1}$ and $A \wedge B = \underline{0}$. Then by De Morgan's Laws, $A^c \wedge B^c = \underline{0}$ and $A^c \vee B^c = \underline{1}$. By (2), $\underline{0} \in \{A^c, B^c\}$. Hence, $\underline{0} \in \{A, B\}$.

(3) $\Rightarrow$ (1), If $\langle \mathcal{F}(X), \delta \rangle$ is not connected, then there exists non-zero separated sets $A, B \in \delta' \setminus \{\underline{0}\}$ such that $A \vee B = \underline{1}$, which contradicts (3).
Hence, $\langle \mathcal{F}(X), \delta \rangle$ is connected. □

# Chapter 4

# Compactness

**Definition 20** (Cover and $C-$compactness)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then, $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a cover of $A$ if $A \subseteq \vee \mathcal{A}$.

- $\langle \mathcal{F}(X), \delta \rangle$ is called $C-$compact if every open cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite subcover.
- $\mathcal{A}$ is called an open cover of $A$, if $\mathcal{A} \subseteq \delta$ and if $\mathcal{A}$ is a cover of $A$.
- $\mathcal{B} \subseteq \mathcal{A}$ is called a subcover if $\mathcal{B}$ is still a cover of $A$.

In particularly, $\mathcal{A}$ is a cover of $\langle \mathcal{F}(X), \delta \rangle$ if $\mathcal{A}$ is a cover of $\underline{1}$.

**Definition 21** ($\alpha-$cover and $\alpha-$compactness)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an $\alpha-$cover, if for very $x \in X \exists A \in \mathcal{A} \ni A(x) > \alpha$.

- $ft$ is called an $\alpha-$compact, if for every open $\alpha-$cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite sub$-\alpha-$cover where $\alpha \in [0, 1)$.

**Definition 22** (Strong Compact)**.** A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called strongly compact if it is $\alpha-$compact for every $\alpha \in [0, 1)$.

**Definition 23** ($\alpha^*-$cover and $\alpha^*-$compactness)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an $\alpha^*-$cover, if for every $x \in X$, there exists $A \in \mathcal{A}$ such that, $A(x) \geq \alpha$.

- For $\alpha \in [0, 1)$, $\langle \mathcal{F}(X), \delta \rangle$ is called an $\alpha^*-$compact, if for every open $\alpha^*-$cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite sub $\alpha^*-$cover.

**Example.** Given $X = \{a, b, c\}$, $\mathcal{A} = \{A, B, C\}$, $\alpha \in [0, 1)$, $\delta = \{\underline{0}, \underline{1}, A, B, C\}$ where,

$$\begin{aligned} A &: \ a \mapsto 0.2, \ \ b \mapsto 0.4, \ \ c \mapsto 0.6; \\ B &: \ a \mapsto 0.4, \ \ b \mapsto 0.6, \ \ c \mapsto 0.8; \\ C &: \ a \mapsto 0.6, \ \ b \mapsto 0.8, \ \ c \mapsto 0.9; \end{aligned}$$

Check whether $\mathcal{A}$ is $\alpha-$compact or, $\alpha^*-$compact corresponding to the given value of $\alpha$.

**Solution.**

1. Let $\alpha = 0.7$
   $a \in X : \alpha = 0.7 > A(a),\ B(a), C(a)$.
   Hence, for $\alpha = 0.7$, $\mathcal{A}$ is not an $\alpha-$cover.

2. Let $\alpha = 0.3$
   $a \in X : \alpha = 0.3 < C(a) = 0.6,\ B(a) = 0.4$
   $b \in X : \alpha = 0.3 < A(b) = 0.4,\ B(b) = 0.6,\ C(b) = 0.8$
   $c \in X : \alpha = 0.3 < A(c) = 0.6,\ B(c) = 0.8,\ C(c) = 0.9$
   $\therefore \mathcal{A}$ is an $\alpha-$compact space for $a = 0.3$.

3. Let $\alpha = 0.6$
   For, $a \in X : \alpha = 0.6 = C(a)$
   For, $b \in X : \alpha = 0.6 = B(b),\ \alpha = 0.6 < C(b) = 0.8$
   For, $c \in X : \alpha = 0.6 = A(c) = 0.6,\ \alpha = 0.6 < B(c) = 0.8,\ C(c) = 0.9$
   $\therefore \mathcal{A}$ is an $\alpha^*-$compact space for $a = 0.6$.

**Definition 24** ($Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a $Q-$cover of $A$ if for every $x \in Supp(A)$, there exists $U \in \mathcal{A}$ such that $x_{A(x)} \propto U$.

**Definition 25** ($Q-$compact)**.** A fuzzy set $A$ is called $Q-$compact if every open $Q-$cover of $A$ has a finite sub $Q-$cover. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called $Q-$compact if $\underline{1}$ is $Q-$compact.

