Chapter 1

Curve Fitting and Spline Interpolation

1.1 Curve Fitting by Least Squares Method

The method of least squares may be one of the most systematic procedure to fit a curve through given data points.
Consider the problem of fitting a set of data points

$$(x_r, y_r), \qquad r = 1, 2, 3, \ldots, m$$

to a curve $y = f(x)$ whose values depend on n parameters $c_1, c_2, c_3, \ldots, c_n$. The values of the function at a point depends on the values of the parameter involved. In least square method we determine a set of values of the parameter $c_1, c_2, c_3, \ldots, c_n$ such that the sum of the squares of the error

$$E(c_1, c_2, \ldots, c_n) = \sum_{i}^{m} \left[f\left(x_i, c_1, c_2, \ldots, c_n\right) - y_i\right]^2$$

is minimum.
The necessary conditions for E to have a minimum is that

$$\frac{\partial E}{\partial c_i} = 0, \qquad i = 1, 2, 3, \ldots, n$$

This condition gives a system of n equations, called *normal equations*, in n unknowns $c_1, c_2, c_3, \ldots, c_n$. If the parameters appear in the function in non-linear form, the normal equations become non-linear and is difficult to solve. This difficulty may be avoided if $f(x)$ is transformed to a form which is linear in parameter.

1.1.1 Parameters in Nonlinear Form

(a) **Power function:** Let the curve

$$y = ax^b$$

be fitted to the given data.
Taking logarithm of both sides, we get

$$\ln y = \ln a + b \ln x$$

which can be written in the form

$$Y = A + bX$$

where,

$$Y = \ln y, \quad A = \ln a, \quad X = \ln x.$$

(b) Let the curve

$$y = \frac{400}{1 + ce^{bx}}$$

be fitted to the given data.
The equation of the curve can be rewritten as

$$\frac{400}{y} - 1 = ce^{bx}$$

Taking logarithm of both sides, we get

$$\ln\left(\frac{400}{y} - 1\right) = \ln c + bx$$

which can be written in the form

$$Y = A + bx \qquad \text{where} \qquad Y = \ln\left(\frac{400}{y} - 1\right), \quad A = \ln c.$$

Problem 1.1.1. The average price, P, of a certain type of second-hand car is believed to be related to its age, x years, by an equation of the form

$$P = 50 + ae^{bx}$$

where a and b are constants. Data from a recent newspaper give the following average price (in Taka) for used car of this type,

x	P (in thousands)
1	774.4
2	603.4
3	439.2
4	360.0
5	328.3

(a) Estimate the values of a and b rounded to 3 significant figures.

(b) Estimate the price of a car of this type that is 6 years old and the original new price of that car.

Solution. (a) The curve $P = 50 + ae^{b}x$ is to be fitted to the given data.
The equation of the curve can be rewritten as $P - 50 = ae^{b}x$
Taking logarithm of both sides, we get $\ln(P - 50) = \ln a + bx$
which can be written in the form

$$Y = A + bx \qquad \text{where} \quad Y = \ln(P - 50), \quad A = \ln a.$$

$$E(A, b) = \sum_{i=1}^{5}(A + bx_i - Y_i)^2$$

At minimum,

$$\frac{\partial E}{\partial A} = 0 \qquad \text{and} \qquad \frac{\partial E}{\partial b} = 0$$

which give

$$2\sum[(A + bx_i - Y_i)]1 = 0$$
$$2\sum(A + bx_i - Y_i)x_i = 0$$

which can be rearranged as

$$A\sum 1 + b\sum x_i = \sum Y_i$$
$$A\sum x_i + b\sum x_i{}^2 = \sum x_x Y_i$$

The sum can be calculated in a tabular form as below:

n	x	P	Y	xY	x^2
1	1	774.4	6.585	6.585	1
2	2	603.4	6.316	12.632	4
3	3	439.2	5.964	17.892	9
4	4	360.0	5.737	22.948	16
5	5	328.3	5.629	28.145	25
Sum	15		30.231	88.202	55

The normal equations are

$$5A + 15b = 30.231$$
$$15A + 55b = 88.202$$

Dividing 1st eq. by 5 and 2nd by 15, we have

$$A + 3b = 6.046$$
$$A + 3.667b = 5.880$$

Subtracting

$$0.667b = -0.166 \qquad \text{or} \quad b = -0.249$$

and

$$A = 5.88 - 3.667(-0.249) = 6.793$$

and hence

$$a = e^6.793 \approx 892$$

The required best fitting curve is

$$P = 50 + 892e^{-0.249x}$$

(b) When $x = 6$, we have

$$P = 50 + 892e^{-0.249\times 6} \approx 250.2$$

The price of the 6 years old car is Tk. 250.2 thousand. New price corresponds to $x = 0$ and is $50 + 892 = 942$ thousand taka.

