Chapter 1

Parabolic Partial Differential Equation

We consider the numerical solution of the wave Equation which is an example of hyperbolic partial differential equation. The wave equation is given by the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0; \qquad 0 < x < l, \quad t > 0 \tag{1.1}$$

subject to the conditions

$$\begin{aligned} &u(0,t) = u(l,t) = 0, \quad t > 0 \\ &u(x,0) = f(x), \quad 0 \le x \le l \end{aligned}$$

and

$$\frac{\partial u(x,0)}{\partial t} = u_t(x,0) = g(x); \quad 0 \le x \le l$$

where α is a constant. For the finite difference method, let an integer $m > 0$ and time-step size $k > 0$, with $h = l/m$. The mesh points (x_i, t_j) are

$$x_i = ih \qquad \text{for each} \quad i = 0, 1, 2, \dots, m$$

and

$$t_j = jk \qquad \text{for each} \quad j = 0, 1, 2, \dots$$

At any interior mesh point (x_i, t_j), the wave equation becomes

$$\frac{\partial^2 u(x_i,t_j)}{\partial t^2} - \alpha^2 \frac{\partial^2 u(x_i,t_j)}{\partial x^2} = 0 \tag{1.2}$$

The difference method is obtained using the center-difference quotient for the second partial derivatives given by,

$$\frac{\partial^2 u(x_i,t_j)}{\partial t^2} = \frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2} - \frac{k^2}{12}\frac{\partial^4 u(x_i,\mu_j)}{\partial t^4}$$

where $\mu_j \in (t_{j-1}, t_{j+1})$ and

$$\frac{\partial^2 u(x_i,t_j)}{\partial x^2} = \frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u(\xi_i,t_j)}{\partial x^4}$$

where $\xi_i \in (x_{i-1}, x_{i+1})$. Substituting these into equation (1.2) gives,

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2} - \alpha^2 \frac{u(x_{i+1,t_j}) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2} = \frac{1}{12}\left[k^2 \frac{\partial^4 u(x_i,\mu_j)}{\partial t^4} - \alpha^2 h^2 \frac{\partial^4 u(\xi_i,t_j)}{\partial x^4}\right]$$

Neglecting the error term

$$T_{i,j} = \frac{1}{12}\left[k^2 \frac{\partial^4 u(x_i,\mu_j)}{\partial t^4} - \alpha^2 h^2 \frac{\partial^4 u(\xi_i,t_j)}{\partial x^4}\right]$$

leads to the difference equation

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2} - \alpha^2 \frac{u(x_{i+1,t_j}) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2} = 0$$

If $\lambda = \alpha k/h^2$, we can write the difference equation as

$$\begin{aligned} & u_{i,j+1} - 2u_{i,j} + u_{i,j-1} - \lambda^2 u_{i+1,j} + 2\lambda u_{i,j} - \lambda^2 u_{i-1,j} = 0 \\ \Rightarrow\ & u_{i,j+1} = 2(1-\lambda^2)u_{i,j} + \lambda^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} = 0 \end{aligned} \tag{1.3}$$

This equation holds for each $i = 1, 2, 3, \ldots, m-1$ and $j = 1, 2, 3, \ldots$. The boundary conditions give

$$u_{0,j} = u_{m,j} = 0 \quad \text{for each } j = 1, 2, 3, \ldots \tag{1.4}$$

and the initial condition implies that

$$u_{i,0} = f(x_i) \quad \text{for each } i = 1, 2, 3, \ldots, m-1 \tag{1.5}$$

Writing this set of equations in matrix form gives

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \ldots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{m-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ u_{m,j-1} \end{bmatrix} \tag{1.6}$$

Equations (1.3) and (1.4) imply that the $(j+1)$st time step requires values from the jth and $(j-1)$th time steps. This produces a minor starting problem since values for $j = 0$ are given by (1.5), but values for $j = 1$ which are needed in equation (1.3) to compute $u_{i,2}$ must be obtained from the initial velocity condition

$$\frac{\partial u(x,0)}{\partial t} = g(x); \qquad 0 \le x \le l$$

Forward-difference approximation is used for $\frac{\partial u}{\partial t}$,

$$\frac{\partial u(x,0)}{\partial t} = \frac{u(x_i,t_1) - u(x_i,0)}{k} - \frac{k}{2}\frac{\partial^2 u(x_i,\bar{\mu_j})}{\partial t^2}; \quad 0 < \bar{\mu_j} < t_1 \tag{1.7}$$

$$\Rightarrow\ u(x_i,t_1) = u(x_i,0) + k\frac{\partial u(x_i,0)}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u(x_i,\bar{\mu_j})}{\partial t^2}$$

$$\Rightarrow\ u(x_i,t_1) = u(x_i,0) + kg(x_i) + \frac{k^2}{2}\frac{\partial^2 u(x_i,\bar{\mu_j})}{\partial t^2} \tag{1.8}$$

$$\tag{1.9}$$

As a consequence

$$u_{i,1} = u_{i,0} + kg(x_i) \quad \text{for each } i = 1, 2, 3, \ldots, m-1 \tag{1.10}$$