Chapter 1

Metric Spaces

Definition 1 (Group)**.** A group G is a non-empty set of elements for which a binary operation $*$ is defined. This operation satisfies the following axioms:

(i) *Closure:* If $a, b \in G$ implies that $a * b \in G$

(ii) *Associativity:* If $a, b, c \in G$ implies that $(a * b) * c = a * (b * c)$

(iii) *Identity:* There exists a unique element $e \in G$(called the identity element) such that $a*e = e*a = a$ for all $a \in G$.

(iv) *Inverse:* For every $a \in G$ there exists an element $a' \in G$(called the inverse of a) such that $a * a' = a' * a = e$.

Note. When the binary operation is addition, G is called an additive group and when the binary operation is multiplication, G is called a multiplicative group.

Definition 2. A group G is called Abelian (or commutative) if for every $a, b \in G$, $a * b = b * a$.

Example. The set of all integers i.e., $\{0, \mp 1, \pm 2, \pm 3, \dots\}$ is a group with respect to the binary operation of addition.

Example. The set $\{\pm 1, \pm i\}$ where $i = \sqrt{-1}$ is a group with respect to the binary operation of multiplication.

Definition 3 (Ring)**.** An additive Abelian (or commutative) group $(G, +)$ with the following properties is said to be a ring:

(i) The group G is closed with respect to the binary operation of multiplication. i.e., for $a, b \in G \Rightarrow a \cdot b \in G$

(ii) Multiplication is associative, i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c$ for all $a, b, c \in G$.

(iii) Multiplication is distributive with respect to addition on both left and the right, that is $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in G$.

Example. The set $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ is a ring under binary operations of ordinary addition and multiplication.

Example. Consider the set $\bar{Z} = \{0, 1, 2, 3, 4, 5\}$, $\bar{Z}$ is a ring under the binary operation of addition and multiplication modulo 6.

Definition 4 (Field)**.** A field F is a commutative ring with unit element in which every non-zero element has a multiplicative inverse.

Example. Examples of fields are the ting of rational numbers, the ring of real numbers and the ring of complex numbers.

1.1 Metric Space

Definition 5. Euclidean space (or Euclidean $n-$space) denoted $\mathbb{R}^n$, consists of all ordered $n-$ tuples of real numbers. Symbolically, $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$.

Thus, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}(n \text{ times})$ is the Cartesian product of $\mathbb{R}$ with itself n times.

Example. The real line $\mathbb{R}$, two-dimensional plane $\mathbb{R}^2$, three-dimensional space $\mathbb{R}^3$ are examples of Euclidean spaces.

Definition 6 (Metric space)**.** A metric space (M, d) is a set M and a function $d : M \times M \to \mathbb{R}$ such that

(i) *Positivity:* $d(x, y) \geq 0$ for all $x, y \in M$

(ii) *Non degeneracy (identity of indiscernibles):* $d(x, y) = 0$ if and only if $x = y$

(iii) *Symmetry:* $d(x, y) = d(y, x)$ for every $x, y \in M$

(iv) *Triangle inequality:* $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$

Thus, a metric space M is a set equipped with a function $d : M \times M \to \mathbb{R}$ that gives a reasonable way of measuring the distance between two elements of M.

Example. The real line $\mathbb{R}$ is a metric space with the metric defined by $d(x, y) = |x - y|$. Similarly, the complex plane $\mathbb{C}$ and the Euclidean space $\mathbb{R}^n$ are metric spaces together with the metric $d(z, w) = |z - w|$ and the standard metric respectively.

Definition 7 (Discrete metric)**.** Let M be any set and let $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Then d is a metric on M.

Definition 8 (Bounded metric)**.** If d is a metric on a set M and $\rho(x, y)$ is defined by $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$, then ρ is a metric called bounded metric. Observe that $\rho(x, y) < 1$ for all $x, y \in M$ i.e., ρ is bounded by 1.

Note. The distance function d on $\mathbb{R}^n$ is given by $d(x, y) = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}$.

1.2 Vector Space

A vector space over an arbitrary field F is a non-empty set V, whose elements are called vectors for which two operations are prescribed. The first operation, called *vector addition*, assigns to each pair of vectors u and v a vector denoted by $u + v$, called their sum. The second operation, called scalar multiplication assigns to each vector u in V and each scalar $\alpha \in F$ a vector denoted by αu which is in V.

