Mathematical Modeling in Biology

MAT431

Asst. Prof. S. M. Saydur Rahman

Shahjalal University of Science and Technology

Edited by
Mehedi Hasan

Preface

This is a compilation of lecture notes with some books and my own thoughts. If there are any mistake/typing error or, for any query mail me at mehedi12@student.sust.edu.

Contents

Syllabus

Graphical, qualitative and stability theory: Representations of solutions of linear and nonlinear differential equation by graphs; the method of isoclines, Phase-plane, critical points, Classification and nature of critical points, Stability of critical point of linear and nonlinear systems. Simple discrete and continuous type population models: Malthusian model, Logistic growth model, Prey-Predator mode, Host-Parasite model & Harvesting model. Epidemic models: SI, SIS and SIR models, HIV/AIDs models. Cells and tumor growth: Concept of cells, CD4 T-cells, Modeling of tumor growth. Basic concepts of microbial growth models: Chemostat, Bacteria growth model, Enzyme kinetics.

Books Recommended:

- S. L. Ross. Differential Equation
- J.D. Murray. Mathematical Biology I. An Introduction
- J.D. Murray. Mathematical Biology II. Spatial Models and Biomedical Applications.
- J.C. Frauenthal. Introduction to population modeling
- Britton Nicholas. Essential Mathematical Biology.
- Brain Ingalls. Mathematical Modeling in Systems Biology: An Introduction
- H.F. Freedman. Deterministic Mathematical models in population.

Part I

Sheet

Chapter 1

Graphical Theory For Solution of ODE

1.1 Integral Curves or Solution Curves

Let

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y) \tag{1.1}$$

be a first order ordinary differential equation, then the graph of the explicit solution of (1.1) in the xy plane are called integral curves or solution curves.

1.2 Line Element

Let

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y) \tag{1.2}$$

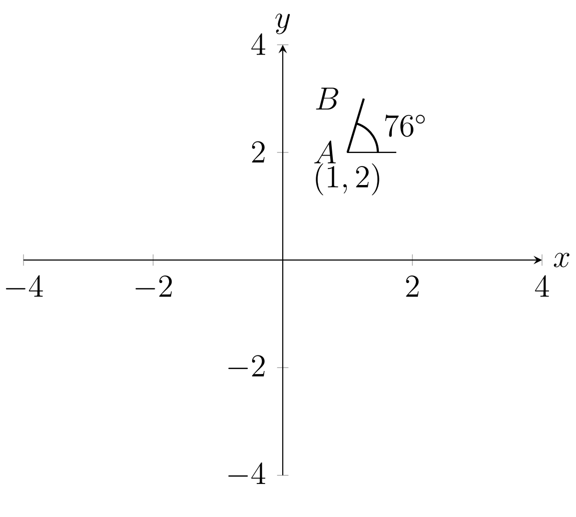
be a first order ordinary differential equation, then a short segment of the tangent line through the point (a, b) and with the slope $f(a, b)$ is called the line element.

1.2.1 Example of Line Element

Suppose

$$y' = 2x + y \tag{1.3}$$

be an ordinary differential equation.
Here, $f(x, y) = 2x + y$
The slope of (1.3) at $(1, 2)$ is 4.
Thus, through $(1, 2)$ we can draw short line AB with an inclination $\approx 76°$.



Here, AB is line Element at $(1, 2)$.

1.2.2 Line Element Configuration

If we draw a large number of line element for a large number of points then we obtain a configuration called line element configuration.

1.3 Direction Field

The totality of the line element together with the corresponding directions constitute a field which is called direction field of the differential equation.

1.4 Graphical Method

A procedure which yields the line element configuration of the direction field of a differential equation is called the graphical method. It provides approximate graphs of solution curves.

Problem 1.1. Construct a line element configuration (direction field) of the differential equation $y' = \frac{y}{x}$ and sketch the several integral curves.

Solution. We have

$$y' = \frac{y}{x} \tag{1.4}$$

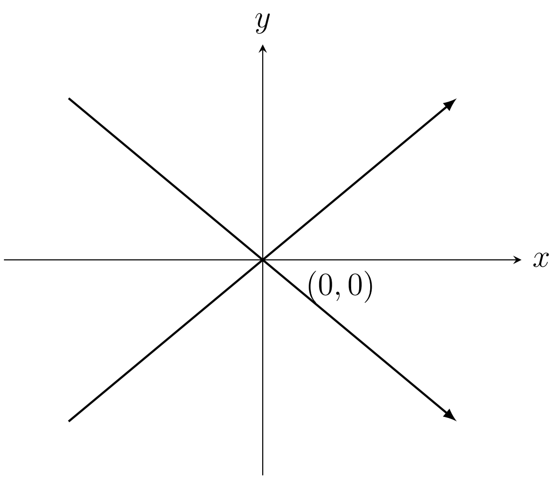
Here, the slope of the differential equation is

$$m = \tan\theta = \frac{y}{x}$$

The slope of the approximate integral curves of (1.4) are calculated at some selected points are given below.

x	-1	1	-2	2	1	-1	2	3	3	-2	-3	-3
y	-1	1	-2	2	-1	1	-2	3	-3	2	3	-3
m	1	1	1	1	-1	-1	-1	1	-1	-1	-1	-1
θ	$45°$	$45°$	$45°$	$45°$	$-45°$	$-45°$	$-45°$	$45°$	$-45°$	$-45°$	$-45°$	$-45°$

Now construct the line element at the selected points and sketch several smooth curves.



We observe that the integral curves represent a family of straight line passing through the origin.

Problem 1.2. Construct a line element configuration (direction field) of the differential equation $y' = -\frac{x}{y}$ and sketch several integral curves.

Solution. We have

$$y' = -\frac{x}{y} \tag{1.5}$$

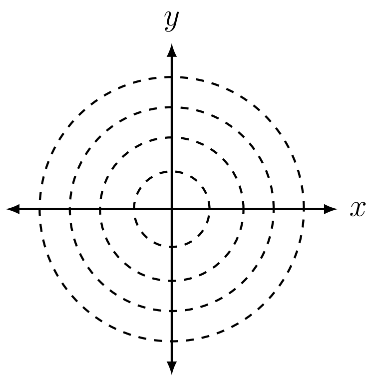
Here, the slope of the differential equation is

$$m = \tan\theta = -\frac{x}{y}$$

The slope of the approximate integral curves of (1.5) are calculated at some selected points are given below.

x	1	-1	-1	2	-2	-2	0	0	0	0	3	-3
y	1	1	-1	2	2	-2	1	2	-2	-1	3	3
m	-1	1	-1	-1	1	-1	0	0	0	0	-1	1
θ	$-45°$	$45°$	$-45°$	$-45°$	$45°$	$-45°$	$0°$	$0°$	$0°$	$0°$	$-45°$	$45°$

We now construct the line element at the selected points and sketch several integral curves.



We observe that the integral curves represent a family of circles which are centered as $(0, 0)$.

1.5 Method of Isoclines

Let us consider the differential equation

That is, the isocline of (1.6)

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y) \tag{1.6}$$

A curve along which the slope $f(x, y)$ has a constant value c, is called a isocline of the differential equation (1.6) are curves $f(x, y) = c$ for different values of c.

Problem 1.3. Employ the method of isocline to sketch the several approximate integral curves of $y' = 3x - y$

Solution.

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 3x - y \tag{1.7}$$

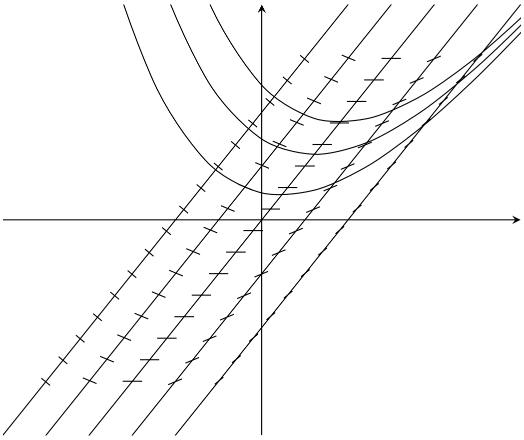
and the isocline of (1.7) is given by

$$\begin{aligned} 3x - y &= c \\ \Rightarrow y &= 3x - c \end{aligned} \tag{1.8}$$

For different values of c (1.8) represent a family of straight line.
We construct the line (1.8) for $c = 0, \pm1, \pm2, \pm3, \ldots$ etc.

On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = 3x$,	$\theta = \tan^{-1} c = 0°$
When $c = 1$,	then $y = 3x - 1$,	$\theta = \tan^{-1} c = 45°$
When $c = -1$,	then $y = 3x + 1$,	$\theta = \tan^{-1} c = -45°$
When $c = 2$,	then $y = 3x - 2$,	$\theta = \tan^{-1} c = 63.43°$
When $c = -2$,	then $y = 3x + 2$,	$\theta = \tan^{-1} c = -63.43°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.7)

Problem 1.4. Employ the method of isocline to sketch the several approximate integral curves of $y' = 2x + y$

Solution.

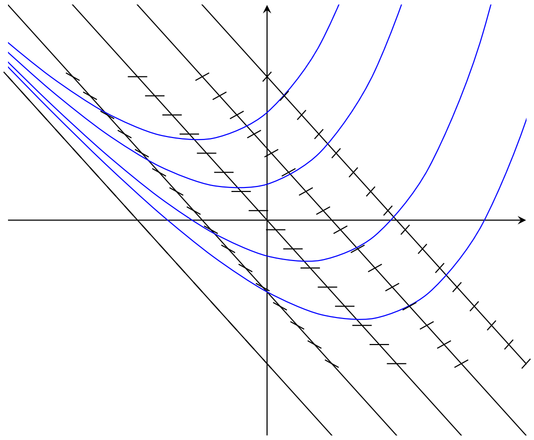
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 2x + y \tag{1.9}$$

The isocline of (1.9) is given by

$$2x + y = c \tag{1.10}$$

For different values of c (1.10) represent a family of straight line.
We construct the line (1.10) for $c = 0, \pm1, \pm2, \pm3, \ldots$ etc.
On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = -2x$,	$\theta = 0°$
When $c = 1$,	then $2x + y = 1$,	$\theta = 45°$
When $c = -1$,	then $2x + y = -1$,	$\theta = -45°$
When $c = 2$,	then $2x + y = 2$,	$\theta = 63.43°$
When $c = -2$,	then $2x + y = -2$,	$\theta = -63.43°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.9)

Problem 1.5. Employ the method of isocline to sketch the several approximate integral curves of $y' = \frac{y-x}{y+x}$

Solution.

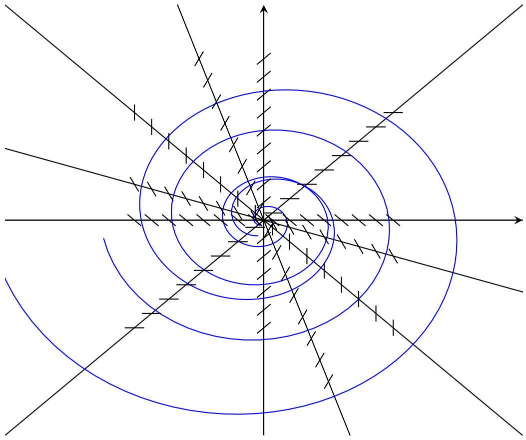
$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = \frac{y-x}{y+x} \tag{1.11}$$

The isocline of (1.11) is given by

$$\begin{aligned} &\frac{y-x}{y+x} = c \\ \Rightarrow\, & y - x = cy + cx \\ \Rightarrow\, & (1+c)x = (1-c)y \\ \Rightarrow\, & y = \frac{1+c}{1-c}x \end{aligned} \tag{1.12}$$

For different values of c (1.12) represent a family of straight line passing through origin.
We construct the line (1.12) for $c = 0, \pm 1, \pm 2, \pm 3, \ldots$ etc.
On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = x$,	$\theta = 0°$
When $c = 1$,	then $x = 0$,	$\theta = 45°$
When $c = -1$,	then $y = 0$,	$\theta = -45°$
When $c = 2$,	then $y = -3x$,	$\theta = 63.43°$
When $c = -2$,	then $y = -\frac{1}{3}x$,	$\theta = -63.43°$
When $c = \infty$,	then $y = -x$,	$\theta = 90°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.11)

Problem 1.6. Employ the method of isocline to sketch the several approximate integral curves of $y' = \frac{3x-y}{x+y}$

Solution.

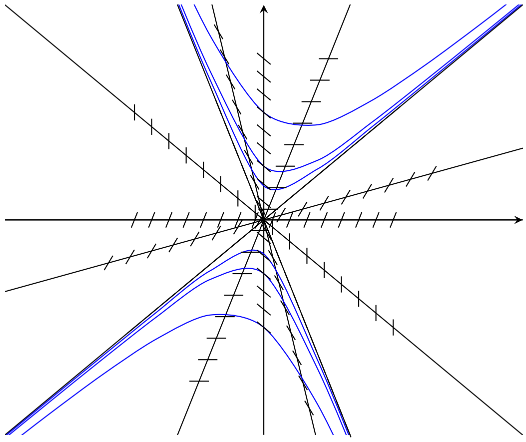
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{3x-y}{x+y} \tag{1.13}$$

The isocline of (1.13) is given by

$$\begin{aligned} &\frac{3x-y}{x+y} = c \\ &\Rightarrow 3x - y = cy + cx \\ &\Rightarrow (3-c)x = (c+1)y \\ &\Rightarrow y = \frac{3-c}{c+1}x \end{aligned} \tag{1.14}$$

For different values of c (1.14) represent a family of straight line passing through origin.
We construct the line (1.14) for $c = 0, \pm 1, \pm 2, \pm 3, \ldots$ etc.
On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = 3x$,	$\theta = 0°$
When $c = 1$,	then $y = x$,	$\theta = 45°$
When $c = -1$,	then $x = 0$,	$\theta = -45°$
When $c = 2$,	then $y = \frac{1}{3}x$,	$\theta = 63.43°$
When $c = -2$,	then $y = -5x$,	$\theta = -63.43°$
When $c = 3$,	then $y = 0$,	$\theta = 71.57°$
When $c = -3$,	then $y = -3x$,	$\theta = -71.57°$
When $c = \infty$,	then $y = -x$,	$\theta = 90°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.13)

Chapter 2

Non-Linear Differential Equations

The general 2nd order non-linear differential equation is of the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} = F\left(x, \frac{\mathrm{d} x}{\mathrm{d} t}\right) \tag{2.1}$$

2.1 Van der Pol Equation

A special example of 2nd order non-linear differential is

$$\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} + \mu(x^2 - 1)\frac{\mathrm{d} x}{\mathrm{d} t} + x = 0 \tag{2.2}$$

This equation is called Van der Pol equation.

$$\Rightarrow \frac{\mathrm{d}^2 x}{\mathrm{d} t^2} = -\mu(x^2 - 1)\frac{\mathrm{d} x}{\mathrm{d} t} - x$$
$$\Rightarrow F\left(x, \frac{\mathrm{d} x}{\mathrm{d} t}\right) = -\mu(x^2 - 1)\frac{\mathrm{d} x}{\mathrm{d} t} - x$$

We can replace the above differential equation (2.1) by the following system by supposing $y = \frac{\mathrm{d} x}{\mathrm{d} t}$.
So, $\frac{\mathrm{d} x}{\mathrm{d} t} = y$ and $\frac{\mathrm{d} y}{\mathrm{d} t} = F(x, y)$
More generally,

$$\frac{\mathrm{d} x}{\mathrm{d} t} = P(x, y)$$
$$\frac{\mathrm{d} y}{\mathrm{d} t} = Q(x, y)$$

2.2 Dynamical System

If a system of ODE

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

describe a physical problem then it is called a dynamical system.

2.3 Phase Plane

Let us suppose that the differential equation $\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} = F\left(x, \frac{\mathrm{d} x}{\mathrm{d} t}\right)$ describes a certain dynamical system having one degree of freedom. The state of this system at time t is determined by the value of x (position) and $\frac{\mathrm{d} x}{\mathrm{d} t}$ (velocity). The plane of variables x and $\frac{\mathrm{d} x}{\mathrm{d} t}$ is called a phase plane.

2.4 Autonomous System

A system of the form

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x, y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x, y)$$

where P and Q have continuous 1st partial derivatives for all (x, y). Such a system in which the independent variable t appears only in the differential $\mathrm{d}\,t$ of the left members and not explicitly in the function P and Q on the right is called autonomous system or time independent system.

2.5 Critical Point

A point (x_0, y_0) of the autonomous system

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x, y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x, y)$$

at which $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$ is called a critical point or equilibrium point or singular point or stationary point.

2.6 Path

Consider the system

$$\left.\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= P(x, y) \\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Q(x, y) \end{aligned}\right\} \tag{2.3}$$

For given t_0 and any pair (x_0, y_0) of real numbers. There exists a unique solution

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \tag{2.4}$$

of the system (2.3) such that

$$f(t_0) = x_0$$
$$g(t_0) = y_0$$

If f and g are not both constant function then (2.4) defines a curve in the xy plane called path (orbit/trajectory).

2.7 Isolated Critical Point

A critical point (x_0, y_0) if the system

of

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x, y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x, y)$$

is called isolated critical point if there exists a circle $(x-x_0)^2+(y-y_0)^2 = r^2$ about the point (x_0, y_0) such that (x_0, y_0) is the only critical point of the system within this circle.

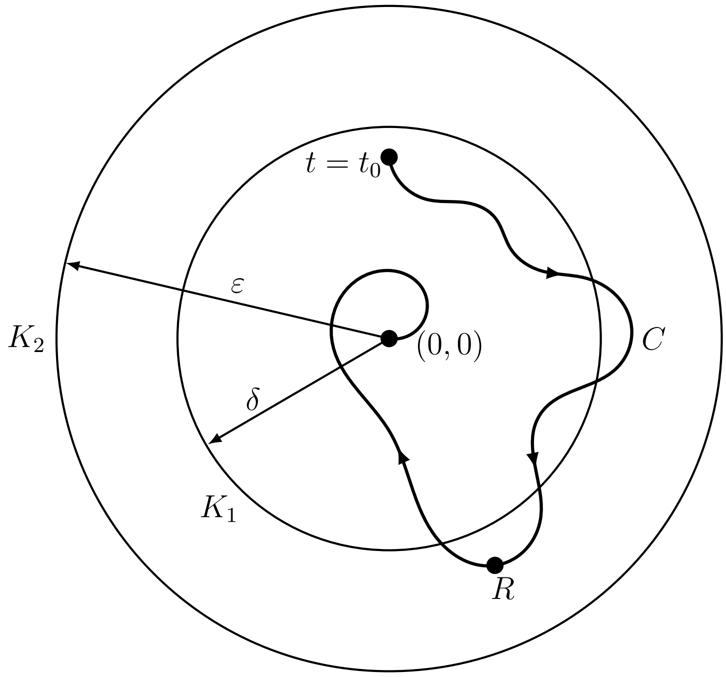
2.8 Center

The isolated critical point $(0,0)$ of the system

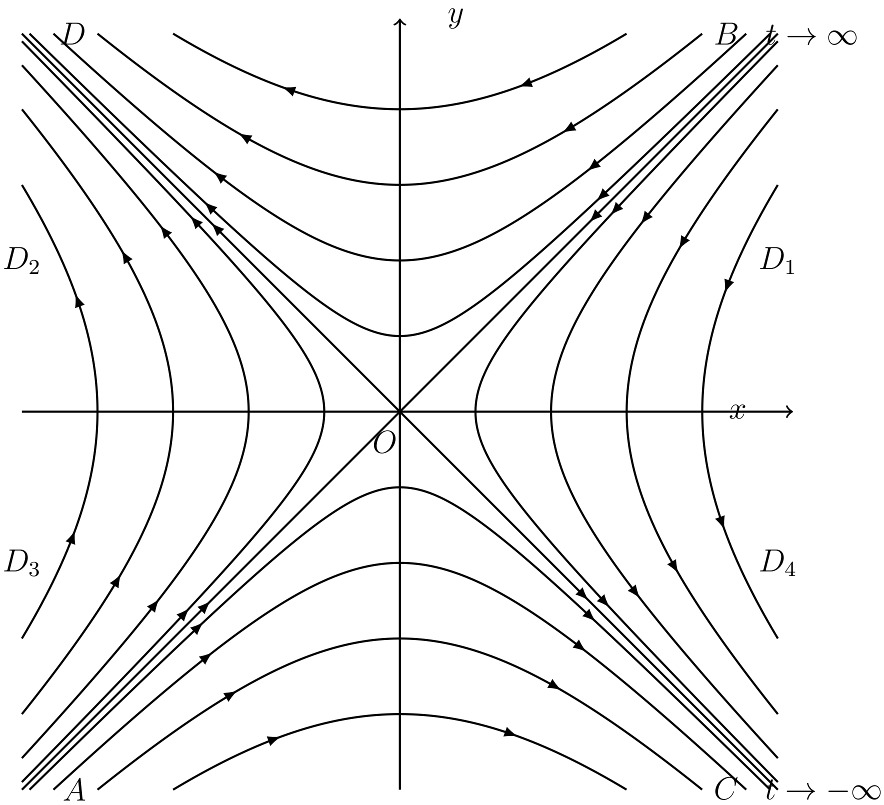
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x,y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x,y)$$

contains (0,0) in its interior and which

is called a center if there exist a neighborhood of $(0,0)$ which contains a countably ~~infinite~~ number of closed paths $P_n(n = 1,2,\dots)$ each of which are such that the diameter of the path approaches 0 as $n \to \infty$ [but $(0,0)$ is not approached by any path either as $t \to \infty$ or as $t \to -\infty$.]



2.9 Saddle Point



The isolated critical point $(0, 0)$ of the system

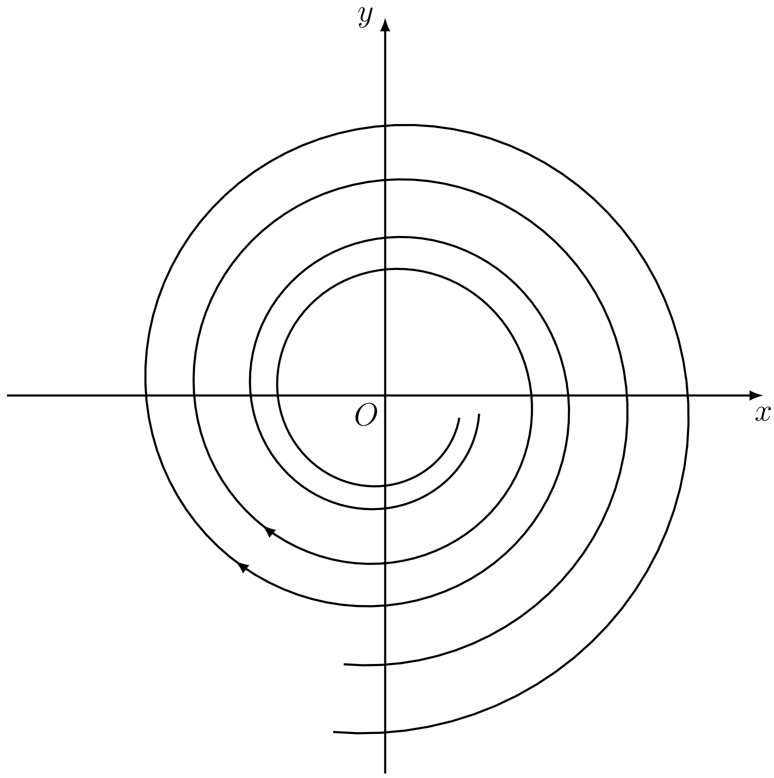
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x, y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x, y)$$

is called a saddle point if there exists a neighborhood of $(0, 0)$ in which the following two condition holds.

(i) There exists two paths which approach and enter $(0, 0)$ from a pair of opposite directions as $t \to \infty$ and there exist two paths which approach and enter $(0, 0)$ from a different pair of opposite direction as $t \to -\infty$.

(ii) In each of the four domains between any two of the four direction in (i) there are infinity many paths which are arbitrary close to $(0, 0)$ but do not approach $(0, 0)$ either as $t \to \infty$ or $t \to -\infty$.

2.10 Spiral Point



The isolated critical point $(0,0)$ of the system

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x,y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x,y)$$

is called a spiral point (focal point) if there exist a neighborhood of $(0,0)$ such that every path P in this neighborhood has the following properties.

(i) P is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0

(ii) P approach $(0,0)$ as $t \to \infty$ (or as $t \to -\infty$)

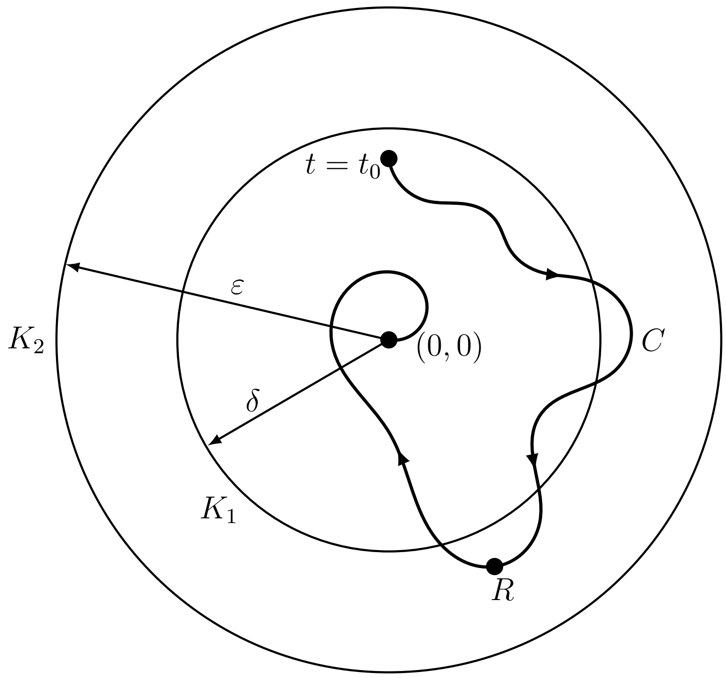
(iii) P approaches $(0,0)$ in a spiral-like manner, winding around $(0,0)$ an infinite number of times as $t \to \infty$ (or as $t \to -\infty$)

2.11 Stability

Consider the autonomous system

$$\left.\begin{aligned} \tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= P(x,y) \\ \tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Q(x,y) \end{aligned}\right\} \tag{2.5}$$

Assume that $(0,0)$ is an isolated critical point of the system (2.5). Let c be a path of (2.5). Let $x = f(t)$, $y = g(t)$ be a solution of (2.5) define c parametrically.



Let

$$D(t) = \sqrt{\{f(t)\}^2 + \{g(t)\}^2} \tag{2.6}$$

denote the distance between the critical point and the point $R : [f(t), g(t)]$ on c. The critical point $(0, 0)$ is called stable if for every number $\varepsilon > 0$ there exist a number $\delta > 0$ such that the following is true.

Every path C for which

$$D(t_0) < \delta \qquad \text{for some value } t_0 \tag{2.7}$$

is defined for all $t \geq t_0$ and is such that

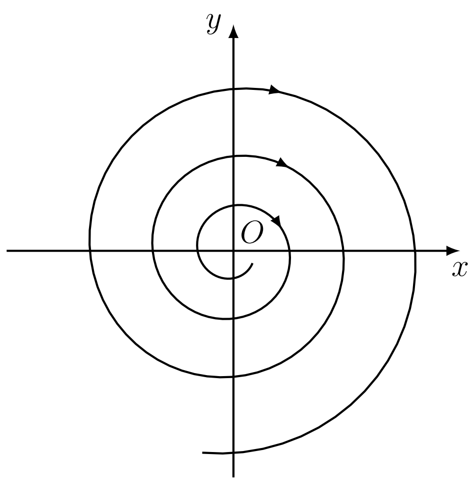
$$D(t) < \varepsilon \qquad \text{for } t_0 \leq t < \infty \tag{2.8}$$

2.11.1 Asymptotically Stable

The isolated critical point $(0, 0)$ is called asymptotically stable if

(i) it is stable and

(ii) There exist a number $\delta_0 > 0$ such that if $D(t_0) < \delta_0$ for some value t_0 then $\lim_{t\to\infty} f(t) = 0$, $\lim_{t\to\infty} g(t) = 0$

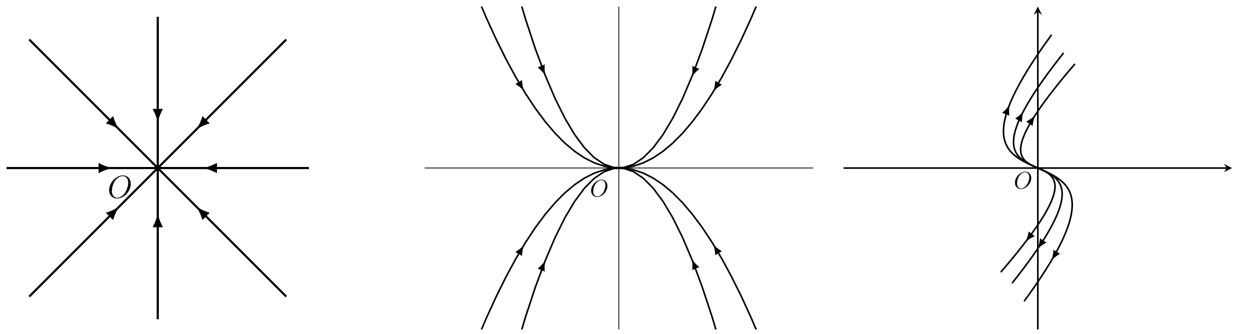


2.11.2 Unstable

A critical point is called unstable if it is not stable.

