Chapter 1

The Laplace Transform

1.1 Definition of The Laplace Transform

Let $F(t)$ be a function of t specified for $t > 0$. The *Laplace transform* of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,\mathrm{d}\,t \tag{1.1}$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex.

The Laplace transform of $F(t)$ is said to *exist* if the integer (1.1) *converges* for some values of s; otherwise it does not exist.

1.2 Laplace Transforms of Some Elementary Functions

$\mathbf{F(t)}$	$\mathcal{L}\{\mathbf{F(t)}\} = \mathbf{f(s)}$	
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
$t^n \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a\rvert$

1.3 Some Important Properties of Laplace Transforms

1. First translation or shifting property.

   Theorem 1.3.1. If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$.

   Example. Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$, we have

$$\mathcal{L}\left\{e^{-t}\cos 2t\right\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

2. Second translation or shifting property.

   Theorem 1.3.2. If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$ then $\mathcal{L}\{G(t)\} = e^{-as} f(s)$.

   Example. Since $\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$, the Laplace transform of the function $G(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$ is $\frac{6e^{-2s}}{s^4}$.

3. Change of scale property.

   Theorem 1.3.3. If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

   Example. Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, we have $\mathcal{L}\{\sin 3t\} = \frac{1}{3}\frac{1}{(s/3)^2+1} = \frac{3}{s^2+9}$.

4. Laplace transform of derivatives.

   Theorem 1.3.4. If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s\, f(s) - F(0)$.
   If $F(t)$ is continuous for $0 \leqq t \leq N$ and of exponential order for $t > N$ while $F'(t)$ is sectionally continuous for $0 \leqq t \leqq N$.

   Example. If $F(t) = \cos 3t$, then $\mathcal{L}\{F'(t)\} = \frac{s}{s^2+9}$ and we have

   $$\mathcal{L}\{F'(t)\} = \mathcal{L}\{-3\sin 3t\} = s\left(\frac{s}{s^2+9}\right) - 1 = \frac{-9}{s^2+9}$$

   This method is useful in finding Laplace transforms without integration.

   Theorem 1.3.5. If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F''(t)\} = s^2\, f(s) - s\, F(0)$.
   If $F(t)$ and $F'(t)$ is continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ while $F''(t)$ is sectionally continuous for $0 \leq t \leqq N$.

5. Laplace transform of integrals.

   Theorem 1.3.6. If $\mathcal{L}\{F(t)\} = f(s)$ then

   $$\mathcal{L}\left\{\int_0^t F(u)\,\mathrm{d}\,u\right\} = \frac{f(s)}{s}$$

   Example. Since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, we have

   $$\mathcal{L}\left\{\int_0^t \sin 2u\,\mathrm{d}\,u\right\} = \frac{2}{s(s^2+4)}$$

   as can be verified directly.

6. Multiplication by t^n.

   Theorem 1.3.7. If $\mathcal{L}\{F(t)\} = f(s)$ then

   $$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{\mathrm{d}^{\,n}}{\mathrm{d}\,s^n} f(s) = (-1)^n f^{(n)}(s)$$

   Example. Since $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$, we have

   $$\mathcal{L}\left\{te^{2t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$
   $$\mathcal{L}\left\{t^2e^{2t}\right\} = \frac{\mathrm{d}^{\,2}}{\mathrm{d}\,s^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^2}$$

1.4 Solved Problems

1.4.1 Laplace Transforms of Some Elementary Functions

Problem 1.4.1. Prove that

(a) $\mathcal{L}\{1\} = \frac{1}{s}$, $s > 0$

(b) $\mathcal{L}\{t\} = \frac{1}{s^2}$, $s > 0$

(c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-1}$, $s > a$

Solution. (a)

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \int_0^P e^{-st}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^P \\
&= \lim_{P\to\infty} \frac{1-e^{-sP}}{s} \\
&= \frac{1}{s} \qquad \text{if } s > 0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty e^{-st}(t)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \int_0^P te^{-st}(t)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right|_0^P \\
&= \lim_{P\to\infty} \left(\frac{1}{s^2} - \frac{e^{-st}}{s^2} - \frac{Pe^{-sP}}{s}\right) \\
&= \frac{1}{s^2} \qquad \text{if } s > 0
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st}\left(e^{at}\right)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \int_0^P e^{-(s-a)t}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^P \\
&= \lim_{P\to\infty} \frac{1-e^{-(s-a)P}}{s-a} \\
&= \frac{1}{s-a} \qquad \text{if } s > a
\end{aligned}$$

Problem 1.4.2. Prove that

(a) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$

(b) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$

Solution.