Definition 9. A vector space (or a linear space) V is a set of elements called vectors, with given operations of vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$ such that:

A(i) *Commutativity:* $u + v = v + u$ for every $u, v \in V$.

A(ii) *Associativity:* $(u + v) + w = u(v + W)$

A(iii) *Zero vector:* There is a zero vector 0 such that $u + 0 = y$ for every $u \in V$.

A(iv) *Negatives:* For each $u \in V$ there is a vector $-u$ such that $u + (-u) = 0$.

M(i) *Distributivity:* For $\alpha \in F$ and $u, v \in V$, $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$

M(ii) *Distributivity:* For any $\alpha, \beta \in F$ and $u \in V$, $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$

M(iii) *Associativity:* For any $\alpha, \beta \in F$ and $u \in V$, $(\alpha\beta) \cdot u = \alpha(\beta \cdot u)$

M(iv) *Multiplicative unity* For each $u \in V$ there is a unit scalar $e \in F$ such that $eu = u$.

If the field $F = \mathbb{R}$, then the linear space V is called a real linear space, similarly if $F = \mathbb{C}$, then the linear space is called a complex linear space. The subset S of a vector space V is called a subspace of V if S itself is a vector space.

1.3 Normed Linear Space (NLS)

A normed linear space $(V, ||\cdot||)$ is a vector space V and a function $||\cdot|| : V \to \mathbb{R}$ called a norm such that

(i) *Positivity:* $||u|| \geq 0$ for all $u \in V$.

(ii) *Non degeneracy:* $||u|| = 0$ if and only if $u = 0$.

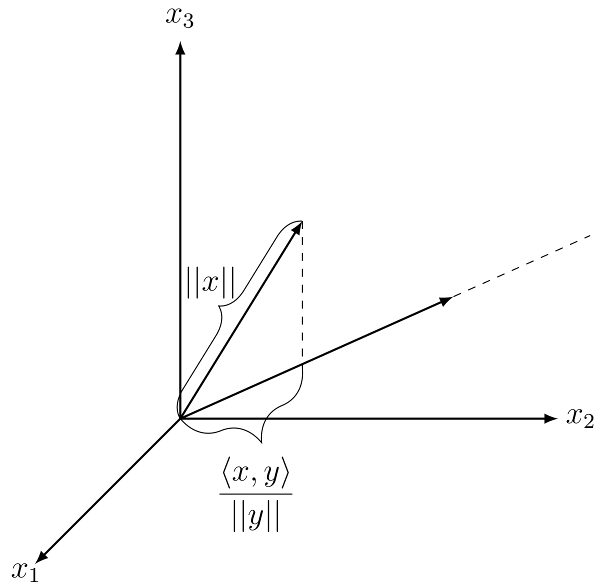
(iii) *Multiplicativity:* $||\alpha u|| = |\alpha| \ ||u||$ for every $u \in V$ and every scalar α.

(iv) *Triangle inequality:* $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in V$.

Definition 10. The norm or length of a vector x in $\mathbb{R}^n$ is defined by $||x|| = \left\{\sum_{i=1}^n x_i^2\right\}^{1/2}$, where $x = (x_1, x_2, \ldots, x_n)$. The distance between two vectors x and y in $\mathbb{R}^n$ is the real number

$$d(x, y) = ||x - y|| = \left\{\sum_{i=1}^n (x_i - y_i)^2\right\}^{1/2}$$

The inner product of x and y in $\mathbb{R}^n$ is defined by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Thus, $||x||^2 = \langle x, x\rangle$.
In $\mathbb{R}^3$, we are also familiar with $\langle x, y\rangle = ||x|| \ ||y|| \cos\theta$ where θ is the angle between x and y.



