





Fuxxy Topology

MAT514

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Preface

This is a compilation of lecture notes with some books and my own thoughts. If there are any mistake/typing error or, for any query mail me at mehedi12@student.sust.edu.

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Part I Sheet

Fuzzy Sets

Definition 1 (Characteristic function). Let X be a universal set and $A \subseteq X$. Then the function¹

$$\chi_A(x) = \begin{cases} 1; & x \in A \\ 0; & x \notin A \end{cases}$$

is characteristic function of A in X.

Definition 2 (Fuzzy Set). A fuzzy set² $A \subseteq X$ is a mapping $A: X \to [0,1]$, where, $A(x) = y \in [0,1]$ is called the membership function or, grade of membership of x in A. The collection of all fuzzy sets of X is denoted by $\mathcal{F}(X)$.

Definition 3 (Fuzzy subset). A fuzzy set A is called a fuzzy subset of another fuzzy set B if $A(x) \leq B(x)$ $\forall x \in X$. We denote it by $A \leq B$.

Definition 4 (Empty fuzzy set). A fuzzy set A is called empty fuzzy set if $\forall x \in X \ A(x) = 0$. The empty fuzzy set is denoted by $\underline{0}$. Thus, $\underline{0}(x) = 0 \ \forall x \in X$.

Definition 5 (Total fuzzy set). The total fuzzy set $\underline{1}$ is defined by $\underline{1}(x) = 1 \ \forall x \in X$.

Definition 6 (Equality of two fuzzy sets). Two fuzzy sets A and B of X is said to be equal iff $A \leq B$ and $B \leq A$.

Example (Empty and Total fuzzy set). Suppose, $A: X \to [0,1]$ where X = [20,90]. Then,

$$\underline{0}(x) = \begin{cases} 0 & \text{if } 15 < x < 90 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \underline{1}(x) = \begin{cases} 1 & \text{if } 20 \le x < 90 \\ 0 & \text{otherwise} \end{cases}$$

Example (Fuzzy subset). Suppose, $A: X \to [0,1]$ where, X = [0,100] defined by

$$A(x) = \begin{cases} 0; & \text{if } 0 \le x < 40\\ \frac{x}{75}; & \text{if } 40 \le x < 75\\ 1; & \text{if } 75 \le x \le 100 \end{cases}$$

and $B: X = [0, 100] \to [0, 1]$ defined by

$$B(x) = \begin{cases} 0; & \text{if } 0 \le x < 40\\ \frac{x}{95}; & \text{if } 40 \le x < 95\\ 1; & \text{if } 95 \le x \le 100 \end{cases}$$

Then, B(x) is a subset of A(x). Since, $B(x) \le A(x) \ \forall x \in X$.

¹Some authors use μ as characteristic function.

²Sometimes fuzzy set is denoted by A.

1.1 Fuzzy Set Operations

Definition 7 (Union of Fuzzy Sets). Let $A, B \in \mathcal{F}(X)$. Then the union of A and B is denoted and defined by, $(A \vee B)(x) = \max\{A(x), B(x)\}$, $\forall x \in X$.

Definition 8 (Intersection of Fuzzy Sets). Let $A, B \in \mathcal{F}(X)$. Then the intersection of A and B is denoted and defined by, $(A \wedge B)(x) = \min \{A(x), B(x)\}, \forall x \in X$.

Definition 9 (Complement of Fuzzy Set). Let A be a fuzzy set of X. Then, the complement of A is denoted by A^c and defined by $A^c(x) = 1 - A(x)$, $\forall x \in X$.

Example. Given,

$$A_1 = \begin{cases} 1; & \text{if } 40 \le x < 50 \\ 1 - \frac{x - 50}{10}; & \text{if } 50 \le x < 60 \\ 0; & \text{if } 60 \le x \le 100 \end{cases} \quad \text{and} \quad A_2 = \begin{cases} 0; & \text{if } 40 \le x < 50 \\ \frac{x - 50}{10}; & \text{if } 50 \le x < 60 \\ 1 - \frac{x - 60}{10}; & \text{if } 60 \le x < 70 \\ 0; & \text{if } 70 \le x \le 100 \end{cases}$$

- 1. Find the complement of A_1 and A_2 .
- 2. Find $(A_1 \wedge A_2)(x)$ and $(A_1 \vee A_2)(x)$

Solution:

1. Complement of A_1 ,

$$A_1{}^c = \begin{cases} 0; & \text{if } 40 \le x < 50\\ \frac{x - 50}{10}; & \text{if } 50 \le x < 60\\ 1; & \text{if } 60 \le x \le 100 \end{cases}$$

Complement of A_2 ,

$$A_2{}^c = \begin{cases} 1; & \text{if } 40 \le x < 50 \\ \frac{60 - x}{10}; & \text{if } 50 \le x < 60 \\ \frac{x - 60}{10}; & \text{if } 60 \le x < 70 \\ 1; & \text{if } 70 \le x \le 100 \end{cases}$$

2.

$$(A_1 \wedge A_2)(x) = \begin{cases} 0; & \text{if } 40 \le x < 50 \\ \frac{x - 50}{10}; & \text{if } 50 \le x \le 55 \\ 1 - \frac{x - 50}{10}; & \text{if } 55 \le x \le 60 \\ 0; & \text{if } 60 \le x \le 100 \end{cases}$$

$$(A_1 \vee A_2)(x) = \begin{cases} 1; & \text{if } 40 \le x \le 50 \\ 1 - \frac{x - 50}{10}; & \text{if } 50 \le x \le 55 \\ \frac{x - 50}{10}; & \text{if } 55 \le x < 60 \\ 1 - \frac{x - 60}{10}; & \text{if } 60 \le x < 70 \end{cases}$$

$$0; & \text{if } 70 \le x \le 100$$

Definition 10 (Level Set). Let $A: X \to [0,1]$ be a fuzzy set. The α level set of A is denoted and defined by, A_{α} or $\alpha_A = \{x \in X | A(x) \ge \alpha\}$ where, $0 < \alpha \le 1$.

Remark. A_{α} is a classical set not a fuzzy set.

Definition 11 (Core level of a fuzzy set). When $\alpha = 1$, then $A_1 = \{x \in X | A(x) = 1\}$ is called the core level of A.

Definition 12 (Support of a fuzzy set). Support of a fuzzy set A is denoted and defined by, $S_A = \{x \in X | A(x) > 0\}$.

Example. Given,

$$A = \begin{cases} 0; & \text{if } x \le 20 \text{ or, } x \ge 60\\ \frac{x - 20}{15}; & \text{if } 20 < x < 35\\ \frac{60 - x}{15}; & \text{if } 45 < x < 60\\ 1; & \text{if } 35 \le x \le 45 \end{cases}$$
 and
$$B = \begin{cases} 0; & \text{if } x \le 45\\ \frac{x - 45}{15}; & \text{if } 45 < x < 60\\ 1; & \text{if } x \ge 60 \end{cases}$$

- 1. (a) Core level of A?
 - (b) Support of A?
 - (c) Half level of A?
 - (d) $\frac{3}{4}$ level of A?
- 2. (a) Core level of B?
 - (b) Support of B?
 - (c) Half level of B?

Solution. 1. (a) Core level of A is $A_1 = \{x \in X | 35 \le x \le 45\}$.

- (b) Support level of *A* is $S_A = \{x \in X | 20 < x < 60\}.$
- (c) Half level of A is $A_{\frac{1}{2}} = \{x \in X | 27.5 \le x \le 52.5\}.$
- (d) $\frac{3}{4}$ level of A is $A_{\frac{3}{4}} = \{x \in X | 31.25 \le x \le 48.75\}.$
- 2. (a) Core level of B is $B_1 = \{x \in X | x \ge 60\}$.
 - (b) Support level of B is $S_B = \{x \in X | x > 45\}.$
 - (c) Half level of B is $B_{\frac{1}{2}}=\{x\in X|x\geq 52.5\}.$

Example. $A: X \to [0,1]$ defined by

$$A(x) = \begin{cases} 1; & \text{if } x \le 20\\ \frac{35 - x}{20}; & \text{if } 20 \le x < 35\\ 0; & \text{if } x > 35 \end{cases}$$

Then find $\frac{1}{2}$ level of A.

