

Chapter 1

Power Series Solutions to the Legendre Equation

1.1 The Legendre equation

The equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (1.1)$$

where α is any real constant, is called Legendre's equation. When $\alpha \in \mathbb{Z}^+$, the equation has polynomial solutions called Legendre polynomials. In fact, these are the same polynomial that encountered earlier in connection with the Gram-Schmidt process.

The equation (1.1) can be rewritten as

$$\left[(x^2 - 1)y'\right]' = \alpha(\alpha + 1)y,$$

which has the form $T(y) = \lambda y$, where $T(f) = (pf')'t'$, with $p(x) = x^2 - 1$ and $\lambda = \alpha(\alpha + 1)$.

Note. The nonzero solutions of (1.1) are eigenfunctions of T corresponding to the eigenvalue $\alpha(\alpha + 1)$.

Since $p(1) = p(-1) = 0$, T is symmetric with respect to the inner product

$$(f, g) = \int_{-1}^1 f(x)g(x) \, dx.$$

Thus, eigenfunctions belonging to distinct eigenvalues are orthogonal.

1.2 Power series solution for the Legendre equation

The Legendre equation can be put in the form

$$y'' + p(x)y' + q(x)y = 0,$$

where

$$p(x) = \frac{2x}{1 - x^2} \quad \text{and} \quad q(x) = \frac{\alpha(\alpha + 1)}{1 - x^2} \quad \text{if } x^2 \neq 1$$

Since $\frac{1}{1 - x^2} = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$, both $p(x)$ and $q(x)$ have power series expansions in the open interval $(-1, 1)$.

Thus, seek a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (-1, 1).$$

Differentiating term by term, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Thus,

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n = \sum_{n=0}^{\infty} 2n a_n x^n$$

and

$$\begin{aligned}
 (1-x^2)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\
 &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n \\
 &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n] x^n
 \end{aligned}$$

Substituting in (1.1), we obtain

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha+1)a_n = 0, n \geq 0,$$

which leads to a recurrence relation

$$a_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+1)(n+2)}a_n$$

Thus, we obtain

$$\begin{aligned}
 a_2 &= -\frac{\alpha(\alpha+1)}{1 \cdot 2}a_0 \\
 a_4 &= -\frac{(\alpha-2)(\alpha+3)}{3 \cdot 4}a_2 = (-1)^2 \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}a_0 \\
 &\vdots \\
 a_{2n} &= (-1)^n \frac{\alpha(\alpha-2) \dots (\alpha-2n+2) \cdot (\alpha+1)(\alpha+3) \dots (\alpha+2n-1)}{(2n)!}a_0
 \end{aligned}$$

Similarly, we can compute a_3, a_5, a_7, \dots , in terms of a_1 and obtain

$$\begin{aligned}
 a_3 &= \frac{(\alpha-1)(\alpha+2)}{2 \cdot 3}a_1 \\
 a_5 &= -\frac{(\alpha-3)(\alpha+4)}{4 \cdot 5}a_3 = (-1)^2 \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}a_1 \\
 &\vdots \\
 a_{2n+1} &= (-1)^n \frac{(\alpha-1)(\alpha-3) \dots (\alpha-2n+1)(\alpha+2)(\alpha+4) \dots (\alpha+2n)}{(2n+1)!}a_1
 \end{aligned}$$

Therefore, the series for $y(x)$ can be written as

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where,

$$\begin{aligned}
 y_1(x) &= 1 + \sum_{n=0}^{\infty} (-1)^n \frac{\alpha(\alpha-2) \dots (\alpha-2n+2) \cdot (\alpha+1)(\alpha+3) \dots (\alpha+2n-1)}{(2n)!} x^{2n}, \text{ and} \\
 y_2(x) &= x + \sum_{n=0}^{\infty} (-1)^n \frac{(\alpha-1)(\alpha-3) \dots (\alpha-2n+1) \cdot (\alpha+2)(\alpha+4) \dots (\alpha+2n)}{(2n+1)!} x^{2n+1}
 \end{aligned}$$

Note. The ratio test shows that $y_1(x)$ and $y_2(x)$ converges for $|x| < 1$. These solutions $y_1(x)$ and $y_2(x)$ satisfy the initial conditions

$$y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = 1.$$

Since $y_1(x)$ and $y_2(x)$ are independent, the general solution of the Legendre equation over $(-1, 1)$ is

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

with arbitrary constants a_0 and a_1 .

