Chapter 1

Simplex Method

1.1 Various Forms and Conversion

1.1.1 General Form

$$
\begin{aligned}
&\max / \min Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&\text{subject to } \; a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n(\leq \text{or} = \text{or} \geq)b_1 \\
&\qquad\qquad\quad a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n(\leq \text{or} = \text{or} \geq)b_2 \\
&\qquad\qquad\quad \ldots \qquad \ldots \qquad \ldots \qquad\qquad \ldots \\
&\qquad\qquad\quad a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n(\leq \text{or} = \text{or} \geq)b_m \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_j \geq 0 (j = 1, 2, \ldots, n)
\end{aligned}
$$

1.1.2 Compact Form

$$
\begin{aligned}
&\max / \min Z = \sum_{j=1}^{n} c_jx_j \\
&\text{subject to } \sum a_{ij}x_j (\leq \text{or} = \text{or} \geq) b_i (i = 1, 2, \ldots, m) \\
&\qquad\qquad\qquad\qquad x_j \geq 0 (j = 1, 2, \ldots, n)
\end{aligned}
$$

Here c_j = cost coefficient

1.1.3 Matrix Form

$$
\begin{aligned}
&\max / \min Z = cX \\
&\text{subject to } AX (\leq \text{or} \; = \text{or} \geq) b
\end{aligned}
$$

$$
A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}_{m \times n} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1} \quad c = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \end{pmatrix}_{1 \times n} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m \times 1}
$$

1.1.4 Canonical Form

$$
\begin{aligned}
&\text{maximize } Z = \sum_{j=1}^{n} c_jx_j \\
&\text{subject to } \sum a_{ij}x_j \leq \; b_i \quad (i = 1, 2, \ldots, m) \\
&\qquad\qquad\qquad x_j \geq 0 \quad (j = 1, 2, \ldots, n)
\end{aligned}
$$

The Characteristic of the Canonical Form

1. All decision variables are non-negative.
2. All constraints are of the ($\leq$) type, and
3. Objective function is of maximization type.

1.1.5 Conversion between forms

Between maximization and minimization

We can change the type of objective function by multiplying the objective function with -1.

$$\min\ Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \Leftrightarrow \max\ G = -Z = -c_1x_1 - c_2x_2 - \cdots - c_nx_n$$

Between $\leq$ and $\geq$

We can change the type of constraints by multiplying the constraints with -1.

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b_1 \Leftrightarrow -a_1x_1 - a_2x_2 - \cdots - a_nx_n \geq b_1$$

Between equations and inequalities

We can transform an equation to two inequalities.

$$a_1x_1 + a_2x_2 = b_1 \Leftrightarrow a_1x_1 + a_2x_2 \leq b_1 \text{ and } a_1x_1 + a_2x_2 \geq b_1 = -a_1x_1 - a_2x_2 \leq -b_1$$

Changing an unrestricted variable to restricted variable

We can change an unrestricted variable to restricted variable by introducing two additional variables. For example, let x_3 is unrestricted then by introducing two additional variables we get $x_3 = x_3' - x_3"$ then we can add the restriction to the newly introduced variable, i.e., $x_3', x_3" \geq 0$.

1.1.6 Standard Form

1. All the decision variables are non-negative.
2. All the constraints are expressed in the form of equation, except the non-negativity constraints which remain inequality (≥ 0).
3. The right-hand side of each constraint equation is non-negative.
4. The objective function is of the form maximization or minimization.

1.2 Basic Variable

Let there are n variables and there are m constants and ($n \geq m$). If we get a unique solution by solving the system of equation of order m (by assuming remaining $n - m$ variables to 0), then these m variables are called basic variables. Otherwise, these are called non-basic variables. Let us look at an example.

Example.

$$\begin{aligned} x_1 + x_2 + 4x_3 + 2x_4 + 3x_5 &= 8 \\ 4x_1 + 2x_2 + 2x_3 + x_4 + 6x_5 &= 4 \end{aligned}$$

Here, $n = 5$ and $m = 2$, so here 2 variables can be basic variable and 5-2=3 variables are non-basic variable.

Case 1: Let $x_2 = x_4 = x_5 = 0$. Then the system of equations become

$$x_1 + 4x_3 = 8$$
$$4x_1 + 2x_3 = 4$$

By solving this system we get a unique solution: $x_1 = 0$ and $x_3 = 2$. So here, $x_1,\ x_3$ are feasible[1] basic variables and $x_2,\ x_4,\ x_5$ are non-basic variables.

Case 2: Let $x_3 = x_4 = x_5 = 0$. Then the system of equations become

$$x_1 + 2x_2 = 8$$
$$4x_1 + 2x_2 = 4$$

By solving this system we get a unique solution: $x_1 = -6$ and $x_3 = 14$. So here, $x_1,\ x_2$ are basic infeasible $(x_1 \ngeq 0)$ variables and $x_3,\ x_4,\ x_5$ are non-basic variables.

