Questions from Previous Years

2014-2015 (2017)

1. Marks: $5 + 6 + 3 = 14$

   (a) Show that $P_n(x)$ is the coefficient of z^n in the expansion of $(1 - 2xz + z^2)^{-1/2}$ in ascending powers of z.

   (b) Prove that $P_n(x) = \dfrac{1}{2^n n!} \dfrac{\mathrm{d}^n}{\mathrm{d}\, x^n}(x^2 - 1)^n$.

   (c) Show that $\int_{-1}^{1} P_n(x)\,\mathrm{d}\, x = 0$ except when $n = 0$; in which case the value of the integral is 2.

2. Marks: $8 + 6 = 14$

   (a) Prove that

   $$\int_0^\infty e^{-x} L_n(x) L_m(x)\,\mathrm{d}\, x = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

   (b) Prove that

   i. $H_n'(x) = 2nH_{n-1}(x), \quad (n \geq 1), \quad H_0'(x) = 0;$

   ii. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), n \geq 1.$

3. Marks: $8 + 6 = 14$

   (a) Define Fourier series in complex form. If

   $$f(x) = \begin{cases} -\cos x & \text{when } -\pi \leq x < 0 \\ \cos x & \text{when } 0 \leq x < \pi \end{cases}$$

   then show that its Fourier series is $f(x) = \dfrac{8}{\pi}\left[\dfrac{\sin 2x}{1 \cdot 3} + \dfrac{2 \sin 4x}{3 \cdot 5} + \dfrac{3 \sin 6x}{5 \cdot 7} + \dots\right]$.
   Hence, deduce $\dfrac{\pi\sqrt{2}}{16} = \dfrac{1}{1 \cdot 3} - \dfrac{3}{5 \cdot 7} + \dfrac{5}{9 \cdot 11} - \dots$.

   (b) Find the cosine transform of a function of x which is unity for $0 < x < a$ and zero for $x \geq a$. What is the function whose Fourier cosine transform is $\dfrac{\sin na}{n}$?

4. Marks: $10 + 4 = 14$

   (a) Solve in series and find the region of convergence of the D.E: $(2x + x^3)\dfrac{\mathrm{d}^2\, y}{\mathrm{d}\, x^2} - \dfrac{\mathrm{d}\, y}{\mathrm{d}\, x} - 6xy = 0.$

(b) If $\mathcal{L}\{F(t)\} = f(s)$, then prove that $\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots$.

5. Marks: $7 + 7 = 14$

(a) Show that $J_n(x)J'_{-n}(x) - J'_n(x)J_{-n}(x) = \dfrac{2\sin \pi x}{\pi x}$.

(b) If α, β are roots of $J_n(x) = 0$, then prove that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,\mathrm{d}\,x = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}J_{n+1}^2(\alpha), & \text{if } \alpha = \beta \end{cases}$$

6. Marks: $5 + 5 + 4 = 14$

(a) If $F(t)$ has period $T > 0$ then prove that $\mathcal{L}\{F(t)\} = \dfrac{\int_0^T e^{-st}F(t)\,\mathrm{d}\,t}{1 - e^{-sT}}$.

(b) Find $\mathcal{L}\{J_0(t)\}$ where $J_0(t)$ is the Bessel function of order zero.

(c) Find $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$.

7. Marks: $(5 + 5) + 4 = 14$

(a) Solve the following equations by using Laplace transforms:

(i) $y'' + 2y' + 5y = e^{-t}\sin t, \qquad y(0) = 0 \quad y'(0) = 1.$

(ii) $y'' + 9y = \cos 2t, \qquad y(0) = 1 \quad y'(\pi/2) = -1.$

(b) Find the Fourier integral of the function $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$ for $k > 0$ and hence prove that $\displaystyle\int_0^\infty \frac{\cos ux}{k^2+u^2}\,\mathrm{d}\,u = \frac{\pi}{2k}e^{-kx}$.

8. Marks: $7 + 7 = 14$

(a) Define Eigenvalue and Eigenfunction. Find the Eigenvalues and corresponding Eigenfunctions of the BVP $y'' + \lambda y = 0$ when $y(0) = y'(\pi) = 0$.

(b) Define Green's function by the relations of the BVP $y'' + \lambda y = f(x), \qquad y(0) = y(\pi) = 0.$

2015-2016 (2018)

1. Marks: $6 + (3 + 5) = 14$

   (a) Establish the relation $P_n(x) = \dfrac{1}{2^n\, n!} \dfrac{\mathrm{d}^n}{\mathrm{d}\, x^n} \left(x^2 - 1\right)^n$ for Legendre's polynomial.

   (b) Prove that

   (i) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$;

   (ii) $\displaystyle\int_0^{\pi} P_n(\cos\theta) \cos n\theta \,\mathrm{d}\,\theta = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}\pi.$

2. Marks: $10 + 4 = 14$

   (a) Derive Bessel's equation from Legendre differential equation.

   (b) Show that $J_n(x) = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \cos(n\phi - x\sin\phi) \,\mathrm{d}\,\phi$.

3. Marks: $7 + (3 + 4) = 14$

   (a) Prove that $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^2} \{H_n(x)\}^2 \,\mathrm{d}\,x = \sqrt{\pi} 2^n\, n! \left(n + \frac{1}{2}\right)$.

