Chapter 1

Lattice

Definition 1. An algebra $\langle L; \wedge, \vee \rangle$ i.e., a set equipped with two binary operation $\wedge$ and $\vee$ where, $\wedge$ and $\vee$ are maps from L^2 to L.

Definition 2. An algebra $\langle L; \wedge, \vee \rangle$ is called a lattice if L is a non-empty set and both $\wedge$ and $\vee$ satisfy the following conditions:

(i) $a \wedge a = a, \qquad a \vee a = a \qquad$ [Idempotency]

(ii) $a \wedge b = b \wedge a, \qquad a \vee b = b \vee a \qquad$ [Commutative]

(iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c), \qquad (a \vee b) \vee c = a \vee (b \vee c) \qquad$ [Associativity]

(iv) $a \wedge (a \vee b) = a, \qquad a \vee (a \wedge b) = a \qquad$ [Absorption]

Now we want to characterize $\langle L; \leq \rangle$ as $\langle L; \wedge, \vee \rangle$. Because if we can treat lattices as algebras then all concepts and methods of universal algebra will become applicable.
We will use the notations:

$$\inf \{a, b\} = a \wedge b \rightarrow \text{ read `}a\text{ meet }b\text{'}$$

$$\sup \{a, b\} = a \vee b \rightarrow \text{ read `}a\text{ join }b\text{'}$$

Definition 3. Let P be a non-empty ordered set.

(i) If $x \vee y$ and $x \wedge y$ exists $\forall x, y \in P$, then P is called a lattice.

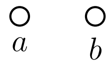
(ii) If $\vee S$ and $\wedge S$ exists $\forall S \subseteq P$, then P is called a complete lattice.

1.1 Remark on $\wedge$ and $\vee$

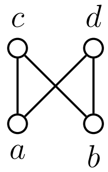
1. Let P be an ordered set. If $x, y \in P$ and $x \leq y$, then $\{x, y\}^u = \uparrow y$ and $\{x, y\}^\ell = \downarrow x$. Since the least element of $\uparrow y$ is y and the greatest element of $\downarrow x$ is x. We have $x \vee y = y$ and $x \wedge y = x; \;\; x \leq y$.

2. In fig (i) we have, $\{a, b\}^u = \varnothing$ and hence, $a \vee b$ does not exist. In fig (ii), $\{a, b\}^u = \{c, d\}$ and thus $a \vee b$ does bot exist as $\{a, b\}^u$ has no least element.

3. Here, $\{b, c\}^u = \{\top, h, i\}$. Since $\{b, c\}^u$ has distinct minimal element namely h and i, it can not have a least element. Hence, $b \vee c$ does not exist. On the other hand, $\{a, b\}^u = \{\top, h, i, f\}$ has a least element f, so $a \vee b = f$.



(i)



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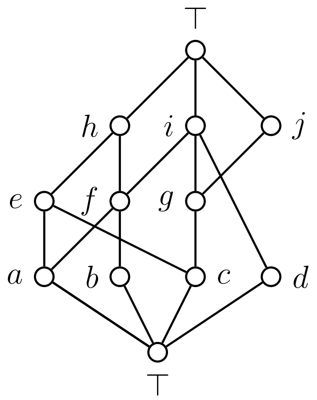
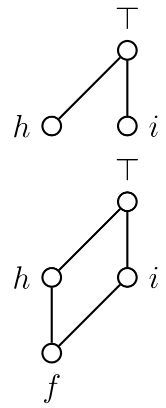


(iii)



$\{b, c\}^u = \{\top, h, i\}$

$\{a, b\}^u = \{\top, h, i, f\}$

Definition 4. Let P be a non-empty ordered set. If $x \vee y$ and $x \wedge y$ exist $\forall x, y \in P$, then P is called a lattice.

Theorem 1.1.1. Let the algebra $\mathcal{L} = \langle L; \wedge, \vee \rangle$ be a lattice. Set $a \leq b$ iff $a \wedge b = a$. Then, $\mathcal{L}^p = \langle L; \leq \rangle$ is a poset and the poset $\mathcal{L}^p$ is a lattice.

