Classical Mechanics

MAT411

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Preface

This is a compilation of lecture notes with some books and my own thoughts. This document is not a holy text. So, if there is a mistake, solve it by your own judgement. Currently, the following topics are not included

- Fourier Transform and Applications

- Boundary value problems

- Eigenfunctions

- Green's functions

- Strum-Liouville problems

- Laplace Equation

Contents

Syllabus

- Topology and topological space

    - open sets and closed sets
    - closure of a set
    - interior, exterior and boundary
    - neighborhoods and neighborhoods systems
    - weak and strong topology
    - topology of the real line and plane
    - cofinite and cocountable topology
    - subspaces
    - relative topology
    - bases and subbases for a topology
    - continuity and topological equivalence
    - homeomorphic spaces

- Metric and normed spaces

    - Metric topologies
    - properties of metric spaces
    - metrizable space
    - Hilbert space
    - convergence and continuity in metric space
    - normed spaces

- Countability

    - First countable spaces
    - second countable spaces and related theorems

- Compactness

    - Covers
    - compact sets
    - subset of a compact space
    - finite intersection property
    - Bolzano-Weierstrass theorem
    - locally compact spaces

- Connectedness

    - Separated sets
    - connected sets
    - connected spaces

- – components
- – locally connected spaces and simply connected spaces

- Separation axioms

  - – T1-spaces
  - – Hausdorff spaces
  - – regular spaces
  - – normal spaces
  - – completely normal spaces and completely regular spaces

Books Recommended:

- Simmons, G.F.: Introduction to Topology and Modern Analysis

- Gal, S.: Point Set Topology

- Lipschutz, S.: General Topology

- Kelley, J.L.: General Topology

- Hockling and Young: Topology

Part I

Class Note/Sheet

Chapter 1

Non-Linear Partial Differential Equation of First Order

Suppose the non-linear partial differential equation is

$$f(x, y, z, p, q) = 0$$

Type of solution of PDE of first order

1. *Complete solution/integral*: $z = (x + a)(y + b)$ is a complete solution/integral of $z = pq$.

2. *Particular solution:* $z = (x + 3)(y - 4)$ is a particular solution of $z = pq$.

3. *General solution:* The complete solution/integral $\Phi(x, y, z, a, b)$ of PDE $f(x, y, z, p, q) = 0$ is said to be general solution if we can write as $\Phi(x, y, z, a, \boxed{b}) = 0$

4. *Singular solution:* Let $z = ax + by + ab$ is a complete solution of $z = xp + qy + pq$.
   partial differential with respect to a and b gives

$$\frac{\partial z}{\partial a} = x + b$$
$$\Rightarrow 0 = x + b$$
$$\Rightarrow b = -x$$

$$\frac{\partial z}{\partial b} = 0 + y + a$$
$$\Rightarrow a = -y$$

Putting in complete solution $z = -yx - \cancel{xy} + \cancel{xy}$
$z = -xy$ is a singular solution.

Type - I: First standard form $f(p, q) = 0$ which does not contain x, y, z explicitly.
   Working rule:

   1 Write

$$f(p, q) = 0 \tag{1.1}$$

   Assume the solution of (1.1) is

$$z = ax + by + c \tag{1.2}$$

   2 Differentiate (1.2) with respect to x and y partially, we get

$$\frac{\partial z}{\partial x} = a \quad \Rightarrow \quad p = a$$
$$\frac{\partial z}{\partial y} = b \quad \Rightarrow \quad q = b$$

   3 Putting the values of p, q in equation (1.1), we get

$$f(a, b) = 0$$
$$\Rightarrow b = \Phi(a)$$

   4 Putting the values in (1.2)

$$z = ax + \Phi(a)y + c$$

   This is the required complete solution.

Problem 1.0.1. Find the complete solution of PDE $p^2 + q^2 = m^2$

Solution. The given equation

$$p^2 + q^2 = m^2 \tag{1.3}$$

is the first standard form.
Suppose the solution is

$$z = ax + by + c \tag{1.4}$$

partial Differentiate with respect to x and y the equation (1.4) gives,

$$\frac{\partial z}{\partial x} = a \quad \Rightarrow \ p = a$$

$$\frac{\partial z}{\partial y} = b \quad \Rightarrow \ q = b$$

put in (1.3), gives

$$a^2 + b^2 = m^2$$

$$\Rightarrow b = \pm\sqrt{m^2 - a^2}$$

Hence from (1.4), we get

$$z = ax \pm \sqrt{m^2 - a^2}y + c \qquad \text{(contains 2 arbitrary constants)}$$

This is the required complete solution.