Solution.

$$A_{\frac{1}{2}} = \{x \in X | x \le 25\}$$

Problem 1.1. Consider, the two fuzzy sets $A, B: X = [0, 100] \rightarrow [0, 1]$ defined by

$$A(x) = \begin{cases} 0; & \text{if } 0 \le x < 40\\ \frac{x}{75}; & \text{if } 40 \le x < 75\\ 1; & \text{if } 75 \le x < 100 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0; & \text{if } 0 \le x < 40\\ \frac{x}{95}; & \text{if } 40 \le x < 95\\ 1; & \text{if } 95 \le x \le 100 \end{cases}$$

Then find $(A \wedge B)(x)$ and $(A \vee B)(x)$.

Solution.

$$(A \land B)(x) = \begin{cases} 0; & \text{if } 0 \le x < 40\\ \frac{x}{95}; & \text{if } 40 \le x < 95\\ 1; & \text{if } 95 \le x \le 100 \end{cases} \quad \text{and} \quad (A \lor B)(x) = \begin{cases} 0; & \text{if } 0 \le x \le 40\\ \frac{x}{75}; & \text{if } 40 \le x < 75\\ 1; & \text{if } 75 \le x \le 100 \end{cases}$$

Suppose, $X = \mathbb{R}$ and the fuzzy set of real numbers much greater than 5 in X, that could be defined by,

$$A(x) = \begin{cases} 0; & \text{if } x \le 5\\ \frac{x-5}{50}; & \text{if } 5 < x \le 55\\ 1; & \text{if } x \ge 55 \end{cases}$$

Example. Consider, the two fuzzy sets A and B of $\mathcal{F}(X)$, where X = [0, 100]

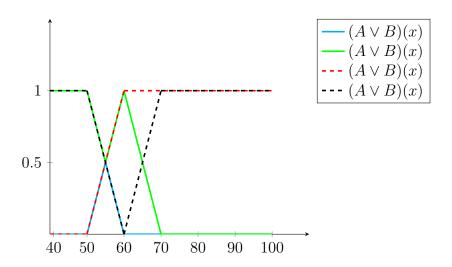
$$A(x) = \begin{cases} 1; & \text{if } 40 \le x \le 50 \\ 1 - \frac{x - 50}{10}; & \text{if } 50 \le x \le 60 \\ 0; & \text{if } 60 \le x \le 100 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0; & \text{if } 40 \le x \le 50 \\ \frac{x - 50}{10}; & \text{if } 50 \le x \le 60 \\ 1 - \frac{x - 60}{10}; & \text{if } 60 \le x \le 70 \\ 0; & \text{if } 70 \le x \le 100 \end{cases}$$

Draw $(A \vee B)(x)$, $(A \wedge B)(x)$, A', B'.

Solution. Here,

$$(A \lor B)(x) = \begin{cases} 1; & \text{if } 40 \le x \le 50 \\ 1 - \frac{x - 50}{10}; & \text{if } 50 \le x \le 55 \\ \frac{x - 50}{10}; & \text{if } 55 \le x \le 60 \\ 1 - \frac{x - 60}{10}; & \text{if } 60 \le x \le 70 \\ 0; & \text{if } 70 \le x \le 100 \end{cases} \quad \text{and} \quad (A \land B)(x) = \begin{cases} 0; & \text{if } 40 \le x \le 50 \\ \frac{x - 50}{10}; & \text{if } 50 \le x \le 55 \\ 1 - \frac{x - 50}{10}; & \text{if } 55 \le x \le 60 \\ 0; & \text{if } 60 \le x \le 100 \end{cases}$$

$$A^{c}(x) = \begin{cases} 0; & \text{if } 40 \le x \le 50 \\ \frac{x - 50}{10}; & \text{if } 50 \le x \le 60 \\ 1; & \text{if } 60 \le x \le 100 \end{cases} \quad \text{and} \quad B^{c}(x) = \begin{cases} 1; & \text{if } 40 \le x \le 50 \\ 1 - \frac{x - 50}{10}; & \text{if } 50 \le x \le 60 \\ \frac{x - 60}{10}; & \text{if } 60 \le x \le 70 \\ 1; & \text{if } 70 \le x \le 100 \end{cases}$$



1.2 Fuzzy Relation

Definition 13 (Fuzzy Relation). Let X and Y be two non-empty classical(Fuzzy) sets. Then a fuzzy relation R on $X \times Y$ is a mapping, $R: X \times Y \to [0,1]$ where, the number $R(x,y) \in [0,1]$ is called the degree of relationship between x and y.

Example. Let $X = \{a, b, c\}$, $Y = \{c, d\}$. Then $X \times Y = \{(a, c), (a, d), (b, c), (b, d), (c, c), (c, d)\}$ where R(a, c) = R(a, d) = 0, R(b, c) = R(b, d) = R(c, c) = 1 and R(c, d) = 0.8. For the fuzzy relation:

- 1. Core of R?
- 2. Support of R?
- 3. 0.7 of R?

Solution.

- 1. Core of $R = \{(b, c), (b, d), (c, c)\}$ Since, R(x, y) = 1 for $x \in X$ and $y \in Y$.
- 2. Support of $R = \{(b, c), (b, d), (c, c), (c, d)\}$ Since, R(x, y) > 0 for $x \in X$ and $y \in Y$.
- 3. 0.7 of $R = \{(b, c), (b, d), (c, c), (c, d)\}$ Since, R(x, y) > 0.7 for $x \in X$ and $y \in Y$.

Definition 14 (Domain). If R(x,y) is a fuzzy relation, its domain is the fuzzy set $dom\ R(x,y)$ whose membership function is

$$\chi_{dom}R(x) = \max \chi_R(x, y) \forall x \in X$$

Definition 15 (Range). If R(x, y) is a fuzzy relation, its range is the fuzzy set ran R(x, y) whose membership function is

$$\chi_{ran}R(y) = \max \chi_R(x, y) \forall y \in y$$

Example. Consider $X = \{x_1, x_2, x_3, x_4\}$ and

$$R(x,x) = \begin{pmatrix} 0.2 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0.0 & 0.4 & 0.0 \\ 0.0 & 0.6 & 0.0 & 0.1 \end{pmatrix}$$

Then the domain is $dom R = \{0.5, 0.8, 0.4, 0.6\}$ and the range is $ran R = \{0.2, 0.6, 0.7, 0.8\}$.

Definition 16 (Max-min and Min-max Composition). Let R be a fuzzy relation on $X \times Y$ i.e., $R \in \mathcal{F}(X \times Y)$ and S be a fuzzy relation on $Y \times Z$ i.e., $S \in \mathcal{F}(Y \times Z)$. Then $R \circ S \in \mathcal{F}(X \times Z)$ defined by

$$(R \circ S)(x, z) = \bigvee_{y \in Y} R(x, y) \land S(y, z)$$

is called the Max-Min composition of R and S on $X \times Z$. And

$$(R \circ S)(x, z) = \bigwedge_{y \in Y} R(x, y) \vee S(y, z)$$

is called the Min-Max composition of R and S on $X \times Z$

Problem 1.2. Consider, $X = \{a, b\}, Y = \{c, d, e\} \text{ and } Z = \{u, v\} \text{ where,}$

$$R(x,y) = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1.0 & 0.0 & 0.9 \end{pmatrix}$$
 and $S(y,z) = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.0 \\ 0.5 & 0.6 \end{pmatrix}$

then find the max-min and min-max composition of R and S.

Solution. Max-min composition of R and S

$$(R \circ S)(x, z) = \bigvee_{y \in Y} R(x, y) \land S(y, z) = \begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}$$

Min-max composition of R and S

$$(R \circ S)(x,z) = \bigwedge_{y \in Y} R(x,y) \lor S(y,z) = \begin{pmatrix} 0.5 & 0.3 \\ 0.1 & 0.0 \end{pmatrix}$$

Definition 17 (Reflexive). Let R be a fuzzy relation in $X \times X$. R is called reflexive if

$$\chi_R(x,x) = 1 \quad \forall x \in X$$

Definition 18 (Symmetric). Let R be a fuzzy relation in $X \times X$. R is called symmetric if

$$R(x,y) = R(y,x) \quad \forall x,y \in X$$

Example. The following relation is a symmetric relation:

$$R(x,y) = \begin{pmatrix} 0.0 & 0.1 & 0.0 & 0.1 \\ 0.1 & 1.0 & 0.2 & 0.2 \\ 0.0 & 0.2 & 0.8 & 0.8 \\ 0.1 & 0.3 & 0.8 & 1.0 \end{pmatrix}$$

Definition 19 (Antisymmetric). Let R be a fuzzy relation in $X \times X$. R is called antisymmetric if for

$$x \neq y$$
 either $\chi_R(x, y) \neq \chi_R(y, x)$
or $\chi_R(x, y) = \chi_R(y, x) = 0$ $\forall x, y, \in X$

Example. The following relation is an antisymmetric relation:

$$R(x,y) = \begin{pmatrix} 0.4 & 0.0 & 0.1 & 0.8 \\ 0.8 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.6 & 0.7 & 0.0 \\ 0.0 & 0.2 & 0.0 & 0.0 \end{pmatrix}$$

Definition 20. A fuzzy relation R is called (max-min) transitive if

$$R\circ R\subseteq R$$

Example. The fuzzy relation

$$R = \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}$$

is a transitive relation.

