Chapter 1

Finite Difference Method For BVP

1.1 Finite Difference Method for Linear BVP

The linear second-order boundary value problem

$$y''(x) + f(x)y'(x) + g(x)y(x) = r(x); \quad a \le x \le b; \quad y(a) = \alpha, \;\; y(b) = \beta \tag{1.1}$$

To obtain the approximate finite difference approximations to the derivatives, we proceed as follows: Expanding $y(x+h)$ in Taylor's series, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) + \dots \tag{1.2}$$

from which we obtain

$$y'(x) = \frac{y(x+h) - y(x)}{h} - \frac{h}{2}y''(x) - \dots$$

Thus we have,

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \tag{1.3}$$

which is the forward difference approximation for $y'(x)$. Similarly, expansion of $y(x-h)$ in Taylor's series gives,

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2}y''(x) - \frac{h^3}{6}y'''(x) + \dots \tag{1.4}$$

from which we obtain

$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h) \tag{1.5}$$

which is the backward difference approximation for $y'(x)$.
Subtracting (1.4) from (1.2) we get

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2) \tag{1.6}$$

which is the central difference approximation for $y'(x)$.

It is clear that (1.6) is a better approximation to $y'(x)$ than either (1.3) or (1.5).
Again, adding (1.4) and (1.2) we have

$$y''(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{h^2} + O(h^2) \tag{1.7}$$

To solve the BVP defined by (1.1), we divide the range $[x_0, x_n]$ i.e., $[a, b]$ (Here $a = x_0$, $b = x_n$) into n equal subintervals of width h so that

$$x_i = x_0 + ih; \qquad i = 0, 1, 2, 3, \dots, n$$

The corresponding values of y at these points are denoted by

$$y(x_i) = y_i = y(x_0 + ih); \qquad i = 0, 1, 2, 3, \dots, n$$

From equations (1.6) and (1.7), values of $y'(x)$ and $y''(x)$ at the point $x = x_i$ can now be written as

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)$$

and

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + O(h^2)$$

Satisfying the differential equation at the point $x = x_i$ we get

$$y_i'' + f_i y_i' + g_i y_i = r_i$$

Substituting the expressions for y_i' and y_i'', this gives

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + f_i \frac{y_{i+1} - y_{i-1}}{2h} + g_i y_i = r_i; \quad i = 1, 2, 3, \ldots, n-1 \quad \text{where } y_i = y(x_i),\ g_i = g(x_i) \text{ etc.}$$

Multiplying through h^2 and simplifying we obtain

$$\left(1 - \frac{h}{2} f_i\right) y_{i-1} + \left(-2 + g_i h^2\right) y_i + \left(1 + \frac{h}{2} f_i\right) y_{i+1} = r_i h^2; \quad i = 1, 2, 3, \ldots, n-1 \tag{1.8}$$

with

$$y_0 = \alpha,\ y_n = \beta \tag{1.9}$$

Equations (1.8) and (1.9) comprise a tridiagonal system. The solution of this tridiagonal system constitutes an approximate solution of the boundary value problem defined by (1.1).

Error: To estimate the error in the numerical solution, we define the local truncation error τ given by

$$\tau = \left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i''\right) + f_i \left(\frac{y_{i+1} - y_{i-1}}{2h} - y_i'\right)$$

Expanding y_{i-1} and y_{i+1} by Taylor's series and simplifying, the above formula gives

$$\tau = \frac{h^2}{12}\left(y_i^{iv} + 2 f_i y_i'''\right) + O(h^4) \tag{1.10}$$

Thus, the finite difference approximation defined by (1.1) has second order accuracy for functions with continuous 4th derivatives on $[a, b]$. Further, it follows that $\tau \to 0$ as $h \to 0$, implying that greater accuracy in the result can be achieved by using a smaller value of h. In such a case, of course more computation effort would be required since the number of equations become larger.

1.2 Calculation

$$\left(1 - \frac{h}{2} f_i\right) y_{i-1} + \left(-2 + g_i h^2\right) y_i + \left(1 + \frac{h}{2} f_i\right) y_{i+1} = r_i h^2; \quad i = 1, 2, 3, \ldots, n-1 \quad \text{with } y_0 = \alpha \text{ and } y_n = \beta$$

Let $i = 1, 2, 3$. So,
For $i = 1$:

$$\left(1 - \frac{h}{2} f_1\right) y_0 + \left(-2 + g_1 h^2\right) y_1 + \left(1 + \frac{h}{2} f_1\right) y_2 = r_1 h^2$$

$$\Rightarrow \left(-2 + g_1 h^2\right) y_1 + \left(1 + \frac{h}{2} f_1\right) y_2 = r_1 h^2 - \left(1 - \frac{h}{2} f_1\right) \alpha \quad [\because y_0 = \alpha]$$

For $i = 2$:

$$\left(1 - \frac{h}{2} f_2\right) y_1 + \left(-2 + g_2 h^2\right) y_2 + \left(1 + \frac{h}{2} f_2\right) y_3 = r_2 h^2$$