Problem 1.0.2. Find the complete solution of $p^2 + q^2 = npq$

Solution. The is

$$p^2 + q^2 = m^2 \tag{1.5}$$

first standard form.
Suppose the solution of (1.5) is

$$z = ax + by + c \tag{1.6}$$

partial Differentiate with respect to x and y the equation (1.6) gives,

$$\frac{\partial z}{\partial x} = a \quad \Rightarrow \ p = a$$

$$\frac{\partial z}{\partial y} = b \quad \Rightarrow \ q = b$$

put in (1.5), gives

$$a^2 + b^2 = nab$$

$$\Rightarrow b^2 - nab + a^2 = 0$$

$$\Rightarrow b = \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}$$

Hence from (1.6), we get

$$z = ax + \left[\frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}\right] y + c$$

This is the required complete solution.

<u>Type - II:</u> Second standard form

$$f(z, p, q) = 0 \tag{1.7}$$

which does not contain x, y explicitly.
Working rule:

   1 Suppose

$$z = f(x + ay) \tag{1.8}$$

     is a solution of (1.7)

2 Let $u = x + ay$ so that[1]

$$z = f(u) \qquad\qquad \left[\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a\right]$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\mathrm{d}\,z}{\mathrm{d}\,u} \cdot \frac{\partial u}{\partial x}$$

$$\Rightarrow p = \frac{\mathrm{d}\,z}{\mathrm{d}\,u} \cdot 1$$

$$\Rightarrow p = \frac{\mathrm{d}\,z}{\mathrm{d}\,u}$$

and

$$\Rightarrow q = \frac{\partial z}{\partial y} = \frac{\mathrm{d}\,z}{\mathrm{d}\,u} \cdot \frac{\partial u}{\partial y}$$

$$\Rightarrow q = a \cdot \frac{\mathrm{d}\,z}{\mathrm{d}\,u}$$

$$\Rightarrow q = a \frac{\mathrm{d}\,z}{\mathrm{d}\,u}$$

3 Put in (1.7)

$$f\left(z,\ \frac{\mathrm{d}\,z}{\mathrm{d}\,u},\ a\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right) = 0$$

4 Solve the above equation for z. Which will be the required complete solution.

Problem 1.0.3. Solve $z = p^2 + q^2$ (Find the complete solution)

Solution. Given

$$z = p^2 + q^2 \tag{1.9}$$

This is second standard form.
We assume the solution

$$z = f(x + ay)$$

and $u = x + ay$, $p = \dfrac{\mathrm{d}\,z}{\mathrm{d}\,u}$, $q = a\dfrac{\mathrm{d}\,z}{\mathrm{d}\,u}$ Put in (1.9)

$$z = \left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2 + a^2\left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2$$

$$\Rightarrow z = \left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2\left(1 + a^2\right)$$

$$\Rightarrow \sqrt{z} = \frac{\mathrm{d}\,z}{\mathrm{d}\,u}\sqrt{1 + a^2}$$

$$\Rightarrow \frac{\mathrm{d}\,z}{\sqrt{z}} = \frac{\mathrm{d}\,u}{\sqrt{1 + a^2}}$$

integrating we get,

$$\Rightarrow 2\sqrt{z} = \frac{u}{\sqrt{1 + a^2}} + b$$

$$\Rightarrow 2\sqrt{z} = \frac{x + ay}{\sqrt{1 + a^2}}$$

This is the required solution.

Problem 1.0.4. Solve $p(1 + q^2 = q(z - a))$

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[1]Note to self: check other resources to confirm

Solution. Given

$$p(1 + q^2 = q(z - a))$$ (1.10)

This is second standard form.

We assume the solution

$$z = f(x + ay)$$

and $u = x + ay$, $p = \dfrac{\mathrm{d}z}{\mathrm{d}u}$, $q = a\dfrac{\mathrm{d}z}{\mathrm{d}u}$

Put in (1.10)

$$\frac{\mathrm{d}z}{\mathrm{d}u}\left[1 + a^2\left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2\right] = b\frac{\mathrm{d}z}{\mathrm{d}u}(z - a)$$

$$\Rightarrow \left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 = \frac{b(z - a) - 1}{b^2}$$

$$\Rightarrow \frac{b\,\mathrm{d}z}{\sqrt{b(z - a)}} = \mathrm{d}u$$

integrating we get,

$$\Rightarrow b\int \frac{\mathrm{d}z}{\sqrt{b(z - a)}} = u + c$$

put $b(z - a) = t$, $b\,\mathrm{d}z = \mathrm{d}t$

$$\Rightarrow \int \frac{\mathrm{d}t}{\sqrt{t}} = u + c$$

$$\Rightarrow 2\sqrt{t} = u + c$$

$$\Rightarrow 2\sqrt{b(z - a)} = x + by + c$$

This is the required solution.

