Chapter 1

Dual-Simplex Method

Problem 1.0.1. Solve the LPP by the dual-simplex method:

$$\begin{aligned}\text{maximize} \quad & Z = -(2x_1 + x_2 + x_3)\\ \text{subject to} \quad & 4x_1 + 6x_2 + 3x_3 \le 8\\ & x_1 - 9x_2 + x_3 \le -3\\ & 2x_1 + 3x_2 - 5x_3 \ge 4\\ & x_1,\, x_2,\, x_3 \ge 0\end{aligned}$$

Solution. Here, the 3rd constraint is of $\ge$ type. To convert into $\le$ type, let us multiply it by -1. Adding s_1, s_2, s_3 as the slack variables to the 1st, 2nd and 3rd constraints respectively, the given LPP is expressed as

$$\begin{aligned}\text{maximize} \quad & Z = -(2x_1 + x_2 + x_3)\\ \text{subject to} \quad & 4x_1 + 6x_2 + 3x_3 + s_1 = 8\\ & x_1 - 9x_2 + x_3 + s_2 = -3\\ & -2x_1 - 3x_2 + 5x_3 + s_3 = -4\\ & x_1,\, x_2,\, x_3,\, s_1,\, s_2 \ge 0\end{aligned}$$

Putting $x_1 = x_2 = x_3 = 0$, the initial basic solution is $s_1 = 8$, $x_2 = -3$, $s_3 = -4$ which is infeasible. The above information is expressed in tab-I called staring dual simplex table.

Tab	C_B	$c_j \rightarrow$ / Basis	-2	-1	-1	0	0	0	Constant/
			x_1	x_2	x_3	s_1	s_2	s_3	Solution
	0	s_1	4	6	3	1	0	0	8
I	0	s_2	-1	-9	1	0	1	0	-3
	0	s_3	-2	-3	5	0	0	1	-4
		$\bar{c}_j$ row	-2	-1	-1	0	0	0	$Z = 0$
	0	s_1	0	0	13	1	0	2	0
II	0	s_2	7	0	-14	0	1	-3	9
	-1	x_2	$2/3$	1	$-5/3$	0	0	$-1/3$	$4/3$
		$\bar{c}_j$ row	$-4/3$	0	$-8/3$	0	0	$-1/3$	$Z = -4/3$

In these tableaux, basis refers to the basic variables in the basic solution. The values of the basic variables are given under the column solutions. Here, c_j denotes the coefficients of the variables in the objective function. C_B denotes the coefficient of the basic variables only and $\bar{c}_j$ denotes the relative cost coefficients of the variables which is given by

$$\begin{aligned}\bar{C}_j &= C_j - [\text{inner product of } C_B \text{ and the column corresponding the } j\text{-th variable in the canonical system}]\\ &= C_j - C_B A_j, \text{where } A_j \text{ is the } j\text{-th column of the matrix } A = (a_{ij}), \text{ which is found by the coefficients of the}\\ &\quad\ \text{basis and non-basis variables of constraints equation.}\end{aligned}$$

In tableau-I, all $\bar{c}_j$ row entry are either negative of zero and the 'solution' column shows that s_2 and s_3 are negative; this solution is optimal but infeasible.
Since the basic variable s_3 has the most negative value, so it will be chosen to leave the basis. Since the variable x_1 and x_2 have negative coefficient in row-3, so we take the ratios of these with corresponding relative cost row in $\bar{c}_j$, which are $\frac{-2}{-2}$, $\frac{-1}{-3}$. $\left(\frac{-2}{-2} \text{ i.e., } 1 \text{ and } \frac{-1}{-3} \text{ i.e., } \frac{1}{3}\right)$

The minimum ratio occurs corresponding to the non-basic variable x_2. Thus, s_3 will be replaced by x_2 in the basis. Hence, the pivot element is $a_{32} = -3$. We construct tab-II, using pivot operation.

Now we see that tableau-II is optimal and the solution is feasible. The optimal solution is

$$x_1 = 0, \qquad x_2 = \frac{40}{3}, \qquad x_3 = 0$$

and the optimum value is $Z = -\frac{4}{3}$.