Here using a max-min composition,

$$R \circ R = \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \circ \begin{pmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.2 \\ 0.4 & 0.3 \end{pmatrix}$$

 \therefore 7; $R \circ R \subseteq R$. Hence, R is a transitive relation.

Definition 21. The max-min composition is called associative if

$$R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3$$

Fuzzy Topology

Definition 22 (Fuzzy Topology). Let X be a non-empty set. A collection δ of fuzzy sets on X is called the fuzzy topology on X if it satisfies the following conditions:

- (i) $\underline{0}, \underline{1} \in \delta$.
- (ii) If $A, B \in \delta$, then $A \wedge B \in \delta$.
- (iii) If $A_i \in \delta$, then $\forall_{i \in I} A_i \in \delta$.

If δ is a topology on X then, $\langle \mathcal{F}(X), \delta \rangle$ is called a fuzzy topological space.

Example. Let $X = \{a, b\}$ and A be a fuzzy set defined by A(a) = 0.5 and A(b) = 0.4. Then $\delta = \{\underline{0}, \underline{1}, A\}$ be a fuzzy topology and $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space.

Example. Let A, B be a fuzzy sets of I = [0, 1] defined as

$$A(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{2} \\ 2x - 1; & \text{if } \frac{1}{2} \le x \le 1 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 1; & \text{if } 0 \le x \le \frac{1}{4} \\ -4x + 2; & \text{if } \frac{1}{4} \le x \le \frac{1}{2} \\ 0; & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

Then $\delta = \{\underline{0}, \underline{1}, A, B, A \vee B\}$ is a fuzzy topology on I.

Definition 23 (Open and Closed Fuzzy Sets). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, the member of δ i.e., each $A \in \delta$ is called the fuzzy open set. A fuzzy set B is called a fuzzy closed set if $B^c \in \delta$.

Example. Let $X = \{a, b\}$, $B : X \to [0, 1]$ such that B(a) = 0.5, B(b) = 0.6. Then, $B^c(a) = 0.5$, $B^c(b) = 0.4$, $\delta = \{\underline{0}, \underline{1}, A\}$, A(a) = 0.5, A(b) = 0.4.

∴ B is closed under δ/δ -closed. i.e., B^c is open.

<u>Difference between classical and fuzzy sets:</u> Classical set contains elements that satisfy precise properties of membership while fuzzy set contains elements that satisfy imprecise properties of membership.

Definition 24 (Interior and Closure of fuzzy sets). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and A be a non-empty subset of X.

The interior of A is denoted by A° and defined as the union of all open sets contained in A. i.e., $A^{\circ} = \bigcup \{G \in \delta \mid G \leq A\}$. (Largest open set contained in A).

The closure of A is denoted by \bar{A} and defined as the intersection of all closed sets containing A. i.e., $\bar{A} = \bigcap \{F \mid F^c \in \delta \text{ and } A \leq F\}$. (Smallest closed set containing A).

Example. Consider, $X = \{a, b, c\}$ and

$$A: a \mapsto 0.2, b \mapsto 0.4, c \mapsto 0.8$$

 $B: a \mapsto 0.4, b \mapsto 0.6, c \mapsto 0.8$
 $C: a \mapsto 0.6, b \mapsto 0.8, c \mapsto 1.0$

Then, $\delta = \{\underline{0}, \underline{1}, A, B, C\}$ be a fuzzy topology on X. Here $U: X \to [0, 1]$ and $U: a \mapsto 0.8, b \mapsto 0.7, c \mapsto 0.8$. Find U° and \bar{U} .

Solution. 1. We know that, $U^{\circ} = \bigcup \{G \in \delta : g \leq U\} = \bigcup \{\underline{0}, A, B\} = B$. Since, $\underline{0} \leq A \leq B$.

2. At first, $A^c: a \mapsto 0.8, b \mapsto 0.6, c \mapsto 0.2$ $B^c: a \mapsto 0.6, b \mapsto 0.4, c \mapsto 0.2$ $C^c: a \mapsto 0.4, b \mapsto 0.2, c \mapsto 0.0$ $0^c = 1$ and $1^c = 0$

We know that $\bar{U} = \bigcap \{F \mid F^c \in \delta \text{ and } U \leq F\} = \underline{1}$.

Theorem 2.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, the following conditions hold:

- (i) $\underline{0}^{\circ} = \underline{0}$ and $\underline{1}^{\circ} = \underline{1}$
- (ii) $\forall A \in \mathcal{F}(X), A^{\circ} \leq A$
- (iii) $\forall A \in \mathcal{F}(X), A^{\circ \circ} = A^{\circ}$
- (iv) for $A, B \in \mathcal{F}(X)$ with $A \leq B$ implies $A^{\circ} \leq B^{\circ}$
- (v) for $A, B \in \mathcal{F}(X)$, $(A \wedge B)^{\circ} = A^{\circ} \wedge B^{\circ}$

Proof.

- (i) By definition, $\underline{0}^{\circ} = \bigcup \{G \in \delta \mid G \leq \underline{0}\} = \underline{0} \text{ and } \underline{1}^{\circ} = \bigcup \{G \in \delta \mid G \leq \underline{1}\} = \underline{1}$
- (ii) By definition, $A^{\circ} = \bigcup \{G \in \delta \mid G \leq A\}$. Since, the arbitrary union of open sets is open, A° is the open set of $\mathcal{F}(X)$ and also, A° is the largest open set contained in A. $A^{\circ} \leq A$.
- (iii) From (ii), $A^{\circ} \leq A \Rightarrow A^{\circ \circ} \leq A^{\circ}$. But A° is the largest open set contained in A. So, $A^{\circ} \leq A^{\circ \circ}$. Hence, $A^{\circ \circ} = A^{\circ}$.
- (iv) Let $A, B \in \mathcal{F}(X)$ such that $A \leq B$. Now, since $A^{\circ} \leq A$, hence $A^{\circ} \leq B$. But B° is the set of all open sets contained in B. So, $B^{\circ} \leq B$. Therefore, $A^{\circ} \leq B^{\circ}$.
- (v) Let $A, B \in \mathcal{F}(X)$. Then,

$$A^{\circ} \leq A, \ B^{\circ} \leq B$$

$$\Rightarrow A^{\circ} \wedge B^{\circ} \leq A \wedge B$$

$$\Rightarrow (A^{\circ} \wedge B^{\circ})^{\circ} \leq (A \wedge B)^{\circ}$$
(2.1)

Here, A° is the largest open set contained in A and B° is the largest open set contained in B. Hence, $A^{\circ} \wedge B^{\circ}$ is also an open set of X. So, $(A^{\circ} \wedge B^{\circ})^{\circ} \leq (A \wedge B)^{\circ}$. From (2.1),

$$A^{\circ} \wedge B^{\circ} \le (A \wedge B)^{\circ} \tag{2.2}$$

Again, Since,

$$A \wedge B \leq A, B$$

$$\Rightarrow (A \wedge B)^{\circ} \leq A^{\circ}, B^{\circ}$$

$$\Rightarrow (A \wedge B)^{\circ} \leq A^{\circ} \wedge B^{\circ}$$
(2.3)

From, (2.2) and (2.3), $A^{\circ} \wedge B^{\circ} = (A \wedge B)^{\circ}$.

Note. If A be a fuzzy open set of the topological space $\langle X, \delta \rangle$, then $A^{\circ} = A$, $\bar{A} = A$ iff A is closed.

Definition 25 (Fuzzy Point). A fuzzy set x_a on X is called a fuzzy point on X if $\forall y \in X$,

$$x_a(y) = \begin{cases} a; & \text{if } x = y \\ 0; & \text{if } x \neq y \end{cases}; \quad \text{where, } 0 < a \le 1$$

The set of all fuzzy points on X is denoted by P(X). The fuzzy points x_{1-a} is called the dual point of the fuzzy points x_a .