Problem 1.0.5. Solve $x^2p^2 + y^2q^2 = z^2$ (which is reducible to 2nd...)

Solution. Given

$$(xp)^2 + (yp)^2 = z^2$$ (1.11)

Put $X = \ln x$; $Y = \ln y$

$$\frac{\partial X}{\partial x} = \frac{1}{x} \qquad \frac{\partial Y}{\partial y} = \frac{1}{y}$$

$$p = \frac{\partial z}{\partial x}$$

$$\Rightarrow p = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$\Rightarrow p = \frac{1}{x}\frac{\partial z}{\partial X}$$

$$\Rightarrow xp = \frac{\partial z}{\partial X}$$

$$\Rightarrow xp = P$$

$$q = \frac{\partial z}{\partial y}$$

$$\Rightarrow q = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$$

$$\Rightarrow q = \frac{1}{y}\frac{\partial z}{\partial Y}$$

$$\Rightarrow yq = \frac{\partial z}{\partial Y}$$

$$\Rightarrow yp = Q$$

From (1.11) we get,

$$P^2 + Q^2 = z^2 \tag{1.12}$$

This is 2nd standard form.

Assume the solution is

$$z = f(X + aY)$$

and $u = X + aY$, $P = \dfrac{\mathrm{d}z}{\mathrm{d}u}$, $Q = a\dfrac{\mathrm{d}z}{\mathrm{d}u}$

Put in (1.12)

$$\left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 + a^2\left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 = z^2$$

$$\Rightarrow \left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2 (1 + a^2) = z^2$$

$$\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}u} = \frac{z}{\sqrt{1+a^2}}$$

$$\Rightarrow \frac{\mathrm{d}z}{z} = \frac{\mathrm{d}u}{\sqrt{1+a^2}}$$

Integrating,

$$\ln z = \frac{u}{\sqrt{1+a^2}} + b$$

$$\Rightarrow \ln z = \frac{\ln x + a \ln y}{\sqrt{1+a^2}} + b$$

This is the required complete solution.

Problem 1.0.6. Solve $xp^2 + yq^2 = z^2$

Solution. Given

$$(\sqrt{x}p)^2 + (\sqrt{y}p)^2 = z^2 \tag{1.13}$$

Put $X = 2x^{\frac{1}{2}}$; $Y = 2y^{\frac{1}{2}}$

$$\frac{\partial X}{\partial x} = 2 \cdot \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{\sqrt{x}} \qquad \frac{\partial Y}{\partial y} = \frac{1}{\sqrt{y}}$$

Now,

$$p = \frac{\partial z}{\partial x}$$

$$\Rightarrow p = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x}$$

$$\Rightarrow p = \frac{1}{\sqrt{x}}\frac{\partial z}{\partial X}$$

$$\Rightarrow \sqrt{x}p = \frac{\partial z}{\partial X}$$

$$\Rightarrow \sqrt{x}p = P$$

$$q = \frac{\partial z}{\partial y}$$

$$\Rightarrow q = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$$

$$\Rightarrow q = \frac{1}{\sqrt{y}}\frac{\partial z}{\partial Y}$$

$$\Rightarrow \sqrt{y}q = \frac{\partial z}{\partial Y}$$

$$\Rightarrow \sqrt{y}p = Q$$

continue...

Problem 1.0.7. Solve $\dfrac{p^2}{x^2} + \dfrac{q^2}{y^2} = z^2$

Solution. Given

$$\left(\frac{p}{x}\right)^2 + \left(\frac{q}{y}\right)^2 = z^2 \tag{1.14}$$

Put $X = \frac{1}{2}x^2 \ Y = \frac{1}{2}y^2$
continue...

<u>Type - III</u>: Third standard form $f_1(x, p) = f_2(y, q)$
  Working rule:

    1 Let

$$f_1(x, p) = f_2(y, q) = a$$
$$f_1(x, p) = a \quad \text{and} \quad f_2(y, q) = a$$
$$\Rightarrow p = \Phi_1(x, a) \quad \text{and} \quad q = \Phi_2(y, a)$$

    2 Solve

$$\mathrm{d}z = p\,\mathrm{d}u + q\,\mathrm{d}u$$
$$= \Phi_1(x, a)\,\mathrm{d}u + \Phi_2(y, a)\,\mathrm{d}u$$

    Integrate on both side

$$z = \int \Phi_1(x, a)\,\mathrm{d}u + \Phi_2(y, a)\,\mathrm{d}u + c$$

    This is the required complete solution.