Theorem 1.3.1. For vectors in $\mathbb{R}^n$, we have

1. Properties of the inner product:
   (i) *Positivity:* $\langle x, x \rangle \geq 0$
   (ii) *Non degeneracy:* $\langle x, x \rangle = 0$ if and only if $x = 0$
   (iii) *Distributivity:* $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
   (iv) *Multiplicativity:* $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for $\alpha \in \mathbb{R}$
   (v) *Symmetry:* $\langle x, y \rangle = \langle y, x \rangle$
2. Properties of the norm:
   (i) $||x|| \geq 0$
   (ii) $||x|| = 0$ if and only if $x = 0$.
   (iii) $||\alpha x|| = |\alpha| \; ||x||$ for $\alpha \in \mathbb{R}$.
   (iv) $||x + y|| \leq ||x|| + ||y||$
3. Properties of the distance:
   (i) $d(x, y) \geq 0$
   (ii) $d(x, y) = 0$ if and only if $x = y$
   (iii) $d(x, y) = d(y, x)$
   (iv) $d(x, y) \leq d(x, z) + d(z, y)$
4. The Cauchy Schwarz inequality:
   $|\langle x, y \rangle| \leq ||x|| \; ||y||$ (Also, named Cauchy-Bunyakovskii-Schwarz inequality).

1.3.1 Examples of normed linear space (NLS)

Example. The real line $\mathbb{R}$ is a NLS with the norm $||x|| = |x|$. Similarly, the set of complex numbers $\mathbb{C}$ is a NLS with $||z|| = |z|$.

Example (Taxicab norm)**.** Consider the space $\mathbb{R}^2$, but instead of the usual norm on it, set $||(x, y)||_1 = |x| + |y|$. Then $||\cdot||_1$ is a norm on $\mathbb{R}^2$, called the taxicab norm. If $P = (x, y)$ and $Q = (a, b)$, then $d_1(P, Q) = ||P - Q||_1 = |x - a| + |y + b|$. This is the sum of the vertical and horizontal separations. You must travel this distance to get from P to Q if you always travel parallel to the axes (stay on the streets in a taxicab).

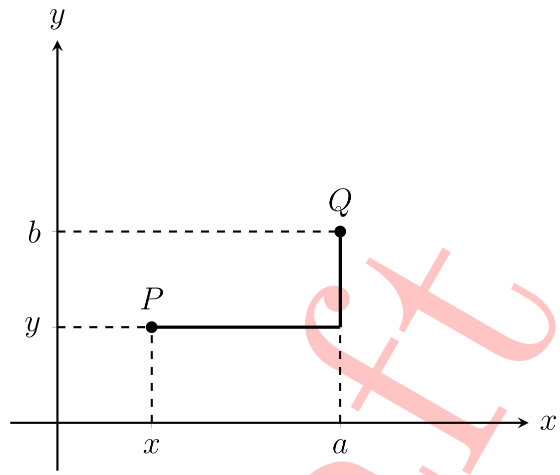


Figure 1.1: The taxicab metric

Example (Supremum norm)**.** Let M = all real-valued functions on the interval $[0, 1]$ that are bounded. That is, let $M = \{f : [0, 1] \to \mathbb{R} \mid$ there is a number B with $|f(x)| \leq B$ for every $x \in [0, 1]\}$. For each f in M, $f([0, 1])$ is a bounded subset of $\mathbb{R}$, and so $\{|f(x) \mid x \in [0, 1]|\}$ is also. It then has a fine least upper bound and $||f||_\infty = \sup\{|x| \mid x \in [0, 1]\}$ defines a function $||\cdot||_\infty : M \to \mathbb{R}$. The set M is a vector space and $||\cdot||_\infty$ is a norm on it, called supremum norm.

The metric in the space M of all bounded functions on $[0, 1]$ is thus defined by $d(f, g) = ||f - g||_\infty = \sup\{|f(x) - g(x)| \mid 0 \leq x \leq 1\}$. Thus, the metric given by the sup norm is the largest vertical separation between the graphs:

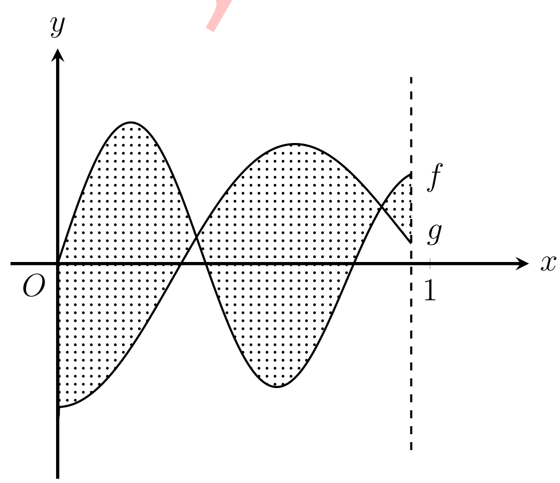


Figure 1.2: The sup distance between function is the largest distance between their graphs.