Center, spiral point and the node are all stable. Of these three spiral point and the node are asymptotically stable.

2.12 Node



The isolated critical point $(0,0)$ of the system

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x,y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x,y)$$

is called a node if there exist a neighborhood of $(0,0)$ such that every path P in this neighborhood has the following properties.

(i) P is defined for all $t > t_0$ (or all $t < t_0$) for some number t_0

(ii) P approach $(0,0)$ as $t \to \infty$ (or as $t \to -\infty$)

(iii) P enters $(0,0)$ as $t \to \infty$ (or as $t \to -\infty$)

or simply, A critical point of an autonomous system which is approached and entered by both the rectilinear and non-rectilinear paths of the autonomous system as $t \to \infty$ or $t \to -\infty$ is called a node.

2.13 Nature and Stability of a Critical Point of a Linear Autonomous System

We consider the linear system

$$\left.\begin{aligned} \tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= ax + by \\ \tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= cx + dy \end{aligned}\right\} \tag{2.9}$$

where a, b, c, d are real constants. The origin $(0,0)$ is clearly a critical point of (2.9).

We assume that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \tag{2.10}$$

We know that the solution of (2.9) is of the form

$$\left.\begin{aligned} x &= Ae^{\lambda t} \\ y &= Be^{\lambda t} \end{aligned}\right\} \tag{2.11}$$

where A, B and λ are constant.
We know that if (2.11) is a solution of (2.9), then λ must satisfy the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \tag{2.12}$$

Because of (2.10) i.e., for $ad - bc \neq 0$, the equation (2.12) has two non-zero solutions. Let λ_1 and λ_2 be the two roots of (2.12).

2.13.1 Stability

(i) If both roots be $-$ve then the critical point is asymptotically stable.

(ii) If both or one roots be positive then the critical point is unstable.

(iii) If both or one roots be purely imaginary then the critical point is stable.

(iv) If the real part of complex roots be $-$ve then the critical point is asymptotically stable.

(v) If the real part of complex roots be +ve then the critical point is unstable.

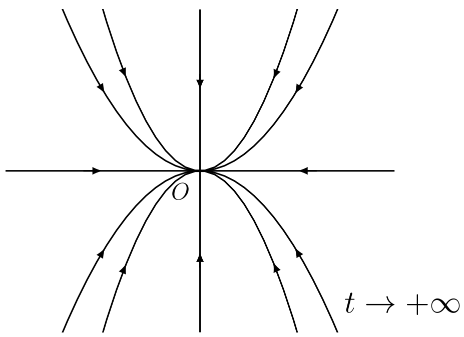
2.13.2 Nature of the Roots

We now consider the following cases.

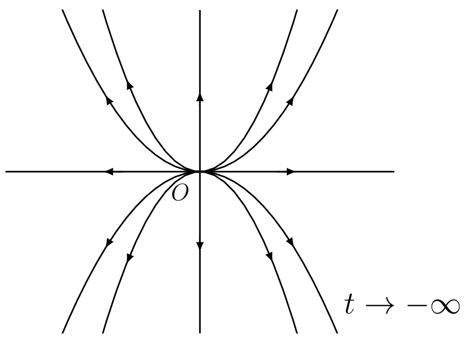
Case I: λ_1 and λ_2 are real and unequal.
In this case the general solution of (2.9) is

$$\left.\begin{aligned} x(t) = x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t} \\ y(t) = y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t} \end{aligned}\right\} \tag{2.13}$$

Subcase 1(a): $\lambda_1 < 0$, $\lambda_2 < 0$
A qualitative picture of the paths (2.13) appears as follow.



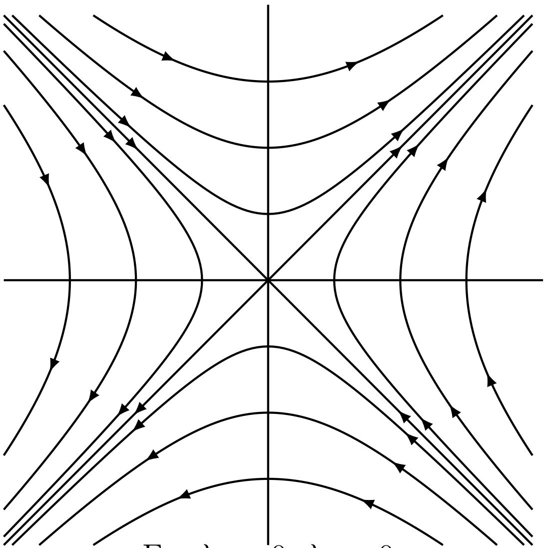
Here the critical point $(0,0)$ is an asymptotically stable node.
Subcase 1(b): $\lambda_1 > 0$, $\lambda_2 > 0$
A qualitative picture of the paths (2.13) appears as follow.



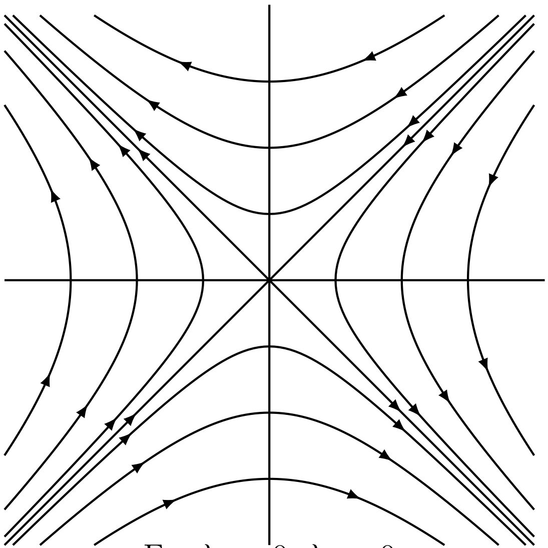
Here the critical point $(0, 0)$ is an unstable node.

Subcase 1(c): $\lambda_1 > 0,\ \lambda_2 < 0$ *or* $\lambda_1 < 0,\ \lambda_2 > 0$

A qualitative picture of the paths (2.13) appears as follow.



For $\lambda_1 > 0$, $\lambda_2 < 0$



For $\lambda_1 < 0$, $\lambda_2 > 0$

Here the critical point $(0, 0)$ is an unstable saddle point.

Case II: λ_1 *and* λ_2 *are real and equal*

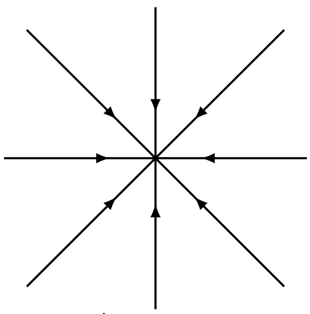
i.e., $\lambda_1 = \lambda_2 = \lambda$

Here the general solution of (2.9) is of the form

$$x = c_1 A e^{\lambda t} + c_2(A_1 t + A_2)e^{\lambda t}$$
$$y = c_1 B e^{\lambda t} + c_2(B_1 t + B_2)e^{\lambda t}$$

Subcase 2(a): $\lambda < 0$

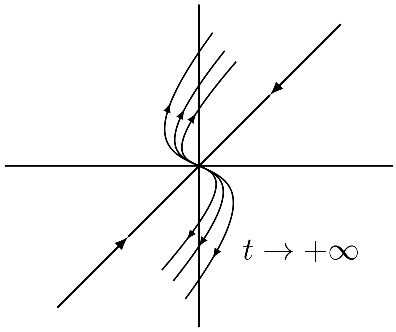
(i) Here the family of half lines approach and enter $(0, 0)$ as $t \to \infty$ where $a = d \neq 0$, $b = c = 0$



$t \to +\infty$

Thus $(0, 0)$ us an asymptotically stable node.

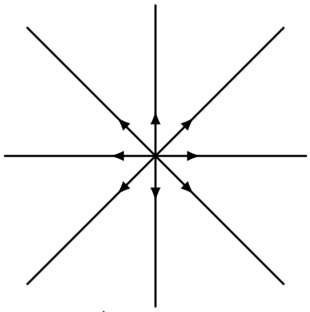
(ii) Two half line paths and family of non-rectilinear paths approach and enter $(0, 0)$ where $a = d \neq 0$ and $b = c = 0$ are not satisfied.



Thus, $(0,0)$ is an asymptotically stable node.

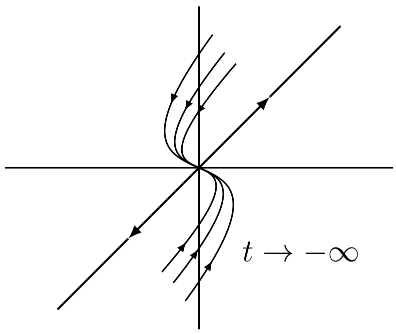
Subcase 2(b): $\lambda > 0$

(i) Here the family of half lines approach and enter $(0,0)$ as $t \to -\infty$ where $a = d \neq 0$, $b = c = 0$



Thus, $(0,0)$ us an unstable node.

(ii) Two half line paths and family of non-rectilinear paths approach and enter $(0,0)$ as $t \to -\infty$ where the conditions $a = d \neq 0$ and $b = c = 0$ are not satisfied.

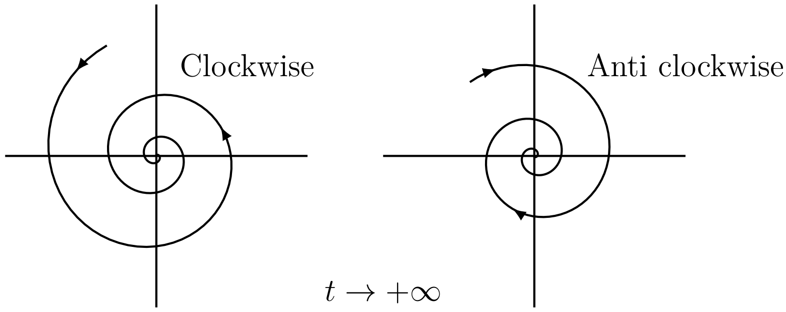


(iii) The critical point $(0,0)$ is an unstable node.

Case III:
Let $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\beta \neq 0$
Here the general solution of (2.9) has the form

$$x = e^{\alpha t}[c_1 \cos \beta t + c_2 \sin \beta t]$$
$$y = e^{\alpha t}[c_3 \cos \beta t + c_4 \sin \beta t]$$

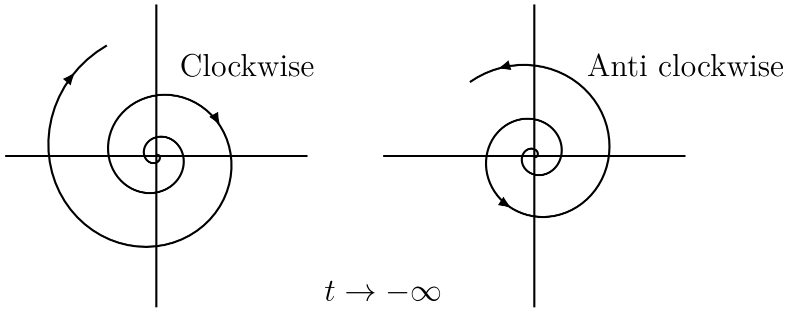
Subcase 3(a): $\alpha < 0$, $\beta \neq 0$
For $\alpha < 0$, the paths approach $(0,0)$ spirally as $t \to \infty$.



The critical point $(0,0)$ is an asymptotically stable spiral.

Subcase 3(b): $\alpha > 0$, $\beta \neq 0$

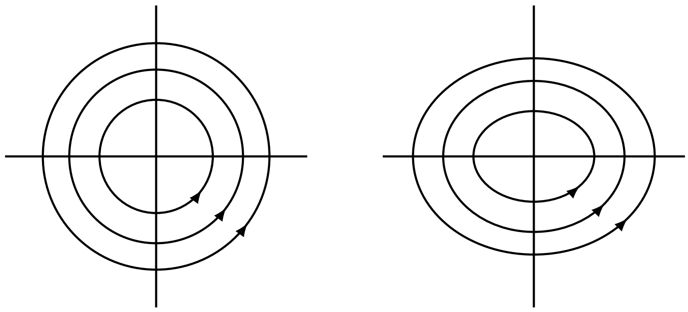
For $\alpha > 0$, the paths approach $(0,0)$ spirally as $t \to -\infty$.



The critical point $(0,0)$ is an unstable spiral.

Subcase 3(c): $\alpha = 0$, $\beta \neq 0$

For $\alpha = 0$, the paths are closed curve surrounding $(0,0)$ and do not approach $(0,0)$.



The critical point $(0,0)$ is a center.

2.14 Nature and Stability of a Critical Point of An Autonomous System

We consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \tag{2.14}$$

where a, b, c and d are the real constants. We assume that

$$ad - bc \neq 0 \tag{2.15}$$

Clearly $(0,0)$ is a critical point of (2.14). We know that solution of (2.14) is of the form

$$\left.\begin{aligned} x &= Ae^{\lambda t} \\ y &= Be^{\lambda t} \end{aligned}\right\} \tag{2.16}$$

If (2.16) is the solution of (2.14) the λ must satisfy the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \tag{2.17}$$

Let λ_1 and λ_2 be two roots of (2.17).

Nature of roots λ_1 and λ_2 of characteristic equation $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$	Nature of critical point of linear system $\dot{x} = ax + by$, $\dot{y} = cx + dy$	Stability of critical point $(0,0)$
real, unequal and of same sign	node	asymptotically stable if roots are negative; unstable if roots are positive
real, unequal and of opposite sign	saddle point	unstable
real and equal	node	asymptotically stable if roots are negative; unstable if roots are positive
conjugate complex but not purely imaginary	spiral point	asymptotically stable if the real parts of roots are negative; unstable if the real parts of roots are positive
pure imaginary	center	stable but not asymptotically stable

2.15 Problems

Problem 2.1. Employ the method of isoclines to sketch several approximate integral curves of the following differential equations.

(a) $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = x^2 + y^2$

(b) $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = y^3 - x^2$

(c) $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{y}{x^2}$

Solution (a).

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = x^2 + y^2 \tag{2.18}$$

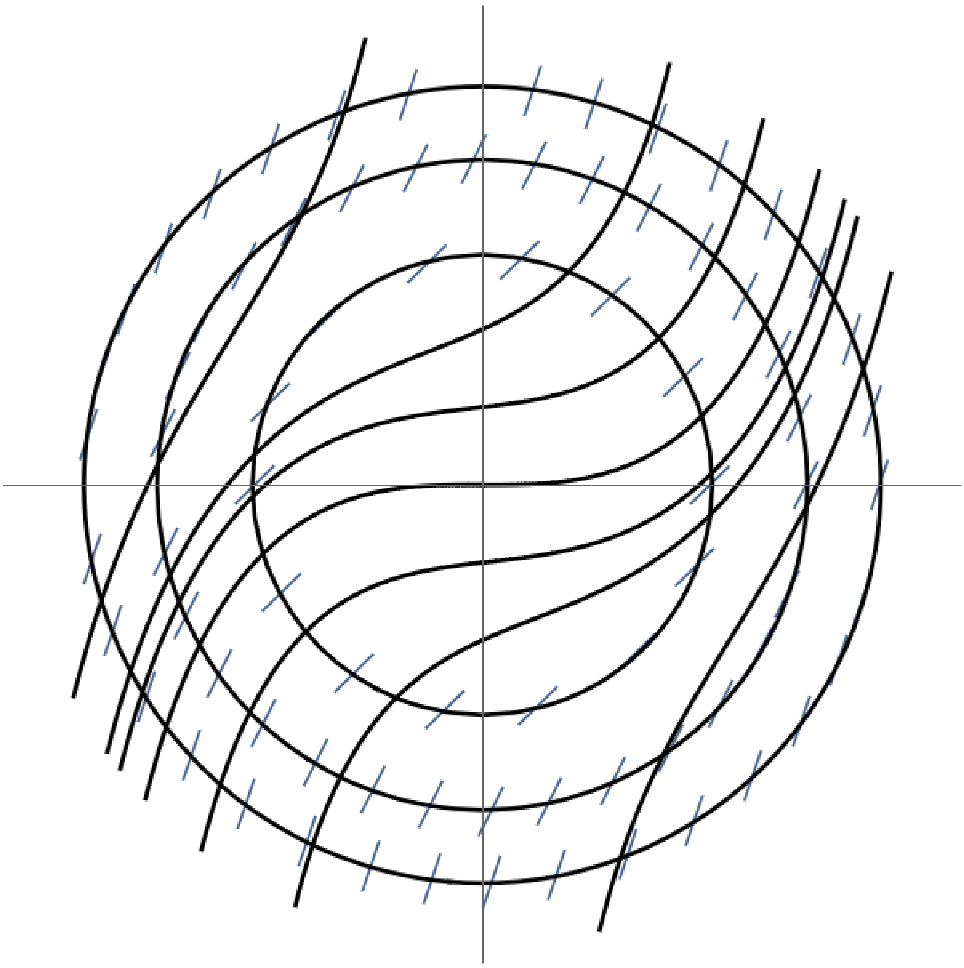
and the isocline of (2.18) is given by

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c$$
$$\Rightarrow x^2 + y^2 = c \tag{2.19}$$

Where c is the slope of the line element of the curves obtained for different values of c.
For different values of parameter c (2.19) represent a family of circle.
We construct the curve (2.19) for $c = 0,\ 0.5,\ 0.75,\ 1,\ 1.5,\ 2,\ 3$ etc.

On each of these circles we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $x^2 + y^2 = 0$,	$\theta = \tan^{-1} c = 0°$
When $c = 0.5$,	then $x^2 + y^2 = 0.5$,	$\theta = \tan^{-1} c = 26.56°$
When $c = 0.75$,	then $x^2 + y^2 = 0.75$,	$\theta = \tan^{-1} c = 36.86°$
When $c = 1$,	then $x^2 + y^2 = 1$,	$\theta = \tan^{-1} c = 45°$
When $c = 1.5$,	then $x^2 + y^2 = 1.5$,	$\theta = \tan^{-1} c = 56.3°$
When $c = 2$,	then $x^2 + y^2 = 2$,	$\theta = \tan^{-1} c = 63.43°$
When $c = 3$,	then $x^2 + y^2 = 3$,	$\theta = \tan^{-1} c = 71.56°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (2.18).

Solution (b)**.**

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = y^3 - x^2 \tag{2.20}$$

and the isocline of (2.20) is given by

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c$$

$$\Rightarrow y^3 - x^2 = c \tag{2.21}$$

Where c is the slope of the line element of the curves obtained for different values of c.
For different values of parameter c (2.21) represent a family of curves.

We construct the curve (2.21) for $c = 0, \pm 1, \pm 23$ etc.
On each of these circles we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$. When $c = 0$, then $y^3 = x^2$, $\theta = \tan^{-1} c = 0°$

$c=0$	x	0	± 0.35	± 1	± 1.84	2.83	± 3.95
	y	0	0.5	1	1.5	2	2.5

When $c = 1$, then $y^3 = x^2 + 1$, $\theta = \tan^{-1} c = 45°$

$c=1$	x	0	± 1.54	± 2.65	± 3.82
	y	1	1.5	2	2.5

When $c = -1$, then $y^3 = x^2 - 1$, $\theta = \tan^{-1} c = -45°$

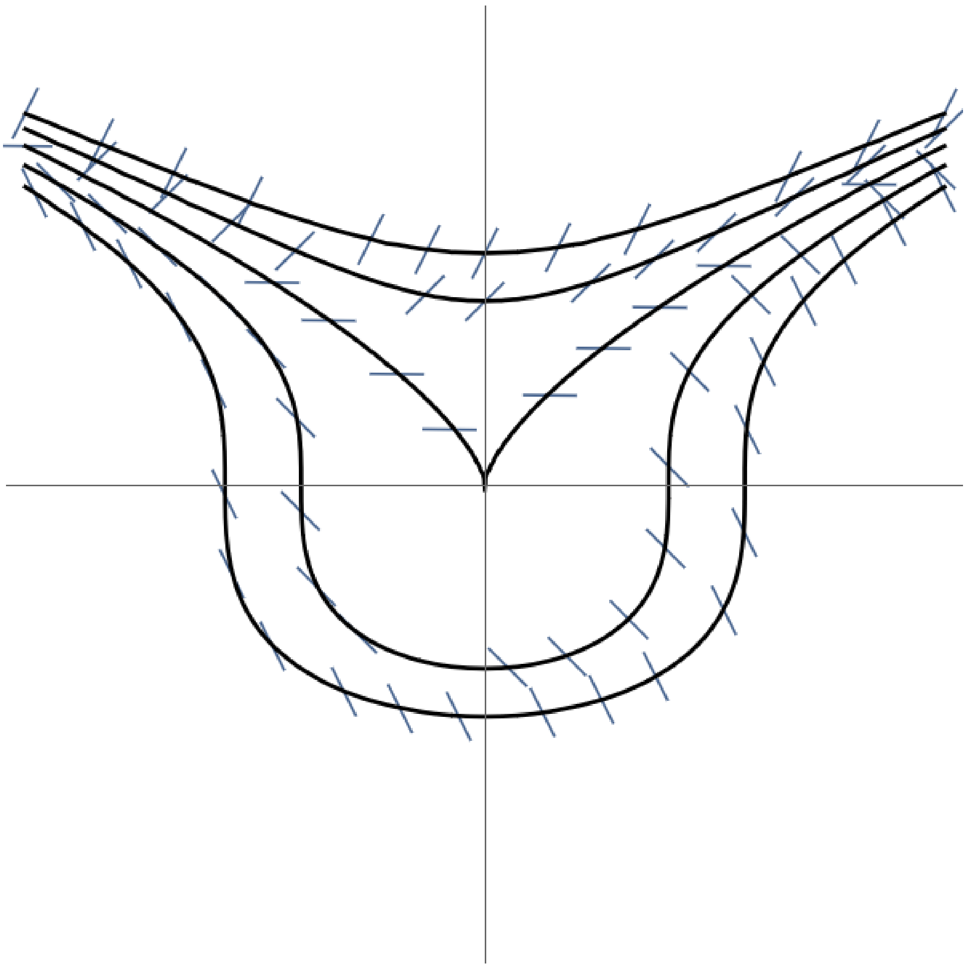
$c=-1$	x	0	± 0.94	± 1	± 1.06	± 1.4	± 2.03	± 3
	y	-1	-0.5	0	0.5	1	1.5	2

When $c = 2$, then $y^3 = x^2 + 2$, $\theta = \tan^{-1} c = 63.43°$

$c=2$	x	0	± 0.5	± 1	± 1.5	± 2	± 2.44	± 3	± 3.69	± 4
	y	1.25	1.31	1.44	1.61	1.81	2	2.22	2.5	2.62

When $c = -2$, then $y^3 = x^2 - 2$, $\theta = \tan^{-1} c = -63.43°$

$c=-2$	x	0	± 1	± 1.5	± 2	± 2.3	± 3.16	± 3.65	± 4
	y	-1.25	-1	0.62	1.25	1.5	2	2.25	2.41



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (2.20).

Solution (c).

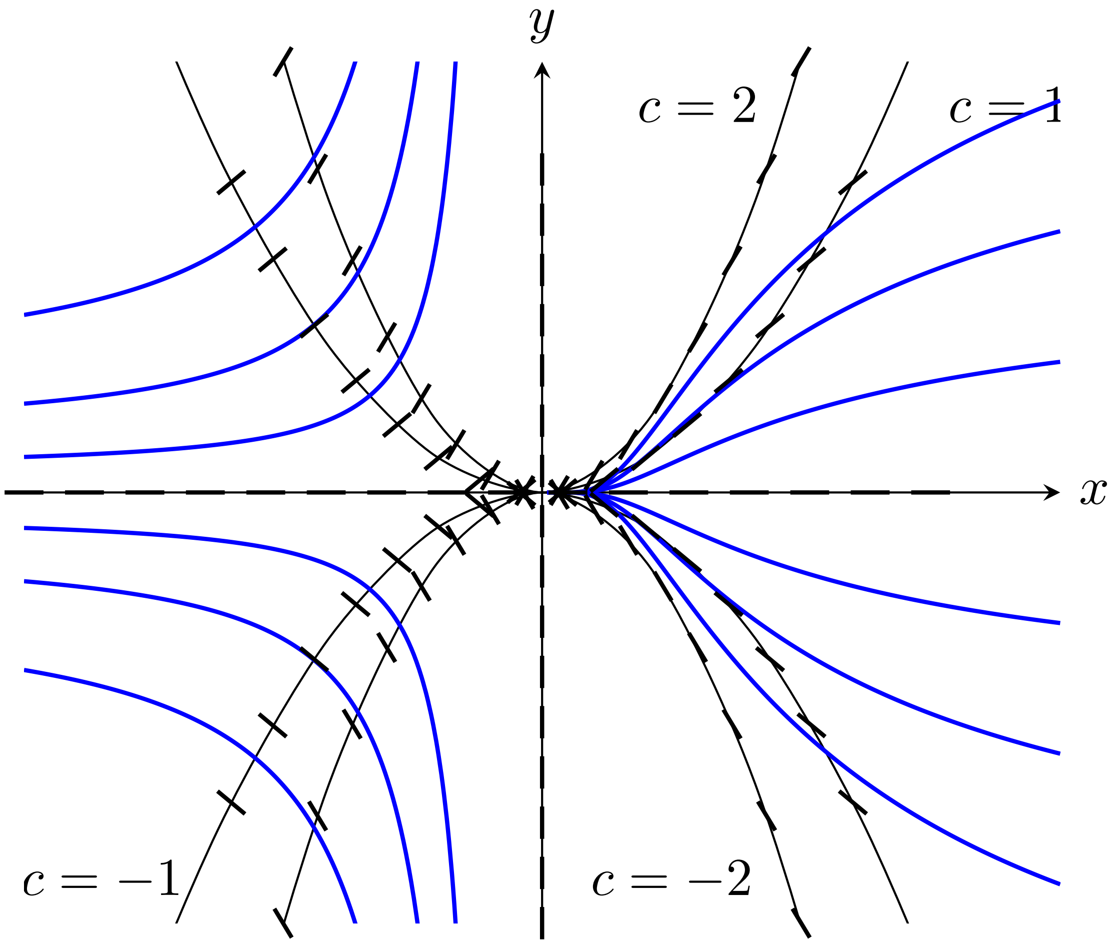
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{y}{x^2} \tag{2.22}$$

and the isocline of (2.22) is given by

$$\begin{aligned}
&\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c \\
\Rightarrow &\frac{y}{x^2} = c \\
\Rightarrow &\, y = cx^2
\end{aligned} \tag{2.23}$$

Where c is the slope of the line element of the curves obtained for different values of c.
For different values of parameter c (2.21) represent a family of parabola.
We construct the curve (2.21) for $c = 0, \pm 1, \pm 2$ etc.
On each of these circles we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = 0$,	$\theta = 0°$
When $c = 1$,	then $y = x^2$,	$\theta = 45°$
When $c = -1$,	then $y = x^2$,	$\theta = -45°$
When $c = 2$,	then $y = 2x^2$,	$\theta = 63.43°$
When $c = -2$,	then $y = -2x^2$,	$\theta = -63.43°$
When $c = \infty$,	then $x = 0$,	$\theta = 90°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (2.22).

Problem 2.2. Examine the nature and stability of the critical points of the following autonomous system and sketch the phase portrait of each cases.

(a) $\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} = x + x^2 - 3xy$

$\dfrac{\mathrm{d}\,y}{\mathrm{d}\,t} = -2x + y + 3y^2$

(b) $\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} = x(4 - 2x - 4y)$

$\dfrac{\mathrm{d}\,y}{\mathrm{d}\,t} = y(x - 1)$

(c) $\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} = y - x^2$

$\dfrac{\mathrm{d}\,y}{\mathrm{d}\,t} = 8x - y^2$

Solution (a)**.** We have

$$\left.\begin{aligned}\dot{x} &= x + x^2 - 3xy\\ \dot{y} &= -2x + y + 3y^2\end{aligned}\right\} \tag{2.24}$$

The critical point of the system (2.24) are given by $\dot{x} = 0$ and $\dot{y} = 0$. i.e.,

$$x + x^2 - 3xy = 0 \tag{2.25}$$

$$-2x + y + 3y^2 = 0 \tag{2.26}$$

From (2.25) we get,

$$\begin{aligned}x = 0, \qquad 1 + x - 3y &= 0\\ \Rightarrow\; x &= 3y - 1\end{aligned}$$

Putting $x = 0$ in (2.26) we get,

$$\begin{aligned}&y + 3y^2 = 0\\ \Rightarrow\; &y(1 + 3y) = 0\\ \Rightarrow\; &y = 0, \quad y = -\frac{1}{3}\end{aligned}$$

Again put $x = 3y - 1$ in (2.26) we get,

$$\begin{aligned}&-6y + 2 + y + 3y^2 = 0\\ \Rightarrow\; &3y^2 - 5y + 2 = 0\\ \Rightarrow\; &y = 1, \quad y = \frac{2}{3}\end{aligned}$$

Hence the critical points are $(0, 0)$, $(0, -\frac{1}{3})$, $(2, 1)$, $(1, \frac{2}{3})$.

