Mathematical Programming

MAT413

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Preface

This is a compilation of lecture notes with some books and my own thoughts. This document is not a holy text. So, if there is a mistake, solve it by your own judgement. Currently, the following topics are not included

- Fourier Transform and Applications
- Boundary value problems
- Eigenfunctions
- Green's functions
- Strum-Liouville problems
- Laplace Equation

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Syllabus

- Linear programming:
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  - graphical solution of linear program;
  - solution of linear program by simplex method;
  - algebraic basis and computational set up;
  - duality problem-duality theorem;
  - transportation problems;
  - assignment problem and simple applications;
  - two-person zero-sum game;
  - simple inventory problems.
- Non-linear programming
  - Definiteness of matrix;
  - general optimization problem;
  - concave and convex functions;
  - optimization of convex functions;
  - general nonlinear programming problem;
  - tangent plane;
  - regular point;
  - equality constraint;
  - Lagrangian for equality and inequality constraints;
  - Kuhn-Tucker condition;
  - standard extermination problem of convex and concave programming;
  - saddle point.

Books Recommended:

- Haldey, G.: Linear Programming
- Gass, S.I.: Mathematical Programming
- Saaty, T.L.: Mathematical Methods of Operational Research
- Lieberman: Operational Research
- Luenberger: Linear and Nonlinear Programming
- Taha: Operation Research

Part I

Sheet

Chapter 1

Linear Programming

Linear programming (LP) is a mathematical modeling technique designed to optimize the usage of limited resources.

In other words, linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equalities and/or inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different sources such as market demand, production processes and equipment, storage capacity, raw material availability, etc.

By '*linearity*' is meant a mathematical expression in which the variables do not have powers.

Successful applications of LP exist in the areas of military, industry, agriculture, transportation, economics, health systems, and even behavioral and social sciences.

1.1 Construction of the LP model

Example (The Reddy Mikks Company)**.** Reddy Mikks produces both interior and exterior paints from two raw materials, M1 and M2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily
	Exterior paint	Interior paint	availability (tons)
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton	5	4	

A market survey restricts the maximum daily demand of interior paint to 2 tons. Additionally, the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Reddy Mikks want to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

The LP model includes three basic elements.

1. Decision variables that we seek to determine.

2. Objective (goal) that we aim to optimize.

3. Constraints that the solution must satisfy.

Step 1: Here, we need to determine the amounts to be produced of exterior and interior paint. The variables of the model are thus defined as

x_1 = tons produced daily of exterior paint x_2 = tons produced daily of interior paint.

Step 2: The next task is to construct the objective function. A logical objective for the company is to increase as much as possible (i.e., maximize) the total daily profit from both exterior and interior paints. Letting Z represent the daily profit, we get,

$$Z = 5x_1 + 4x_2$$

The objective of the company is

$$\text{Maximize } Z = 5x_1 + 4x_2$$

Step 3: The last element of the model deals with the constraints that restrict raw materials usage and demand. The raw materials restrictions are expressed verbally as

$$(\text{Usage of raw material by both paints}) \;\geq\; (\text{Maximum raw material availability})$$

From the data of the problem that

$$\begin{aligned}\text{Usage of raw material } M1 &= 6x_1 + 4x_2 \text{ tons/day}\\ \text{Usage of raw material } M2 &= 1x_1 + 2x_2 \text{ tons/day}\end{aligned}$$

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$\begin{aligned}6x_1 + 4x_2 &\le 24 \quad \text{(raw materials M1)}\\ x_1 + 2x_2 &\le 6 \quad \text{(raw materials M2)}\end{aligned}$$

There are two types of demand restrictions :

(a) Maximum daily demand of interior paint is limited to 2 tons, and

(b) Excess of daily production of interior paint over that of exterior paint is at most 1 tons.

$$\begin{aligned}\text{Thus, first restriction, } & x_2 \le 2\\ \text{second restriction, } & x_2 - x_1 \le 1\end{aligned}$$

An implicit (or, "understood to be") restriction on the model is that the variables x_1 and x_2 must not be negative. We thus add the non-negative restrictions, $x_1 \ge 0$ and $x_2 \ge 0$, to account for this requirement.

Here, the complete Reddy Mikks model is written as

$$\begin{aligned}\text{Maximize} \quad & Z = 5x_1 + 4x_2 \quad \text{(objective function)}\\ \text{Subject to} \quad & \left.\begin{aligned}6x_1 + 4x_2 &\le 24\\ x_1 + 2x_2 &\le 6\\ -x_1 + x_2 &\le 1\\ x_2 &\le 2\end{aligned}\right\} \text{Constraints (or, Restriction)}\\ & x_1, x_2 \ge 0 \quad \text{Non-negativity restriction on decision variable}\end{aligned}$$

Note. Profit problem $\longrightarrow$ maximization
Cost problem $\longrightarrow$ minimization

Example (Production Allocation problem). A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines given in the table below:

Machine	Time per unit (minutes)			Machine capacity (minutes/day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Tk. 4, Tk. 3 and Tk. 6 respectively. It is assumed that all the amounts produced are consumed in the market.

Solution: Formulation of Linear Programming Model:

Step 1: Identify the decision variable:
Let the amount of products 1, 2, and 3 manufactured daily be x_1, x_2 and x_3 respectively.

Step 2: Identify the constraints:
Here constrains are on the machine capacities and can be mathematically expressed as

$$\begin{aligned}2x_1 + 3x_2 + 2x_3 &\le 440\\ 4x_1 \qquad\quad + 3x_3 &\le \;\;\, 6\\ 2x_1 + 5x_2 \qquad\quad &\le 430\end{aligned}$$

Also, since it is not possible to produce negative units, non-negative restriction will be $x_1 \ge 0$, $x_2 \ge 0$ and $x_3 \ge 0$.

Step 3: Identify the objective function:
The objective is to maximize the total profit from sales. Assuming that a perfect market exists for the product such that all that is produced can be sold. The total profit from sale is

$$Z = 4x_1 + 3x_2 + 6x_3$$

Hence, the linear programming model for our production allocation problem becomes

$$\begin{aligned} \text{maximize} \quad & Z = 4x_1 + 3x_2 + 6x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 2x_3 \leq 440 \\ & 4x_1 + 3x_3 \leq 470 \\ & 2x_1 + 5x_2 \leq 430 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Example (Inspection problem)**.** A company has two grades of inspectors, 1 and 2 to undertake quality control inspection. At least 1500 pieces must be inspected in an 8-hour day. Grade 1 inspector can check 20 pieces in an hour with an accuracy of 96%. Grade 2 inspector checks 14 pieces an hour with an accuracy of 92%. The daily wages of grade 1 inspector are Tk.5 per hour while those of grade 2 inspector are Tk. 4 per hour. Any error made by an inspector costs Tk. 3 to the company.

If there are, in all, 10 grade 1 inspectors and 15 grade 2 inspectors in the company, formulate an LP for the optimal assignment of inspectors that minimize the daily inspection cost.

Formulation of LP Model:

Step 1: Identify the decision variables.

We have to determine the number of grade 1 and grade 2 inspectors for assignment. Let x_1, and x_2 represent the number of grade 1 & grade 2 inspectors, respectively. Here $x_1 \geq 0$ and $x_2 \geq 0$.

Step 2: Identify the constraints.

Here, the number of pieces that inspected daily by grade 1 inspector is $x_1 \times 20 \times 8$ and that of grade 2 inspector is $x_2 \times 14 \times 8$. Hence, the constraints are

On the number of pieces to be inspected daily : $(x_1 \times 20 \times 8) + (x_2 \times 14 \times 8) \geq 1500$
or, $160x_1 + 112x_2 \geq 1500$
On the number of grade 1 inspector : $x_1 \leq 10$
On the number of grade 2 inspector : $x_2 \leq 15$

Step 3: Identify the objective function.

The objective is to minimize the daily cost of inspection. Now the company had to incur two types of costs: wage paid to the inspectors and the cost of their inspection errors.

The cost of grade 1 inspector/hour is Tk. $(5 + 20 \times 0.04 \times 3) =$ Tk.7.40
Similarly, cost of grade 2 inspector/hour is Tk. $(4 + 14 \times 0.08 \times 3) =$ Tk.7.36

The objective function is

$$\text{Minimize} \quad Z = 8(7.40x_1 + 7.36x_2)$$

Hence, the linear programming model for our inspection problem becomes

$$\begin{aligned} \text{minimize} \quad & Z = 8(7.40x_1 + 7.36x_2) \\ \text{subject to} \quad & 160x_1 + 112x_2 \geq 1500 \\ & x_1 \leq 10 \\ & x_2 \leq 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

1.2 Some definitions

Solution: The set of values of decision variables x_j $(j = 1, 2, \ldots, n)$ which satisfy all the constraints of a linear programming problem, is called the solution of that linear programming problem.

Feasible solution: Any solution that satisfies all the constraints and non-negativity conditions of a linear programming problem simultaneously, is a feasible solution.

Feasible region/solution space: Set of all feasible solution or the area bounded by all the constraints is called the solution space or the region of feasible solutions.

Optimum solution: A solution of a model that optimizes (maximizes or minimizes) the value of the objective function while satisfying all the constraints, is referred to as the optimum solution.
Note: The term *optimal solution* is also used for *optimum solution*.

Unbounded solution: A solution which can increase or decrease the value of objective function of the LP problem indefinitely is called an unbounded solution.

Parameters: The constants (namely, the coefficients and right-hand sides) in the constraints and the objective function are called the parameters of the LP problem.

Chapter 2

Linear Programming: The Graphical Method

For LP problems that have only two variables, it is possible that the entire set of feasible solutions can be displayed graphically by plotting linear constraints on a graph paper to locate the best (optimal) solution. Although most real-world problems have more than two decision variables, and hence can not be solved graphically, this solution approach provides valuable understanding of how to solve LP problems involving more than two variables algebraically.

Here, we shall discuss two graphical solution methods or approaches:

(i) Extreme point solution method

(ii) Iso-profit (cost) function line method

to find the optimal solution to an LP problem.

To solve an LPP by using graphical method:

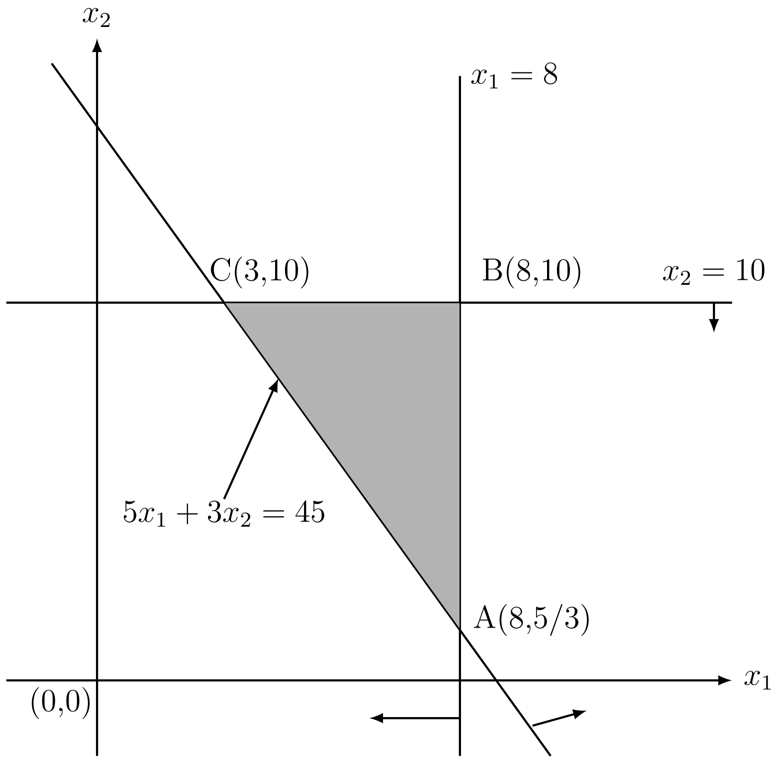
1. Consider the constraints as equalities.
2. Draw the lines in plane corresponding to each equation and non-negative restriction.
3. Find the feasible region for the values of the variables, which is the region, bounded by the lines.
4. Find the feasible point in the region, which gives the optimal value of Z.

2.1 Extreme point solution method

Problem 2.1.1.

$$\begin{aligned} \text{minimize} \quad & Z = 40x_1 + 36x_2 \\ \text{subject to} \quad & x_1 \leq 8 \\ & x_2 \leq 10 \\ & 5x_1 + 3x_2 \geq 45 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution. Let us consider x_1 along the horizontal axis and x_2 along the vertical axis. Now, each constraint have to be plotted on the graph by treating it as a linear equation appropriate and then inequality conditions have to use to make the area of feasible solutions.



Area $ABC \rightarrow$ Solution space or feasible region

From the graph, we have the shaded region which is a feasible region or solution space. The optimal value of the objective function occurs at one of the corner points of the feasible regions. We now compute the Z-values corresponding to each corner points.

$$\begin{aligned} \text{For } A\left(8, \frac{5}{3}\right), &\qquad Z = 380 \\ \text{For } B(8,10), &\qquad Z = 680 \\ \text{For } C(3,10), &\qquad Z = 480 \end{aligned}$$

Since A is minimum at point $A\left(8, \frac{5}{3}\right)$, hence the optimal solution is $x_1 = 8$, $x_2 = \frac{5}{3}$ and optimal value is $Z = 380$.

Note. (i) The optimum solution is identified always by one of the corner point.

(ii) To determine which side of a constraint equation is in the feasible region, examine whether the origin (0,0) satisfies the constraints. If it does, then all points on the constraint equation and all points on the side of the constraint at which origin is situated are feasible points. If it does not, then all points on the constraint equation and all points on the opposite side of the origin are feasible points.

2.2 Iso-profit (cost) function line method

Problem 2.2.1 (Reddy Mikks Model)**.**

$$\begin{aligned} \text{maximize} \quad & Z = 5x_1 + 4x_2 \\ \text{subject to} \quad & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution. Let us consider x_1 along the horizontal axis and x_2 along the vertical axis. Now, each constraint have to be plotted on the graph by treating it as a linear equation and then appropriate inequality conditions have to use to make the area of feasible solutions.

From the graph we have the shaded region, which is a feasible region or solution space.
To determine the optimum solution, we will identify the direction in which the profit function $Z = 5x_1 + 4x_2$ increase (since we are maximizing Z). We can do so by assigning Z the (arbitrary) increasing values of 10 and 15, which would be equivalent to plotting the lines

$$5x_1 + 4x_2 = 10 \qquad \text{and} \qquad 5x_1 + 4x_2 = 15$$

Figure superimposes these two lines on the solution space of the model. The profit Z thus can be increased in the direction shown in the figure until we reach the point in the solution space beyond which any further increase will put us outside the boundaries of $ABCDEF$. Such a point is the optimum.

From the figure, the optimum solution is given by point C. The values of x_1 and x_2 are thus determined by solving the equations associated with lines 1 and 2, i.e.,

$$6x_1 + 4x_2 = 24 \qquad \text{and} \qquad x_1 + 2x_2 = 6$$

The solution yields $x_1 = 3$ and $x_2 = 1.5$ with $Z = (5 \times 3) + (4 \times 1.5) = 21$
This means that the optimum daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint will yield a daily profit of 21 unit.