Example. X = [0, 1], where $X = \{x, y, z\}$. We need to find $x_a(y)$ where $y \in X$.

$$x_a: x \to a$$
 $y_a: x \to 0$ $z_a: x \to 0$ dual of $x_a, x_{1-a}: x \to (1-a)$
 $y \to 0$ $y \to a$ $y \to 0$ $y \to 0$ $y \to 0$
 $z \to 0$ $z \to 0$ $z \to a$ $z \to 0$

Definition 26 (Neighborhood of a fuzzy point). Let $\langle X, \delta \rangle$ be a fuzzy topological space and $x_a \in P(X)$. Then $U \in \delta$ is called a fuzzy neighborhood of x_a if $x_a \in U$.

The set of all fuzzy neighborhood of x_a is denoted by $\mathcal{N}_{\delta}(x_a)$.

Example.
$$X = \{a, b, c\}, \ \delta = \{\underline{0}, \underline{1}, A, B\}, \ A : a \to 0.0, \ B : a \to 0.2.$$
 Find the neighborhood of $a_{0.4}, b_{0.7}, c_{0.8}$. $b \to 0.2$ $b \to 0.4$ $c \to 0.7$ $c \to 0.8$

Solution.

- 1. $a_{0.4}: a \to 0.4$; Fuzzy neighborhood of $a_{0.4}: \{\underline{1}\}$. $b \to 0.0$ $c \to 0.0$
- 2. $b_{0.7}: a \to 0.0$; Fuzzy neighborhood of $b_{0.7}: \{\underline{1}\}$. $b \to 0.7$ $c \to 0.0$
- 3. $c_{0.8}: a \to 0.0$; Fuzzy neighborhood of $c_{0.8}: \{B,\underline{1}\}.$ $b \to 0.0$ $c \to 0.8$

Theorem 2.0.2. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $A \subseteq X$. Then a fuzzy point $x_a \in A^{\circ} \Leftrightarrow x_a$ has a neighborhood U such that $U \subseteq A$.

Proof. Suppose, $x_a \in A^{\circ}$. By the definition of A° , $A^{\circ} = \bigcup \{G \in \delta \mid G \subseteq A\}$. $\therefore x_a \in \bigcup \{G \in \delta \mid G \subseteq A\}$. Thus we have $x_a \in U$ for some $U \in \delta \ni U \subseteq A$. \therefore There exists a neighborhood U of x_a such that $U \subseteq A$.

Conversely, suppose, U be a neighborhood of a fuzzy point $x_a \ni U \subseteq A$. This implies, $x_a \in U \subseteq A$. Now, since A° is the largest open set contained in A, we have $U \subseteq A^{\circ}$. Thus, $x_a \in A^{\circ}$.

Definition 27 (Quasi-Coincident of a fuzzy point). Let $\langle X, \delta \rangle$ be a fuzzy topological space. A fuzzy point x_a is called quasi-coincident of a fuzzy set A denoted by $x_a \propto A$ iff $x_a \not\leq A^c$ i.e., $a > A^c(x) \Rightarrow a + A(x) > 1$,

Definition 28 (Quasi-Coincident of a fuzzy set). A fuzzy set A is said to be quasi-coincident with a fuzzy set B iff there exists an $x \in X$ such that $A(x) > B^c(x)$ i.e., A(x) + B(x) > 1 for some $x \in X$.

Definition 29 (Quasi-neighborhood). An open set $U \in \delta$ is called a quasi-neighborhood of a fuzzy point x_a if x_a is a quasi-coincident of U. The set of all quasi-coincident of x_a is denoted by $\mathcal{Q}_{\delta}(x_a)$.

Example. Consider, $X = \{a, b, c\}, \delta = \{\underline{0}, \underline{1}, A, B\},$

$$\begin{array}{c} A: \ a \mapsto 0.0, \ b \mapsto 0.2, \ c \mapsto 0.7 \\ B: \ a \mapsto 0.6, \ b \mapsto 0.4, \ c \mapsto 0.8 \\ \text{Given, } P: \ a \mapsto 0.0, \ b \mapsto 0.4, \ c \mapsto 0.9 \end{array}$$

Find the quasi-neighborhood of x_a at a = 0.4.

Solution. Here, B^c : $a \mapsto 0.4$, $b \mapsto 0.6$, $c \mapsto 0.2$. Since, $a = 0.4 \ge B^c(a) = 0.4$ so, $x_{0.4}$ is a quasi-coincident of B and $\mathcal{Q}_{\delta}(x_a) = \{\underline{1}, B\}$.

Theorem 2.0.3. A quasi-neighborhood of x_a is exactly a neighborhood of x_{1-a} .

Proof. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $U \in \delta$ be a quasi-neighborhood of x_a . By the definition of quasi-neighborhood of x_a ,

$$a > U^c(x)$$
, for some $x \in X$,
 $\Leftrightarrow a > 1 - U(x)$, for some $x \in X$,
 $\Leftrightarrow a + U(x) > 1$, for some $x \in X$,
 $\Leftrightarrow 1 - a < U(x)$, for some $x \in X$,
 $\Leftrightarrow x_{1-a} \in U$,
 $\Leftrightarrow U$ is a neighborhood of x_{1-a}

Proposition 1. Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $A, B \subseteq X$. Then $A \leq B$ iff A and B^c are not quasi-coincident.

Proof. Suppose, $A \leq B$, then, $A(x) \leq B(x)$, for all $x \in X$. Now, $A(x) + B^c(x) = A(x) + 1 - B(x) \leq 1$, for all $x \in X$ [Since, $A(x) \leq B(x)$]

Hence, A and B^c are not quasi-coincident.

Conversely, suppose A(x) and $B^{c}(x)$ are not quasi-coincident. Then,

$$A(x) + B^{c}(x) \le 1$$

$$\Rightarrow A(x) + 1 - B(x) \le 1$$

$$\Rightarrow A(x) - B(x) \le 0$$

$$\Rightarrow A(x) \le B(x)$$

Theorem 2.0.4. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then, the following conditions hold:

- 1. $x_a \in A^{\circ}$ iff $x_{1-a} \notin \bar{A}^c$.
- 2. $x_a \in \bar{A}$ iff each neighborhood of its dual point x_{1-a} is quasi-coincident with A.

Proof.

1. Let $x_a \in A^{\circ}$. Then by definition of A° , there exists $B \in \delta$ such that $x_a \in B \subseteq A$ i.e., B is a neighborhood of x_a and hence B is a quasi-neighborhood of x_{1-a} . Hence $x_{1-a} \not\leq B^c$ i.e., $x_{1-a} \not\in B^c$. Since, $B \subseteq A$ and \bar{A} is the smallest closed set containing A, we have, $B \subseteq A \subseteq \bar{A}$ implies $\bar{A}^c \subseteq B^c$. Hence we can show that $x_{1-a} \not\in \bar{A}^c$.

Conversely, suppose $x_{1-a} \notin \bar{A}^c$. Then there is a neighborhood B of x_a which is not quasi-coincident with A^c . Thus,

$$B(x) + A^{c}(x) \le 1 \quad \forall x \in X$$

$$\Rightarrow B(x) \le A(x) \quad \forall x \in X$$

 $\therefore B^c \subseteq A \text{ and so } x_a \in B \subseteq A \text{ i.e., } x_a \in A^\circ.$

2. Let N be the neighborhood of x_{1-a} . Now, N is a quasi-coincident with A implies

$$N(x) + A(x) > 1, \ \forall x \in X$$

 $\Rightarrow N \text{ and } A \text{ intersect at } x$
 $\Rightarrow x_a \in \bar{A}$

Conversely, suppose $x_a \in \bar{A}$. Then, N and A intersect at x. This implies,

$$N(x) + A(x) > 1$$
, $\forall x \in X$
 $\Rightarrow N$ is a quasi-coincident with A at x
 \Rightarrow each neighborhood N of x_{1-a} is quasi-coincident with A

Definition 30 (Subspace). Let $\langle X, \delta \rangle$ be a fuzzy topological space and let Y be a nonempty set $(Y \neq \emptyset)$ such that $Y \subseteq X$. Define $\delta_{\upharpoonright_Y} = \{U_{\upharpoonright_Y} \mid U \in \delta\}$ [where, $U_{\upharpoonright_Y} = U \cap Y = U$ restricted in Y]. Then $\delta_{\upharpoonright_Y}$ is a fuzzy topology on Y. The fuzzy topological space $\langle Y, \delta_{\upharpoonright_Y} \rangle$ is called a subspace of $\langle X, \delta \rangle$.