(I) Investigation for critical point $(0, 0)$*:*
The corresponding linearized system of (2.24) is

$$\left.\begin{aligned}\dot{x} &= x\\ \dot{y} &= -2x + y\end{aligned}\right\} \tag{2.27}$$

Here we observe that

(i) $\begin{vmatrix} a & b\\ c & d\end{vmatrix} = \begin{vmatrix} 1 & 0\\ -2 & 1\end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{P(x,y)}{\sqrt{x^2+y^2}} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{Q(x,y)}{\sqrt{x^2+y^2}} = 0$

Where $P(x,y) = x^2 - 3xy$; $Q(x,y) = 3y^2$

Hence the behavior of the paths of the system (2.24) near $(0,0)$ would be similar to that of the paths of the related linearized system (2.27).
The characteristic equation of (2.27) is

$$\begin{aligned}
&\lambda^2 - (1+1)\lambda + 1 - 0 = 0\\
\Rightarrow\ &\lambda^2 - 2\lambda + 1 = 0\\
\Rightarrow\ &(\lambda - 1)^2 = 0\\
\Rightarrow\ &\lambda = 1,\quad 1
\end{aligned}$$

The characteristic roots are real, equal and positive. Thus, the critical points $(0,0)$ is unstable node.

(II) Investigation for critical point $(0, -\frac{1}{3})$*:*
For the critical point $(0, -\frac{1}{3})$ we make the transformation $x = \xi$, $y = \eta - \frac{1}{3}$. So that $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{\eta}$. Which transforms the critical point $x = 0$, $y = -\frac{1}{3}$ to $\xi = 0$, $\eta = 0$ in the $\xi\eta$ plane.
With this transformation, we get from (2.24)

$$\begin{aligned}
&\dot{\xi} = \xi + \xi^2 - 3\xi\left(\eta - \frac{1}{3}\right)\\
&\dot{\eta} = -2\xi + \eta - \frac{1}{3} + 3\left(\eta - \frac{1}{3}\right)^2\\
\Rightarrow\ &\left.\begin{aligned}\dot{\xi} &= 2\xi + \xi^2 - 3\xi\eta\\ \dot{\eta} &= -2\xi - \eta + 3\eta^2\end{aligned}\right\}
\end{aligned}\tag{2.28}$$

The corresponding linearized system of (2.28) is

$$\left.\begin{aligned}\dot{\xi} &= 2\xi\\ \dot{\eta} &= -2\xi - \eta\end{aligned}\right\}\tag{2.29}$$

Here we observe that

(i) $\begin{vmatrix} a & b\\ c & d\end{vmatrix} = \begin{vmatrix} 2 & 0\\ -2 & -1\end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{P_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = \lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{Q_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = 0$

Where $P_1(\xi,\eta) = \xi^2 - 3\xi\eta$; $Q_1(\xi,\eta) = 3\eta^2$

Hence the behavior of the paths of the system (2.28) near $(0,0)$ would be similar to that of the paths of the related linearized system (2.27).
The characteristic equation of (2.29) is

$$\begin{aligned}
&\lambda^2 - (2-1)\lambda + (-2) - 0 = 0\\
\Rightarrow\ &\lambda^2 - \lambda - 2 = 0\\
\Rightarrow\ &\lambda^2 - 2\lambda + \lambda - 2 = 0\\
\Rightarrow\ &\lambda = 2,\quad -1
\end{aligned}$$

Since the characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (2.29) but also an unstable saddle point of the system (2.28). So the critical point $(0, -\frac{1}{3})$ is an unstable saddle point of the system (2.24).

(III) Investigation for critical point $(2, 1)$*:*
For the critical point $(2, 1)$ we make the transformation $x = \xi_1 + 2$, $y = \eta_1 + 1$. So that $\dot{x} = \dot{\xi}_1$, $\dot{y} = \dot{\eta}_1$. Which transforms the critical point $x = 2$, $y = 1$ to $\xi_1 = 0$, $\eta_1 = 0$ in the $\xi_1\eta_1$ plane.
With this transformation, we get from (2.24)

$$\begin{aligned}
&\dot{\xi}_1 = \xi_1 + 2 + (\xi_1 + 2)^2 - 3(\xi_1 + 2)(\eta_1 + 1) \\
&\dot{\eta}_1 = -2(\xi_1 + 2) + (\eta_1 + 1) + 3(\eta_1 + 1)^2 \\
\Rightarrow &\left.\begin{aligned} \dot{\xi}_1 &= 2\xi_1 - 6\eta_1 + \xi_1^2 - 3\xi_1\eta_1 \\ \dot{\eta}_1 &= -2\xi_1 + 7\eta_1 + 3\eta_1^2 \end{aligned}\right\}
\end{aligned} \tag{2.30}$$

The corresponding linearized system of (2.30) is

$$\left.\begin{aligned} \dot{\xi}_1 &= 2\xi_1 - 6\eta_1 \\ \dot{\eta}_1 &= -2\xi_1 + 7\eta_1 \end{aligned}\right\} \tag{2.31}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2 & -6 \\ -2 & 7 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{P_2(\xi_1, \eta_1)}{\sqrt{\xi_1^2 + \eta_1^2}} = \lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{Q_2(\xi_1, \eta_1)}{\sqrt{\xi_1^2 + \eta_1^2}} = 0$

Where $P_2(\xi_1, \eta_1) = \xi_1^2 - 3\xi_1\eta_1$; $Q_2(\xi_1, \eta_1) = 3\eta_1^2$

Hence the behavior of the paths of the system (2.30) near $(0, 0)$ would be similar to that of the paths of the related linearized system (2.31).
The characteristic equation of (2.29) is

$$\begin{aligned}
&\lambda^2 - (2 + 7)\lambda + 14 - 12 = 0 \\
\Rightarrow &\lambda^2 - 9\lambda + 2 = 0 \\
\Rightarrow &\lambda = \frac{3}{2} \pm \sqrt{\frac{73}{4}}
\end{aligned}$$

Since the characteristic roots are real, unequal and of positive sign. Hence, not only $(0, 0)$ is an unstable node of the system (2.31) but also an unstable node of the system (2.30). So the critical point $(2, 1)$ is an unstable node of the system (2.24).

(IV) Investigation for critical point $(1, \frac{2}{3})$*:*
For the critical point $(1, \frac{2}{3})$ we make the transformation $x = \xi_2 + 1$, $y = \eta_2 + \frac{2}{3}$. So that $\dot{x} = \dot{\xi}_2$, $\dot{y} = \dot{\eta}_2$. Which transforms the critical point $x = 1$, $y = \frac{2}{3}$ to $\xi_2 = 0$, $\eta_2 = 0$ in the $\xi_2\eta_2$ plane.
With this transformation, we get from (2.24)

$$\begin{aligned}
&\dot{\xi}_2 = (\xi_2 + 1) + (\xi_2 + 1)^2 - 3(\xi_2 + 1)\left(\eta_2 + \frac{2}{3}\right) \\
&\dot{\eta}_2 = -2(\xi_2 + 1) + (\eta_2 + \frac{2}{3}) + 3\left(\eta_2 + \frac{2}{3}\right)^2 \\
\Rightarrow &\left.\begin{aligned} \dot{\xi}_2 &= \xi_2 - 3\eta_2 + \xi_2^2 - 3\xi_2\eta_2 \\ \dot{\eta}_2 &= -2\xi_2 + 5\eta_2 + 3\eta_2^2 \end{aligned}\right\}
\end{aligned} \tag{2.32}$$

The corresponding linearized system of (2.32) is

$$\left.\begin{aligned} \dot{\xi}_2 &= \xi_2 - 3\eta_2 \\ \dot{\eta}_2 &= -2\xi_2 + 5\eta_2 \end{aligned}\right\} \tag{2.33}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi_2 \to 0 \\ \eta_2 \to 0}} \dfrac{P_3(\xi_2, \eta_2)}{\sqrt{\xi_2^2 + \eta_2^2}} = \lim\limits_{\substack{\xi_2 \to 0 \\ \eta_2 \to 0}} \dfrac{Q_3(\xi_2, \eta_2)}{\sqrt{\xi_2^2 + \eta_2^2}} = 0$

Where $P_3(\xi_2, \eta_2) = \xi_2^2 - 3\xi_2\eta_1$ and $Q_3(\xi_2, \eta_2) = 3\eta_2^2$

Hence the behavior of the paths of the system (2.32) near $(0, 0)$ would be similar to that of the paths of the related linearized system (2.33).
The characteristic equation is

$$\begin{aligned} &\lambda^2 - (1 + 5)\lambda + 5 - 6 = 0 \\ \Rightarrow\ &\lambda^2 - 6\lambda - 1 = 0 \\ \Rightarrow\ &\lambda = 3 \pm \sqrt{10} \end{aligned}$$

Since the characteristic roots are real, unequal and of opposite sign. Hence, not only $(0, 0)$ is an unstable saddle point of the system (2.33) but also an unstable saddle point of the system (2.32). So the critical point $(1, \frac{2}{3})$ is an unstable saddle point of the system (2.24).

Finally, we get the critical point

(i) $(0, 0)$ is unstable node of (2.24)

(ii) $(0, -\frac{1}{3})$ is unstable saddle point of (2.24)

(iii) $(2, 1)$ is unstable node of (2.24)

(iv) $(1, \frac{2}{3})$ is unstable saddle point of (2.24)

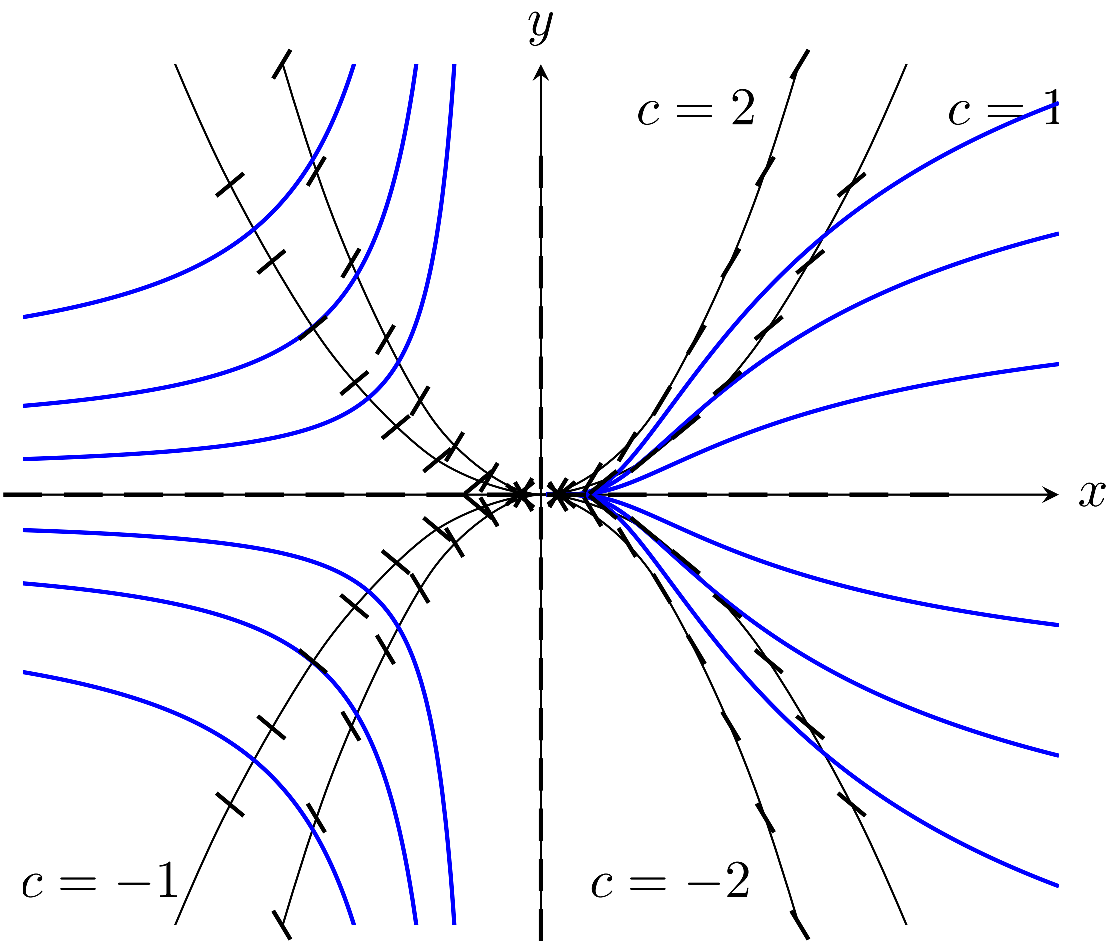
Phase Portrait: From (2.24) we get

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{-2x + y + 3y^2}{x + x^2 - 2xy}$$

The isocline of the above differential equation are given by

$$\begin{aligned} &\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c \\ \Rightarrow\ &\frac{-2x + y + 3y^2}{x + x^2 - 2xy} = c \end{aligned}$$

Where c is the slope of the line element of the canvas obtained for different values of c.
For $c = 0$, $-2x + y + 3y^2 = 0 \quad \Rightarrow \quad \left(y + \frac{1}{6}\right)^2 = \frac{2}{3}\left(x + \frac{1}{24}\right)$ then $\theta = 0°$
For $c = \infty$, $x(1 + x - 2y) = 0 \quad \Rightarrow \quad x = 0, \quad \frac{x}{-1} + \frac{y}{1/3} = 1$ then $\theta = 90°$



Finally we draw several smooth curves. These smooth curves complete the phase portrait of (2.24).

Solution (b)**.** We have

$$\left.\begin{aligned} \dot{x} &= x(4-2x-4y) \\ \dot{y} &= y(x-1) \end{aligned}\right\} \tag{2.34}$$

The critical point of the system (2.34) are given by $\dot{x}=0$ and $\dot{y}=0$. i.e.,

$$x(4-2x-4y)=0 \tag{2.35}$$

$$y(x-1)=0 \tag{2.36}$$

From (2.36) we get, $y=0$, $x=1$
Putting $y=0$ in (2.35) we get, $x=0$, $x=2$
Putting $x=1$ in (2.35) we get, $y=1/2$

Hence, the critical points are $(0,0)$, $(2,0)$, $(1,\frac{1}{2})$.

(I) Investigation for critical point $(0,0)$*:*
The corresponding linearized system of (2.34) is

$$\left.\begin{aligned} \dot{x} &= 4x \\ \dot{y} &= -y \end{aligned}\right\} \tag{2.37}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & -1 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{P(x,y)}{\sqrt{x^2+y^2}} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{Q(x,y)}{\sqrt{x^2+y^2}} = 0$

Where $P(x,y) = -2x^2 - 4xy$ and $Q(x,y) = xy$

Hence the behavior of the paths of the system (2.34) near $(0,0)$ would be similar to that of the paths of the related linearized system (2.37).
The characteristic equation of (2.37) is

$$\begin{aligned} &\lambda^2 - (4-1)\lambda - 4 - 0 = 0 \\ \Rightarrow\ &\lambda^2 - 3\lambda - 4 = 0 \\ \Rightarrow\ &\lambda^2 - 4\lambda + \lambda - 4 = 0 \\ \Rightarrow\ &\lambda = 4, \quad -1 \end{aligned}$$

The characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (2.37) but also an unstable saddle point of the system (2.34).

(II) Investigation for critical point $(2,0)$*:*
For the critical point $(2,0)$ we make the transformation $x = \xi + 2$, $y = \eta$. So that $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{\eta}$. Which transforms the critical point $x = 2$, $y = 0$ to $\xi = 0$, $\eta = 0$ in the $\xi\eta$ plane.
With this transformation, we get from (2.34)

$$\begin{aligned} &\dot{\xi} = 4(\xi+2) - 2(\xi+2)^2 - 4(\xi+2)\eta \\ &\dot{\eta} = \eta(\xi+2) - \eta \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} &\dot{\xi} = -4\xi - 8\eta - 2\xi^2 - 4\xi\eta \\ &\dot{\eta} = \eta + \xi\eta \end{aligned}\right\} \tag{2.38}$$

The corresponding linearized system of (2.28) is

$$\left.\begin{aligned} &\dot{\xi} = -4\xi - 8\eta \\ &\dot{\eta} = \eta \end{aligned}\right\} \tag{2.39}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -4 & -8 \\ 0 & 1 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{P_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = \lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{Q_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = 0$

Where $P_1(\xi,\eta) = -2\xi^2 - 4\xi\eta$, and $Q_1(\xi,\eta) = \xi\eta$

Hence the behavior of the paths of the system (2.38) near $(0,0)$ would be similar to that of the paths of the related linearized system (2.37).
The characteristic equation of (2.29) is

$$\begin{aligned} &\lambda^2 - (-4+1)\lambda - 4 = 0 \\ \Rightarrow\ &\lambda^2 + 3\lambda - 4 = 0 \\ \Rightarrow\ &\lambda^2 + 4\lambda - \lambda - 4 = 0 \\ \Rightarrow\ &\lambda = -4, \quad 1 \end{aligned}$$

Since the characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (2.39) but also an unstable saddle point of the system (2.38). So the critical point $(0, -\frac{1}{3})$ is an unstable saddle point of the system (2.34).

(III) Investigation for critical point $(1, 1/2)$*:*
For the critical point $(1, 1/2)$ we make the transformation $x = \xi_1 + 1$, $y = \eta_1 + \frac{1}{2}$. So that $\dot{x} = \dot{\xi}_1$, $\dot{y} = \dot{\eta}_1$. Which transforms the critical point $x = 1$, $y = \frac{1}{2}$ to $\xi_1 = 0$, $\eta_1 = 0$ in the $\xi_1\eta_1$ plane.
With this transformation, we get from (2.34)

$$\begin{aligned} \dot{\xi} &= 4(\xi_1 + 1) - 2(\xi_1 + 1)^2 - 4(\xi_1 + 1)\left(\eta_1 + \frac{1}{2}\right) \\ \dot{\eta} &= \left(\eta_1 + \frac{1}{2}\right)(\xi_1 + 1) - \left(\eta_1 + \frac{1}{2}\right) \\ \Rightarrow \quad & \left.\begin{aligned} \dot{\xi} &= -2\xi_1 - 4\eta_1 - 2\xi_1^2 - 4\xi_1\eta_1 \\ \dot{\eta} &= \tfrac{1}{2}\xi_1 + \xi_1\eta_1 \end{aligned}\right\} \end{aligned} \tag{2.40}$$

The corresponding linearized system of (2.40) is

$$\left.\begin{aligned} \dot{\xi}_1 &= 2\xi_1 - 4\eta_1 \\ \dot{\eta}_1 &= \tfrac{1}{2}\xi_1 \end{aligned}\right\} \tag{2.41}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -2 & -4 \\ \frac{1}{2} & 0 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{P_2(\xi_1, \eta_1)}{\sqrt{\xi_1^2 + \eta_1^2}} = \lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{Q_2(\xi_1, \eta_1)}{\sqrt{\xi_1^2 + \eta_1^2}} = 0$

Where $P_2(\xi_1, \eta_1) = -2\xi_1^2 - 4\xi_1\eta_1$ and $Q_2(\xi_1, \eta_1) = \xi_1\eta_1$

Hence the behavior of the paths of the system (2.40) near $(0, 0)$ would be similar to that of the paths of the related linearized system (2.41).
The characteristic equation of (2.29) is

$$\begin{aligned} \lambda^2 + 2\lambda + 2 &= 0 \\ \Rightarrow \lambda &= \frac{-2 \pm \sqrt{4 - 8}}{2} \\ \Rightarrow \lambda &= -1 \pm i \end{aligned}$$

Since the characteristic roots are conjugate complex with negative real parts. Hence, $(0, 0)$ is an asymptotically stable spiral point of the system (2.41) and hence of (2.40). So the critical point $(2, 1)$ is an asymptotically stable spiral point of (2.24).

Finally, we get the critical point

(i) $(0, 0)$ is an unstable saddle point of (2.34)

(ii) $(2, 0)$ is an unstable saddle point of (2.34)

(iii) $(1, 1/2)$ is an asymptotically stable spiral point of (2.34)

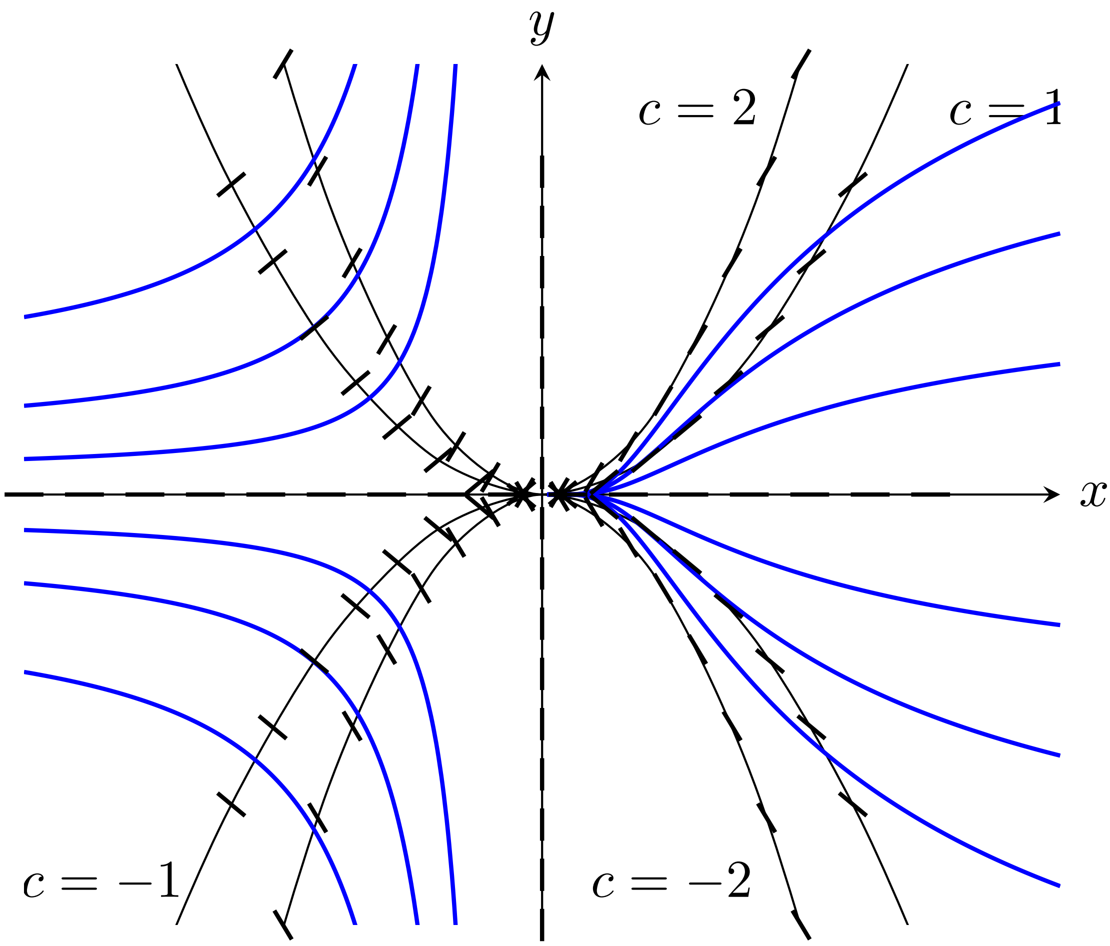
Phase Portrait: From (2.34) we get

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{y(x - 1)}{x(4 - 2x - 4y)}$$

The isocline of the above differential equation are given by

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c$$
$$\Rightarrow \frac{y(x-1)}{x(4-2x-4y)} = c$$

Where c is the slope of the line element of the canvas obtained for different values of c.
When $c = 0$, then $y = 0,\ x = 1$ and $\theta = 0^\circ$
When $c = \infty$, then $x = 0\ \ 2x + 4y = 4 \Rightarrow \frac{x}{2} + \frac{y}{1}$ and $\theta = 90^\circ$



Finally we draw several smooth curves. These smooth curves complete the phase portrait of (2.34).

Solution (c)**.** We have

$$\left.\begin{aligned} \dot{x} &= y - x^2 \\ \dot{y} &= 8x - y^2 \end{aligned}\right\} \tag{2.42}$$

The critical point of the system (2.42) are given by $\dot{x} = 0$ and $\dot{y} = 0$. i.e.,

$$y - x^2 = 0 \tag{2.43}$$
$$8x - y^2 = 0 \tag{2.44}$$

Substituting (2.43) in (2.44) we get

$$8x - x^4 = 0$$
$$\Rightarrow x(8 - x^3) = 0$$
$$\Rightarrow x = 0,\ \ x = 2$$

Putting $x = 2$ in (2.43) we get, $y = 0$, $y = 4$

Hence, the critical points are $(0, 0)$, $(2, 4)$.

(I) Investigation for critical point $(0, 0)$*:*
The corresponding linearized system of (2.42) is

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= 8x \end{aligned}\right\} \tag{2.45}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 8 & 0 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{P(x, y)}{\sqrt{x^2 + y^2}} = \lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{Q(x, y)}{\sqrt{x^2 + y^2}} = 0$

Where $P(x, y) = -x^2$ and $Q(x, y) = -y^2$

Hence the behavior of the paths of the system (2.42) near $(0, 0)$ would be similar to that of the paths of the related linearized system (2.45).
The characteristic equation of (2.45) is

$$\begin{aligned} \lambda^2 - 8 &= 0 \\ \Rightarrow \lambda &= \pm 2\sqrt{2} \end{aligned}$$

The characteristic roots are real, unequal and of opposite sign. Hence, not only $(0, 0)$ is an unstable saddle point of the system (2.45) but also an unstable saddle point of the system (2.42).

(II) Investigation for critical point $(2, 4)$*:*
For the critical point $(2, 4)$ we make the transformation $x = \xi + 2$, $y = \eta + 4$. So that $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{\eta}$. Which transforms the critical point $x = 2$, $y = 4$ to $\xi = 0$, $\eta = 0$ in the $\xi\eta$ plane.
With this transformation, we get from (2.42)

$$\begin{aligned} \dot{\xi} &= 4 + \eta - (\xi + 2)^2 \\ \dot{\eta} &= 8(\xi + 2) - (\eta + 4)^2 \\ \Rightarrow \left.\begin{aligned} \dot{\xi} &= -4\xi + \eta - \xi^2 \\ \dot{\eta} &= 8\xi - 8\eta - \eta^2 \end{aligned}\right\} \end{aligned} \tag{2.46}$$

The corresponding linearized system of (2.46) is

$$\left.\begin{aligned} \dot{\xi} &= -4\xi + \eta \\ \dot{\eta} &= 8\xi - 8\eta \end{aligned}\right\} \tag{2.47}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 8 & -8 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi \to 0 \\ \eta \to 0}} \dfrac{P_1(\xi, \eta)}{\sqrt{\xi^2 + \eta^2}} = \lim\limits_{\substack{\xi \to 0 \\ \eta \to 0}} \dfrac{Q_1(\xi, \eta)}{\sqrt{\xi^2 + \eta^2}} = 0$

Where $P_1(\xi, \eta) = -\xi^2$, and $Q_1(\xi, \eta) = -\eta^2$

Hence the behavior of the paths of the system (2.47) near $(0,0)$ would be similar to that of the paths of the related linearized system (2.46).
The characteristic equation of (2.47) is

$$\begin{aligned} &\lambda^2 - (-4-8)\lambda + 32 - 8 = 0 \\ \Rightarrow\ &\lambda^2 + 12\lambda + 24 = 0 \\ \Rightarrow\ &\lambda = -6 \pm \sqrt{12} \end{aligned}$$

Since the characteristic roots are real, unequal and of negative sign. Hence, not only $(0,0)$ is an asymptotically stable node of the system (2.47) but also an asymptotically stable node of the system (2.46). So the critical point $(2,4)$ is an asymptotically stable node of the system (2.42).