1.1.2 Exercise

1. Find the least square line $y = ax + b$ to the following data

x	y
-1	1.5
0	3.7
2	6.2
4	8.5
5	12.8

2. Students collected the following set of data to find the gravitational constant g. Use the relation $d = (gt^2)/2$, where d is the distance in metres and t is time in seconds, to find the value of g.

Time (t)	Distance (d)
0.2	0.2142
0.4	0.7789
0.6	1.7676
0.8	3.1365
5.0	4.9075

3. Fit a curve of the form $y = ax^2 + be^{-x}$ to the following data.

x	y
1	5.18
2	6.70
4	21.31
5	33.07
8	84.48

4. The temperature in a metal strip was measured at various time intervals. Given that the relation between the temperature $T(°C)$ and time t (min) is of the form $T = a + be^{t/2}$. Six pairs of observations of the two variables T and t gave the following results.

$$\sum T = 165.5, \quad \sum e^{t/2} = 48.71, \quad \sum e^{t} = 656.6, \quad \sum T e^{t/2} = 1425$$

Find the temperature after 7 minutes.

5. Given the following set of values of x and y :

x	y
2	1.14
3	1.45
6	1.97
7	2.41
10	2.99

(a) Fit the power equation $y = ax^b$ to the given data.

(b) Find the best fitting curve of the form $y = ae^{bx}$.

(c) Fit the saturation growth rate model $y = \frac{ax}{b+x}$ to the above data.

By finding errors determine which one of the above is the best fitting curve.

6. The following table gives the population of a certain country from 1960 to 2000 at ten yearly interval

Year	Population P (in Lac)
1970	20.5
1980	26.4
1990	33.1
2000	40.4
2010	48.2

It is known that if environmental factors remain constant, the population size, P, is given by

$$P(t) = \frac{200}{1 + ce^{at}}$$

where c and a are constants.
Estimate, to 3 significant figures, the values of c and a.
Hence, predict the population in the year 2015.

7. A bowl of hot water is kept in a room of constant temperature $25°C$. At 5 minutes interval temperature of the water is recorded and listed as given below.

t (in min)	T (in $°C$)
5	75.3
10	70.0
15	63.4
20	58.5
25	54.0

The law of cooling can be assumed to be of the form $T = 25 + ae^{-kt}$.
Find, to 2 significant figures, the best values of a and k.
Estimate the initial temperature.
Hence, find the time, to the nearest minute, when the temperature of the water in the bowl will be $51°C$.

8. The pressure p and volume V of a fixed mass of gas are believed to be related by an equation of the form

$$pV^{\gamma} = c$$

where γ and c are constants.
In six set of experiments on the fixed mass of gas, in each of which p was controlled and V measured. The results are as follows:

$p\,(Nm^{-2})$	$V\,(m^3)$
0.4	1.894
0.6	1.426
0.8	1.166
1.0	0.998
1.2	0.878
1.4	0.789

Estimate to 2 decimal places

(i) the value of γ,

(ii) the value of V when $p = 0.75Nm^{-2}$.

9. The method of least squares is used to estimate the constants a, b, c in the formula

$$y = a + b\sin x + ce^{-x/5}$$

Eight pairs of data points leads to the following normal equations, where the missing numerical values are to be determined.

$$\begin{aligned} \_\_\_\_\_a - 3.616b + 5.784c &= 100.06 \\ \_\_\_\_\_a + 3.261b - 2.737c &= -46.796 \\ \_\_\_\_\_a + \_\_\_\_\_b + 4.403c &= 73.060 \end{aligned}$$

The method of Gaussian elimination leads to the following equations:

$$\begin{aligned} a - 0.452b + \_\_\_\_\_c &= \_\_\_\_\_ \\ 0.450b - 0.034c &= -0.433 \\ 1.734c &= 4.895 \end{aligned}$$

Supply the missing values and complete the solution of the set of equations, giving a, b and c to two decimal places.
Using the estimated values of a, b and c, find the value of y given by the formula to one decimal place when $x = 4$.

