Chapter 1

Fuzzy Sets

Definition 1 (Characteristic function)**.** Let X be a universal set and $A \subseteq X$. Then the function[1]

$$\chi_A(x) = \begin{cases} 1; & x \in A \\ 0; & x \notin A \end{cases}$$

is characteristic function of A in X.

Definition 2 (Fuzzy Set)**.** A fuzzy set[2] $A \subseteq X$ is a mapping $A : X \to [0,1]$, where, $A(x) = y \in [0,1]$ is called the membership function or, grade of membership of x in A. The collection of all fuzzy sets of X is denoted by $\mathcal{F}(X)$.

Definition 3 (Fuzzy subset)**.** A fuzzy set A is called a fuzzy subset of another fuzzy set B if $A(x) \leq B(x)$ $\forall x \in X$. We denote it by $A \leq B$.

Definition 4 (Empty fuzzy set)**.** A fuzzy set A is called empty fuzzy set if $\forall x \in X$ $A(x) = 0$. The empty fuzzy set is denoted by $\underline{0}$. Thus, $\underline{0}(x) = 0$ $\forall x \in X$.

Definition 5 (Total fuzzy set)**.** The total fuzzy set $\underline{1}$ is defined by $\underline{1}(x) = 1$ $\forall x \in X$.

Definition 6 (Equality of two fuzzy sets)**.** Two fuzzy sets A and B of X is said to be equal iff $A \leq B$ and $B \leq A$.

Example (Empty and Total fuzzy set)**.** Suppose, $A : X \to [0,1]$ where $X = [20, 90]$. Then,

$$\underline{0}(x) = \begin{cases} 0 & \text{if } 15 < x < 90 \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \underline{1}(x) = \begin{cases} 1 & \text{if } 20 \leq x < 90 \\ 0 & \text{otherwise} \end{cases}$$

Example (Fuzzy subset)**.** Suppose, $A : X \to [0,1]$ where, $X = [0, 100]$ defined by

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \frac{x}{75}; & \text{if } 40 \leq x < 75 \\ 1; & \text{if } 75 \leq x \leq 100 \end{cases}$$

and $B : X = [0, 100] \to [0,1]$ defined by

$$B(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \frac{x}{95}; & \text{if } 40 \leq x < 95 \\ 1; & \text{if } 95 \leq x \leq 100 \end{cases}$$

Then, $B(x)$ is a subset of $A(x)$. Since, $B(x) \leq A(x)$ $\forall x \in X$.

[1] Some authors use μ as characteristic function.

[2] Sometimes fuzzy set is denoted by $\underset{\sim}{A}$.

1.1 Fuzzy Set Operations

Definition 7 (Union of Fuzzy Sets)**.** Let $A,\ B \in \mathcal{F}(X)$. Then the union of A and B is denoted and defined by, $(A \vee B)(x) = \max\{A(x),\ B(x)\},\ \forall x \in X$.

Definition 8 (Intersection of Fuzzy Sets)**.** Let $A,\ B \in \mathcal{F}(X)$. Then the intersection of A and B is denoted and defined by, $(A \wedge B)(x) = \min\{A(x),\ B(x)\},\ \forall x \in X$.

Definition 9 (Complement of Fuzzy Set)**.** Let A be a fuzzy set of X. Then, the complement of A is denoted by A^c and defined by $A^c(x) = 1 - A(x),\ \forall x \in X$.

Example. Given,

$$A_1 = \begin{cases} 1; & \text{if } 40 \leq x < 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases} \qquad \text{and} \qquad A_2 = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 0; & \text{if } 70 \leq x \leq 100 \end{cases}$$

1. Find the complement of A_1 and A_2.
2. Find $(A_1 \wedge A_2)(x)$ and $(A_1 \vee A_2)(x)$

<u>Solution:</u>

1. Complement of A_1,

$$A_1{}^c = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x < 60 \\ 1; & \text{if } 60 \leq x \leq 100 \end{cases}$$