Example. Let, $X = \{a, b, c\}$ and $Y = \{b, c\}$. Let $\delta = \{\underline{0}, \underline{1}, A, B\}$ where

$$A: a \mapsto 0.2, b \mapsto 0.4, c \mapsto 1.0$$

 $B: a \mapsto 0.1, b \mapsto 0.4, c \mapsto 0.8$

Then $\delta_{\uparrow_Y} = \{\underline{0}_{\uparrow_Y}, \underline{1}_{\uparrow_Y}, A_{\uparrow_Y}, B_{\uparrow_Y}\}$ where,

$$A_{\uparrow_Y}: b \mapsto 0.4, c \mapsto 1.0$$

 $B_{\downarrow_Y}: b \mapsto 0.4, c \mapsto 0.8$

is a fuzzy topology on Y and hence $\langle Y, \delta_{\upharpoonright_{V}} \rangle$ is a fuzzy subspace of $\langle X, \delta \rangle$.

Example. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 3, 4\}$. Find a non-trivial fuzzy topology on X and hence, find a fuzzy subspace of $\langle X, \delta \rangle$.

Solution. Let $\delta = \{\underline{0}, \underline{1}, A, B\}$ be a fuzzy topology on X where,

$$A: 1 \mapsto 0.3, 2 \mapsto 0.1, 3 \mapsto 0.6, 4 \mapsto 0.2$$

 $B: 1 \mapsto 0.7, 2 \mapsto 0.4, 3 \mapsto 0.1, 4 \mapsto 0.2$

Then $\delta_{\uparrow_Y} = \{\underline{0}_{\uparrow_Y}, \underline{1}_{\uparrow_Y}, A_{\uparrow_Y}, B_{\uparrow_Y}\}$ where,

$$A_{|Y|}: 1 \mapsto 0.3, 3 \mapsto 0.6, 4 \mapsto 0.2$$

 $B_{|Y|}: 1 \mapsto 0.7, 3 \mapsto 0.1, 4 \mapsto 0.2$

is a fuzzy topology on Y and hence $\langle Y, \delta_{\upharpoonright Y} \rangle$ is a fuzzy subspace of $\langle X, \delta \rangle$.

Remark. Let $\langle X, \tau \rangle$ be a fuzzy topological space. The two fuzzy sets A and B in X are said to be intersecting \Leftrightarrow there exists a point $x \in X$ such that $(A \wedge B)(x) \neq 0$.

For such a case, we say that, A and B intersect at x.

Again, if A and B are quasi-coincident at x, then, A(x) + B(x) > 1 i.e., both A(x) and B(x) are not zero and here A and B intersect at x.

- $x_a \to \text{quasi-coincident of } A \text{ if } a > A^c(y) \text{ for some } y \in X.$
- $U \in \delta \to \text{quasi-neighborhood if } x_a \text{ is a quasi-coincident of } U.$

Definition 31 (Adherent point). A fuzzy point x_a is called an adherent point of a fuzzy set A iff every quasi-neighborhood of x_a is a quasi-coincident with A.

Problem 2.1. Give an example of an adherent point.

Definition 32 (Accumulation Point). A fuzzy point x_a is called an accumulation point of a fuzzy set A iff x_a is an adherent point of A and every quasi-neighborhood of x_a and A are quasi-coincident at some point different from sup x_a , whenever, $x_a \in A$.

Definition 33 (Base). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then \mathcal{B} is a base for δ iff every open set $G \in \delta$ is the union of members of \mathcal{B} i.e., $G = \bigcup B_i, \forall B_i \in \mathcal{B}$.

Definition 34 (Subbase). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $\mathcal{S} \in X$ is called a subbase iff finite intersection of member of \mathcal{S} form a base for δ .

Separation Axioms

Definition 35 (Quasi T_0 -space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a quasi T_0 -space, if for every two distinct fuzzy points x_a and x_b with same support point x, there exists $U \in Q_{\delta}(x_a)$ such that $x_b \not\propto U$ or, there exists $V \in Q_{\delta}(x_b)$ such that $x_a \not\propto V$.

Definition 36 (Sub T_0 -space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a sub T_0 -space, if for every two distinct $x, y \in X$, there exists $a \in [0, 1]$ such that either $\exists U \in Q_{\delta}(x_a)$ with $y_a \not\propto U$ or, $\exists V \in Q_{\delta}(y_a)$ with $x_a \not\propto V$.

Definition 37 (T_0 -space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_0 -space, if for every two distinct fuzzy points x_a and y_b , $\exists U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, $\exists V \in Q_\delta(y_b)$ with $x_a \not\propto V$.

Definition 38 $(T_1$ —space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_1 —space, if for every two distinct fuzzy points x_a and y_b such that $x_a \not\leq y_b$ then there exists $U \in Q_{\delta}(x_a)$ such that $y_b \not\propto U$ and, $\exists V \in Q_{\delta}(y_b)$ such that $x_a \not\propto V$.

Definition 39 (T_2 -space). Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_2 -space, if for every two distinct fuzzy points x_a and y_b (i.e., $x_a \neq y_b$) then there exists $U \in Q_\delta(x_a)$ and, $V \in Q_\delta(y_b)$ such that $U \wedge V = 0$.

Theorem 3.0.1. Quasi T_0 property is heriditary. or, Every subspace of a Quasi T_0 space is Quasi T_0 space.

Proof. Suppose, $\langle X, \delta \rangle$ be a fuzzy topological space which is Quasi T_0 -space. Let $\langle Y, \mu \rangle$ be the subspace of $\langle X, \delta \rangle$. We have to prove that, $\langle Y, \mu \rangle$ be a Q- T_0 -space.

Now, since, $Y \subseteq X$ so every $V \in \mu$, $V = U_{|Y}$ for some $U \in \delta$. Let y_a and y_b be two distinct fuzzy points in Y such that, $y_a \neq y_b$. Then as $Y \subseteq X$, we have y_a and y_b are in X with $y_a \neq y_b$.

Again, since $\langle X, \delta \rangle$ is a Quasi T_0 -space there exist $U \in Q_{\delta}(y_a)$ such that $y_b \not\propto U$ or, there exist $V \in Q_{\delta}(y_b)$ such that $y_a \not\propto V$. This implies, there is $U_{|y|} \in Q_{\delta|y}(y_a)$ such that $y_b \not\propto U_{|Y|}$ or there is $V_{|Y|} \in Q_{\delta|y}(y_b)$ such that $y_a \not\propto V_{|Y|}$.

Thus, by definition of a Q- T_0 -space $\langle Y, \mu \rangle$ is a Q- T_0 -space.

Theorem 3.0.2. Every subspace of a T_0 -space is T_0 -space.

Proof. Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $\langle Y, \mu \rangle$ be a subspace of $\langle X, \delta \rangle$. Let x_a and y_b be two distinct points in Y. Then since, $Y \subseteq X$, we have, x_a and y_b in X with $x_a \neq y_b$. Now since $\langle X, \delta \rangle$ is a fuzzy T_0 —space. We have either there is $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, there is $V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.

Now, $U_{\uparrow Y} \in Q_{\delta \uparrow Y}(x_a)$ such that $y_b \not\propto U_{\uparrow Y}$ as $x_a, y_b \in Y$ and $V_{\uparrow Y} \in Q_{\delta \uparrow Y}(y_b)$ such that $x_a \not\propto V_{\uparrow Y}$. Thus, $\langle Y, \mu \rangle$ is a T_0 -space.

Thus, $\langle Y, \mu \rangle$ is a T_0 -space.

Theorem 3.0.3. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is a quasi- T_0 -space iff for every $x \in X$ and $a \in [0, 1]$ there exists $B \in \delta$ such that B(x) = a.

Proof. Suppose, $\langle \mathcal{F}(X), \delta \rangle$ be a quasi T_0 -space. If a = 0, then it suffices to take $B = \underline{0}$. If 0 < a < 1, we take a strictly monotonic increasing sequence of positive real numbers converging to a. Let $\Delta_n = (a_n, a_{n+1}]$, $n = 1, 2, 3, \ldots$

Since $\langle \mathcal{F}(X), \delta \rangle$ be a quasi T_0 -space, then for any $x \in X$ and $\Delta = (a_1, a_2)$ with $0 \le a_1 < a_2 < 1$, there exists $B \in \delta$ such that $B(x) \in \Delta$.