   (b) Prove that

   i. $H_n'(x) = 2nH_{n-1}(x), \qquad (n \geq 1), \quad H_0'(x) = 0$;

   ii. $\displaystyle\int_0^{\infty} e^{-ax} J_0(bx) \,\mathrm{d}\,x = \frac{1}{\sqrt{a^2 + b^2}},\ a > 0$

4. Marks: 14
   Define regular and irregular singular points. Locate and classify the singular points of the following differential equation:

$$x(1-x)\frac{\mathrm{d}^2\, y}{\mathrm{d}\, x^2} + \{\gamma - (1 + \alpha + \beta)\, x\}\, \frac{\mathrm{d}\, y}{\mathrm{d}\, x} - \alpha\beta\gamma = 0,$$

   where α, β, γ are parametric constants.
   By the method of Frobenius obtain the solution of the above differential equation.

5. Marks: $6 + 8 = 14$

   (a) Obtain Fourier's series for the expansion of $f(x) = x\sin x$ in the interval $-\pi < x < \pi$. Hence, deduce that $\dfrac{\pi}{4} = \dfrac{1}{2} + \dfrac{1}{1 \cdot 3} - \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} - \ldots$

   (b) Using Fourier transform solve $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$ with conditions
   $U(0,t) = 0$; $U(\pi, t) = 0$, $U(x, 0) = 0.1\sin x + 0.001\sin 4x$ and $U_t(x, 0) = 0$ for $0 < x < \pi$, $t > 0$.

6. Marks: $5 + 3 + 6 = 14$

   (a) Define Laplace transformation. If $\mathcal{L}\{F(t)\} = f(s)$, then prove that
   $\mathcal{L}\{t^n F(t)\} = (-1)^n \dfrac{\mathrm{d}^{\,n}}{\mathrm{d}\, s^n} f(s) = (-1) f^n(s)$ where $n = 1, 2, 3, \ldots$

(b) Find $\mathcal{L}\{J_0(t)\}$ where $J_0(t)$ is the Bessel function of order zero.

(c) Find

i. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$;

ii. $\mathcal{L}^{-1}\left\{\dfrac{4s+12}{s^2+8s+16}\right\}$

7. Marks: $6+8=14$

(a) State and prove the convolution theorem.

(b) Solve the following equations by using Laplace transforms:

i. $y''-3y'+2y=4e^{2t}, \qquad y(0)=-3, \quad y'(0)=5;$

ii. $y''+9y=\cos 2t, \qquad y(0)=1, \quad y(\pi/2)=-1.$

8. Marks: $7+7=14$

(a) Prove that the solution of the boundary value problem

$$\frac{\partial U}{\partial t}=3\frac{\partial^2 U}{\partial x^2} \qquad U(0,t)=U(2,t)=0,\ t>0 \qquad U(x,0)=x,\ 0<x<2$$

is $U(x,t)=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}e^{-\frac{3}{4}n^2\pi^2 t}$

(b) Define eigenvalue and eigenfunction. Find the eigenvalues and the corresponding eigenfunctions of the Strum-Liouville problem $y''+\lambda y=0$ when $y(0)=y'(\pi)=0$.

2016-2017(2019)

1. Marks: $8+6=14$

   (a) Define Fourier series for the function $f(x)$ in the interval $(-l,l)$. Find the Fourier series expansion of the function $f(x)=x+x^2$ in the interval $-\pi<x<\pi$. Hence, deduce that

   $$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots$$

   (b) Find the Fourier integral of the function $f(x)=e^{-kx}$ when $x>0$ and $f(-x)=f(x)$ for $k>0$ and hence prove that

   $$\int_0^\infty \frac{\cos ux}{k^2+u^2}\,\mathrm{d}\,u=\frac{\pi}{2k}e^{-kx}$$

2. Marks: $6+8=14$

   (a) Find the Fourier transform of the function

   $$f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$$

   and find the reciprocal relation.

   (b) Using finite Fourier transforms solve

   $$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2},\qquad U(0,t)=0,\quad U(\pi,t)=0,\quad U(x,0)=2x \text{ where } 0<x<\pi,\ t>0$$

3. Marks: $6+8=14$

   (a) Define Laplace transform and inverse Laplace transform. Find the Laplace transform of $e^{4t}\cos 5t$ and the inverse Laplace transform of $\frac{3s+1}{(s-1)(s^2+6)}$.

   (b) Solve the following differential equation by Laplace transform

   $$X''(t)+4X'(t)+4X(t)=4e^{-2t},\qquad X(0)=-1,\ ,X'(0)=4$$

   verify that your solution satisfies the above differential equation and the given function.

4. Marks: $4+6+4=14$

   (a) If $\mathcal{L}\{F(t)\}=f(s)$ then prove that $\mathcal{L}\{F'(t)\}=s^2f(s)-sF(0)-F'(0)$.

   (b) If $F(t)$ has a period $T>0$ then prove that $\mathcal{L}\{F(t)\}=\dfrac{\int_0^T e^{-st}F(t)\,\mathrm{d}\,t}{1-e^{-sT}}$.

   (c) State the convolution theorem for the inverse Laplace transform. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$ by using the convolution theorem.

5. Marks: $6+8=14$

   (a) Define singular point and regular singular point of a differential equation. Find the singular points of the differential equation $2x^2\dfrac{\mathrm{d}^2\,y}{\mathrm{d}\,x^2}-x\dfrac{\mathrm{d}\,y}{\mathrm{d}\,x}+(x-5)y=0$.

(b) Find the series solution of the differential equation $y'' + xy' + (x^2 + 2)y = 0$ in powers of x.

6. Marks: $5 + 4 + 5 = 14$

7. Marks: $6 + 5 + 3 = 14$

8. Marks: $7 + 7 = 14$