Chapter 1

Conjugacy and Class Equation

Definition 1. Let G be a group. The *normalizer* of a non-empty subset $S \subseteq G$ is defined by $N_S = \{x \in G : xS = Sx\}$.

Definition 2. Let G be a group and $a \in G$. Then the set $N_a = \{x \in G : ax = xa\}$ is called the *normalizer* of $a \in G$ in G.

Thus, N_a is the set of those elements of G which commute with a.

Example. N_a is a subgroup of G.

Proof. We know that $N_a = \{x \in G : ax = xa\}$ when $a \in G$. Let $x, y \in N_a$. Then $ax = xa$ and $ay = ya$. Hence, we have

$$a(xy) = (ax)y = (xa)y = x(ay) = x(ya) = (xy)a.$$

Besides, we also get

$$x^{-1}(ax)x^{-1} = x^{-1}(xa)^{-1} \text{ which implies that } \quad x^{-1}a = ax^{-1}.$$

Thus, it follows that $xy \in N_a$ and $x^{-1} \in N_a$ for all $x, y \in N_a$.
So, N_a is a subgroup of G. $\square$

Note. $N_a = G \iff a = Z$.

Example. N_a need not be a normal subgroup[1] of G.

Proof. In order to show that N_a need not be normal in G, consider an element (23) of the symmetric group S_3.
It is easy to verify that $N_{(23)} = \{(1), (23)\}$ is a subgroup of S_3.
But $(12) \circ N_{(23)} = \{(12), (123)\}$ and $N_{(23)} \circ (12) = \{(12), (132)\}$
Thus $(12) \circ N_{(23)} \neq N_{(23)} \circ (12)$.
This shows that $N_{(23)}$ is not a normal subgroup of S_3. $\square$

[1]Normal subgroup: A subgroup N of a group G is called normal subgroup iff $aN = Na$ holds $\forall a \in G$. Denoted by $N \triangleleft G$.