Chapter 3

Convexity

The ***line segment*** joining the points $x_1, x_2 \in \mathbb{R}^n$ is the set of points given by

$$\{x : x = \lambda x_1 + (1-\lambda)x_2, 0 \leq \lambda \leq 1\}$$

The points x_1 and x_2 are called the *end points* of this segments and, for each $\lambda,\ 0 < \lambda < 1$, the point $\lambda x_1 + (1-\lambda)x_2$ is called an in-between point (or, internal point) of the line segment.

A vector $x \in \mathbb{R}^n$ is called a ***linear combination*** of vectors $x^1, x^2, \ldots, x^m$ in $\mathbb{R}^n$ if there exists real numbers $\lambda_i (i = 1, 2, \ldots, m)$ such that $x = \sum_{i=1}^{m} \lambda_i x^i$.

A vector $x \in \mathbb{R}^n$ is called a ***convex combination*** of vectors $x^1, x^2 \ldots, x^m$ $(m \leq n)$ if there exists real numbers λ_i satisfy

$$\lambda_i \geq 0 \ (i = 1, 2, \ldots, m), \ \sum_{i=0}^{m} \lambda_i = 1 \quad \text{and} \quad x = \sum_{i=1}^{m} \lambda_i x^i$$

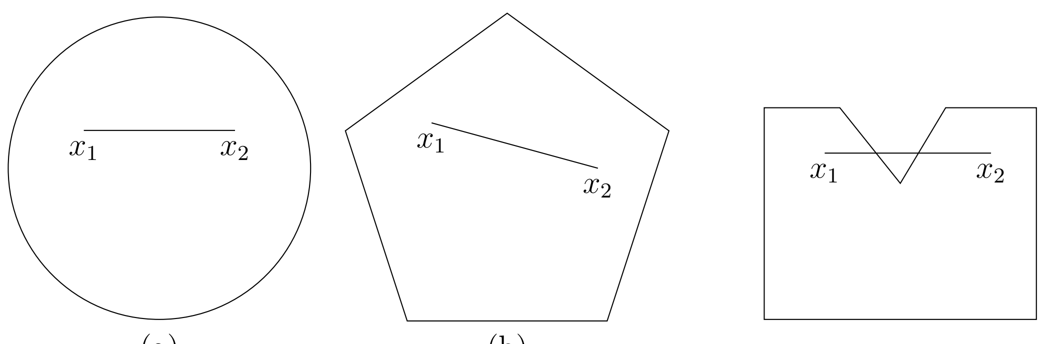
3.1 Convex set

A set S in the n-dimensional space is said to be *convex* if the *line segment* joining any two distinct points in the set lies entirely in S.

Mathematically, this means that if x^1 and x^2 are two distinct points in S, then every points

$$x = \lambda x^1 + (l - \lambda)x^2, 0 \leq \lambda \leq 1$$

must also be in S.
i.e. A set $S \subset \mathbb{R}^n$ is called convex set if $x^1, x^2 \in S \Rightarrow \lambda x^1 + (1-\lambda)x^2 \in S$, for all $0 \leq \lambda 1$.



(a) (b) (c)

The diagrams above illustrate the definition where sets (a) and (b) are convex, and set (c) is non-convex.

- A set $S \subseteq \mathbb{R}^n$ is called *line variety* if

$$x^1, x^2 \in S \Rightarrow \lambda x^1 + (1-\lambda)x^2 \in S, \quad \text{for all } \lambda \in \mathbb{R}$$

- The *extreme points* of a convex set are those points in the set that cannot be expressed as a convex combination of any two distinct points in the same set.

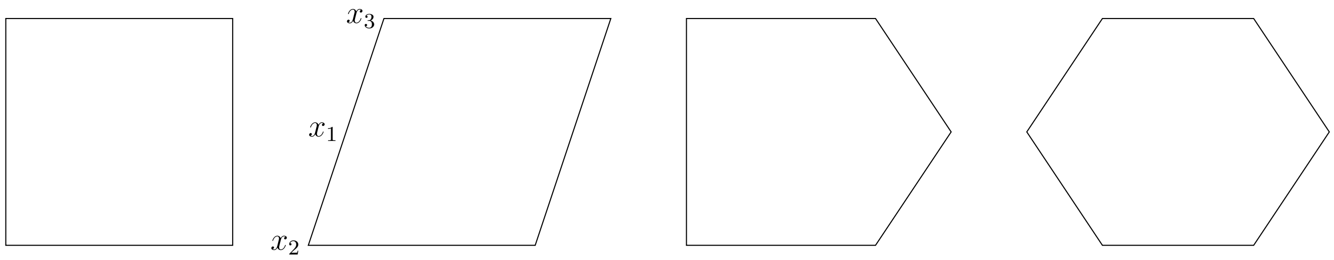
In the figure, the convex set (a) has an infinite number of extreme points (namely, the tangent points of the circle) and the set (b), which is typical LP solution space, has only a finite number (=6).

3.2 Extreme Points (or vertex):

Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $x \in S$ is called an extreme point or vertex of S if there exists no distinct points x_1 and x_2 in S such that,

$$x = \lambda x^1 + (1-\lambda)x^2 \quad \text{for} 0 < \lambda < 1$$

Remark. Note that strict inequalities are imposed on λ. By definition stipulates that an extreme point cannot be "between" any other two points of the set. Clearly, an extreme point is a boundary point of a convex set are necessarily extreme points. Some boundary points may be between two other boundary points.



The polygons in the above are convex sets and, the extreme points are the vertices. Point x_1 is not an extreme point because it can be represented as a convex combination of x_2 and x_3 with $0 < \lambda < 1$.

Problem 3.2.1. Show that $S = \{(x_1, x_2) : 2x_1 + 3x_2 = 7\} \subset \mathbb{R}^2$ is a convex set.

Solution. Let $U, V \in S$, where $U = (u_1, u_2)$, $V = (v_1, v_2)$.

$$\therefore 2u_1 + 3u_2 = 7 \qquad \text{and} \qquad 2v_1 + 3v_1 = 7$$

The line segment joining U and V is the set

$$\{Z : Z = \lambda U + (1-\lambda)V,\ 0 \leq \lambda \leq 1\}$$

For some λ, $0 \leq \lambda \leq 1$, let $Z = (z_1, z_2)$ be a point of this set, so that

$$z_1 = \lambda u_1 + (1-\lambda)v_1, \quad z_2 = \lambda u_2 + (1-\lambda)v_2$$

Now,

$$\begin{aligned} 2z_1 + 3z_2 &= 2\left[\lambda u_1 + (1-\lambda)v_1\right] + 3\left[\lambda u_2 + (1-\lambda)v_2\right] \\ &= \lambda(2u_1 + 3u_2) + (1-\lambda)(2v_1 + 3v_2) \\ &= 7\lambda + 7(1-\lambda) \\ &= 7 \\ \therefore Z &= (z_1, z_2) \ \text{ is a point of } S \end{aligned}$$

Since Z is any point of the line segment joining U and V,

$$\therefore U, V \in S \Rightarrow \lambda u + (1-\lambda)V \in S \ \ \forall\, 0 \leq \lambda \leq 1$$

Hence, S is a convex set.

Chapter 4

Theory of Simplex Method

Simplex method, also called simplex technique or simplex algorithm was developed by G.B. Dantzig, an American mathematician in 1974. It has the advantage of being universal, i.e., any linear model for which the solution exists, can be solved by it. In principle, it consists of starting with a certain solution of which all that we know is that it is feasible, i.e., it satisfies the non-negativity conditions ($x_j \geq 0, j = 1, 2, 3, \ldots, n$). We, then, improve upon this solution at consecutive stage, until, after a certain finite number of stages, we arrive at the optimal solution.
The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is less (or more as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, this method leads to an optimal vertex in a finite number of steps. The basis of the simplex method consists of two fundamental conditions:

1. The feasibility condition : It ensures that if the starting solution is basic feasible, only basic feasible solutions will be obtained during computation.

2. The optimality condition : It guarantees that only better solutions (as compared to the current solution) will be encountered.

The simplex method or technique is an iterative procedure for solving the linear programming problems. It consists in

(i) having a basic feasible solution,

(ii) testing whether it is an optimal solution or not, and

(iii) improving the first trial solution by a set of rules, and repeating the sequences till an optimal solution is obtained.

Problem 4.0.1. Solve the following L.P.P by simplex method.

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 14 \\ & x_1 - x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution. In order to solve the linear programming problem by simplex method, the given problem has to convert into its standard form. For this the inequalities are converted into equations by including the slack variable in the 1st, 2nd and 3rd inner respectively. Since all the variables are non-negative. Thus, the standard L.P.P is as follows:

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 + s_1 = 4 \\ & 3x_1 + 2x_2 + s_2 = 14 \\ & x_1 - x_2 + s_3 = 3 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Since all the slack variables are basic variables, we shall start with a basic feasible solution, which we shall get by assuming the profit is zero. This will be when non-basic variables $x_1 = x_2 = 0$. We get $s_1 = 4,\ s_2 = 14,\ s_3 = 3$ as the first basic feasible solution. The above information can be put in the form of a simplex matrix tableau-1.
In these tableaux, *basis* refers to the basic variables in the current basic feasible solution. The values of the basic variables are given under the column *constant.* The symbol C_j denotes the coefficients of the variables in the objective function while C_B denotes the coefficients of the basic variables only. $\bar{C_j}$ denotes the *relative profit*(cost for minimization problem)*coefficient* of the variables given by

$$\begin{aligned} \bar{C_j} &= C_j - [\text{inner product of column } C_B \text{ and the column corresponding the } j\text{−th variable in the canonical system}] \\ &= C_j - C_B A_j, \text{where } A_j \text{ is the } j\text{−th column of the matrix } A = (a_{ij}), \text{ which is found by the coefficients of the} \\ &\quad \text{basis and non-basis variables of constrains equation.} \end{aligned}$$

Tab	C_B	basis \ $c_j \rightarrow$	3 x_1	2 x_2	0 s_1	0 s_2	0 s_3	Constant/ Solution
I	0	s_1	-1	2	1	0	0	4
	0	s_2	3	2	0	1	0	14
	0	s_3	1	-1	0	0	1	3
		$\bar{c_j}$ row	3	2	0	0	0	Z=0
II	0	s_1	0	1	1	0	1	7
	0	s_2	0	5	0	1	-3	5
	3	x_1	1	-1	0	0	1	3
		$\bar{c_j}$ row	0	5	0	0	-3	Z=9
III	0	s_1	0	0	1	-1/5	8/5	6
	2	x_2	0	1	0	1/5	-3/5	1
	3	x_1	1	0	0	1/5	2/5	4
		$\bar{c_j}$ row	0	0	0	-1	0	Z=14
IV	0	s_3	0	0	5/8	-1/8	1	15/4
	2	x_2	0	1	3/8	1/8	0	13/4
	3	x_1	1	0	-1/4	1/4	0	5/2
		$\bar{c_j}$ row	0	0	0	-1	0	Z=14

In tableau-I, the non-basic variable x_1 has the largest relative profit in the $\bar{C}_j$-row. Hence x_1 enters into basic. Considering 1st (the column corresponding x_1) as the *pivot column*, we take the ratios of the positive entries to the constant column (otherwise set to ∞). The ratios are 14/3, 3/1. By the minimum ratio rule (here, minimum ratio is 3), 3rd row is the *pivot row* and $a_{31} = 1$ is the *pivot element*. The pivot element $a_{31} = 1$ in the tab-I denotes the non-basic variable x_1 enters into the basis and the basic variable s_3 leaves the basis in tab-II. By using the pivot operations, we prepare tab-II.

Since tab-II is not optimal (because $\bar{C}_2 > 0$), proceeding the above way in tab-II, the column under x_2 is pivot column and by the minimum ratio rule x_2 enters into the basis and s_2 leaves the basis. By using the pivot operations, we prepare tab-III.

In tab-III, none of coefficients in $\bar{C}_j$-row are positive, so tab-III is optimal. Thus, the solution $(x_1, x_2) = (4, 1)$ is the optimal solution. The optimal value is $Z_{\max} = 14$.

In tab-III, the non-basic variable s_3, has a relative profit of zero. This means that any increase in s_3 will produce no change in the objective function value. In other words, s_3 can be made a basic variable and the resulting basic feasible solution will also have Z is 14. To find the other solution, consider the pivot column under s_3. By the minimum ratio rule 1st row is the pivot row and $a_{15} = \frac{8}{5}$ is the pivot element in tab-III. Since s_3 us a basic variable, enters into the basis and s_1 leaves the basis. Using the pivot operation we form the table-IV. Tab-IV is optimal and the optimal solution is $(x_1, x_2) = \left(\frac{5}{2}, \frac{13}{4}\right)$. The optimal value is $Z_{\max} = 14$.

Now, we know that the set of all feasible solution is convex, it follows from the optimal solution that all points on the line joining the points $(4, 1)$ and $\left(\frac{5}{2}, \frac{13}{4}\right)$ are optimal solution of the L.P.P. with optimal solution is

$$\left\{(x_1, x_2) = \lambda(4, 1) + (1 - \lambda)\left(\frac{5}{2}, \frac{13}{4}\right),\ 0 \leq \lambda \leq 1\right\}$$

Chapter 5

Duality

The term 'duality' implies that every linear programming problem (LPP) whether if maximization or minimization has associated with it another linear programming problem based on the same data. The original (given) problem is called the **primal problem**, while the other is called its **dual problem**.

A solution to the dual linear programming problem may be found in a manner similar to that used for the primal. The two problems have very closely related properties so that optimum solution of the dual gives the complete information about the optimal solution of the primal and vice versa.

5.1 An Algorithm for primal-dual constructions

To construct a dual linear programming problem from a primal (given) LPP, we have to follow the following steps:

Step 1: Express the primal problem into its standard form.
(All primal constrains are equations with non-negative R.H.S., and all primal variables are non-negative)

Step 2: Identify the variables to be used in the dual problem.
The number of new variables required in the dual problem equals the number of Constraints in the primal.

Step 3: Objective function for the dual problem:
Using the right-hand side values of the primal constraints write down the objective function of the dual.
If the primal problem is of maximization (minimization), the dual will be minimization (maximization) problem.

Step 4: Constraints for the dual problem:
Using the dual variables identified in *step 2*, write the constraints for the dual problem.

(i) If the primal is a maximization problem, the constraints in the dual must be all of $\geq$ type. On the other hand, if the primal is a minimization problem, the constraints in the dual must be $\leq$ type. [Note to self check these signs]
(ii) The row coefficients of the primal constraints become the column coefficients of the dual constraints.
(iii) The coefficients of the primal objective function become the R.H.S. of the dual constrains set.
(iv) The dual variables are defined to be unrestricted.

Step 5: Making use of *steps 3* and *4*, write the dual problem. This is the required dual problem of the given LPP.

Remark. The dual constraint associated with an artificial variable in the standard form is always redundant, hence it is never necessary to consider the dual constrains associated with an artificial variable.

