Chapter 1

Hyperbolic Equations

1.1 Hyperbolic Partial differential equation

We consider the boundary value problem defined by

$$u_{tt} - u_{xx} = 0, \quad 0 < x < l \tag{1.1}$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= \phi(x) \end{aligned}\right\} 0 \leq x \leq l \tag{1.2}$$

$$\left.\begin{aligned} u(0,t) &= \Psi_1(t) \\ u(l,t) &= \Psi_2(t) \end{aligned}\right\} 0 \leq x \leq T \tag{1.3}$$

which models the transverse vibration of a stretched string.

We use the following difference(central-difference) approximations for the derivatives with $x_i = ih$ for each $i = 0, 1, \ldots, m$ and $t_j = jk$ for each $j = 0, 1, 2, \ldots$

$$u_{xx} = \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] + O(h^2) \tag{1.4}$$

and

$$u_{tt} = \frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right] + O(k^2) \tag{1.5}$$

Further, $u_t(x,t)$ is approximated as follows:

$$u_t(x,t) = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \tag{1.6}$$

substituting (1.4) and (1.5) in (1.1), we obtain,

$$\frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right] = \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Putting $\alpha = k/h$ in the above equation and rearrange the terms, we get

$$u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i-1,j} + u_{i+1,j}) + 2(1-\alpha^2)u_{i,j} \tag{1.7}$$

for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2, 3, \ldots$

Which shows that the function values at the jth and $(j-1)$th time levels are required in order to determine those at the $(j+1)$th time level. Such difference schemes are called three level explicit difference schemes.

Note. Formula (1.7) holds good if $\alpha < 1$, which is the condition for stability.

The boundary conditions gives

$$\left.\begin{aligned} u_{0,j} &= \Psi_1(t_j) \\ u_{m,t} &= \Psi_2(t_j) \end{aligned}\right\} \tag{1.8}$$

for each $j = 1, 2, 3, \ldots$ and the intial condition implies that

$$u_{i,0} = f(x_i) \quad \text{for each} i = 1, 2, \ldots, m-1 \tag{1.9}$$

Writing this set of equations in matrix form gives

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\alpha^2) & \alpha^2 & 0 & \dots & 0 \\ \alpha^2 & 2(1-\alpha^2) & \alpha^2 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha^2 & 2(1-\alpha^2) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{m-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ u_{m,j-1} \end{bmatrix} \tag{1.10}$$

Example. Solve the equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ subject to the following conditions

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}\right\} \quad (t > 0)$$

and

$$\left.\begin{aligned} u_t(x,0) &= 0 \\ u(x,0) &= \sin^3 \pi x \end{aligned}\right\} \quad \text{for all } x \text{ in } 0 \le x \le 1.$$

Note. Exact solution

$$u(x,t) = \frac{3}{4}\sin \pi x \cos \pi t - \frac{1}{4}\sin 3\pi x \cos 3\pi t$$

Solution. We use the explicit formula given by (1.7), viz,

$$u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i-1,j} + u_{i+1,j}) + 2(1-\alpha^2)u_{i,j} \quad \text{where } \alpha = h/k < 1 \tag{1.11}$$

Let $h = 0.25$ and $k = 0.2$. Hence, $\alpha = 0.8$, so that the stability condition is satisfied. Let $u_{i,j} = u(ih, jk)$, so that the boundary conditions become

$$u_{0,j} = 0 \qquad u_{4,j} = 0 \tag{1.12}$$

$$u_{i,0} = \sin^3 \pi i h \quad i = 1, 2, 3 \tag{1.13}$$

and from

$$\begin{aligned} u_t(x,t) &= \frac{u_{i,j+1} - u_{i,j-1}}{2k} \\ \Rightarrow \frac{u_{i,j+1} - u_{i,j-1}}{2k} &= 0 \\ \Rightarrow u_{i,j+1} - u_{i,j-1} &= 0 \end{aligned}$$

for $j = 0$, $u_{i,1} - U_{i,-1} = 0$

$$\Rightarrow u_{i,1} = u_{i,-1} \tag{1.14}$$

substituting the value of $\alpha = 0.8$ equation (1.11) becomes

$$u_{i,j+1} = -u_{i,j-1} + 0.64(u_{i-1,j} + u_{i+1,j}) + 2(0.36)u_{i,j} \tag{1.15}$$

At the 1st step, $j = 0$, we have from (1.16)

$$u_{i,1} = -u_{i,-1} + 0.64(u_{i-1,0} + u_{i+1,0}) + 2(0.36)u_{i,0} \tag{1.16}$$

$$\Rightarrow 2u_{i,1} = 0.64(u_{i-1,0} + u_{i+1,0}) + 2(0.36)u_{i,0} \quad [\because u_{i,1} = u_{i,-1}]$$

$$\Rightarrow u_{i,1} = 0.32(u_{i-1,0} + u_{i+1,0}) + (0.36)u_{i,0} \tag{1.17}$$

For $i = 1$,

$$\begin{aligned} & u_{1,1} = 0.32(u_{0,0} + u_{2,0}) + 0.36u_{1,0} \\ \Rightarrow\ & u_{1,1} = 0.32(0+1) + 0.36 \times 0.3537 [\because u_{i,0} = \sin^3 \pi i h \text{ so, } u_{1,0} = \sin^3 \pi(0.25) \text{ and } u_{2,0} = \sin^3 2\pi(0.25) = 1] \\ \Rightarrow\ & u_{1,1} = 0.4473 \end{aligned}$$

The exact value $u(0.25, 0.2) = 0.4838$
Again, for $i = 2$,

$$\begin{aligned} u_{2,1} &= 0.32(u_{1,0} + u_{3,0}) + 0.36u_{2,0} \\ &= 0.32(0.3537 + 0.3537) + 0.36(1) \\ &= 0.5867 \end{aligned}$$

Exact value $= 0.5296$
Finally,

$$\begin{aligned} u_{3,1} &= 0.32(1+1) + 0.36 \times (0.3537) \\ &= 0.4473 \end{aligned}$$

Exact value $= 0.4838$
The computations can be continued for $j = 1, 2, 3$

H.W.

1. (**)

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0; \quad 0 < x < 1, \ \ 0 < t \\ &u(0,t) = u(1,t) = 0 \quad \text{ for } 0 < t \\ &u(x,0) = \sin \pi x \quad 0 \le x \le 1 \\ &u_t(x,0) = 0 \quad 0 \le x \le 1 \end{aligned}$$

2.

$$\begin{aligned} &u_{tt} - u_{xx} = 0; \quad 0 < x < 1 \\ &u(0,t) = u(1,t) = 0 \\ &u(x,0) = x - x^2 \\ &u_t(x,0) = 0 \end{aligned}$$

take $h = 0.25$ and $k = 0.2$.