10. Use a suitable substitution to derive a linearized form for the following functions:

(i)

$$y = \frac{x}{ax + b}$$

(ii)

$$y = (ax + b)^{-1}$$

(iii)

$$x = a^x b^y + 2e^y$$

(iv)

$$y = \frac{5 + ax}{b + x^3}$$

(v)

$$y = \frac{x^2}{(a+bx)^2}$$

(vi)

$$y = \frac{1}{a(2^y) + b(2^{-x})}$$

(vii)

$$y = \frac{x^2}{(ax+1)(bx+2)}$$

(viii)

$$y = \frac{x}{1 + ae^{bx}}$$

(ix)

$$y = \frac{1}{\sqrt{(x+a)(x+b)}}$$

1.2 Spline Interpolation

Spline interpolation is a form of interpolation where the interpolant is a special type of piecewise polynomial called a **spline**. Spline interpolation is preferred over polynomial interpolation because the interpolation error can be made small even when using low degree polynomials for the spline.
Divide the interval containing the tabular points as subintervals $x_0 < x_1 < x_2 < \cdots < x_n$ and replace the function $f(x)$ by some lower degree interpolating polynomial in each of the subinterval. The tabular points $x_0, x_1, x_3, \ldots, x_n$ at which the function changes its character is termed as **knots** in the theory of spline.

A function $S(x)$ of the form

$$S(x) = \begin{cases} f_0(x) & x \in [x_0, x_1] \\ f_1(x) & x \in [x_1, x_2] \\ \vdots \\ f_{n-1}(x) & x \in [x_{n-1}, x_n] \end{cases}$$

is called a spline of degree m if

(i) the domain of $S(x)$ is the interval $[x_0, x_n]$

(ii) $S(x)$, $S'(x)$, $S''(x), \ldots, S^{(m-1)}(x)$ are all continuous functions on $[x_0, x_n]$

(iii) $S(x)$ is a polynomial of degree less than equal to m on each subintervals $[x_k, x_{k+1}]$, $k = 0, 1, 2, 3, \ldots, n-1$.

1.2.1 Linear Spline Interpolation

The simplest polynomial to use, a polynomial of degree one, produces a polygon path that consists of line segments that pass through the points. The point-slope formuala for the line segment may be used to represent this piecewise linear curve:

$$S(x) = f_k(x) = a_k(x - x_k) + b_k, \qquad \text{for } x_k \le x \le x_{k+1}(k = 0, 1, 2, \ldots, n-1)$$

Since the line passes through (x_k, y_k) and (x_{k+1}, y_{k+1}) we have

$$f_k(x_k) = y_k = b_k$$

and

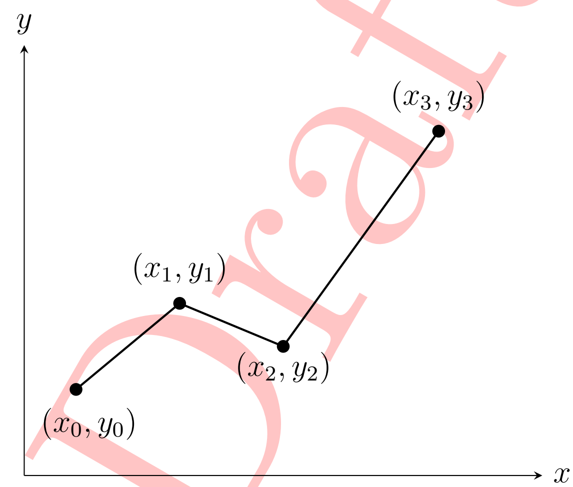
$$f_k(x_{k+1}) = y_{k+1} = a_k(x_{k+1} - x_k) + b_k$$

or

$$a_k = \frac{y_{k+1} - y_k}{x_k + 1 - x_k} = \frac{\Delta y_k}{h_k}$$

where

$$\Delta y_k = y_{k+1} - y_k \qquad \text{and} \quad h_k = x_{k+1} - x_k$$



The resulting linear spline curve $f_k(x)$ in $[x_k, x_{k+1}]$ can be written as

$$f_k(x) = \frac{\Delta y_k}{h_k}(x - x_k) + y_k, \qquad (k = 0, 1, 2, \ldots, n-1)$$

The resulting curve looks like a "broken line" as shown in the diagram.