**Example.** Consider, $X = \{a, b, c\}$, $\delta = \{\underline{0}, \underline{1}, U, V, W\}$ where

$$\begin{aligned} U &: a \mapsto 0.3,\ b \mapsto 0.5,\ c \mapsto 0.7; \\ V &: a \mapsto 0.4,\ b \mapsto 0.6,\ c \mapsto 0.8; \\ W &: a \mapsto 0.6,\ b \mapsto 0.8,\ c \mapsto 0.9; \end{aligned}$$

Consider $\mathcal{A} = \{U, V\} \subseteq \delta$ and let, $A: a \mapsto 0.1,\ b \mapsto 0.2,\ c \mapsto 0.3$. Then, find the $Q-$cover of $A$.

**Solution.** Here, $Supp(A) = \{a, b, c\}$
For, $x = a$, $a_{A(a)} = a_{0.1} = 0.1$
For, $x = b$, $b_{A(b)} = b_{0.2} = 0.2$
For, $x = c$, $c_{A(c)} = c_{0.3} = 0.3$
For $x = a$, we have $U_a, : 0.3 + 0.1 < 1$, $V_a = 0.4 + 0.1 < 1$. Hence $\mathcal{A}$ is not a $Q-$cover of $A$.

# If $A: a \mapsto 0.7,\ b \mapsto 0.6,\ c \mapsto 0.5$.
Then, For $x = a$, $a_{A(a)} = a_{0.7} = 0.7$
For, $x = b$, $b_{A(b)} = b_{0.6} = 0.6$
For, $x = c$, $c_{A(c)} = c_{0.5} = 0.5$
For, $x = a, 0.3 + 0.7 \geq 1$, $0.4 + 0.7 > 1$
For, $x = b, 0.5 + 0.6 > 1$, $0.6 + 0.6 > 1$
For, $x = c, 0.7 + 0.5 > 1$, $0.8 + 0.5 > 1$
Hence, for every $x \in Supp(A)$, $x_{A(x)} \propto U$.
$\therefore \mathcal{A}$ is a $Q-$cover of $A$.

**Definition 26** ($\alpha - Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\varphi \subseteq \mathcal{F}(X)$ is called an $\alpha - Q-$cover of $A$, if for every $x_a \subseteq A$, there exists $U \in \varphi$ such that $x_a \propto U$. It is denoted by $\vee\varphi\hat{q}A(\alpha)$.

**Definition 27** ($\bar{\alpha} - Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\varphi \subseteq \mathcal{F}(X)$ is called an $\bar{\alpha} - Q-$cover of $A$, if there exists $\gamma \in B^*(\alpha)$ such that $\gamma$ is a $\gamma - Q-$cover of $A$.

- $B(b) = \{a \in L : a \propto b\}$, where the binary relation $\propto$ is defined as, for $a, b \in L$, $a \propto b \Leftrightarrow$ for every subset $D \subseteq L$, $b \leq Sup\, D$ implies the existence of $d \in D$ with $a \leq$
- $B^*(b) = B(b) \cap M(L)$, where, $M(L) = (0, 1]$.

**Definition 28.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. $A$ is called $N-$compact if for every $\alpha \in (0, 1] - M([0, 1])$, every open $\alpha - Q-$cover of $A$ has a finite subfamily which is an $\bar{\alpha} - Q-$cover of $A$. $\langle \mathcal{F}(X), \delta \rangle$ is called $N-$compact, if $\underline{1}$ is compact.

*Theorem* 4.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. Then $A$ is $N-$compact iff the following conditionds hold:

(a) For every $\alpha \in (0, 1]$, every open $\alpha - Q-$cover of $A$ has a finite sub $\alpha - Q-$cover.

(b) For every $\alpha \in (0, 1]$, every open $\alpha - Q-$cover of $A$ which consists of just one subset is an $\bar{\alpha} - Q-$cover of $A$.

*Proof.* (a) Let, $A$ be $N-$compact, $\alpha \in (0, 1]$ and $\varphi$ is an open $\alpha - Q-$cover of $A$. By the definition of $N-$compact, $\varphi$ has a finite subfamily $\psi$ such that, $\psi$ is an $\bar{\alpha} - Q-$cover of $A$. Hence, $\vee\psi\hat{q}A(\alpha)$ i.e., $\psi$ is an $\alpha - Q-$cover of $A$.

(b) Suppose, $U \in \delta$ and $\varphi = \{U\}$ is an open $\alpha - Q-$cover of $A$. Then, by the $N-$compactness of $A$, $\varphi$ has a subfamily $\psi$ such that $\psi$ is an $\bar{\alpha} - Q-$cover of $A$. But, clearly, $\varphi = \psi$. Hence, $\psi$ is an open $\alpha - Q-$cover of $A$.

