Chapter 1

Fuzzy Topology

Definition 1 (Fuzzy Topology)**.** Let X be a non-empty set. A collection δ of fuzzy sets on X is called the fuzzy topology on X if it satisfies the following conditions:

(i) $\underline{0}, \underline{1} \in \delta$.

(ii) If $A, B \in \delta$, then $A \wedge B \in \delta$.

(iii) If $A_i \in \delta$, then $\vee_{i \in I} A_i \in \delta$.

If δ is a topology on X then, $\langle \mathcal{F}(X), \delta \rangle$ is called a fuzzy topological space.

Example. Let $X = \{a, b\}$ and A be a fuzzy set defined by $A(a) = 0.5$ and $A(b) = 0.4$. Then $\delta = \{\underline{0}, \underline{1}, A\}$ be a fuzzy topology and $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space.

Example. Let A, B be a fuzzy sets of $I = [0, 1]$ defined as

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x \leq \dfrac{1}{2} \\ 2x - 1; & \text{if } \dfrac{1}{2} \leq x \leq 1 \end{cases} \qquad \text{and} \qquad B(x) = \begin{cases} 1; & \text{if } 0 \leq x \leq \dfrac{1}{4} \\ -4x + 2; & \text{if } \dfrac{1}{4} \leq x \leq \dfrac{1}{2} \\ 0; & \text{if } \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

Then $\delta = \{\underline{0}, \underline{1}, A, B, A \vee B\}$ is a fuzzy topology on I.

Definition 2 (Open and CLosed Fuzzy Sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, the member of δ i.e., each $A \in \delta$ is called the fuzzy open set. A fuzzy set B is called a fuzzy closed set if $B^c \in \delta$.

Example. Let $X = \{a, b\}$, $B : X \to [0, 1]$ such that $B(a) = 0.5$, $B(b) = 0.6$. Then, $B^c(a) = 0.5$, $B^c(b) = 0.4$, $\delta = \{\underline{0}, \underline{1}, A\}$, $A(a) = 0.5$, $A(b) = 0.4$.
$\therefore$ B is closed under $\delta/\delta-$closed. i.e., B^c is open.

Difference between classical and fuzzy sets: Classical set contains elements that satisfy precise properties of membership while fuzzy set contains elements that satisfy imprecise properties of membership.

Definition 3 (Interior and Closure of fuzzy sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological spaceand A be a non-empty subset of X.

The interior of A is denoted by A° and defined as the union of all open sets contained in A. i.e., $A^\circ = \cup\{G \in \delta | G \leq A\}$. (Largest open set contained in A).

The closure of A is denoted by $\bar{A}$ and defined as the intersection of all closed sets containing A. i.e., $\bar{A} = \cap\{F | F^c \in \delta \text{ and } A \leq F\}$. (Smallest closed set containing A).

Example. Consider, $X = \{a, b, c\}$ and

$$\begin{aligned} A &: \ a \mapsto 0.2, \ b \mapsto 0.4, \ c \mapsto 0.8 \\ B &: \ a \mapsto 0.4, \ b \mapsto 0.6, \ c \mapsto 0.8 \\ C &: \ a \mapsto 0.6, \ b \mapsto 0.8, \ c \mapsto 1.0 \end{aligned}$$

Then, $\delta = \{\underline{0}, \underline{1}, A, B, C\}$ be a fuzzy topology on X. Here $U : X \to [0, 1]$ and $U : a \mapsto 0.8$, $b \mapsto 0.7$, $c \mapsto 0.8$. Find U° and $\bar{U}$.