Chapter 1

Bessel's Equation and Bessel's Function

1.1 Bessel's Equation and Bessel's Function

The differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} + x \frac{\mathrm{d}\, y}{\mathrm{d}\, x} + \left(x^2 - n^2\right) y = 0$$

where n is a positive constant (not necessarily an integer) is known as the Bessel's equation.

Since it is a linear differential equation of second order, it must have two linearly independent solutions.

Case 1 : *n is not an integer*

The complete solution of the Bessel's equation can be expressed as

$$y = AJ_n(x) + BJ_{-n}(x)$$

Where $J_n(x)$ is called the Bessel function of the first kind of order n and $J_{-n}(x)$ is called the Bessel function of first kind of order $-n$.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

and,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,(-n+r)!}$$

Case 2 : *n is an integer*

The complete solution of the Bessel's equation can be expressed as

$$y = AJ_n(x) + BY_n(x)$$

Where $Y_n(x)$ is called Bessel function of second kind of order n,

$$Y_n(x) = J_n(x) \int \frac{\mathrm{d}\, x}{x\,(J_n(x))^2}$$

Problem 1.1.1. Prove that $J_{-n}(x) = (-1)^n J_n(x)$

Proof. We have,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,(-n+r)!} \tag{1.1}$$

Let,

$$r - n = s$$
$$\Rightarrow r = n + s$$

From (1.1),

$$\begin{aligned} J_{-n}(x) &= \sum \frac{(-1)^{n+s} \left(\frac{x}{2}\right)^{-n+2(n+s)}}{(n+s)!\,(-n+n+s)!} \\ &= (-1)^n \sum \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{s!\,(n+s)!} \\ &= (-1)^n J_n(x) \end{aligned}$$

□

Problem 1.1.2. Prove the following

(i) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(ii) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

(iii) $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$

(iv) $J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x\right)$

Proof. We have

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

Putting $r = 0, 1, 2, \dots$

$$\begin{aligned} J_n(x) &= \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{\left(\frac{x}{2}\right)^2}{(n+1)!} + \frac{\left(\frac{x}{2}\right)^4}{2!\,(n+2)!} - \dots\right] \\ &= \frac{\left(\frac{x}{2}\right)^n}{n!} \left[1 - \frac{\left(\frac{x}{2}\right)^2}{n+1} + \frac{\left(\frac{x}{2}\right)^4}{2!\,(n+2)(n+1)} - \dots\right] \end{aligned} \tag{1.2}$$

(i) Putting $n = \frac{1}{2}$ in (1.2) we get,

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!} \left[1 - \frac{\left(\frac{x}{2}\right)^2}{\frac{1}{2}+1} + \frac{\left(\frac{x}{2}\right)^4}{2!\,\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+1\right)} - \dots\right] \\ &= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!} \left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots\right] \\ &= \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right] \\ &= \sqrt{\frac{x}{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right] \\ &= \sqrt{\frac{2}{\pi x}} \sin x \end{aligned}$$

(ii) Putting $n = -\frac{1}{2}$ in (1.2) we get,

$$\begin{aligned} J_{-\frac{1}{2}}(x) &= \frac{\left(\frac{x}{2}\right)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} \left[1 - \frac{\left(\frac{x}{2}\right)^2}{-\frac{1}{2}+1} + \frac{\left(\frac{x}{2}\right)^4}{2!\,\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+1\right)} - \dots\right] \\ &= \frac{\left(\frac{x}{2}\right)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} \left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right] \\ &= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \cdot \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right] \\ &= \sqrt{\frac{2}{\pi x}} \cos x \end{aligned}$$

$\Gamma(n+1) = n!$
$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

(iii) From the recurrence relation we have,

$$\frac{2n}{x}J_n(x) = \{J_{n+1}(x) + J_{n-1}(x)\}$$

Putting $n = \frac{1}{2}$ we get

$$\begin{aligned}
J_{\frac{1}{2}}(x) &= x\left\{J_{\frac{3}{2}}(x) + J_{-\frac{1}{2}}(x)\right\}\\
\Rightarrow J_{\frac{3}{2}}(x) &= \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)\\
\Rightarrow J_{\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)
\end{aligned}$$

(iv) Again putting $n = -\frac{1}{2}$ in the recurrence relation we get

$$\begin{aligned}
-J_{-\frac{1}{2}}(x) &= x\left\{J_{\frac{1}{2}}(x) + J_{-\frac{3}{2}}(x)\right\}\\
\Rightarrow J_{-\frac{3}{2}}(x) &= -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)\\
\Rightarrow J_{-\frac{3}{2}}(x) &= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)
\end{aligned}$$