Note. The rules for determining the sense of optimization, the type of the constraint, and the sign of the variables in the dual problem are summarized in the following table:

Standard primal Objective	**Dual Problem** Objective	Constraints Type	Variable sign
Maximization	Minimization	$\geq$	unrestricted
Minimization	Maximization	$\leq$	unrestricted

The characteristics of the primal-dual construction relationship may be summed as below

Primal problem	Dual problem
Minimize $Z = 12x_1 + 20x_2$	Maximize $W = 100y_1 + 120y_2$
Subject to	Subject to
$6x_1 + 8x_2 \geq 100$	$6y_1 + 7y_2 \leq 12$
$7x_1 + 12x_2 \geq 120$	$8y_1 + 12y_2 \leq 20$
$x_1,\, x_2 \geq 0$	$y_1,\, y_2 \geq 0$

In details:
Standard form of the given L.P.P. is

$$\begin{aligned}
\text{minimize} \quad & Z = 12x_1 + 20x_2 + 0x_3 + 0x_4 \\
\text{subject to} \quad & 6x_1 + 8x_2 - x_3 = 100 \\
& 7x_1 + 12x_2 - x_4 = 120 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Notice that x_3 and x_4 are surplus variables in the 1st and 2nd constraints respectively; hence both have zero coefficient in the objective function. Also, the coefficients of x_4 and x_3 will be zero in the first and second constraints respectively.

Chapter 6

Dual-Simplex Method

Problem 6.0.1. Solve the LPP by the dual-simplex method:

$$\begin{aligned} \text{maximize} \quad & Z = -(2x_1 + x_2 + x_3) \\ \text{subject to} \quad & 4x_1 + 6x_2 + 3x_3 \leq 8 \\ & x_1 - 9x_2 + x_3 \leq -3 \\ & 2x_1 + 3x_2 - 5x_3 \geq 4 \\ & x_1,\, x_2,\, x_3 \geq 0 \end{aligned}$$

Solution. Here, the 3rd constraint is of $\geq$ type. To convert into $\leq$ type, let us multiply it by -1. Adding s_1, s_2, s_3 as the slack variables to the 1st, 2nd and 3rd constraints respectively, the given LPP is expressed as

$$\begin{aligned} \text{maximize} \quad & Z = -(2x_1 + x_2 + x_3) \\ \text{subject to} \quad & 4x_1 + 6x_2 + 3x_3 + s_1 = 8 \\ & x_1 - 9x_2 + x_3 + s_2 = -3 \\ & -2x_1 - 3x_2 + 5x_3 + s_3 = -4 \\ & x_1,\, x_2,\, x_3,\, s_1,\, s_2 \geq 0 \end{aligned}$$

Putting $x_1 = x_2 = x_3 = 0$, the initial basic solution is $s_1 = 8$, $x_2 = -3$, $s_3 = -4$ which is infeasible. The above information is expressed in tab-I called staring dual simplex table.

Tab	C_B	$c_j \rightarrow$ / Basis	-2	-1	-1	0	0	0	Constant/
			x_1	x_2	x_3	s_1	s_2	s_3	Solution
	0	s_1	4	6	3	1	0	0	8
	0	s_2	-1	-9	1	0	1	0	-3
I	0	s_3	-2	-3	5	0	0	1	-4
		$\bar{c}_j$ row	-2	-1	-1	0	0	0	$Z = 0$
	0	s_1	0	0	13	1	0	2	0
	0	s_2	7	0	-14	0	1	-3	9
II	-1	x_2	$2/3$	1	$-5/3$	0	0	$-1/3$	$4/3$
		$\bar{c}_j$ row	$-4/3$	0	$-8/3$	0	0	$-1/3$	$Z = -4/3$

In these tableaux, basis refers to the basic variables in the basic solution. The values of the basic variables are given under the column solutions. Here, c_j denotes the coefficients of the variables in the objective function. C_B denotes the coefficient of the basic variables only and $\bar{c}_j$ denotes the relative cost coefficients of the variables which is given by

$$\begin{aligned} \bar{C}_j &= C_j - [\text{inner product of } C_B \text{ and the column corresponding the } j\text{−th variable in the canonical system}] \\ &= C_j - C_B A_j, \text{where } A_j \text{ is the } j\text{−th column of the matrix } A = (a_{ij}), \text{ which is found by the coefficients of the} \\ &\quad \text{basis and non-basis variables of constraints equation.} \end{aligned}$$

In tableau-I, all $\bar{c}_j$ row entry are either negative of zero and the 'solution' column shows that s_2 and s_3 are negative; this solution is optimal but infeasible.
Since the basic variable s_3 has the most negative value, so it will be chosen to leave the basis. Since the variable x_1 and x_2 have negative coefficient in row-3, so we take the ratios of these with corresponding relative cost row in $\bar{c}_j$, which are $\frac{-2}{-2}$, $\frac{-1}{-3}$. $\left(\frac{-2}{-2} \text{ i.e., } 1 \text{ and } \frac{-1}{-3} \text{ i.e., } \frac{1}{3}\right)$

The minimum ratio occurs corresponding to the non-basic variable x_2. Thus, s_3 will be replaced by x_2 in the basis. Hence, the pivot element is $a_{32} = -3$. We construct tab-II, using pivot operation.

Now we see that tableau-II is optimal and the solution is feasible. The optimal solution is

$$x_1 = 0, \qquad x_2 = \frac{40}{3}, \qquad x_3 = 0$$

and the optimum value is $Z = -\frac{4}{3}$.

From tab-II, we see that the none of the non-basic variables has zero relative cost factors in the $\bar{c}_j$ row. So, there is no alternative optimal of the given LPP. Hence, the optimal solution is unique at $(x_1, x_2, x_3) = (0, \frac{4}{3}, 0)$

Problem 6.0.2. Solve by Dual-Simplex method

$$\begin{aligned} \text{minimize} \quad & Z = x_1 + 4x_2 + 3x_4 \\ \text{subject to} \quad & x_1 + 2x_2 - x_3 + x_4 \geq 3 \\ & -2x_1 - x_2 + 4x_3 + x_4 \geq 2 \\ & x_i \geq 0 \end{aligned}$$

Solution. Multiplying both constraints by -1 and then adding x_5 and x_6 as the slack variables to the 1st and 2nd constraints respectively, we get

$$\begin{aligned} \text{minimize} \quad & Z = x_1 + 4x_2 + 3x_4 \\ \text{subject to} \quad & -x_1 - 2x_2 + x_3 - x_4 + x_5 = -3 \\ & 2x_1 + x_2 - 4x_3 - x_4 + x_6 = -2 \\ & x_i \geq 0 \end{aligned}$$

Putting $x_1 = x_2 = x_3 = x_4 = 0$, the initial basic solution is $x_5 = -3$, $x_6 = -2$; which is infeasible. The above information is expressed in tab-I called starting dual simplex table.

Tab	C_B	$c_j \rightarrow$ / Basis	1	4	0	3	0	0	Constant/
			x_1	x_2	x_3	x_4	x_5	x_6	Solution
I	0	x_5	-1	-2	1	-1	1	0	-3
	0	x_6	2	1	-4	-1	0	1	-2
		$\bar{c}_j$ row	1	4	0	3	0	0	$Z = 0$
II	1	x_1	1	2	-1	1	-1	0	3
	0	x_6	0	-3	-2	-3	2	1	-8
		$\bar{c}_j$ row	0	2	1	2	1	0	$Z = 3$
III	1	x_1	1	$7/2$	0	$5/2$	-2	$-1/2$	7
	0	x_3	0	$3/2$	1	$3/2$	-1	$-1/2$	4
		$\bar{c}_j$ row	0	$1/2$	0	$1/2$	2	$1/2$	$Z = 7$

In these tableaux, basis refers to the basic variables in the basic solution. The values of the basic variables are given under the column solutions. Here, c_j denotes the coefficients of the variables in the objective function. C_B denotes the coefficient of the basic variables only and $\bar{c}_j$ denotes the relative cost coefficients of the variables which is given by

$$\begin{aligned} \bar{C}_j &= C_j - [\text{inner product of } C_B \text{ and the column corresponding the } j\text{-th variable in the canonical system}] \\ &= C_j - C_B A_j, \text{where } A_j \text{ is the } j\text{-th column of the matrix } A = (a_{ij}), \text{ which is found by the coefficients of the} \\ &\quad \text{basis and non-basis variables of constraints equation.} \end{aligned}$$

In tab-I, the basic solution is given by $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = -3$, $x_6 = -2$. This is infeasible though it satisfies the optimality condition.

Since the basic variable x_5 has the most negative value, so it will be chosen to leave the basis. Since the variable x_1, x_2 and x_4 have negative coefficient in row-1, so we take the ratios of these with corresponding relative cost row in $\bar{c}_j$, which are

$$\left|\frac{1}{-1}\right| = 1, \qquad \left|\frac{4}{-2}\right| = 2, \qquad \left|\frac{3}{-1}\right| = 3$$

The minimum ratio occurs corresponding to the non-basic variable x_1. Thus, x_1 will be replaced by x_5 in the basis. Hence, the pivot element is $a_{11} = -1$. We construct tab-II, using pivot operation.

Tab-II is optimal but the basic variable x_6 has negative value. So, x_6 will leave the basis. By the same procedure, we see that x_3 will replace x_6 in the basis; we construct tab-III.

Tab-III is optimal and the solution is feasible. The optimal solution is

$$x_1 = 7, \qquad x_2 = 0, \qquad x_3 = 4, \qquad x_4 = 0$$

and the optimum value is $Z = 7$.

Since in $\bar{c}_j$ row there is no zero values corresponding to non-basic variable, hence the solution is unique.

Chapter 7

Transportation Problem

The transportation model is a special class of the linear programming problem. It deals with the situation in which a commodity is shipped from *sources* (e.g., factories) to *destinations* (e.g., warehouses). The objective is to determine the amounts shipped from each sources to each destination that minimize the total shipping cost while satisfying both the supply limits and the demand requirements. The model assumes that the shipping cost on a given route is directly proportional to the number of units shipped on that route.

Suppose that there are m sources and n destinations. Let a_i be the number of supply units available at source $i\,(i=1,2,\ldots,m)$ and let b_j be the number of demand units required at destination $j\,(j=1,2,\ldots,n)$. Let c_{ij} represent the unit transportation cost for transporting the units from source i to destination j. Then, if x_{ij} $(x_{ij}>0)$ is the number of units shipped from source i to destination j, the problem is to determine the Transportation schedule so as to minimize the total transportation cost satisfying the supply and demand conditions.

Mathematically, the problem may be stated as follows:

$$\begin{aligned}
\text{minimize} \quad & Z=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to} \quad & x_{i1}+x_{i2}+\cdots+x_{in}=a_i; \quad i=1,2,\ldots,m \quad \Rightarrow \sum_{j=1}^{n} x_{ij}=a_i \\
& x_{1j}+x_{2j}+\cdots+x_{mj}=b_i; \quad j=1,2,\ldots,n \quad \Rightarrow \sum_{i=1}^{m} x_{ij}=b_j \\
& x_{ij}\geq 0; \quad \text{for all } i \text{ and } j
\end{aligned}$$

For a feasible solution to exist, it is necessary that total supply equals total requirement i.e.,

$$\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$$

This restriction causes one of the constraints to be redundant (and hence it can be deleted) so that the problem will have $(m+n-1)$ independent constraints and $(m\times n)$ unknowns.

Note. The standard transportation problem has $(m+n)$ constraint, mn variables. In general, the number of basic variables in a basic feasible solution is given by the number of constraints. Nut in the TP, the number of variables that can take positive values is limited $(m+n-1)$, since

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}=\sum_{i=1}^{m} a_i \qquad \text{and} \qquad \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}=\sum_{j=1}^{n} b_j$$

Also, we have that $\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$ and hence, the transportation model have only $(m+n-1)$ independent constraints.

7.1 Starting Basic Feasible Solution

To solve a standard TP, we shall use the following methods (for starting basic feasible solution):

1. North-west corner rule (NCR)
2. Least-cost rule (LCR)
3. Vogel's approximation rule/method (VAM)

Problem 7.1.1. Consider a transportation problem with 3 warehouses and 4 markets. The warehouse capacities are $a_1=3,\ a_2=7,\ a_3=5$. The market demands are $b_1=4,\ b_2=3,\ b_4=4$. The unit cost of shipping is given by the following table:

Markets

Warehouse	M_1	M_2	M_3	M_4	Supply
W_1	2	2	2	1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
Demand	4	3	4	4	15

15 → Supply=Demand
∴ Standard TP

Solution. In this transportation problem,

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 15$$

So, it is a standard TP.
The transportation table is as follows:

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	$\cancel{3}$
W_2	1 $\boxed{10}$	3 $\boxed{8}$	3 $\boxed{5}$	$\boxed{4}$	$\cancel{7}\ \cancel{6}\ \cancel{3}$
W_3	$\boxed{7}$	$\boxed{6}$	1 $\boxed{6}$	4 $\boxed{8}$	$\cancel{5}\ \cancel{4}$
Demand	$\cancel{4}\ \cancel{1}$	$\cancel{3}$	$\cancel{4}\ \cancel{1}$	4	

Here, we shall refrain from writing the variable names x_{ij}.
A basic feasible solution to this problem will have at most $(3 + 4 - 1) = 6$ positive variables.

7.1.1 North-west corner Rule

This rule generates a feasible solution with no more than $(m + n - 1)$ basic variables.

Here x_{11} is selected as the 1st basic variable and is assigned a value as much as possible consistent with the supply and demand restriction.
Set $x_{11} = \min\{3, 4\} = 3$. We set the variable $x_{12} = x_{13} = x_{14} = 0$. The remaining demand of the market M_1 is 1 unit. The next variable x_{21} and $x_{21} = \min\{7, 1\} = 1$.
Similarly, $x_{22} = \min\{6, 3\} = 3$
In the same manner, finally we obtain the table-2.

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	3
W_2	1 $\boxed{10}$	3 $\boxed{8}$	3 $\boxed{5}$	$\boxed{4}$	7
W_3	$\boxed{7}$	$\boxed{6}$	1 $\boxed{6}$	4 $\boxed{8}$	5
Demand	4	3	4	4	

Table 7.1: tab-2

Since we have six positive variables in the solution, so for the initial basic feasible solution, the total cost is given by

$$Z = 3 \cdot 2 + 1 \cdot 10 + 3 \cdot 8 + 3 \cdot 5 + 1 \cdot 6 + 4 \cdot 8 = 93$$

7.1.2 Least Cost Rule (LCR)

In this rule, the variable with the least shipping cost will be chosen as the basic variable.

According to this rule, in our present problem x_{14} must be chosen as the 1st basic variable, and $x_{14} = \min\{3, 4\} = 3$.

	M_1	M_2	M_3	M_4	Supply
W_1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	3 $\boxed{1}$	$\cancel{3}$
W_2	2 $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	1 $\boxed{4}$	$\cancel{7}$ $\cancel{6}$ $\cancel{2}$
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	$\cancel{5}$ $\cancel{2}$
Demand	$\cancel{4}$ $\cancel{2}$	$\cancel{3}$	$\cancel{4}$	$\cancel{4}$ $\cancel{1}$	

Now, out of the remaining unassigned cells, the variable x_{24} has the least cost and $x_{24} = \min\{7, 1\} = 1$. In this way, finally, we obtain the following table which gives the initial basic feasible solution.

	M_1	M_2	M_3	M_4	Supply
W_1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	3 $\boxed{1}$	3
W_2	2 $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	1 $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

The total cost of transportation for the above basic feasible solution is

$$Z = 3 \cdot 1 + 2 \cdot 10 + 4 \cdot 5 + 1 \cdot 4 + 2 \cdot 7 + 3 \cdot 6 = 79$$

7.1.3 Vogel's Approximation Method (VAM)

In Vogel's approximation method, we compute a penalty for each row and column. Vogel defined the penalty as the absolute difference between the smallest and the next smallest cost in a row or a column. In two or more cells lie for the minimum cost, then the penalty is set to zero.