From this property, we can say that, $\exists B_n \in \delta$ such that $B_n(x) \in \Delta_n$, for each n

$$\therefore B = \bigvee_{n=1}^{\infty} B_n \in \delta \quad \text{and} \quad B(x) = a.$$

Conversely, suppose x_a and x_b are two fuzzy points with b < a where $a, b \in [0, 1]$. Then by hypothesis, there is an open set B such that B(x) = 1 - b > 1 - a.

This implies, B is an open Q-nbd of x_a but not quasi-conincident with x_b [since, B is a nbd of x_{1-a}]. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a quasi T_0 -space.

Theorem 3.0.4. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is T_1 -space iff for every $x \in X$ and each $a \in [0, 1]$ there exists $B \in \delta$ such that B(x) = 1 - a and B(y) = 1 for $y \neq x$.

Or, $\langle \mathcal{F}(X), \delta \rangle$ is a T_1 -space \Leftrightarrow every fuzzy point in $\langle X, \delta \rangle$ is closed.

Proof. Suppose $\langle \mathcal{F}(X), \delta \rangle$ be a T_1 -space. If a = 0 then it suffices to take $B = \underline{1}$.

Suppose, a > 0 and x_a is a fuzzy point. Since, every fuzzy point in a T_1 -space is closed, so, x_a is a closed set. \therefore We have, $B = 1 - x_a \in \delta$ and hence B(x) = 1 - a and B(y) = 1. if $y \neq x$.

Conversely, let x_a be a fuzzy point. Then by hypothesis there exists $B \in \delta$ such that B(x) = 1 - a and B(y) = 1 with $y \neq x$. This implies, $B = 1 - x_a$ and hence $B^c = x_a$ which is closed. Thus, $B \in \delta$. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is a T_1 -space.

Definition 40 (Purely T_2 -space). $\langle \mathcal{F}(X), \delta \rangle$ is called purely T_2 -space if for every two zero-meet fuzzy points x_a and y_b , $\exists U \in Q_\delta(x_a)$ and $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

Definition 41. A fuzzy topological space $\langle X, \delta \rangle$ is said to be fuzzy regular iff for each $x \in X$ and each closed fuzzy set U with U(x) = 0 there exists $V, W \in \delta$ such that V(x) = 1 and $V \subseteq 1 - W$.

Theorem 3.0.5. For a fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ the following statements are equivalent

- 1. $\langle X, \delta \rangle$ is a fuzzy T_0 -space.
- 2. For $x, y \in X$, $x \neq y$, $\exists U \in \delta$ such that U(x) > 0, U(y) = 0 or U(y) > 0, U(x) = 0.

Proof. (1) \Rightarrow (2), Suppose $\langle X, \delta \rangle$ is a fuzzy T_0 -space. Thus, we have $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$.

Connected Fuzzy Topological Space

Definition 42 (Separated Fuzzy Sets). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $A, B \in \mathcal{F}(X)$ are called separated sets if $\bar{A} \wedge B = \underline{0} = A \wedge \bar{B}$.

Lemma. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A, B, C \in \mathcal{F}(X)$. If B and C are separated sets then $A \wedge B$ and $A \wedge C$ are separated.

Proof. Since B and C are separated, we have, $\bar{B} \wedge C = \underline{0} = B \wedge \bar{C}$.

We have, $A \wedge B < B \implies \overline{A \wedge B} < \overline{B}$ and $A \wedge C < C$.

This implies, $\overline{(A \wedge B)} \wedge (A \wedge C) \leq \overline{B} \wedge C = \underline{0}$. Similarly, $(A \wedge B) \wedge \overline{(A \wedge C)} \leq B \wedge \overline{C} = \underline{0}$.

Hence, $A \wedge B$ and $A \wedge C$ are separated.

Definition 43 (Connected Fuzzy Sets). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. A fuzzy set A on X is called connected if there do not exist $C, D \in \mathcal{F}(X) \setminus \{\underline{0}\}$ such that $A = C \vee D$. Or, A set A is connected if $A = B \vee C$ then either $B = \underline{0}$ or, $C = \underline{0}$.

Theorem 4.0.1. Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then the following are equivalent:

- 1. A is connected.
- 2. $B, C \in \mathcal{F}(X)$ are separated, $A \leq B \vee C$ implies $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$.
- 3. $B, C \in \mathcal{F}(X)$ are separated, $A < B \lor C$ implies A < B or, A < C.

Proof. (1) \Rightarrow (2), Since, B and C are separated set. By the above lemma, we have $(A \land B)$ and $(A \land C)$ are separated. Since, A is connected and $A \leq B \lor C$ implies

$$A = A \wedge (B \vee C)$$

= $(A \wedge B) \vee (A \wedge C)$

then by definition of connectedness, either $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$. Hence, (2) holds.

 $(2) \Rightarrow (3)$, Suppose, $A \wedge B = \underline{0}$, then,

$$A = (A \land B) \lor (A \land C)$$

= $\underline{0} \lor (A \land C)$
= $(A \land C)$.

So, $A \leq C$. Similarly, if $A \wedge C = 0$, then we can prove that $A \leq B$. Thus, (3) holds.

Finally, (3) \Rightarrow (1), Suppose, (3) holds, we need to show that, A is connected. Let $B, C \in \mathcal{F}(X)$ are two separated fuzzy sets such that $A = B \vee C$.

We need to prove that, either, $B = \underline{0}$ or, $C = \underline{0}$. By (3), we have either $A \leq B$ or, $A \leq C$. Now if $A \leq B$ then $C \wedge A \leq C \wedge B \leq C \wedge \bar{B}$. But since, B, C are separated sets so, $C \wedge \bar{B} = \underline{0}$. $\therefore C \wedge A = \underline{0}$. Again, $C \wedge A = C \wedge (B \vee C) = C$. So $C = \underline{0}$.

Now if $A \leq C$, we can similarly prove that B = 0. Thus, A is connected.

Theorem 4.0.2. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$ is connected such that $A \leq B \leq \bar{A}$. Show that B is connected.

Proof. Suppose, C and D are two separated fuzzy sets such that, $B = C \vee D$. To show that, B is connected we need only to show either, C=0 or, D=0.

By the lemma, we have $(A \wedge C)$ and $(A \wedge D)$ are separated sets. Let $F = A \wedge C$, $G = A \wedge D$. Now

$$F \lor G = (A \land C) \lor (A \land D)$$
$$= A \lor (C \land D)$$
$$= A \lor B$$
$$= A$$

Since, A is connected, we have either $F = \underline{0}$ or, $G = \underline{0}$.

Suppose, F = 0. Then, $A = F \vee G = G = A \wedge D$. This implies, A < D. Thus, $\bar{A} < \bar{D}$. i.e., $B < \bar{A} < \bar{D}$. Now, $C \wedge B < C \wedge \bar{A} < C \wedge \bar{D} = 0$. i.e.,

$$C \land B \leq \underline{0}$$

$$\Rightarrow C \land (C \lor D) \leq \underline{0}$$

$$\Rightarrow C = 0$$

Similarly, if G=0, then we can show that D=0. Hence, B is connected.

Definition 44 (Connected Fuzzy Topological Space). If the fuzzy set $\underline{1}$ is connected i.e., there does not exist separated sets $C, D \in \mathcal{F}(X) \setminus \{0\}$ such that $1 = C \vee D$, then the fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called a connected fuzzy topological space.

Theorem 4.0.3 (Characterization Theorem). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then the followings are equivalent

- 1. $\langle \mathcal{F}(X), \delta \rangle$ is connected.
- 2. $A, B \in \delta, A \vee B = 1, A \wedge B = 0, \text{ implies } 0 \in \{A, B\}.$
- 3. $A, B \in \delta', A \vee B = 1, A \wedge B = 0, \text{ implies } 0 \in \{A, B\}.$

Proof. (1) \Rightarrow (2), Suppose, (2) is false. Then there are $A, B \in \delta \setminus \{0\}$ such that

$$A \lor B = \underline{1}, \qquad A \land B = \underline{0}$$

 $\Rightarrow A^c \land B^c = \underline{0}, \text{ and } A^c \lor B^c = \underline{1} \text{ [By De Morgan's Law]}$
 $\Rightarrow \bar{A}^c \land B^c = 0, \text{ and } A^c \land \bar{B}^c = 0 \text{ [Since, } A^c, B^c \text{ are closed.]}$

 \therefore We have by definition, A^c and B^c are two separated sets. Therefore, we have $A^c \vee B^c = \underline{1}$ and A^c , B^c are two separated sets. Hence, $\langle \mathcal{F}(X), \delta \rangle$ is disconnected. Hence, (2) is true.