1.2.1 Observations

Case I.

When $\alpha = 0$ or $\alpha = 2m$, we note that

$$\alpha(\alpha - 2) \dots (\alpha - 2n + 2) = 2m(2m - 2) \dots (2m - 2n + 2) = \frac{2^n m!}{(m - n)!}$$

and

$$(\alpha + 1)(\alpha + 3) \dots (\alpha + 2n - 1) = (2m + 1)(2m + 3) \dots (2m + 2n - 1) = \frac{(2m + 2n)! m!}{2^n (2m)! (m + n)!}$$

Then, in this case, $y_1(x)$ becomes

$$y_1(x) = 1 + \frac{(m!)^2}{(2m)!} \sum_{k=0}^m (-1)^k \frac{(2m + 2k)!}{(m - k)! (m + k)! (2k)!} x^{2k}$$

which is a polynomial of degree $2m$. In particular, for $\alpha = 0, 2, 4 (m = 0, 1, 2)$, the corresponding polynomials are

$$y_1(x) = 1, 1 - 3x^2, 1 - 10x^2 + \frac{35}{3}x^4$$

Note that the series $y_2(x)$ is not a polynomial when α is even because the coefficients of x^{2n+1} is never zero.

Case II.

When $\alpha = 2m + 1$, $y_2(x)$ becomes a polynomial and $y_1(x)$ is not a polynomial.

In this case,

$$y_2(x) = x + \frac{(m!)^2}{(2m + 1)!} \sum_{k=0}^m (-1)^k \frac{(2m + 2k + 1)!}{(m - k)! (m + k)! (2k + 1)!} x^{2k+1}$$

For example, when $\alpha = 1, 3, 5 (m = 0, 1, 2)$, the corresponding polynomials are

$$y_2(x) = x, x - \frac{5}{3}x^3, x - \frac{14}{3}x^3 + \frac{21}{5}x^5$$

1.3 The Legendre polynomial

Let,

$$P_n(x) = \frac{1}{2^n} \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r (2n - 2r)!}{r! (n - r)! (n - 2r)!} x^{n-2r} \quad (1.2)$$

where $\lfloor n/2 \rfloor$ denotes the greatest integer $\leq n/2$.

- When n is even, it is a constant multiple of the polynomial $y_1(x)$.
- When n is odd, it is a constant multiple of the polynomial $y_2(x)$.

The first five Legendre polynomials are

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2} (3x^2 - 1), P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x)$$