□

Problem 1.1.3. Show that

$$J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x) = \frac{2\sin n\pi}{\pi x}$$

Solution. The Bessel's differential equation is

$$\begin{aligned}
&x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - n^2\right)y = 0\\
\Rightarrow &\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{n^2}{x^2}\right)y = 0
\end{aligned}$$

Since $J_n(x)$ and $J_{-n}(x)$ satisfies the Bessel's differential equation,

$$J_n''(x) + \frac{1}{x}J_n'(x) + \left(1 - \frac{n^2}{x^2}\right)J_n(x) = 0 \tag{1.3}$$

And,

$$J_{-n}''(x) + \frac{1}{x}J_{-n}'(x) + \left(1 - \frac{n^2}{x^2}\right)J_{-n}(x) = 0 \tag{1.4}$$

Now (1.3)$\times J_{-n}(x)-$(1.4)$\times J_n x$,

$$J_n''(x)J_{-n}(x) - J_{-n}''(x)J_n(x) + \left[J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x)\right] = 0 \tag{1.5}$$

Put $\quad z = J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x)$

$\quad\therefore z' = J_n''(x)J_{-n}(x) + J_n'(x)J_{-n}'(x) - J_{-n}''(x)J_n(x) - J_{-n}'(x)J_n'(x)$

From (1.5),

$$\begin{aligned}
&z' + \frac{1}{x}z = 0\\
\Rightarrow &\frac{z'}{z} + \frac{1}{x} = 0\\
\Rightarrow &\log z + \log x = \log c\\
\Rightarrow &zx = c\\
\Rightarrow &J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x) = \frac{c}{x}
\end{aligned} \tag{1.6}$$

But

$$\begin{aligned}
J_n(x) &= \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} \qquad \text{and,}\\
J_{-n}(x) &= \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}\\
\therefore\ J_n'(x) &= \sum_{r=0}^{\infty}(-1)^r \frac{(n+2r)}{2\cdot r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1}\\
\therefore\ J_{-n}'(x) &= \sum_{r=0}^{\infty}(-1)^r \frac{(-n+2r)}{2\cdot r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r-1}
\end{aligned}$$

From (1.6),

$$\begin{aligned}
&\sum_{r=0}^{\infty}(-1)^r \frac{(n+2r)}{2\cdot r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1}\cdot\sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}\\
&\quad-\sum_{r=0}^{\infty}(-1)^r \frac{(-n+2r)}{2\cdot r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r-1}\cdot\sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = \frac{c}{x}\\
\Rightarrow &\sum_{r=0}^{\infty}(-1)^{2r}\frac{(n+2r)x^{4r-1}}{2^{4r}(r!)^2(n+r)!(-n+r)!} - \sum_{r=0}^{\infty}(-1)^{2r}\frac{(-n+2r)x^{4r-1}}{2^{4r}(r!)^2(n+r)!(-n+r)!} = \frac{c}{x}
\end{aligned}$$

Equating the coefficient of $\frac{1}{x}$ from both sides,

$$\begin{aligned}
&\frac{1}{n!\,(-n)!}\{n-(-n)\} = c\\
\Rightarrow &\frac{2n}{\Gamma(n+1)\,\Gamma(-n+1)} = c\\
\Rightarrow &c = \frac{2}{\Gamma(n)\,\Gamma(1-n)}\\
\Rightarrow &c = \frac{2\sin n\pi}{\pi}
\end{aligned}$$

$$\therefore\ J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x) = \frac{2\sin n\pi}{\pi x}$$

1.2 Orthogonality of Bessel Functions

Problem 1.2.1. Show that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,\mathrm{d}\,x = 0$$

where α and β are different roots of $J_n(x)=0$.

Proof. We have $u = J_n(\alpha x)$ and $v = J_n(\beta c)$ respectively be the solution of

