Chapter 1

Inverse Laplace Transform

1.1 Definition of Inverse Laplace Transform

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e., if $\mathcal{L}\left\{F(t)\right\} = f(s)$, then $F(t)$ is called an inverse Laplace Transform of $f(s)$, and we write symbolically $F(t) = \mathcal{L}^{-1}\left\{f(s)\right\}$ where $\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator.

1.2 Some Inverse Laplace Transforms

Here is a table of some inverse Laplace transforms

$f(s)$	$\mathcal{L}^{-1}\left\{f(s)\right\} = F(t)$
$\dfrac{1}{s}$	1
$\dfrac{1}{s^2}$	t
$\dfrac{1}{s^{n+1}}, n = 0, 1, 2, \ldots$	$\dfrac{t^n}{n!}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{s^2+a^2}$	$\dfrac{\sin at}{a}$
$\dfrac{s}{s^2+a^2}$	$\cos at$
$\dfrac{1}{s^2-a^2}$	$\dfrac{\sinh at}{a}$
$\dfrac{s}{s^2-a^2}$	$\cosh at$

1.3 Properties

TODO :: Check class

1.4 The Convolution Theorem

The convolution theorem can be used to solved integral and integral-differential equations.

Theorem 1.4.1. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$ then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G.$$

We call $F * G$ the convolution or faulting of F and G and the theorem is called the convolution theorem. [Here, $*$ (asterisk) denotes convolution in this context, not standard multiplication.]

The formulation is especially useful for implementing a numerical convolution on a computer. The standard convolution algorithm has quadratic computational complexity. With the help of convolution theorem and the fast Fourier transform the complexity of the convolution can be reduced from $O(n^2)$ to $O(n \log n)$.

Problem 1.4.1. Prove the convolution theorem:
If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$ then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G.$$

Proof. The required results follows if we can prove that

$$\mathcal{L}\left\{\int_0^t F(u)G(t-u)\,\mathrm{d}\,u\right\} = f(s)g(s) \tag{1.1}$$

Where,

$f(s) = \mathcal{L}\{F(t)\}$ and
$g(s) = \mathcal{L}\{G(t)\}$

To show this we note the left side of (1.1) is

$$\begin{aligned}
&\int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t} F(u)G(t-u)\,\mathrm{d}\,u\right\}\mathrm{d}\,t \\
&= \int_{t=0}^{\infty}\int_{u=0}^{\infty} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t \\
&= \lim_{M\to\infty} s_M
\end{aligned}$$

where,

$$s_M = \int_{t=0}^{M}\int_{u=0}^{t} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t \tag{1.2}$$

The region in the tu plane over which the integration (1.2) is performed is shown shaded in figure 1.1.

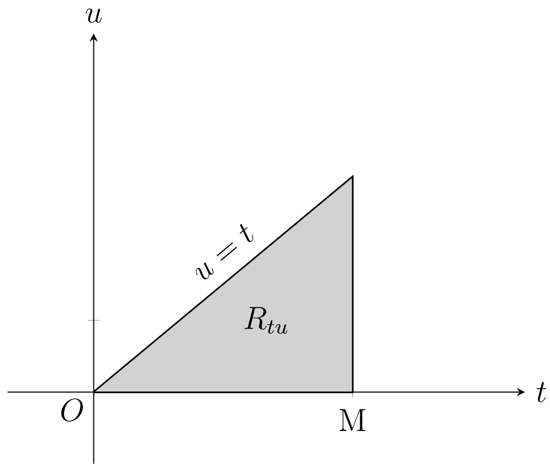

Figure 1.1:

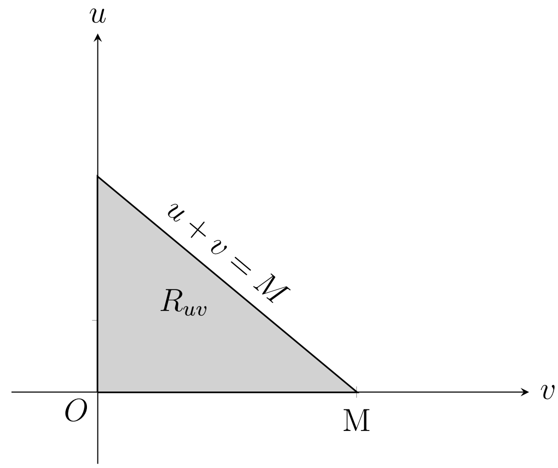

Figure 1.2:

