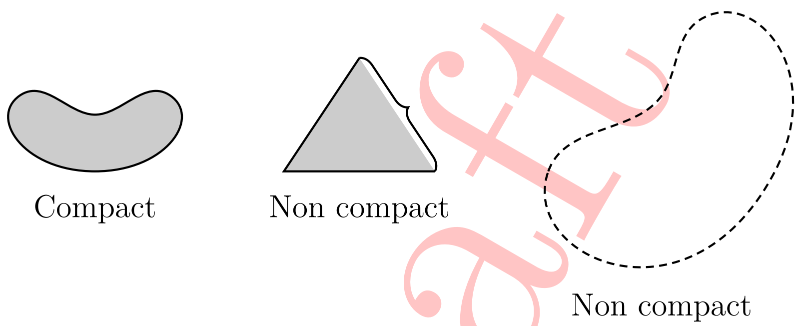
Chapter 1

Compact and Connected Sets

Definition 1. Let M be a metric space. A subset $A \subset M$ is called *sequentially compact* if every sequence in A has a subsequence that converges to a point in A.



Definition 2 (Some useful definitions)**.** Let M be a metric space and $A \subset M$ a subset. A *cover of* A is collection $\{U_i\}$ of sets whose union contains A; it is an *open cover* if each U_i is open. A sub-cover of a given cover is a sub-collection of $\{U_i\}$ whose union also contains A; it is a *finite sub-cover* if the sub-collection contains only a finite number of sets.

Open covers are not necessarily countable collections of open sets. For example, the uncountable set of disks $\{D_\varepsilon((x,0))\} = \{D_1((x,0)) \mid x \in \mathbb{R}^1\}$ in $\mathbb{R}^2$ covers the real axis, and the sub-collection of all disks $D_1((n,0))$ centered at integer points on the real line forms a countable sub-cover. Note that the set of disks $D_1((2n,0))$ centered at even integer points on the real line does not form a sub-covering (why?).

Definition 3 (Compact set)**.** A subset A of a metric space M is called *compact* if every open cover of A has a finite sub-cover.

1.1 Bolzano-Weierstrass Theorem

Theorem 1.1.1 (Bolzano-Weierstrass Theorem)**.** *A subset of a metric space is compact if and only if it is sequentially compact.*

We will provide the proof of the theorem later on. Some simple observations will help give a feel for compactness and for the theorem.

First, a sequentially compact set must be closed:
Indeed, if $x_n \in A$ converges to $x \in M$, then by assumption there is a subsequence converging to a point $x_0 \in A$; by uniqueness of limits $x = x_0$, and so A is closed.

Secondly, a sequentially compact set A must be bounded:
For if not, there is a point $x_0 \in A$ and a sequence $x_n \in A$ with $d(x_n, x_0) \geq n$. Then x_n cannot have any convergent subsequence. To show directly that a compact set is bounded, use the fact that for any $x_0 \in A$, the open balls $D_n(x_0), n = 1, 2, \ldots$ cover A, so there is a finite sub-cover.

Definition 4 (Totally Bounded)**.** A set A in a metric space M is called totally bounded if for each $\varepsilon > 0$ there is a finite set $\{x_1, x_2, \ldots, x_N\}$ in M such that $A \subset \bigcup_{i=1}^{N} D(x_i, \varepsilon)$.

Note that, a totally bounded set is bounded:
If A is totally bounded, then for each $\varepsilon > 0$, there is a finite set $\{x_1, x_2, \ldots, x_N\}$ in a metric space M such that $A \subset \bigcup_{i=1}^{N} D(x_i, \varepsilon)$. Observe that $D(x_i, \varepsilon) \subset D(x_1, \varepsilon + d(x_i, x_1))$, so that if $R = \varepsilon + \max\{d(x_2, x_1), \ldots, d(x_N, x_1)\}$, then $A \subset D(x_1, R)$ and so a totally bounded set is bounded.

Example. The entire real line is *not* compact, for it is unbounded. Another reason is that $\{D(n, 1) = (n-1, n+1) \mid n = 0, \pm 1, \pm 2, \pm 3, \ldots\}$ is on open cover of $\mathbb{R}$ but does not have a finite sub-cover (why?).

Problem 1.1.1. Let $A = (0, 1]$. Find an open cover with no finite sub-cover.

Solution. Consider the open cover $\left\{(\frac{1}{n}, 2) \mid n = 1, 2, 3, \ldots\right\}$. Then we have $A = (0, 1] \subset (1, 2) \cup (\frac{1}{2}, 2) \cup (\frac{1}{3}, 2) \cup \cdots = (0, 2)$. Clearly, this open cover cannot have a finite sub-cover. This time compactness fails because A is not closed; the point 0 is "missing" from A.

