Discrete assignment (Question 7)

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Problem. State and prove Euler's theorem for connected graph.

Solution.

Theorem. A finite connected graph G is Eulerian if and only if each vertex has even degree.

Proof. Suppose G is Eulerian and T is a closed Eulerian trail. For any vertex v of G, the trail T enters and leaves v the same number of times without repeating any edge. Hence v has even degree.

Suppose conversely that each vertex of G has even degree. We construct an Eulerian trail. We begin a trail T_1 at any edge e. We extend T_1 by adding one edge after the other. If T_1 is not closed at any step, say, T_1 begins at u but ends at $v \neq u$, then only an odd number of the edges incident on v appear in T_1; hence we can extend T_1 by another edge incident on v. Thus we can continue to extend T_1 until T_1 returns to its initial vertex u, i.e., until T_1 is closed. If T_1 includes all the edges of G, then T_1 is our Eulerian trail.

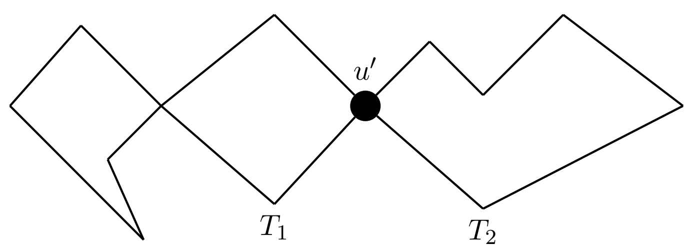


Figure: 1

Suppose T_1 does not include all edges of G. Consider the graph H obtained by deleting all edges of T_1 from G. H may not be connected, but each vertex of H has even degree since T_1 contains an even number of the edges incident on any vertex. Since G is connected, there is an edge e' of H which has an endpoint u' in T_1. We construct a trail T_2 in H beginning at u' and using e'. Since all vertices in H have even degree, we can continue to extend T_2 in H until T_2 returns to u' as pictured in Fig. 1. We can clearly put T_1 and T_2 together to form a larger closed trail in G. We continue this process until all the edges of G are used. We finally obtain an Eulerian trail, and so G is Eulerian. $\square$

Problem. Establish that $K_{3,3}$ is always non-planar.

Solution. Planar graph: A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

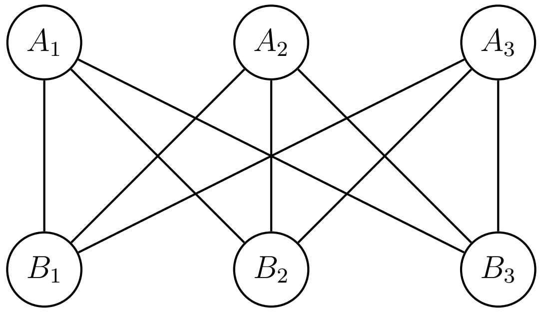
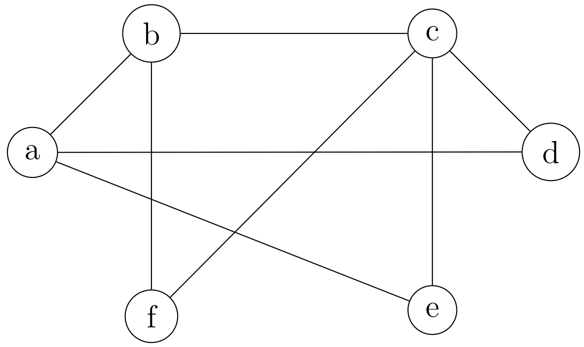


Figure: $K_{3,3}$

Above graph is $K_{3,3}$ and it has $p = 6$ vertices and $q = 9$ edges. Let us assume this graph is planar. Euler's formula for planar graph is $V - E + R = 2$ (here, V is the number of vertices, E is the number of edges, and R is the number of regions). So by Euler's formula a planar representation for this graph has $r = 5$ regions. But here no three vertices are connected to each other; hence the degree of each region must be 4 or more and so the sum if degrees of the regions must be 20 or more. But we know that, the sum of the degrees of the regions of a map is equal to twice the number of edges. So the graph must have 10 or more edges. This contradicts the fact that the graph has $q = 9$ edges. Thus, the graph $K_{3,3}$ is always non-planar.

Problem. Define bipartite graph. Is the following graph bipartite? Explain why?



Solution. Bipartite graph: A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 so that no edge in G connects either two vertices in V_1 or two vertices in V_2. When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

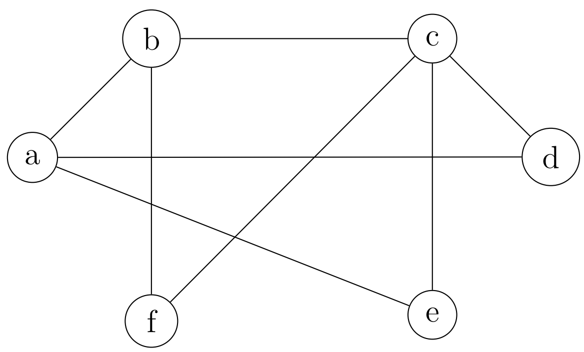


Figure: 1

We can check if a graph is bipartite by using graph coloring. The graph in figure: 1 is not a bipartite graph.

Let V_1 and V_2 be two vertex sets and assign red color to vertices in V_1 and blue color to vertex set V_2. For $a \in V_1$, color it with red. From the graph, we can see that vertex a is connected to $\{b, d, e\}$ so the must be in vertex set V_2. We assign blue color to vertices $\{b, d, e\}$. Vertex c must be in the same set V_1 as it is connected to b and e. We color it red. As vertex c is connected to vertex f so f must be in vertex set V_2 and assign blue color to f but this is not possible as f is connected to b.

Therefore, the graph in figure: 1 is not bipartite.