From tab-II, we see that the none of the non-basic variables has zero relative cost factors in the $\bar{c}_j$ row. So, there is no alternative optimal of the given LPP. Hence, the optimal solution is unique at $(x_1, x_2, x_3) = (0, \frac{4}{3}, 0)$

Problem 1.0.2. Solve by Dual-Simplex method

$$\begin{aligned} \text{minimize} \quad & Z = x_1 + 4x_2 + 3x_4 \\ \text{subject to} \quad & x_1 + 2x_2 - x_3 + x_4 \geq 3 \\ & -2x_1 - x_2 + 4x_3 + x_4 \geq 2 \\ & x_i \geq 0 \end{aligned}$$

Solution. Multiplying both constraints by -1 and then adding x_5 and x_6 as the slack variables to the 1st and 2nd constraints respectively, we get

$$\begin{aligned} \text{minimize} \quad & Z = x_1 + 4x_2 + 3x_4 \\ \text{subject to} \quad & -x_1 - 2x_2 + x_3 - x_4 + x_5 = -3 \\ & 2x_1 + x_2 - 4x_3 - x_4 + x_6 = -2 \\ & x_i \geq 0 \end{aligned}$$

Putting $x_1 = x_2 = x_3 = x_4 = 0$, the initial basic solution is $x_5 = -3$, $x_6 = -2$; which is infeasible. The above information is expressed in tab-I called starting dual simplex table.

Tab	C_B	Basis \ $c_j \rightarrow$	1	4	0	3	0	0	Constant/
			x_1	x_2	x_3	x_4	x_5	x_6	Solution
I	0	x_5	-1	-2	1	-1	1	0	-3
	0	x_6	2	1	-4	-1	0	1	-2
		$\bar{c}_j$ row	1	4	0	3	0	0	$Z = 0$
II	1	x_1	1	2	-1	1	-1	0	3
	0	x_6	0	-3	-2	-3	2	1	-8
		$\bar{c}_j$ row	0	2	1	2	1	0	$Z = 3$
III	1	x_1	1	$7/2$	0	$5/2$	-2	$-1/2$	7
	0	x_3	0	$3/2$	1	$3/2$	-1	$-1/2$	4
		$\bar{c}_j$ row	0	$1/2$	0	$1/2$	2	$1/2$	$Z = 7$

In these tableaux, basis refers to the basic variables in the basic solution. The values of the basic variables are given under the column solutions. Here, c_j denotes the coefficients of the variables in the objective function. C_B denotes the coefficient of the basic variables only and $\bar{c}_j$ denotes the relative cost coefficients of the variables which is given by

$$\begin{aligned} \bar{C}_j &= C_j - [\text{inner product of } C_B \text{ and the column corresponding the } j\text{−th variable in the canonical system}] \\ &= C_j - C_B A_j, \text{where } A_j \text{ is the } j\text{−th column of the matrix } A = (a_{ij}), \text{ which is found by the coefficients of the} \\ &\quad \text{basis and non-basis variables of constraints equation.} \end{aligned}$$

In tab-I, the basic solution is given by $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = -3$, $x_6 = -2$. This is infeasible though it satisfies the optimality condition.

Since the basic variable x_5 has the most negative value, so it will be chosen to leave the basis. Since the variable x_1, x_2 and x_4 have negative coefficient in row-1, so we take the ratios of these with corresponding relative cost row in $\bar{c}_j$, which are

$$\left|\frac{1}{-1}\right| = 1, \qquad \left|\frac{4}{-2}\right| = 2, \qquad \left|\frac{3}{-1}\right| = 3$$

The minimum ratio occurs corresponding to the non-basic variable x_1. Thus, x_1 will be replaced by x_5 in the basis. Hence, the pivot element is $a_{11} = -1$. We construct tab-II, using pivot operation.

Tab-II is optimal but the basic variable x_6 has negative value. So, x_6 will leave the basis. By the same procedure, we see that x_3 will replace x_6 in the basis; we construct tab-III.

Tab-III is optimal and the solution is feasible. The optimal solution is

$$x_1 = 7, \qquad x_2 = 0, \qquad x_3 = 4, \qquad x_4 = 0$$

and the optimum value is $Z = 7$.

Since in $\bar{c}_j$ row there is no zero values corresponding to non-basic variable, hence the solution is unique.