<u>Proof of Bolzano-Weierstrass Theorem</u>
We begin with two lemmas:

Lemma 1. *A compact set $A \subset M$ is closed.*

Proof. We will show that $M \setminus A$ is open. Let $x \in M \setminus A$ and consider the following collection of open sets: $U_n = \{y \mid d(y, x) > {}^1/_n\}$. Since every $y \in M$ with $y \neq x$ has $d(y, x) > 0$, y lies in some U_n. Thus, the U_n cover A, and since A is compact, so there must be a finite sub-cover. One of these has the largest index, say, U_N. If $\varepsilon = \frac{1}{N}$, then, by conclusion(?/ contradiction), $D(x, \frac{1}{N}) \subset M \setminus A$, and so $M \setminus A$ is open. $\square$

Lemma 2. *If M is a compact metric space and $B \subset M$ is closed, then B is compact.*

Proof. Let $\{U_i\}$ be an open covering of B and let $V = M \setminus B$, so that V is open. Thus, $\{U_i, V\}$ is an open cover of M. Therefore, M has a finite cover, say $\{U_1, U_2, \ldots, U_N, V\}$. Then $\{U_1, U_2, \ldots, U_N\}$ is a finite open cover of B. Hence, B is compact. $\square$

Bolzano-Weierstrass Theorem Proof. Let A be compact. Assume that there exists a sequence $x_k \in A$ that has no convergent subsequences. In particular, this means that x_k has infinitely many distinct points, say $y_1, y_2, \ldots$. Since there are no convergent subsequences, there is some neighborhood U_k of y_k containing no other y_i. This is because if every neighborhood of y_k contained another y_i we could, by choosing the neighborhoods $D(y_k, {}^1/_m), m = 1, 2, 3, \ldots$, select a subsequence converging to y_k. We claim that the set $\{y_1, y_2, y_3, \ldots\}$ is closed. Indeed, it has no accumulation points, by the assumption that there are no convergent subsequences. Applying lemma (2) to $\{y_1, y_2, y_3, \ldots\}$ as a subset of A, we find that $\{y_1, y_2, y_3, \dots\}$ is compact. But $\{U_k\}$ is an open cover that has no finite sub-cover, a contradiction. Thus, x_k has a convergent subsequence. The limit lies in A, since A is closed, by lemma (1).

Conversely, suppose that A is sequentially compact. To prove that A is compact, let $\{U_i\}$ be an open cover of A. We need to prove that this has a finite sub-cover. To show this we proceed in several steps. $\square$

Lemma 3. *There is an $r > 0$ such that for each $y \in A$, $D(y, r) \subset U_i$ for some U_i.*

Proof. If not, then for every integer n, there is some y_n such that $D(y_n, {}^1/_n)$ is not contained in any U_i. By hypothesis, y_n has a convergent subsequence, say $z_n \to z \in A$. Since the U_i cover A, $z \in Ui_0$. Choosing $\varepsilon > 0$ such that $D(z, \varepsilon) \subset U_{i_0}$[1], which is possible since U_{i_0} is open. Choose N large enough so that [2]$d(z_N, z) < {}^\varepsilon/_2$ and ${}^1/_N < {}^\varepsilon/_2$. Then $D(z_n, {}^1/_N) \subset U_{i_0}$, a contradiction. $\square$

Lemma 4. *A is totally bounded.*

Proof. If A is not totally bounded, then some $\varepsilon > 0$, we cannot cover A with finitely many disks. Choose $y_1 \in A$ and $y_2 \in A \setminus D(y_1, \varepsilon)$. By assumption, we can repeat; choose $y_n \in A \setminus [D(y_1, \varepsilon) \cup \cdots \cup D(y_{n-1}, \varepsilon)]$. This is a sequence with $d(y_n, y_m) \geq \varepsilon$ for all n and m, and so y_n has no convergent subsequence, a contradiction to the assumption that A is sequentially compact. $\square$

Bolzano-Weierstrass Theorem Proof (continued). To complete our proof, let r be as in lemma (3). By lemma (4) we can write $A \subset D(y_1, r) \cup D(y_2, r) \cup \cdots \cup D(y_n, r)$ for finitely many y_i. By lemma (3), $D(y_i, r) \subset U_{i_j}$, $j = 1, 2, \dots, n$ for some index j. Then $U_{i_1}, U_{i_2}, \dots, U_{i_n}$ cover A. $\square$

1.2 Heine-Borel Theorem

Theorem 1.2.1 (Heine-Borel Theorem)**.** *A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. (In fact, a compact set is closed and bounded in any metric space.)*

Proof. We have already proved that compact sets are closed and bounded. We must now show that a set $S \subset \mathbb{R}^n$ is compact if it is closed and bounded. In fact, we shall prove that a closed and bounded set A is sequentially compact.

