Chapter 1

Topological spaces and metric spaces

The topology on $\mathbb{R}^n$ is defined in terms of open balls, which in turn are defined in terms of distance between points. There are many other spaces whose topology can be defined in a similar way in terms of a suitable notion of distance between points in the space.

Definition 1.1. A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A metric space (X, d) is a set equipped with a metric d on X. We denote a metric space simply by X.

The **open ball** of radius r centered at x in a metric space (X, d) is defined as

$$B_r(x) = \{y \in X | d(x, y) < r\}$$

that is, the points in X within r in distance from x. It is also known as **open sphere**, or $r-$ **neighborhood** of x.

Definition 1.2. A subset U of a metric space (X, d) is open if for any $x \in U$ there is an $r > 0$ so that $B_r(x) \subseteq U$. We note the following properties of open subsets of metric spaces.

1. An open ball $B_r(x)$ is an open set in (X, d)
2. An arbitrary union of open subsets is open.
3. The finite intersection of open subsets is open.

Theorem 1.1. Let (X, d) be a metric space. For any $x \in X$ and $r > 0$, let

$$B_r(x_= \{y \in X | d(x, y) < r\})$$

Define $\tau_d = \{A \subseteq X : \forall X \in A \exists r > 0 \text{ such that } B_r \subseteq A\} \cup \varnothing$. Then τ_d is a topology on X.

Proof. (i) By definition, $\varnothing \in \tau_d$. Also, for any $x \in X$, there exist an $r > 0$ such that $B_r(x) \subseteq X$. Hence $X \in \tau_d$.

(ii) Let $A, B \in \tau_d$. If $A \cap B = \varnothing$, then clearly $A \cap B \in \tau_d$. If $A \cap B \neq \varnothing$, then for $x \in A \cap B$ we get $x \in A$ and $x \in B$. Now, $x \in A$ implies there exists $r_1 > 0$ such that $B_{r_1}(x) \subseteq A$ and $x \in B$ implies there exists $r_2 > 0$ such that $B_{r_2}(x) \subseteq B$. Let $r = \min(r_1, r_2)$. Then $B_r(x) \subseteq B_{r_1}(x)$ and $B_r(x) \subseteq B_{r_2}(x)$. Thus $B_r(x) \subseteq A \cap B$. Hence $A \cap B \in \tau_d$.

(iii) Let $\{A_\alpha\}$ be a family of members of τ_d. Let $x \in \cup_{\alpha\in\Omega} A_\alpha$. Then $x \in A_{\alpha_0}$ for some $\alpha_0 \in \tau_d$, there exist $r > 0$ such that $B_r(x) \subseteq A_{\alpha_0}$ and hence $B_r(x) \subseteq \cup_{\alpha\in\Omega} A_\alpha$. Therefore $\cup_{\alpha\in\Omega} A_\alpha \in \tau_d$.

From (i), (ii) and (iii) τ_d is a topology on X. □

This topology τ_d is called the topology induced by the metric d on X.

Let (X, d) be a metric space and τ be the collection of all open sets in (X, d). Then τ is a topology on X, called a **metric topology** generated by (induced by) the metric d and the open balls of all points are a basis for this topology.

Example 1.1. Let d be a usual metric on the real line $\mathbb{R}$, i.e., $d(x, y) = |x - y|$, then the open balls in $\mathbb{R}$ are precisely the finite open intervals and these open intervals forms a topology on $\mathbb{R}$ called usual topology. Hence the usual metric on $\mathbb{R}$ induces the usual topology on $\mathbb{R}$. Similarly, the usual metric on the plane $\mathbb{R}^2$ induces the usual topology on $\mathbb{R}^2$.

Proposition 1.1. The collection of all open balls $B_r(x)$ for $r > 0$ and $x \in X$ forms a base for a topology on X.

Proof. First a preliminary observation: For a point $y \in B_r(X)$ the ball $B_s(y)$ is contained in $B_r(x)$ if $s \leq r - d(x, y)$, since for $z \in B_s(y)$, we have $d(z, y) < s$ and hence

$$d(z, x) \leq d(z, y) + d(y, z) < s + d(x, y) \leq r$$

Now to show the condition to have a basis is satisfied, suppose $y \in B_{r_1}(x_1) \cap B_{r_2}(x_2)$. Then the observation in the preceding paragraph implies that $B_s(y) \subseteq B_{r_1}(x_1) \cap B_{r_2}(x_2)$, for any $s \leq \min\{r_1 - d(x_1, y), r_2 - d(x_2, y)\}$. Therefore the collection of all open balls $B_r(x)$ is a base for a topology on X. □

A topological space (X, τ) together with a metric d that induces the topology τ is called a **metric topological space** or a **metric space** and it is denoted by (X, d).