$$x^2u'' + xu' + \left(\alpha^2x^2 - n^2\right)u = 0 \tag{1.7}$$

$$x^2v'' + xv' + \left(\alpha^2x^2 - n^2\right)v = 0 \tag{1.8}$$

Now (1.7) $\times \frac{v}{x}$ − (1.8) $\times \frac{u}{x}$,

$$\begin{aligned}
&x\left(u''v - uv''\right) + \left(u'v - uv'\right) + \left(\alpha^2-\beta^2\right)xuv = 0\\
\Rightarrow &\frac{\mathrm{d}\,\{x\left(u'v-uv'\right)\}}{\mathrm{d}\,x} = \left(\beta^2-\alpha^2\right)xuv\\
\Rightarrow &= \int_0^1\left(\beta^2-\alpha^2\right)xuv\,\mathrm{d}\,x = \left[x\left(u'v-uv'\right)\right]_0^1\\
\Rightarrow &= \int_0^1\left(\beta^2-\alpha^2\right)xuv\,\mathrm{d}\,x = \left[\left(u'v-uv'\right)\right]_{x=1}\\
\Rightarrow &= \int_0^1 (xuv)\ \mathrm{d}\,x = \frac{1}{\beta^2-\alpha^2}\left[\left(u'v-uv'\right)\right]_{x=1}
\end{aligned} \tag{1.9}$$

But $u' = \alpha J_n'(\alpha x)$, $v' = \beta J_n'(\beta x)$
From (1.9)

$$\Rightarrow \int_0^1 (xuv)\,\mathrm{d}\,x = \frac{\alpha J_n'(\alpha x) J_n(\beta) - \beta J_n(\alpha) J_n'(\beta x)}{\beta^2 - \alpha^2} \tag{1.10}$$

If α and β are distinct roots of $J_n(x) = 0$ then $J_n(\alpha) = J_n(\beta) = 0$
From (1.10),

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,\mathrm{d}\,x = 0$$

Note. The Bessel's equation is

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} + x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + \left(x^2 - n^2\right) y = 0$$

Let $x = \alpha r$, we get

$$r^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\,r} + r\frac{\mathrm{d}\,y}{\mathrm{d}\,r} + \left(\alpha^2 r^2 - n^2\right) y = 0$$

$$\Rightarrow x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} + x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + \left(\alpha^2 x^2 - n^2\right) y = 0$$

$$x = \alpha r$$

$$\therefore \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\mathrm{d}\,y}{\mathrm{d}\,r} \cdot \frac{\mathrm{d}\,r}{\mathrm{d}\,x}$$

$$\Rightarrow \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{1}{\alpha}\frac{\mathrm{d}\,y}{\mathrm{d}\,r}$$

$$\therefore x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\alpha r}{\alpha}\frac{\mathrm{d}\,y}{\mathrm{d}\,r}$$

$$\therefore x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = r\frac{\mathrm{d}\,y}{\mathrm{d}\,r}$$

□

Problem 1.2.2. Show that

$$\int_0^x x^n J_{n-1}(x)\,\mathrm{d}\,x = x^n J_n(x)$$

Proof. We have,

$$x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!}(x)^{2n+2r}\frac{1}{2^{n+2r}}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}\,x}\left(x^n J_n(x)\right) = x^n J_{n-1}(x) \tag{1.11}$$

Integrating (1.11) with respect to x from 0 to x we get,

$$\begin{aligned}\int_0^x x^n J_{n-1}(x)\,\mathrm{d}\,x &= \left[x^n J_n(x)\right]_0^x \\ &= x^n J_n(x) + \lim_{x\to 0} x^n J_n(x) \\ &= x^n J_n(x) + 0 \\ &= x^n J_n(x)\end{aligned}$$

□

Problem 1.2.3. Show that

$$\int_0^x x^{-n} J_{n+1}(x)\,\mathrm{d}\,x = \frac{1}{2^n\,n!} - x^{-n} J_n(x)$$

Proof. We have,

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!}\frac{1}{2^{n+2r}}(x)^{2r}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}\,x}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) \tag{1.12}$$

Integrating (1.12) with respect to x from 0 to x we get,

$$\begin{aligned}\int_0^x -x^{-n} J_{n+1}(x)\,\mathrm{d}\,x &= \left[-x^{-n} J_n(x)\right]_0^x \\ &= -x^{-n} J_n(x) + \lim_{x\to 0} x^{-n} J_n(x) \qquad (1.13)\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{x\to 0}\left(x^{-n}J_n(x)\right) &= \lim_{x\to 0}\frac{J_n(x)}{x^n} \\
&= \lim_{x\to 0}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+2r)!}\frac{1}{2^{n+2r}}x^{2n} \\
&= \lim_{x\to 0}\left[\frac{1}{2^n\,n!}-\frac{1}{2^{n+2}(n+2)!}x^2+\dots\right] \\
&= \frac{1}{2^n\,n!}
\end{aligned}$$

From (1.13)

$$\int_0^x -x^{-n}J_{n+1}(x)\,\mathrm{d}\,x = \frac{1}{2^n n!} - x^{-n}J_n(x)$$

□

1.3 Recurrence Relation

Problem 1.3.1. Prove the following recurrence formula for $J_n(x)$

(i) $\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^nJ_n(x)\right] = x^nJ_{n-1}x$

(ii) $\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}x$

(iii) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]$

(iv) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$

(v) $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

Proof.

