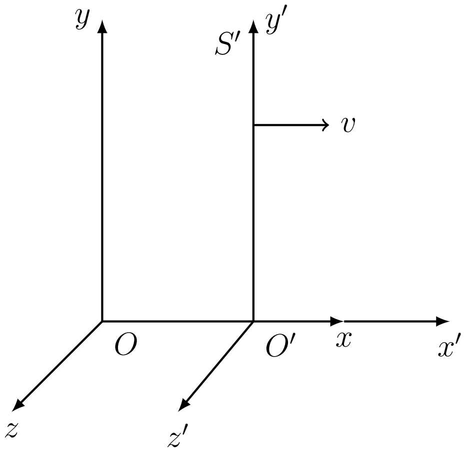
## 0.1 Lorentz Transformation




we observe an event in one inertial frame $S$. The location and time of the event are described by the coordinates $(x, y, z, t)$.

In a second inertial frame $S'$, the same event is recorded as the time-space coordinates $(x', y', z', t')$.

Let

$$x' = x'(x, y, z, t)$$
$$y' = y'(x, y, z, t)$$
$$z' = z'(x, y, z, t)$$
$$t' = t'(x, y, z, t)$$

We use the assumptions:

(i) Space is isotropic, i.e., all spatial direction are equivalent.

(ii) Space and time are homogenous, i.e., all points in space and time are equivalent.

(iii) $S$ and $S'$ coincide at $t = 0$, $t' = 0$.

Let $S'-$frame moves with relative velocity $v$ along the common $x - x'$ axis.
The homogeneity of space and time implies that the transformation equations must be linear:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

where the subscripted coefficients are constant.

*Note.* If $x' = a_{11}x^2$, then $x'_2 - x'_1 = a_{11}(x_2^2 - x_1^2)$;
For a rod of unit length in $S$ with end points at

(i) $x_1 = 1$ and $x_2 = 2$, we get $x'_2 - x'_1 = 3a_{11}$;

(ii) $x_1 = 4$ and $x_2 = 5$, we get $x'_2 - x'_1 = 9a_{11}$;

i.e., the measured length of the rod depends on there it is in space. Similar is the situation for $t$.

If $v = 0$, then $a_{11} = a_{22} = a_{33} = a_{44} = 1$, all other coefficients being zero. The $x-$ axis coincides continuously with $x'-$axis. This gives $y' = 0$, $z' = 0$ for $y = 0$, $z = 0$. Then we have,

$$y' = a_{22}y + a_{23}z$$
$$z' = a_{32}y + a_{33}z$$
$$\text{i.e., } a_{21} = a_{24} = a_{31} = a_{34} = 0$$

Again, the plane $z = 0$ should transform to $z' = 0$ and the plane $y = 0$ to $y' = 0$. Hence,

$$y' = a_{22}y$$
$$z' = a_{33}z$$
$$\text{i.e., } a_{23} = 0 = a_{32}$$

Consider a rod at rest of unit length lying along the $y-$axis in $S$. According to the $S'$ observer, the rod's length will be[1]

$$y' = a_{22} \times 1 = a_{22}$$

Consider the same rod at rest along the $y'$ axis in S'. To the S observer, the rod's length will be[2]

$$y = \frac{1}{a_{22}}y' = \frac{1}{a_{22}} \times 1 = \frac{1}{a_{22}}$$

The first postulate of special relativity implies that these measurements are identical. Therefore,

$$\frac{1}{a_{22}} = a_{22} \qquad \Rightarrow \quad a_{22} = 1$$

With the similar argument, $a_{33} = 1$.
Thus,

$$y' = y$$
$$z' = z$$

Other two transformation equations are

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Since space is isotropic, we get that $t'$ does not depend on $y$ and $z$.[3]
Hence, $a_{42} = 0 = a_{43}$.

[1] $y' = a_{22}y$, $y'_2 - y'_1 = a_{22}(y_2 - y_1)$, $y' = a_{22} \times 1$
[2] $y'_2 - y'_1 = a_{22}(y_2 - y_1) = a_{22}y$, $y = \frac{1}{a_{22}}$
[3]

*Note.* Otherwise, if we place clocks at $+y$, $-y$, then $t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \neq a_{11}x - a_{12}y + a_{13}z + a_{14}t$ Similar is the case at $+z$, $-z$. That is, clocks placed symmetrically in the $y - z$ plane about the $x-$axis would appear to disagree as observed from $S'$, which contradicts the isotropy of space.

Also, a point with $x' = 0$ appears to move in the positive $x-$axis with speed $v$.
So, $x' = 0$ corresponds to $x = vt$, and we expect

$$\begin{aligned} x' &= a_{11}(x - vt) \\ &= a_{11}x - a_{11}vt \\ &= a_{11}x + a_{14}t \\ \text{i.e.,} \quad a_{14} &= -va_{11} \end{aligned}$$

Therefore, the transformation equations reduce to

$$\left.\begin{aligned} x' &= a_{11}(x - vt) \\ y' &= y \\ z' &= z \\ t' &= a_{41}x + a_{44}t \end{aligned}\right\} \tag{1}$$

We now recall the second postulate of special relativity i.e., the speed of light in free space has the same value $c$ in all inertial frames.

Consider a spherical electromagnetic wave leaving the origin at $t = 0$. The wave propagation is described by

$$x^2 + y^2 + z^2 = c^2t^2 \quad \text{for } S \tag{2}$$

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \text{for } S' \tag{3}$$

Substituting (1) into (3), we get

$$\begin{aligned} & a_{11}^2(x - vt)^2 + y^2 + z^2 = c^2\left(a_{41}x + a_{44}t\right)^2 \\ \Rightarrow \ & \left(a_{11}^2 - c^2a_{41}^2\right)x^2 + y^2 + z^2 - (2a_{11}^2v + 2c^2a_{41}a_{44})xt = (c^2a_{44}^2 - a_{11}^2v^2)t^2 \end{aligned}$$

This must be the same as (2)