Finally, we get the critical point

(i) $(0,0)$ is an unstable saddle point of (2.34)

(ii) $(2,4)$ is an asymptotically stable node of (2.42)

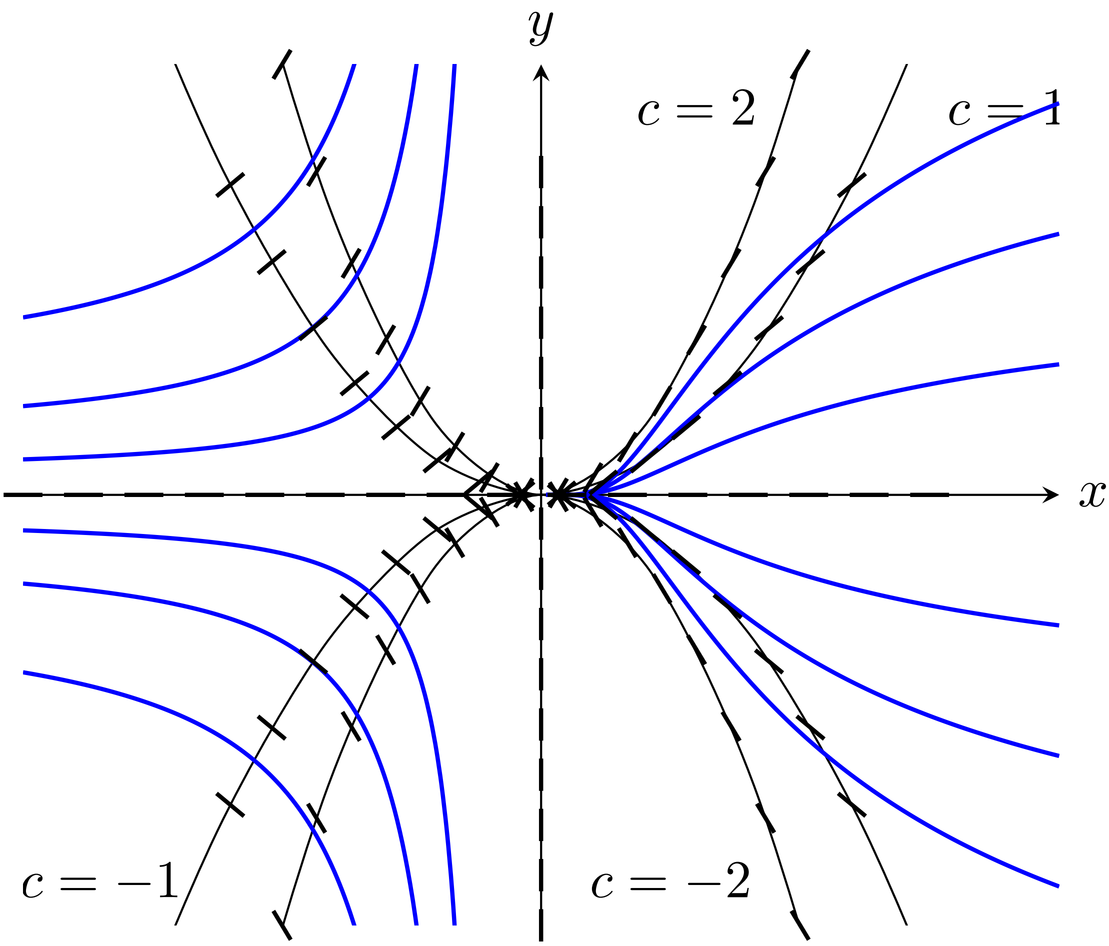
Phase Portrait: From (2.42) we get

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{8x - y^2}{y - x^2}$$

The isocline of the above differential equation are given by

$$\begin{aligned} &\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c \\ \Rightarrow\ &\frac{8x - y^2}{y - x^2} = c \end{aligned}$$

Where c is the slope of the line element of the canvas obtained for different values of c.
When $c = 0$, then $y^2 = 8x$ and $\theta = 0°$
When $c = \infty$, then $y = x^2$ and $\theta = 90°$



Finally we draw several smooth curves. These smooth curves complete the phase portrait of (2.42).

Chapter 3

Population Models

Problem 3.1. What is mathematical model? Discuss the steps in building a good mathematical model. Discuss briefly the limitations of mathematical modelling.

Solution. Mathematical model: A differential equation that describes some physical process is often called a mathematical model of the process.
Steps in building a good mathematical model: The following steps are necessary for building a good mathematical model.

(i) To identify the independent and dependent variables. The independent variable is often time.

(ii) To choose the limits of measurement for each variable.

(iii) To articulate the basic principle that underlies or governs the problem.

(iv) To express the principle or law in terms of the variables.

(v) To make sure that each term in the equation has the same physical units.

Limitations of mathematical modeling:
There are still an equally large or even a larger number of situations which have not yet been mathematically modeled either because the situations are sufficiently complex or because mathematical models formed are mathematically intractable.
However, successful guidelines are not available for choosing the number of parameter and of estimating the values of these parameters. Mathematical modelling of large systems presents its own special problems.

Problem 3.2. Discuss the mathematical model for population. What are the necessity and techniques of mathematical model?

Solution. A differential equation that describes some physical process is often called a mathematical model of the process.

It is necessary first to formulate the appropriate differential equation that describes or models, the problem being investigated. Sometimes it is difficult to construct a satisfactory model.

The following are necessity and techniques of mathematical model.

(i) Identify the dependent and independent variables. The independent variable is often time.

(ii) Choose the units of measurement for each variable.

(iii) Articulate the basic principle that underlies or governs the problem.

(iv) Express the principle or law in terms of the variables.

(v) Make sure that each term in the equation has same physical units.

Problem 3.3. Discuss Malthusian model for single species population growth.
Discuss the limiting behavior of the model as $t \to \infty$. Comment on the appropriateness of the model and suggest some improvement.
or,
Describe the mathematical model of a single species population. Comment on the plausibility of the model and suggest some improvement.

Solution. Let $N(t)$ be the population of species at time t. Then the rate of change,

$$\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = \text{births} \ - \text{deaths} \ + \text{migration} \tag{3.1}$$

is a conservation equation for the population.

The simplest model has no migration and the birth and death terms are proportional to N. Thus, (3.1) takes the form,

$$\begin{aligned} \frac{\mathrm{d}\,N}{\mathrm{d}\,t} &= (b-d)N \\ &= rN \qquad \text{Where } r = b-d \\ \Rightarrow \frac{\mathrm{d}\,N}{N} &= r\,\mathrm{d}\,t \\ \Rightarrow \ln N &= rt + \ln c \quad \text{[Integrating]} \\ \Rightarrow N &= ce^{rt} \end{aligned} \tag{3.2}$$

Let the initial condition, $N = N_0$ at $t = t_0$
From (3.2) we have,

$$\begin{aligned} N(0) &= ce^{rt_0} \\ \Rightarrow c &= N_0 e^{-rt_0} \\ \therefore N &= N_0 e^{r(t-t_0)} \end{aligned}$$

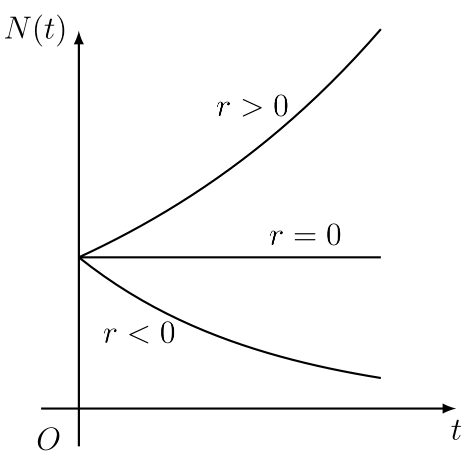
This is the Malthusian model for population growth.

Behaviour of the solution:
For different values of r, we obtain three different solution of Malthusian model.

Case 1: If $r < 0$, then $\lim_{t\to\infty} N(t) = 0$. This is called the case of extinction decays (exponentially).

Case 2: If $r = 0$, then $N(t) = N_0$. The population remains unchanged. (constant)

Case 3: If $r > 0$, then $\lim_{t\to\infty} N(t) = \infty$. The population grows exponentially. (unlimited growth)



Taking $t_0 = 0$, then the solution reduces to

$$N(t) = N_0 e^{rt}$$

In the absence of birth ($b = 0$), the population is by death at rated and consequently the average life span of a number of this problem $\frac{1}{d}$.

If $b > 0$, then the average number of offspring over the lifetime of an average individual under the Malthusian model would be $\frac{b}{d}$.

If the ratio is greater than 1, then birth exceeds death and the population explodes.

If the ratio is less than 1, then death exceeds birth and the population dies out.

Problem 3.4. Describe the Verhulst logistic growth model for the dynamics of a single species population. Find the complete solution of the model. Discuss the behavior of the population as $t \to \infty$. Discuss the stability of the equilibrium states of the logistic model.
Or, Describe the single species discrete time population model. Show that according to this model the population tends to be a constant (carrying capacity).

Solution. *1st part:* For the dynamics of a single species population Verhulst suggested the model

$$\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = rN\left(1 - \frac{N}{K}\right)$$

where $N = N(t)$ = population of single species at time t and r and K are constants. Here, $r\left(1 - \frac{N}{K}\right)$ is the per capita birth rate. The constant K is the carrying capacity of the environment.

2nd part: We have,

$$\begin{aligned}
&\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = rN\left(1 - \frac{N}{K}\right) \\
\Rightarrow\ &\frac{k\,\mathrm{d}\,N}{N(K-N)} = r\,\mathrm{d}\,t \\
\Rightarrow\ &\left(\frac{1}{N} + \frac{1}{K-N}\right)\mathrm{d}\,N = r\,\mathrm{d}\,t
\end{aligned}$$

Integrating both sides, we get

$$\begin{aligned}
&\Rightarrow \log N - \log(K-N) = rt + \log c \\
&\Rightarrow \log\frac{N}{K-N} = \log ce^{rt} \qquad [\because \log e^{rt} = rt] \\
&\Rightarrow \frac{N}{K-N} = ce^{rt} \qquad (3.3)
\end{aligned}$$

Using the initial condition, $N = N_0$ at $t = 0$

$$\begin{aligned}
&\frac{N_0}{K-N_0} = ce^0 \\
\Rightarrow\ &c = \frac{N_0}{K-N_0}
\end{aligned}$$

Equation (3.3) becomes,

$$\begin{aligned}
&\Rightarrow \frac{N}{K-N} = \frac{N_0}{K-N_0}e^{rt} \\
&\Rightarrow N(K-N_0) = N_0(K-N)e^{rt} \\
&\Rightarrow N(K-N_0+N_0e^{rt}) = N_0Ke^{rt} \\
&\Rightarrow N = \frac{N_0Ke^{rt}}{(K-N_0)+N_0e^{rt}}
\end{aligned}$$

$$\therefore\ N = \frac{N_0 K}{N_0 + (K - N_0)e^{-rt}} \tag{3.4}$$

3rd part: From (3.4) we have

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rt}}$$

when, $t \to \infty$, then,

$$\begin{aligned} N(t) &\to \frac{KN_0}{N_0 + (K - N_0)\cdot 0} \\ \Rightarrow N(t) &\to \frac{KN_0}{N_0} \\ \Rightarrow N(t) &\to K \end{aligned}$$

This is the limiting behavior of the model as $t \to \infty$.

4th part: For stability and equilibrium:
For stability and equilibrium,
we have,

$$\begin{aligned} &\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 0 \\ \Rightarrow\ & rN\left(1 - \frac{N}{K}\right) = 0 \\ \Rightarrow\ & N = 0 \quad \text{or,} \quad \left(1 - \frac{N}{K}\right) = 0 \\ & \qquad\qquad \Rightarrow \ \frac{N}{K} = 1 \\ & \qquad\qquad \Rightarrow \ N = K \end{aligned}$$

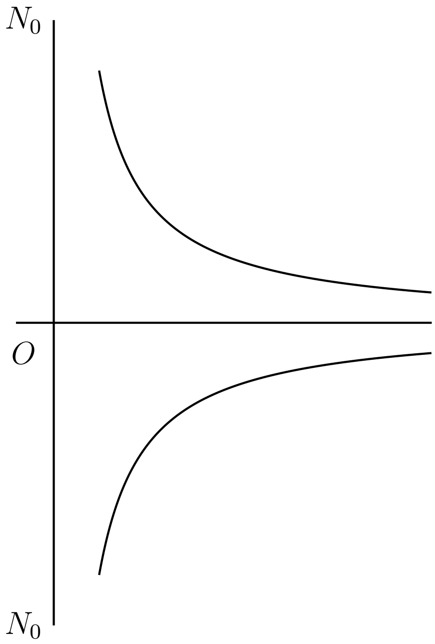
Thus there are two equilibrium states $N = 0$ and $N = K$.

(i) $N = 0$ is unstable linearization about it gives $\frac{\mathrm{d}\,N}{\mathrm{d}\,t} \approx rN$ and so N grows exponentially from any small initial values.

(ii) For linearization about $N = k$, we put $N = k + \varepsilon$ with $|\varepsilon|$ small so that,

$$\begin{aligned} &\frac{\mathrm{d}\,(k+\varepsilon)}{\mathrm{d}\,t} = r(k+\varepsilon)\left(1 - \frac{(k+\varepsilon)}{k}\right) \\ \Rightarrow\ & \frac{\mathrm{d}\,\varepsilon}{\mathrm{d}\,t} = r(k+\varepsilon)\left(\frac{(-\varepsilon)}{k}\right) \\ \Rightarrow\ & \frac{\mathrm{d}\,\varepsilon}{\mathrm{d}\,t} \approx -r\varepsilon \quad \text{to the first order} \\ \Rightarrow\ & \frac{\mathrm{d}\,(N-k)}{\mathrm{d}\,t} \approx -r(N-k) \end{aligned}$$

which gives $N \to k$ as $t \to \infty$.
Hence, $N = k$ is stable.



Problem 3.5. A population grows according to the equation

$$\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 1 - e^{-r\left(1-\frac{N}{k}\right)}$$

where r and k are positive constant.

(i) Determine the equilibrium population size.

(ii) Decide whether the equilibrium is stable, unstable or neutral.

Solution. The given equation is

$$\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 1 - e^{-r\left(1-\frac{N}{k}\right)}, \qquad r, k > 0$$

(i) For equilibrium, we have

$$\begin{aligned}
&\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 0 \\
&\Rightarrow 1 - e^{-r\left(1-\frac{N}{k}\right)} = 0 \\
&\Rightarrow e^{-r\left(1-\frac{N}{k}\right)} = 1 \\
&\Rightarrow e^{-r\left(1-\frac{N}{k}\right)} = 1 = e^0 \\
&\Rightarrow -r\left(1 - \frac{N}{k}\right) = 0 \\
&\Rightarrow 1 - \frac{N}{k} = 0 \\
&\Rightarrow \frac{N}{k} = 1 \\
&\Rightarrow N = k
\end{aligned}$$

Thus the equilibrium population size is $N = N(t) = k = \text{constant}$.

(ii) We linearize the given equation by putting $N = k + \varepsilon$ with $|\varepsilon| \ll 1$. We have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\,t}[k+\varepsilon] = 1 - e^{-r\left(1-\frac{k+\varepsilon}{k}\right)}\\
\Rightarrow\ &\frac{\mathrm{d}\,\varepsilon}{\mathrm{d}\,t} = 1 - e^{-r\left(\frac{-\varepsilon}{k}\right)}\\
\Rightarrow\ &\frac{\mathrm{d}\,\varepsilon}{\mathrm{d}\,t} = 1 - e^{\left(\frac{r\varepsilon}{k}\right)}\\
\Rightarrow\ &\frac{\mathrm{d}\,\varepsilon}{\mathrm{d}\,t} = 1 - \left[1 + \frac{r\varepsilon/k}{1!} + \frac{(r\varepsilon/k)^2}{2!} + \dots\right]\\
\Rightarrow\ &\frac{\mathrm{d}\,\varepsilon}{\mathrm{d}\,t} \approx -\frac{r\varepsilon}{k} \qquad \text{as } |\varepsilon| \ll 1\\
\Rightarrow\ &\frac{\mathrm{d}\,(N-k)}{\mathrm{d}\,t} \approx -\frac{r}{k}(N-k)
\end{aligned}$$

This shows the population decreases.
Hence, the equilibrium $N = k$ is stable.

Problem 3.6. Discuss the continuous/ discontinuous

$$u_{t+1} = u_t \exp[r(1-u_t)] \qquad 0 < r < 1$$

Discuss its total qualitative behavior.

Solution. We have, the given discontinuous model.

$$u_{t+1} = u_t \exp[r(1-u_t)] \qquad 0 < r < 1 \tag{3.5}$$

For the steady states, we have, $u_{t+1} = u_t = u^*$ and (3.5) becomes

$$\begin{aligned}
&u^* = u^* \exp[r(1-u^*)]\\
\Rightarrow\ &u^*[1 - \exp\{r(1-u^*)\}] = 0\\
\Rightarrow\ &u^* = 0 \quad \text{or} \quad 1 - \exp\{r(1-u^*)\} = 0\\
&\qquad\qquad \Rightarrow\ \exp\{r(1-u^*)\} = e^0 \quad [\because e^0 = 1]\\
&\qquad\qquad \Rightarrow\ r(1-u^*) = 0\\
&\qquad\qquad \Rightarrow\ (1-u^*) = 0, \quad r > 0\\
&\qquad\qquad \Rightarrow\ u^* = 1
\end{aligned}$$

The steady states are $u^* = 0$, $u^* = 1$
Here,

$$\begin{aligned}
u_{t+1} = f(u_t; r) &= u_t \exp\{r(1-u_t)\}\\
&= u_t e^{r(1-u_t)}\\
\Rightarrow u_t &= e^{r(1-u_t)} + u_t e^{r(1-u_t)}(-1)r\\
\Rightarrow (u_t) &= (1 - r u_t) e^{r(1-u_t)}
\end{aligned}$$

The corresponding eigenvalues are, $u^* = 0, 1$, then

$$\begin{aligned}
\lambda = f'(u^* = 0) &= (1-u^*)e^{r(1-u^*)}\\
&= (1-0)e^{r(1-0)}\\
&= e^r > 0 \quad [\because r > 0]
\end{aligned}$$

when, $u^* = 1$, then,

$$\begin{aligned}
\lambda &= f'(u^* = 1)\\
&= (1-r)e^{r(1-1)}\\
&= (1-r)e^0 > 0\\
&= (1-r)
\end{aligned}$$

Hence, $u^* = 0$ is unstable and $u^* = 1$ is stable for $0 < r < 2$.

Problem 3.7. Suppose that a certain population obey the Verhulst model with intrinsic growth rate r and carrying capacity K. Find a complete solution of the model. Obtain a formula for the time t where the population size $p(t) = \beta K$, given that initial population $p(0) = \alpha K$ with $0 < \alpha < \beta < 1$.

Solution. The considered population obeys the Verhulst model with intrinsic growth rate r and carrying capacity k. So the model is,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right) \tag{3.6}$$

where, $N = N(t)$ is the population size at time t and r is the intrinsic growth and k is the carrying capacity.
From (3.6)

$$\begin{aligned}
&\frac{\mathrm{d}N}{N(K-N)} = \frac{r}{K}\,\mathrm{d}t \\
\Rightarrow\ &\frac{1}{K}\left(\frac{1}{N} + \frac{1}{K-N}\right)\mathrm{d}N = \frac{r}{K}\,\mathrm{d}t
\end{aligned}$$

Taking integration,

$$\Rightarrow \frac{1}{K}\{\ln N + \ln(K-N)\} = \frac{r}{K}t + A \qquad \text{[Where } A \text{ is an arbitrary constant]}$$

$$\Rightarrow \frac{1}{K}\ln\left(\frac{N}{K-N}\right) = \frac{1}{K}rt + A \tag{3.7}$$

If initially at $t = 0$, $N = N(0) = N_0$, then

$$\frac{1}{K}\ln\left(\frac{N_0}{K-N_0}\right) = A$$

putting the value of A in (3.7), we get

$$\begin{aligned}
&\frac{1}{K}\ln\left(\frac{N}{K-N} \times \frac{K-N_0}{N_0}\right) = \frac{1}{K}rt \\
\Rightarrow\ &\frac{KN - NN_0}{KN_0 - NN_0} = e^{rt} \\
\Rightarrow\ &KN - NN_0 = KN_0e^{rt} - NN_0e^{rt} \\
\Rightarrow\ &KN - NN_0 + NN_0e^{rt} = KN_0e^{rt} \\
\Rightarrow\ &N(K - N_0 + N_0e^{rt}) = KN_0e^{rt} \\
\Rightarrow\ &N = \frac{KN_0e^{rt}}{K - N_0 + N_0e^{rt}}
\end{aligned}$$

$$\therefore N(t) = \frac{KN_0}{N_0 + (K-N_0)e^{-rt}} \tag{3.8}$$

Which is the complete solution of the model.

2nd part: Let $N(t) = P(t)$. If $P(0) = \alpha K$, then (3.8) gets,

$$\begin{aligned}
\alpha K &= \frac{kP_0}{P_0 + (K-P_0)e^0} \\
&= \frac{kP_0}{P_0 + K - P_0} \\
&= \frac{kP_0}{K} \\
&= P_0
\end{aligned}$$

$$\therefore\ \alpha K = P_0 \tag{3.9}$$

Thus from equation (3.8)

$$
\begin{aligned}
P(t) &= \frac{K\,P_0}{P_0 + (K - P_0)e^{-rt}} \\
\Rightarrow \beta\,K &= \frac{K\,P_0}{P_0 + (K - P_0)e^{-rt}} \\
\Rightarrow \beta\,K &= \frac{K\,P_0}{\alpha\,K + (K - \alpha\,K)e^{-rt}} \\
\Rightarrow \beta\,\alpha\,K &+ (\beta\,K - \alpha\,\beta\,K)e^{-rt} = \alpha\,K \quad \text{[dividing both sides by } K\text{]} \\
\Rightarrow e^{-rt} &= \frac{\alpha\,K - \beta\,\alpha\,K}{\beta\,K - \alpha\,\beta\,K} \\
\Rightarrow e^{-rt} &= \frac{K(\alpha - \beta\,\alpha)}{K(\beta - \alpha\,\beta)} \\
\Rightarrow e^{rt} &= \frac{\beta - \alpha\,\beta}{\alpha - \alpha\,\beta} \\
\Rightarrow e^{rt} &= \frac{\alpha\,\beta(\frac{1}{\alpha} - 1)}{\alpha\,\beta(\frac{1}{\beta} - 1)} \\
\Rightarrow rt &= \ln\left\{\frac{1 - \frac{1}{\alpha}}{1 - \frac{1}{\beta}}\right\} \\
\therefore t &= \frac{1}{r}\ln\left\{\frac{1 - \frac{1}{\alpha}}{1 - \frac{1}{\beta}}\right\}
\end{aligned}
$$

Which is the required formula for time t.

Problem 3.8. Determine the outcome of a competition modeled by the system.

$$
\begin{aligned}
\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= x(80 - 3x - 2y) \\
\frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= y(80 - x - y)
\end{aligned}
$$

Solution. Here, $\quad f(x, y) = 80 - 3x - 2y$ and $\quad g(x, y) = 80 - x - y$

$\therefore fy = -2 < 0 \qquad\qquad \therefore gx = -1 < 0$

Since, $fy < 0$, $gx < 0$, hence the given system is competition model.
For equilibrium,

$$
\begin{aligned}
x(80 - 3x - 2y) &= 0 \\
y(80 - x - y) &= 0
\end{aligned}
$$

A consistence equilibrium is found by solving this system.

$$
(80 - 3x - 2y) = 0 \qquad \Rightarrow 3x + 2y = 80 \tag{3.10}
$$

$$
(80 - x - y) = 0 \qquad \Rightarrow x + y = 80
$$

$$
\therefore y = 80 - x \tag{3.11}
$$

From (3.10)

$$
\begin{aligned}
&3x + 2(80 - x) = 80 \\
\Rightarrow\ &3x - 2x = 80 - 160 \\
\Rightarrow\ &x = -80 \\
\therefore\ &y = 80 + 80 = 160
\end{aligned}
$$

$\therefore$ The coenistence equilibrium $(-80, 160)$, this is not possible, because the species x can not be negative.
When $x = 0$, then $y = 80$
When $y = 0$, then $3x = 80 \Rightarrow \; x = \frac{80}{3}$

$\therefore$ The equilibrium points are $(0, 0)$, $(0, 80)$, $(\frac{80}{3}, x)$.

Thus, there is no equilibrium within $x > 0, \; y > 0$.

Here, $F(x, y) = x(80 - 3x - 2y)$
$G(x, y) = y(80 - x - y)$

$$\therefore F_x = 80 - 6x - 2y, \qquad F_y = -2x$$
$$G_x = -y, \qquad G_y = 80 - x - 2y$$

For $(0, 0)$: $F_x(0, 0) = 80$, $\quad G_x(0, 0) = 0$,
$F_y(0, 0) = 0 \quad G_y(0, 0) = 80$

$\therefore$ The community matrix is $A = \begin{pmatrix} 80 & 0 \\ 0 & 80 \end{pmatrix}$

$$\therefore |A| = 6400 > 0$$
$$\text{tra } A = 80 + 80 = 160 > 0$$
$$\Delta = (80 - 80)^2 + 4 \cdot 0 \cdot 0 = 0$$

So the equilibrium is unstable point $(0, 0)$ is node.

For $(0, 80)$: $F_x(0, 80) = 80 - 2 \cdot 80 = -80$, $\quad G_x(0, 80) = -80$,
$F_y(0, 80) = 0 \quad G_y(0, 80) = 80 - 0 - 2 \cdot 80 = -80$

$\therefore$ The community matrix is $A = \begin{pmatrix} -80 & 0 \\ -80 & -80 \end{pmatrix}$

$$\therefore |A| = 6400 > 0$$
$$\text{tra } A = -80 - 80 = -160 < 0$$
$$\Delta = (-80 + 80) + 4 \cdot 0 \cdot 0 = 0$$

So the equilibrium is asymptotically stable node point.
From Dulac's criterion with $\beta(x, y) = \frac{1}{xy}$

$$\therefore \; \frac{\partial}{\partial x}\left(\frac{80 - 3x - 2y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{80 - x - y}{x}\right) = \frac{-3}{y} - \frac{1}{x} < 0$$

So there is no periodic orbit, this means that every point approaches $(0, 80)$.

For $(\frac{80}{3}, 0)$: $F_x(\frac{80}{3}, 0) = -80$, $\quad G_x(\frac{80}{3}, 0) = 0$,
$F_y(\frac{80}{3}, 0) = \frac{-160}{3} \quad G_y(\frac{80}{3}, 0) = \frac{160}{3}$

$\therefore$ The community matrix is $A = \begin{pmatrix} -80 & \frac{-160}{3} \\ 0 & \frac{160}{3} \end{pmatrix}$

$$\therefore |A| = \frac{-3200}{3} < 0$$
$$\text{tr}(A) = \frac{160}{3} - 80 = \frac{-80}{3} < 0$$
$$\Delta = (-80 - \frac{160}{3}) + 4 \cdot 0 < 0$$

So the equilibrium point is unstable saddle point and every orbit approaches $(\frac{80}{3}, 0)$.

Problem 3.9. The following two-dimensional NLODE has been proposed as a model for call differentiation.

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = y - x$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = \frac{5x^2}{4+x^2} - y$$

(i) Sketch the graph $y = x$, $y = \frac{5x^2}{4+x^2}$ in the positive quadrant of the (x, y) plane.

(ii) Determine the equilibrium points.

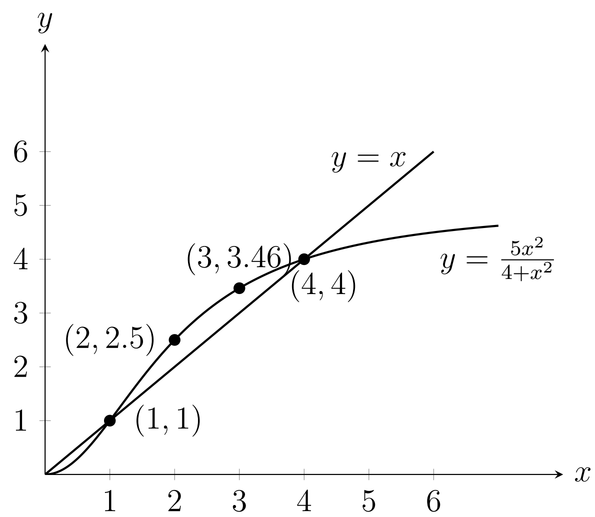
(iii) Determine the local stability of each positive equilibrium point and classify the equilibrium points.

Solution. Given, $y = x$

$$y = \frac{5x^2}{4+x^2}$$

For $y = \frac{5x^2}{4+x^2}$

x	0	0.5	1	2	3	4	5
y	0	0.3	1	2.5	3.4	4	4.3



(ii) The equilibrium points are the solution of

$$y - x = 0 \tag{3.12}$$

$$\frac{5x^2}{4+x^2} - y = 0 \tag{3.13}$$

putting x in (3.13), we get

$$\frac{5x^2}{4+x^2} - x = 0$$
$$\Rightarrow 5x^2 - 4x - x^3 = 0$$
$$\Rightarrow x(x^2 - 5x + 4) = 0$$
$$\Rightarrow x(x^2 - 4x - x + 4) = 0$$
$$\Rightarrow x\{x(x-4) - 1(x-4)\} = 0$$
$$\Rightarrow x(x-4)(x-1) = 0$$
$$\therefore x = 0,\ 1\ 4$$

So the equilibrium points are $(0,0)$, $(1,1)$, $(4,4)$.

(iii) Here,

$$F(x,y) = y - x, \qquad G(x,y) = \frac{5x^2}{4+x^2} - y$$

$$\begin{aligned}
Fx &= -1, & Gx &= \frac{(4+x^2)10x - 5x^2(2x)}{(4+x^2)^2} \\
Fy &= -1, & &= \frac{40x + 10x^3 - 10x^3}{(4+x^2)^2} \\
& & &= \frac{40x}{(4+x^2)^2} \\
& & Gy &- 1
\end{aligned}$$

The linearization at an equilibrium (x_∞, y_∞) is

$$\begin{aligned}
u' &= Fx(x_\infty, y_\infty)u + Fy(x_\infty, y_\infty)v \\
v' &= Gx(x_\infty, y_\infty)u + Gy(x_\infty, y_\infty)v
\end{aligned}$$

Thus,

$$\begin{aligned}
u' &= -u + v \\
v' &= \frac{40x_\infty}{(4+x_\infty^2)^2}u - v
\end{aligned}$$

For equilibrium point $(0,0)$ with linearization

$$\begin{aligned}
u' &= -u + v \\
v' &= -v
\end{aligned}$$

For equilibrium point $(1,1)$ with linearization

$$\begin{aligned}
u' &= -u + v \\
v' &= \frac{40}{(4+1)^2}u - v = \frac{8}{5}u - v
\end{aligned}$$

For equilibrium point $(4,1)$ with linearization

$$\begin{aligned}
u' &= -u + v \\
v' &= \frac{40 \cdot 4}{(4+4)^2}u - v = \frac{2}{5}u - v
\end{aligned}$$

The community matrix at the equilibrium point (x_∞, y_∞) is,

$$\begin{aligned}
A &= \begin{bmatrix} Fx & Fy \\ Gx & Gy \end{bmatrix} \\
&= \begin{bmatrix} -1 & 1 \\ \frac{40x_\infty}{(4+x_\infty^2)^2} & -1 \end{bmatrix}
\end{aligned}$$

For equilibrium point $(0,0)$*:*

The community matrix, $A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = 1 > 1 \therefore |A_1| = 1 > 0$ and tr $(A) = -1 - 1 = -2 < 0$

The critical point $(0,0)$ is asymptotically stable.
The characteristic equation is,

$$\begin{aligned}
&\lambda^2-(-1-1)\lambda+1-0=0\\
&\Rightarrow \lambda^2+2\lambda+1=0\\
&\Rightarrow (\lambda+1)^2=0\\
&\Rightarrow \lambda=-1,-1
\end{aligned}$$

Since, the roots are real, equal and both have $-$ve sign. Thus, the point $(0,0)$ is an asymptotically stable node.