(i) From the Bessel function of the first kind of order n
We have,

$$\begin{aligned}
J_n(x) &= \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!} \\
\therefore\ x^nJ_n(x) &= \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{2n+2r}}{2^{n+2r}r!\,(n+r)!}
\end{aligned}$$

$$\begin{aligned}
\therefore\ \frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^nJ_n(x)\right] &= \sum_{r=0}^{\infty}(-1)^r\frac{1}{2^{n+2r}\,n!\,(n+r)!}\cdot 2(n+r)x^{2(n+r)-1} \\
&= \sum(-1)^r\frac{1}{r!\,(n+r-1)!}\frac{x^n - x^{n+2r-1}}{2^{n+2r-1}} \\
&= x^n\sum(-1)^r\frac{1}{r!\,(n-1+r)!}\left(\frac{x}{2}\right)^{(n-1)+2r} \\
&= x^nJ_{n-1}(x)
\end{aligned}$$

(ii)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left[\sum(-1)^r\frac{x^{2r}}{r!\,(n+r)!\,2^{n+2r}}\right] \\
&= \sum(-1)^r\frac{2r\cdot x^{2r-1}}{2^{n+2r}\,n!\,(n+r)!} \\
&= \sum(-1)^r\frac{x^{2r-1}}{r!\,(n+r-1)!2^{n-1+2r}} \\
&= -\sum(-1)^{r-1}\frac{1}{(r-1)!\,(n+r)!}\frac{x^{n+2r-1}\cdot x^{-n}}{2^{n-1+2r}} \\
&= -x^{-n}\sum_{r=0}^{\infty}(-1)^{r-1}\frac{1}{(r-1)!\,(n+r)!}\cdot\frac{x^{n+1+2(r-1)}}{x^{n+1+2(r-1)}}
\end{aligned}$$

When $r = 0$ $(r-1)! = (-1)! = \infty$
i.e., $\frac{1}{(r-1)!} = 0$
$\therefore$ When $r = 0$, the first term vanishes. So,

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] = -x^{-n}\sum_{r=1}^{\infty}(-1)^r\frac{1}{(r-1)!\,(n+1+r-1)!}\cdot\left(\frac{x}{2}\right)^{n+2r-1}$$

Putting $r - 1 = k$ i.e., $r = k + 1$

$$\begin{aligned}\therefore\ \frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] &= -x^{-n}\sum_{k=0}^{\infty}(-1)^r\frac{1}{k!\,(n+1+k)!}\cdot\frac{x}{2}^{n+1+2k} \\ &= -x^{-n}J_{n+1}(x)\end{aligned} \quad \left|\quad \begin{array}{l}\text{When,}\\ \quad r=1,\ k=0\end{array}\right.$$

(iii) We have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)\\ &\Rightarrow x^n J_n^{'}(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x)\\ &\Rightarrow J_n^{'}(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)\end{aligned}\tag{1.14}$$

Also,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)\\ &\Rightarrow x^{-n}J_n^{'}(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)\\ &\Rightarrow -J_n^{'}(x) + \frac{n}{x}J_n(x) = J_{n+1}(x)\end{aligned}\tag{1.15}$$

Adding (1.14) with (1.15) with we get,

$$\begin{aligned}\frac{2n}{x}J_n(x) &= J_{n-1}(x) + J_{n+1}(x)\\ \therefore J_n(x) &= \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]\end{aligned}$$

(iv) Subtracting (1.15) from (1.14) we get,

$$\begin{aligned}2J_n^{'}(x) &= J_{n-1}(x) - J_{n+1}(x)\\ \therefore J_n^{'}(x) &= \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\end{aligned}$$

(v) We have

$$\begin{aligned}J_n(x) &= \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]\\ \Rightarrow \frac{2n}{x}J_n(x) &= \left[J_{n-1}(x) + J_{n+1}(x)\right]\end{aligned}\tag{1.16}$$

Again

$$\begin{aligned}J_n^{'}(x) &= \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\\ \Rightarrow 2J_n^{'}(x) &= J_{n-1}(x) - J_{n+1}(x)\end{aligned}\tag{1.17}$$