Problem 1.0.8. Solve $p^2 - q^2 = x - y$

Solution. Given,
$$p^2 - x = q^2 - y \tag{1.15}$$

This is 3rd standard form.
Let

$$p^2 - x = q^2 - y = a$$
$$\Rightarrow p = \sqrt{a + x} \quad \text{and} \quad q = \sqrt{a + y}$$

Since,

$$\mathrm{d}z = p\,\mathrm{d}x + q\,\mathrm{d}y$$
$$\mathrm{d}z = \sqrt{a + x}\,\mathrm{d}x + \sqrt{a + y}\,\mathrm{d}y$$

Integrating,

$$z = \int \sqrt{a + x}\,\mathrm{d}x + \int \sqrt{a + y}\,\mathrm{d}y$$
$$z = \frac{2}{3}(a + x)^{\frac{3}{2}} + \frac{2}{3}(a + y)^{\frac{3}{2}} + c$$

This is the required complete solution.

Problem 1.0.9. Solve $yp + xq + pq = 0$

Solution. Given,

$$yp + xq = -pq$$
$$\Rightarrow \frac{yp}{-pq} + \frac{xq}{-pq} = 1$$
$$\Rightarrow \frac{x}{p} = 1 + \frac{y}{q}$$

This is 3rd standard form.

Let

$$\frac{x}{p} = 1 + \frac{y}{q} = a$$

$$\Rightarrow p = \frac{x}{a} \quad \text{and} \quad q = \frac{y}{a-1}$$

Since,

$$\mathrm{d}\,z = p\,\mathrm{d}\,x + q\,\mathrm{d}\,y$$

$$\mathrm{d}\,z = \frac{x}{a}\,\mathrm{d}\,x + \frac{y}{a-1}\,\mathrm{d}\,y$$

Integrating,

$$z = \frac{1}{2a}x^2 + \frac{1}{2(a-1)}y^2 + c$$

This is the required complete solution.

Problem 1.0.10. Solve $z^2(p^2 + q^2) = x^2 + y^2$

Solution. Given,

$$(zp)^2 + (zq)^2 = x^2 + y^2 \tag{1.16}$$

Taking $Z = \frac{z^2}{2}$ and $\frac{\partial Z}{\partial z} = z$

Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}$$

$$p = \frac{1}{z}P$$

and,

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y}$$

$$zq = Q \qquad \left[\text{Where } P = \frac{\partial Z}{\partial x},\ Q = \frac{\partial Z}{\partial y}\right]$$

From (1.16)

$$P^2 + Q^2 = x^2 + y^2$$

$$\Rightarrow P^2 - x^2 = y^2 - Q^2$$

This is 3rd standard form.

Let

$$P^2 - x^2 = y^2 - Q^2 = a^2$$

$$\Rightarrow P^2 = a^2 + x^2 \quad \text{and} \quad Q^2 = y^2 - a^2$$

Since,

$$\mathrm{d}\,Z = P\,\mathrm{d}\,x + Q\,\mathrm{d}\,y$$

$$\Rightarrow \mathrm{d}\,Z = (a^2 + x^2)\,\mathrm{d}\,x + (y^2 - a^2)\,\mathrm{d}\,y$$

Integrating,

$$Z = \frac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2\log(x + \sqrt{x^2 + a^2})\right] + \frac{1}{2}\left[y\sqrt{y^2 - a^2} - a^2\log(y + \sqrt{y^2 - a^2})\right] + c$$

$$\Rightarrow \frac{z^2}{2} = \frac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2\log(x + \sqrt{x^2 + a^2})\right] + \frac{1}{2}\left[y\sqrt{y^2 - a^2} - a^2\log(y + \sqrt{y^2 - a^2})\right] + c$$

$$\Rightarrow z^2 = \left[x\sqrt{x^2 + a^2} + a^2\log(x + \sqrt{x^2 + a^2})\right] + \left[y\sqrt{y^2 - a^2} - a^2\log(y + \sqrt{y^2 - a^2})\right] + c$$

This is the required complete solution.