Case 3: Let $x_1 = x_2 = x_5 = 0$. Then the system of equations become

$$4x_3 + 2x_4 = 8$$
$$2x_3 + 2x_4 = 4$$

By solving this system we get infinitely many solutions. So these are not basic variables.

Case 4: Let $x_1 = x_3 = x_4 = 0$. Then the system of equations become

$$x_2 + 3x_5 = 8$$
$$2x_3 + 6x_4 = 4$$

There are no solution of this system. So these are not basic variables.

Definition 1 (Degenerate Solution)**.** If at least one of the basic feasible solution is zero then that solution is degenerate solution.

So, Case 1: of above example is a degenerate solution $[\because x_1 = 0]$ and Case 2: of above example is a non-degenerate solution $[\because x_1, x_2 \neq 0]$.

1.3 Simplex Method

Problem 1.3.1.

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 14 \\ & x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Steps:

- Transform the problem into standard form.
- For basic variable: check if the slack variables are basic or not
- $c_B \to$ coefficient of basis in objective function
- Tab: table number
- Basis: basic variables
- c_j constants of objective function (cost coefficient)

[1] If a variable/solution satisfies all constraints and non-negativity condition then it will be a feasible solution.

- $\bar{c_j}$ row $= c_j - \sum c_B A_j$ (here A_j is the coefficient in the j-th column)
- Z calculation: $Z = \sum c_B Z_j$
- Pivot column: maximization: max in $\bar{c_j}$ row; minimization: min in $\bar{c_j}$ row (purple in table)
- Pivot row: check positive ratio of constants and pivot column, i.e., ratio $= \frac{\text{constants}}{\text{pivot column}}$ and select minimum ratio as pivot row, same for maximization and minimization problem. (green in table)
- Intersection of pivot column and pivot row is pivot element (blue in table)
- For next iteration make the pivot element 1 and other element in pivot column 0 by doing row operations.
- Unbounded check: If ratio in pivot column is all negative then the problem is unbounded.
- For Optimal solution check:
  - In maximization, no positive element should appear in $\bar{c_j}$ row
  - In minimization, no negative element should appear in $\bar{c_j}$ row
- Alternative solution check: If any non-basic variable is zero in final $\bar{c_j}$ row then alternative solution exists.

Solution. First we need to transform the problem into standard form.
Standard form:

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 + s_1 = 4 \\ & 3x_1 + 2x_2 + s_2 = 14 \\ & x_1 - 2x_2 + s_3 = 3 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Taking $x_1 = x_2 = 0,\ S_1 = 4,\ S_2 = 14,\ S_3 = 3$ are the initial basic feasible variables.

Tab	c_B	$c_j \rightarrow$ / basis	3 x_1	2 x_2	0 S_1	0 S_2	0 S_3	Constant/ Solution
I	0	S_1	-1	2	1	0	0	4
	0	S_2	3	2	0	1	0	14
	0	S_3	1	-1	0	0	1	3
		$\bar{c_j}$ row	3	2	0	0	0	Z=0
II	0	S_1	0	1	1	0	1	7
	0	S_2	0	5	0	1	-3	5
	3	x_1	1	-1	0	0	1	3
		$\bar{c_j}$ row	0	5	0	0	-3	Z=9
III	0	S_1	0	0	1	-1/5	8/5	6
	2	x_2	0	1	0	1/5	-3/5	1
	3	x_1	1	0	0	1/5	2/5	4
		$\bar{c_j}$ row	0	0	0	-1	0	Z=14
IV	0	S_1	0	0	5/8	-1/8	1	15/4
	2	x_2	0	1	3/8	1/8	0	13/4
	3	x_1	1	0	-1/4	1/4	0	5/2
		$\bar{c_j}$ row	0	0	0	-1	0	Z=14

So, $(x_1, x_2) = (4, 1)$, $\quad Z_{max} = 14$
$(x_1, x_2) = (5/2, 13/4)$, $\quad Z_{max} = 14$
Solution:

$$\{(x_1, x_2) = \lambda(4, 1) + (1 - \lambda)(5/2, 13/2), 0 \leq \lambda \leq 1\}$$

Problem 1.3.2.

$$\begin{aligned} \text{maximize} \quad & Z = 5x_1 + 4x_2 \\ \text{subject to} \quad & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution. First we need to transform the problem into standard form.
Standard form:

$$\begin{aligned} \text{maximize} \quad & Z = 5x_1 + 4x_2 \\ \text{subject to} \quad & 6x_1 + 4x_2 + S_1 = 24 \\ & x_1 + 2x_2 + S_2 = 6 \\ & -x_1 + x_2 + S_3 = 1 \\ & x_2 + S_4 = 2 \\ & x_1, x_2, S_1, S_2, S_3, S_4 \geq 0 \end{aligned}$$

Taking $x_1 = x_2 = 0$ we get, $S_1 = 24$, $S_2 = 6$, $S_3 = 1$, $S_4 = 2$.
So S_1, S_2, S_3, S_4 are initial basic feasible variables.