	M_1	M_2	M_3	M_4	Supply	Penalty 1st	2nd	3rd	4th
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	$\cancel{3}$	$(2-1)=1$			
W_2	$\boxed{10}$	$\boxed{8}$	3 $\boxed{5}$	4 $\boxed{4}$	$\cancel{7}$ $\cancel{3}$	1	1	3	3
W_3	1 $\boxed{7}$	3 $\boxed{6}$	1 $\boxed{6}$	$\boxed{8}$	$\cancel{5}$ $\cancel{4}$ $\cancel{3}$	0	0	0	0
Demand	$\cancel{4}$ $\cancel{1}$	3	$\cancel{4}$ $\cancel{1}$	$\cancel{4}$					
1st Penalty	$7-2=5$	4	3	3					
2nd Penalty	3	2	1	4					
3rd Penalty	3	2	1						
4th Penalty		2	1						

Now, the row or column with the largest penalty is identified and the variable that has the smallest cost in that row or column is selected as the basic variable.

In this problem, column 1 has the largest penalty and so, the variable x_{11} will be selected as the 1st basic variable. We set $x + 11 = \min\{3, 4\} = 3$.

The 2nd set of penalties are shown in table. Applying the same procedure, the variable x_{24} is selected as basic variable and $x_{24} = \min\{7, 4\} = 4$.

Similarly, we take the 3rd penalty and so on. Finally, the is as follows:

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	3 $\boxed{5}$	4 $\boxed{4}$	7
W_3	1 $\boxed{7}$	3 $\boxed{6}$	1 $\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

The total cost of shipping is

$$Z = 3 \cdot 2 + 3 \cdot 5 + 4 \cdot 4 + 1 \cdot 7 + 3 \cdot 6 + 1 \cdot 6 = 68$$

Note. From the starting basic solution, we observe that the best starting solution is obtained by VAM having the least transportation cost of 68.

Remark. Vogel's approximation method (VAM) yields a very good initial solution, which, sometimes may be the optimal solution.

7.2 Improving the initial basic feasible solution

7.2.1 U-V method or MODI (Modified Distribution Method)

For any basic feasible solution find numbers u_i for warehouses i and market j such that

$$u_i + v_j = c_{ij} \qquad \text{for every basic } x_{ij}$$

These number can be positive, negative or zero. Then $\bar{c_{ij}} = c_{ij} - (u_i + v_j)$ for all non-basic variables x_{ij}

If all $\bar{c_{ij}}$ are non-negative (cost T.P.) then the current basic feasible solution is optimal. If not, there exists a non-basic variable x_{pq} such that $\bar{c_{pq}} = \min\{\bar{c_{ij}} < 0\}$ and x_{pq} is made as basic variable to improve the objective function.

	M_1	M_2	M_3	M_4	Supply
W_1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	3 $\boxed{1}$	3
W_2	2 $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	1 $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Table 7.2: Tab-1

To apply U-V method, we have to compute the numbers u_1, u_2, u_3, v_1, v_2, v_3, v_4 from the initial basic feasible solution table of LCR.

$$\begin{aligned}
u_1 + v_4 &= 1 \\
u_2 + v_1 &= 10 \\
u_2 + v_3 &= 5 \\
u_2 + v_4 &= 4 \\
u_3 + v_1 &= 7 \\
u_3 + v_2 &= 6
\end{aligned}$$

Let us suppose that

$$\begin{aligned}
u_1 &= 0 &&\Rightarrow v_4 = 1 \\
u_2 &= 3 &&\Rightarrow v_1 = 7 \\
u_2 &= 3 &&\Rightarrow v_3 = 2 \\
u_2 &= 0 &&\Rightarrow v_2 = 6
\end{aligned}$$

Now,

$$\begin{aligned}
\bar{c_{11}} &= c_{11} - (u_1 + v_1) = 2 - 7 = -5 \\
\bar{c_{12}} &= c_{12} - (u_1 + v_2) = 2 - (0 + 6) = -4 \\
\bar{c_{13}} &= c_{13} - (u_1 + v_3) = 2 - (0 + 2) = 0 \\
\bar{c_{22}} &= c_{22} - (u_2 + v_2) = -1 \\
\bar{c_{33}} &= c_{33} - (u_3 + v_3) = -4 \\
\bar{c_{34}} &= c_{34} - (u_3 + v_4) = 7
\end{aligned}$$

Since $\min\{\bar{c_{ij}} < 0\} = -5$. So, the non-basic variable x_{11} is introduced into the basis.

To determine the maximum increase in x_{11}. We assign an unknown non-negative value θ. We add or subtract θ from the basic variables so that the row sums and column sums are equal to the corresponding supply and demand respectively, which is shown in the table 2.

	M_1	M_2	M_3	M_4	Supply
W_1	θ $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$3-\theta$ $\boxed{1}$	3
W_2	$2-\theta$ $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	$1+\theta$ $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Now θ is increased as long as the solution remains non-negative. In this problem, the maximum value of θ is 2, and the basic variable x_{21} is removed from the basis, i.e., x_{21} is replaced by x_{11}. The new basic feasible solution is given by the following table.

	M_1	M_2	M_3	M_4	Supply
W_1	2 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	1 $\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	3 $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Now, we proceed as before to find the relative cost coefficients of non-basic variables.

$$\begin{aligned} u_1 + v_1 &= 2 \\ u_1 + v_4 &= 1 \\ u_2 + v_3 &= 5 \\ u_2 + v_4 &= 4 \\ u_3 + v_1 &= 7 \\ u_3 + v_2 &= 6 \end{aligned}$$

Let

$$\begin{aligned} u_1 = 0 \quad & \Rightarrow v_1 = 2 \\ & \Rightarrow v_4 = 1 \\ u_2 = 3 \quad & \Rightarrow v_3 = 2 \\ u_2 = 5 \quad & \Rightarrow v_2 = 1 \end{aligned}$$

Now,

$$\begin{aligned} \bar{c_{12}} &= 2 - (0 + 1) = 1 \\ \bar{c_{13}} &= 2 - (0 + 2) = 0 \\ \bar{c_{21}} &= 10 - (3 + 2) = 5 \\ \bar{c_{22}} &= 4 \\ \bar{c_{33}} &= -1 \\ \bar{c_{34}} &= 2 \end{aligned}$$

Since the relative cost-coefficients of non-basic variable is x_{33} is -1. Hence, x_{33} is introduced as a basic variable at a non-negative θ. This produces the following change of the variables:

	M_1	M_2	M_3	M_4	Supply
W_1	$2+\theta$ $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$1-\theta$ $\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	$4-\theta$ $\boxed{5}$	$3+\theta$ $\boxed{4}$	7
W_3	$2-\theta$ $\boxed{7}$	3 $\boxed{6}$	θ $\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Since the maximum value of θ is 1 and x_{33} replace x_{14}.
The new basic feasible solution is given by the following table.

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	3 $\boxed{5}$	4 $\boxed{4}$	7
W_3	$\boxed{7}$	3 $\boxed{6}$	1 $\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Now proceeding as above, we get

$$\begin{aligned} u_1 + v_1 &= 2 \\ u_2 + v_3 &= 5 \\ u_2 + v_4 &= 4 \\ u_3 + v_1 &= 7 \\ u_3 + v_2 &= 6 \\ u_3 + v_3 &= 6 \end{aligned}$$

Let

$$\begin{aligned} u_1 = 0 \quad &\Rightarrow v_1 = 2 \\ u_2 = 4 &\Rightarrow v_2 = 1 \\ &\Rightarrow v_3 = 1 \\ u_3 = 5 \quad &\Rightarrow v_4 = 0 \end{aligned}$$

Now,

$$\begin{aligned} \bar{c_{12}} &= 1 \\ \bar{c_{13}} &= 1 \\ \bar{c_{14}} &= 1 \\ \bar{c_{21}} &= 4 \\ \bar{c_{22}} &= 3 \\ \bar{c_{34}} &= 3 \end{aligned}$$

Since all $\bar{c_{ij}} > 0$ for all non-basic variables. Thus, the above table represents unique optimal solution. The optimal shipping schedule is to ship

3 units from warehouse w_1 to market M_1
3 units from warehouse w_2 to market M_3
4 units from warehouse w_2 to market M_4
1 units from warehouse w_3 to market M_1
3 units from warehouse w_3 to market M_2
1 units from warehouse w_3 to market M_3

The least cost of shipping is

$$Z = 3 \cdot 2 + 3 \cdot 5 + 4 \cdot 4 + 1 \cdot 7 + 3 \cdot 6 + 1 \cdot 6 = 68 \text{ units}$$

Chapter 8

Game Theory

It was in 1928 when **Von Neumann** (called the father of game theory) developed the theory of games. After 1944, when **Von Neumann and Morgenstern** published their work named "Theory of Games and Economic Behaviour", the theory received the proper attention. The theory of games (or game theory) deals with mathematical analysis of competitive problems and is based on the **minimax** principle put forward by Von Neumann which implies that each competitor will act so as to minimize his maximum loss (or maximize his minimum gain).
This theory does not describe how a game should be played. It describes only the procedures and principles by which plays should be selected. It is, therefore, a decision theory applicable to competitive situations.

Definition 1 (Game & Player)**.** A conflicting or competitive situation involving two or more participants is said to be a ***game***.
The participants are called ***players***. The player may be an individual, a group of individuals, an organization or the nature.

Definition 2 (Strategy)**.** The decision rule by which a player determines his course of action(moves) is called a ***strategy***. To reach the decision regarding which strategy to use, neither player needs to know the other's strategy.

8.1 Characteristics of a competitive game

A competitive game has the following characteristics:

(i) There are a *finite* number of participants. The number of participants is $n \geq 2$, If $n = 2$, the game is called a ***two-person game***; if $n > 2$, it is called $n-$person game.

(ii) Each participant has a finite number of possible courses of action.

(iii) Each participant must know all the course of action available to others, but must not know which of these will be chosen.

(iv) A play of the game is said to occur when each player chooses one of his course of action. The choices are assumed to be made simultaneously, so that no participant knows the choice of other until he has decided his own.

(v) The outcome of the game is affected by the choices made by the competitors.

(vi) All combinations of courses of action chosen by various players always result in some outcome of the game denoting gain (or loss) of an individual player.
This outcome can be represented by a single **pay off** number that can be zero (no gain - no loss), positive (gain) or negative (loss).

8.2 Payoff

The ***payoff*** is a connecting link between the sets of strategies open to all the players. In other words, payoff is the outcome of playing the game.

Suppose at the end of a play of a game, player $p_i (i = 1, 2, \dots, n)$ is expected to obtain an amount v_i called the payoff or return to the player p_i. So, the total payoff to all the players in a play of the game is equal to $\sum_{i=1}^{n} v_i$.

The game is called a **zero-sum game** if $\sum_{i=1}^{n} v_i = 0$, at each play of the game. Thus, in a zero-sum game, the players make payments only to each other, i.e., the loss of one is the gain of others, and nothing comes from outside.
A game with two players, where a gain of one player equals the loss of other is known as **two-person zero-sum game**. If there are n players and the sum of game is zero, then it is called **n**$-$***person/players zero-sum game.***

8.2.1 Payoff matrix

In a two-person zero-sum game, the loss (gain) of one player is exactly equal to the gain (loss) of the opponent and each player knows the outcome for all possible strategies that he and his opponent may use during a play of the game. The resulting outcomes, representing gain (or loss) to a particular player, can be conveniently displayed in the form of a payoff matrix $A = \left(a_{ij}\right)$, where a_{ij} is the payoff to player-I (say), when he employs his $i-$th move while player-II (the opponent) employs his $j-$th move.
For a given payoff matrix, we adopt the following:

(i) Row designations are the courses of action available to player-I, who will be called the *row player.*

(ii) Column designations are the courses of action available to player-II, who will be called the *column player.*

(iii) The various payoffs are the payoff to the row player.

Thus, if player-I has m strategies (moves) available to him and player-II has n moves available to him, then the payoffs for various strategy combinations nay be represented by an $m \times n$ payoff matrix $\left(a_{ij}\right)$. For this reason, the two-person zero-sum games are also called **matrix games**.

To illustrate, suppose a two-person zero-sum game has two players. A and B with respective available strategies A_1, A_2 and B_1, B_2, B_3 respectively. Let the payoffs to the player A be expressed in terms of gains to him. Let the payoffs to player A be given by the following 2×3 payoff matrix:

$$
\begin{array}{ccc}
 & & \text{Player } B \\
 & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\
\text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \left[\begin{array}{ccc} -1 & 3 & 0 \\ 2 & -4 & 1 \end{array}\right] \\
 & & A\text{'s payoff matrix}
\end{array}
$$

Then the following explanation may be given for the various payoffs:

Strategy of A	Strategy of B B_1	B_2	B_3
A_1	A losses 1 unit	A gains 3 units	None gains
A_2	A gains 2 units	A losses 4 units	A gains 1 unit

Since the game is zero-sum, every gain of player A is an equal loss of player B and vice-versa. Thus, the payoff matrix for the player B will be just the negative of the payoff matrix of A, so that the sum of the payoff matrices of the two players is ultimately a **null matrix**.

Remark. Though the payoff matrix for the column player can be obtained just by negating the payoff matrix of the row player, we hardly need to do so. This is so because the optimal course of action for the column player can be determined from the payoff matrix of the row player alone.

Summary: A **payoff matrix** (also known as a **gain matrix** or a **game matrix**) is a table showing the amounts received by the player named at the left-hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

8.3 Pure and Mixed Strategy

If a player decides to use only one particular course of action during every play, then it is said to use ***pure strategy***. In other words, a decision to play a certain row (column) with probability zero, is called pure strategy for player-I (player-II). Otherwise, the strategy is called *mixed strategy* i.e., if a player decides in advance, to use all or some of his available courses of action in some fixed proportion, called to use ***mixed strategy***. Thus, a mixed strategy is a selection among pure strategies with some fixed probabilities (proportions).

A mixed strategy if a player with possible course of action is denoted by a set $\underline{X}$of m non-negative numbers. The sum of these numbers is unity and each number represents the probability with which each course of action is chosen. Thus, if x_i is the probability of choosing the course of action i, we have

$$\underline{X} = (x_1, x_2, \ldots, x_m) \qquad \text{where } \sum_{i=1}^{m} x_i = 1 \quad \text{and } x_i \geq 1;\ i = 1, 2, \ldots, m$$

A player may be able to choose only m pure strategies, but he has an infinite number of mixed strategies to choose them.

8.4 Maximin and minimax principle (Lower and upper bound)

The selection of an optimal strategy by each player, without the knowledge of the competitor's strategy, is the basic problem of playing games. So, the objective of the study is to know how these must select their respective strategies so that they are able to optimize their payoff. Such a decision-making criterion is referred to as the **minimax-maximin principle**. Such principle in pure strategies game always leads to the best possible selection of a strategy for both players.

Maximin principle: In a two-person zero-sum game, player-I (row player) examines each row in the payoff matrix and selects the minimum element in each row, say p_{ij} with $i = 1, 2, \ldots, m$. He then selects the maximum of these minimum elements, say p_{rs}. Mathematically,

$$p_{rs} = \max_i \left[\min_j p_{ij}\right]$$

The element p_{rs} is called the **maximin (lower bound)** of the game, and decision to play row r is called the *maximin pure strategy* (or *maximin principle*).