Definition 1.3. A topological space (X, τ) is said to be **metrizable** if there is a metric d on X which induces the topology τ.

Example 1.2. $(\mathbb{R}, \tau_u)$ is a metrizable space.

Example 1.3. Discrete topological space is a metrizable space.

Example 1.4. Let $X = \{x, y\}$ and τ be the indiscrete topology. Then τ is not metrizable. Indeed, assume that τ is a metric topology for some metric d. Let $r = d(x, y)$. Then $B_r(x) = \{x\}$ is an open set. But $\{x\}$ is not an element of τ. A contradiction.

Example 1.5. Let X be an arbitrary set and let τ be a discrete topology on X. Let d be a metric on X defined by

$$d(x, y) = \begin{cases} 0, & \text{for } x = y \\ 1, & \text{for } x \neq y \end{cases}$$

Then $B_{\frac{1}{2}}(x) = \{x\}$; so, singleton subsets are open and hence d induces the discrete topology on X. Thus, we find a trivial metric d on X which induces the given topology τ. Accordingly, (X, τ) is metrizable.

1.0.1 Distance between Sets, Diameters

Let d be a metric on a set X. The **distance** between two non-empty sets A and B is denoted and defined by

$$d(A, B) = \inf \{d(a, b) : a \in A \text{ and } b \in B\}$$

The distance between a point $p \in X$ and a non-empty subset B of X is denoted and defined by

$$d(p, B) = \inf \{d(p, b) : b \in B\}$$

The **diameter** of a non-empty subset E of X is denoted and defined by

$$d(E) = \sup \{d(a, b) : a, b \in E\}$$

If the diameter of a non-empty subset E of X is finite, i.e., $d(E) < \infty$, then E is said to be bounded. If $d(E) = \infty$, then E is said to be unbounded. Clearly a set has diameter 0 iff it is a singleton set.

Example 1.6. Let d be a trivial metric on X defined by

$$d(x, y) = \begin{cases} 0, & \text{for } x = y \\ 1, & \text{for } x \neq y \end{cases}$$

Then for any $p \in X$ and $A, B \subseteq X$.

$$d(p, A) = \begin{cases} 0, & \text{for } p \in A \\ 1, & \text{for } p \notin A \end{cases} \qquad d(A, B) = \begin{cases} 0, & \text{if } A \cap B = \varnothing \\ 1, & \text{if } A \cap B \neq \varnothing \end{cases}$$

Theorem 1.2. Let d be a metric on a set X. For any nonempty subset E of X, $d(\bar{E}) = d(E)$.

Proof. We know that $E \subset \bar{E}$. Now, if $d(\bar{E})$ is infinite, then there is nothing to prove. So, let $d(E) = r$ which is finite. If $d(\bar{E}) = r'$, then $r' \geq r$. Suppose $r' \neq r$ and let $r' - r = s > 0$. Then any point $x_0 \in \bar{E} - E$ must be a limit point of E and any open sphere centered at x_0 contains some points of E. But the open sphere $N_{\frac{s}{2}}(x_0)$ does not contain any point of E.

Hence our assumption that $r' > r$ is wrong and therefore $r' = r$, i.e., $d(\bar{E}) = d(E)$. □

Theorem 1.3. For any non-empty set A of a metric space X, the closure $\bar{A}$ of A is the set of points whose distance from A is 0.

This theorem can be stated as:
Let A be a non-empty subset of a metric space X. Then $d(x, A) = 0$ iff $x \in \bar{A}$.

Proof. Let $d(x, A) = 0$. Then every open sphere with center at x contains at least one point of A and therefore every open set G containing x also contains at least one point of A. Hence, x is a limit point of A and so $x \in \bar{A}$.

Conversely, let $x \in \bar{A}$. Suppose that $d(x, A) \neq 0$ and $d(x, A) = \varepsilon > 0$. Then the open sphere $S_{\frac{\varepsilon}{2}}(x)$ with center x contains no points of A and so x is an exterior point of A; i.e., $x \notin \bar{A}$, a contradiction. Hence, $\bar{A} = \{x : d(x, A) = 0\}$. □

Theorem 1.4. Let A and B be closed disjoint subset of a metric space X. Then there exist disjoint open subsets G and H in X such that $A \subseteq G$ and $B \subseteq H$.