(a)

$$
\begin{aligned}
\mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st}\sin at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\int_0^P e^{-st}\sin at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\left.\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\right|_0^P \\
&= \lim_{P\to\infty}\left\{\frac{a}{s^2+a^2} - \frac{e^{-sP}(a\sin aP + a\cos aP)}{s^2+a^2}\right\} \\
&= \frac{a}{s^2+a^2} \qquad \text{if } s>0
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st}\cos at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\int_0^P e^{-st}\cos at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\left.\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right|_0^P \\
&= \lim_{P\to\infty}\left\{\frac{s}{s^2+a^2} - \frac{e^{-sP}(a\cos aP - a\sin aP)}{s^2+a^2}\right\} \\
&= \frac{s}{s^2+a^2} \qquad \text{if } s>0
\end{aligned}
$$

We have used here the results

$$
\begin{aligned}
\int e^{\alpha t}\sin\beta t \,\mathrm{d}\,t &= \frac{e^{at}(\alpha\sin\beta t - \beta\cos\beta t)}{\alpha^2+\beta^2} \\
\int e^{\alpha t}\cos\beta t \,\mathrm{d}\,t &= \frac{e^{at}(\alpha\cos\beta t + \beta\sin\beta t)}{\alpha^2+\beta^2}
\end{aligned}
$$

1.4.2 Translation and Change of Scale Properties

Problem 1.4.3. Prove the first translation or shifting property: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$.

Solution. We have, $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,\mathrm{d}\,t$
$= f(s)$

Then

$$
\begin{aligned}
\mathcal{L}\left\{e^{at}F(t)\right\} &= \int_0^\infty e^{-st}\left\{e^{at}F(t)\right\}\,\mathrm{d}\,t \\
&= \int_0^\infty e^{-(s-a)t}F(t)\,\mathrm{d}\,t \\
&= f(s-a)
\end{aligned}
$$

Problem 1.4.4. Find

(a) $\mathcal{L}\{t^2e^{3t}\}$

(b) $\mathcal{L}\{e^{-2t}\sin 4t\}$

(c) $\mathcal{L}\{e^{4t}\cosh 5t\}$

(d) $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

Solution.

(a) $\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$. Then $\mathcal{L}\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}$.

(b) $\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16}$. Then $\mathcal{L}\{e^{-2t}\sin 4t\} = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$.

(c) $\mathcal{L}\{\cosh 5t\} = \frac{s}{s^2-25}$. Then $\mathcal{L}\{e^{4t}\cosh 5t\} = \frac{s-4}{(s-4)^2-25} = \frac{s-4}{s^2-8s-9}$.

Another method.

$$\begin{aligned}\mathcal{L}\left\{e^{4t}\cosh 5t\right\} &= \mathcal{L}\left\{e^{4t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\right\}\\ &= \frac{1}{2}\mathcal{L}\left\{e^{9t}+e^{-t}\right\}\\ &= \frac{1}{2}\left\{\frac{1}{s-9}+\frac{1}{s+1}\right\}\\ &= \frac{s-4}{s^2-8s-9}\end{aligned}$$

(d)

$$\begin{aligned}\mathcal{L}\{3\cos 6t - 5\sin 6t\} &= 3\mathcal{L}\{\cos 6t\} - 5\mathcal{L}\{\sin 6t\}\\ &= 3\left(\frac{s}{s^2+36}\right) - 5\left(\frac{6}{s^2+36}\right)\\ &= \frac{3s-30}{s^2+36}\end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}\left\{e^{-2t}(3\cos 6t - 5\sin 6t)\right\} &= \frac{3(s+2)-30}{(s+2)^2+36}\\ &= \frac{3s-24}{s^2+4s+40}\end{aligned}$$

Problem 1.4.5. Prove the second translation or shifting property:
If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then $\mathcal{L}\{G(t)\} = e^{-as}f(s)$.

Solution.

$$\begin{aligned}\mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st}G(t)\,\mathrm{d}\,t\\ &= \int_0^a e^{-st}G(t)\,\mathrm{d}\,t + \int_a^\infty e^{-st}G(t)\,\mathrm{d}\,t\\ &= \int_0^a e^{-st}(0)\,\mathrm{d}\,t + \int_a^\infty e^{-st}F(t-a)\,\mathrm{d}\,t\\ &= \int_a^\infty e^{-st}F(t-a)\,\mathrm{d}\,t\\ &= \int_a^\infty e^{-s(u+a)}F(u)\,\mathrm{d}\,u\\ &= e^{-as}\int_a^\infty e^{-su}F(u)\,\mathrm{d}\,u\\ &= e^{-as}f(s)\end{aligned}$$

Where we have used the substitution $t = u + a$.