Let $x_k = (x_k^1, x_k^2, \dots, x_k^n) \in \mathbb{R}^n$ be a sequence. Since A is bounded x_k^1 has a convergent subsequence, say, $x_{f_1(k)}^1$. Then x_k^2 has a convergent subsequence, say, $x_{f_2(k)}^2$. Continuing, we get a further subsequence $x_{f_n(k)} = \left(x_{f_1(k)}^1, \dots, x_{f_n(k)}^n\right)$, all of whose components converge. This, $x_{f_n(k)}$ converges in $\mathbb{R}^n$. The limit lies in A since A is closed. Thus, A is sequentially compact, and so is compact. $\square$

Theorem 1.2.2 (Nested Set Property)**.** *Let F_k be a sequence of compact non-empty sets in a metric space M such that $F_{k+1} \subset F_k$ for all $k = 1, 2, 3, \dots$. Then there is at least one point in $\bigcap_{k=1}^\infty F_k$.*

1.3 Path Connected Sets

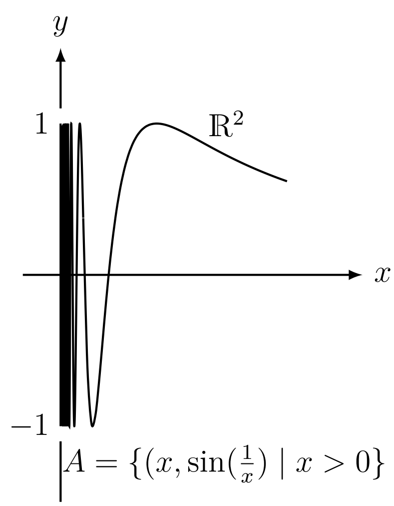
Definition 5. We call a map $\varphi : [a, b] \to M$ of an interval $[a, b]$ into a metric space M *continuous* if $(t_k \to t)$ implies $(\varphi(t_k) \to \varphi(t))$ for every sequence t_k in $[a, b]$ converging to some $t \in [a, b]$. A *continuous path* joining two points x, y in a metric space M is a mapping $\varphi : [a, b] \to M$ such that

[1] $D_\varepsilon(z) = \{z^* \colon d(z, z^*) < \varepsilon\}$

[2] $D_{{}^\varepsilon/_2(z) \subset D_\varepsilon(z)}$ when ${}^1/_N < {}^\varepsilon/_2$
$\Rightarrow D_{{}^1/_N}(z_n) \subset D_{{}^\varepsilon/_2}(z_n) \subset D_\varepsilon(z_n) \subset U$

$\varphi(a) = x$, $\varphi(b) = y$, and φ is continuous: the x may or may not be equal y, and $b \geq a$. A path φ is said to lie in a set A if $\varphi(t) \in A$ for all $t \in [a, b]$.

We say a set is *path-connected* if every two points in the set can be joined by a continuous path lying in the set.



$A = \{(x, \sin(\frac{1}{x}) \mid x > 0\} \cup \{(0, y) \mid y \in [-1, 1]\}$

Figure 1.1: A is not path-connected

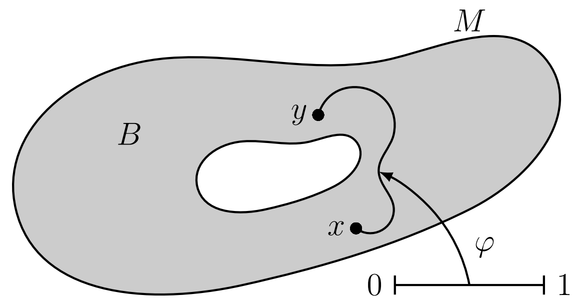


Figure 1.2: B is path-connected

Example. $[0, 1] \subset \mathbb{R}^1$ is *path-connected*: To prove this, let $x, y \in [0, 1]$ and define $\varphi : [0, 1] \to \mathbb{R}$ by $\varphi(t) = (y - x)t + x$. This is a continuous path connecting x and y, and it lies in $[0, 1]$.

Example (H.W.)**.** Which of the sets are path-connected?

