Chapter 1

Inverse Laplace Transform

1.1 Definition of Inverse Laplace Transform

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e., if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace Transform of $f(s)$, and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator.

Example. Since $\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$ we can write

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

1.2 Some Inverse Laplace Transforms

Here is a table of some inverse Laplace transforms

$\mathbf{f(s)}$	$\mathcal{L}^{\mathbf{-1}}\{\mathbf{f(s)}\} = \mathbf{F(t)}$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s^{n+1}}, n = 0, 1, 2, \ldots$	$\frac{t^n}{n!}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
$\frac{s}{s^2-a^2}$	$\cosh at$

1.3 Properties

1. Linearity property

   Theorem 1.3.1. If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$$\begin{aligned}\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} &= c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\}\\ &= c_1F_1(s) + c_2F_2(t)\end{aligned}$$

This result easily extended to more than two functions.

Example.

$$\mathcal{L}^{-1}\left\{\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}+5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$
$$= 4e^{2t}-3\cos 4t+\frac{5}{2}\sin 2t$$

Because of this property we can say that $\mathcal{L}^{-1}$ is a *linear operator* or that it has the *linearity property.*

2. First translation or shifting property

   Theorem 1.3.2. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

   $$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have

   $$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}e^t\sin 2t$$

3. Second translation or shifting property

   Theorem 1.3.3. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

   $$\mathcal{L}^{-1}\left\{e^{-as}f(s)\right\} = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, we have

   $$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/3}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi/3) & \text{if } t > \pi/3 \\ 0 & \text{if } t < \pi/3 \end{cases}$$

4. Change of scale property

   Theorem 1.3.4. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

   $$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F\left(\frac{t}{k}\right)$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t$, we have

   $$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos\frac{4t}{2} = \frac{1}{2}\cos 2t$$

   as is verified directly.

5. Inverse Laplace transform of derivatives

   Theorem 1.3.5. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

   $$\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{\mathrm{d}^n}{\mathrm{d}s^n}f(s)\right\} = (-1)^n t^n F(t)$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{-2s}{(s^2+1)^2}$, we have

   $$\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\} = -t\sin t \qquad \text{or} \qquad \mathcal{L}^{-1}\left\{\frac{s}{(s^1+1)^2}\right\} = \frac{1}{2}t\sin t$$

6. Inverse Laplace transform of integrals

   Theorem 1.3.6. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\,\mathrm{d}\,u\right\} = \frac{F(t)}{t}$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$, we have

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)\mathrm{d}\,u\right\} = \mathcal{L}^{-1}\left\{\ln\left(1 + \frac{1}{s}\right)\right\} = \frac{1 - e^{-t}}{t}$$

7. Multiplication by s^n

   Theorem 1.3.7. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t)$$

   Thus multiplication by s has the effect of differentiating $F(t)$.
   If $F(0) \neq 0$, then

$$\mathcal{L}^{-1}\{sf(s) - F(0)\} = F'(t)$$

   or,

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t) + F(0)\delta(t)$$

   where $\delta(t)$ is the Dirac delta function or unit impulse function.

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\sin 0 = 0$, then

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{\mathrm{d}}{\mathrm{d}\,t}(\sin t) = \cos t$$

   Generalizations to $\mathcal{L}^{-1}\{s^n f(s)\}$, $n = 2, 3, \ldots$ are possible

8. Division by s

   Theorem 1.3.8. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,\mathrm{d}\,u$$

   Thus division by s (or multiplication by $1/s$) has the effect of integrating $F(t)$ from 0 to t.

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\,\mathrm{d}\,u = \frac{1}{4}(1 - \cos 2t)$$

   Generalizations to $\mathcal{L}^{-1}\{f(s)/s^n\}$, $n = 2, 3, \ldots$ are possible

9. The convolution property

   Theorem 1.3.9. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$ then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G.$$

   We call $F * G$ the convolution or faulting of F and G and the theorem is called the convolution theorem or property.

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \int_0^t e^u e^{2(t-u)}\,\mathrm{d}\,u = e^{2t} - e^t$$

Problem 1.3.1. Prove $\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = (-1)^n t^n F(t)$, $n = 1, 2, 3, \ldots$

Proof. Since $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\, s^n} f(s) = (-1)^n f^{(n)}(s)$ we have

$$\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = (-1)^n t^n F(t)$$

$\square$

Problem 1.3.2. Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

Solution. We have

$$\frac{\mathrm{d}}{\mathrm{d}\, s}\left\{\frac{1}{s^2+a^2}\right\} = \frac{-2s}{(s^2+a^2)^2}$$

Thus

$$\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{1}{s^2+a^2}\right)$$

Then since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$, we have by property of inverse Laplace transform of derivatives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{1}{s^2+a^2}\right)\right\}\\ &= \frac{1}{2}t\left(\frac{\sin at}{a}\right)\\ &= \frac{t\sin at}{2a}\end{aligned}$$

Another Method.