Problem 1.0.11. Solve $p^2 - q^2 = z(x - y)$

Solution. Given,

$$\left(\frac{p}{\sqrt{z}}\right)^2 - \left(\frac{q}{\sqrt{z}}\right)^2 = x - y \tag{1.17}$$

Taking $Z = 2\sqrt{z}$ and $\frac{\partial Z}{\partial z} = \frac{1}{\sqrt{z}}$

Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}$$
$$\Rightarrow p = \sqrt{z}\frac{\partial Z}{\partial x}$$
$$\Rightarrow \frac{p}{\sqrt{z}} = P$$

and,

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y}$$
$$= \sqrt{z}\frac{\partial Z}{\partial y}$$
$$\Rightarrow \frac{q}{\sqrt{z}} = Q \qquad \left[\text{Where } P = \frac{\partial Z}{\partial x}, \ Q = \frac{\partial Z}{\partial y}\right]$$

From (1.17)

$$P^2 - Q^2 = x - y$$
$$\Rightarrow P^2 - x = Q^2 - y$$

This is 3rd standard form.
Let

$$P^2 - x = Q^2 - y = a$$
$$\Rightarrow P^2 = x + a \quad \text{and} \quad Q^2 = a + y$$
$$\Rightarrow P = \sqrt{x + a} \quad \text{and} \quad Q = \sqrt{a + y}$$

Since,

$$dZ = P\,dx + Q\,dy$$
$$\Rightarrow dZ = \sqrt{x + a}\,dx + \sqrt{y + a}\,dy$$

Integrating,

$$Z = \frac{2}{3}(x + a)^{\frac{3}{2}} + \frac{2}{3}(y + a)^{\frac{3}{2}} + c$$
$$\Rightarrow 2\sqrt{z} = \frac{2}{3}(x + a)^{\frac{3}{2}} + \frac{2}{3}(y + a)^{\frac{3}{2}} + c$$

This is the required complete solution.

Problem 1.0.12. Solve $z(p^2 - q^2) = x - y$

Solution. Given,

$$\left(\sqrt{z}p\right)^2 - \left(\sqrt{z}q\right)^2 = x - y \tag{1.18}$$

Put $Z = \frac{2}{3}z^{\frac{3}{2}}$ and $\frac{\partial Z}{\partial z} = \sqrt{z}$

Problem 1.0.13. Solve $p^2 + q^2 = z^2(x^2 + y^2)$

Solution. Given,

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2 + y^2 \tag{1.19}$$

Put $Z = \log z$ and $\frac{\partial Z}{\partial z} = \frac{1}{z}$

Type - IV: Forth standard form

$$z = px + qy + f(p, q)$$

The complete solution is

$$z = ax + by + f(a, b) \tag{1.20}$$

Singular solution
Partial differentiate (1.20) with respect to a and b

$$0 = x + 0 + f'(a, b) \tag{1.21}$$
$$0 = 0 + y + f'(a, b) \tag{1.22}$$

Solve (1.21) and (1.22) find the value of a, b put in (1.20), which gives the required singular solution.

Problem 1.0.14. Find the complete and singular solution of

$$z = px + qy + p^2 + q^2$$

Solution. Given,

$$z = px + qy + p^2 + q^2 \tag{1.23}$$

This is fourth standard form.
The complete solution is

$$z = ax + by + a^2 + b^2 \tag{1.24}$$

Singular solution
Partial differentiate (1.24) with respect to a and b

$$0 = x + 0 + 2a + 0 \qquad \Rightarrow a = \frac{-x}{2}$$
$$0 = 0 + y + 0 + 2b \qquad \Rightarrow b = \frac{-y}{2}$$

Putting in (1.24)

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{y^2}{4} + \frac{x^2}{4}$$
$$z = -\frac{x^2}{4} - \frac{y^2}{4}$$

$x^2 + y^2 + 4z = 0$ is singular solution.

Chapter 2

Charpit's Method

2.1 Derivation

Let us suppose the partial differential equation of first order is given by

$$f(x, y, z, p, q) = 0 \tag{2.1}$$

Also, we have

$$\mathrm{d}\, z = \frac{\partial\, z}{\partial\, x}\,\mathrm{d}\, x + \frac{\partial\, z}{\partial\, y}\,\mathrm{d}\, y$$
$$\Rightarrow\ \ \mathrm{d}\, z = p\,\mathrm{d}\, x + q\,\mathrm{d}\, y \tag{2.2}$$

We assume that a relation

$$F(x, y, z, p, q) = 0 \tag{2.3}$$

exists such that after solving (2.1) and (2.3) simultaneously for p and q and putting these values of p, q in (2.2), (2.2) becomes integrable.

Thus, z, p, q may be expressed as functions of x and y.

Since these values identically satisfy (2.1) and (2.3) both, their differentiating coefficient with respect to x and y vanish.

We know that,