Proposition 1.3.2. If $(V, ||\cdot||)$ is a normal vector space and $d(u, v)$ is defined by $d(u, v) = ||u - v||$, then d is a metric in V.

1.4 Inner Product Space

A vector space V over an arbitrary field F is called an inner product space if there is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ that associates a scalar $\langle u, v \rangle \in F$ with each pair of vectors u and v in V in such a way that the following axioms are satisfied for all vectors u, v and w in V and all scalars $\alpha, \beta \in F$

(i) *Positivity:* $\langle u, u \rangle \geq 0$

(ii) *Non degeneracy:* $\langle u, u \rangle = 0$ if and only if $u = 0$

(iii) *Hermitian symmetry:* $\langle u, v \rangle = \overline{\langle v, u \rangle}$

(iv) *Distributivity:* $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(v) *Multiplicativity:* $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

Note. The function $\langle \cdot, \cdot \rangle : V \times V \to F$ is called the inner product on V and $(V, \langle \cdot, \cdot \rangle)$ is called the inner product space.

Note. If $F = \mathbb{R}$ (real field), then the inner product space $(V, \langle \cdot, \cdot \rangle)$ is called a real inner product space. In this case the Hermitian symmetry $\langle u, v \rangle = \overline{\langle v, u \rangle}$ becomes simply symmetry $\langle u, v \rangle = \langle v, u \rangle$, and the second distributive property $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ holds by the properties (iii) and (iv).

Similarly, if $F = \mathbb{C}$, the inner product space $(V, \langle \cdot, \cdot \rangle)$ is called a complex inner product space or UNITARY space. With the help of (iii) and (v) we have $\langle \alpha u, v \rangle = \bar{\alpha} \langle u, v \rangle$ if $\alpha \in \mathbb{C}$.

(v) implies that $\langle 0, y \rangle = 0$ for all $y \in V$.

By (i), we may define $||u||$, the norm of the vector $x \in V$ to be the non-negative square roots of $\langle u, u \rangle$. Thus, $||u||^2 = \langle u, u \rangle$. The properties (i) to (v) excluding (ii) imply that $|\langle x, y \rangle| \leq ||x|| \; ||y||$ for all $x, y \in V$.

1.4.1 The Cauchy-Schwarz Inequality

If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, then $|\langle x, y \rangle| \leq ||x|| \; ||y||$ for all x and $y \in V$. The equality holds if and only if x and y are linearly dependent.

Proof. Method 1: If either x and y is 0, then $\langle x, y \rangle = 0$, and so the inequality holds. Therefore, we can assume $x \neq 0$, and $y \neq 0$. Then $\langle x, x \rangle > 0$ and $\langle y, y \rangle > 0$. Then for any α and β in $\mathbb{C}$, we have

$$
\begin{aligned}
0 \leq ||\alpha x + \beta y||^2 &= \langle \alpha x + \beta y, \alpha x + \beta y \rangle \text{ where } \alpha \text{ and } \beta \text{ are not both zero} \\
&= \alpha \bar{\alpha} \langle x, x \rangle + \alpha \bar{\beta} \langle x, y \rangle + \bar{\alpha} \beta \langle y, x \rangle + \beta \bar{\beta} \langle y, y \rangle \\
&= |\alpha|^2 ||x||^2 + \alpha \bar{\beta} \langle x, y \rangle + \overline{\alpha \bar{\beta} \langle x, y \rangle} + |\beta|^2 ||y||^2 \\
&= |\alpha|^2 ||x||^2 + 2\mathrm{Re} \left\{ \alpha \bar{\beta} \langle x, y \rangle \right\} + |\beta|^2 ||y||^2 \\
&\leq |\alpha|^2 ||x||^2 + 2 |\alpha| |\beta| |\langle x, y \rangle| + |\beta|^2 ||y||^2 \qquad [\text{As } \mathrm{Re}(z) \leq |z| \text{ and } \left|\bar{\beta}\right| = |\beta|]
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow |\alpha|^2 a + 2|\alpha||\beta| b + \left|\beta^2\right| c \geq 0 \qquad \text{Where } a = ||x||^2, \ b = |\langle x, y\rangle|, \text{ and } c = ||y||^2 \\
&\Rightarrow \left|\frac{\alpha}{\beta}\right|^2 a + 2\left|\frac{\alpha}{\beta}\right| b + c \geq 0 \qquad \text{If } \beta \neq 0 \\
&\Rightarrow ax^2 + 2bx + c \geq 0 \qquad \text{Where } \left|\frac{\alpha}{\beta}\right| = x, \text{ a real variable} \\
&\Rightarrow 0 \leq a\left(x^2 + 2 \cdot x \cdot \frac{b}{a} + \frac{b^2}{a^2}\right) + c - \frac{b^2}{a} \\
&\Rightarrow 0 = a\left(x + \frac{b}{a}\right)^2 + \frac{ca - b^2}{a} \qquad (1.1)
\end{aligned}
$$

