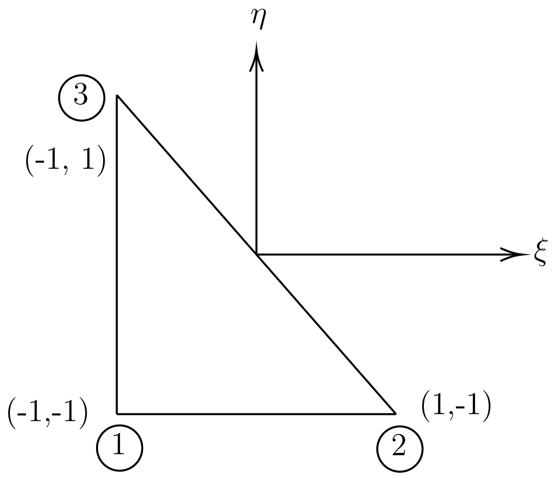
Problem 0.1. Evaluate

$$\iint_{ST} \xi^m \eta^n \,\mathrm{d}\,\xi \,\mathrm{d}\,\eta;$$

where $m, n \geq 0$ and ST= standrad triangle.

Solution. From the figure we see that,



straight line passing through 1, 2 is $\eta = -1$,
straight line passing through 1, 3 is $\xi = -1$,
straight line passing through 2, 3 is $\xi + \eta = 0$.
So the region of the triangle is covered by $\eta = -1$ to $\eta = -\xi$ and $\xi = -1$ to $\xi = 1$. By using this the integral becomes,

$$\iint_{ST} \xi^m \eta^n \,\mathrm{d}\,\xi \,\mathrm{d}\,\eta$$

$$= \int_{\xi=-1}^{1} \int_{\eta=-1}^{\eta=-\xi} \xi^m \eta^n \,\mathrm{d}\,\eta \,\mathrm{d}\,\xi$$

$$= \int_{-1}^{1} \xi^m \left[\frac{\eta^{n+1}}{n+1}\right]_{-1}^{-\xi} \mathrm{d}\,\xi$$

$$= \int_{-1}^{1} \xi^m \left[\frac{(-\xi)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1}\right] \mathrm{d}\,\xi$$

$$= \int_{-1}^{1} \frac{(-1)^{n+1}}{n+1} \left[\xi^{m+n+1} - \xi^m\right] \mathrm{d}\,\xi$$

$$= \frac{(-1)^{n+1}}{n+1} \left[\frac{\xi^{m+n+2}}{m+n+2} - \frac{\xi^{m+1}}{m+1}\right]_{-1}^{1}$$

$$= \frac{(-1)^{n+1}}{n+1} \left[\frac{1^{m+n+2}}{m+n+2} - \frac{1^{m+1}}{m+1} - \frac{(-1)^{m+n+2}}{m+n+2} + \frac{(-1)^{m+1}}{m+1}\right]$$

$$= \frac{(-1)^{n+1}}{n+1} \left[\frac{1}{m+n+2} - \frac{1}{m+1} - \frac{(-1)^{m+n}(-1)^2}{m+n+2} + \frac{(-1)^{m+1}}{m+1}\right]$$

$$= \frac{(-1)^{n+1}}{n+1} \left[\frac{1}{m+n+2} - \frac{(-1)^{m+n}}{m+n+2} + \frac{(-1)^{m+1}}{m+1} - \frac{1}{m+1}\right]$$

$$= \frac{(-1)^{n+1}}{n+1} \left[\frac{1-(-1)^{m+n}}{m+n+2} + \frac{(-1)^{m+1}-1}{m+1}\right]$$

$$\therefore \quad \iint_{ST} \xi^m \eta^n \,\mathrm{d}\,\xi \,\mathrm{d}\,\eta = \frac{(-1)^{n+1}}{n+1} \left[\frac{1-(-1)^{m+n}}{m+n+2} + \frac{(-1)^{m+1}-1}{m+1}\right] \tag{1}$$

Here, four cases arise for the values of $m, n \geq 0$. They are,

(i) $m = 0$ or even, $n = 0$ or even,

(ii) $m = 0$ or even, $n =$odd,

(iii) $m =$odd, $n = 0$ or even,

(iv) $m =$odd, $n =$odd.

Case 1: $(m = 0$ or even, $n = 0$ or even$)$
$(m+n =$ even $, m+1 =$ odd $, n+1 =$ odd .)
So from equation (1),

$$\iint_{ST} \xi^m \eta^n \,\mathrm{d}\,\xi\,\mathrm{d}\,\eta = \frac{-1}{n+1}\left[0 + \frac{-2}{m+1}\right] = \frac{-2}{(m+1)(n+1)}$$

Case 2: $(m = 0$ or even, $n =$ odd$)$
$(m+n =$ odd $, m+1 =$ odd $, n+1 =$ even .)
So from equation (1),

$$\iint_{ST} \xi^m \eta^n \,\mathrm{d}\,\xi\,\mathrm{d}\,\eta = \frac{1}{n+1}\left[\frac{2}{m+n+2} + \frac{-2}{m+1}\right]$$

Case 3: $(m =$ odd, $n = 0$ or even$)$
$(m+n =$ odd $, m+1 =$ even $, n+1 =$ odd .)
So from equation (1),

$$\iint_{ST} \xi^m \eta^n \,\mathrm{d}\,\xi\,\mathrm{d}\,\eta = \frac{-1}{n+1}\left[\frac{2}{m+n+2} + 0\right] = \frac{-2}{(n+1)(m+n+1)}$$

Case 4: $(m =$ odd, $n =$ odd$)$
$(m+n =$ even $, m+1 =$ even $, n+1 =$ even .)
So from equation (1),

$$\iint_{ST} \xi^m \eta^n \,\mathrm{d}\,\xi\,\mathrm{d}\,\eta = 0$$