Proof. If either A or B is empty, say $A = \varnothing$, the $\varnothing$ and X are disjoint open sets such that $A \subseteq \varnothing$ and $B \subseteq X$. Hence, we may assume that A and B are non empty.
Let $a \in A$. THen since A and B are disjoint, $a \notin B$ and so $d(a, B) > 0$. Similarly, if $b \in B$, then $d(b, A) > 0$. Set

$$S_a = S_{\frac{\delta}{3}}(a) \quad \text{and} \quad S_b = S_{\frac{\delta}{3}}(b)$$

Clearly, $a \in S_a$ and $b \in S_b$.

Let $G = \{S_a : a \in A\}$ and $H = \{s_b : b \in B\}$. Then clearly G and H are open because they are the union of open spheres and $A \subseteq G$ and $B \subseteq H$. We now have to show that $G \cap H = \varnothing$.
Suppose $G \cap H \neq \varnothing$ and let $p \in G \cap H$. Then $p \in G$ and $p \in H$ implies $p \in S_{a_0}$ and $p \in S_{b_0}$ for some $a_0 \in A$ and $b_0 \in B$ respectively. Let $d(a_0, b_0) = \varepsilon > 0$. Then $d(a_0, B) < \varepsilon$ and $d(b_0, A) < \varepsilon$. But $d(a_0, p) < \dfrac{\delta}{3}$ and $d(b_0, p) < \dfrac{\delta}{3}$. Therefore, by triangle inequality,

$$\varepsilon = d(a_0, b_0) \leq d(a_0, p) + d(p, b_0) < \frac{\delta}{3} + \frac{\delta}{3} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

which is impossible. Hence, G and H are disjoint. □

1.1 Euclidean $n-$dimensional space

In $\mathbb{R}^n$ space, the function d defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ with $x_i, y_i \in \mathbb{R}$ is a metric called the **Euclidean** metric on $\mathbb{R}^n$ and the space $(\mathbb{R}^n, d)$ is known as **Euclidean $n-$dimensional space**.

Theorem 1.5. Euclidean $n-$dimensional space is a metric space.

1.2 Hilbert Space

Hilbert Space is an immediate generalization of Euclidean $n-$dimensional space $\mathbb{R}^n$ arises when we replace $n-$tuples $x = (x_1, x_2, \ldots, x_n)$ with sequences $x = \langle x_1, x_2, \ldots \rangle$.
Let ℓ_2 denote the set of all sequences of real numbers such that

$$\sum_1^\infty (x_k)^2 < \infty$$

i.e., such that the series $x_1^2 + x_2^2 + \ldots$ converges and define

$$d(x, y) = \sqrt{\sum_1^\infty (x_k - y_k)^2}$$

Then d is a metric on ℓ_2. The resulting metric space ℓ_2 is usually called ℓ_2 space or **Hilbert** space, named after one of the most important and influential mathematician of his time, David Hilbert (1862-1943).

Theorem 1.6. Hilbert space or ℓ_2 space is a metric space.

1.3 Normed Space

A **norm** on a linear space is a function that gives a notion of the ‘length’ of a vector. The formal definition of a norm on a linear space is given below:
A norm on a linear space X is a function $||\cdot|| : X \to \mathbb{R}$ with the following properties:

1. $||x|| \geq 0$, for all $x \in X$ and $||x|| = 0$ implies $x = 0$
2. $||\lambda x|| = |\lambda| \; ||x||$, for all $x \in X$ and $\lambda \in \mathbb{R}$
3. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$

A linear space X together with a norm is called a **normed linear space**.
A normed linear space X is metric space with the metric

$$d(x, y) = ||x - y||$$

And it is known as induced metric on X.
The set of real numbers $\mathbb{R}$ with the absolute value norm $||x|| = |x|$ is a one-dimensional real normed linear space. More generally, $\mathbb{R}^n$, where $n = 1, 2, \ldots,$ is an $n-$dimension linear space. We define **Euclidean norm** of a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}$ by

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

A normed linear space X that is complete (every Cauchy sequence on X converges in X) with respect to the metric d is called a **Banach space**.