Proof. Given $\mathcal{L} = \langle L; \wedge, \vee \rangle$ be a lattice. Set $a \leq b$ to mean $a \wedge b = a$. To show that $\langle L; \leq \rangle$ is a poset, we need to show:

(i) "$\leq$" is reflexive: Since, $\wedge$ is idempotent, i.e., $a \wedge a = a$. So, $\leq$ is reflexive.

(ii) "$\leq$" is antisymmetric: Let $a \leq b$ and $b \leq a$. It means that $a \wedge b = a$ and $b \wedge a = b$. But, $\wedge$ is commutative, therefore,

$$\begin{aligned} & a \wedge b = b \wedge a \\ \Rightarrow\ & a = b \end{aligned}$$

Hence, $\leq$ is antisymmetric.

(iii) "$\leq$" is transitive: Let, $a \leq b$ and $b \leq c$. It means, $a \wedge b = a$ and $b \wedge c = b$.
Now,

$$\begin{aligned} a &= a \wedge b \\ &= a \wedge (b \wedge c) \\ &= (a \wedge b) \wedge c \\ &= a \wedge c \end{aligned}$$

Therefore, $a \leq c$. Hence, "$\leq$" is transitive.

Thus, $\langle L; \leq \rangle$ is a poset.
Conversely, to prove that $\langle L; \leq \rangle$ is a lattice: we will verify that,

$$a \wedge b = \inf \{a, b\} \qquad \text{and} \qquad a \vee b = \sup \{a, b\}$$

Indeed, $a \wedge b \leq a$, since,

$$\begin{aligned} (a \wedge b) \wedge a &= a \wedge (b \wedge a) \\ &= a \wedge (a \wedge b) \\ &= (a \wedge a) \wedge b \\ &= a \wedge b \end{aligned}$$

$\therefore (a \wedge b) \leq a$. Similarly, $(a \wedge b) \leq b$.
Now if $c \leq a$, $c \leq b$, i.e., $c \wedge a = c$ and $c \wedge b = c$ then,

$$\begin{aligned} & c \wedge (a \wedge b) \\ &= (c \wedge a) \wedge b \\ &= c \wedge b \\ &= c \end{aligned}$$

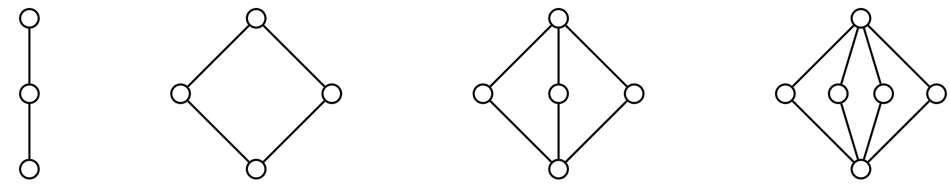
$\therefore c \leq a \wedge b$ and so, $a \wedge b = \inf\{a, b\}$.

Finally, $a, b \leq a \vee b$. Because, $a \wedge (a \vee b) = a$ and also $b = b \wedge (a \vee b)$ by the first absorption identity.
Now if, $a \leq c$, $b \leq c$, i.e., $a \wedge c = a$ and $b \wedge c = b$ then, $a \vee c = (a \wedge c) \vee c = c$ and $b \vee c = (b \wedge c) \vee c = c$ by the second absorption identity.
Now,

$$\begin{aligned} & (a \vee b) \wedge c \\ &= (a \vee b) \wedge (a \vee c) \\ &= (a \vee b) \wedge [a \vee (b \vee c)] \\ &= (a \vee b) \wedge [(a \vee b) \vee c] \\ &= a \vee b \quad \Rightarrow \quad a \vee b \leq c \end{aligned}$$

Hence, $a \vee b = \sup\{a, b\}$.
Therefore, $\langle L; \leq \rangle$ is a lattice. $\square$

Example. The ordered set $M_n (n \geq 1)$ is easily seen to be a lattice. Here for Let $x, y \in M_n$ with $x|y$. Then x and y are in the central antichain of M_n and hence $x \vee y = \top$ and $x \wedge y = \bot$.



M_1 M_2 M_3 M_4