Problem 1.4.6. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$.

Solution.

Method 1.

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \int_0^{\frac{2\pi}{3}} e^{-st}(0)\,\mathrm{d}\,t + \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)\mathrm{d}\,t\\ &= \int_0^{\infty} e^{-s\left(u+\frac{2\pi}{3}\right)}\cos u\,\mathrm{d}\,u\\ &= e^{-\frac{2\pi}{3}}\int_0^{\infty} e^{-su}\cos u\,\mathrm{d}\,u\\ &= \frac{se^{-\frac{2\pi}{3}}}{s^2+1}\end{aligned}$$

Method 2. Since $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, it follows from second translation property, with $a = \frac{2\pi}{3}$, that

$$\mathcal{L}\{F(t)\} = \frac{se^{-\frac{2\pi}{3}}}{s^2+1}$$

1.4.3 Laplace Transform of Derivatives

Problem 1.4.7. Prove Theorem 1.3.4: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F'(t)\} = s\,f(s) - F(0)$.

Solution. Using integration by parts, we have

$$\begin{aligned}\mathcal{L}\{F'(t)\} &= \int_0^{\infty} e^{-st}F'(t)\,\mathrm{d}\,t - \lim_{P\to\infty}\int_0^{P} e^{-st}F'(t)\,\mathrm{d}\,t\\ &= \lim_{P\to\infty}\left\{e^{-st}F(t)\Big|_0^P + s\int_0^P e^{-st}F(t)\,\mathrm{d}\,t\right\}\\ &= \lim_{P\to\infty}\left\{e^{-sP}F(P) - F(0) + s\int_0^P e^{-st}F(t)\,\mathrm{d}\,t\right\}\\ &= s\int_0^{\infty} e^{-st}F(t)\,\mathrm{d}\,t - F(0)\\ &= s\,f(s) - F(0)\end{aligned}$$

Using the fact that $F(t)$ is of exponential order γ as $t\to\infty$, so that $\lim_{P\to\infty} e^{-sP}F(P) = 0$ for $s > \gamma$.

Problem 1.4.8. Prove Theorem 1.3.5: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2\,f(s) - s\,F(0) - F'(0)$.

Solution. By Problem 1.4.3,

$$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0) = s\,g(s) - G(0)$$

Let $G(t) = F'(t)$. Then

$$\begin{aligned}\mathcal{L}\{F''(t)\} &= s\mathcal{L}\{F'(t)\} - F'(0)\\ &= s\left[s\mathcal{L}\{F(t)\} - F(0)\right] - F'(0)\\ &= s^2\mathcal{L}\{F(t)\} - s\,F(0) - F'(0)\\ &= s^2 f(s) - s\,F(0) - F'(0)\end{aligned}$$

The generalization to higher derivatives can be proved by using mathematical induction.

1.4.4 Multiplication By Powers of t

Problem 1.4.9. Prove Theorem 1.3.7: If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{t^nF(t)\} = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}\,s^n}f(s) = (-1)^n f^{(n)}(s) \qquad \text{where } n = 1, 2, 3, \ldots$$

Solution. We have,

$$f(s) = \int_0^\infty e^{-st} F(t)\,\mathrm{d}\,t$$

Then by Leibniz's rule for differentiating under the integral sign,

$$\begin{aligned}\frac{\mathrm{d}\,f}{\mathrm{d}\,s} = f'(s) = \frac{\mathrm{d}}{\mathrm{d}\,s}\int_0^\infty e^{-st}F(t)\,\mathrm{d}\,t &= \int_0^\infty \frac{\partial}{\partial s}e^{-st}F(t)\,\mathrm{d}\,t \\ &= \int_0^\infty -te^{-st}F(t)\,\mathrm{d}\,t \\ &= -\int_0^\infty e^{-st}\{tF(t)\}\,\mathrm{d}\,t \\ &= -\mathcal{L}\{tF(t)\}\end{aligned}$$

Thus,

$$\mathcal{L}\{tF(t)\} = -\frac{\mathrm{d}\,f}{\mathrm{d}\,s} = -f'(s) \tag{1.2}$$

which proves the theorem for $n = 1$.

