0.1 Method For Numerical Integration Along A Characteristic

Let U be specified on the initial curve Γ which must not be a characteristic curve.

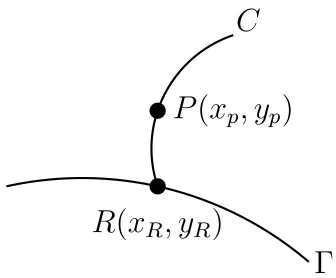


fig: 1

Let $R(x_R, y_R)$ be a point on Γ and $P(x_p, y_p)$ be a point on the characteristic curve C through R such that $x_p - x_R$ is small (fig-1). The difference equation for the characteristic is

$$a\,\mathrm{d}\,y = b\,\mathrm{d}\,x \tag{1}$$

which gives either $\mathrm{d}\,y$ or $\mathrm{d}\,x$ when the other quantities are known.

The differential equation for the solution along a characteristic is either

$$a\,\mathrm{d}\,U = c\,\mathrm{d}\,x \qquad \text{or,} \qquad b\,\mathrm{d}\,U = c\,\mathrm{d}\,y \tag{2}$$

which gives $\mathrm{d}\,U$ for known $\mathrm{d}\,x$ or $\mathrm{d}\,y$ and known a, b and c.

Denote a first approximation to U by $u^{(1)}$ a second approximation by $u^{(2)}$ etc.

<u>First approximations:</u> Assume that x_p is known. Then by the equation (1), we have

$$a_R \left\{ y_p^{(1)} - y_R \right\} = b_R(x_p - x_R)$$

gives a first approximation $y_p^{(1)}$ to y_p and by (2), we will get,

$$a_R \left\{ u_p^{(1)} - u_R \right\} = c_R(x_p - x_R)$$

gives $u_p^{(1)}$.

<u>Second and Subsequent approximations:</u> Replace the coefficient a, b and c by known mean values over the arc R. Then

$$\frac{1}{2}\left(a_R + a_p^{(1)}\right)\left(y_p^{(2)} - y_R\right) = \frac{1}{2}\left(b_R + b_p^{(1)}\right)(x_p - x_R)$$

gives $y_p^{(2)}$
and

$$\frac{1}{2}\left(a_R + a_p^{(1)}\right)\left(U_p^{(2)} - U_R\right) = \frac{1}{2}\left(C_R + C_p^{(1)}\right)(x_p - x_R)$$

gives $u_p^{(2)}$.

This second procedure can be repeated iteratively until successive iterates agree to a specified number of decimal places.

Example. The function U satisfies the equation

$$\sqrt{x}\frac{\partial U}{\partial x} + U\frac{\partial U}{\partial y} = -U^2$$

and the condition $U = 1$ on $y = 0$, $0 < x < \infty$.

Show that the Cartesian equation of the characteristic through the point $R(x_R, 0)$, $x_R > 0$ is $y = \log(2\sqrt{x} + 1 - 2\sqrt{x_R})$. Use a finite difference method to calculate first approximation to the solution and to the value of y at the point $P(1.1, y)$, $y > 0$, on the characteristic through the point $R(1, 0)$.

Calculate a second approximation to these values by an iterative method. Compare the results with those given by the analytical formulae for y and U.

Solution. Given,

$$\sqrt{x}\frac{\partial U}{\partial x} + U\frac{\partial U}{\partial y} = -U^2 \tag{3}$$

comapring with $a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} = c$ we have $a = \sqrt{x}$, $b = U$ and $c = -U^2$.
Hence

$$\frac{\mathrm{d}\, x}{\sqrt{x}} = \frac{\mathrm{d}\, y}{U} = \frac{\mathrm{d}\, u}{-U^2} \tag{4}$$

From,

$$\begin{aligned} \frac{\mathrm{d}\, y}{U} &= \frac{\mathrm{d}\, u}{-U^2} \\ \Rightarrow\ \mathrm{d}\, y &= -\frac{\mathrm{d}\, u}{U} \\ \Rightarrow\ y &= -\log AU \end{aligned}$$

As $U = 1$ at $(x_R, 0)$ then $A = 1$ and so

$$y = \log\left(\frac{1}{U}\right) \tag{5}$$

Similarly, from

$$\begin{aligned} \frac{\mathrm{d}\, x}{\sqrt{x}} &= \frac{\mathrm{d}\, u}{-U^2} \\ \Rightarrow\ 2\sqrt{x} &= \frac{1}{U} + B \end{aligned}$$

As $U = 1$ at $(x_R, 0)$, $B = 2\sqrt{x_R} - 1$.
Therefore

$$\frac{1}{U} = 2\sqrt{x} + 1 - 2\sqrt{x_R} \tag{6}$$

Hence

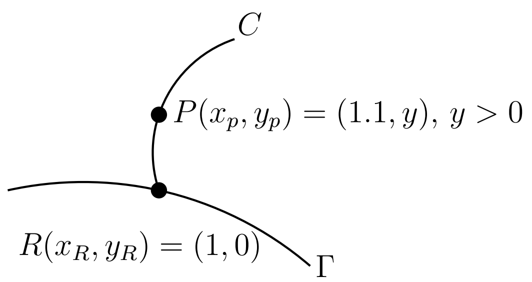
$$y = \log(2\sqrt{x} - 2\sqrt{x_R} + 1) \tag{7}$$

which is the required Cartesian equation. Now from (5) $U = e^{-y}$ and from (6)

$$U = \frac{1}{2\sqrt{x} + 1 - 2\sqrt{x_R}}$$

First approximation at $P(1.1, y)$, $(y > 0)$: We have,

$$\begin{aligned} \frac{\mathrm{d}\, x}{\sqrt{x}} &= \frac{\mathrm{d}\, y}{U} \\ \Rightarrow\ \sqrt{x}\,\mathrm{d}\, y &= U\,\mathrm{d}\, x \end{aligned}$$