1.2.2 Quadratic Spline Interpolation

For a quadratic spline through (x_k, y_k) we may take $f_k(x)$ is of the form

$$f_k(x) = a_k(x - x_k)^2 + b_k(x - x_k) + c_k \tag{1.1}$$

The quadratic spline function $S(x)$ is

$$S(x) = f_k(x)$$

on the interval $[x_k, x_{k+1}]$ for $k = 0, 1, 2, \ldots, n-1$.
Each quadratic polynomial, $f_k(x)$, has three unknown constants, hence there are $3n$ unknown coefficients a_k, b_k, c_k $(k = 0, 1, 2, \ldots, n-1)$.
To find $3n$ unknowns, one needs to set up $3n$ equations and then simultaneously solve them. These $3n$ equations are found as follows:

1. As the splines pass through (x_k, y_k), we have

$$f_k(x_k) = c_k = y_k \qquad \text{for} \quad k = 0, 1, 2, \ldots, n-1$$

   and

$$f_{n-1}(x_n) = y_n$$

2. Continuity of $S(x)$ at the interior points gives

$$f_k(x_{k+1}) = f_{k+1}(x_{k+1}) \tag{1.2}$$

   Since there are $(n-1)$ interior points, we have $(n-1)$ such equations.

3. Continuity of $S'(x)$ at the interior points gives

$$f_k^{'}(x_{k+1}) = f_{k+1}^{'}(x_{k+1}) \tag{1.3}$$

Differentiating eq. (1.1) we have

$$f_k^{'}(x) = 2a_k(x - x_k) + b_k$$

From (1.3) we have

$$2a_k h_k + b_k = b_{k+1}$$

Using the notation,

$$f_k^{'}(x_k) = b_k = Z_k$$

we have

$$a_k = \frac{Z_{k+1} - Z_k}{2h_k}, \qquad k = 0, 1, 2, \ldots, n-1$$

From Eq. (1.2),

$$\begin{aligned}
a_k(h_k^2) + b_k(h_k) + c_k &= c_{k+1} \\
a_k h_k + b_k &= \frac{c_{k+1} - c_k}{h_k} = \frac{y_{k+1} - y_k}{h_k} \\
\frac{Z_{k+1} - Z_k}{2} + Z_k &= \frac{y_{k+1} - y_k}{h_k} \\
Z_{k+1} + Z_k &= 2\frac{\Delta y_k}{h_k}
\end{aligned}$$

Here also $(n-1)$ interior points, we have $(n-1)$ such equations.
So far, the total number of equations are

$$(2n) + (n-1) = (3n-1)$$

We still need one more equation. We can assume that the first spline is linear, that is

$$a_0 = 0$$

This gives us $3n$ equations for $3n$ unknowns. These can be solved by a number of techniques used to solve simultaneous linear equations.
It should be mentioned that the curvature of the quadratic spline changes abruptly at each knot, and the curve may not be pleasing to the eye.

1.2.3 Cubic Spline Interpolation

For the cubic spline through the points (x_k, y_k), $k = 0, 1, 2, \ldots, n$ we may take $f_k(x)$ is of the form

$$f_k(x) = a_k(x - x_k)^3 + b_k(x - x_k)^2 + c_k(x - x_k) + d_k \quad \text{in}[x_k, x_{k+1}] \tag{1.4}$$

Thus the cubic spline function $S(x)$ is of the form

$$S(x) = f_k(x) \qquad \text{on the interval} \quad [x_k, x_{k+1}] \quad \text{for } k = 0, 1, 2, 3, \ldots, n-1$$

with the following properties:

(a)
$$f_k(x_k) = y_k, \qquad k = 0, 1, 2, \ldots, n-1 \quad \text{and} \quad f_{n-1}(x_n) = y_n$$