Complement of A_2,

$$A_2{}^c = \begin{cases} 1; & \text{if } 40 \leq x < 50 \\ \dfrac{60-x}{10}; & \text{if } 50 \leq x < 60 \\ \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 1; & \text{if } 70 \leq x \leq 100 \end{cases}$$

2.

$$(A_1 \wedge A_2)(x) = \begin{cases} 0; & \text{if } 40 \leq x < 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ 1 - \dfrac{x-50}{10}; & \text{if } 55 \leq x \leq 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases}$$

$$(A_1 \vee A_2)(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ \dfrac{x-50}{10}; & \text{if } 55 \leq x < 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x < 70 \\ 0; & \text{if } 70 \leq x < 100 \end{cases}$$

Definition 10 (Level Set)**.** Let $A : X \to [0, 1]$ be a fuzzy set. The α level set of A is denoted and defined by, A_α or $\alpha_A = \{x \in X | A(x) \geq \alpha\}$ where, $0 < \alpha \leq 1$.

Remark. A_α is a classical set not a fuzzy set.

Definition 11 (Core level of a fuzzy set)**.** When $\alpha = 1$, then $A_1 = \{x \in X | A(x) = 1\}$ is called the core level of A.

Definition 12 (Support of a fuzzy set)**.** Support of a fuzzy set A is denoted and defined by,$S_A = \{x \in X | A(x) > 0\}$.

Example. Given,

$$A = \begin{cases} 0; & \text{if } x \leq 20 \text{ or, } x \geq 60 \\ \dfrac{x-20}{15}; & \text{if } 20 < x < 35 \\ \dfrac{60-x}{15}; & \text{if } 45 < x < 60 \\ 1; & \text{if } 35 \leq x \leq 45 \end{cases} \quad \text{and} \quad B = \begin{cases} 0; & \text{if } x \leq 45 \\ \dfrac{x-45}{15}; & \text{if } 45 < x < 60 \\ 1; & \text{if } x \geq 60 \end{cases}$$

1. (a) Core level of A?
   (b) Support of A?
   (c) Half level of A?
   (d) $\frac{3}{4}$ level of A?
2. (a) Core level of B?
   (b) Support of B?
   (c) Half level of B?

Solution. 1. (a) Core level of A is $A_1 = \{x \in X | 35 \leq x \leq 45\}$.
   (b) Support level of A is $S_A = \{x \in X | 20 < x < 60\}$.
   (c) Half level of A is $A_{\frac{1}{2}} = \{x \in X | 27.5 \leq x \leq 52.5\}$.
   (d) $\frac{3}{4}$ level of A is $A_{\frac{3}{4}} = \{x \in X | 31.25 \leq x \leq 48.75\}$.
2. (a) Core level of B is $B_1 = \{x \in X | x \geq 60\}$.
   (b) Support level of B is $S_B = \{x \in X | x > 45\}$.
   (c) Half level of B is $B_{\frac{1}{2}} = \{x \in X | x \geq 52.5\}$.

Example. $A : X \to [0, 1]$ defined by

$$A(x) = \begin{cases} 1; & \text{if } x \leq 20 \\ \dfrac{35-x}{20}; & \text{if } 20 \leq x < 35 \\ 0; & \text{if } x \geq 35 \end{cases}$$

Then find $\frac{1}{2}$ level of A.

Solution.

$$A_{\frac{1}{2}} = \{x \in X | x \leq 25\}$$

Problem 1.1. Consider, the two fuzzy sets $A, B : X = [0, 100] \to [0, 1]$ defined by

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \dfrac{x}{75}; & \text{if } 40 \leq x < 75 \\ 1; & \text{if } 75 \leq x < 100 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \dfrac{x}{95}; & \text{if } 40 \leq x < 95 \\ 1; & \text{if } 95 \leq x \leq 100 \end{cases}$$

Then find $(A \wedge B)(x)$ and $(A \vee B)(x)$.