(i) $[0, 3]$

(ii) $[1, 2] \cup [3, 4]$

(iii) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$

(iv) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$

Example. Let $\varphi : B = [0, 1] \to \mathbb{R}^2$ be a continuous path, and $C = \varphi([0, 1])$. Show that C is path-connected.

Solution. This is intuitively clear, for we can i=use the path φ itself to join two points in C. Precisely, if $x = \varphi(a)$, $y = \varphi(b)$, where $0 \leq a \leq b \leq 1$, let $c : B \to \mathbb{R}^2$ be defined by $c(t) = \varphi(t)$. Then c is path joining x to y and c lies in C.

1.4 Connected Sets

Definition 6. Let A be a subset of a metric space M. Then A is said to be disconnected if there exists two open sets U and V such that

(i) $U \cap V \cap A = \varnothing$

(ii) $U \cap A \neq \varnothing$

(iii) $V \cap A \neq \varnothing$

(iv) $A \subset U \cup V$

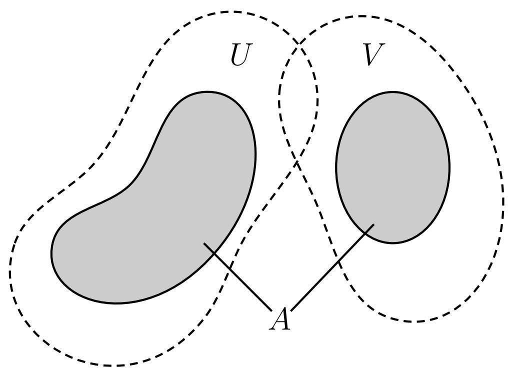


Figure 1.3: A is neither connected nor path-connected

Theorem 1.4.1. *Path-connected sets are connected.*

Problem 1.4.1. Is $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\} \subset \mathbb{R}$ connected?

Solution. No, for if $U = (\frac{1}{2}, \infty)$ and $V = (-\infty, \frac{1}{4})$, then $\mathbb{Z} \subset U \cup V$, $\mathbb{Z} \cap U = \{1, 2, 3, \dots\} \neq \varnothing$, $\mathbb{Z} \cap V = \{\dots, -2, -1, 0\} \neq \varnothing$, and $\mathbb{Z} \cap U \cap V = \varnothing$. Hence, $\mathbb{Z}$ is disconnected(i.e., not connected).

Problem 1.4.2. Is $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is connected?

Solution. Yes, because $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is path-connected and hence is connected by theorem 1.4.1.

Example (H.W.)**.** Are $[0, 1] \cup (2, 3]$ and $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\} \cup \{(x, 0) \mid 1 < x < 2\}$ connected? Prove or disprove.

Problem 1.4.3. Determine the compactness of

(i) finite set $A = \{x_1, x_2, \dots, x_n\}$

(ii) $\mathbb{R}$

(iii) $B = [0, \infty) \to G_n = (-1, n) \Rightarrow B \subset \bigcup_1^\infty G_n \Rightarrow B \not\subset \bigcup_{i=1}^k G_{n_i}$

(iv) $C = (0, 1)$

Solution.

(i) $A = \{x_1, x_2, \dots, x_n\} -$ a finite subset of $\mathbb{R}$,

Let $\mathscr{G} = \{G_\alpha\}$ be any open cover of A, then each x_i is contained in some ..... Then $A \subset \bigcup_{i=1}^n G_{\alpha_i} \Rightarrow \{G_{\alpha_i} : i = 1, 2, \dots, n\}$ is a finite sub-cover of $\mathscr{G}$. Since $\mathscr{G}$ is arbitrary, so A is compact.

Problem 1.4.4. Show that $A = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is compact and connected.

Solution. To show that A is compact, we show it is closed and bounded. To show that it is closed, consider $A^{\complement} = \mathbb{R}^n \setminus A = \{x \in \mathbb{R}^n \mid ||x|| > 1\} = B$. For $x \in B$, $||x|| = 1$, $N_\delta(x) \subset B$, with $\delta = ||x|| - 1$, so that B is open and hence A is closed. It is clear that A is bounded, since $A \subset N_2(0)$ and therefore A is compact.

To show that A is connected, we show that A is path-connected. Let $x, y \in A$. Then the straight line joining x, y is the required path. Explicitly, we use $\varphi : [0,1] \to \mathbb{R}^n, \varphi(t) = (1-t)x + ty$. One sees that $\varphi(t) \in A$, since $||\varphi(t)|| \leq (1-t)\,||x|| + t\,||y|| \leq (1-t) + t = 1$, by triangle inequality.