0.1 First Order Quasi-Linear Equations and Characteristics

Consider the equation

$$a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} = c \tag{1}$$

where a, b and c are, in general functions of x, y and U but not $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$. Such equations are said to be quasi-linear first order partial differential equation.

If we consider $p = \frac{\partial U}{\partial x}$ and $q = \frac{\partial U}{\partial y}$, then (1) can be written as

$$ap + bq = c \tag{2}$$

If we know the solution values of U of equation (2) at every point on a curve c in the $xy-$plane, where c does not coincide with the curve Γ on which initial values of U are specified. Then we can determine the values of p and q on c from the values of U on c so that they satisfy equation (2).

Then in directions tangential to c from points on c, we shall automatically satisfy the differential relationship

$$\begin{aligned} \mathrm{d}\,U &= \frac{\partial U}{\partial x}\,\mathrm{d}\,x + \frac{\partial U}{\partial y}\,\mathrm{d}\,y \\ \Rightarrow \quad \mathrm{d}\,U &= p\,\mathrm{d}\,x + q\,\mathrm{d}\,y \end{aligned} \tag{3}$$

where $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ is the slope of the tangent to c at the point $P(x, y)$ on c.

Eliminating p from (3) by (2), we obtain

$$\begin{aligned} \mathrm{d}\,U &= \frac{c - bq}{a}\,\mathrm{d}\,x + q\,\mathrm{d}\,y \\ \Rightarrow \quad adU &= c\,\mathrm{d}\,x - bq\,\mathrm{d}\,x + q\,\mathrm{d}\,y \\ \Rightarrow \quad c\,\mathrm{d}\,x - adU &+ q(a\,\mathrm{d}\,y - b\,\mathrm{d}\,x) = 0 \end{aligned} \tag{4}$$

This equation is explicitly independent of p because the coefficient a, b and c are functions of x, y and U only. It can be made independent of q by choosing the curve c so that its solope $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ satisfy the equation

$$a\,\mathrm{d}\,y - b\,\mathrm{d}\,x = 0 \tag{5}$$

Then equation (4) becomes

$$c\,\mathrm{d}\,x - adU = 0 \tag{6}$$

Equation (5) is a differential equation for the curve c and equation (6) is a differential equation for the solution values of U along c. The curve c is called a characteristic curve or simply characteristic.

From (5) and (6), we can write

$$\frac{\mathrm{d}\,x}{a} = \frac{\mathrm{d}\,y}{b} = \frac{\mathrm{d}\,u}{c} \tag{7}$$

The equation (7) shows that U may be found from either the equation $\mathrm{d}\,U = (c/a)\,\mathrm{d}\,x$ or the equation $\mathrm{d}\,u = (c/b)\,\mathrm{d}\,y$.

Example. Consider the equation

$$y\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 2$$

where U is known along the initial segment Γ defined by $y = 0,\ 0 \le x \le 1$. Find the characteristic and the solution along the characteristic.

Solution. We have

$$y\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 2 \tag{8}$$

Compairing (8) with (1), we have $a = y$, $b = 1$, $c = 2$. Then,

$$\begin{aligned} &\frac{\mathrm{d}\,x}{a} = \frac{\mathrm{d}\,y}{b} = \frac{\mathrm{d}\,u}{c} \\ \Rightarrow \quad &\frac{\mathrm{d}\,x}{y} = \frac{\mathrm{d}\,y}{1} = \frac{\mathrm{d}\,u}{2} \end{aligned} \tag{9}$$

The differential equation of the family of characteristic curve is

$$\frac{\mathrm{d}\,x}{y} = \frac{\mathrm{d}\,y}{1}$$

$$x = \frac{y^2}{2} + A \tag{10}$$

Where the parameter A is a constant for each characteristic. FOr the characteristic through $R(x_R, 0)$, from (10), $A = x_R$. So the equation of this particular characteristic is

$$x = \frac{y^2}{2} + x_R \tag{11}$$

$$y^2 = 2(x - x_R) \tag{12}$$

The solution along the characteristic curve is given by

$$\frac{\mathrm{d}\,y}{1} = \frac{\mathrm{d}\,u}{2}$$

$$\Rightarrow\ U = 2y + B \tag{13}$$

where B is constant along a particular characteristic. If $U = U_R$ at $R(x_R, 0)$, then $B = U_R$ and hence the solution along the characteristic $y^2 = 2(x - x_R)$ is $U = 2y + U_R$.

Note. Since the initial values for U are known only on the segment of Γ, where $0 \leq x_R \leq 1$, it follows that the solution is defined only in the region bounded by and including the terminal characteristics $y^2 = 2x$ and $y^2 = 2(x - 1)$. In this region the solution is clearly unique and outside this region the solution is undefined.

H.W.

1. G.D Smith page 220

2. G.D Smith page 221