

Chapter 1

Separation Axioms

Definition 1 (Quasi T_0 –space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a quasi T_0 –space, if for every two distinct fuzzy points x_a and x_b with same support point x , there exists $U \in Q_\delta(x_a)$ such that $x_b \not\propto U$ or, there exists $V \in Q_\delta(x_b)$ such that $x_a \not\propto V$.

Definition 2 (Sub T_0 –space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a sub T_0 –space, if for every two distinct $x, y \in X$, there exists $a \in [0, 1]$ such that either $\exists U \in Q_\delta(x_a)$ with $y_a \not\propto U$ or, $\exists V \in Q_\delta(y_a)$ with $x_a \not\propto V$.

Definition 3 (T_0 –space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_0 –space, if for every two distinct fuzzy points x_a and y_b , $\exists U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, $\exists V \in Q_\delta(y_b)$ with $x_a \not\propto V$.

Definition 4 (T_1 –space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_1 –space, if for every two distinct fuzzy points x_a and y_b such that $x_a \not\leq y_b$ then there exists $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ and, $\exists V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.

Definition 5 (T_2 –space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_2 –space, if for every two distinct fuzzy points x_a and y_b (i.e., $x_a \neq y_b$) then there exists $U \in Q_\delta(x_a)$ and, $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

Theorem 1.0.1. Quasi T_0 property is hereditary.

or, Every subspace of a Quasi T_0 space is Quasi T_0 space.

Proof. Suppose, $\langle X, \delta \rangle$ be a fuzzy topological space which is Quasi T_0 –space. Let $\langle Y, \mu \rangle$ be the subspace of $\langle X, \delta \rangle$. We have to prove that, $\langle Y, \mu \rangle$ be a Q- T_0 –space.

Now, since, $Y \subseteq X$ so every $V \in \mu$, $V = U|_Y$ for some $U \in \delta$. Let y_a and y_b be two distinct fuzzy points in Y such that, $y_a \neq y_b$. Then as $Y \subseteq X$, we have y_a and y_b are in X with $y_a \neq y_b$.

Again, since $\langle X, \delta \rangle$ is a Quasi T_0 –space there exist $U \in Q_\delta(y_a)$ such that $y_b \not\propto U$ or, there exist $V \in Q_\delta(y_b)$ such that $y_a \not\propto V$. This implies, there is $U|_Y \in Q_{\delta|_Y}(y_a)$ such that $y_b \not\propto U|_Y$ or there is $V|_Y \in Q_{\delta|_Y}(y_b)$ such that $y_a \not\propto V|_Y$.

Thus, by definition of a Q- T_0 –space $\langle Y, \mu \rangle$ is a Q- T_0 –space. □

Theorem 1.0.2. Every subspace of a T_0 –space is T_0 –space.

Proof. Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $\langle Y, \mu \rangle$ be a subspace of $\langle X, \delta \rangle$. Let x_a and y_b be two distinct points in Y . Then since, $Y \subseteq X$, we have, x_a and y_b in X with $x_a \neq y_b$. Now since $\langle X, \delta \rangle$ is a fuzzy T_0 –space. We have either there is $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, there is $V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.

Now, $U|_Y \in Q_{\delta|_Y}(x_a)$ such that $y_b \not\propto U|_Y$ as $x_a, y_b \in Y$ and $V|_Y \in Q_{\delta|_Y}(y_b)$ such that $x_a \not\propto V|_Y$.

Thus, $\langle Y, \mu \rangle$ is a T_0 –space. □

Theorem 1.0.3. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is a quasi- T_0 –space iff for every $x \in X$ and $a \in [0, 1]$ there exists $B \in \delta$ such that $B(x) = a$.

Proof. Suppose, $\langle \mathcal{F}(X), \delta \rangle$ be a quasi T_0 –space. If $a = 0$, then it suffices to take $B = \underline{0}$. If $0 < a < 1$, we take a strictly monotonic increasing sequence of positive real numbers converging to a . Let $\Delta_n = (a_n, a_{n+1}]$, $n = 1, 2, 3, \dots$

Since $\langle \mathcal{F}(X), \delta \rangle$ be a quasi T_0 -space, then for any $x \in X$ and $\Delta = (a_1, a_2)$ with $0 \leq a_1 < a_2 < 1$, there exists $B \in \delta$ such that $B(x) \in \Delta$.

From this property, we can say that, $\exists B_n \in \delta$ such that $B_n(x) \in \Delta_n$, for each n

$$\therefore B = \bigvee_{n=1}^{\infty} B_n \in \delta \quad \text{and} \quad B(x) = a.$$

Conversely, suppose x_a and x_b are two fuzzy points with $b < a$ where $a, b \in [0, 1]$. Then by hypothesis, there is an open set B such that $B(x) = 1 - b > 1 - a$.

This implies, B is an open Q-nbd of x_a but not quasi-conincident with x_b [since, B is a nbd of x_{1-a}]. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a quasi T_0 -space. \square

Theorem 1.0.4. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is T_1 -space iff for every $x \in X$ and each $a \in [0, 1]$ there exists $B \in \delta$ such that $B(x) = 1 - a$ and $B(y) = 1$ for $y \neq x$.

Or, $\langle \mathcal{F}(X), \delta \rangle$ is a T_1 -space \Leftrightarrow every fuzzy point in $\langle X, \delta \rangle$ is closed.

Proof. Suppose $\langle \mathcal{F}(X), \delta \rangle$ be a T_1 -space. If $a = 0$ then it suffices to take $B = \underline{1}$.

Suppose, $a > 0$ and x_a is a fuzzy point. Since, every fuzzy point in a T_1 -space is closed, so, x_a is a closed set.

\therefore We have, $B = 1 - x_a \in \delta$ and hence $B(x) = 1 - a$ and $B(y) = 1$, if $y \neq x$.

Conversely, let x_a be a fuzzy point. Then by hypothesis there exists $B \in \delta$ such that $B(x) = 1 - a$ and $B(y) = 1$ with $y \neq x$. This implies, $B = 1 - x_a$ and hence $B^c = x_a$ which is closed. Thus, $B \in \delta$. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a T_1 -space. \square

Definition 6 (Purely T_2 -space). $\langle \mathcal{F}(X), \delta \rangle$ is called purely T_2 -space if for every two zero-meet fuzzy points x_a and y_b , $\exists U \in Q_\delta(x_a)$ and $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

Definition 7. A fuzzy topological space $\langle X, \delta \rangle$ is said to be fuzzy regular iff for each $x \in X$ and each closed fuzzy set U with $U(x) = 0$ there exists $V, W \in \delta$ such that $V(x) = 1$ and $V \subseteq 1 - W$.

Theorem 1.0.5. For a fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ the following statements are equivalent

1. $\langle X, \delta \rangle$ is a fuzzy T_0 -space.
2. For $x, y \in X$, $x \neq y$, $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.

Proof. (1) \Rightarrow (2), Suppose $\langle X, \delta \rangle$ is a fuzzy T_0 -space. Thus, we have $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$. \square

Theorem 1.0.6. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called a fuzzy T_0 space iff

- (i) $\forall x, y \in X$, $\exists U \in \delta$ such that $U(x) = 1$, $U(y) = 0$ or $U(y) = 1$, $U(x) = 0$.
- (ii) $\forall x, y \in X$, $x \neq y$, $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$
- (iii) $\forall x, y \in X$, $x \neq y$, such that $U(x) < U(y)$ or $U(y) < U(x)$

Proof. (ii) \rightarrow (i)

Suppose $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy T_0 space. THus we have, $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$. This implies that either \square