For equilibrium point $(1,1)$*:*

$$Fx(1,1)=-1, \qquad Fy(1,1)=1$$
$$Gx(1,1)=\frac{40}{25}=\frac{8}{5}, \qquad Gy(1,1)=-1$$

The community matrix, $A_2=\begin{bmatrix}-1 & 1\\ \frac{8}{5} & -1\end{bmatrix}$ $\therefore$ $|A_2|=\frac{-3}{5}<0$ and tr $(A_2)=-1-1=-2<0$
Hence, $(1,1)$ is unstable saddle point.
The characteristic equation is,

$$\begin{aligned}
&\lambda^2-5(-1-1)\lambda+1-\frac{8}{5}=0\\
&\Rightarrow 5\lambda^2+10\lambda-3=0\\
&\Rightarrow \lambda=\frac{-10\pm\sqrt{100+60}}{2\cdot 5}\\
&\Rightarrow \lambda=\frac{-10\pm\sqrt{10}}{10}
\end{aligned}$$

Since, the roots are real, unequal and opposite sign. Thus, the equilibrium point $(0,0)$ is an unstable node.

For equilibrium point $(4,4)$*:*

$$Fx(4,4)=-1, \qquad Fy(4,4)=1$$
$$Gx(4,4)=\frac{40\cdot 4}{(4+4)^2}=\frac{2}{5}, \qquad Gy(4,4)=-1$$

The community matrix, $A_3=\begin{bmatrix}-1 & 1\\ \frac{2}{5} & -1\end{bmatrix}$ $\therefore$ $|A_3|=\frac{3}{5}>0$ and tr $(A_3)=-1-1=-2<0$
Hence, $(4,4)$ is asymptotically stable point.
The characteristic equation is,

$$\begin{aligned}
&\lambda^2+2\lambda+1-\frac{2}{5}=0\\
&\Rightarrow 5\lambda^2+10\lambda+3=0\\
&\Rightarrow \lambda=\frac{-10\pm\sqrt{100-60}}{2\cdot 5}\\
&\Rightarrow \lambda=\frac{-10\pm\sqrt{40}}{10}
\end{aligned}$$

Since, the roots are real, unequal and both have negative sign. Thus, the equilibrium point $(4,4)$ is an asymptotically stable node.

Malthusian Model : The model is defined by

$$\frac{dN_t}{dt} = KN_t$$

The solution of the model is $N_t = N_0 e^{Kt}$
where, N_t = number of organisms present at time t
K = the average number of offspring born per organism in the population per unit time
t = the independent variable representing time

Logistic Model : This model is given by equation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)$$

where $f(x) = r\left(1 - \frac{x}{k}\right)$
r = constant of proportionality
k = carrying capacity
The solution of the model is

$$x = \frac{Kx_0 e^{rt}}{K - x_0 + x_0 e^{rt}}$$

Problem 3.10. A population N grows accordingly to the differential equation

$$\frac{dN}{dt} = KN$$

where K is a positive constant. Determine how long it takes the population to double in size?

Solution. We have,

$$\begin{aligned} \frac{dN}{dt} &= KN \\ \Rightarrow \frac{dN}{N} &= K\,dt \end{aligned}$$

Integrating,

$$\begin{aligned} &\Rightarrow \log N = Kt + \log c \\ &\Rightarrow N = ce^{Kt} \qquad (3.14) \end{aligned}$$

Applying initial population, $N = N_0$, for $t = t_0$. From (3.14)

$$\begin{aligned} N_0 &= ce^{Kt_0} \\ \Rightarrow c &= N_0 e^{-Kt_0} \end{aligned}$$

Putting the value of c in (3.14), we get,

$$\begin{aligned} N &= N_0 e^{-kt_0} e^{kt} \\ \therefore N &= N_0 e^{k(t-t_0)} \qquad (3.15) \end{aligned}$$

Suppose at time $t = T$, the population will be double in size i.e., $N = 2N_0$.
Now from (3.15)

$$\begin{aligned} 2N_0 &= N_0 e^{K(T-t_0)} \\ \Rightarrow 2 &= e^{K(T-t_0)} \\ \Rightarrow \log 2 &= K(T - t_0) \\ \Rightarrow T &= \frac{1}{K}\log 2 + t_0 \end{aligned}$$

Which is the required time that will make population double in size.

Problem 3.11. World population was estimated to be 1550 million in 1900 and 2500 in 1950. Estimate the population of the world in the year 2000.

Solution. We know, from Malthusian model,

$$N = N_0 e^{K(t-t_0)} \tag{3.16}$$

Here, $N = 2500$, $N_0 = 1550$, $t = 1950$, $t_0 = 1900$
From (3.16)

$$\begin{aligned}
& 2500 = 1500e^{K(1950-1900)} \\
\Rightarrow\ & \frac{2500}{1500} = e^{50K} \\
\Rightarrow\ & 50K = \log\left(\frac{2500}{1500}\right) = .4780358 \\
\Rightarrow\ & K = 0.009
\end{aligned}$$

Now let N be the population in 2000.
Here, $N = ?$, $N_0 = 2500$, $t = 2000$, $t_0 = 1950$
Now from (3.16)

$$\begin{aligned}
N &= 2500e^{.009(2000-1950)} \\
&= 2500e^{.009\times 50} \\
&= 2500e^{.45} \\
&= 2500 \times 1.5683122 \\
\therefore\ N &= 3920.7805 \text{ million}
\end{aligned}$$

Problem 3.12. Assume that the rate of change of human population of the world is proportional to the number of people present at any time and suppose that this population is increasing at the rate of 2% per year. The world population of 1978 was 4219 million. Calculate the world population of

(a) 1950

(b) 2000

Solution. (a) We know, from Malthusian population model,

$$N = N_0 e^{K(t-t_0)} \tag{3.17}$$

Here, $N_0 = 4219$, $t_0 = 1978$, $t = 1950$, $K = \frac{2}{100} = 0.02$
From (3.17)

$$\begin{aligned}
N &= 4219 \times e^{0.02(1950-1978)} \\
&= 4219 \times e^{0.02(-28)} \\
&= 4219 \times e^{-0.56} \\
&= 4219 \times 0.5712 \\
\therefore\ N &= 2409 \text{ million}
\end{aligned}$$

(b) When $t = 2000$ then

$$\begin{aligned}
N &= 4219 \times e^{0.02(2000-1978)} \\
&= 4219 \times e^{0.02\times 22} \\
&= 4219 \times e^{0.44} \\
&= 4219 \times 1.5527072 \\
\therefore\ N &= 6550.8717 \text{ million}
\end{aligned}$$

Problem 3.13. In a certain bacteria culture, the rate of increasing is the number of bacteria is proportional to the number of present.

(a) If the number triples in 5 hrs, how many will be present in 10 hrs.

(b) When will the number present be 10 times the number initially presents?

Solution. We have the solution of Malthusian population model,

$$N = N_0 e^{K(t-t_0)} \tag{3.18}$$

Initially, $N = N_0, \quad t_0 = 0$
Here, $N(5 \text{ hrs.}) = 3N_0$
From (3.18)

$$\begin{aligned}
&3N_0 = N_0 e^{5K}; \qquad t_0 = 0, \quad t = 5 \text{ hrs.}\\
&\Rightarrow 3 = e^{5K}\\
&\Rightarrow 5K = \log 3\\
&\Rightarrow K = \frac{1}{5}\log 3\\
&\therefore K = 0.2197224
\end{aligned}$$

(a) Let in 10 hrs the bacteria will be P time i.e., $N = PN_0$, $t = 10$ hrs. From (3.18)

$$\begin{aligned}
&PN_0 = N_0 e^{.2197224\times 10}\\
&\Rightarrow P = e^{2.197224}\\
&\Rightarrow P = 8.9999948\\
&\therefore P \approx 9 \text{ times}
\end{aligned}$$

(b) Let at time $t = T$, the number will be 10 times i.e., $N = 10N_0$. From (3.18)

$$\begin{aligned}
&10N_0 = N_0 e^{.2197224\times T}\\
&\Rightarrow 10 = e^{2.197224}\\
&\Rightarrow 0.2197224 \times T = \log 10\\
&\therefore T = 10.48 \text{ hours.}
\end{aligned}$$

Problem 3.14. The population of a city increase at a rate proportional to the number of inhabitants present at any time t. If the population of the city was 30,000 in 1970 and 35,000 in 1980. What will be the population in 1990 and 2000?

Solution. We know the solution of Malthusian population model,

$$N = N_0 e^{K(t-t_0)} \tag{3.19}$$

Here, $N_0 = 30000, \quad N = 35000, \quad t_0 = 1970, \quad t = 1980$
From (3.19)

$$\begin{aligned}
&35000 = 30000 \times e^{K(1980-1970)}\\
&\Rightarrow e^{10K} = \frac{7}{6}\\
&\Rightarrow 10K = \log\left(\frac{7}{6}\right)\\
&\Rightarrow K = \frac{1}{10}\log\left(\frac{7}{6}\right)\\
&\therefore K = 0.015
\end{aligned}$$

Now, $N(1990)=?\quad N_0=35000,\quad t_0=1980,\quad t=1990$
From (3.19)

$$\begin{aligned} N(1990) &= 35000\times e^{0.015(1990-1980)}\\ &= 35000\times e^{0.015\times 10}\\ &= 35000\times e^{0.15}\\ &= 35000\times 1.1618342\\ &= 40664.198 \end{aligned}$$

Now, $N(2000)=?\quad N_0=40664.198,\quad t_0=1990,\quad t=2000$
From (3.19)

$$\begin{aligned} N(2000) &= 40664.198\times e^{0.015(2000-1990)}\\ &= 40664.198\times e^{0.015\times 10}\\ &= 40664.198\times e^{0.15}\\ &= 40664.198\times 1.1618342\\ &= 47245.058 \end{aligned}$$

Problem 3.15. The population of a city satisfies the logistic law $\frac{\mathrm{d}N}{\mathrm{d}t}=10^{-2}N-10^{-8}N^2$ where t is measured in years. Given that the population of the city was 100000 in 1980.

(a) Determine the population in the year 2000.

(b) When will the population be 200000?

(c) What would be the maximum population of the city?

Solution. We have,

$$\begin{aligned} &\frac{\mathrm{d}N}{\mathrm{d}t}=10^{-2}N-10^{-8}N^2\\ \Rightarrow\ &\frac{\mathrm{d}N}{10^{-2}N-10^{-8}N^2}=\mathrm{d}t\\ \Rightarrow\ &\frac{10^2\,\mathrm{d}N}{N-10^{-6}N^2}=\mathrm{d}t\\ \Rightarrow\ &\frac{10^2\,\mathrm{d}N}{N(1-10^{-6}N)}=\mathrm{d}t\\ \Rightarrow\ &10^2\left[\frac{1}{N}+\frac{10^{-6}}{1-10^{-6}N}\right]\mathrm{d}N=\mathrm{d}t\\ \Rightarrow\ &10^2\left[\log N-\log(1-10^{-6}N)\right]=t+100\log c \qquad \text{where } 100\log c \text{ is constant}\\ \Rightarrow\ &\log\frac{N}{1-10^{-6}N}=\frac{t}{100}+\log c\\ \Rightarrow\ &\log\frac{N}{1-10^{-6}N}=\log ce^{\frac{t}{100}}\\ \Rightarrow\ &\frac{N}{1-10^{-6}N}=ce^{\frac{t}{100}} \qquad (3.20)\\ \Rightarrow\ &N=ce^{\frac{t}{100}}-10^{-6}ce^{\frac{t}{100}}N\\ \Rightarrow\ &N\left(1+10^{-6}ce^{\frac{t}{100}}\right)=ce^{\frac{t}{100}}\\ \therefore\ &N=\frac{ce^{\frac{t}{100}}}{1+10^{-6}ce^{\frac{t}{100}}} \qquad (3.21) \end{aligned}$$

Again from (3.20),

$$
\begin{aligned}
N &= ce^{\frac{t}{100}} - 10^{-6}ce^{\frac{t}{100}}N \\
\Rightarrow Ne^{\frac{-t}{100}} &= c - 10^{-6}Nc \\
\Rightarrow c &= \frac{N\,e^{\frac{-t}{100}}}{1-10^{-6}N}
\end{aligned} \tag{3.22}
$$

using initial condition, $t = 1980$, $N = 100000$ in (3.22) we get,

$$
\begin{aligned}
c &= \frac{100000e^{\frac{-1980}{100}}}{1-10^{-6}\times 100000} \\
&= \frac{10^5e^{-19.8}}{1-10^{-6}\times 10^5} \\
&= \frac{10^5e^{-19.8}}{1-10^{-1}} \\
&= \frac{10^6e^{-19.8}}{9}
\end{aligned}
$$

putting the value of c in (3.21),

$$
\begin{aligned}
N &= \frac{\frac{10^6e^{-19.8}}{9}e^{\frac{t}{100}}}{1+10^{-6}\frac{10^6e^{-19.8}}{9}e^{\frac{t}{100}}} \\
&= \frac{10^6e^{\frac{t}{100}-19.8}}{9}\times\frac{9}{9+e^{\frac{t}{100}-19.8}} \\
&= \frac{10^6e^{\frac{t}{100}-19.8}}{9+e^{\frac{t}{100}-19.8}} \\
\text{i.e., } N(t) &= \frac{10^6}{1+9e^{19.8-\frac{t}{100}}}
\end{aligned} \tag{3.23}
$$

which represents the population at any time $t > 1980$.

(a) When $t = 2000$ then from (3.23)

$$
\begin{aligned}
N(2000) &= \frac{10^6}{1+9e^{19.8-20}} \\
N(2000) &= \frac{10^6}{1+9e^{-.2}} \\
N(2000) &= \frac{10^6}{8.3685768} \\
N(2000) &= \frac{100000}{8.3685768} \\
N(2000) &= 119494.63
\end{aligned}
$$

(b) Let at time $t = T$, the population will be 200000. i.e., $N(T) = 200000 = N(t)$.

From (3.23),

$$
\begin{aligned}
200000 &= \frac{10^6}{1+9e^{19.8-\frac{T}{100}}} \\
\Rightarrow 200000 &+ 1800000e^{19.8-\frac{T}{100}} = 1000000 \\
\Rightarrow 1800000&e^{19.8-\frac{T}{100}} = 800000 \\
\Rightarrow e^{19.8-\frac{T}{100}} &= \frac{1800000}{800000} = \frac{4}{9} \\
\Rightarrow 19.8 - \frac{T}{100} &= \log\left(\frac{4}{9}\right) = -0.8109302 \\
\Rightarrow \frac{T}{100} &= 19.8 + 0.8109302 = 20.61093 \\
\Rightarrow T &= 2061.093 \\
\therefore T &\approx 2061
\end{aligned}
$$

(c) $t \to \infty$ gives the maximum population. Now from (3.23)

$$
\begin{aligned}
\lim_{t\to\infty} N(t) &= \lim_{t\to\infty} \frac{10^6}{1+9e^{19.8-\frac{t}{100}}} \\
&= 10^6 \\
&= 100000
\end{aligned}
$$

Problem 3.16. The population N of Natore satisfies the logistic law,

$$\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = (0.03)N - 3\times 10^{-8}N^2$$

where time t is measured in years. If the population of Natore was 200000 in 1980. What will be the population in the year 2000?

Solution. We have,

$$
\begin{aligned}
&\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = (.03)N - 3\times 10^{-8}N^2 \\
\Rightarrow &\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 3\times 10^{-2}N - 3\times 10^{-8}N^2 \\
\Rightarrow &\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 3\times 10^{-2}(N - 10^{-6}N^2) \\
\Rightarrow &\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 3\times 10^{-2}N(1 - 10^{-6}N) \\
\Rightarrow &\frac{\mathrm{d}\,N}{N(1-10^{-6}N)} = 3\times 10^{-2}\,\mathrm{d}\,t \\
\Rightarrow &\left[\frac{1}{N} + \frac{10^{-6}}{1-10^{-6}N}\right]\mathrm{d}\,N = 3\times 10^{-2}\,\mathrm{d}\,t
\end{aligned}
$$

Integrating,

$$
\begin{aligned}
&\Rightarrow \log N - \log(1-10^{-6}N) = \frac{3t}{100} + \log c \\
&\Rightarrow \frac{N}{1-10^{-6}N} = ce^{\frac{3t}{100}} \\
&\Rightarrow N = ce^{\frac{3t}{100}} - 10^{-6}ce^{\frac{3t}{100}}N && (3.24) \\
&\Rightarrow N\left(1+10^{-6}ce^{\frac{3t}{100}}\right) = ce^{\frac{3t}{100}} \\
&\therefore N = \frac{ce^{\frac{3t}{100}}}{1+10^{-6}ce^{\frac{3t}{100}}} && (3.25)
\end{aligned}
$$

Again from (3.24)

$$N = ce^{\frac{3t}{100}}\left(1 - 10^{-6}N\right)$$
$$\Rightarrow c = \frac{Ne^{\frac{3t}{100}}}{1 - 10^{-6}N} \tag{3.26}$$

Applying initial condition, $N = 200000 = 2 \times 10^5$, $t = 1980$ in (3.26) we get,

$$\begin{aligned} c &= \frac{2 \times 10^5 e^{-59.4}}{1 - 10^{-6} \times 2 \times 10^5} \\ &= \frac{5}{2} \times 10^5 e^{-59.4} \\ &= \frac{10^6 e^{-59.4}}{4} \end{aligned}$$

putting the value of c in (3.25)

$$N(t) = \frac{\frac{10^6 e^{-59.4}}{4} e^{\frac{3t}{100}}}{1 + 10^{-6}\frac{10^6 e^{-59.4}}{4} e^{\frac{3t}{100}}}$$
$$\therefore\ N(t) = \frac{10^6 e^{\left(\frac{3t}{100} - 59.4\right)}}{4 + e^{\frac{3t}{100} - 59.4}} \tag{3.27}$$

which represents the population at any time $t > 1980$.
Now, $t = 2000$
From (3.27)

$$\begin{aligned} N(2000) &= \frac{10^6\left(e^{\frac{3\times 2000}{100} - 59.4}\right)}{4 + e^{\frac{3\times 2000}{100} - 59.4}} \\ &= \frac{10^6 \times e^{60-59.4}}{4 + e^{60-59.4}} \\ &= 312964.76 \\ \therefore\ N(2000) &\approx 312964 \end{aligned}$$

Problem 3.17. The human population of an island obeys the logistic law,

$$\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = (0.0025)N - 10^{-8}N^2$$

If the initial population of the island is 20,000 in 1980, then,

(i) Find the population in 2000.

(ii) Find the maximum ultimate population.

(iii) When will the population be 40,000.

(iv) Modify the model when 100 people leaves the island every year.

Solution. We have

$$
\begin{aligned}
& \frac{\mathrm{d}\,N}{\mathrm{d}\,t} = 0.0025N - 10^{-8}N^2 \\
\Rightarrow\ & \frac{\mathrm{d}\,N}{\mathrm{d}\,t} = \frac{25}{10000}N - 10^{-8}N^2 \\
\Rightarrow\ & \frac{\mathrm{d}\,N}{\mathrm{d}\,t} = \frac{N}{400} - 10^{-8}N^2 \\
\Rightarrow\ & \frac{\mathrm{d}\,N}{N\left(\frac{1}{400} - N\,10^{-8}\right)} = \mathrm{d}\,t \\
\Rightarrow\ & \frac{\mathrm{d}\,N}{\frac{1}{400}N\,(1 - 4\times 10^{-6}N)} = \mathrm{d}\,t \\
\Rightarrow\ & \frac{\mathrm{d}\,N}{N\,(1 - 4\times 10^{-6}N)} = \frac{\mathrm{d}\,t}{400} \\
\Rightarrow\ & \left[\frac{1}{N} + \frac{4\times 10^{-6}}{1 - 4\times 10^{-6}N}\right]\mathrm{d}\,N = \frac{\mathrm{d}\,t}{400} \\
\Rightarrow\ & \log N - \log(1 - 4\times 10^{-6}N) = \frac{t}{400} + \log c \\
\Rightarrow\ & \frac{N}{1 - 4\times 10^{-6}N} = ce^{\frac{t}{400}} \\
\Rightarrow\ & N = ce^{\frac{t}{400}} - 4\times 10^{-6}ce^{\frac{t}{400}}N \qquad (3.28) \\
\Rightarrow\ & N = \frac{ce^{\frac{t}{400}}}{1 + 4\times 10^{-6}ce^{\frac{t}{400}}} \\
\therefore\ & N = \frac{c}{4c10^{-6} + e^{-\frac{t}{400}}} \qquad (3.29)
\end{aligned}
$$

From (3.29),

$$
\begin{aligned}
ce^{\frac{t}{400}}\left(1 - 4\times 10^{-6}N\right) &= N \\
c &= \frac{Ne^{-\frac{t}{400}}}{1 - 4\times 10^{-6}N} \qquad (3.30)
\end{aligned}
$$

Applying initial condition $N = 20,000$ $t = 1980$ in (3.30)

$$
\begin{aligned}
c &= \frac{2\times 10^4 e^{-\frac{1980}{400}}}{1 - 4\times 10^{-6}\times 2\times 10^4} \\
&= \frac{2\times 10^4 e^{-4.95}}{1 - 8\times 10^{-2}} \\
&= \frac{10^6 e^{-4.95}}{46}
\end{aligned}
$$

Now (3.29) becomes,

$$
\begin{aligned}
N(t) &= \frac{\frac{10^6 e^{-4.95}}{46}}{4\cdot\frac{10^6 e^{-4.95}}{46}10^{-6} + e^{-t/400}} \\
&= \frac{10^6 e^{-4.95}}{4e^{-4.95} + 46e^{-t/400}} \\
\therefore\quad N(t) &= \frac{10^6}{4 + 46e^{4.95 - t/400}} \qquad (3.31)
\end{aligned}
$$

Which represents population at any time $t > 1980$.

(i) When $t = 2000$, then

$$
\begin{aligned}
N(2000) &= \frac{10^6}{4+46e^{4.95-5}} \\
&= \frac{10^6}{4+46e^{-0.05}} \\
&= \frac{10^6}{4+43.76} \\
&= \frac{10^6}{47.76} \\
\therefore\ N(2000) &\approx 20938
\end{aligned}
$$

(ii) The population will be maximum when $t \to \infty$.
From (3.31)

$$
\begin{aligned}
\lim_{t\to\infty} N(t) &= \lim_{t\to\infty} \frac{10^6}{4+46e^{4.95-t/400}} \\
&= \frac{10^6}{4} \\
&= 250000
\end{aligned}
$$

(iii) Let at time $t = T$, the population will be 40,000. i.e., $N(t) = 40,000$.
From (3.31)

$$
\begin{aligned}
40,000 &= \frac{10^6}{4+46e^{4.95-T/400}} \\
\Rightarrow\ 4 &= \frac{10^2}{4+46e^{4.95-T/400}} \\
\Rightarrow\ 184e^{4.95-T/400} &= 100-16 \\
\Rightarrow\ e^{4.95-T/400} &= 0.4565217 \\
\Rightarrow\ 4.95-\frac{T}{400} &= \log(0.4565217) \\
\Rightarrow\ 4.95-\frac{T}{400} &= -0.7841189 \\
\Rightarrow\ \frac{T}{400} &= 5.734119 \\
\Rightarrow\ T &\approx 2293
\end{aligned}
$$

(iv) If 100 peoples leaves the island every year then we can show that $K = 100$. Thus, the model will be,

$$\frac{\mathrm{d}\,N}{\mathrm{d}\,t} = -100N - 10^{-8}N^2$$

Problem 3.18. If the population of a country double in 50 years, in how many years will it triple under the assumption of the Malthusian model? What will be the population in the year 2000?

Solution. We have from Malthusian model,

$$N = N_0 e^{K(t-t_0)} \tag{3.32}$$

Suppose at $t - t_0 = 50$ years, the population will be double in size. i.e., $N = 2N_0$.
From (3.32)

$$\begin{aligned}
& 2N_0 = N_0 e^{K\times 50} \\
\Rightarrow\ & e^{50K} = 2 \\
\Rightarrow\ & 50K = \ln 2 \\
\Rightarrow\ & k = \frac{1}{50}\ln 2 \\
\therefore\ & k = 0.01386
\end{aligned}$$

For population to be tripled $N = 3N_0$, $t - t_0 =?$
From (3.32)

$$\begin{aligned}
& 3N_0 = N_0 e^{K(t-t_0)} \\
\Rightarrow\ & 3 = e^{0.01386(t-t_0)} \\
\Rightarrow\ & (t - t_0) = \frac{\ln 3}{0.01386} = 79.26 \text{ years}
\end{aligned}$$

After 79.26 years, the population will be triple.

Problem 3.19. Use the logistic model with an assumed carrying capacity of 100×10^9 an observed population of 5×10^9 in 1986 and an observed rate of growth of 2% per year when population size is 5×10^9 predict the population of the earth in the year 2008.

Solution. We have, from logistic model,

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}} \tag{3.33}$$

Where, $k = 100 \times 10^9$
$r = 2\% = 0.02$
$t = 2008 - 1968 = 22$
$N_0 = 5 \times 10^9$

Then from (3.33),

$$\begin{aligned}
N(t) &= \frac{100 \times 10^9 \times 5 \times 10^9}{5 \times 10^9 + (100 \times 10^9 - 5 \times 10^9 \times e^{-0.02\times 22})} \\
&= \frac{500 \times 10^{18}}{5 \times 10^9 + 95 \times 10^9 \times e^{-0.44}} \\
&= \frac{500 \times 10^{18}}{10^9(5 + 95 \times 0.6440)} \\
&= 7.55 \times 10^9
\end{aligned}$$

Problem 3.20. The Pacific halibut fishery is modeled by the logistic equation with carrying capacity 80.5×10^6 measured in kilograms and intrinsic growth rate 0.71 per year. If the initial biomass is one fourth the carrying capacity find the biomass one year later and the time required for the biomass to grow to half the carrying capacity.

Solution. The logistic model is

$$x' = rx\left(1 - \frac{x}{K}\right) \tag{3.34}$$

Given,

$$\begin{aligned}
\text{Carrying capacity, } K &= 80.5 \times 10^6 \text{ kg} \\
\text{Intrinsic growth rate, } r &= 0.71 \\
\text{The initial biomass, } x_0 &= \frac{1}{4}K \\
&= \frac{1}{4} \times 80.5 \times 10^6 \text{ kg} \\
&= 20.125 \times 10^6 \text{ kg} \\
\text{time, } x_0 &= 1 \text{ years}
\end{aligned}$$

The biomass one year later, $x(1) = ?$

We know, the solution of logistic model is,

$$x(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}}$$

$$\begin{aligned}
\therefore\ x(t) &= \frac{80.5 \times 10^6 \times 20.125 \times 10^6}{20.125 \times 10^6 + (80.5 \times 10^6 - 20.125 \times 10^6)e^{-.71\times 1}} \\
&= \frac{1620.0625 \times 10^6}{20.125 + (60.375) \times 0.492} \\
&= \frac{1620.063}{49.808} \times 10^6 \\
&= 32.526 \times 10^6 \text{ kilograms}
\end{aligned}$$

Now let, after time t, the biomass be $x(t)$.

$$\therefore\ x(t) = \frac{1}{2}K = \frac{1}{2} \times 10^6 = 40.25 \times 10^6 \text{ kg}$$

Then by solution of logistic model,

$$\begin{aligned}
x(t) &= \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}} \\
\Rightarrow x(t) &= \frac{Kx_0e^{rt}}{x_0e^{rt} + K - x_0} \\
\Rightarrow x\left(x_0e^{rt} + K - x_0\right) &= Kx_0e^{rt} \\
\Rightarrow xx_0e^{rt} + xK - xx_0 - Kx_0e^{rt} &= 0 \\
\Rightarrow e^{rt}\left(xx_0 - Kx_0\right) &= x(x_0 - K) \\
\Rightarrow e^{rt} &= \frac{x(x_0 - K)}{xx_0 - Kx_0} \\
\Rightarrow e^{rt} &= \frac{x(x_0 - K)}{x_0(x - K)} \\
\Rightarrow e^{rt} &= \frac{40.25 \times 10^6(20.125 \times 10^6 - 80.5 \times 10^6)}{20.125 \times 10^6(40.25 \times 10^6 - 80.5 \times 10^6)} \\
\Rightarrow e^{rt} &= \frac{40.25 \times (-60.375)}{20.125 \times (-40.25)} \\
\Rightarrow e^{rt} &= 3 \\
\Rightarrow rt &= \ln 3 \\
\Rightarrow t &= \frac{\ln 3}{r} \\
\Rightarrow t &= \frac{\ln 3}{0.71} \\
\therefore\ t &= 1.547 \text{ years}
\end{aligned}$$

Problem 3.21. Assume that $P(t)$ size of a population obeying an exponential growth law. P_1, P_2 be the values of $P(t)$ at distinct times t_1 and t_2 $(t_1 < t_2)$ respectively. Prove that the growth rate,

$$r = \frac{1}{t_2 - t_1} \ln \frac{P_2}{P_1}$$

Determine how long it takes the population to double its size under the model.