Let, $t - u = v$ or $t = u + v$, the shaded region R_{tu} of the tu plane is transformed into the shaded region R_{uv} of the uv plane shown in figure 1.2. Then by a theorem on transformation on multiple integral, We have

$$\begin{aligned} s_M &= \iint\limits_{R_{tu}} e^{-st} F(u) G(t-u) \,\mathrm{d}\,u \,\mathrm{d}\,t \\ &= \iint\limits_{R_{uv}} e^{-s(u+v)} F(u) G(v) \left| \frac{\partial(u,t)}{\partial(u,v)} \right| \mathrm{d}\,u \,\mathrm{d}\,v \end{aligned} \tag{1.3}$$

where the Jacobian of the transformation is

$$J = \frac{\partial(u,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Thus, the right side of (1.3) is,

$$s_M = \int_{v=0}^{M} \int_{u=0}^{M} e^{-s(u+v)} F(u) G(v) \,\mathrm{d}\,u \,\mathrm{d}\,v \tag{1.4}$$

Let us define a function

$$k(u,v) = \begin{cases} e^{-s(u+v)} F(u) G(v) & \text{if } u + v \leq M \\ 0 & \text{if } u + v > M \end{cases} \tag{1.5}$$

This function is defined over the square of figure 1.3 but as indicated in (1.5), is zero over

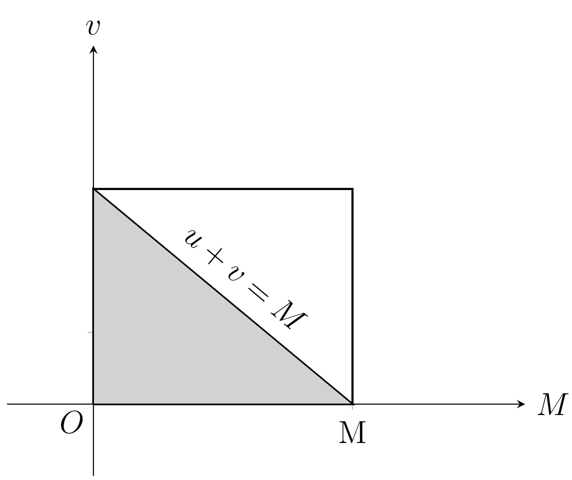

Figure 1.3:

the unshaded portion of the square. In terms of this new function we can write (1.4) as,

$$s_M = \int_{v=0}^{M}\int_{u=0}^{M} k(u,v)\,\mathrm{d}\,u\,\mathrm{d}\,v$$

Then,

$$\begin{aligned}
\lim_{M\to\infty} s_M &= \int_0^\infty\int_0^\infty k(u,v)\,\mathrm{d}\,u\,\mathrm{d}\,v\\
&= \int_0^\infty\int_0^\infty e^{-s(u+v)}F(u)G(v)\,\mathrm{d}\,u\,\mathrm{d}\,v\\
&= \left\{\int_0^\infty e^{-su}F(u)\,\mathrm{d}\,u\right\}\left\{\int_0^\infty e^{-sv}G(v)\,\mathrm{d}\,v\right\}\\
&= f(s)g(s)
\end{aligned}$$

Which establishes the theorem.

We call $\int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G$ the convolution integral or convolution of F and G. □

Problem 1.4.2. Evaluate each of the following by the use of the convolution theorem

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$

Solution. (a) We can write

$$\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}\times\frac{1}{s^2+a^2}$$

Now,

$$\begin{aligned}
\frac{s}{s^2+a^2} &= \cos at \qquad \text{and}\\
\frac{1}{s^2+a^2} &= \frac{\sin at}{a}
\end{aligned}$$

By the convolution theorem we get,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= \int_0^t \cos au\,\frac{\sin a(t-u)}{a}\,\mathrm{d}\,u\\
&= \frac{1}{a}\int_0^t(\cos^2 au)(\sin at\cos au - \cos at\sin au)\,\mathrm{d}\,u\\
&= \frac{1}{a}\sin at\int_0^t\cos^2 au\,\mathrm{d}\,u - \frac{1}{a}\cos at\int_0^t\sin au\cos au\,\mathrm{d}\,u\\
&= \frac{1}{a}\sin at\int_0^t\frac{1+\cos 2au}{2}\,\mathrm{d}\,u - \frac{1}{a}\cos at\int_0^t\frac{\sin 2au}{2}\,\mathrm{d}\,u\\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right) - \frac{1}{a}\cos at\left(\frac{1-\cos 2at}{4a}\right)\\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin at\cos at}{2a}\right) - \frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right)\\
&= \frac{2\sin at}{2a}
\end{aligned}$$

(b) We have,

$$\frac{1}{s^2} = t \qquad \text{and}$$

$$\frac{1}{(s+1)^2} = te^{-t}$$

By the convolution theorem we get,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s+1\right)^2}\right\} &= \int_0^t ue^{-u}(t-u)\,\mathrm{d}\,u \\
&= \int_0^t \left(ut-u^2\right)e^{-u}\,\mathrm{d}\,u \\
&= \left(ut-u^2\right)\left(-e^{-u}\right) - (t-2u)\left(e^{-u}\right) + (-2)\left(-e^{-u}\right)\Big|_0^t \\
&= te^{-t} + 2e^{-t} + t - 2
\end{aligned}$$