Tab	c_B	$c_j \rightarrow$ / basis	5	4	0	0	0	0	Constant/
			x_1	x_2	S_1	S_2	S_3	S_4	Solution
I	0	S_1	6	4	1	0	0	0	24
	0	S_2	1	2	0	1	0	0	6
	0	S_3	-1	1	0	0	1	0	1
	0	S_4	0	1	0	0	0	1	2
		$\bar{c}_j$ row	5	4	0	0	0	0	Z=0
II	5	x_1	1	2/3	1/6	0	0	0	4
	0	S_2	0	4/3	-1/6	1	0	0	2
	0	S_3	0	5/3	1/6	0	1	0	5
	0	S_4	0	1	0	0	0	1	2
		$\bar{c}_j$ row	0	2/3	-5/6	0	0	0	Z=20
III	5	x_1	1	0	1/4	1/2	0	0	3
	4	x_2	0	1	-1/8	3/4	0	0	3/2
	0	S_3	0	0	3/8	-5/4	1	0	5/2
	0	S_4	0	0	1/8	-3/4	0	1	1/2
		$\bar{c}_j$ row	0	0	-3/4	-11/2	0	0	Z=21

So, $(x_1, \ x_2) \ = \ (5, 4), \quad Z_{\max} = 21$

1.3.1 Artificial Variables Technique for Finding the first basic feasible solution

There are two technique to find first basic feasible solutions by using artificial variables. They are

(i) The big M method/M-technique/Method of penalty

(ii) Two phase method

The big M method

When the constraints are of ($\geq$ or $=$) types then we may not get basic feasible solution easily. For this type of problem we use big M method to ensure that we get basic initial feasible solution.

Problem 1.3.3. Solve the following LPP:

$$\begin{aligned} \text{minimize} \quad & Z = -3x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 \le 11 \\ & -4x_1 + x_2 + 2x_3 \ge 3 \\ & 2x_1 - x_3 = -1 \\ & -x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution. Standard form:

$$\begin{aligned} \text{minimize} \quad & Z = -3x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 + x_4 11 \\ & -4x_1 + x_2 + 2x_3 - x_5 = 3 \\ & 2x_1 - x_3 = -1 \\ & -x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Taking $x_1 = x_2 = 0$ we get, $x_3 = 1,\ x_4 = 10,\ x_5 = -1$. But this is not basic feasible solution as $x_5 \not\ge 0$.
So we need to use big M method.
Standard form:

$$\begin{aligned} \text{minimize} \quad & Z = -3x_1 + x_2 + x_3 + M(x_6 + x_7) \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 + x_4 11 \\ & -4x_1 + x_2 + 2x_3 - x_5 + x_6 = 3 \\ & 2x_1 - x_3 + x_7 = -1 \\ & -x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0 \end{aligned}$$

M is a large positive number.
Taking $x_1 = x_2 = x_3 = x_5 = 0$ we get, $x_4 = 11,\ x_6 = 3,\ x_7 = 1$. This is a basic feasible solution.

Tab	c_B	$c_j \rightarrow$ / basis	-3	1	1	0	0	M	M	Constant/
			x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution
I	0	x_4	1	-2	1	1	0	0	0	11
	M	x_6	-4	1	2	0	-1	1	0	3
	M	x_7	2	0	1	0	0	0	1	1
		$\bar{c_j}$ row	$-3+6M$	$1-M$	1-3M	0	M	0	0	$Z = 4M$
II	0	x_4	3	-2	0	1	0	0	1	10
	M	x_6	0	1	0	0	-1	1	-2	1
	1	x_3	-2	0	1	0	0	0	1	1
		$\bar{c_j}$ row	-1	$1-M$	0	0	M	0	$3M-1$	$Z = 1+M$
III	0	x_4	3	0	0	1	-2	2	-5	12
	1	x_2	0	1	0	0	-1	1	-2	1
	1	x_3	-2	0	1	0	0	0	1	1
		$\bar{c_j}$ row	-1	0	0	0	1	$M-1$	$M+1$	$Z = 2$
IV	-3	x_1	1	0	0	1/3	-2/3	2/3	-5/3	4
	1	x_2	0	1	0	0	-1	1	-2	1
	1	x_3	0	0	1	2/3	-4/3	4/3	-7/3	9
		$\bar{c_j}$ row	0	0	0	1/3	1/3	$M-1/3$	$M-2/3$	$Z = -2$

So, $(x_1,\ x_2,\ x_3) = (4, 1, 9), \quad Z_{\min} = -2$

- In minimization: The sign of M in objective function is positive $(+)$
- In maximization: The sign of M in objective function is negative $(-)$

- Add artificial variables in only ($\geq$ or $=$) type of constraints
- There can be three cases in the c_B column
  - No M is present in c_B column − then the solution is optimal solution.
  - M is present in c_B column, but the coefficient is 0 in 'solution' column − then the solution is optimal solution.
  - M is present in c_B column, and the coefficient is non-zero in 'solution' column − then the solution is non-optimal solution even though the solution maintains the optimal test for stopping the simplex method.