Proof. We know that,

$$P(t) = P_0 e^{r(t-t_0)}$$

Then,

$$P_1 = P_0 e^{r(t_1-t_0)}, \quad P(t_1) = P_1$$
$$P_2 = P_0 e^{r(t_2-t_0)}, \quad P(t_2) = P_2$$

$$\begin{aligned}
\therefore\ \frac{P_2}{P_1} &= \frac{P_0 e^{r(t_2-t_0)}}{P_0 e^{r(t_1-t_0)}} \\
\Rightarrow\ \frac{P_2}{P_1} &= e^{r(t_2-t_1)} \\
\Rightarrow\ r(t_2 - t_1) &= \ln \frac{P_2}{P_1} \\
\Rightarrow\ r &= \frac{1}{t_2 - t_1} \ln \frac{P_2}{P_1}
\end{aligned}$$

Let, at time $t = T$, the population will be double in size i.e., $P = 2P_0$

$$\begin{aligned}
&\therefore\ P = P_0 e^{r(t-t_0)} \\
&\Rightarrow\ 2P_0 = P_0 e^{r(T-t_0)} \\
&\Rightarrow\ 2 = e^{r(T-t_0)} \\
&\Rightarrow\ r(T - t_0) = \ln 2 \\
&\Rightarrow\ T = t_0 + \frac{\ln 2}{r}
\end{aligned}$$

$\square$

Chapter 4

Liapunov's Direct Method

Definition 1. Let $E(x,y)$ have continuous first partial derivatives at all points (x,y) in a domain D containing the origin $(0,0)$.

1. The function E is called positive defined in D if $E(0,0)=0$ and $E(x,y)>0$ for all other points in (x,y) in D.
2. The function E is called positive semi-defined in D if $E(0,0)=0$ and $E(x,y)\geq 0$ for all other points in (x,y) in D.
3. The function E is called negative defined in D if $E(0,0)=0$ and $E(x,y)<0$ for all other points in (x,y) in D.
4. The function E is called negative semi-defined in D if $E(0,0)=0$ and $E(x,y)\leq 0$ for all other points in (x,y) in D.

4.1 Liapunov Function

Consider the non-linear autonomous system,

$$\left.\begin{aligned} \tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= P(x,y) \\ \tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Q(x,y) \end{aligned}\right\} \tag{4.1}$$

has isolated critical point at $(0,0)$ and P and Q have continuous first partial derivatives for all (x,y). If there exists a differentiable function $E(x,y)$ such that,

(i) $E(x,y)$ is positive defined and

(ii) $\dot{E}(x,y)$ is negative semi defined.

Then $E(x,y)$ is called Liapunov function for the system (4.1) in D.

4.2 Three theorem on Liapunov

Theorem 4.2.1. Consider the system,

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = P(x,y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = Q(x,y)$$

if

(i) $(0,0)$ is an isolated critical point

(ii) $E(x,y)$ is a Liapunov function

Then $(0,0)$ is called stable critical point.

Theorem 4.2.2. Consider the system,

$$\begin{aligned}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= P(x,y)\\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Q(x,y)\end{aligned}$$

if

(i) $(0,0)$ is an isolated critical point

(ii) $E(x,y)$ is a Liapunov function

(iii) $\dot{E}(x,y)$ is a negative defined

Then $(0,0)$ is asymptotically stable.

Theorem 4.2.3. Consider the system,

$$\begin{aligned}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= P(x,y)\\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Q(x,y)\end{aligned}$$

if there exists a function $E(x,y)$ such that

$$\begin{aligned}&E(0,0) = 0\\ &E(x,y) > 0 \qquad \text{for } x \neq 0,\ y \neq 0\end{aligned}$$

Then $(0,0)$ is unstable.

Problem 4.1. For what value of A, $V(x,y) = x^2 + y^2$ is a Liapunov function for the system.

$$\begin{aligned}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= Ax + xy^2\\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Ay - yx^2\end{aligned}$$

and discuss the stability of the critical point $(0,0)$.

Solution. We have,

$$\begin{aligned}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= Ax + xy^2\\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= Ay - yx^2\end{aligned}$$

Again, we have,

$$\begin{aligned}V(x,y) &= x^2 + y^2\\ \dot{V}(x,y) &= 2x\dot{x} + 2y\dot{y}\\ &= 2x(Ax + xy^2) + 2y(Ay - yx^2)\\ &= 2Ax^2 - 2x^2y^2 + 2Ay^2 - 2x^2y\\ &= 2A(x^2 + y^2) - 4x^2y^2\end{aligned}$$

If $V(x, y)$ is a Liapunov function then,

$$\begin{aligned} & \dot{V}(x,y) \leq 0 \\ \Rightarrow\ & 2A(x^2+y^2) - 4x^2y^2 \leq 0 \\ \Rightarrow\ & A \leq \frac{4x^2y^2}{2(x^2+y^2)} \end{aligned}$$

Now, we observe that

(i) V is a differentiable function of x and y

(ii) V is positive defined

(iii) $\dot{V}(x,y) \leq 0$ if $A \leq \frac{4x^2y^2}{2(x^2+y^2)}$

The critical point $(0, 0)$ is stable for the given system for $A \leq \frac{4x^2y^2}{2(x^2+y^2)}$.

Problem 4.2. For the autonomous system

$$\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= -x - y - x^3 \\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= x - y - y^3 \end{aligned}$$

Construct a Liapunov function of the form $Ax^2 + By^2$ where A and B are the constant and use the function to determine the stability of the trivial solution of the system.

Solution. The given autonomous system,

$$\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= -x - y - x^3 \\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= x - y - y^3 \end{aligned}$$

Let us consider a Liapunov function is,

$$E(x, y) = Ax^2 + By^2$$

which is differentiable function of x and y.
Now,

$$\begin{aligned} \dot{E}(x,y) &= 2Ax\dot{x} + 2By\dot{y} \\ &= 2Ax(-x-y-x^3) + 2By(x-y-y^3) \\ &= -2Ax^2 - 2Axy - 2Ax^4 + 2Bxy - 2By^2 - 2By^4 \\ &= 2(Bxy - Axy) - 2\left\{A(x^2+x^4) + B(y^2+y^4)\right\} \end{aligned}$$

For Liapunov function

$$\begin{aligned} & \dot{E}(x,y) \leq 0 \\ \Rightarrow\ & 2(Bxy - Axy) = 0 \\ \Rightarrow\ & Bxy = Axy \\ \Rightarrow\ & \frac{A}{B} = \frac{1}{1} \\ \therefore\ & A = 1 \\ \therefore\ & B = 1 \end{aligned}$$

Hence, $E(x, y) = x^2 + y^2$
The function E is defined by $E(x, y) = x^2 + y^2$ is positive defined in every domain D containing $(0, 0)$.
Clearly, $E(0, 0) = 0$
Also, $\dot{E}(x, y) < 0$ for all (x, y)
Hence, $(0, 0)$ is asymptotically stable point.

Problem 4.3. For the system

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -x + 2x^2 + y^2$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = -y + xy$$

Construct a Liapunov function of the form $Ax^2 + By^2$ where A and B are the constant and use the function to determine whether the critical point $(0, 0)$ of the system is asymptotically stable or at least stable.

Solution. The given system is,

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -x + 2x^2 + y^2$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = -y + xy$$

Let us consider the Liapunov function,

$$E(x, y) = Ax^2 + By^2$$

Now,

$$\begin{aligned} \dot{E}(x, y) &= 2Ax\dot{x} + 2By\dot{y} \\ &= 2Ax(-x + 2x^2 + y^2) + 2By(-y + xy) \\ &= -2Ax^2 + 4Ax^3 + 2Axy^2 - 2By^2 + 2Bxy^2 \\ &= x^2(-2A + 4Ax) + y^2(2Ax - 2B + 2Bx) \end{aligned}$$

For Liapunov function,

$$\begin{aligned} &\dot{E}(x, y) \leq 0 \\ \Rightarrow\ & -2A + 4Ax = 0 \end{aligned} \tag{4.2}$$

and

$$\Rightarrow\ 2Ax - 2B + 2Bx = 0 \tag{4.3}$$

From (4.2) we get,

$$\begin{aligned} & -2A = -4Ax \\ \Rightarrow\ & 1 = 2x \\ \Rightarrow\ & x = \frac{1}{2} \end{aligned}$$

From (4.3) we get,

$$\begin{aligned} & 2A\frac{1}{2} - 2B + 2B\frac{1}{2} = 0 \\ \Rightarrow\ & A - 2B + B = 0 \\ \Rightarrow\ & A - B = 0 \\ \Rightarrow\ & \frac{A}{B} = \frac{1}{1} \end{aligned}$$

Now,

$$\begin{aligned}
\dot{E}(x,y) &= 2x\dot{x} + 2y\dot{y} \\
&= 2x(-x + 2x^2 + y^2) + 2y(-y + xy) \\
&= -2x^2 + 4x^3 + 2xy^2 - 2y^2 - 2y^2 + 2xy^2 \\
&= -2(x^2 + y^2) + 4x^3 + 4xy^2 \\
&= -2(x^2 + y^2) + 4x(x^2 + y^2)
\end{aligned}$$

Here, $E(0,0) = 0$ and $\dot{E}(x,y) < 0$
Hence, $(0,0)$ is an asymptotically stable point.

Problem 4.4. Find the Liapunov function of the dynamical system

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -y - \frac{x}{2} - \frac{x^3}{4}$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = x - \frac{y}{2} - \frac{y^3}{4}$$

and examine the stability of $(0,0)$.

Solution. The given system is,

$$\left.\begin{aligned}
\tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= -y - \tfrac{x}{2} - \tfrac{x^3}{4} \\
\tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= x - \tfrac{y}{2} - \tfrac{y^3}{4}
\end{aligned}\right\} \tag{4.4}$$

Chapter 5

Lotka-Volterra Prey-Predator, Competition, Symbiosis Model

Problem 5.1. Define prey, predator and competition for two species interaction model.

Solution. The dynamics of two species population can be described by

$$N_1{}' = N_1(t)f_1(t, N_1(t), N_2(t), \lambda)$$
$$N_2{}' = N_2(t)f_2(t, N_1(t), N_2(t), \lambda)$$

Prey: A prey is an organism that is or may be seized by a predator to be eaten.

Predator: A predator is an organism that depends on predation for its food.

Competition: If the growth rate of each population is decreased then it is called competition. In this case, two species compete with each other for the same resource such a way that each tries to inhibit the growth of the other. The conditions for the two species competition are $\frac{\partial f_1}{\partial N_2} < 0$ and $\frac{\partial f_2}{\partial N_1} < 0$.

Problem 5.2. State and explain the two-dimensional Lotka-Volterra predator-prey model. Make a stability analysis of the model. Find an exact solution of the system and illustrate the population dynamics in a phase-space.

Solution. Statement: The system governed by two non-linear ODE as below.

$$\left.\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= x(\alpha - \beta y) \\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= y(-\gamma + \delta x) \end{aligned}\right\} \tag{5.1}$$

where α, $\beta\,\gamma$, δ are positive constants is called 2-D Lotka-Volterra predator-prey model. Here $x(t)$ and $y(t)$ represents the prey and predator population respectively.

Explanation: Growth rate of any species is proportional to the number of that species present at that time. In the absence of predator, they prey population increases exponentially and in the presence of the predator, the growth rate of prey population will slow down. In the absence of prey, the predator population decreases exponentially and in the presence of the prey the predator population increases gradually.

Stability analysis: The equilibrium points of (5.1) are given by the following curves.

$$
\begin{aligned}
& x(\alpha-\beta y)=0 \\
& y(-\gamma+\delta x)=0 \\
\Rightarrow\ & x=0, \quad y=\frac{\alpha}{\beta} \\
& y=0, \quad x=\frac{\gamma}{\delta}
\end{aligned}
$$

The critical points are $(0,0)$, $(\frac{\gamma}{\delta},\frac{\alpha}{\beta})$. But $(0,0)$ is not acceptable because both species are absent at $(0,0)$.
We now investigate the critical point $(\frac{\gamma}{\delta},\frac{\alpha}{\beta})$. For $(\frac{\gamma}{\delta},\frac{\alpha}{\beta})$ we make substitution.

$\xi=x-\frac{\gamma}{\delta}$, $\eta=y-\frac{\alpha}{\beta}$ which transform the critical point $x=\frac{\gamma}{\delta}$, $y=\frac{\alpha}{\beta}$ to $\xi=0$, $\eta=0$ in the $\xi\eta-$plane.
Form (5.1),

$$
\begin{aligned}
\frac{\mathrm{d}\,\xi}{\mathrm{d}\,t} &= \left(\xi+\frac{\gamma}{\delta}\right)\left\{\alpha-\beta\left(\eta+\frac{\alpha}{\beta}\right)\right\} \\
&= \left(\xi+\frac{\gamma}{\delta}\right)(\alpha-\beta\eta-\alpha) \\
&= -\xi\eta\beta-\frac{\gamma\beta}{\delta}\eta\beta \\
&= -\beta\frac{\gamma}{\delta}\eta-\xi\eta\beta
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}\,\eta}{\mathrm{d}\,t} &= \left(\eta+\frac{\alpha}{\beta}\right)\left\{-\gamma+\delta\left(\xi+\frac{\gamma}{\delta}\right)\right\} \\
&= \left(\eta+\frac{\alpha}{\beta}\right)(-\gamma+\delta\xi+\gamma) \\
&= \xi\eta\delta+\frac{\alpha\delta}{\beta}\xi
\end{aligned}
$$

The corresponding linearized system is,

$$
\left.\begin{aligned}
\frac{\mathrm{d}\,\xi}{\mathrm{d}\,t} &= -\frac{\beta\gamma}{\delta}\eta \\
\frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= \frac{\alpha\delta}{\beta}\xi
\end{aligned}\right\} \tag{5.2}
$$

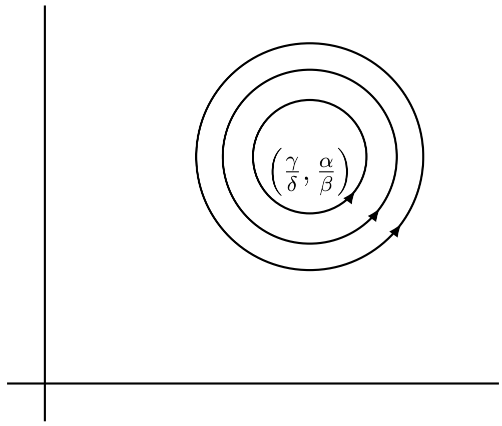
The characteristic equation of (5.2) is,

$$
\lambda^2-(a+d)\lambda+ad-bc=0 \tag{5.3}
$$

where $a=0$, $b=-\frac{\beta\gamma}{\delta}$, $c=\frac{\alpha\delta}{\beta}$, $d=0$.
Now (5.3) becomes

$$
\begin{aligned}
& \lambda^2+\alpha\lambda=0 \\
\Rightarrow\ & \lambda=\pm i\sqrt{\alpha\gamma}
\end{aligned}
$$

Since the roots are purely imaginary, so we say that the equilibrium point $\left(\frac{\gamma}{\delta},\frac{\alpha}{\beta}\right)$ is a stable center.



<u>Exact Solution</u>: From (5.1) we get,

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = \frac{y - (\gamma + \delta x)}{x(\alpha - \beta y)}$$

Separating the variable,

$$\begin{aligned} & \frac{\alpha - \beta y}{y}\, \mathrm{d}\, y = \frac{-\gamma + \delta x}{x}\, \mathrm{d}\, x \\ \Rightarrow\ & \left[\frac{\alpha}{y} - \beta\right] \mathrm{d}\, y = \left[\frac{-\gamma}{x} + \delta\right] \mathrm{d}\, x \end{aligned}$$

Integrating,

$$\begin{aligned} \Rightarrow\ & \alpha \log y - \beta y = -\gamma \log x + \delta x + \log K \qquad K = \text{constant} \\ \Rightarrow\ & \log y^{\alpha} - \log e^{\beta y} = \log \frac{1}{x^{\gamma}} + \log e^{\delta x} + \log K \\ \Rightarrow\ & \log \frac{y^{\alpha}}{e^{\beta y}} = \log \frac{e^{\delta x}}{x^{\gamma}} K \\ \Rightarrow\ & K \frac{e^{\delta x}}{x^{\gamma}} = \frac{y^{\alpha}}{e^{\beta y}} \\ \Rightarrow\ & K = \frac{y^{\alpha}}{e^{\beta y}} \cdot \frac{x^{\gamma}}{e^{\delta x}} \end{aligned} \tag{5.4}$$

Using initial population, $x = x_0$, $y = y_0$ in (5.4) we get,

$$K = \frac{y_0^{\alpha}}{e^{\beta y_0}} \cdot \frac{x_0^{\gamma}}{e^{\delta x_0}}$$

putting it into (5.4)

$$\begin{aligned} & \frac{y^{\alpha}}{e^{\beta y}} \cdot \frac{x^{\gamma}}{e^{\delta x}} = \frac{y_0^{\alpha}}{e^{\beta y_0}} \cdot \frac{x_0^{\gamma}}{e^{\delta x_0}} \\ \Rightarrow\ & f(x) g(y) = K \end{aligned}$$

where,

$$\left.\begin{aligned} f(x) &= x^{\gamma} - e^{-\delta x} \\ g(x) &= y^{\alpha} - e^{-\beta y} \end{aligned}\right\} \tag{5.5}$$

$$K = f(x_0) g(y_0) = \frac{y_0^{\alpha}}{e^{\beta y_0}} \cdot \frac{x_0^{\gamma}}{e^{\delta x_0}}$$

Now when $x = 0$ then $f(0) = 0$
when $x = \infty$ then $f(\infty) = 0$
Now,

$$\begin{aligned}\frac{\partial f}{\partial x} &= f'(x) = x^{\gamma} e^{-\delta x}(-\delta) + e^{-\delta x}\gamma x^{\gamma-1} \\ &= -\delta x^{\gamma} e^{-\delta x} + e^{-\delta x}\gamma x^{\gamma-1} \\ &= e^{-\delta x} x^{\gamma-1}\left[\gamma - \delta x\right]\end{aligned}$$

For critical point,

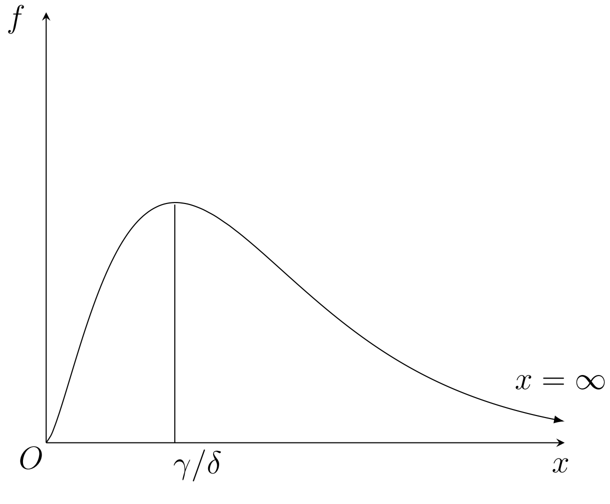
$$\begin{aligned} & f'(x) = 0 \\ \Rightarrow\ & e^{-\delta x} x^{\gamma-1}\left[\gamma - \delta x\right] = 0 \\ \Rightarrow\ & x = \frac{\gamma}{\delta} \qquad \text{where } e^{-\delta x} x^{\gamma-1} \neq 0\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= f''(x) = e^{-\delta x} x^{\gamma-1}(-\delta) + (\gamma - \delta x)\left\{e^{-\delta x}(\gamma-1)x^{\gamma-2} + x^{\gamma-1}e^{-\delta x}(-\delta)\right\} \\ &= -\delta e^{-\delta x} x^{\gamma-1} + \gamma e^{-\delta x}(\gamma-1)x^{\gamma-2} - \gamma\delta x^{\gamma-1}e^{-\delta x} - \delta x e^{-\delta x}(\gamma-1)x^{\gamma-2} + \delta^2 x^{\gamma-2} e^{-\delta x}\end{aligned}$$

when $x = \frac{\gamma}{\delta}$ then,

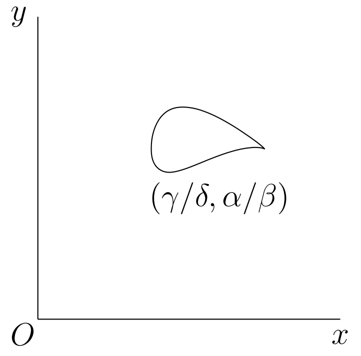
$$\begin{aligned} f''\left(\frac{\gamma}{\delta}\right) &= -\delta e^{-\gamma}\left(\frac{\gamma}{\delta}\right)^{\gamma-1} + \gamma e^{-\gamma}(\gamma-1)\left(\frac{\gamma}{\delta}\right)^{\gamma-2} - \gamma\delta\left(\frac{\gamma}{\delta}\right)^{\gamma-1} e^{-\gamma} - \gamma e^{-\gamma}(\gamma-1)\left(\frac{\gamma}{\delta}\right)^{\gamma-2} \\ &= -\text{ve} \qquad \text{for } \alpha,\ \beta,\ \gamma,\ \delta > 0\end{aligned}$$

$f(x)$ is maximum at $x = \frac{\gamma}{\delta}$.
Thus $f(x)$ looks like below.



Similarly $g(y)$ looks the same.

Illustration: We now try to sketch the trajectory of (5.4) in the population space. To sketch the trajectory of (5.4) assume that we are given the point (x_0, y_0) as the initial population. This fixes the value of K. For any value of y there will be one value of $g(y)$. Since $f(x)g(y) = K$. This determines the value of $f(x)$. It is clear from the figure that for any value of $f(x)$ there are either two values of x or else none. Same argument holds for y also. This implies that the trajectory in the $xy-$space is a closed loop. Hence, the paths of (5.4) will look as follows:



Problem 5.3. Use principle matrix method to solve the following predator-prey model.

$$x' = x + y$$
$$y' = -x + y$$

where $x(0) = y(0) = 1000$. When the species y will be eliminated?

Solution. We have,

$$\left.\begin{aligned} x' &= x + y \\ y' &= -x + y \end{aligned}\right\} \tag{5.6}$$

The system (5.6) can be written as,

$$X'(t) = AX(t)$$

where,

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic equation of A is,

$$\begin{aligned} & |A - \lambda I| = 0 \\ \Rightarrow & \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \\ \Rightarrow & \lambda^2 - 2\lambda + 2 = 0 \\ \Rightarrow & \lambda = 1 \pm i \end{aligned}$$

Case i: When $\lambda = 1 + i$ then the corresponding eigenvector, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$\begin{aligned} & |A - \lambda I|\, X = 0 \\ \Rightarrow & \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\ \Rightarrow & \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad R_2' = R_2(-i) + R_1 \\ \Rightarrow & -ix_1 + x_2 = 0 \end{aligned}$$

Let us choose arbitrary non-trivial solution $x_1 = 1$, $x_2 = i$.
Hence the eigen vector corresponding to $\lambda = 1 + i$ is $\bar{V}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Case ii: When $\lambda = 1 - i$ then the corresponding eigenvector, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that

$$\begin{aligned}
&|A - \lambda I|\, Y = 0 \\
\Rightarrow\ &\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \\
\Rightarrow\ &\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \qquad R_2' = R_2(i) + R_1 \\
\Rightarrow\ &iy_1 + y_2 = 0
\end{aligned}$$

Let us choose arbitrary non-trivial solution $y_1 = 1$, $y_2 = -i$.
Hence the eigen vector corresponding to $\lambda = 1 - i$ is $\bar{V_2} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\begin{aligned}
\therefore\ c &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
\therefore\ c^{-1} &= -\frac{1}{2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
D &= c^{-1}Ac \\
&= \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 1+i & 1-i \\ -1+i & -1-i \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 2+2i & 0 \\ 0 & 2-2i \end{pmatrix} \\
&= \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}
\end{aligned}$$

$$\therefore\ e^{Dt} = \begin{pmatrix} e^{1+i}t & 0 \\ 0 & e^{1-i}t \end{pmatrix}$$

Now,

$$\begin{aligned}
e^{At} &= ce^{Dt}c^{-1} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{(1+i)t} & 0 \\ 0 & e^{(1-i)t} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{(1+i)t} & -ie^{(1+i)t} \\ e^{(1-i)t} & ie^{(1-i)t} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} e^{(1+i)t} + e^{(1-i)t} & -ie^{(1+i)t} + ie^{(1-i)t} \\ ie^{(1+i)t} - ie^{(1-i)t} & e^{(1+i)t} + e^{(1-i)t} \end{pmatrix} \\
&= \frac{1}{2}e^t \begin{pmatrix} \cos t + i\sin t + \cos t - i\sin t & -i\cos t + \sin t + i\cos t + \sin t \\ i\cos t - \sin t - i\cos t - \sin t & \cos t + i\sin t + \cos t - i\sin t \end{pmatrix} \\
&= \frac{1}{2}e^t \begin{pmatrix} 2\cos t & 2\sin t \\ -2\sin t & 2\cos t \end{pmatrix} = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}
\end{aligned}$$

Now,

$$\begin{aligned}
&X(t) = e^{At}c \\
\Rightarrow &\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
\Rightarrow &\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}
\end{aligned}$$

$$\left.\begin{aligned}
\therefore \quad x(t) &= e^t \left(c_1 \cos t + c_2 \sin t\right) \\
\therefore \quad y(t) &= e^t \left(-c_1 \sin t + c_2 \cos t\right)
\end{aligned}\right\} \tag{5.7}$$

Which is the general solution.
Applying initial condition $x(0) = 1000$, $y(0) = 1000$ in (5.7),

$$1000 = 1 \cdot (c_1 \cos 0 + 0) \qquad \Rightarrow \quad c_1 = 1000$$

and

$$1000 = 1 \cdot (0c_2 \cos 0) \qquad \Rightarrow \quad c_2 = 1000$$

So (5.7) becomes

$$\left.\begin{aligned}
\therefore \quad x(t) &= 1000e^t \left(\cos t + \sin t\right) \\
\therefore \quad y(t) &= 1000e^t \left(-\sin t + \cos t\right)
\end{aligned}\right\} \tag{5.8}$$

Now from (5.8), for eliminating y,

$$\begin{aligned}
&1000e^t(\cos t - \sin t) = 0 \\
\Rightarrow \ &\cos t = \sin t \qquad e^t \neq 0 \\
\Rightarrow \ &\tan t = 1 = \tan \pi/4 \\
\Rightarrow \ &t = \pi/4 = \frac{11}{4} \text{ years}
\end{aligned}$$

Problem 5.4 (Compitition Model)**.** Identify the population model and solve

$$\begin{aligned}
\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= 3x - y \\
\frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= -2x + 2y
\end{aligned}$$

where the time t is measured in year with initial population $x(0) = 90$, $y(0) = 150$ and also find

(i) when will the species y be eliminated?

(ii) what will be the population after six months?

Solution. We have,

$$\left.\begin{aligned}
\tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= 3x - y \\
\tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= -2x + 2y
\end{aligned}\right\} \tag{5.9}$$

In the absence of y, x increase and in the presence of y, x decreases. This is true for 1st equation of the system (5.9).

Again, in the absence of x, y increases and in the presence of x, y decreases. This is valid for 2nd equation of the system (5.9).