Hence,

$$\begin{aligned} &\sqrt{x_R}(y_p^{(1)} - y_R) = U_R(x_p - x_R) \\ \Rightarrow\ &\sqrt{1}(y_p^{(1)} - 0) = 1(1.1 - 1) \qquad [\because x_R = 1,\ U_R = 1] \\ \Rightarrow\ &y_p^{(1)} = 0.1 \end{aligned}$$

Again,

$$
\begin{aligned}
\frac{\mathrm{d}\,x}{\sqrt{x}} &= -\frac{\mathrm{d}\,U}{U^2} \\
\Rightarrow\ \sqrt{x}\,\mathrm{d}\,U &= -U^2\,\mathrm{d}\,x \\
\Rightarrow\ \sqrt{x_R}(U_p^{(1)} - U_R) &= -U_R^2(x_p - x_R) \\
\Rightarrow\ \sqrt{1}(U_p^{(1)} - 1) &= (-1)^2(1.1 - 1) \\
\Rightarrow\ U_p^{(1)} &= 0.9
\end{aligned}
$$

Second Approximation: Using average values for the coefficients,

$$
\begin{aligned}
&\frac{1}{2}\left(\sqrt{x_R} + \sqrt{x_p}\right)\left(y_p^{(2)} - y_R\right) = \frac{1}{2}\left(U_R + U_p^{(1)}\right)(x_p - x_R) \\
\Rightarrow\ &\frac{1}{2}\left(\sqrt{1} + \sqrt{1.1}\right)\left(y_p^{(2)} - 0\right) = \frac{1}{2}\left(1 + 0.9\right)(1.1 - 1.0) \\
\Rightarrow\ &y_p^{(2)} = 0.19
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\left(\sqrt{x_R} + \sqrt{x_p}\right)\left(U_p^{(2)} - U_R\right) = \frac{1}{2}\left(U_R^2 + (U_p^{(1)})^2\right)(x_p - x_R) \\
\Rightarrow\ &\frac{1}{2}\left(\sqrt{1} + \sqrt{1.1}\right)\left(U_p^{(2)} - 1\right) = \frac{1}{2}\left(1^2 + 0.9^2\right)(1.1 - 1.0) \\
\Rightarrow\ &U_p^{(2)} = 0.9117
\end{aligned}
$$

Analytical Value: By equation (7), we have

$$
\begin{aligned}
y_p &= \log(2\sqrt{x} - 2\sqrt{x_R} + 1) \\
&= \log(2\sqrt{1.1} - 2\sqrt{1} + 1) \\
&= 0.0931
\end{aligned}
$$

and

$$
\begin{aligned}
U_p &= \frac{1}{2\sqrt{x} - 2\sqrt{x_R} + 1} \\
&= \frac{1}{2\sqrt{1.1} - 2\sqrt{1} + 1} \\
&= 0.9111
\end{aligned}
$$

Note. Characteristic Curves and Equations:

$$
au_x + bu_y = c; \quad x, y \rightarrow \text{ independent variable}, \quad u = u(x,y) \rightarrow \text{ dependent variable}, \quad u_x = \frac{\partial u}{\partial x}, \ \ u_y = \frac{\partial u}{\partial y}
$$

(a, b, c) tangent vector to the solution integral surface at (x, y, u). [Direction of (a, b, c) is characteristic direction]
Characteristic curve: tangent at any point, tangent direction must coincide with characteristic direction.
Parametric form of characteristic curve:
Let,

$$
\begin{aligned}
x &= x(t) \\
y &= y(t) \\
u &= u(t)
\end{aligned}
$$

't' parameter/unknown.
Tangent vector: $\left(\frac{\mathrm{d}\,x}{\mathrm{d}\,t}, \frac{\mathrm{d}\,y}{\mathrm{d}\,t}, \frac{\mathrm{d}\,u}{\mathrm{d}\,t}\right) = (a, b, c)$ [must coincide]

$$
\begin{aligned}
&\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = a, \quad \frac{\mathrm{d}\,y}{\mathrm{d}\,t} = b, \quad \frac{\mathrm{d}\,u}{\mathrm{d}\,t} = c \\
\Rightarrow\ &\frac{\mathrm{d}\,x}{a} = \frac{\mathrm{d}\,y}{b} = \frac{\mathrm{d}\,u}{c} \qquad (8)
\end{aligned}
$$

which is characteristic equation.
(8) is a system of equation, where two independent variables, so we will get two solutions.

$$\begin{aligned}
&(x, y, u) + \text{ one arbitrary constant} \\
&(x, y, u) + \text{ another arbitrary constant}
\end{aligned}$$

these are characteristic curves.
From $\frac{\mathrm{d}\,x}{a} = \frac{\mathrm{d}\,y}{b} \quad \Rightarrow \quad \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{b}{a}$ which is the slope of the characteristic curve.

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{b(x, y, u)}{a(x, y, u)}$$