Solution.

$$(A \wedge B)(x) = \begin{cases} 0; & \text{if } 0 \leq x < 40 \\ \dfrac{x}{95}; & \text{if } 40 \leq x < 95 \\ 1; & \text{if } 95 \leq x \leq 100 \end{cases} \quad \text{and} \quad (A \vee B)(x) = \begin{cases} 0; & \text{if } 0 \leq x \leq 40 \\ \dfrac{x}{75}; & \text{if } 40 \leq x < 75 \\ 1; & \text{if } 75 \leq x \leq 100 \end{cases}$$

Suppose, $X = \mathbb{R}$ and the fuzzy set of real numbers much greater than 5 in X, that could be defined by,

$$A(x) = \begin{cases} 0; & \text{if } x \leq 5 \\ \dfrac{x-5}{50}; & \text{if } 5 < x \leq 55 \\ 1; & \text{if } x \geq 55 \end{cases}$$

Example. Consider, the two fuzzy sets A and B of $\mathcal{F}(X)$, where $X = [0, 100]$

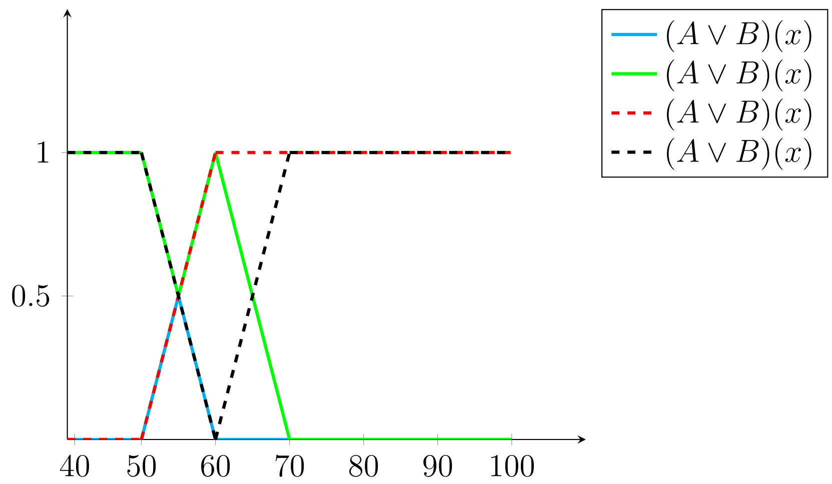
$$A(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0; & \text{if } 40 \leq x \leq 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x \leq 70 \\ 0; & \text{if } 70 \leq x \leq 100 \end{cases}$$

Draw $(A \vee B)(x)$, $(A \wedge B)(x)$, A', B'.

Solution. Here,

$$(A \vee B)(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ \dfrac{x-50}{10}; & \text{if } 55 \leq x \leq 60 \\ 1 - \dfrac{x-60}{10}; & \text{if } 60 \leq x \leq 70 \\ 0; & \text{if } 70 \leq x \leq 100 \end{cases} \quad \text{and} \quad (A \wedge B)(x) = \begin{cases} 0; & \text{if } 40 \leq x \leq 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 55 \\ 1 - \dfrac{x-50}{10}; & \text{if } 55 \leq x \leq 60 \\ 0; & \text{if } 60 \leq x \leq 100 \end{cases}$$

$$A^c(x) = \begin{cases} 0; & \text{if } 40 \leq x \leq 50 \\ \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ 1; & \text{if } 60 \leq x \leq 100 \end{cases} \quad \text{and} \quad B^c(x) = \begin{cases} 1; & \text{if } 40 \leq x \leq 50 \\ 1 - \dfrac{x-50}{10}; & \text{if } 50 \leq x \leq 60 \\ \dfrac{x-60}{10}; & \text{if } 60 \leq x \leq 70 \\ 1; & \text{if } 70 \leq x \leq 100 \end{cases}$$



1.2 Fuzzy Relation

Definition 13 (Fuzzy Relation)**.** Let X and Y be two non-empty classical(Fuzzy) sets. Then a fuzzy relation R on $X \times Y$ is a mapping, $R : X \times Y \to [0, 1]$ where, the number $R(x, y) \in [0, 1]$ is called the degree of relationship between x and y.