Differentiating with respect to the parameter a, we find

$$\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{s}{s^2+a^2}\right) = \frac{-2as}{(s^2+a^2)^2}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{s}{s^2+a^2}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\, s}\left\{\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}\right\} = -2a\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

i.e.,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}\, a}(\cos at) = -\frac{1}{2a}(-t\sin at) = \frac{t\sin at}{2a}$$

Problem 1.3.3. Find $\mathcal{L}^{-1}\left\{\ln\left(1+\dfrac{1}{s^2}\right)\right\}$.

Solution. Let $f(s) = \ln\left(1+\dfrac{1}{s^2}\right) = \mathcal{L}\{F(t)\}$.
Then $f'(s) = \dfrac{-2}{s(s^2+1)} = -2\left\{\dfrac{1}{s} - \dfrac{s}{s^2+1}\right\}$.
Thus, since $\mathcal{L}^{-1}\{f'(s)\} = -2(1-\cos t) = -tF(t)$, $F(t) = \frac{2(1-\cos t)}{t}$.

1.4 The Convolution Theorem

The convolution theorem can be used to solved integral and integral-differential equations.

Theorem 1.4.1. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$ then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\, u = F * G.$$

We call $F * G$ the convolution or faulting of F and G and the theorem is called the convolution theorem. [Here, $*$ (asterisk) denotes convolution in this context, not standard multiplication.]

The formulation is especially useful for implementing a numerical convolution on a computer. The standard convolution algorithm has quadratic computational complexity. With the help of convolution theorem and the fast Fourier transform the complexity of the convolution can be reduced from $O(n^2)$ to $O(n \log n)$.

Problem 1.4.1. Prove the convolution theorem:
If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$ then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G.$$

Proof. The required result follows if we can prove that

$$\mathcal{L}\left\{\int_0^t F(u)G(t-u)\,\mathrm{d}\,u\right\} = f(s)g(s) \tag{1.1}$$

Where,

$$f(s) = \mathcal{L}\{F(t)\} \qquad \text{and}$$
$$g(s) = \mathcal{L}\{G(t)\}$$

To show this we note the left side of (1.1) is

$$\begin{aligned}
&\int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t} F(u)G(t-u)\,\mathrm{d}\,u\right\}\mathrm{d}\,t \\
&= \int_{t=0}^{\infty}\int_{u=0}^{\infty} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t \\
&= \lim_{M\to\infty} s_M
\end{aligned}$$

where,

$$s_M = \int_{t=0}^{M}\int_{u=0}^{t} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t \tag{1.2}$$

The region in the tu plane over which the integration (1.2) is performed is shown shaded in figure 1.1.

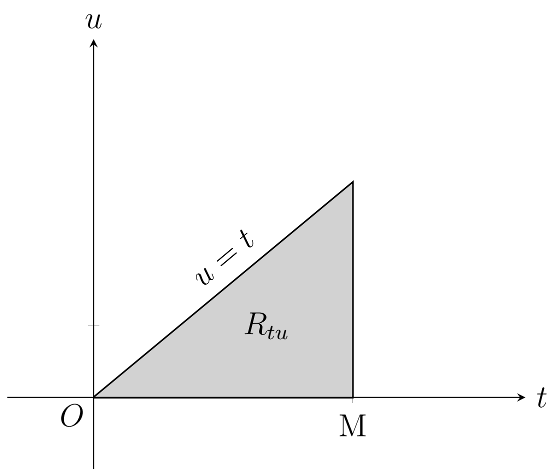

Figure 1.1:

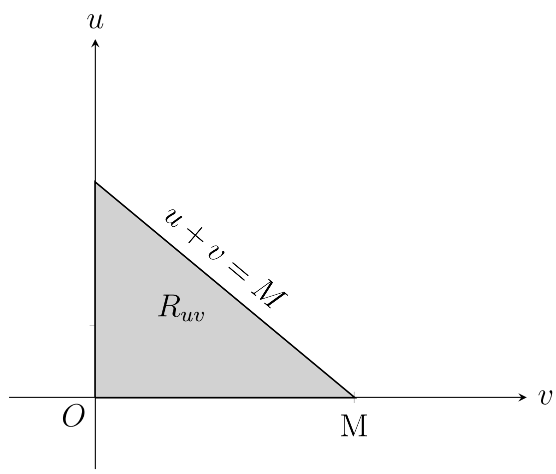

Figure 1.2:

Let, $t - u = v$ or $t = u + v$, the shaded region R_{tu} of the tu plane is transformed into the shaded region R_{uv} of the uv plane shown in figure 1.2. Then by a theorem on transformation on multiple integral,
We have

$$\begin{aligned}
s_M &= \iint_{R_{tu}} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t \\
&= \iint_{R_{uv}} e^{-s(u+v)}F(u)G(v)\left|\frac{\partial(u,t)}{\partial(u,v)}\right|\mathrm{d}\,u\,\mathrm{d}\,v
\end{aligned} \tag{1.3}$$

where the Jacobian of the transformation is

$$J = \frac{\partial(u,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Thus, the right side of (1.3) is,

$$s_M = \int_{v=0}^{M} \int_{u=0}^{M} e^{-s(u+v)} F(u) G(v) \, \mathrm{d}\, u \, \mathrm{d}\, v \tag{1.4}$$