Inequality (1.1) holds if and only if $\frac{ca-b^2}{a} \geq 0$ since $\left(x + \frac{b}{a}\right)^2 \geq 0$

$$
\begin{aligned}
&\Rightarrow b^2 \leq ac \\
&\Rightarrow |\langle x, y\rangle| \leq ||x|| \ ||y||
\end{aligned}
$$

For equality, there must be a value of x of which $ax^2 + 2bx + c = 0$, which is possible if and only if $\alpha x + \beta y = 0$ where not bot of α and β are zero, which implies that x and y are linearly dependent.[1] □

Method 2: Let $a = ||x||^2$, $b = |\langle x, y\rangle|$, and $c = ||y||^2$.
There is a complex number α such that $|\alpha| = 1$ and $\alpha \langle y, x\rangle = b$.
For any real r, we then have

$$
\begin{aligned}
0 &\leq \langle x - r\alpha y, x - r\alpha y\rangle = \langle x, x\rangle - r\alpha\langle y, x\rangle - r\bar{\alpha}\langle x, y\rangle + r^2\langle y, y\rangle \\
&= cr^2 - 2br + a \qquad \overline{\alpha\langle y, x\rangle} = \bar{\beta} \\
\text{i.e., } f(r) &= cr^2 - 2br + a \geq 0 \qquad \bar{\alpha}\langle x, y\rangle = b \text{ as } b \text{ is real}
\end{aligned}
$$

Here $\frac{\mathrm{d}f}{\mathrm{d}r} = 2cr - 2b$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} = 2c > 0$.
Since, $\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} > 0$ so the quadratic expression $f(r)$ has a minimum which occurs when $\frac{\mathrm{d}f}{\mathrm{d}r} = 0$ i.e., $r = \frac{b}{c}$.
Therefore, we insert the value of r and obtain,

$$
\begin{aligned}
&c \cdot \frac{b^2}{c^2} - 2b \cdot \frac{b}{c} + a \geq 0 \\
&\Rightarrow \frac{b^2}{c} \leq a \\
&\Rightarrow b^2 \leq ac \\
&\Rightarrow |\langle x, y\rangle| \leq ||x|| \ ||y||
\end{aligned}
$$

The second part is followed if and only if $x - r\alpha y = 0$, so x and y are linearly dependent. □

Note. The above inequality also variously known as the Schwarz, the Cauchy-Schwarz or the Cauchy-Buniakowsky-Schwarz inequality.

Remark. A consequence of this remark is that the linear function $f(x) = \langle x, y\rangle$ $[f : V \to F$ (here field $F = \mathbb{C}$)] is bounded by $||y||$, and from this it follows that $\langle x, y\rangle$ is a continuous function from $V \times V$ to $\mathbb{C}$.

[1] If $\alpha \neq 0$, $x = \frac{-\beta}{\alpha} y$ and if $\beta \neq 0$, $y = \frac{-\alpha}{\beta} x$

Theorem 1.4.1. If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space and $||\cdot||$ is defined for $v \in V$ by $||v|| = \sqrt{\langle v, v \rangle}$ then $||\cdot||$ is a norm on V.

Proof. Hints for triangle inequality,

$$\begin{aligned} ||v + w||^2 &= \langle v + w, v + w \rangle \\ &= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle \\ &= ||v||^2 + 2 \langle v, w \rangle + ||w||^2 \\ &\leq ||v||^2 + 2 ||v|| \; ||w|| + ||w||^2 \\ &= (||v|| + ||w||)^2 \qquad \text{and so } \; ||v + w|| \leq ||v|| + ||w|| \end{aligned}$$

□