 $(2) \Rightarrow (3)$, Let $A, B \in \delta'$ such that $A \vee B = \underline{1}$ and $A \wedge B = \underline{0}$. Then by De Morgan's Laws, $A^c \wedge B^c = \underline{0}$ and $A^c \vee B^c = 1$. By (2), $0 \in \{A^c, B^c\}$. Hence, $0 \in \{A, B\}$.

 $(3) \Rightarrow (1)$, If $\langle \mathcal{F}(X), \delta \rangle$ is not connected, then there exists non-zero separated sets $A, B \in \delta' \setminus \{0\}$ such that $A \vee B = \underline{1}$, which contradicts (3).

Hence, $\langle \mathcal{F}(X), \delta \rangle$ is connected.

Compactness

Definition 45 (Cover and C-compactness). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then, $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a cover of A if $A \subseteq \vee \mathcal{A}$.

- $\langle \mathcal{F}(X), \delta \rangle$ is called C-compact if every open cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite subcover.
- \mathcal{A} is called an open cover of A, if $\mathcal{A} \subseteq \delta$ and if \mathcal{A} is a cover of A.
- $\mathcal{B} \subseteq \mathcal{A}$ is called a subcover if \mathcal{B} is still a cover of A.

In particularly, \mathcal{A} is a cover of $\langle \mathcal{F}(X), \delta \rangle$ if \mathcal{A} is a cover of $\underline{1}$.

Definition 46 (α -cover and α -compactness). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an α -cover, if for very $x \in X \exists A \in \mathcal{A} \ni A(x) > \alpha$.

• ft is called an α -compact, if for every open α -cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite sub- α -cover where $\alpha \in [0, 1)$.

Definition 47 (Strong Compact). A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called strongly compact if it is α -compact for every $\alpha \in [0, 1)$.

Definition 48 (α^* -cover and α^* -compactness). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an α^* -cover, if for every $x \in X$, there exists $A \in \mathcal{A}$ such that, $A(x) \geq \alpha$.

• For $\alpha \in [0,1)$, $\langle \mathcal{F}(X), \delta \rangle$ is called an α^* -compact, if for every open α^* -cover of $\langle \mathcal{F}(X), \delta \rangle$ has a finite sub α^* -cover.

Example. Given $X = \{a, b, c\}, A = \{A, B, C\}, \alpha \in [0, 1), \delta = \{\underline{0}, \underline{1}, A, B, C\}$ where,

$$A: a \mapsto 0.2, b \mapsto 0.4, c \mapsto 0.6;$$

 $B: a \mapsto 0.4, b \mapsto 0.6, c \mapsto 0.8;$
 $C: a \mapsto 0.6, b \mapsto 0.8, c \mapsto 0.9;$

Check whether \mathcal{A} is α -compact or, α^* -compact corresponding to the given value of α .

Solution.

- 1. Let $\alpha = 0.7$ $a \in X : \alpha = 0.7 > A(a), B(a), C(a)$. Hence, for $\alpha = 0.7$, \mathcal{A} is not an α -cover.
- 2. Let $\alpha = 0.3$ $a \in X : \alpha = 0.3 < C(a) = 0.6$, B(a) = 0.4 $b \in X : \alpha = 0.3 < A(b) = 0.4$, B(b) = 0.6, C(b) = 0.8 $c \in X : \alpha = 0.3 < A(c) = 0.6$, B(c) = 0.8, C(c) = 0.9 $\therefore A$ is an α -compact space for a = 0.3.
- 3. Let $\alpha = 0.6$ For, $a \in X : \alpha = 0.6 = C(a)$ For, $b \in X : \alpha = 0.6 = B(b), \alpha = 0.6 < C(b) = 0.8$ For, $c \in X : \alpha = 0.6 = A(c) = 0.6, \alpha = 0.6 < B(c) = 0.8, C(c) = 0.9$ $\therefore A$ is an α^* -compact space for a = 0.6.

Definition 49 (Q-cover). Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a Q-cover of A if for every $x \in Supp(A)$, there exists $U \in \mathcal{A}$ such that $x_{A(x)} \propto U$.

Definition 50 (Q-compact). A fuzzy set A is called Q-compact if every open Q-cover of A has a finite sub Q-cover. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called Q-compact if $\underline{1}$ is Q-compact.

Example. Consider, $X = \{a, b, c\}, \delta = \{\underline{0}, \underline{1}, U, V, W\}$ where

 $U: a \mapsto 0.3, b \mapsto 0.5, c \mapsto 0.7;$ $V: a \mapsto 0.4, b \mapsto 0.6, c \mapsto 0.8;$ $W: a \mapsto 0.6, b \mapsto 0.8, c \mapsto 0.9;$

Consider $\mathcal{A} = \{U, V\} \subseteq \delta$ and let, $A: a \mapsto 0.1, b \mapsto 0.2, c \mapsto 0.3$. Then, find the Q-cover of A.

Solution. Here, $Supp(A) = \{a, b, c\}$

For, x = a, $a_{A(a)} = a_{0.1} = 0.1$

For, x = b, $b_{A(b)} = b_{0.2} = 0.2$

For, x = c, $c_{A(c)} = c_{0.3} = 0.3$

For x = a, we have U_a : 0.3 + 0.1 < 1, $V_a = 0.4 + 0.1 < 1$. Hence \mathcal{A} is not a Q-cover of A.

If $A: a \mapsto 0.7$, $b \mapsto 0.6$, $c \mapsto 0.5$.

Then, For x = a, $a_{A(a)} = a_{0.7} = 0.7$

For, x = b, $b_{A(b)} = b_{0.6} = 0.6$

For, x = c, $c_{A(c)} = c_{0.5} = 0.5$

For, $x = a, 0.3 + 0.7 \ge 1, 0.4 + 0.7 > 1$

For, x = b, 0.5 + 0.6 > 1, 0.6 + 0.6 > 1

For, x = c, 0.7 + 0.5 > 1, 0.8 + 0.5 > 1

Hence, for every $x \in Supp(A)$, $x_{A(x)} \propto U$.

 \therefore \mathcal{A} is a Q-cover of A.

Definition 51 $(\alpha - Q - \text{cover})$. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\varphi \subseteq \mathcal{F}(X)$ is called an $\alpha - Q - \text{cover}$ of A, if for every $x_a \subseteq A$, there exists $U \in \varphi$ such that $x_a \propto U$. It is denoted by $\vee \varphi \hat{q} A(\alpha)$.

Definition 52 $(\bar{\alpha} - Q - \text{cover})$. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\varphi \subseteq \mathcal{F}(X)$ is called an $\bar{\alpha} - Q - \text{cover}$ of A, if there exists $\gamma \in B^*(\alpha)$ such that γ is a $\gamma - Q - \text{cover}$ of A.

- $B(b) = \{a \in L : a \propto b\}$, where the binary relation \propto is defined as, for $a, b \in L$, $a \propto b \Leftrightarrow$ for every subset $D \subset L$, b < Sup D implies the existence of $d \in D$ with a <
- $B^*(b) = B(b) \cap M(L)$, where, M(L) = (0, 1].

Definition 53. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. A is called N-compact if for every $\alpha \in (0,1] - M([0,1])$, every open $\alpha - Q$ -cover of A has a finite subfamily which is an $\bar{\alpha} - Q$ -cover of A. $\langle \mathcal{F}(X), \delta \rangle$ is called N-compact, if $\underline{1}$ is compact.

Theorem 5.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. Then A is N-compact iff the following conditionds hold:

- (a) For every $\alpha \in (0,1]$, every open αQ -cover of A has a finite sub αQ -cover.
- (b) For every $\alpha \in (0,1]$, every open αQ -cover of A which consists of just one subset is an $\bar{\alpha} Q$ -cover of A.
- *Proof.* (a) Let, A be N-compact, $\alpha \in (0,1]$ and φ is an open αQ -cover of A. By the definition of N-compact, φ has a finite subfamily ψ such that, ψ is an $\bar{\alpha} Q$ -cover of A. Hence, $\vee \psi \hat{q} A(\alpha)$ i.e., ψ is an αQ -cover of A.
- (b) Suppose, $U \in \delta$ and $\varphi = \{U\}$ is an open αQ -cover of A. Then, by the N-compactness of A, φ has a subfamily ψ such that ψ is an $\bar{\alpha} Q$ -cover of A. But, clearly, $\varphi = \psi$. Hence, ψ is an open αQ -cover of A.