Let us define a function

$$k(u,v) = \begin{cases} e^{-s(u+v)} F(u) G(v) & \text{if } u+v \leq M \\ 0 & \text{if } u+v > M \end{cases} \tag{1.5}$$

This function is defined over the square of figure 1.3 but as indicated in (1.5), is zero over

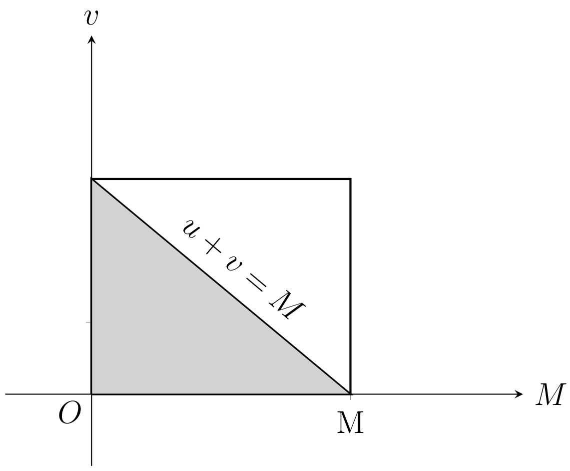

Figure 1.3:

the unshaded portion of the square. In terms of this new function we can write (1.4) as,

$$s_M = \int_{v=0}^{M} \int_{u=0}^{M} k(u,v) \, \mathrm{d}\, u \, \mathrm{d}\, v$$

Then,

$$\begin{aligned} \lim_{M \to \infty} s_M &= \int_0^{\infty} \int_0^{\infty} k(u,v) \, \mathrm{d}\, u \, \mathrm{d}\, v \\ &= \int_0^{\infty} \int_0^{\infty} e^{-s(u+v)} F(u) G(v) \, \mathrm{d}\, u \, \mathrm{d}\, v \\ &= \left\{ \int_0^{\infty} e^{-su} F(u) \, \mathrm{d}\, u \right\} \left\{ \int_0^{\infty} e^{-sv} G(v) \, \mathrm{d}\, v \right\} \\ &= f(s) g(s) \end{aligned}$$

Which establishes the theorem.

We call $\int_0^t F(u) G(t-u) \, \mathrm{d}\, u = F * G$ the convolution integral or convolution of F and G. □

Problem 1.4.2. Prove that $F * G = G * F$.

Proof. Letting $t - u = v$ or $u = t - v$ we have

$$\begin{aligned} F * G &= \int_0^t F(u) G(t-u) \, \mathrm{d}\, u \\ &= \int_0^t F(t-v) G(v) \, \mathrm{d}\, v \\ &= \int_0^t G(v) F(t-v) \, \mathrm{d}\, v \\ &= G * F \end{aligned}$$

This shows that convolution of F and G obeys the commutative law of algebra. It also obeys the associative law and distributive law. □

Problem 1.4.3. Evaluate each of the following by the use of the convolution theorem

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{\left(s^2+a^2\right)^2}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2\left(s+1\right)^2}\right\}$

Solution. (a) We can write

$$\frac{s}{\left(s^2+a^2\right)^2}=\frac{s}{s^2+a^2}\times\frac{1}{s^2+a^2}$$

Now,

$$\frac{s}{s^2+a^2}=\cos at \qquad \text{and}$$
$$\frac{1}{s^2+a^2}=\frac{\sin at}{a}$$

By the convolution theorem we get,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\} &= \int_0^t \cos au\,\frac{\sin a(t-u)}{a}\,\mathrm{d}\,u\\
&= \frac{1}{a}\int_0^t (\cos^2 au)(\sin at\cos au-\cos at\sin au)\,\mathrm{d}\,u\\
&= \frac{1}{a}\sin at\int_0^t \cos^2 au\,\mathrm{d}\,u-\frac{1}{a}\cos at\int_0^t \sin au\cos au\,\mathrm{d}\,u\\
&= \frac{1}{a}\sin at\int_0^t \frac{1+\cos 2au}{2}\,\mathrm{d}\,u-\frac{1}{a}\cos at\int_0^t \frac{\sin 2au}{2}\,\mathrm{d}\,u\\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right)-\frac{1}{a}\cos at\left(\frac{1-\cos 2at}{4a}\right)\\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin at\cos at}{2a}\right)-\frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right)\\
&= \frac{2\sin at}{2a}
\end{aligned}$$

(b) We have,

$$\frac{1}{s^2}=t \qquad \text{and}$$
$$\frac{1}{(s+1)^2}=te^{-t}$$

By the convolution theorem we get,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s+1\right)^2}\right\} &= \int_0^t ue^{-u}(t-u)\,\mathrm{d}\,u\\
&= \int_0^t \left(ut-u^2\right)e^{-u}\,\mathrm{d}\,u\\
&= \left(ut-u^2\right)\left(-e^{-u}\right)-(t-2u)\left(e^{-u}\right)+(-2)\left(-e^{-u}\right)\Bigg|_0^t\\
&= te^{-t}+2e^{-t}+t-2
\end{aligned}$$