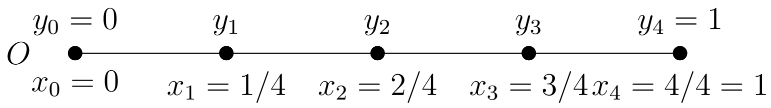
For $i = 3$:

$$\left(1 - \frac{h}{2} f_3\right) y_2 + \left(-2 + g_3 h^2\right) y_3 + \left(1 + \frac{h}{2} f_3\right) y_4 = r_3 h^2$$

$$\Rightarrow \left(1 - \frac{h}{2} f_3\right) y_2 + \left(-2 + g_3 h^2\right) y_3 = r_3 h^2 - \left(1 + \frac{h}{2} f_3\right) \beta \quad [\because y_n = \beta]$$

$$\begin{bmatrix} -2+g_1h^2 & 1+\frac{h}{2}f_1 & 0 \\ 1-\frac{h}{2}f_2 & -2+g_2h^2 & 1+\frac{h}{2}f_2 \\ 0 & 1-\frac{h}{2}f_3 & -2+g_3h^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} r_1h^2 - \left(1-\frac{h}{2}f_1\right)\alpha \\ r_2h^2 \\ r_3h^2 - \left(1+\frac{h}{2}f_3\right)\beta \end{bmatrix}$$
$$\Rightarrow AY = B$$

Which is a tridiagonal system of equations.



Example. Consider the equation $y'' + y + 1 = 0$ with the boundary conditions $y(0) = 0$, $y(1) = 0$.

Solution. Here $nh = 1$. The differential equation is approximated as

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i + 1 = 0$$

and this gives after simplification,

$$y_{i-1} - (2 - h^2)y_i + y_{i+1} = -h^2; \qquad i = 1, 2, 3, \ldots, n-1$$

Choose $h = 1/4$ i.e., $n = 4$, we obtain the equations

$$\begin{aligned} y_0 - \frac{31}{16}y_1 + y_2 &= -\frac{1}{16} \quad \Rightarrow \quad -\frac{31}{16}y_1 + y + 2 = -\frac{1}{16} \quad [\because y_0 = 0] \\ y_1 - \frac{31}{16}y_2 + y_3 &= -\frac{1}{16} \\ y_2 - \frac{31}{16}y_3 + y_4 &= -\frac{1}{16} \quad \Rightarrow \quad y_2 - \frac{31}{16}y_3 = -\frac{1}{16} \quad [\because y_4 = 0] \end{aligned}$$

Solving the above system, we get,

$$y_1 = 0.104677, \quad y_2 = 0.140312, \quad y_3 = 0.104677$$

Hence $y_2 = y(0.5) = 0.1140312$.

Example. Consider the equation

$$y'' = y; \qquad y(0) = 0, \quad y(2) = 3.627$$

The finite difference approximation is written as

$$\begin{aligned} \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} &= y_i \\ \Rightarrow y_{i-1} - (2 + h^2)y_i + y_{i+1} &= 0; \qquad i = 1, 2, 3, \ldots, n-1 \end{aligned} \tag{1.11}$$

Taking $h = 0.5$, we have $n = 4$ and from (1.11)

$$\begin{aligned} 4(y_0 - 2y_1 + y_2) &= y_1 \\ 4(y_1 - 2y_2 + y_3) &= y_2 \\ 4(y_2 - 2y_3 + y_4) &= y_3 \end{aligned}$$

using $y_0 = 0$ and $y_4 = 3.627$, the above system becomes

$$\begin{aligned} -9y_1 + 4y_2 \qquad &= 0 \\ 4y_1 - 9y_2 + 4y_3 &= 0 \\ 4y_2 - 9y_3 &= -14.508 \end{aligned}$$

The solution of which is given in table below:

x	Computed value of y	Exact value $y = \sinh x$	Error
0.5	0.5262	0.5261	0.0051
1.0	1.1843	1.1752	0.0091
1.5	2.1382	2.1293	0.0089

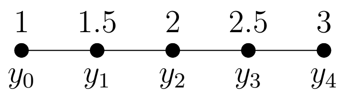
Problem 1.1 (H.W.)**.** Derive the finite difference approximation for nonlinear BVP.

Problem 1.2 (H.W.)**.** Solve

1.
$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\log x)}{x^2}; \quad 1 \le x \le 2 \quad y(1) = 1,\ y(2) = 2$$
by taking $h = 0.25$

2.
$$y'' = y' + 2y + \cos x; \quad 0 \le x \le \frac{\pi}{2} \quad y(0) = -0.3,\ y\left(\frac{\pi}{2}\right) = -0.1$$
by using $h = \frac{\pi}{4}$ and $h = \frac{\pi}{6}$

3.
$$y'' = t + \left(1 - \frac{t}{5}\right)y; \quad y(1) = 2,\ y\,(3) = -1$$
by using $h = 0.5$



Ans: $y_1 = y(1.5) = 0.552$, $y_2 = y(2.0) = -0.424$, $y_3 = y(2.5) = -0.964$.