Conversely, suppose $(a)$ and $(b)$ holds.
Let $\alpha \in (0,1]$ and $\varphi$ is an open $\alpha - Q-$cover of $A$.
By $(a)$, $\varphi$ has a finite sub $\alpha - Q-$cover $\psi$ of $A$. Take $U = \vee\psi$. Then $\{U\}$ is an $\alpha - Q-$cover of $A$.
By $(b)$, $\{U\}$ is also an $\bar{\alpha} - Q-$cover of $A$. By the definition of $\bar{\alpha} - Q-$cover, there exists $\gamma \in B^*(\alpha)$ such that $x_\gamma$ is a quasi-coincident with $U$ for every $x_\gamma \subseteq A$. Hence, $\gamma + U(x) > 1 \Rightarrow \gamma > 1 - U(x)$
i.e., $\gamma \leq (U(x))' \;\Rightarrow \gamma \nleq (U\psi(x))' = \wedge\{(W(x))'|W \in \psi\}$
i.e., $W \in Q_\gamma(x_\gamma)$. So, $\psi$ is an $\bar{\alpha} - Q-$cover of $A$. Hence, $A$ is $N-$compact. □

*Theorem* 4.0.2. Continuous image of an $N-$compact space is $N-$compact.

*Proof.* Let $f^{\rightarrow} : \langle \mathcal{F}(X), \delta\rangle \rightarrow \langle \mathcal{F}(Y), \mu\rangle$ be a continuous fuzzy mapping and $A$ be a $N-$compact fuzzy set in $\mathcal{F}(X)$. For $\alpha \in (0,1]$, let $\mathcal{A}$ be an open $\alpha - Q-$cover of $f^{\rightarrow}(A)$. Then for every $x_\alpha \leq A$, $f^{\rightarrow}(x_\alpha) = f(x)_\alpha \leq f^{\rightarrow}(A)$, there exists $U \in \mathcal{A}$ such that $f(x)_\alpha \propto U \;\Rightarrow f(x)_\alpha \not\propto U^c \;\Rightarrow \alpha \nleq U^c(f(x)) \;\Rightarrow \alpha \nleq f^{\leftarrow}(U^c)(x) = f^{\leftarrow}(U)^c(x)$. That is $x_a \propto f^{\leftarrow}(U)$. Since, $f^{\rightarrow}$ is continuous, $f^{\leftarrow}(U) \in \delta$ and hence $f^{\leftarrow}(U) \in Q(x_\alpha)$. Thus, $f^{\leftarrow}(A)$ is an open $\alpha - Q-$cover of $A$.

Since $A$ is $N-$compact, $\mathcal{A}$ has a finite subfamily $\mathcal{A}_n = \{U_i : 1 \leq i \leq n\}$ such that $f^{\leftarrow}(\mathcal{A}_n)$ is an $\bar{\alpha}-Q-$cover of $A$.

Now, we show that, $\mathcal{A}_n$ is an $\bar{\alpha} - Q-$cover of $f^{\rightarrow}(A)$. Since, $f^{\leftarrow}(\mathcal{A})_n$ is an open $\bar{\alpha} - Q-$cover of $A$, there exists $\gamma \in \mathcal{B}(\alpha)$ such that $f^{\leftarrow}(\mathcal{A}_\backslash)$ is $\gamma - Q-$cover of $A$. This implies, $\gamma \sqsubseteq a$ and hence $\exists\lambda \in (0,1]$ such that $\gamma \sqsubseteq \lambda \sqsubseteq \alpha$. So, $\lambda \in \mathcal{B}(\alpha)$ and hence we have, $\lambda \leq f^{\leftarrow}(A)(y) = \vee\{A(x) : x \in X, f(x) = y\}$. Now, $\gamma \sqsubseteq \lambda$ implies, $\gamma \nleq (f^{\leftarrow}(U_i))^c(x) = f^{\leftarrow}(U_i^c)(x) = U_i^x(f(x)) = U_i^c(y)$, for some $1 \leq i \leq n$ such that $x_\gamma \propto f^{\leftarrow}(U_i)$.
By $\gamma \sqsubseteq \lambda$ and hence $\gamma \leq \lambda$, we have $\lambda \nleq U_i^c(y)$. Thus $y_\lambda \propto U_i$ for some $1 \leq i \leq n$. So, $\mathcal{A}_n$ is an open $\lambda - Q-$cover of $f^{\rightarrow}(A)$ and hence $\mathcal{A}_n$is an $\bar{\alpha} - Q-$cover of $f^{\leftarrow}(A)$.
Therefore, $f^{\rightarrow}(A)$ is an $N-$compact. □

**Definition 29** (Net in $X$)**.** Let $X$ be a non-empty ordinary set and $D$ be a directed set then every mapping $S : D \rightarrow X$ is called a net in $X$ and $D$ is called the index set of $S$.

*Theorem* 4.0.3. Let $\langle \mathcal{F}(X), \delta\rangle$ be a fuzzy topological space. Let $A, B, C \in \mathcal{F}(X)$ such that $A$ be a $N-$compact and $B$ be closed. Then $A \wedge B$ is $N-$compact.

*Proof.* Let $S$ be an $\alpha-$net in $A \wedge B$. Then $S$ is also an $\alpha-$net in $A$. Since, $A$ is $N-$compact, $S$ has a cluster point $x_\alpha$ in $A$ such that $ht(\alpha) = \alpha$. But, $S$ is also a net in closed subset $B$, we have $x_\alpha \leq B$.
So, $x_a \leq A \wedge B$, i.e., $x_a$ is a cluster point of $\delta$ in $A \wedge B$ such that $ht(\alpha) = \alpha$. Hence, $A \wedge B$ is $N-$compact. □