Hence the system (5.9) is a competition model.
The system (5.9) can be written as,

$$X'(t) = AX(t)$$

where,

$$A = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic equation of A is,

$$\begin{aligned} & |A - \lambda I| = 0 \\ \Rightarrow & \begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = 0 \\ \Rightarrow & \lambda^2 - 5\lambda + 4 = 0 \\ \Rightarrow & \lambda = 1, \ 4 \end{aligned}$$

Case i: When $\lambda = 1$ then the corresponding eigenvector, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$\begin{aligned} & [A - \lambda I]X = 0 \\ \Rightarrow & \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\ \Rightarrow & \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad R_2' = R_2 + R_1 \\ \Rightarrow & \ 2x_1 - x_2 = 0 \end{aligned}$$

Let us choose arbitrary non-trivial solution $x_1 = 1$, $x_2 = 2$.
Hence the eigen vector corresponding to $\lambda = 1$ is $\bar{V_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Case ii: When $\lambda = 4$ then the corresponding eigenvector, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that

$$\begin{aligned} & [A - \lambda I]Y = 0 \\ \Rightarrow & \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \\ \Rightarrow & \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \qquad R_2' = -\frac{1}{2}R_2 + R_1 \\ \Rightarrow & - y_1 - y_2 = 0 \\ \Rightarrow & + y_1 + y_2 = 0 \end{aligned}$$

Let us choose arbitrary non-trivial solution $y_1 = 1$, $y_2 = -1$.
Hence the eigenvector corresponding to $\lambda = 4$ is $\bar{V_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{aligned} & \therefore \ c = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\ & \therefore \ c^{-1} = -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
D &= c^{-1}Ac \\
&= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & -4 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 12 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}
\end{aligned}
$$

$$
\therefore \; e^{Dt} = \begin{pmatrix} e^t & 0 \\ 0 & e^{4t} \end{pmatrix}
$$

Now,

$$
\begin{aligned}
e^{At} &= ce^{Dt}c^{-1} \\
&= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^t & e^t \\ 2e^{4t} & -e^{4t} \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} e^t + 2e^{4t} & e^t - e^{4t} \\ 2e^t - 2e^{4t} & 2e^t + e^{4t} \end{pmatrix}
\end{aligned}
$$

Now,

$$
\begin{aligned}
X(t) &= e^{At}c \\
\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} e^t + 2e^{4t} & e^t - e^{4t} \\ 2e^t - 2e^{4t} & 2e^t + e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
\end{aligned}
$$

$$
\left.
\begin{aligned}
\therefore \; x(t) &= \tfrac{1}{3}\left(c_1(e^t + 2e^{4t}) + c_2(e^t - e^{4t})\right) \\
\therefore \; y(t) &= \tfrac{1}{3}\left(c_1(2e^t - 2e^{4t}) + c_2(2e^t + e^{4t})\right)
\end{aligned}
\right\} \tag{5.10}
$$

Which is the general solution.
Applying initial condition $x(0) = 90$, $y(0) = 150$ in (5.10),

$$
90 = \frac{1}{3} \times 3c_1 \qquad \Rightarrow \quad c_1 = 90
$$

and

$$
150 = \frac{1}{3} \times 3c_2 \qquad \Rightarrow \quad c_2 = 150
$$

So (5.7) becomes

$$
\begin{aligned}
x(t) &= \frac{1}{3}\left\{90(e^t + 2e^{4t}) + 150(e^t - e^{4t})\right\} \\
y(t) &= \frac{1}{3}\left\{90(2e^t - 2e^{4t}) + 150(2e^t + 4e^{4t})\right\}
\end{aligned}
$$

$$
\left.
\begin{aligned}
\therefore \; x(t) &= 80e^t + 10e^{4t} \\
\therefore \; y(t) &= 160e^t - 10e^{4t}
\end{aligned}
\right\} \tag{5.11}
$$

which is the population at any time.

(i) For eliminating y, putting $y = 0$ in 2nd equation of the system (5.11),

$$\begin{aligned} 0 &= 160e^{t} - 10e^{4t} \\ \Rightarrow\ e^{3t} &= 16 \\ \Rightarrow\ t &= .92 \text{ years} \qquad \text{i.e., 11 months} \end{aligned}$$

(ii) When $t = \frac{1}{2}$ years (six month) then from (5.11)

$$\begin{aligned} x(t) &= 80e^{\frac{1}{2}} + 10e^{2} = 206 \\ y(t) &= 160e^{\frac{1}{2}} - 10e^{2} = 190 \end{aligned}$$

Problem 5.5 (Symbiotic Model)**.** Identify the population model and solve

$$\begin{aligned} \frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= -2x + 4y \\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= x - 2y \end{aligned}$$

where the time t is measured in year with initial population $x(0) = 100$, $y(0) = 300$.

Solution. We have,

$$\left.\begin{aligned} \tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} &= -2x + 4y \\ \tfrac{\mathrm{d}\,y}{\mathrm{d}\,t} &= x - 2y \end{aligned}\right\} \tag{5.12}$$

In the absence of y, x decreases and in the presence of y, x increases. This is true for 1st equation of the system (5.12).

Again, in the absence of x, y decreases and in the presence of x, y increases. This is valid for 2nd equation of the system (5.12).

Hence, the system (5.9) described symbiotic model.

The system (5.9) can be written as,

$$X'(t) = AX(t)$$

where,

$$A = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The characteristic equation of A is,

$$\begin{aligned} &|A - \lambda I| = 0 \\ \Rightarrow\ &\begin{vmatrix} -2-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = 0 \\ \Rightarrow\ &\lambda^2 + 4\lambda = 0 \\ \Rightarrow\ &\lambda = 0,\ -4 \end{aligned}$$

Case i: When $\lambda = 0$ then the corresponding eigenvector, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$\begin{aligned} &[A - \lambda I]X = 0 \\ \Rightarrow\ &\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \\ \Rightarrow\ &\begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad R_2' = R_2 + 2R_1 \\ \Rightarrow\ &-x_1 + 2x_2 = 0 \end{aligned}$$

Let us choose arbitrary non-trivial solution $x_1 = 2$, $x_2 = 1$.
Hence, the eigenvector corresponding to $\lambda = 0$ is $\bar{V}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Case ii: When $\lambda = -4$ then the corresponding eigenvector, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that

$$\begin{aligned}
&[A - \lambda I]Y = 0 \\
\Rightarrow\ & \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \\
\Rightarrow\ & \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \qquad R_2' = R_2 - \frac{1}{2} R_1 \\
\Rightarrow\ & y_1 + 2y_2 = 0
\end{aligned}$$

Let us choose arbitrary non-trivial solution $y_1 = 2$, $y_2 = -1$.
Hence, the eigenvector corresponding to $\lambda = -4$ is $\bar{V}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\begin{aligned}
&\therefore\ c = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \\
&\therefore\ c^{-1} = -\frac{1}{4} \begin{pmatrix} -1 & -2 \\ -1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
D &= c^{-1} A c \\
&= \frac{1}{4} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & -8 \\ 0 & 4 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 0 & -16 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}
\end{aligned}$$

$$\therefore\ e^{Dt} = \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-4t} \end{pmatrix}$$

Now,

$$\begin{aligned}
e^{At} &= c e^{Dt} c^{-1} \\
&= \frac{1}{4} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-4t} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ e^{-4t} & -2e^{-4t} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 2 + 2e^{-4t} & 4 - 4e^{-4t} \\ 1 - 2e^{-4t} & 2 + 2e^{-4t} \end{pmatrix}
\end{aligned}$$

Now,

$$\begin{aligned}
& X(t) = e^{At} c \\
\Rightarrow\ & \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 + 2e^{-4t} & 4 - 4e^{-4t} \\ 1 - 2e^{-4t} & 2 + 2e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
\end{aligned}$$

$$\left.\begin{aligned} \therefore\quad x(t) &= \tfrac{1}{4}\left(c_1(2+2e^{-4t}) + c_2(4-4e^{-4t})\right) \\ \therefore\quad y(t) &= \tfrac{1}{4}\left(c_1(1-e^{-4t}) + c_2(2+2e^{-4t})\right) \end{aligned}\right\} \tag{5.13}$$

Which is the general solution.
Applying initial condition $x(0) = 100$, $y(0) = 300$ in (5.13),

$$c_1 = 100, \qquad c_2 = 300$$

So (5.13) becomes

$$\begin{aligned} x(t) &= \frac{1}{4}\left\{100(2+2e^{-4t}) + 300(4-4e^{-4t})\right\} \\ y(t) &= \frac{1}{3}\left\{100(1-e^{-4t}) + 300(2+2e^{-4t})\right\} \end{aligned}$$

$$\begin{aligned} \therefore\; x(t) &= 350 - 250e^{-4t} \\ \therefore\; y(t) &= 175 + 125e^{-4t} \end{aligned}$$

which is the population at any time.

Chapter 6

Equilibrium and Stability

Problem 6.1. Define and discuss all the equilibria and linearized of the model of two interacting species. Also find the community matrix of each possible equilibria.

Solution. We will consider populations of two interacting species with population size $x(t)$ and $y(t)$ respectively.

We will assume that $x(t)$ and $y(t)$ are continuously differentiable function of t whose derivatives are function of two population sizes at the same time.
Thus our model will be systems of 1sr order differential equations

$$\left.\begin{aligned} x' &= F(x, y) \\ y' &= G(x, y) \end{aligned}\right\} \tag{6.1}$$

An equilibrium is a solution (x_∞, y_∞) of the pair of equations, $F(x_\infty, y_\infty) = 0$, $G(x_\infty, y_\infty) = 0$. Thus an equilibrium is a constant solution of the system of differential equation. Geometrically, an equilibrium is point in the phase plane that is the orbit of a constant solution.

If (x_∞, y_∞) is an equilibrium, we make the change of variables, $u = x - x_\infty$, $v = y - y_\infty$ obtaining the system

$$\begin{aligned} u' &= F(x_\infty + u, y_\infty + v) \\ v' &= G(x_\infty + u, y_\infty + v) \end{aligned}$$

Using Taylor's theorem for function of two variables, we may write

$$\begin{aligned} F(x_\infty + u, y_\infty + v) &= F(x_\infty, y_\infty) + F_x(x_\infty, y_\infty)u + F_y(x_\infty, y_\infty)v + h_1 \\ G(x_\infty + u, y_\infty + v) &= G(x_\infty, y_\infty) + G_x(x_\infty, y_\infty)u + G_y(x_\infty, y_\infty)v + h_2 \end{aligned}$$

where h_1 and h_2 are functions that are small for small uv in the sense that

$$\lim_{\substack{u\to 0\\ v\to 0}} \frac{h_1(u,v)}{\sqrt{u^2+v^2}} = \lim_{\substack{u\to 0\\ v\to 0}} \frac{h_2(u,v)}{\sqrt{u^2+v^2}} = 0$$

The linearized of the system obtained by using, $F(x_\infty, y_\infty) = 0$, $G(x_\infty, y_\infty) = 0$ and neglecting the higher order term $h_1(u, v)$ and $h_2(u, v)$ is defined to be the 2-D linear system.

$$\left.\begin{aligned} u' &= F_x(x_\infty, y_\infty)u + F_y(x_\infty, y_\infty)v \\ v' &= G_x(x_\infty, y_\infty)u + G_y(x_\infty, y_\infty)v \end{aligned}\right\} \tag{6.2}$$

The coefficient matrix of the system (6.2)

$$\begin{pmatrix} F_x(x_\infty, y_\infty) & F_y(x_\infty, y_\infty) \\ G_x(x_\infty, y_\infty) & G_y(x_\infty, y_\infty) \end{pmatrix}$$

is called the community matrix of the system at the equilibrium (x_∞, y_∞).

Again, consider the system,

$$\begin{aligned} x' &= xf(x,y) \\ y' &= yg(x,y) \end{aligned}$$

so that $f(x,y)$ and $g(x,y)$ are the per capita growth rates of the two species. The community matrix at the equilibrium then has the form

$$\begin{pmatrix} x_\infty f_x(x_\infty, y_\infty) + f(x_\infty, y_\infty) & x_\infty f_y(x_\infty, y_\infty) \\ y_\infty g_x(x_\infty, y_\infty) & y_\infty g_y(x_\infty, y_\infty) + g(x_\infty, y_\infty) \end{pmatrix}$$

There are four distinct kinds of possible equilibria as follows:

(i) $(0,0)$ the community matrix

$$\begin{pmatrix} f(0,0) & 0 \\ 0 & g(0,0) \end{pmatrix}$$

(ii) $(k,0)$ with $k > 0$, $f(k,0) = 0$ having community matrix

$$\begin{pmatrix} kf_x(0,0) & kf_y(k,0) \\ 0 & g(k,0) \end{pmatrix}$$

(iii) $(0,M)$ with $M > 0$, $g(0,M) = 0$ having community matrix

$$\begin{pmatrix} f(0,M) & 0 \\ Mg_x(0,M) & Mg_y(0,M) \end{pmatrix}$$

(iv) (x_∞, y_∞) with $x_\infty > 0$, $y_\infty > 0$, $f(x_\infty, y_\infty) = 0$, $g(x_\infty, y_\infty) = 0$ having community matrix

$$\begin{pmatrix} x_\infty f_x(x_\infty, y_\infty) & x_\infty f_y(x_\infty, y_\infty) \\ y_\infty g_x(x_\infty, y_\infty) & y_\infty g_y(x_\infty, y_\infty) \end{pmatrix}$$

The term $F_x(x_\infty, y_\infty)$ and $g_y(x_\infty, y_\infty)$ in the community matrix are self-regulating terms which are normally non-positive.

The term $f_y(x_\infty, y_\infty)$ and $g_x(x_\infty, y_\infty)$ are interacting term.

- If both interacting terms are $-$ve the two species are said to be competition.
- If there is one $+$ve and one $-$ve interaction term, the two species are said to be in a predator-prey solution.
- If both interacting terms are $+$ve is called mutualistic.

An equilibrium (x_∞, y_∞) is said to be stable if every solution $(x(t), y(t))$ with $x(0), y(0)$ sufficiently close to the equilibrium remains close to the equilibrium for all $t \geq 0$.

An equilibrium (x_∞, y_∞) is said to be asymptotically stable if it is stable and if solution with $x(0), y(0)$ sufficiently close to the equilibrium tends to the equilibrium as $t \to \infty$.

Theorem 6.0.1. If (x_∞, y_∞) is an equilibrium of the system $x' = F(x,y)$, $y' = G(x,y)$ and if all eigenvalues of the coefficient of the linearized of this equilibrium have $-$ve real part. Specifically if,

$$\begin{aligned} \operatorname{tr} A(x_\infty, y_\infty) &= F_x(x_\infty, y_\infty) + G_y(x_\infty, y_\infty) < 0 \\ \det A(x_\infty, y_\infty) &= F_xG_y - F_yG_x > 0 \end{aligned}$$

then the equilibrium (x_∞, y_∞) is asymptotically stable.

$$\begin{aligned} \Delta &= (a+d)^2 - 4(ad - bc) \\ &= (a-d)^2 + 4(bc) < 0 \end{aligned}$$

1. If $\det A = (ad - bc) < 0$, the origin is a saddle point.
2. If $\det A > 0$ and $\text{tr } A = a + d < 0$, the origin is asymptotically stable, a node if $\Delta \geq 0$ and spiral point if $\Delta < 0$.
3. If $\det A > 0$ and $\text{tr } A > 0$, the origin is unstable, a node if $\Delta \geq 0$ and spiral point if $\Delta < 0$.
4. If $\det A > 0$ and $\text{tr } A = 0$, the origin is center.

Problem 6.2. Determine whether each equilibrium of the system,

$$\begin{aligned} x' &= y \\ y' &= 2(x^2 - 1)y - x \end{aligned}$$

is asymptotically stable or not.

Solution. The equilibrium are the solution of $x' = 0$ and $y' = 0$.
Then $y = 0$ and $2(x^2 - 1)y - x = 0$
Thus the only equilibrium point is $(0, 0)$.
Now let us consider,

$$\begin{aligned} F(x, y) &= y \\ G(x, y) &= 2(x^2 - 1)y - x \end{aligned}$$

then

$$\begin{aligned} F_x &= 0, & F_y &= 1 \\ G_x &= 2 \cdot 2xy - 1 = 4xy - 1, & G_y &= 2(x^2 - 1) \end{aligned}$$

So the community matrix is

$$\begin{pmatrix} 0 & 1 \\ 4xy - 1 & 2(x^2 - 1) \end{pmatrix}$$

For equilibrium point $(0, 0)$ the community matrix is,

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

Here,

$$\begin{aligned} \det A &= 1 > 0 \\ \text{trace } A &= -2 < 0 \end{aligned}$$

Thus, the equilibrium point $(0, 0)$ is asymptotically stable.

Problem 6.3. Determine the qualitative behavior of solutions of the system

$$\begin{aligned} x' &= x\left(1 - \frac{x}{30}\right) - \frac{xy}{x + 10} \\ y' &= y\left(\frac{x}{x + 10} - \frac{1}{3}\right) \end{aligned}$$

Solution. Equilibria are the solution of the pair of equations

$$x\left(1 - \frac{x}{30}\right) - \frac{xy}{x + 10} = 0 \tag{6.3}$$

$$y\left(\frac{x}{x + 10} - \frac{1}{3}\right) = 0 \tag{6.4}$$

there is an equilibrium (0,0).
If $y = 0$, then equation (6.3) gives, $x\left(1 - \frac{x}{30}\right) = 0$

$$\therefore \quad x = 0 \quad \text{or} \quad 1 - \frac{x}{30} = 0 \quad \Rightarrow \quad x = 30$$

$\therefore$ The 2nd equilibrium is $(30, 0)$
If $x \neq 0$, $y \neq 0$, we must solve,

$$1 - \frac{x}{30} = \frac{xy}{x+10} \tag{6.5}$$

and

$$\frac{x}{x+10} = \frac{1}{3} \tag{6.6}$$

From (6.6),

$$\begin{aligned} \frac{x}{x+10} &= \frac{1}{3} \\ \Rightarrow\ 3x &= x + 10 \\ \Rightarrow\ 2x &= 10 \\ \Rightarrow\ x &= 5 \end{aligned}$$

From (6.5),

$$\begin{aligned} 1 - \frac{5}{30} &= \frac{y}{5+10} \\ \Rightarrow\ y &= 12.5 \end{aligned}$$

$\therefore$ The 3rd equilibrium is $(5, 12.5)$.
Let,

$$\begin{aligned} F(x,y) &= x\left(1 - \frac{x}{30}\right) - \frac{xy}{x+10} = \left(x - \frac{x^2}{30}\right) - \frac{xy}{x+10} \\ G(x,y) &= y\left(\frac{x}{x+10} - \frac{1}{3}\right) \end{aligned}$$

Then,

$$\begin{aligned} F_x &= \left(1 - \frac{2x}{30}\right) - \frac{(x+10)y - xy \cdot 1}{(x+10)^2} \\ &= \left(1 - \frac{x}{15}\right) - \frac{10y}{(x+10)^2} \\ F_y &= -\frac{x}{x+10} \\ G_x &= y\left[\frac{x + 10 - x}{(x+10)^2}\right] = \frac{10y}{(x+10)^2} \\ G_y &= \left(\frac{x}{x+10} - \frac{1}{3}\right) \end{aligned}$$

The community matrix,

$$\begin{pmatrix} 1 - \frac{x}{15} - \frac{10y}{(x+10)^2} & \frac{-x}{x+10} \\ \frac{10y}{(x+10)^2} & \frac{x}{x+10} - \frac{1}{3} \end{pmatrix}$$

The community matrix at $(0, 0)$ is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \qquad \det A = -\frac{1}{3} < 0$$

Since, the determinant is negative, hence the equilibrium point is unstable saddle point.

The community matrix at $(30, 0)$ is

$$B = \begin{pmatrix} -1 & -\frac{3}{4} \\ 0 & -\frac{5}{12} \end{pmatrix} \qquad \det B = -\frac{5}{12} < 0$$

Since, the determinant is negative, hence the equilibrium point is unstable saddle point.

The community matrix at $(5, 12.5)$ is

$$C = \begin{pmatrix} \frac{1}{9} & -\frac{1}{3} \\ \frac{5}{9} & 0 \end{pmatrix} \qquad \det C = \frac{5}{27} > 0 \quad \text{trace } C = \frac{1}{9} > 0$$

The equilibrium point $(5, 12.5)$ is unstable.
Every orbit approaches a periodic orbit around the unstable equilibrium $(5, 12.5)$. Thus, the two species co-exist with oscillations.

Problem 6.4. Determine the equilibrium behavior of predator-prey system model by,

$$x' = x\left(1 - \frac{x}{30}\right) - \frac{xy}{x+10}$$
$$y' = y\left(\frac{x}{x+10} - \frac{3}{5}\right)$$

Solution. Equilibria are the solution of the pair of equations

$$x\left(1 - \frac{x}{30}\right) - \frac{xy}{x+10} = 0$$
$$\Rightarrow\ x = 0, \qquad 1 - \frac{x}{30} - \frac{y}{x+10} = 0$$

and

$$y\left(\frac{x}{x+10} - \frac{1}{3}\right) = 0$$
$$\Rightarrow\ \frac{x}{x+10} - \frac{1}{3} = 0, \qquad y = 0$$
$$\Rightarrow\ \frac{5x - 3x - 30}{5(x+10)} = 0$$
$$\Rightarrow\ x = 0$$

Thus the equilibrium points are (0,0), (30,0), (15,12.5)
Here,

$$F(x, y) = x\left(1 - \frac{x}{30} - \frac{y}{x+10}\right) = x - \frac{x^2}{30} - \frac{xy}{x+10}$$
$$G(x, y) = y\left(\frac{x}{x+10} - \frac{3}{5}\right)$$

Then,

$$F_x = 1 - \frac{x}{15} - \frac{10y}{(x+10)^2}$$
$$F_y = -\frac{x}{x+10}$$
$$G_x = \frac{10y}{(x+10)^2}$$
$$G_y = \left(\frac{x}{x+10} - \frac{3}{5}\right)$$

For $(0,0)$:

$$F_x(0,0) = 1, \qquad F_y(0,0) = 0$$
$$G_x(0,0) = 0, \qquad G_y(0,0) = -\frac{3}{5}$$

The community matrix at $(0,0)$ is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{3}{5} \end{pmatrix} \qquad \det A = -\frac{3}{5} < 0$$

Thus $(0,0)$ is unstable saddle point.

For $(30,0)$:

$$F_x(30,0) = -1, \qquad F_y(30,0) = -\frac{3}{4}$$
$$G_x(30,0) = 0, \qquad G_y(30,0) = \frac{3}{20}$$

The community matrix at $(30,0)$ is

$$A = \begin{pmatrix} -1 & -\frac{3}{4} \\ 0 & \frac{3}{20} \end{pmatrix} \qquad \det A = -\frac{3}{20} < 0$$

Thus $(30,0)$ is also unstable saddle point.

For $(15,12.5)$:

$$F_x(15,12.5) = -\frac{1}{5}, \qquad F_y(15,12.5) = -\frac{3}{5}$$
$$G_x(15,12.5) = \frac{1}{5}, \qquad G_y(15,12.5) = 0$$

The community matrix at $(15,12.5)$ is

$$A = \begin{pmatrix} -\frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & 0 \end{pmatrix} \qquad \det A = \frac{3}{25} > 0, \;\; \operatorname{trace} A = -\frac{1}{5} < 0$$

Thus $(15,12.5)$ is asymptotically stability and every orbit approaches the equilibrium.

Problem 6.5. Determine the behavior as $t \to \infty$ of solution in the 1st quadrant of the system.

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x(100 - 4x - 2y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = y(60 - x - y)$$

Solution. Equilibria are the solution of

$$x(100 - 4x - 2y) = 0$$
$$y(60 - x - y) = 0$$

then $x = 0$ and $-4x - 2y = -100 \quad \Rightarrow 4x + 2y = 100$
and $y = 0$ and $-x - y = -60 \quad \Rightarrow x + y = 60$
If $y = 0$ then $4x = 100 \quad \Rightarrow x = 25$

If $x = 0$ then $-y = -60 \quad \Rightarrow y = 60$
then the equilibrium points are $(0,0)$, $(25,0)$, $(0,60)$.
Now,

$$\begin{aligned} 4x + 2y &= 100 \\ x + y &= 60 \end{aligned}$$

Solving this we get $x = -70$, $y = 70$.
Another equilibrium point is $(-10, 70)$
The equilibria at $(0,0)$, $(0,60)$, $(25,0)$ but not at $(-10,70)$
Let

$$\begin{aligned} F(x,y) &= x(100 - 4x - 2y) \\ G(x,y) &= y(60 - x - y) \end{aligned}$$

The community matrix is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 100 - 8x - 2y & -2x \\ -y & 60 + x - 2y \end{pmatrix}$$

For point $(0,0)$: The community matrix is

$$A = \begin{pmatrix} 100 & 0 \\ 0 & 60 \end{pmatrix} \qquad \begin{aligned} \det A &= 6000 > 0 \\ \text{trace } A &= 160 > 0 \end{aligned}$$

Hence, the equilibrium point is unstable.

For point $(25,0)$:

$$\begin{aligned} F_x &= 100 - 8 \cdot 25 = -100, & F_y &= -2 \times 25 = -100 \\ G_x &= 0, & G_y &= 60 - 25 = 35 \end{aligned}$$

The community matrix is

$$A = \begin{pmatrix} -100 & -50 \\ 0 & 35 \end{pmatrix} \qquad \det A = -3500 < 0$$

Hence, the equilibrium point is unstable saddle point.

For point $(0,60)$:

$$\begin{aligned} F_x &= -120, & F_y &= 0 \\ G_x &= -60, & G_y &= -60 \end{aligned}$$

The community matrix is

$$A = \begin{pmatrix} -120 & 0 \\ -60 & -60 \end{pmatrix} \qquad \begin{aligned} \det A &= -1200 > 0 \\ \text{trace } A &= -80 < 0 \end{aligned}$$

So the equilibrium is asymptotically stable. Since $\Delta > 0$, so $(0,60)$ is a node.

In order to show every orbit approaches $(0,60)$, we must show that there is no periodic orbits.

$$\beta(x,y) = \frac{1}{xy}, \quad \frac{\partial}{\partial x}\left(\frac{100 - 4x - 2y}{y}\right) + \frac{\partial}{\partial y}\left(\frac{60 - x - y}{x}\right) = \frac{-4}{y} - \frac{1}{x} < 0$$

Problem 6.6. Determine the qualitative behavior of solution (outcome or nature) of the system.

$$\begin{aligned} x' &= x(100 - 4x - y) \\ y' &= y(60 - x - 2y) \end{aligned}$$

Problem 6.7. Find the equilibrium population of

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x(10 - y); \quad \frac{\mathrm{d}\,y}{\mathrm{d}\,t} = y(10 - x)$$

Discuss the stability of the model and sketch some trajectories.

Solution. The equilibrium are the solution of

$$\begin{aligned} x(10 - y) = 0 \qquad &\Rightarrow x = 0, \quad 10 - y = 0 \\ &\quad\; x = 0, \quad y = 10 \\ y(10 - x) = 0 \qquad &\Rightarrow y = 0, \quad 10 - x = 0 \\ &\quad\; y = 0, \quad x = 10 \end{aligned}$$

The equilibrium points are $(0, 0)$, $(10, 10)$.
Here,

$$\begin{aligned} F(x, y) &= x(10 - y), & G(x, y) &= y(10 - x) \\ F_x = 10 - y,\ F_y &= -x & G_x &= -y,\ G_y = 10 - x \end{aligned}$$

The community matrix at the equilibrium (x_∞, y_∞) is,

$$A = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 10 - y_\infty & -x_infty \\ -y_infty & 10 - x_infty \end{bmatrix}$$

The community matrix at $(0, 0)$ is,

$$A_1 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

and the characteristic equation is,

$$\begin{aligned} &\lambda^2 - (10 + 10)\lambda + 100 = 0 \\ \Rightarrow\ &\lambda^2 - 20\lambda + 100 = 0 \\ \Rightarrow\ &(\lambda - 10)^2 = 0 \\ \Rightarrow\ &\lambda = 10, 10 \end{aligned}$$

Since, the roots are real, equal and has positive sign.
Hence, $(0, 0)$ is an asymptotically unstable node.

The community matrix at $(10, 10)$ is,

$$A_2 = \begin{bmatrix} 0 & -10 \\ -10 & 0 \end{bmatrix}, \qquad |A_2| = -100 < 0$$

and the characteristic equation is,

$$\begin{aligned} &\lambda^2 - (0)\lambda + (0 - 100) = 0 \\ \Rightarrow\ &\lambda^2 - 100 = 0 \\ \Rightarrow\ &\lambda = 10, -10 \end{aligned}$$

Since, the roots are real, unequal and has opposite sign.

Hence, $(10, 10)$ is an unstable saddle point.

2nd part

Hence,

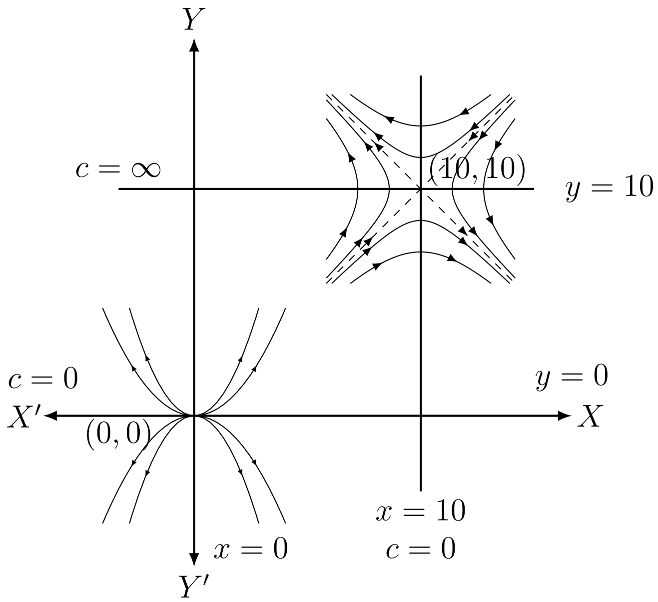
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{y(10-x)}{x(10-y)} \tag{6.7}$$

The isoclines of (6.7) are,

$$\frac{y(10-x)}{x(10-y)} = c$$

If $c = 0$, then $y(10-x) = 0 \Rightarrow\ y = 0,\ x = 10,\ \theta = \tan^{-1} 0 = 0$

If $c = \infty$, then $x(10-y) = 0 \Rightarrow\ x = 0,\ y = 10,\ \theta = \tan^{-1}\infty = 90°$



6.1 Chemostat

Problem 6.8. Define and discuss the chemostat.

Solution. A chemostat is a piece of laboratory apparatus used to cultivate bacteria. It consists of a reservoir containing a nutrient, a culture vessel in which the bacteria are cultivated and an output receptacle. Nutrient is pumped from the reservoir to the culture vessel at a constant rate and the bacteria are collected in the receptacle by pumping the contents of the culture vessel at same rate. This process is called a continuous culture of bacteria, in contrast with a batch in which a fixed quantity of nutrient is supplied and bacteria are harvested after a growth period.