Subtracting (1.17) from (1.16) we get,

$$\begin{aligned}\frac{2n}{x}J_n(x) - 2J_n^{'} &= 2J_{n+1}(x)\\ \therefore xJ_n^{'}(x) &= nJ_n(x) - xJ_{n+1}(x)\end{aligned}$$

$\square$

Problem 1.3.2. Show that
$$xJ_n^{'} = -nJ_n + xJ_{n-1}$$

Solution.
$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$
$$\therefore J_n^{'} = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$
$$\Rightarrow xJ_n^{'} = \sum \frac{(-1)^r (2n+2r-n)}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{2}$$
$$\Rightarrow xJ_n^{'} = -n \sum \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r} + \sum \frac{(-1)^r 2(n+r)}{r!\,(n+r)!} \cdot \frac{x^{n+2r}}{2}$$
$$\Rightarrow xJ_n^{'} = -nJ_n + x \sum \frac{(-1)^r}{r!\,(n+r-1)!} \cdot \frac{x^{n+2r-1}}{2}$$
$$\Rightarrow xJ_n^{'} = -nJ_n + xJ_{n-1}$$

1.4 Generating Function of the Bessel Function $J_n(x)$

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Proof. From the exponential series, we have
$$e^z = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \tag{1.18}$$

$$\begin{aligned} e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} &= e^{\frac{1}{2}tx} \cdot e^{\frac{-x}{2t}} \\ &= \left[1 + \frac{tx}{2 \cdot 1!} + \frac{t^2x^2}{2^2 \cdot 2!} + \frac{t^3x^3}{2^3 \cdot 3!} + \dots + \frac{t^nx^n}{2^n \cdot n!} + \frac{t^{n+1}x^{n+1}}{2^{n+1} \cdot (n+1)!} + \dots\right] \times \\ &\left[1 - \frac{x}{2t \cdot 1!} + \frac{x^2}{2^2 \cdot t^2 \cdot 2!} - \frac{x^3}{2^3 \cdot t^3 \cdot 3!} + \dots + (-1)^n \frac{x^n}{2^n \cdot t^n \cdot n!} + (-1)^{n+1} \frac{x^{n+1}}{2^{n+1} \cdot t^{n+1} \cdot (n+1)!} + \dots\right] \end{aligned}$$

In this product the coefficient of t^n is
$$\begin{aligned} &\frac{x^n}{2^n \cdot n!} - \frac{x^{n+1}}{2^{n+1} \cdot (n+1)!} \cdot \frac{x}{2} + \frac{x^{n+2}}{2^{n+2} \cdot (n+2)!} \cdot \frac{x^2}{2^2 \cdot 2!} - \frac{x^{n+3}}{2^{n+3} \cdot (n+3)!} \cdot \frac{x^3}{2^3\,3!} + \dots \\ =&\frac{1}{n!} \left(\frac{x}{2}\right)^n - \frac{1}{1!\,(n+1)!} \left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\,(n+2)!} \left(\frac{x}{2}\right)^{n+4} - \frac{1}{3!\,(n+3)!} \left(\frac{x}{2}\right)^{n+6} + \dots \\ =&\sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\,(n+m)!} \left(\frac{x}{2}\right)^{n+2m} \\ =&J_n(x) \end{aligned}$$

Also in the product, the coefficient of t^{-n} is
$$\begin{aligned} &(-1)^n \left[2^n \frac{x^n}{n!} - \frac{x^{n+1}}{2^{n+1} \cdot (n+1)!} \cdot \frac{x}{2} + \frac{x^{n+2}}{2^{n+2} \cdot (n+2)!} \cdot \frac{x^2}{2^2 \cdot 2!} - \frac{x^{n+3}}{2^{n+3} \cdot (n+3)!} \cdot \frac{x^3}{2^3\,3!} + \dots\right] \\ =&(-1)^n \left[\frac{1}{n!} \left(\frac{x}{2}\right)^n - \frac{1}{1!\,(n+1)!} \left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\,(n+2)!} \left(\frac{x}{2}\right)^{n+4} - \frac{1}{3!\,(n+3)!} \left(\frac{x}{2}\right)^{n+6} + \dots\right] \\ =&(-1)^n J_n(x) \\ =&J_{-n}(x) \end{aligned}$$

Thus all the integral powers of t both positive and negative occur in the product.
Hence, we have
$$\begin{aligned} e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} &= J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \dots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \dots \\ &= \sum_{n=\infty}^{\infty} t^n J_n(x) \end{aligned}$$

For this reason $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$ is called the generating function of Bessel function. □