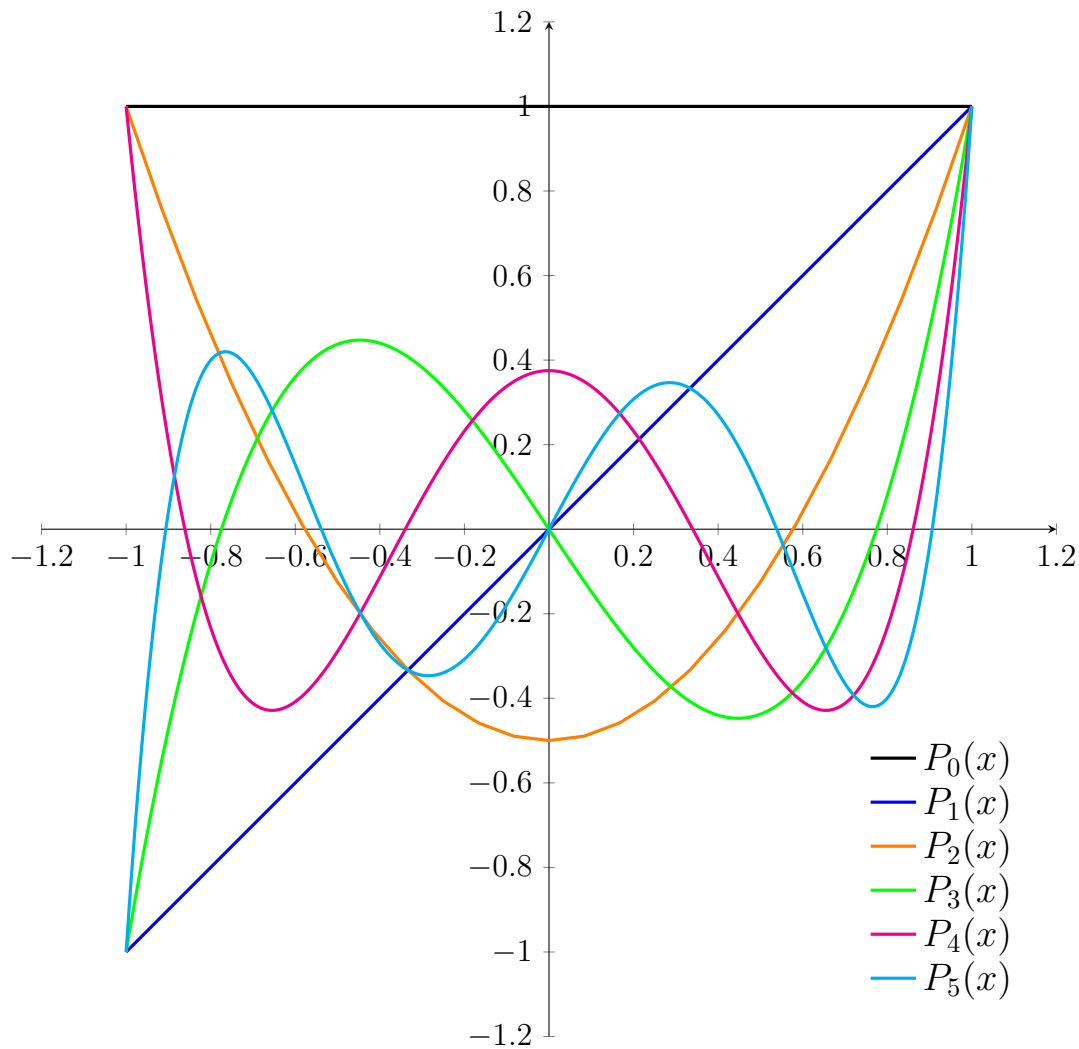


Figure 1.1: Legendre polynomial $(P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x))$ over the interval $[-1, 1]$

1.4 Rodrigues's formula for the Legendre polynomials

Note that

$$\frac{(2n-2r)!}{(n-2r)!} x^{n-2r} = \frac{d^n}{dx^n} x^{2n-2r} \quad \text{and} \quad \frac{1}{r!(n-r)!} = \frac{1}{n!} \binom{n}{r}$$

Thus, $P_n(x)$ in (1.2) can be expressed as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n}{r} x^{2n-2r}$$

When $\lfloor n/2 \rfloor < r \leq n$, the term x^{2n-2r} has degree less than n , so its n th derivative is zero. This gives

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{r=0}^n (-1)^r \binom{n}{r} x^{2n-2r} = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

which is known as Rodrigues' formula.

1.5 Properties of the Legendre polynomials $P_n(x)$

- For each $n \geq 0$, $P_n(1) = 1$. Moreover, $P_n(x)$ is the only polynomial which satisfies the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

and $P_n(1) = 1$.

- For each $n \geq 0$, $P_n(-x) = (-1)^n P_n(x)$.

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$$\int_{-1}^1 P_n(x)P_m(x) \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

- If $f(x)$ is a polynomial of degree n , we have

$$f(x) = \sum_{k=0}^n c_k P_k(x), \text{ where}$$

$$c_k = \frac{2k+1}{2} \int_{-1}^1 f(x)P_k(x) \, dx$$

- It follows from the orthogonality relation that

$$\int_{-1}^1 g(x)P_n(x) \, dx = 0$$

for every polynomial $g(x)$ with $\deg(g(x)) < n$.