(b)
$$f_k(x_{k+1}) = f_{k+1}(x_{k+1}), \qquad k = 0, 1, 2, \ldots, n-1$$

(c)
$$f_k^{'}(x_{k+1}) = f_{k+1}^{'}(x_{k+1}), \qquad k = 0, 1, 2, \ldots, n-1$$

(d)
$$f_k^{''}(x_{k+1}) = f_{k+1}^{''}(x_{k+1}), \qquad k = 0, 1, 2, \ldots, n-1$$

Each cubic polynomial, $f_k(x)$, has four unknown constants, hence there are $4n$ coefficients to be determined. The data points supply $(n+1)$ conditions, and properties (b), (c) and (d) each supply $(n-1)$ conditions. Hence, $n + 1 + 3(n-1) = 4n - 2$ conditions are specified. Two more conditions are needed which will be discussed later.

The conditions (a) then gives

$$d_k = y_k, \qquad k = 0, 1, 2, \ldots, n-1$$

From (b) we have

$$\begin{aligned} y_{k+1} &= a_k(x_{k+1} - x_k)^3 + b_k(x_{k+1} - x_k)^2 + c_k(x_{k+1} - x_k) + y_k \\ &= a_k {h_k}^3 + b_k {h_k}^2 + c_k h_k + y_k, \qquad k = 0, 1, 2, \ldots, n-1 \end{aligned} \tag{1.5}$$

where $h_k = (x_{k+1} - x_k)$.
Differentiating (1.4) we have

$$f_k^{'}(x) = 3a_k(x - x_k)^2 + 2b_k(x - x_k) + c_k \tag{1.6}$$

$$f_k^{''}(x) = 6a_k(x - x_k) + 2b_k \tag{1.7}$$

Development is simplified if we write the equations in terms of the second derivatives-that is, if we use

$$M_k = f_k^{''}(x_k) \qquad \text{for } k = 0, 1, 2, \ldots, n-1 \quad \text{and } M_n = f_k^{''}(x_k)$$

From Eq.(1.7), we have

$$\begin{aligned} M_k &= 6a_k(x_k - x_k) + 2b_k = 2b_k \\ M_{k+1} &= 6a_k(x_{k+1} - x_k) + 2b_k = 6a_k h_k + 2b_k \end{aligned}$$

Hence we can write

$$b_k = \frac{M_k}{2}$$
$$a_k = \frac{M_{k+1} - M_k}{6h_k}$$

and from Eq.(1.5), we have

$$y_{k+1} = \left(\frac{M_{k+1} - M_k}{6h_k}\right) {h_k}^3 + \frac{M_k}{2} {h_k}^2 + c_k h_k + y_k$$
$$c_k = \frac{y_{k+1} - y_k}{h_k} - \frac{h_k}{6}(M_{k+1} + 2M_k)$$

In order to get the cubic splines, it is required to determine the second derivatives

$$M_0, M_1, M_2, \ldots, M_n$$

at the knots and can be evaluated by the continuity of the second derivatives.
From Eq. (1.4),

$$f_k^{'}(x) = 3a_k(x - x_k)^2 + 2b_k(x - x_k) + c_k$$

At the common knot (x_{k+1}, y_{k+1}) the first derivatives $f_k^{'}(x)$ and $f_(k+1)^{'}(x)$ should be equal i.e.

$$f_{k+1}^{'}(x_{k+1}) = f_k^{'}(x_{k+1})$$

But

$$f_{k+1}^{'}(x_{k+1}) = c_{k+1} = \frac{y_{k+2} - y_{k+1}}{h_{k+1}} - \frac{h_{k+1}}{6}(M_{k+2} + 2M_{k+1}) \tag{1.8}$$

and

$$\begin{aligned} f_k^{'}(x_{k+1}) &= 3a_k h_k^2 + 2b_k h_k + c_k \\ &= 3\left(\frac{M_{k+1} - M_k}{6h_k}\right) {h_{k-1}}^2 + 2\left(\frac{M_k}{2}\right) h_k + \frac{y_{k+1} - y_k}{h_k} - \frac{h_k}{6}(M_{k+1} + 2M_k) \\ &= \frac{y_{k+1} - y_k}{h_k} + \frac{h_k}{6}(M_{k+1} + 2M_k) \end{aligned} \tag{1.9}$$

Eq. (1.8) with Eq.(1.9),

$$h_k M_k + 2(h_k + h_{k+1})M_{k+1} + h_{k+1}M_{k+2} = 6\left[\frac{\Delta y_{k+1}}{h_{k+1}} - \frac{\Delta y_k}{h_k}\right], \qquad k = 0, 1, 2, \ldots, n-2 \tag{1.10}$$

1.2.4 End Points Constraints

We need to impose suitable end-conditions to get a unique cubic spline. The standard end points constraints are mentioned below.