Minimax principle: Likewise, player-II (column player) examines each column in the payoff matrix to determine the maximum loss. Since player-II wants to minimize his losses, he then selects the strategy that gives the minimum loss among the column maximum values. Mathematically, this can be expressed as

$$p_{tu} = \min_j \left[\max_i p_{ij}\right]$$

The element p_{tu} is called the **minimax (upper bound)** of the game, and decision to play column u is called the *minimax pure strategy* (or *minimax principle*).

8.5 Value of the game

This is the expected payoff at the end of the game, when each player uses his optimal strategy.
The value of the game v, in general, satisfies the equation:

$$\text{maximin value} \leq v \leq \text{minimax value}$$

If the maximin value is equal to the minimax value, then the game is said to have a **saddle (equilibrium) point** and the corresponding strategies are called ***optimal strategies***. The amount of payoff at an equilibrium point is the ***value of the game***.

- A game is said to be ***fair*** game if maximin=minimax=0, and is said to be ***strictly determinable*** if maximin=minimax≠0. In words, a game is said to be ***fair*** game if the lower (maximin) and upper (minimax) values of the game are equal and both equals zero.

Definition 3 (Saddle point and Value of the game)**.** A *saddle point* (or *equilibrium point*) of a payoff matrix is that position in the payoff matrix where the maximum of row minima coincides with the minimum of the column maxima. The payoff at the saddle point is called the *value of the game* and is obviously equal to the maximin and minimax values of the game. Thus, (k, r)th position of the payoff matrix (a_{ij}) will be a saddle point iff

$$a_{kr} = \max_i \left[\min_j \{a_{ij}\}\right] = \min_j \left[\max_i \{a_{ij}\}\right]$$

Remark.

- The saddle point, and hence the value of the game, need not be unique.
- We shall denote a value of the game by v.
- The importance of the saddle point arises from the fact that, in general, the optimum play consists in sticking to the strategies which correspond to the saddle point. To solve a game, we therefore merely need to look for the saddle point of the payoff matrix. If it exists, the game is solved. But, unfortunately, most payoff matrices do not possess any saddle point. A game with no saddle point is solved by employing mixed strategies.
- By solving a game we mean determining the optimal strategies for both the players and the value of the game.

We summarize below the steps required to detect a saddle point:

(i) Select the minimum (lowest) element in each row of the payoff matrix and write them under 'row minima' heading at the right of the corresponding row. Then, ring the largest of them.

(ii) Select the maximum (largest) element in each column of the payoff matrix and write them under 'column maxima' heading at the bottom of the corresponding column. Then, ring the smallest of them.

(iii) If these two ringed elements are **same**, the cell where the corresponding row and column meet is a ***saddle point*** and the element in that cell is the *value of the game.*
If the two ringed elements are **unequal**, there is no saddle point, and the value of game lies between these two values.

(iv) If there are more than one saddle points, then there will be more than one solution, each solution corresponding to each saddle point.

8.6 Rules for Game Theory

8.6.1 Rule 1. Pure strategy (Look for a saddle point)

In a certain game, player A has three possible choices L, M and N, while player B has two possible choices P and Q. Payments are to be made according to the choices made

Choices	Payment
L, P	A pays B Tk. 3
L, Q	B pays A Tk. 3
M, P	A pays B Tk. 2
M, Q	B pays A Tk. 4
N, P	B pays A Tk. 2
N, Q	B pays A Tk. 3

What are the best strategies for player A and B in this game? What is the value of the game for A and B? Is this game (i) fair? (ii) strictly determinable?

Solution. Let (+)ve number represent a payment from B to A and (−)ve number a payment from A to B. We have the payoff matrix shown below:

$$\begin{array}{cc}
 & \text{Player } B \\
 & \begin{array}{cc} P & Q \end{array} \quad \text{Row Minima:} \\
\text{Player } A \begin{array}{c} L \\ M \\ N \end{array} & \begin{bmatrix} -3 & 3 \\ -2 & 4 \\ 2 & 3 \end{bmatrix} \quad \begin{array}{c} -3 \\ -2 \\ (2) \leftarrow \textbf{maximin} \end{array} \\
\text{Column Maxima:} & \begin{array}{cc} (2) & 4 \end{array} \\
 & \uparrow \\
 & \textbf{minimax}
\end{array}$$

Here, maximin=2=minimax
Thus the matrix has a saddle point at position $(3, 2)$. The payoff amount in the saddle point position is 2, which is the *value of the game.*
So, the optimal solution to the game is given by

(i) the best strategy for player A is N;

(ii) the best strategy for player B is p; and

(iii) the value of the game is Tk. 2 for player A, and Tk. −2 for player B.

Since the value of the game is not zero, the game is not fair. The game is strictly determinable.

Let us consider a few more examples of games:

Example.

$$\left.\begin{array}{cc}
 & \text{Player } B \\
 & \qquad\qquad \text{Row Minima:} \\
\text{Player } A & \begin{bmatrix} -4 & 3 \\ -3 & -7 \end{bmatrix} \quad \begin{array}{c} (-4) \\ -7 \end{array} \\
\text{Column Maxima:} & \begin{array}{cc} (-3) & 3 \end{array}
\end{array}\right\} \text{No saddle point exists.}$$

Example.

$$
\begin{array}{c c c}
 & B & \\
A & \begin{bmatrix} 3 & 2 \\ -2 & -3 \\ -4 & -5 \end{bmatrix} & \begin{array}{c} (2) \\ -3 \\ -5 \end{array} \\
 & \begin{array}{cc} 3 & (2) \end{array} &
\end{array}
$$

Saddle point: $(1, 2)$
Strategies : $A \to$ row 1; $B \to$ column 2.
Game value : $+2$

Example.

$$
\begin{array}{c c c}
 & B & \\
A & \begin{bmatrix} 1 & 13 & 11 \\ -9 & 5 & -11 \\ 0 & -3 & 13 \end{bmatrix} & \begin{array}{c} (1) \\ -11 \\ -3 \end{array} \\
 & \begin{array}{ccc} (1) & 13 & 13 \end{array} &
\end{array}
$$

Saddle point: $(1, 2)$
Strategies : $A \to$ row 1; $B \to$ column 2.
Game value : $+2$

Example.

$$
\begin{array}{c c c}
 & B & \\
A & \begin{bmatrix} 16 & 4 & 0 & 14 & -2 \\ 10 & 8 & 6 & 10 & 12 \\ 2 & 6 & 4 & 8 & 14 \\ 8 & 10 & 2 & 2 & 0 \end{bmatrix} & \begin{array}{c} -2 \\ (6) \\ 2 \\ 0 \end{array} \\
 & \begin{array}{ccccc} 16 & 10 & (6) & 14 & 14 \end{array} &
\end{array}
$$

Saddle point: $(2, 3)$
Strategies : $A \to$ row 2; $B \to$ column 3.
Game value : $+6$

Note.

- Always look for a saddle point before attempting to solve a game.
- If there is no saddle point, neither player can optimize his chances by using a pure strategy; they must mix some or all of their course of action, resulting in mixed strategy.

8.6.2 Rule 2. Mixed Strategy: Game without saddle point (For 2×2 games)

There are many solution procedures for determining the optimal strategies and the value of the game. In the most of the situation, the given rectangular game can be reduced to a much smaller 2×2 game. It is, therefore, worthwhile to determine formulae for the optimal strategies and the value of the game in the case of a 2×2 game.
To find the optimum strategies as well as game value for a 2×2 game, we can use **algebraic** and **arithmetic method**.

Algebraic Method

Let us consider a two-person zero-sum game where the optimal strategies are not pure(i.e., mixed) and for which A's payoff matrix is given by

$$
\begin{array}{c c c}
 & & \text{Player } B \\
 & & \begin{array}{cc} \text{I} & \text{II} \end{array} \\
\text{Player } A & \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\end{array}
$$

If (x_1^*, x_2^*) and (y_1^*, y_2^*) are the optimal strategies for A and B respectively, then

$$
x_1^* = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \qquad y_1^* = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}
$$

$$
x_2^* = 1 - x_1^* \qquad y_2^* = 1 - y_1^*
$$

Value of the game, $v = \sum_{i=1}^{2} \sum_{j=1}^{2} x_i^* y_i^* a_{ij}$.

Remark. Here x_1^* and x_2^* are the *probabilities* with which player A chooses his strategies I and II respectively. Also, y_1^* and y_2^* are the *probabilities* with which player B chooses his strategies I and II respectively.

Note.

Value of game = (Expected profits to player A when player B uses strategy I) × Prob. (player B using strategy I)+ (Expected profits to player A when player B uses strategy II) × Prob. (player B using strategy II)

$$= (a_{11}x_1^* + a_{21}x_2^*)y_1^* + (a_{12}x_1^* + a_{22}x_2^*)y_2^*$$

$$\text{i.e.,} v = \sum_{i=1}^{2}\sum_{j=1}^{2} x_i^* y_j^* a_{ij}$$

Arithmetic Method

This method consists of the following steps:

(i) Subtract the two digits in column 1 and write them under column 2, ignoring sign.

(ii) Subtract the two digits in column 2 and write them under column 1, ignoring sign.

(iii) Similarly proceed for the two rows.

These values are called **oddments**. They are the *frequencies* with which the players must use their courses of action in their optimum strategies. These may be converted to *probabilities* by dividing each of them by their sum.

Example. In a game of matching coins, player A wins Tk. 2 if there are two heads, wins nothing if there are two tails and losses Tk. 1 when there are one head and one tail. Determine the payoff matrix, best strategies for each player and the value of game to A.

Solution. The payoff matrix for A will be

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} H & T \end{array} \\ \text{Player } A \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \end{array}$$

Since there is no saddle point, the optimal strategies will be mixed strategies.
Using **arithmetic method**, we get

$$\begin{array}{cccc} & \text{Player } B & & \\ & \begin{array}{cc} H & T \end{array} & \textbf{oddment} & \text{Probability} \\ \text{Player } A \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 3 \end{array} & \begin{array}{c} \frac{1}{1+3} = \frac{1}{4} = 0.25 \\ \frac{3}{1+3} = \frac{3}{4} = 0.75 \end{array} \\ \textbf{oddment} & \begin{array}{cc} 1 & 3 \end{array} & & \\ \text{Probability} & \begin{array}{cc} \frac{1}{4} & \frac{3}{4} \\ = 0.25 & = 0.75 \end{array} & & \end{array}$$

Thus for optimum gains, player A should use strategy H for 25% of the time and strategy T for 75% of the time, while player B should use strategy H 25% of the time and strategy T 75% of the time.

To obtain the value of the game any of the following expressions may be used:

Using A's oddments:

$$B \text{ plays } H : \text{Value of the game, } v = \text{Tk. } \left(\frac{1 \times 2 + 3 \times (-1)}{3+1}\right) = \text{Tk. } \left(-\frac{1}{4}\right)$$

$$B \text{ plays } T : \text{Value of the game, } v = \text{Tk. } \left(\frac{1 \times (-1) + 3 \times 0}{3+1}\right) = \text{Tk. } \left(-\frac{1}{4}\right)$$

Using B's oddments:

$$A \text{ plays } H : \text{Value of the game, } v = \text{Tk. } \left(\frac{1 \times 2 + 3 \times (-1)}{3+1}\right) = \text{Tk. } \left(-\frac{1}{4}\right)$$

$$A \text{ plays } T : \text{Value of the game, } v = \text{Tk. } \left(\frac{1 \times (-1) + 3 \times 0}{3+1}\right) = \text{Tk. } \left(-\frac{1}{4}\right)$$

The above values of v are equal only if sum of the oddments vertically and horizontally are equal.
Thus, the full solution of the game is

$$\begin{aligned}&\text{Strategies: } A(1,3) \text{ and } B(1,3);\\ &\text{Game value, } v = \text{ Tk. } \left(-\frac{1}{4}\right).\end{aligned}$$

This is the value of the game to A i.e., A gains Tk. $-\frac{1}{4}$ i.e., he loses Tk. $\frac{1}{4}$ which B, in turn, gets.

Algebraic Method:

$$\begin{aligned}x_1^* &= \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{0-(-1)}{(2+0)-(-1-1)} = \frac{1}{4},\\ x_2^* &= 1-x_1^* = 1-\frac{1}{4} = \frac{3}{4}.\\ y_1^* &= \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{0-(-1)}{(2+0)-(-1-1)} = \frac{1}{4},\\ y_2^* &= 1-y_1^* = 1-\frac{1}{4} = \frac{3}{4}.\end{aligned}$$

$$\begin{aligned}\text{Value of the game, } v &= \sum_{i=1}^{2}\sum_{j=1}^{2} x_i^* y_j^* a_{ij}\\ &= \sum_{i=1}^{2} x_i^* y_1^* a_{i1} + x_i^* y_2^* a_{i2}\\ &= x_1^*y_1^*a_{11} + x_2^*y_1^*a_{21} + x_1^*y_2^*a_{12} + x_2^*y_2^*a_{22}\\ &= \frac{1}{4}\cdot\frac{1}{4}\cdot 2 + \frac{3}{4}\cdot\frac{1}{4}\cdot(-1) + \frac{1}{4}\cdot\frac{3}{4}\cdot(-1) + \frac{3}{4}\cdot\frac{3}{4}\cdot 0\\ &= -\frac{1}{4}\end{aligned}$$

8.6.3 Rule 3. Reduce game by dominance

If no pure strategies exist, the next step is to eliminate certain strategies (row and/or column) by dominance. The principle of dominance states:
"If one pure strategy of a player is better or superior than another, irrespective of the strategy employed by his opponent, then the inferior strategy may be simply ignored (i.e., assigned a zero probability) while searching for optimal strategies."

The superior strategies are said to dominate the inferior ones. A player would have no incentive to use inferior strategies, which are dominated by some other(s).
The rules (principles) of dominance can be summarized as below:

(i) The dominance rule for row:
Every value in the dominating row(s) must be greater than or equal to the corresponding value of the dominated row.

(ii) The dominance rule for column: Every value in the dominating column(s) must be less than or equal to the corresponding value of the dominated column.

(iii) Dominance need no he based on the superiority of pure strategies only. A given strategy can be dominated if it is inferior to an ***average*** (or a ***convex combination***) of two or more other pure strategies.

After reducing the size of the payoff matrix, we will then obtain the solution of the game by applying any of the methods used for mixed-strategy game.

Note. Roughly speaking, **small rows** and **large columns** can be safely removed form the payoff matrix.

Remark.

1. It should be noted that a game reduced by ***dominance*** may disclose a saddle point which was not found in the original matrix under rule 1 (look for a pure strategy or saddle point). This is not necessarily a true saddle point since it may not be the least value in its row and the highest value in its column as per the original matrix. Therefore, this pseudo-saddle point is ignored and the reduced game should be solved for ***mixed strategies***.

2. The rules of dominance discussed above are used when the payoff matrix is a profit matrix for player A (and a loss matrix for player B); if otherwise, the rules get reversed.

