Chapter 1

Connected Fuzzy Topological Space

Definition 1 (Separated Fuzzy Sets)**.** Let $\langle \mathcal{F}(x), \delta \rangle$ be a fuzzy topological space. Then $A, B \in \mathcal{F}(x)$ are called separated sets if $\bar{A} \wedge B = \underline{0} = A \wedge \bar{B}$.

Lemma. Let $\langle \mathcal{F}(x), \delta \rangle$ be a fuzzy topological spaceand A, B, $C \in \mathcal{F}(x)$. If B and C are separated sets then $A \wedge B$ and $A \wedge C$ are separated.

Proof. Since B and C are separated, we have, $\bar{B} \wedge C = \underline{0} = B \wedge \bar{C}$.
We have, $A \wedge B < B \;\; \Rightarrow \overline{A \wedge B} < \overline{B}$ and $A \wedge C < C$.
This implies, $\overline{(A \wedge B)} \wedge (A \wedge C) \leq \bar{B} \wedge C = \underline{0}$. Similarly, $(A \wedge B) \wedge \overline{(A \wedge C)} \leq B \wedge \bar{C} = \underline{0}$.
Hence, $A \wedge B$ and $A \wedge C$ are separated. □

Definition 2 (Connected Fuzzy Sets)**.** Let $\langle \mathcal{F}(x), \delta \rangle$ be a fuzzy topological space. A fuzzy set A on X is called connected if there do not exist $C, D \in \mathcal{F}(x) \setminus \{\underline{0}\}$ such that $A = C \vee D$.
Or, A set A is connected if $A = B \vee C$ then either $B = \underline{0}$ or, $C = \underline{0}$.

Theorem 1.0.1. Let, $\langle \mathcal{F}(x), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(x)$. Then the following are equivalent:

1. A is connected.
2. B, $C \in \mathcal{F}(x)$ are separated, $A \leq B \vee C$ implies $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$.
3. B, $C \in \mathcal{F}(x)$ are separated, $A \leq B \vee C$ implies $A \leq B$ or, $A \leq C$.

Proof. (1) $\Rightarrow$ (2), Since, B and C are separated set. By the above lemma, we have $(A \wedge B)$ and $(A \wedge C)$ are separated. Since, A is connected and $A \leq B \vee C$ implies

$$\begin{aligned} A &= A \wedge (B \vee C) \\ &= (A \wedge B) \vee (A \wedge C) \end{aligned}$$

then by definition of connectedness, either $A \wedge B = \underline{0}$ or, $A \wedge C = \underline{0}$. Hence, (2) holds.

(2) $\Rightarrow$ (3), Suppose, $A \wedge B = \underline{0}$, then,

$$\begin{aligned} A &= (A \wedge B) \vee (A \wedge C) \\ &= \underline{0} \vee (A \wedge C) \\ &= (A \wedge C). \end{aligned}$$

So, $A \leq C$. Similarly, if $A \wedge C = \underline{0}$, then we can prove that $A \leq B$. Thus, (3) holds.

Finally, (3) $\Rightarrow$ (1), Suppose, (3) holds, we need to show that, A is connected. Let $B, C \in \mathcal{F}(x)$ are two separated fuzzy sets such that $A = B \vee C$.
We need to prove that, either, $B = \underline{0}$ or, $C = \underline{0}$. By (3), we have either $A \leq B$ or, $A \leq C$. Now if $A \leq B$ then $C \wedge A \leq C \wedge B \leq C \wedge \bar{B}$. But since, B, C are separated sets so, $C \wedge \bar{B} = \underline{0}$. $\therefore$ $C \wedge A = \underline{0}$. Again, $C \wedge A = C \wedge (B \vee C) = C$. So $C = \underline{0}$.
Now if $A \leq C$, we can similarly prove that $B = \underline{0}$. Thus, A is connected. □

Theorem 1.0.2. Let $\langle \mathcal{F}(x), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(x)$ is connected such that $A \leq B \leq \bar{A}$. Show that B is connected.