Let y represent the number of bacteria and c the concentration of nutrient in the chemostat, both are functions of t.

Let v be the volume of the chemostat and Q be the rate of flow into the chemostat from the nutrient to reservoir and also the rate of flow out from the chemostat. The fixed concentration of nutrient of the reservoir is a constant $C^{(0)}$.

We assume that the average per capita bacterial birth rate is a function $r(c)$ of the nutrient concentration and that the rate of nutrient consumption of an individual bacterium is proportional to $r(c)$ say $\alpha r(c)$.

Then the rate of change of population size is the birth rate $r(c)y$ of bacteria minus the outflow rate $\frac{Qy}{v}$. It is convenient to let $q = \frac{Q}{v}$, so that this outflow rate becomes qy.

The rate of change of nutrient volume is the replenishment rate $QC^{(0)}$ minus the outflow rate Q minus the consumption rate $\alpha r(c)y$.

This gives the pair of differential equations

$$\left.\begin{aligned} \frac{\mathrm{d}\, y}{\mathrm{d}\, t} &= r(c)y - qy \\ \frac{\mathrm{d}\, cv}{\mathrm{d}\, t} &= Q(c^{(0)} - c) - \alpha r(c)y \end{aligned}\right\} \tag{6.8}$$

we divide the 2nd equation by the constant v and let $\beta = \frac{\alpha}{v}$ to give the system

$$\left.\begin{aligned} y' &= r(c)y - qy \\ c' &= q(c^{(0)} - c) - \beta r(c)y \end{aligned}\right\} \tag{6.9}$$

The system (6.9) describes the chemostat. Assume that the function $r(c)$ is zero if $c = 0$ and approaches a limit when c become large.

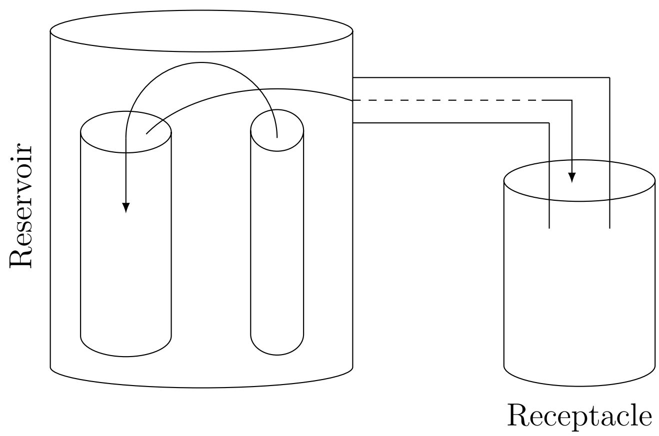
The simplest function with these properties is,

$$r(c) = \frac{ac}{c + A} \tag{6.10}$$

where a and A are constants and this was the choice originally made by Monod.
The explicit chemostat model is now,

$$\left.\begin{aligned} y' &= \frac{acy}{c + A} - qy \\ c' &= q(c^{(0)} - c) - \frac{\beta acy}{c + A} \end{aligned}\right\} \tag{6.11}$$

where a, A, q and β are constants.



Chapter 7

Epidemic Models and Dynamics of Infectious Diseases

7.1 Epidemiology

Epidemiology is the study of the distribution and determinants of disease frequency in human population (or in the group of population). It is the cornerstone of public health and informs policy decisions and evidence-based medicine by identifying risk factors for disease and targets for preventive medicine.

7.2 Endemic

Endemic is the habitual presence of a disease within a given geographic area.

7.3 Epidemic

The epidemic is the occurrence in a community or region of a group of illnesses of similar nature in excess of normal expectancy and distributed from a common or propagated source.

7.4 Pandemic

The pandemic is a worldwide epidemic.

There are various types of epidemiological models. We can classify them into two classes:

(i) disease with removal,

(ii) disease without removal.

7.5 Epidemic models with removal

The model considers the diseases which have the property that individuals once infected by these diseases will be removed from the disease through recovery or death. The individuals removed through recovery are immune temporarily or permanently.

In this case we have three classes of individuals.

(i) The susceptible class (S)

(ii) The infective class (I)

(iii) The removal class (R)

7.6 Susceptible class

The susceptible class consists of those individuals who are not infective but who are capable of catching the disease.

7.7 Infective class

The infective class consists of those individuals who are capable of transmitting the disease to others.

7.8 Removal class

The removal class consists of those individuals who had the disease are dead or recovered or permanently immune or isolated until recovery.

Problem 7.1. Assume that $t_0 < t_1 < t_2$ are equally spaced time values. Let the corresponding population size are P_0, P_1, P_2 respectively. Then the growth rate a and the carrying capacity k of logistic population are

$$a = \frac{1}{t_0 - t_1} \ln \left[\frac{\frac{1}{P_2} - \frac{1}{P_1}}{\frac{1}{P_1} - \frac{1}{P_0}} \right]$$

$$K = \frac{\frac{2}{P_1} - \frac{1}{P_0} - \frac{1}{P_2}}{\frac{1}{P_1^2} - \frac{1}{P_0 P_2}}$$

Solution. We have

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right) e^{-a(t-t_0)}} \tag{7.1}$$

$$\begin{aligned} \Rightarrow \frac{1}{P(t)} &= \frac{1}{K} \left[1 + \left(\frac{K}{P_0} - 1 \right) e^{-a(t-t_0)} \right] \\ &= \frac{1}{K} \left[1 - e^{-a(t-t_0)} \right] + \frac{1}{P_0} e^{-a(t-t_0)} \end{aligned}$$

$$\therefore \frac{1}{P_1} = \frac{1}{K} \left[1 - e^{-a(t_1-t_0)} \right] + \frac{1}{P_0} e^{-a(t_1-t_0)} \tag{7.2}$$

and

$$\therefore \frac{1}{P_2} = \frac{1}{K} \left[1 - e^{-a(t_2-t_0)} \right] + \frac{1}{P_0} e^{-a(t_2-t_0)} \tag{7.3}$$

Now (7.3)-(7.2), we have

$$\begin{aligned} \frac{1}{P_2} - \frac{1}{P_1} &= \frac{1}{K} \left[1 - e^{-a(t_2-t_0)} - 1 + e^{-a(t_1-t_0)} \right] + \frac{1}{P_0} e^{-a(t_2-t_0)} - \frac{1}{P_0} e^{-a(t_1-t_0)} \\ &= \left(\frac{1}{P_1} - \frac{1}{P_0} \right) e^{-a(t_1-t_0)} \quad [\text{Since } t_0 < t_1 < t_1 \text{ are equally spaced. So } t_1 = t_2,\ t_0 = t_1] \end{aligned}$$

$$e^{-a(t_1-t_0)} = \frac{\frac{1}{P_2} - \frac{1}{P_1}}{\left(\frac{1}{P_1} - \frac{1}{P_0} \right)} \tag{7.4}$$

$$\begin{aligned} \Rightarrow a(t_0 - t_1) &= \ln \left[\frac{\frac{1}{P_2} - \frac{1}{P_1}}{\left(\frac{1}{P_1} - \frac{1}{P_0} \right)} \right] \\ \therefore a &= \frac{1}{t_0 - t_1} \ln \left[\frac{\frac{1}{P_2} - \frac{1}{P_1}}{\left(\frac{1}{P_1} - \frac{1}{P_0} \right)} \right] \end{aligned}$$

From (7.2) we have,

$$\begin{aligned}
\frac{1}{P_1} - \frac{1}{P_0}e^{-a(t_1-t_0)} &= \frac{1}{K}\left[1 - e^{-a(t_1-t_0)}\right] \\
\Rightarrow K &= \frac{1 - e^{-a(t_1-t_0)}}{\frac{1}{P_1} - \frac{1}{P_0}e^{-a(t_1-t_0)}} \\
\Rightarrow K &= \frac{1 - \dfrac{\frac{1}{P_2} - \frac{1}{P_1}}{\frac{1}{P_1} - \frac{1}{P_0}}}{\dfrac{1}{P_1} - \dfrac{1}{P_0}\left(\dfrac{\frac{1}{P_2} - \frac{1}{P_1}}{\frac{1}{P_1} - \frac{1}{P_0}}\right)} \qquad \text{[Using (7.4)]} \\
\Rightarrow K &= \frac{\dfrac{\frac{1}{P_1} - \frac{1}{P_0} - \frac{1}{P_2} + \frac{1}{P_1}}{\frac{1}{P_1} - \frac{1}{P_0}}}{\dfrac{\frac{1}{P_1^2} - \frac{1}{P_0P_1} - \frac{1}{P_0P_2} + \frac{1}{P_0P_1}}{\frac{1}{P_1} - \frac{1}{P_0}}} \\
\therefore K &= \frac{\dfrac{2}{P_2} - \dfrac{1}{P_0} + \dfrac{1}{P_2}}{\dfrac{1}{P_1^2} - \dfrac{1}{P_0P_2}}
\end{aligned}$$

Problem 7.2. What do you mean by disease with removal and without removal?

Solution. The epidemic logistic models are various in type such as

(i) Disease without removal

(ii) Disease with removal

Disease without removal: In this case, it is assumed that persons are infected can never be removed from the disease. Therefore, the total population always remains either in S class or in I class.

Example: SI and SIS models etc. are disease without removal model.

Disease with removal: In this model, we consider those diseases which are of the nature, individuals once infected can be removed from the disease through recovery or death. This removal may be temporary or permanent.

Example: Several types of this model are SIR model, $SIRS$, $SEIR$ etc.

Problem 7.3. Discuss the SI model with limiting behavior.

Solution. Without removals, we have

$$S(t) + I(t) = N(t) = \text{ Constant} \tag{7.5}$$

where, $S(t) =$ The number of susceptible

$I(t) =$ The number of infective person in the population

$N(t) =$ The total population size

Let n be the initial number of susceptible in the population in which one infected person has been introduced, so that,

$$\left.\begin{aligned}
&S(t) + I(t) = n + 1 \\
&S(0) = S_0 = n \\
&I(0) = I_0 = 1
\end{aligned}\right\} \tag{7.6}$$

Due to infection, the number of susceptibles decreases and the number of infected persons increases. The epidemic model is,

$$\frac{\mathrm{d}\,S}{\mathrm{d}\,t} = -\beta SI \tag{7.7}$$

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = \beta SI \tag{7.8}$$

From (7.8),

$$\begin{aligned}
&\frac{\mathrm{d}\,S}{\mathrm{d}\,t} = -\beta SI = -\beta S(n+1-S) \quad \text{by (7.6)}\\
\Rightarrow\ &\frac{-1\,\mathrm{d}\,S}{S(n+1-S)} = \beta\,\mathrm{d}\,t\\
\Rightarrow\ &-\frac{1}{n+1}\left(\frac{1}{S}+\frac{1}{n+1-S}\right) = \beta\,\mathrm{d}\,t\\
\Rightarrow\ &-\int\frac{1}{S}\,\mathrm{d}\,S - \int\frac{1}{n+1-S}\,\mathrm{d}\,S = \int(n+1)\beta\,\mathrm{d}\,t\\
\Rightarrow\ &-\ln S + \ln(n+1-S) = (n+1)\beta t + \ln c \quad \text{where } c \text{ is a constant}
\end{aligned} \tag{7.9}$$

By using the initial conditions (7.6), we have

$$\begin{aligned}
&-\ln S_0 + \ln(n+1-S_0) = (n+1)\beta\cdot 0 + \ln c\\
\Rightarrow\ &-\ln n + \ln(n+1-n) = \ln c\\
\Rightarrow\ &-\ln n = \ln c
\end{aligned}$$

Putting the value of *lnc* in (7.9) we get,

$$\begin{aligned}
\Rightarrow\ &-\ln S + \ln(n+1-S) = (n+1)\beta t - \ln n\\
\Rightarrow\ &\ln\frac{n(n+1-S)}{S} = (n+1)\beta t\\
\Rightarrow\ &\frac{n(n+1-S)}{S} = e^{(n+1)\beta t}\\
\Rightarrow\ &n(n+1) - nS = Se^{(n+1)\beta t}\\
\Rightarrow\ &n(n+1) = nS + Se^{(n+1)\beta t}\\
\Rightarrow\ &S = S(t) = \frac{n(n+1)}{n+e^{(n+1)\beta t}}
\end{aligned} \tag{7.10}$$

From (7.8), we have

$$\begin{aligned}
&\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = \beta SI = \beta I(n+1-I) \quad [\text{by (7.6)}]\\
\Rightarrow\ &\frac{\mathrm{d}\,I}{I(n+1-I)} = \beta\,\mathrm{d}\,t\\
\Rightarrow\ &\int\frac{1}{n+1}\left(\frac{1}{I}+\frac{1}{n+1-I}\right)\mathrm{d}\,I = \int\beta\,\mathrm{d}\,t\\
\Rightarrow\ &\int\left(\frac{1}{I}+\frac{1}{n+1-I}\right)\mathrm{d}\,I = \int(n+1)\beta\,\mathrm{d}\,t\\
\Rightarrow\ &\ln I - \ln(n+1-I) = (n+1)\beta t + \ln A \quad \text{where } \ln A \text{ is a constant}
\end{aligned} \tag{7.11}$$

Initially, $t = 0$, $I_0 = 1$, so we get,

$$\begin{aligned}
&\ln I_0 - \ln(n+1-I_0) = (n+1)\beta\cdot 0 + \ln A\\
\Rightarrow\ &\ln 1 - \ln(n+1-1) = 0 + \ln A\\
\Rightarrow\ &-\ln n = \ln A\\
\Rightarrow\ &\ln A = -\ln n
\end{aligned}$$

Putting this value in (7.11), we get.

$$
\begin{aligned}
& \ln I - \ln(n+1-I) = (n+1)\beta t - \ln n \\
\Rightarrow\ & \ln \frac{nI}{n+1-I} = (n+1)\beta t \\
\Rightarrow\ & \frac{nI}{n+1-I} = e^{(n+1)\beta t} \\
\Rightarrow\ & nI = (n+1)e^{(n+1)\beta t} - Ie^{(n+1)\beta t} \\
\Rightarrow\ & \left[n + e^{(n+1)\beta t}\right] I = (n+1)e^{(n+1)\beta t} \\
\Rightarrow\ & I = \frac{(n+1)e^{(n+1)\beta t}}{n + e^{(n+1)\beta t}} \qquad (7.12)
\end{aligned}
$$

From (7.10) and (7.12), we have,

$$\lim_{t\to\infty} S(t) = \frac{n(n+1)}{n+e^{\infty}} = \frac{n(n+1)}{\infty} = 0$$

and

$$
\begin{aligned}
\lim_{t\to\infty} I(t) &= \lim_{t\to\infty} \frac{(n+1)e^{(n+1)\beta t}}{n+e^{(n+1)\beta t}} \\
&= \lim_{t\to\infty} \frac{(n+1)}{ne^{-(n+1)\beta t}+1} \\
&= \frac{(n+1)}{ne^{-\infty}+1} \\
&= n+1
\end{aligned}
$$

Thus, ultimately all persons will be infected.

Problem 7.4. Consider the epidemic model

$$
\begin{aligned}
S' &= -\alpha SI \\
I' &= \alpha SI - \gamma I \\
R' &= \gamma I
\end{aligned}
$$

Interpret the state variables $S(t)$, $I(t)$, $R(t)$ and the model parameters.
Find the co-ordinate on which the infection (disease) will ultimately die out.

Solution. Let,

$S(t) =$ The number of susceptibles who can catch the disease.
$I(t) =$ The number of infected persons in the population.
$R(t) =$ The number of those removed from the population by recovery, death or by any other means.
$N(t) =$ The total number of population size.

The progress of individuals is schematically represented by $S \to I \to R$. Such models are often called SIR models.
Here we assume that

(i) The gain in the infective class is at a rate proportional to the number of infectives and susceptibles, that is αSI, where $\alpha > 0$ is a constant parameter.

(ii) The rate of removed of infectives to the removal class is proportional to the number of infectives, that is γI where $\gamma > 0$ is a constant.

(iii) The incuration period is short enough to be negligible.

The model based on the above assumption is,

$$\frac{\mathrm{d}\,S}{\mathrm{d}\,t} = -\alpha SI \tag{7.13}$$

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = \alpha SI - \gamma I \tag{7.14}$$

$$\frac{\mathrm{d}\,R}{\mathrm{d}\,t} = \gamma I \tag{7.15}$$

where, $\alpha > 0$ is the infection rate and $\gamma > 0$ is the removal rate of infectives.
The above model has initial conditions

$$S(0) = S_0 > 0, \qquad I(0) = I_0 > 0, \qquad R(0) = 0 \tag{7.16}$$

From (7.14), we write,

$$\left[\frac{\mathrm{d}\,I}{\mathrm{d}\,t}\right]_{t=0} = I_0(\alpha S_0 - \gamma) \begin{cases} > 0 & \text{if } S_0 > \frac{\gamma}{\alpha} \\ < 0 & \text{if } S_0 < \frac{\gamma}{\alpha} \end{cases}$$

where $\frac{\gamma}{\alpha}$ is relative removal rate.
Since from (7.13) we have, $\frac{\mathrm{d}\,S}{\mathrm{d}\,t} \leq 0, \quad S \leq S_0$.
If $S_0 < \frac{\gamma}{\alpha}$, then

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = I(\alpha S - \gamma) \leq 0 \tag{7.17}$$

for all $t > 0$ in which case $I_0 > I \to 0$ as $t \to \infty$ and so the infection dies out that is no epidemic can occur.

On the other hand, if $S_0 > \frac{\gamma}{\alpha}$ then $I(t)$ initially increases and we have an epidemic. The term epidemic means that, $I(t) > I_0$ for some $t > 0$.
Again from (7.13) and (7.14), we have

$$\begin{aligned}
&\frac{\mathrm{d}\,I}{\mathrm{d}\,S} = \frac{\alpha SI - \gamma I}{-\alpha SI} = -1 + \frac{\gamma}{\alpha S} \\
\Rightarrow\ &\int \mathrm{d}\,I = -\int \mathrm{d}\,S + \rho \int \frac{1}{S}\,\mathrm{d}\,S, \quad \text{where } \rho = \frac{\gamma}{\alpha} \\
\Rightarrow\ &\int \mathrm{d}\,I = -\int \mathrm{d}\,S + \rho \int \frac{1}{S}\,\mathrm{d}\,S, \quad \text{where } \rho = \frac{\gamma}{\alpha} \\
\Rightarrow\ &I = -S + \rho \ln S + \text{ constant} \\
\Rightarrow\ &I + S - \rho \ln S = \text{ constant } = I_0 + S_0 - \rho \ln S_0
\end{aligned} \tag{7.18}$$

Here $R(0) = 0$, so $0 \leq S + I < N$
From (7.17), I will be maximized if

$$\begin{aligned}
&\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = 0 \\
\Rightarrow\ &I(\alpha S - \gamma) = 0 \\
\Rightarrow\ &S = \frac{\gamma}{\alpha} = \rho \qquad \text{since, } I \neq 0
\end{aligned}$$

Putting, $S = \rho$ in (7.18) we get,

$$\begin{aligned}
&I_{\max} + \rho - \rho \ln \rho = I_0 + S_0 - \rho \ln S_0 \\
\Rightarrow\ &I_{\max} = \rho \ln \rho - \rho + I_0 + S_0 - \rho \ln S_0 \\
\Rightarrow\ &I_{\max} = (I_0 - S_0) - \rho + \rho \ln\left(\frac{\rho}{S_0}\right) \qquad (7.19) \\
\Rightarrow\ &I_{\max} = N - \rho + \rho \ln\left(\frac{\rho}{S_0}\right), \qquad N = I_0 + S_0
\end{aligned}$$

If $I_0 > 0$, and $S_0 > \rho$, then the phase trajectory start with $S > \rho$, also in this case I increases from I_0 and hence an epidemic ensure.

If $S_0 < \rho$ then I decreases from I_0 and as such no epidemic occurs.

Problem 7.5. Describe the SIR model for an epidemic. Discuss the asymptotic behavior of $S(t)$, $I(t)$, $R(t)$.
or,
Describe the deterministic epidemic model with removal. Find the condition on which the infection die out or spread throughout the population.
or,
Discuss the Kermack-Mckendric epidemic model. Analyze the asymptotic behavior of the solution of the model.

Solution. The SIR model is given by,

$$\frac{\mathrm{d}\,S}{\mathrm{d}\,t} = S' = -\alpha SI \tag{7.20}$$

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = I' = \alpha SI - \gamma I \tag{7.21}$$

$$\frac{\mathrm{d}\,R}{\mathrm{d}\,t} = R' = \gamma I \tag{7.22}$$

Here,

$S(t) =$ The number of susceptibles who can catch the disease.
$I(t) =$ The number of infected persons in the population.
$R(t) =$ The number of those removed from the population by recovery, death or by any other means.
$\alpha =$ The infection rate which is positive.
$\gamma =$ The removal rate of infective which is positive.
$\rho = \dfrac{\gamma}{\alpha} =$ It is a pure number which is the ratio of removal rate of infectives with infection rate.

The above model has initial conditions

$$S(0) = S_0 > 0, \qquad I(0) = I_0 > 0, \qquad R(0) = 0 \tag{7.23}$$

From (7.14), we write,

$$\left[\frac{\mathrm{d}\,I}{\mathrm{d}\,t}\right]_{t=0} = I_0(\alpha S_0 - \gamma) \begin{cases} > 0 & \text{if } S_0 > \rho = \frac{\gamma}{\alpha} \\ < 0 & \text{if } S_0 < \rho = \frac{\gamma}{\alpha} \end{cases}$$

Since from (7.20) we have

$$\frac{\mathrm{d}\,s}{\mathrm{d}\,t} \leq 0, \quad S \leq S_0$$

If $S_0 < \frac{\gamma}{\alpha}$, then $\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = I(\alpha S - \gamma) \leq 0$ for all $t > 0$ in which case $I_0 > I(t) \to 0$ as $t \to \infty$. So the infection (disease) ultimately die out.

From (7.20), $S(t)$ is monotonic decreasing function of t. So that $S(t) \leq S_0$.

This shows that $S(t)$ is bounded below $(S(t) \leq 0)$, we find that $\lim_{t\to\infty} S(t) = S(\infty)$ exist.

From (7.22), we have $R(t)$ is a monotonic increasing function of t and is bounded above $R(t) \leq N$, we see that $\lim_{t\to\infty} R(t) = R(\infty)$ exist.

Again, since $S(t) + I(t) + R(t) = N$ for all t.
We find that $\lim_{t\to\infty} I(t) = I(\infty)$ also exist.

Problem 7.6. Describe SIS model.

Solution. The SIS model is given by

$$\frac{\mathrm{d}\,S}{\mathrm{d}\,t} = -\alpha SI - \beta I \tag{7.24}$$
$$\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = \alpha SI - \beta I \tag{7.25}$$

where, $S(t) =$ the number of susceptible individuals at time t
$I(t) =$ the number of infected individuals at time t
where α and β are the constant and

$$S + I = N \tag{7.26}$$

where N is the size of the population.

In this model we assume that a susceptible person becomes infected at a rate proportional to SI and then an infected person recovers and again becomes susceptible at rate proportional to I_0.
The above model has the initial condition, $S(0) = S_0$, $I(0) = I_0$ at $t = 0$.
We have, $S(t) + I(t) = S_0 + I_0 = \text{constant} = N$.
From (7.25),

$$\begin{aligned}
&\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = \alpha(N - I)I - \beta I \\
\Rightarrow\ &\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = \alpha NI - \alpha I^2 - \beta I \\
\Rightarrow\ &\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = I(\alpha N - \beta) - \alpha I^2 \\
\Rightarrow\ &\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = KI - \alpha I^2 \qquad \text{where, } K = \alpha N - \beta \\
\Rightarrow\ &\frac{\mathrm{d}\,I}{\mathrm{d}\,t} = KI\left(1 - \frac{\alpha}{K}I\right) \\
\Rightarrow\ &\frac{\mathrm{d}\,I}{I\left(1 - \frac{\alpha}{K}I\right)} = K\,\mathrm{d}\,t \\
\Rightarrow\ &\left[\frac{1}{I} + \frac{1}{\frac{K}{\alpha} - I}\right]\mathrm{d}\,I = K\,\mathrm{d}\,t \\
\Rightarrow\ &\ln I + \ln\left[\frac{K}{\alpha} - I\right] = Kt + \ln A \\
\Rightarrow\ &\ln\frac{I}{A\left[\frac{K}{\alpha} - I\right]} = Kt \\
\Rightarrow\ &\frac{I}{A\left[\frac{K}{\alpha} - I\right]} = e^{Kt} \\
\Rightarrow\ &A = \frac{I}{\left[\frac{K}{\alpha} - I\right]e^{Kt}} && (7.27) \\
\Rightarrow\ &A\frac{K}{\alpha}e^{Kt} - AIe^{kt} = I \\
\Rightarrow\ &I = \frac{A\frac{K}{\alpha}e^{Kt}}{1 + Ae^{Kt}} && (7.28)
\end{aligned}$$

Initially, $t = 0$, $I = I_0$, $A = \frac{I_0}{\frac{k}{\alpha} - I_0}$

From (7.28),

$$
\begin{aligned}
I &= \frac{I_0\left[\frac{K}{\alpha}\right]e^{Kt}}{\left(\frac{k}{\alpha}-I_0\right)\left[1+\frac{I_0}{\frac{k}{\alpha}-I_0}\right]e^{Kt}} \\
&= \frac{I_0\frac{K}{\alpha}e^{Kt}}{\frac{k}{\alpha}-I_0+I_0e^{Kt}} \\
&= \frac{\frac{K}{\alpha}I_0e^{kt}}{I_0\frac{k}{\alpha}\left[\frac{1}{I_0}+\left(e^{kt}-1\right)\frac{\alpha}{k}\right]} \\
&= \frac{e^{kt}}{\frac{\alpha}{k}\left(e^{kt}-1\right)+\frac{1}{I_0}} \qquad \text{where } k \neq 0
\end{aligned}
$$

When $k = 0$,

$$
\begin{aligned}
\frac{\mathrm{d}\,I}{\mathrm{d}\,t} &= -\alpha I^2 \\
\Rightarrow -\frac{\mathrm{d}\,I}{I^2} &= \alpha\,\mathrm{d}\,t \\
\Rightarrow \frac{1}{I} &= \alpha + B
\end{aligned}
$$

Initially, $t = 0$, $I = I_0$

$$
\begin{aligned}
\therefore B &= \frac{1}{I_0} \\
\therefore \frac{1}{I} &= \alpha + \frac{1}{I_0} \\
\Rightarrow I &= \frac{1}{\alpha t + \frac{1}{I_0}}
\end{aligned}
$$

$$
I(t) = \begin{cases} \frac{e^{kt}}{\frac{\alpha}{k}\left(e^{kt}-1\right)+\frac{1}{I_0}}, & k \neq 0 \\ \frac{1}{\alpha t+\frac{1}{I_0}}, & k = 0 \end{cases}
$$

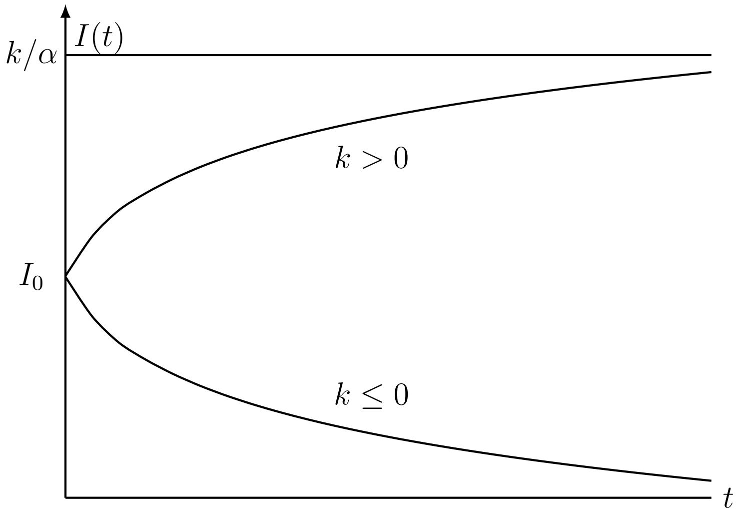
Since, $S(t) + I(t) = N$, i.e., $S(t) = N - I(t)$
We get,

$$
S(t) = \begin{cases} N - \frac{e^{kt}}{\frac{\alpha}{k}\left(e^{kt}-1\right)+\frac{1}{I_0}}, & k \neq 0 \\ N - \frac{1}{\alpha t+\frac{1}{I_0}}, & k = 0 \end{cases}
$$

We have, $k = \alpha N - \beta \quad \Rightarrow \frac{K}{\alpha} = N - \frac{\beta}{\alpha} = N - \rho$ where $\rho = \frac{\beta}{\alpha}$ is known relative removal rate.
Now, as $t \to \infty$,

$$
\begin{aligned}
I(t) &\to \frac{k}{\alpha} = N - \rho \quad \text{if } k > 0, \text{i.e., } N > \rho \\
\text{and } I(t) &\to 0 \quad \text{if } k \leq 0, \text{i.e., } N \leq \rho
\end{aligned}
$$

These results are shown in the diagram.


$I(t)$
k/α
$k > 0$
I_0
$k \leq 0$
t