Problem 8.6.1. Solve the following problem using dominance rule.

$$\begin{array}{cc|cccccc|}
 & & \multicolumn{6}{c}{\text{Player } B} \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
 & 1 & 4 & 2 & 0 & 2 & 1 & 1 \\
 & 2 & 4 & 3 & 1 & 3 & 2 & 2 \\
\text{Player } A & 3 & 4 & 3 & 7 & -5 & 1 & 2 \\
 & 4 & 4 & 3 & 4 & -1 & 2 & 2 \\
 & 5 & 4 & 3 & 3 & -2 & 2 & 2 \\
\hline
\end{array}$$

Solution.

$$\begin{array}{cc|cccccc|cl}
 & & \multicolumn{6}{c}{\text{Player } B} & & \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & \text{Row Minima} & \\
\hline
 & 1 & 4 & 2 & 0 & 2 & 1 & 1 & 0 & \\
 & 2 & 4 & 3 & 1 & 3 & 2 & 2 & (\mathbf{1}) & \longleftarrow \text{maximin} \\
\text{Player } A & 3 & 4 & 3 & 7 & -5 & 1 & 2 & -5 & \\
 & 4 & 4 & 3 & 4 & -1 & 2 & 2 & -1 & \\
 & 5 & 4 & 3 & 3 & -2 & 2 & 2 & -2 & \\
\hline
\multicolumn{2}{c}{\text{Column Maxima:}} & 4 & 3 & 7 & 3 & (\mathbf{2}) & 2 & & \\
 & & & & & & \uparrow & & & \\
 & & & & & & \text{minimax} & & & \\
\end{array}$$

Table−1

From table-1, we see that maximin$\neq$minimax.
So, there is no saddle point. Now, we try to reduce the size of the given payoff matrix by dominance rule.
From player A's point of view, row 1 is dominated by row 2. Similarly, row 5 is dominated by row 4. Therefore, row 1 and row 5 can be eliminated and the payoff matrix is given by in Table-2.

$$\begin{array}{cc|cccccc|}
 & & \multicolumn{6}{c}{\text{Player } B} \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
 & 2 & 4 & 3 & 1 & 3 & 2 & 2 \\
\text{Player } A & 3 & 4 & 3 & 7 & -5 & 1 & 2 \\
 & 4 & 4 & 3 & 4 & -1 & 2 & 2 \\
\hline
\end{array}$$

Table−2

$$\begin{array}{cc|ccc|}
 & & \multicolumn{3}{c}{\text{Player } B} \\
 & & 3 & 4 & 5 \\
\hline
 & 2 & 1 & 3 & 2 \\
\text{Player } A & 3 & 7 & -5 & 1 \\
 & 4 & 4 & -1 & 2 \\
\hline
\end{array}$$

Table−3

Now, from B's point of view, column 1 and column 2 are dominated by column 4 and column 5 respectively. Also, column 6 is dominated by column 5. So, the column 1, 2 and 6 are eliminated from the payoff table and the resulting table is given by Table-3.

Column 5 is dominated by the average of the column 3 and 4 which is $\begin{pmatrix} 2 \\ 1 \\ \frac{3}{2} \end{pmatrix}$, so the column 5 is deleted from Table-3, and the resulting table is given by Table-4.

$$\begin{array}{cc|cc|}
 & & \multicolumn{2}{c}{\text{Player } B} \\
 & & 3 & 4 \\
\hline
 & 2 & 1 & 3 \\
\text{Player } A & 3 & 7 & -5 \\
 & 4 & 4 & -1 \\
\hline
\end{array}$$

Table−4

Further, row 4 is obtained by the average of row 2 and 3. Hence, row 4 is deleted. The result 2×2 game is

$$
\begin{array}{cc|cc|cc}
 & & \multicolumn{2}{c}{\text{Player } B} & & \\
 & & 3 & 4 & \text{oddments} & \\
\text{Player } A & 2 & 1 & 3 & 12 & \frac{6}{7} \\
 & 3 & 7 & -5 & 2 & \frac{1}{7} \\
 & \text{oddments:} & 8 & 6 & & \\
 & & \frac{4}{7} & \frac{3}{7} & &
\end{array}
$$

Let x_2^* and x_3^* be strategies for A. Then by the arithmetic method, we have

$$x_2^* = \frac{6}{7}, \qquad x_3^* = \frac{1}{7}$$

while y_3^* and y_4^* be the strategies for B such that

$$y_3^* = \frac{4}{7}, \qquad y_4^* = \frac{3}{7}$$

Hence, the optimal strategies for A are $\left(0, \ \frac{6}{7}, \ \frac{1}{7}, \ 0, \ 0\right)$; and for B are $\left(0, \ 0, \ \frac{4}{7}, \ \frac{3}{7}, \ 0\right)$.
So, the value of the game is

$$v = \frac{4}{7} \cdot 1 + \frac{3}{7} \cdot 3 = \frac{13}{7}$$

- Try to solve it (table-4) by algebraic method.

8.6.4 Rule 4. Mixed Strategies ($2 \times n$ games or $m \times 2$ games)

These are the games in which one player has only two courses of action open to him while his opponent may have any number.
To solve such games, the first step is to look for a saddle point; if there is one, the game is readily solved. If not, next step is to reduce the given matrix to 2×2 size matrix by the rules of dominance. If the matrix can be reduced to 2×2 size, it can be easily solved by the arithmetic or algebraic method.

Part II

Class Note

Chapter 9

First Chapter

Mathematical programming is also known as operation research. Mathematical programming is divided into two parts. Namely, Linear mathematical programming and Non-linear mathematical programming.

Let us look at an example.

Example. Pen Industry

Model:	A_1	A_2	Total Availability (per day)
Man power/labor:	10 h	5 h	100 h
Raw materials:	6 kg	10 kg	120 kg
Profit(Tk) :	5	5	

Model this problem.

Chapter 10

Simplex Method

10.1 Various Forms and Conversion

10.1.1 General Form

$$\begin{aligned}
&\max / \min\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&\text{subject to}\ \ a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n(\ \leq \text{or} = \text{or} \geq)b_1 \\
&\qquad\qquad\ \ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n(\ \leq \text{or} = \text{or} \geq)b_2 \\
&\qquad\qquad\quad \ldots \qquad \ldots \qquad \ldots \qquad\qquad\qquad \ldots \\
&\qquad\qquad\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n(\ \leq \text{or} = \text{or} \geq)b_m \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_j \geq 0(j = 1, 2, \ldots, n)
\end{aligned}$$

10.1.2 Compact Form

$$\begin{aligned}
&\max / \min\ Z = \sum_{j=1}^{n} c_jx_j \\
&\text{subject to} \sum a_{ij}x_j(\leq \text{or} = \text{or} \geq)b_i(i = 1, 2, \ldots, m) \\
&\qquad\qquad\qquad\qquad x_j \geq 0(j = 1, 2, \ldots, n)
\end{aligned}$$

Here c_j = cost coefficient

10.1.3 Matrix Form

$$\begin{aligned}
&\max / \min\ Z = cX \\
&\text{subject to}\ AX(\leq \text{or} \ = \text{or} \geq)b
\end{aligned}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}_{m\times n} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \quad c = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \end{pmatrix}_{1\times n} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$$

10.1.4 Canonical Form

$$\begin{aligned}
&\text{maximize}\ Z = \sum_{j=1}^{n} c_jx_j \\
&\text{subject to} \sum a_{ij}x_j \leq \ b_i \quad (i = 1, 2, \ldots, m) \\
&\qquad\qquad\qquad x_j \geq 0 \quad (j = 1, 2, \ldots, n)
\end{aligned}$$

The Characteristic of the Canonical Form

1. All decision variables are non-negative.

2. All constraints are of the ($\leq$) type, and

3. Objective function is of maximization type.

10.1.5 Conversion between forms

Between maximization and minimization

We can change the type of objective function by multiplying the objective function with -1.

$$\min\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \Leftrightarrow \max\ G = -Z = -c_1x_1 - c_2x_2 - \cdots - c_nx_n$$

Between $\leq$ and $\geq$

We can change the type of constraints by multiplying the constraints with -1.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b_1 \Leftrightarrow -a_1x_1 - a_2x_2 - \cdots - a_nx_n \geq b_1$$

Between equations and inequalities

We can transform an equation to two inequalities.

$$a_1x_1 + a_2x_2 = b_1 \Leftrightarrow a_1x_1 + a_2x_2 \leq b_1 \text{ and } a_1x_1 + a_2x_2 \geq b_1 = -a_1x_1 - a_2x_2 \leq -b_1$$

Changing an unrestricted variable to restricted variable

We can change an unrestricted variable to restricted variable by introducing two additional variables. For example, let x_3 is unrestricted then by introducing two additional variables we get $x_3 = x_3' - x_3$" then we can add the restriction to the newly introduced variable, i.e., x_3', x_3" ≥ 0.

10.1.6 Standard Form

1. All the decision variables are non-negative.
2. All the constraints are expressed in the form of equation, except the non-negativity constraints which remain inequality (≥ 0).
3. The right-hand side of each constraint equation is non-negative.
4. The objective function is of the form maximization or minimization.

10.2 Basic Variable

Let there are n variables and there are m constants and ($n \geq m$). If we get a unique solution by solving the system of equation of order m (by assuming remaining $n - m$ variables to 0), then these m variables are called basic variables. Otherwise, these are called non-basic variables. Let us look at an example.

Example.

$$\begin{aligned} x_1 + x_2 + 4x_3 + 2x_4 + 3x_5 &= 8 \\ 4x_1 + 2x_2 + 2x_3 + x_4 + 6x_5 &= 4 \end{aligned}$$

Here, $n = 5$ and $m = 2$, so here 2 variables can be basic variable and 5-2=3 variables are non-basic variable.

Case 1: Let $x_2 = x_4 = x_5 = 0$. Then the system of equations become

$$\begin{aligned} x_1 + 4x_3 &= 8 \\ 4x_1 + 2x_3 &= 4 \end{aligned}$$

By solving this system we get a unique solution: $x_1 = 0$ and $x_3 = 2$. So here, $x_1,\ x_3$ are feasible[1] basic variables and $x_2,\ x_4,\ x_5$ are non-basic variables.

Case 2: Let $x_3 = x_4 = x_5 = 0$. Then the system of equations become

$$\begin{aligned} x_1 + 2x_2 &= 8 \\ 4x_1 + 2x_2 &= 4 \end{aligned}$$

By solving this system we get a unique solution: $x_1 = -6$ and $x_3 = 14$. So here, $x_1,\ x_2$ are basic infeasible ($x_1 \ngeq 0$) variables and $x_3,\ x_4,\ x_5$ are non-basic variables.

[1] If a variable/solution satisfies all constraints and non-negativity condition then it will be a feasible solution.

Case 3: Let $x_1 = x_2 = x_5 = 0$. Then the system of equations become

$$\begin{aligned} 4x_3 + 2x_4 &= 8 \\ 2x_3 + 2x_4 &= 4 \end{aligned}$$

By solving this system we get infinitely many solutions. So these are not basic variables.

Case 4: Let $x_1 = x_3 = x_4 = 0$. Then the system of equations become

$$\begin{aligned} x_2 + 3x_5 &= 8 \\ 2x_3 + 6x_4 &= 4 \end{aligned}$$

There are no solution of this system. So these are not basic variables.

Definition 4 (Degenerate Solution)**.** If at least one of the basic feasible solution is zero then that solution is degenerate solution.

So, Case 1: of above example is a degenerate solution $[\because x_1 = 0]$ and Case 2: of above example is a non-degenerate solution $[\because x_1, x_2 \neq 0]$.

10.3 Simplex Method

Problem 10.3.1.

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 14 \\ & x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Steps:

- Transform the problem into standard form.
- For basic variable: check if the slack variables are basic or not
- $c_B \rightarrow$ coefficient of basis in objective function
- Tab: table number
- Basis: basic variables
- c_j constants of objective function (cost coefficient)
- $\bar{c_j}$ row $= c_j - \sum c_B A_j$ (here A_j is the coefficient in the j-th column)
- Z calculation: $Z = \sum c_B Z_j$
- Pivot column: maximization: max in $\bar{c_j}$ row; minimization: min in $\bar{c_j}$ row (purple in table)
- Pivot row: check positive ratio of constants and pivot column, i.e., ratio $= \frac{\text{constants}}{\text{pivot column}}$ and select minimum ratio as pivot row, same for maximization and minimization problem. (green in table)
- Intersection of pivot column and pivot row is pivot element (blue in table)
- For next iteration make the pivot element 1 and other element in pivot column 0 by doing row operations.
- Unbounded check: If ratio in pivot column is all negative then the problem is unbounded.
- For Optimal solution check:
  - In maximization, no positive element should appear in $\bar{c_j}$ row
  - In minimization, no negative element should appear in $\bar{c_j}$ row
- Alternative solution check: If any non-basic variable is zero in final $\bar{c_j}$ row then alternative solution exists.

Solution. First we need to transform the problem into standard form.
Standard form:

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 + s_1 = 4 \\ & 3x_1 + 2x_2 + s_2 = 14 \\ & x_1 - 2x_2 + s_3 = 3 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Taking $x_1 = x_2 = 0,\ S_1 = 4,\ S_2 = 14,\ S_3 = 3$ are the initial basic feasible variables.

Tab	c_B	$c_j \rightarrow$ / basis	3	2	0	0	0	Constant/
			x_1	x_2	S_1	S_2	S_3	Solution
I	0	S_1	-1	2	1	0	0	4
	0	S_2	3	2	0	1	0	14
	0	S_3	1	-1	0	0	1	3
		$\bar{c}_j$ row	3	2	0	0	0	Z=0
II	0	S_1	0	1	1	0	1	7
	0	S_2	0	5	0	1	-3	5
	3	x_1	1	-1	0	0	1	3
		$\bar{c}_j$ row	0	5	0	0	-3	Z=9
III	0	S_1	0	0	1	-1/5	8/5	6
	2	x_2	0	1	0	1/5	-3/5	1
	3	x_1	1	0	0	1/5	2/5	4
		$\bar{c}_j$ row	0	0	0	-1	0	Z=14
IV	0	S_1	0	0	5/8	-1/8	1	15/4
	2	x_2	0	1	3/8	1/8	0	13/4
	3	x_1	1	0	-1/4	1/4	0	5/2
		$\bar{c}_j$ row	0	0	0	-1	0	Z=14

So, $(x_1, x_2) = (4, 1)$, $\quad Z_{max} = 14$
$(x_1, x_2) = (5/2, 13/4)$, $\quad Z_{max} = 14$
Solution:

$$\{(x_1, x_2) = \lambda(4, 1) + (1 - \lambda)(5/2, 13/2), 0 \leq \lambda \leq 1\}$$

Problem 10.3.2.

$$\begin{aligned} \text{maximize} \quad & Z = 5x_1 + 4x_2 \\ \text{subject to} \quad & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution. First we need to transform the problem into standard form.
Standard form:

$$\begin{aligned} \text{maximize} \quad & Z = 5x_1 + 4x_2 \\ \text{subject to} \quad & 6x_1 + 4x_2 + S_1 = 24 \\ & x_1 + 2x_2 + S_2 = 6 \\ & -x_1 + x_2 + S_3 = 1 \\ & x_2 + S_4 = 2 \\ & x_1, x_2, S_1, S_2, S_3, S_4 \geq 0 \end{aligned}$$

Taking $x_1 = x_2 = 0$ we get, $S_1 = 24,\ S_2 = 6,\ S_3 = 1,\ S_4 = 2$.
So $S_1,\ S_2,\ S_3,\ S_4$ are initial basic feasible variables.