Conversely, suppose (a) and (b) holds.

Let $\alpha \in (0,1]$ and φ is an open $\alpha - Q$ -cover of A.

By (a), φ has a finite sub $\alpha - Q$ -cover ψ of A. Take $U = \vee \psi$. Then $\{U\}$ is an $\alpha - Q$ -cover of A.

By (b), $\{U\}$ is also an $\bar{\alpha} - Q$ -cover of A. By the definition of $\bar{\alpha} - Q$ -cover, there exists $\gamma \in B^*(\alpha)$ such that x_{γ} is a quasi-coincident with U for every $x_{\gamma} \subseteq A$. Hence, $\gamma + U(x) > 1 \Rightarrow \gamma > 1 - U(x)$

i.e., $\gamma \leq (U(x))' \Rightarrow \gamma \not\leq (U\psi(x))' = \wedge \{(W(x))' | W \in \psi\}$

i.e., $W \in Q_{\gamma}(x_{\gamma})$. So, ψ is an $\bar{\alpha} - Q$ -cover of A. Hence, A is N-compact.

Theorem 5.0.2. Continuous image of an N-compact space is N-compact.

Proof. Let $f^{\to}: \langle \mathcal{F}(X), \delta \rangle \to \langle \mathcal{F}(Y), \mu \rangle$ be a continuous fuzzy mapping and A be a N-compact fuzzy set in $\mathcal{F}(X)$. For $\alpha \in (0,1]$, let \mathcal{A} be an open $\alpha - Q$ -cover of $f^{\to}(A)$. Then for every $x_{\alpha} \leq A$, $f^{\to}(x_{\alpha}) = f(x)_{\alpha} \leq f^{\to}(A)$, there exists $U \in \mathcal{A}$ such that $f(x)_{\alpha} \propto U \Rightarrow f(x)_{\alpha} \not\propto U^{c} \Rightarrow \alpha \not\leq U^{c}(f(x)) \Rightarrow \alpha \not\leq f^{\leftarrow}(U^{c})(x) = f^{\leftarrow}(U)^{c}(x)$. That is $x_{\alpha} \propto f^{\leftarrow}(U)$. Since, f^{\to} is continuous, $f^{\leftarrow}(U) \in \delta$ and hence $f^{\leftarrow}(U) \in Q(x_{\alpha})$. Thus, $f^{\leftarrow}(A)$ is an open $\alpha - Q$ -cover of A.

Since A is N-compact, A has a finite subfamily $A_n = \{U_i : 1 \le i \le n\}$ such that $f^{\leftarrow}(A_n)$ is an $\bar{\alpha} - Q$ -cover of A.

Now, we show that, \mathcal{A}_n is an $\bar{\alpha} - Q$ -cover of $f^{\rightarrow}(A)$. Since, $f^{\leftarrow}(\mathcal{A})_n$ is an open $\bar{\alpha} - Q$ -cover of A, there exists $\gamma \in \mathcal{B}(\alpha)$ such that $f^{\leftarrow}(\mathcal{A}_{\setminus})$ is $\gamma - Q$ -cover of A. This implies, $\gamma \sqsubseteq a$ and hence $\exists \lambda \in (0,1]$ such that $\gamma \sqsubseteq \lambda \sqsubseteq \alpha$. So, $\lambda \in \mathcal{B}(\alpha)$ and hence we have, $\lambda \leq f^{\leftarrow}(A)(y) = \vee \{A(x) : x \in X, f(x) = y\}$. Now, $\gamma \sqsubseteq \lambda$ implies, $\gamma \not\leq (f^{\leftarrow}(U_i))^c(x) = f^{\leftarrow}(U_i^c)(x) = U_i^x(f(x)) = U_i^c(y)$, for some $1 \leq i \leq n$ such that $x_{\gamma} \propto f^{\leftarrow}(U_i)$. By $\gamma \sqsubseteq \lambda$ and hence $\gamma \leq \lambda$, we have $\lambda \not\leq U_i^c(y)$. Thus $y_{\lambda} \propto U_i$ for some $1 \leq i \leq n$. So, \mathcal{A}_n is an open $\lambda - Q$ -cover of $f^{\rightarrow}(A)$ and hence \mathcal{A}_n is an $\bar{\alpha} - Q$ -cover of $f^{\leftarrow}(A)$. Therefore, $f^{\rightarrow}(A)$ is an N-compact.

Definition 54 (Net in X). Let X be a non-empty ordinary set and D be a directed set then every mapping $S:D\to X$ is called a net in X and D is called the index set of S.

Theorem 5.0.3. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Let $A, B, C \in \mathcal{F}(X)$ such that A be a N-compact and B be closed. Then $A \wedge B$ is N-compact.

Proof. Let S be an α -net in $A \wedge B$. Then S is also an α -net in A. Since, A is N-compact, S has a cluster point x_{α} in A such that $ht(\alpha) = \alpha$. But, S is also a net in closed subset B, we have $x_{\alpha} \leq B$.

So, $x_a \leq A \wedge B$, i.e., x_a is a cluster point of δ in $A \wedge B$ such that $ht(\alpha) = \alpha$. Hence, $A \wedge B$ is N-compact. \square

Fuzzy Mapping

Definition 55 (Fuzzy Mapping). Let X and Y be two non-empty set and let $f: X \to Y$ be an ordinary mapping. A fuzzy mapping $f^{\to}: \langle \mathcal{F}(X), \delta \rangle \to \langle \mathcal{F}(Y), \mu \rangle$ is defined by $f^{\to}(A)(y) = \bigvee \{A(x) | x \in X, f(x) = y\} \forall y \in Y$, and a fuzzy reverse mapping $f^{\leftarrow}: \langle \mathcal{F}(Y), \mu \rangle \to \langle \mathcal{F}(X), \delta \rangle$ is defined by $f^{\leftarrow}(B)(x) = B(f(x)) \forall x \in X$.

Definition 56 (Continuous Fuzzy Mapping). Let $\langle \mathcal{F}(X), \delta \rangle$ and $\langle \mathcal{F}(Y), \mu \rangle$ be two fuzzy topological space. A fuzzy mapping $f^{\rightarrow} : \langle \mathcal{F}(X), \delta \rangle \rightarrow \langle \mathcal{F}(Y), \mu \rangle$ is called continuous if for each $v \in \mu$, $f^{\rightarrow}(v) \in \delta$.

Definition 57 (Open Fuzzy Mapping). Let $\langle \mathcal{F}(X), \delta \rangle$ and $\langle \mathcal{F}(Y), \mu \rangle$ be two fuzzy topological space. A fuzzy mapping f^{\rightarrow} is called open if for each $u \in \delta$, $f^{\rightarrow}(u) \in \mu$.

Definition 58 (Closed Fuzzy Mapping). Let $\langle \mathcal{F}(X), \delta \rangle$ and $\langle \mathcal{F}(Y), \mu \rangle$ be two fuzzy topological space. A fuzzy mapping f^{\rightarrow} is called closed if for each closed set $F \in \delta$, $f^{\rightarrow}(F)$ is closed in μ .

Theorem 6.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ and $\langle \mathcal{F}(Y), \mu \rangle$ be two fuzzy topological spaces and $f: X \to Y$ be an ordinary mapping. Then for each $a \in [0, 1]$ and every $A \in \mathcal{F}(X)$, $f^{\to}(aA) = af^{\to}(A)$.

Proof. For all $a \in [0,1], \forall A \in \mathcal{F}(X)$ and $\forall y \in Y$ we have,

$$\begin{split} f^{\to}(aA)(y) &= \vee \{ (aA)(x) | x \in X, \ f(x) = y \} \\ &= \vee \{ a \wedge (A)(x) | x \in X, \ f(x) = y \} \\ &= a \wedge (\vee \{ (A)(x) | x \in X, \ f(x) = y \}) \\ &= a \wedge f^{\to}(A)(y) \\ &= (af^{\to}(A))(y) \end{split}$$

Thus, $f^{\rightarrow}(aA) = af^{\rightarrow}(A)$.