To establish the theorem in general, we use mathematical induction. Assume the theorem is true for $n = k$, i.e., assume

$$\int_0^\infty e^{-st}\left\{t^k F(t)\right\}\,\mathrm{d}\,t = (-1)^k f^{(k)}(s) \tag{1.3}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\int_0^\infty e^{-st}\left\{t^k F(t)\right\}\,\mathrm{d}\,t = (-1)^k f^{(k+1)}(s)$$

or by Leibniz's rule,

$$-\int_0^\infty e^{-st}\left\{t^{k+1} F(t)\right\}\,\mathrm{d}\,t = (-1)^k f^{(k+1)}(s)$$

i.e.,

$$\int_0^\infty e^{-st}\left\{t^{k+1} F(t)\right\}\,\mathrm{d}\,t = (-1)^{k+1} f^{(k+1)}(s) \tag{1.4}$$

It follows that if (1.3) is true, i.e., if the theorem holds for $n = k$, then (1.4) is true, i.e., the theorem holds for $n = k + 1$. But by (1.2) the theorem is true for $n = 1$. Hence, it is true for $n = 1 + 1 = 2$ and $n = 2 + 1 = 3$, etc., and thus for all positive integer values of n.

Problem 1.4.10. Find

(a) $\mathcal{L}\{t\sin at\}$

(b) $\mathcal{L}\{t^2\cos at\}$

Solution.

(a) Since $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, we have by multiplication by the powers of t

$$\mathcal{L}\{t\sin at\} = -\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

Another method
Since $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,\mathrm{d}\,t = \dfrac{s}{s^2+a^2}$
We have by differentiating with respect to the parameter a [using Leibniz's rule],

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,a}\int_0^\infty e^{-st}\cos at\,\mathrm{d}\,t &= \int_0^\infty e^{-st}\{-t\sin at\}\,\mathrm{d}\,t = -\mathcal{L}\{t\sin at\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\,a}\left(\frac{s}{s^2+a^2}\right) = -\frac{2as}{(s^2+a^2)^2}\end{aligned}$$

from which

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

Note that the result is equivalent to $\frac{\mathrm{d}}{\mathrm{d}\,a}\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}\,a}\cos at\right\}$.

(b) Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, we have by multiplication by the powers of t

$$\mathcal{L}\left\{t^2 \cos at\right\} = -\frac{\mathrm{d}^2}{\mathrm{d}\, s^2}\left(\frac{s}{s^2+a^2}\right) = \frac{2s^3 - 6a^2 s}{(s^2+a^2)^3}$$

We can also use the second method of part (a) by writing

$$\mathcal{L}\left\{t^2 \cos at\right\} = \mathcal{L}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}\, a^2}\cos at\right\} = -\frac{\mathrm{d}^2}{\mathrm{d}\, a^2}\mathcal{L}\{\cos at\}$$

which gives the same result.

1.4.5 Evaluation of Integral

Problem 1.4.11. Evaluate

(a) $\displaystyle\int_0^\infty te^{-2t}\cos t\,\mathrm{d}\, t$,

(b) $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,\mathrm{d}\, t$

Solution. (a) By multiplication by the powers of t,

$$\begin{aligned}
\mathcal{L}\{t\cos t\} &= \int_0^\infty te^{-st}\cos t\,\mathrm{d}\, t \\
&= -\frac{\mathrm{d}}{\mathrm{d}\, s}\mathcal{L}\{\cos t\} \\
&= -\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{s}{s^2+1}\right) \\
&= \frac{s^2-1}{(s^2+1)^2}
\end{aligned}$$

Then letting $s = 2$, we find

$$\int_0^\infty te^{-2t}\cos t\,\mathrm{d}\, t = \frac{3}{25}$$

(b) If $F(t) = e^{-t} - e^{-3t}$, then

$$f(s) = \mathcal{L}\{F(t)\} = \frac{1}{s+1} - \frac{1}{s+3}$$

Thus by division by powers of t,

$$\begin{aligned}
\mathcal{L}\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} &= \int_0^\infty \left\{\frac{1}{u+1} - \frac{1}{u+3}\right\}\mathrm{d}\, u \\
\Rightarrow \int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-3t}}{t}\right)\mathrm{d}\, t &= \ln\left(\frac{s+3}{s+1}\right)
\end{aligned}$$

Taking the limit as $s \to 0^+$, we find

$$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,\mathrm{d}\, t = \ln 3$$