**Problem 1.** Define: modular lattice, distributive lattice, complement, atom, boolean lattice, relative complement, join irreducible element, complete lattice, congruence, equivalence relation.

**Solution.** *Modular lattice:* Let $L$ be a lattice. $L$ is said to be modular if it satisfies the modular law,

$$(\forall a, b, c \in L) a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c.$$

*Distributive lattice:* Let $L$ be a lattice. $L$ is said to be distributive if it satisfies the distributive law,

$$(\forall a, b, c \in L) a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

*Complement:* Let $L$ be a lattice with 0 and 1. For $a \in L$, we say $b \in L$ is a complement of $a$ if $a \wedge b = 0$ and $a \vee b = 1$. If $a$ has a unique complement, we denote this complement by $a'$.

*Atom:* Let $L$ be a lattice with least element 0. Then $a \in L$ is called an atom if $0 \prec a$. The set of atoms of $L$ is denoted by $\mathcal{A}(L)$. The lattice $L$ is called atomic if, given $a \neq 0$ in $L$, there exists $x \in \mathcal{A}(L)$ such that $x \leq a$. Every finite lattice is atomic.

*Boolean Lattice:* A lattice $L$ is called a Boolean lattice if

(i) $L$ is distributive,

(ii) $L$ has 0 and 1,

(iii) each $a \in L$ has a (necessarily unique) complement $a' \in L$

*Relative complement:*

*Join irreducible element:* Let $L$ be a lattice. An element $x \in L$ is join irreducible if

(i) $x \neq 0$ (in case $L$ has a zero),

(ii) $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in L$.

Condition (ii) is equivalent to the more pictorial, (ii)' $a < x$ and $b < x$ imply $a \vee b < x$ for all $a, b \in L$. A meet-irreducible element is defined dually.

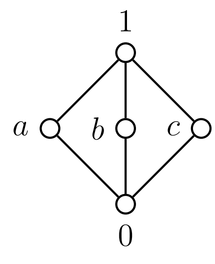
*Complete Lattice:* Let $P$ be a non-empty ordered set. If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then $P$ is called a complete lattice.

*Congruence:* An equivalence relation on a lattice $L$ which is compatible with both join and meet is called a congruence on $L$.

*Equivalence Relation:* An equivalence relation on a set $A$ is a binary relation on $A$ which is reflexive, symmetric and transitive. We write $a \equiv b (mod \theta)$ or $a\ \theta\ b$ to indicate that $a$ and $b$ are related under the relation $\theta$.

**Problem 2.** Show that $M_3$ is a modular lattice but not distributive lattice.

**Solution.** Consider the $M_3$ lattice in the fig,



Let us take 1, $a$, 0 with $1 \geq 0 \Rightarrow\ 1 \wedge (a \vee 0) = 1 \wedge a = a = (1 \wedge a) \vee 0$.
Similarly, 1, $b$, 0 with $1 \geq 0 \Rightarrow\ 1 \wedge (b \vee 0) = 1 \wedge b = b = (1 \wedge b) \vee 0$.
and, 1, $c$, 0 with $1 \geq 0 \Rightarrow\ 1 \wedge (c \vee 0) = 1 \wedge c = c = (1 \wedge c) \vee 0$.
Hence, $M_3$ is modular.
Now, we need to prove that, $M_3$ is not distributive.
Consider, $a, b, c \in M_3$, $a \wedge (b \vee c) = a \wedge 1 = a$, but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$.
Thus, $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$ and hence, $M_3$ is not distributive.

**Problem 3.** $N_5$ is not modular and (also not distributive).

**Problem 4.** The $M_3 - N_5$ theorem.

**Solution.**

*Theorem* 0.1. Let $L$ be a lattice.

(i) $L$ is non-modular if and only if $N_5 \mapsto L$.

(ii) $L$ is non-distributive if and only if $N_5 \mapsto L$ or $M_3 \mapsto L$.

**Problem 5.** Show that every distributive lattice is modular, but the converse is not true.

**Problem 6.** Let $L$ be a lattice, then $L$ is modular iff it has no sub-lattice isomorphic to $N_5$.

**Problem 7.** Prove: Let $f : B \to C$ where $B$ and $C$ boolean algebras.

**Problem 8.** Zorn's Lemma.

**Problem 9.** A lattice $L$ is distributive iff for any two ideals $I$ and $J$ of $L$

$$I \vee J = \{x \in L | x = i \vee j, \quad \text{for some } i \in I \text{ and } j \in J\}$$