Example. Let $X = \{a, b, c\}$, $Y = \{c, d\}$. Then $X \times Y = \{(a, c), (a, d), (b, c), (b, d), (c, c), (c, d)\}$ where $R(a, c) = R(a, d) = 0$, $R(b, c) = R(b, d) = R(c, c) = 1$ and $R(c, d) = 0.8$. For the fuzzy relation:

1. Core of R?
2. Support of R?
3. 0.7 of R?

Solution.

1. Core of $R = \{(b, c), (b, d), (c, c)\}$ Since, $R(x, y) = 1$ for $x \in X$ and $y \in Y$.
2. Support of $R = \{(b, c), (b, d), (c, c), (c, d)\}$ Since, $R(x, y) > 0$ for $x \in X$ and $y \in Y$.
3. 0.7 of $R = \{(b, c), (b, d), (c, c), (c, d)\}$ Since, $R(x, y) > 0.7$ for $x \in X$ and $y \in Y$.

Definition 14 (Domain)**.** If $R(x, y)$ is a fuzzy relation, its domain is the fuzzy set *dom* $R(x, y)$ whose membership function is

$$\chi_{dom}R(x) = \max \chi_R(x, y) \forall x \in X$$

Definition 15 (Range)**.** If $R(x, y)$ is a fuzzy relation, its range is the fuzzy set *ran* $R(x, y)$ whose membership function is

$$\chi_{ran}R(y) = \max \chi_R(x, y) \forall y \in y$$

Example. Consider $X = \{x_1, x_2, x_3, x_4\}$ and

$$R(x, x) = \begin{pmatrix} 0.2 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0.0 & 0.4 & 0.0 \\ 0.0 & 0.6 & 0.0 & 0.1 \end{pmatrix}$$

Then the domain is *dom* $R = \{0.5, 0.8, 0.4, 0.6\}$ and the range is *ran* $R = \{0.2, 0.6, 0.7, 0.8\}$.

Definition 16 (Max-min and Min-max Composition)**.** Let R be a fuzzy relation on $X \times Y$ i.e., $R \in \mathcal{F}(X \times Y)$ and S be a fuzzy relation on $Y \times Z$ i.e., $S \in \mathcal{F}(Y \times Z)$. Then $R \circ S \in \mathcal{F}(X \times Z)$ defined by

$$(R \circ S)(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z)$$

is called the Max-Min composition of R and S on $X \times Z$. And

$$(R \circ S)(x, z) = \bigwedge_{y \in Y} R(x, y) \vee S(y, z)$$

is called the Min-Max composition of R and S on $X \times Z$

Problem 1.2. Consider, $X = \{a, b\}$, $Y = \{c, d, e\}$ and $Z = \{u, v\}$ where,

$$R(x, y) = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1.0 & 0.0 & 0.9 \end{pmatrix} \qquad \text{and} \qquad S(y, z) = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.0 \\ 0.5 & 0.6 \end{pmatrix}$$

then find the max-min and min-max composition of R and S.

Solution. Max-min composition of R and S

$$(R \circ S)(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z) = \begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}$$

Min-max composition of R and S

$$(R \circ S)(x, z) = \bigwedge_{y \in Y} R(x, y) \vee S(y, z) = \begin{pmatrix} 0.5 & 0.3 \\ 0.1 & 0.0 \end{pmatrix}$$

Definition 17 (Reflexive)**.** Let R be a fuzzy relation in $X \times X$. R is called reflexive if

$$\chi_R(x, x) = 1 \quad \forall x \in X$$

Definition 18 (Symmetic)**.** Let R be a fuzzy relation in $X \times X$. R is called symmetric if

$$R(x, y) = R(y, x) \quad \forall x, y \in X$$

Definition 19 (Antisymmetic)**.** Let R be a fuzzy relation in $X \times X$. R is called antisymmetric if for

$$x \neq y \begin{array}{r} \text{either } \chi_R(x, y) \neq \chi_R(y, x) \\ \text{or } \chi_R(x, y) = \chi_R(y, x) = 0 \end{array} \Bigg\} \forall x, y, \in X$$