Proof. Suppose, C and D are two separated fuzzy sets such that, $B = C \vee D$. To show that, B is connected we need only to show either, $C = \underline{0}$ or, $D = \underline{0}$.
By the lemma, we have $(A \wedge C)$ and $(A \wedge D)$ are separated sets. Let $F = A \wedge C$, $G = A \wedge D$. Now

$$\begin{aligned} F \vee G &= (A \wedge C) \vee (A \wedge D) \\ &= A \vee (C \wedge D) \\ &= A \vee B \\ &= A \end{aligned}$$

Since, A is connected, we have either $F = \underline{0}$ or, $G = \underline{0}$.
Suppose, $F = \underline{0}$. Then, $A = F \vee G = G = A \wedge D$. This implies, $A \leq D$. Thus, $\bar{A} \leq \bar{D}$. i.e., $B \leq \bar{A} \leq \bar{D}$.
Now, $C \wedge B \leq C \wedge \bar{A} \leq C \wedge \bar{D} = \underline{0}$. i.e.,

$$\begin{aligned} &C \wedge B \leq \underline{0} \\ \Rightarrow &C \wedge (C \vee D) \leq \underline{0} \\ \Rightarrow &C = \underline{0} \end{aligned}$$

Similarly, if $G = \underline{0}$, then we can show that $D = \underline{0}$. Hence, B is connected. $\square$

Definition 3 (Connected Fuzzy Topological Space)**.** If the fuzzy set $\underline{1}$ is connected i.e., there does not exist separated sets $C, D \in \mathcal{F}(x) \setminus \{\underline{0}\}$ such that $\underline{1} = C \vee D$, then the fuzzy topological space$\langle \mathcal{F}(x), \delta \rangle$ is called a connected fuzzy topological space.

Theorem 1.0.3 (Characterization Theorem). Let $\langle \mathcal{F}(x), \delta \rangle$ be a fuzzy topological space. Then the followings are equivalent

1. $\langle \mathcal{F}(x), \delta \rangle$ is connected.
2. $A, B \in \delta$, $A \vee B = \underline{1}$, $A \wedge B = \underline{0}$, implies $\underline{0} \in \{A, B\}$.
3. $A, B \in \delta'$, $A \vee B = \underline{1}$, $A \wedge B = \underline{0}$, implies $\underline{0} \in \{A, B\}$.

Proof. (1) $\Rightarrow$ (2), Suppose, (2) is false. Then there are $A, B \in \delta \setminus \{\underline{0}\}$ such that

$$\begin{aligned} &A \vee B = \underline{1}, \qquad A \wedge B = \underline{0} \\ \Rightarrow &A^c \wedge B^c = \underline{0}, \text{ and } A^c \vee B^c = \underline{1} \quad \text{[By De Morgan's Law]} \\ \Rightarrow &\bar{A}^c \wedge B^c = \underline{0}, \text{ and } A^c \wedge \bar{B}^c = \underline{0} \quad \text{[Since, } A^c, B^c \text{ are closed.]} \end{aligned}$$

$\therefore$ We have by definition, A^c and B^c are two separated sets. Therefore, we have $A^c \vee B^c = \underline{1}$ and A^c, B^c are two separated sets. Hence, $\langle \mathcal{F}(x), \delta \rangle$ is disconnected. Hence, (2) is true.

(2) $\Rightarrow$ (3), Let $A, B \in \delta'$ such that $A \vee B = \underline{1}$ and $A \wedge B = \underline{0}$. Then by De Morgan's Laws, $A^c \wedge B^c = \underline{0}$ and $A^c \vee B^c = \underline{1}$. By (2), $\underline{0} \in \{A^c, B^c\}$. Hence, $\underline{0} \in \{A, B\}$.

(3) $\Rightarrow$ (1), If $\langle \mathcal{F}(x), \delta \rangle$ is not connected, then there exists non-zero separated sets $A, B \in \delta' \setminus \{\underline{0}\}$ such that $A \vee B = \underline{1}$, which contradicts (3).
Hence, $\langle \mathcal{F}(x), \delta \rangle$ is connected. $\square$