Tab	c_B	$c_j \rightarrow$ / basis	5	4	0	0	0	0	Constant/
			x_1	x_2	S_1	S_2	S_3	S_4	Solution
I	0	S_1	6	4	1	0	0	0	24
	0	S_2	1	2	0	1	0	0	6
	0	S_3	-1	1	0	0	1	0	1
	0	S_4	0	1	0	0	0	1	2
		$\bar{c}_j$ row	5	4	0	0	0	0	Z=0
II	5	x_1	1	2/3	1/6	0	0	0	4
	0	S_2	0	4/3	-1/6	1	0	0	2
	0	S_3	0	5/3	1/6	0	1	0	5
	0	S_4	0	1	0	0	0	1	2
		$\bar{c}_j$ row	0	2/3	-5/6	0	0	0	Z=20
III	5	x_1	1	0	1/4	1/2	0	0	3
	4	x_2	0	1	-1/8	3/4	0	0	3/2
	0	S_3	0	0	3/8	-5/4	1	0	5/2
	0	S_4	0	0	1/8	-3/4	0	1	1/2
		$\bar{c}_j$ row	0	0	-3/4	-11/2	0	0	Z=21

So, $(x_1,\ x_2)\ =\ (5,4), \quad Z_{\max} = 21$

10.3.1 Artificial Variables Technique for Finding the first basic feasible solution

There are two technique to find first basic feasible solutions by using artificial variables. They are

(i) The big M method/M-technique/Method of penalty

(ii) Two phase method

The big M method

When the constraints are of ($\geq$ or $=$) types then we may not get basic feasible solution easily. For this type of problem we use big M method to ensure that we get basic initial feasible solution.

Problem 10.3.3. Solve the following LPP:

$$\begin{aligned} \text{minimize} \quad & Z = -3x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 \leq 11 \\ & -4x_1 + x_2 + 2x_3 \geq 3 \\ & 2x_1 - x_3 = -1 \\ & -x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution. Standard form:

$$\begin{aligned} \text{minimize} \quad & Z = -3x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 + x_4 11 \\ & -4x_1 + x_2 + 2x_3 - x_5 = 3 \\ & 2x_1 - x_3 = -1 \\ & -x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Taking $x_1 = x_2 = 0$ we get, $x_3 = 1,\ x_4 = 10,\ x_5 = -1$. But this is not basic feasible solution as $x_5 \ngeq 0$.
So we need to use big M method.
Standard form:

$$\begin{aligned} \text{minimize} \quad & Z = -3x_1 + x_2 + x_3 + M(x_6 + x_7) \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 + x_4 11 \\ & -4x_1 + x_2 + 2x_3 - x_5 + x_6 = 3 \\ & 2x_1 - x_3 + x_7 = -1 \\ & -x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

M is a large positive number.
Taking $x_1 = x_2 = x_3 = x_5 = 0$ we get, $x_4 = 11,\ x_6 = 3,\ x_7 = 1$. This is a basic feasible solution.

Tab	c_B	$c_j \rightarrow$ / basis	-3	1	1	0	0	M	M	Constant/
			x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution
I	0	x_4	1	-2	1	1	0	0	0	11
	M	x_6	-4	1	2	0	-1	1	0	3
	M	x_7	2	0	1	0	0	0	1	1
		$\bar{c}_j$ row	$-3+6M$	$1-M$	1-3M	0	M	0	0	$Z = 4M$
II	0	x_4	3	-2	0	1	0	0	1	10
	M	x_6	0	1	0	0	-1	1	-2	1
	1	x_3	-2	0	1	0	0	0	1	1
		$\bar{c}_j$ row	-1	$1-M$	0	0	M	0	$3M-1$	$Z = 1+M$
III	0	x_4	3	0	0	1	-2	2	-5	12
	1	x_2	0	1	0	0	-1	1	-2	1
	1	x_3	-2	0	1	0	0	0	1	1
		$\bar{c}_j$ row	-1	0	0	0	1	$M-1$	$M+1$	$Z = 2$
IV	-3	x_1	1	0	0	1/3	-2/3	2/3	-5/3	4
	1	x_2	0	1	0	0	-1	1	-2	1
	1	x_3	0	0	1	2/3	-4/3	4/3	-7/3	9
		$\bar{c}_j$ row	0	0	0	1/3	1/3	$M-1/3$	$M-2/3$	$Z = -2$

So, $(x_1,\ x_2,\ x_3) = (4, 1, 9)$, $\quad Z_{\min} = -2$

- In minimization: The sign of M in objective function is positive $(+)$
- In maximization: The sign of M in objective function is negative $(-)$
- Add artificial variables in only $(\geq$ or $=)$ type of constraints
- There can be three cases in the c_B column
  - No M is present in c_B column − then the solution is optimal solution.
  - M is present in c_B column, but the coefficient is 0 in 'solution' column − then the solution is optimal solution.
  - M is present in c_B column, and the coefficient is non-zero in 'solution' column − then the solution is non-optimal solution even though the solution maintains the optimal test for stopping the simplex method.

Chapter 11

Non-Linear Programming

11.1 Preliminary Concepts

Let a model be

$$\begin{aligned}
\text{maximize} \quad & Z = f(x_1, x_2, \ldots, x_n) \\
\text{subject to} \quad & g^1(x_1, x_2, \ldots, x_n) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_1 \\
& g^2(x_1, x_2, \ldots, x_n) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_2 \\
& \qquad \vdots \qquad\qquad\qquad \vdots \\
& g^m(x_1, x_2, \ldots, x_n) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_m \\
& \qquad\qquad x_i \geq 0 \qquad i = 1, 2, \ldots, n
\end{aligned}$$

If f or g^i or both of them have one or more non-linear expression (i.e., have a variable that has degree 2 or above) then this type of problems are called non-linear programming problem.
Matrix form

$$\begin{aligned}
\text{maximize} \quad & Z = f(\underline{x}) \\
\text{subject to} \quad & g^i(\underline{x}) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_i \qquad i = 1, 2, \ldots, m \\
& \Rightarrow h^i(\underline{x}) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad 0 \qquad \text{where } h^i(\underline{x}) = g^i(\underline{x}) - b_i
\end{aligned}$$

11.1.1 Principal minor

Let $Q_{n\times n}$ a matrix. It's k-th order ($k \leq n$) principal minor is a matrix that is obtained from Q by removing $(n-k)$ corresponding rows and column.
Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

its principal minors are

$$\begin{aligned}
\text{Order 1}: & \begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 5 \end{pmatrix}, \begin{pmatrix} 9 \end{pmatrix} \\
\text{Order 2}: & \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \\
\text{Order 3}: & \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\end{aligned}$$

Leading principal minor: Remove last $n-k$ corresponding row and column.
Principal determinant:

$$\begin{aligned}
\text{Order 1}: & \begin{vmatrix} 1 \end{vmatrix}, \begin{vmatrix} 5 \end{vmatrix}, \begin{vmatrix} 9 \end{vmatrix} \\
\text{Order 2}: & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \\
\text{Order 3}: & \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}
\end{aligned}$$

11.1.2 Quadratic form

$Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x_i x_j$
e.g., $Q(x) = x_1^2 + 7x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$ is a quadratic form because every term has degree of two.

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 7 & 6 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 & 7x_2 & x_1 + 6x_2 + 2x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 7 & 6 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 7x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$$

So, $Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x_i x_j = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ (here $\mathbf{A}$ can be symmetric) where $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Again,

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 7 & 3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 7x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$$

The quadratic form $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ is said to be

1. Positive definite: if $Q(x) > 0 \, \forall \underline{x} \neq 0$
2. Positive semi-definite: if $Q(x) \geq 0$ for all x such that there exists at least one $x \neq 0$ satisfying $Q(x) = 0$
3. Negative definite: $-Q(x) > 0$ or $Q(x) < 0$
4. Negative semi-definite: $-Q(x) \geq 0$ or $Q(x) \leq 0$
5. Indefinite: if quadratic form does not fall into above categories.

Example.

- Positive definite: $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2$
- Positive semi-definite: $Q(x) = (x_1 - x_2)^2 + 2x_3^2 \; x_1 = x_2$ and $x_3 = 0$
- Negative definite: $Q(x) = -x_1^2 - 3x_2^2$
- Indefinite: $Q(x) = x_1^2 - 3x_2^2$

The necessary and sufficient condition:

1. Positive definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if leading principal determinant > 0
2. Positive semi-definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if leading principal determinant ≥ 0
3. Negative definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if sign of $k-$th leading principal determinant $= (-1)^k$
4. Negative semi-definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if sign of $k-$th leading principal determinant $= (-1)^k$ or zero
5. Indefinite: there must be two opposite sign in diagonal.

Example. $Q(x) = 2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3$
Here, $A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

The leading principal determinants of A: $\begin{vmatrix} 2 \end{vmatrix} = 2$, $\begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} = 4$, $\begin{vmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 12$

So $Q(x)$ is positive definite.

11.1.3 Hessian matrix:

Let $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ be a function that is continuous has double derivative. Then Hessian matrix of $f(\underline{x})$ is

$$H(\underline{x}) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

$$f(x_1, x_2, x_3) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_3} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2\,\partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_3\,\partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{pmatrix}$$

Hessian matrix is a symmetric matrix [$\because f_{ij} = f_{ji}$].

Problem 11.1.1. Find the Hessian matrix of $f(x, y) = x^3 - 2xy - y^6$ at the point $(1, 2)$

Solution.

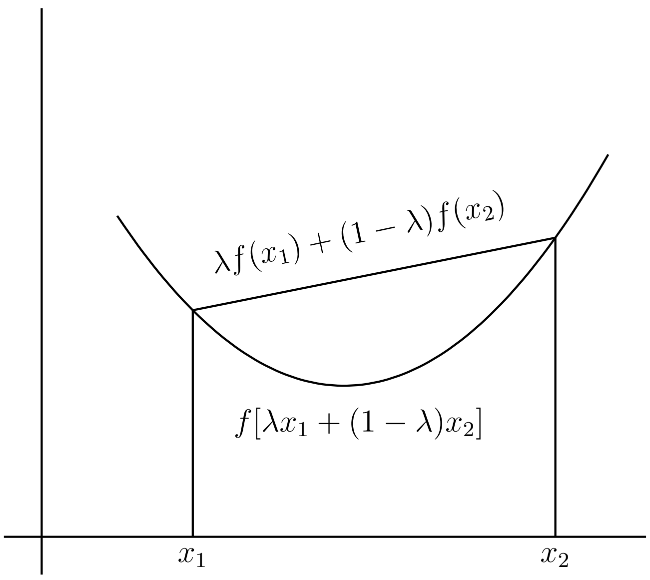
$$\frac{\partial f}{\partial x} = 3x^2 - 2y \qquad \frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial f}{\partial x\,\partial y} = -2 \qquad \frac{\partial f}{\partial y\,\partial x} = -2 \qquad \frac{\partial^2 f}{\partial y^2} = -30y^4$$

$$H = \begin{pmatrix} 6x & -2 \\ -2 & -30y^4 \end{pmatrix}$$

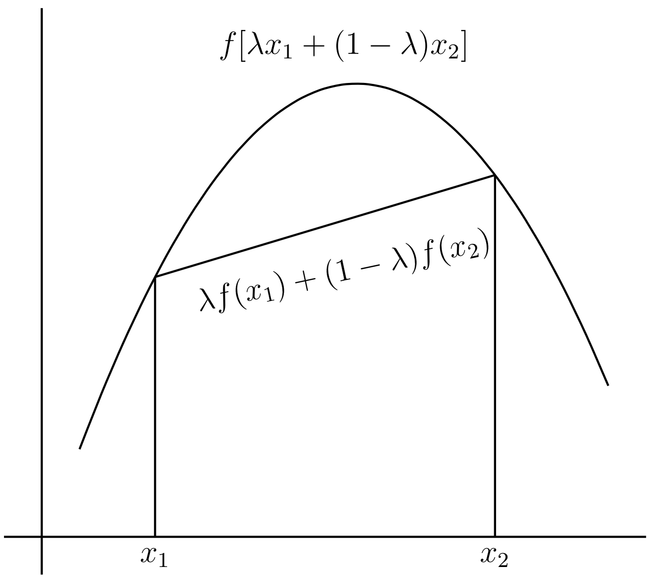
$$H_{(1,2)} = \begin{pmatrix} 6 & -2 \\ -2 & -480 \end{pmatrix}$$

11.1.4 Convex and concave function:

If $f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ where x_1, x_2 are points and $0 \leq \lambda \leq 1$, then $f(x)$ is a convex function. $f(x)$ is strictly convex if $f[\lambda x_1 + (1-\lambda)x_2] < \lambda f(x_1) + (1-\lambda)f(x_2)$ where $0 < \lambda < 1$.



Convex function



Concave function

If $f[\lambda x_1 + (1-\lambda)x_2] \geq \lambda f(x_1) + (1-\lambda)f(x_2)$ where x_1, x_2 are points and $0 \leq \lambda \leq 1$, then $f(x)$ is a concave function. $f(x)$ is strictly concave if $f[\lambda x_1 + (1-\lambda)x_2] > \lambda f(x_1) + (1-\lambda)f(x_2)$ where $0 < \lambda < 1$.

Tests for a function to be convex/concave: $H(x)$ is the Hessian matrix of $f(x)$

- Convex: if $H(x)$ is positive semi-definite
- Strictly convex: if $H(x)$ is positive definite
- Concave: if $H(x)$ is negative semi-definite
- Strictly concave: if $H(x)$ is negative definite

Problem 11.1.2. Test the convexity of the function $f(\underline{x}) = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 6x_1 - 4x_2 - 2x_3$ at the point (x_1, x_2, x_3).

Solution. The given function is $f(\underline{x}) = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 6x_1 - 4x_2 - 2x_3$

$$\begin{array}{l|l|l}
\dfrac{\partial f}{\partial x_1} = 6x_1 - 2x_2 - 2x_3 - 6 & \dfrac{\partial^2 f}{\partial x_1^2} = 6 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} = -2 = \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} \\
\dfrac{\partial f}{\partial x_2} = 4x_2 - 2x_1 + 2x_3 - 4 & \dfrac{\partial^2 f}{\partial x_2^2} = 4 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_3} = -2 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_1} \\
\dfrac{\partial f}{\partial x_3} = 3x_3 - 2x_1 + 2x_2 - 2 & \dfrac{\partial^2 f}{\partial x_3^2} = 2 & \dfrac{\partial^2 f}{\partial x_2\,\partial x_3} = 2 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_2}
\end{array}$$

Then the Hessian matrix of the function $f(x)$ is

$$H(\underline{x}) = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$

Since,

(i) $H(\underline{x})$ is symmetric

(ii) All diagonal elements are positive

(iii) The leading principal minor determinant $\left|6\right| > 0$, $\begin{vmatrix} 6 & -2 \\ -2 & 4 \end{vmatrix} = 20 > 0$, $\left|H(\underline{x})\right| = 16 > 0$

So, $H(\underline{x})$ is positive definite for all values of (x_1, x_2, x_3) which implies that $f(\underline{x})$ is strictly convex function.

Problem 11.1.3. Test the convexity of the function $f(\underline{x}) = -x_1^2 - 3x_2^2 - 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$.

Solution. The given function is $f(\underline{x}) = -x_1^2 - 3x_2^2 - 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$

$$\begin{array}{l|l|l}
\dfrac{\partial f}{\partial x_1} = -2x_1 + 4x_2 + 2x_3 & \dfrac{\partial^2 f}{\partial x_1^2} = -2 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} = 4 = \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} \\
\dfrac{\partial f}{\partial x_2} = -6x_2 + 4x_1 + 4x_3 & \dfrac{\partial^2 f}{\partial x_2^2} = -6 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_3} = 2 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_1} \\
\dfrac{\partial f}{\partial x_3} = -4x_3 + 2x_1 + 4x_2 & \dfrac{\partial^2 f}{\partial x_3^2} = -4 & \dfrac{\partial^2 f}{\partial x_2\,\partial x_3} = 4 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_2}
\end{array}$$

Then the Hessian matrix of the function $f(x)$ is

$$H(\underline{x}) = \begin{pmatrix} -2 & 4 & 2 \\ 4 & -6 & 4 \\ 2 & 4 & -4 \end{pmatrix}$$

Since,

(i) $H(\underline{x})$ is symmetric

(ii) All diagonal elements are negative

(iii) The leading principal minor determinant $\left|-2\right| < 0$, $\begin{vmatrix} -2 & 4 \\ 4 & -6 \end{vmatrix} = -4 < 0$, $\left|H(\underline{x})\right| = 136 > 0$

So, $H(\underline{x})$ is an indefinite matrix which implies that $f(\underline{x})$ is neither convex nor concave function.

11.2 Unconstrained Optimization:

To find stationary point: $\nabla f(\underline{x}) = 0$. Let $\underline{x}_0$ is a stationary point.

(i) If $H(\underline{x}_0$ is positive definite then x_0 is a minimum point.

(ii) If $H(\underline{x}_0$ is negative definite then x_0 is a maximum point.

- If $H(\underline{x}_0)$ is indefinite then $x = x_0$ is a point of inflection (saddle point).

Problem 11.2.1. Determine the local maximum or minimum (if any) of the function $f(\underline{x}) = x_1^2 + 2x_2^2 + x_3^2 + x_1x_2 - 2x_3 - 7x_1 + 12$.

Solution. The given function is $f(\underline{x}) = x_1^2 + 2x_2^2 + x_3^2 + x_1x_2 - 2x_3 - 7x_1 + 12$.
The necessary condition to obtain the maximum or minimum is $\nabla f(\underline{x}) = 0$.
Now,

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= 0 \qquad \Rightarrow 2x_1 + x_2 - 7 = 0\\
\frac{\partial f}{\partial x_2} &= 0 \qquad \Rightarrow 4x_2 + x_1 = 0\\
\frac{\partial f}{\partial x_3} &= 0 \qquad \Rightarrow 2x_3 - 2 = 0
\end{aligned}$$

Solving these three equation we get $x_1 = 4,\ x_2 = -1,\ x_3 = 1$ i.e., $\underline{x}_0 = (x_1^0,\ x_2^0,\ x_3^0) = (4,\ -1, 1)$.
For the sufficient condition, let us find the Hessian matrix $H(\underline{x}^0)$.

$$H(\underline{x}^0) = \begin{pmatrix} 2 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 2 \end{pmatrix}$$

Since,

(i) $H(\underline{x}_0)$ is symmetric.

(ii) All diagonal elements are positive.

(iii) The leading principal minor determinants are $\begin{vmatrix}2\end{vmatrix} > 0,\ \begin{vmatrix}2 & 1\\ 1 & 4\end{vmatrix} = 7 > 0,\ \begin{vmatrix}H(\underline{x})\end{vmatrix} = 14 > 0$

Thus, $H(\underline{x}_0)$ is positive definite, so the function $f(\underline{x})$ is strictly convex. Hence, $f(\underline{x})$ is minimum at $x^0 = (4, -1, 1)$.

11.3 Constrained Problems

11.3.1 Lagrangian Method

We can only use this method if the constraints are equation i.e., of $=$ type.
Consider the problem

$$\begin{gathered}\text{optimize } f(\underline{x})\\ \text{subject to } \underline{g}(\underline{x}) = 0\end{gathered}$$

where $\underline{x} = (x_1,\ x_2, \ldots, x_n)$ and $\underline{g} = (g_1,\ g_2, \ldots, g_m)^T$. The functions $f(\underline{x})$ and $g_i(\underline{x}),\ i = 1,\ 2\ldots, m$ are assumed to be twice differentiable.
Let $L(\underline{x},\ \underline{\lambda}) = f(\underline{x}) - \underline{\lambda}\underline{g}(\underline{x})$. Here $L =$ Lagrangian function, $\underline{\lambda} =$ Lagrangian multipliers.
The necessary conditions for determining the stationary points of $f(\underline{x})$ are subject to $\underline{g}(\underline{x}) = 0$ is given by

$$\frac{\partial L}{\partial \underline{x}} = 0 \qquad \frac{\partial L}{\partial \underline{\lambda}} = 0$$

To establish the sufficient condition:
Define

$$H^B = \left(\begin{array}{c|c} O & P\\ \hline P^T & Q \end{array}\right)_{(m+n)+(m+n)}$$

where

$$P = \begin{pmatrix} \nabla g_1(\underline{x})\\ \nabla g_2(\underline{x})\\ \vdots\\ \nabla g_m(\underline{x}) \end{pmatrix}_{m\times n} \quad \text{and } Q = \left(\frac{\partial^2 L(\underline{x},\ \underline{\lambda})}{\partial x_i\,\partial x_j}\right)_{n\times n} \quad \text{for all } i \text{ and } j$$

$O =$ Null matrix whose order is adjusted to make H^B a square matrix $\quad H^B =$ Bordered Hessian matrix

Given stationary point $(\underline{x}^0, \underline{\lambda}^0)$ for the Lagrangian function $L(\underline{x},\ \underline{\lambda})$ and bordered Hessian matrix H^B evaluated at $(\underline{x}^0, \underline{\lambda}^0)$, then $\underline{x}^0$ is

1. a *maximum* point, if starting with the principal minor determinants of order $(2m + 1)$ the last $(n - m)$ principal minor determinants of H^B form an alternating sign pattern starting with $(-1)^{m+1}$.

2. a *minimum* point, if starting with the principal minor determinants of order $(2m+1)$ the last $(n-m)$ principal minor determinants of H^B have the sign of $(-1)^m$.

Problem 11.3.1. Solve the following problem by using Lagrangian method.

$$\begin{aligned} \text{minimize} \quad & f(\underline{x}) = x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} \quad & x_1 + x_2 + 3x_3 = 2 \\ & 5x_1 + 2x_2 + x_3 = 5 \end{aligned}$$

Solution. Suppose that

$$\begin{aligned} f(x_1,\, x_2,\, x_3) &= x_1^2 + x_2^2 + x_3^2 \\ g_1(x_1,\, x_2,\, x_3) &= x_1 + x_2 + 3x_3 - 2 = 0 \\ g_2(x_1,\, x_2,\, x_3) &= 5x_1 + 2x_2 + x_3 - 5 = 0 \end{aligned}$$

$$L(\underline{x}, \underline{\lambda}) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

where

$$\underline{x} = (x_1,\, x_2,\, x_3) \quad \text{and} \quad \underline{\lambda} = (\lambda_1,\, \lambda_2)$$

The necessary condition:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda = 0 \tag{11.1}$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda = 0 \tag{11.2}$$
$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - 2\lambda = 0 \tag{11.3}$$
$$\frac{\partial L}{\partial \lambda_1} = -x_1 - x_2 - 3x_3 + 2 = 0 \tag{11.4}$$
$$\frac{\partial L}{\partial \lambda_2} = -5x_1 - 2x_2 - x_3 + 5 = 0 \tag{11.5}$$

From (11.1) and (11.2)

$$2(x_1 - x_2) = 3\lambda_2 \quad \Rightarrow \lambda_2 = \frac{2}{3}(x_1 - x_2) \quad \text{and} \quad \lambda_1 = \frac{1}{3}(-4x_1 + 10x_2) \tag{11.6}$$

From (11.3) and (11.6)

$$2x_3 - 3 \times \frac{1}{3}(-4x_1 + 10x_2) - \frac{2}{3}(x_1 - x_2) = 0$$
$$\Rightarrow 5x_1 - 14x_2 + 3x_3 = 0 \tag{11.7}$$

From (11.4) and (11.5)

$$3x_1 - 5x_3 = 1 \tag{11.8}$$

From (11.4) and (11.7)

$$4x_1 - 15x_2 = 2 \tag{11.9}$$

Solving (11.7), (11.8) and (11.9) we get

$$x_1 = \frac{37}{46}, \qquad x_2 = \frac{16}{46}, \qquad x_3 = \frac{13}{46}$$
$$\therefore \lambda_1 = \frac{4}{46}, \qquad \lambda_2 = \frac{14}{46}$$

So, the stationary point is given by

$$\underline{x}^0 = \left(\frac{37}{46}, \frac{16}{46}, \frac{13}{46}\right) \quad \underline{\lambda}^0 = \left(\frac{4}{46}, \frac{14}{46}\right)$$

For the sufficient condition the Lagrangian function:

$$\begin{aligned} \frac{\partial^2 L}{\partial x_1^2} &= 2 & \frac{\partial^2 L}{\partial x_1\, \partial x_2} &= 0 & \frac{\partial^2 L}{\partial x_1\, \partial x_3} &= 0 \\ \frac{\partial^2 L}{\partial x_2\, \partial x_1} &= 0 & \frac{\partial^2 L}{\partial x_2^2} &= 2 & \frac{\partial^2 L}{\partial x_2\, \partial x_3} &= 0 \\ \frac{\partial^2 L}{\partial x_3\, \partial x_1} &= 0 & \frac{\partial^2 L}{\partial x_3\, \partial x_2} &= 0 & \frac{\partial^2 L}{\partial x_3^2} &= 2 \end{aligned}$$

$$\therefore Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\nabla g_1(\underline{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}$$
$$\nabla g_2(\underline{x}) = \begin{pmatrix} \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix} \qquad P^T = \begin{pmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$\therefore \text{ Bordered Hessian matrix, } H^B(\underline{x}^0, \underline{\lambda}^0) = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ \hline 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{array}\right) \quad \text{taking } O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Here, $m = 2$, $n = 3 \therefore 2m + 1 = 5$. So, $\left|H^B(x^0,\ \lambda^0)\right| = 460 > 0$ i.e., having the sign $(-1)^2$.
So this point is sufficient.

Problem 11.3.2. Consider the problem

$$\begin{aligned} \text{minimize} \quad & Z = x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} \quad & 4x_1 + x_2^2 + 2x_3 - 14 = 0 \end{aligned}$$

11.3.2 Inequality Constraints [Karush-Kuhn-Tucker (KKT) conditions]

Problem 11.3.3. Use KKT conditions to solve

$$\begin{aligned} \text{maximize} \quad & f(\underline{x}) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 2 \\ & 2x_1 + 3x_2 \leq 12 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

Solution. We have

$$\begin{aligned} f(\underline{x}) &= -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 \\ h^1(\underline{x}) &= x_1 + x_2 - 2 \\ h^2(\underline{x}) &= 2x_1 + 3x_2 - 12 \end{aligned}$$

The KKT conditions for maximization problem are

$$\begin{aligned} \nabla f(\underline{x}) - \underline{\lambda} \nabla \underline{h}(\underline{x}) &= 0 \\ \lambda_i h^i(\underline{x}) &= 0 \\ h^i(\underline{x}) &\leq 0 \\ \underline{\lambda} &\geq 0 \end{aligned}$$

Applying these conditions we get,

$$-2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0 \tag{11.10}$$
$$-2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0 \tag{11.11}$$
$$-2x_3 = 0 \tag{11.12}$$
$$\lambda_1(x_1 + x_2 - 2) = 0 \tag{11.13}$$
$$\lambda_2(2x_1 + 3x_2 - 12) = 0 \tag{11.14}$$
$$x_1 + x_2 \leq 2 \tag{11.15}$$
$$2x_1 + 3x_2 \leq 12 \tag{11.16}$$
$$x_1,\ x_2,\ x_3 \geq 0 \ \lambda_1,\ \lambda_2 \ \geq 0 \tag{11.17}$$

Now these arise following four cases:

Case 1: If $\lambda_1 = 0$, $\lambda_2 = 0$, then from (11.10), (11.11) and (11.12) we get,

$$\begin{aligned} -2x_1 + 4 &= 0 \\ -2x_2 + 6 &= 0 \\ -2x_3 &= 0 \end{aligned}$$

Solving these equations we get $x_1 = 2$, $x_2 = 3$ and $x_3 = 0$. This solution violates the equation (11.15) and (11.16). So, it is rejected.

Case 2: If $\lambda_1 = 0$, $\lambda_2 \neq 0$, then from (11.14) we get

$$2x_1 + 3x_2 - 12 = 0 \tag{11.18}$$

and from (11.10) and (11.11) we get

$$\begin{aligned} -2x_1 + 4 - 2\lambda_2 &= 0 \\ -2x_2 + 6 - 3\lambda_2 &= 0 \end{aligned}$$

By manipulating these two equation we get

$$\begin{aligned} 6x_1 - 4x_2 &= 0 \\ \Rightarrow x_1 &= \frac{2}{3}x_2 \end{aligned} \tag{11.19}$$

From (11.18) and (11.19) we get $x_1 = \frac{24}{13}$, $x_2 = \frac{36}{13}$ and from (11.12) we get $x_3 = 0$. This solution violates the inequality (11.15). This is also rejected.

Case 3: If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, then from (11.13), (11.14) we get,

$$\begin{aligned} x_1 + x_2 - 2 &= 0 \\ -2x_1 + x_2 - 12 &= 0 \end{aligned}$$

Solving these equations we get $x_1 = -6$, $x_2 = 8$. This solution violates the inequality (11.17). So, this solution is also rejected.

Case 4: If $\lambda_1 \neq 0$, $\lambda_2 = 0$, then from (11.13) we get

$$x_1 + x_2 - 2 = 0$$

and from (11.10) and (11.11) we get

$$-2x_1 + 2x_2 - 2 = 0$$

By solving these two equation we get $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{2}$
Again from (11.10) we get $\lambda_1 = 3$ and (11.12) we get $x_3 = 0$.
Observe that the solution $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{2}$, $x_3 = 0$ and $\lambda_1 = 3$, $\lambda_2 = 0$ satisfies all the KKT conditions.

So the optimum solution of the given non-linear programming problem is

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{3}{2}, \quad x_3 = 0 \qquad \text{and} \qquad f(\underline{x})_{\max} = \frac{17}{2}$$

For sufficient condition the objective function and constraints must be concave functions.

$$f(\underline{x}) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

$$H(\underline{x}^0) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The leading principal determinants are -2, 4, -8. So $H(\underline{x}^0)$ is negative definite and hence the function is concave. Here the constraints are linear, so they are also concave. Hence, this point is sufficient.

Remark. Non negativity is not mandatory for non-linear problems.

Let a problem be

$$\begin{aligned} \text{max or min } Z &= f(\underline{x}) \\ \text{subject to } g_i(\underline{x}) &\leq 0 \quad i = 1, 2, \ldots, r \\ g_i(\underline{x}) &\geq 0 \quad i = r+1, r+2, \ldots, p \\ g_i(\underline{x}) &= 0 \quad i = p+1, p+2, \ldots, m \end{aligned}$$

Sense of Optimization	Required Conditions		
	$f(\underline{x})$	$g_i(\underline{x})$	λ_i
Maximization	Concave	Convex Concave Linear	$\geq 0 \quad (1 \leq i \leq r)$ $\leq 0 \quad (r+1 \leq i \leq p)$ Unrestricted $(p+1 \leq i \leq m)$
Minimization	Convex	Convex Concave Linear	$\leq 0 \quad (1 \leq i \leq r)$ $\geq 0 \quad (r+1 \leq i \leq p)$ Unrestricted $(p+1 \leq i \leq m)$