Description of the strategy	Equations involving M_0 and M_n
Natural cubic spline "a relaxed curve": $S^{'}(x_0)$ and $S^{''}(x_n)$.	$M_0 = 0$, $M_n = 0$
Clamped cubic spline: specify $S^{'}(x_0) = A$ and $S^{'}(x_n) = B$.	$2M_0 + M_1 = \frac{6}{h_0}\left[\frac{\Delta y_0}{h_0} - A\right]$ $M_n + 2M_{n-1} = \frac{6}{h_{n-1}} 6\left[B - \frac{\Delta y_{n-1}}{h_{n-1}}\right]$
Extrapolated cubic spline: M_0 as linear extrapolation from M_1 and M_2 $: \frac{M_1 - M_0}{h_0} = \frac{M_2 - M_1}{h_1}$ M_n as linear extrapolation from M_{n-1} and M_{n-2}: $\frac{M_n - M_{n-1}}{h_{n-1}} = \frac{M_{n-1} - M_{n-2}}{h_{n-2}}$	$M_0 = M_1 - \frac{h_0(M_2 - M_1)}{h_1}$ $M_n = M_{n-1} - \frac{h_{n-1}(M_{n-1} - M_{n-2})}{h_{n-2}}$
Parabolically terminated spline ($S^{''}(x)$ is constant near the end points)	$M_0 = M_1, \quad M_n = M_{n-1}$

Problem 1.2.1. Find the linear spline for the following data:

x	y
0	0.0
1	0.5
2	2.0
3	1.5

Solution. Here, $h_0 = h_1 = h_2 = 1$.
Linear spline functions are

$$f_0(x) = \frac{\Delta y_0}{h_0}(x - x_0) + y_0 = 0.5x \qquad 0 \le x \le 1$$

$$f_1(x) = \frac{\Delta y_1}{h_1}(x - x_1) + y_1 = 1.5(x - 1) + 0.5 \qquad 1 \le x \le 2$$

$$f_2(x) = \frac{\Delta y_2}{h_2}(x - x_2) + y_2 = -0.5(x - 2) + 2 \qquad 2 \le x \le 3$$

Linear spline function is

$$S(x) = \begin{cases} 0.5x, & 0 \le x \le 1 \\ 1.5(x-1) + 0.5, & 1 \le x \le 2 \\ -0.5(x-2) + 2, & 2 \le x \le 3 \end{cases}$$

Problem 1.2.2. Find the quadratic spline for the following data:

x	y
0	0.0
1	0.5
2	2.0
3	1.5

Solution. Here, $h_0 = h_1 = h_2 = 1$.

$$Z_{k+1} + Z_k = 2\frac{\Delta y_k}{h_k} \qquad k = 0, 1, 2$$

Using the recurrence relation, we obtain the equations

$$\begin{aligned} Z_1 + Z_0 &= 2(0.5) = 1 \\ Z_2 + Z_1 &= 2(2 - 0.5) = 3 \\ Z_3 + Z_2 &= 2(1.5 - 2) = -1 \end{aligned}$$

Using the end condition $a_0 = 0$, we have

$$\frac{Z_1 - Z_0}{2(1)} = 0 \qquad \text{or} \quad Z_1 = Z_0$$

Solving above equations, we have

$$\begin{aligned} Z_0 = Z_1 &= \frac{1}{2} = 0.5 \\ Z_2 &= 3 - 0.5 = 2.5 \\ Z_3 &= -1 - 2.5 = -3.5 \end{aligned}$$

With these values of Z's the spline coefficients the spline coefficients can be obtained as follows:
With k = 0,

$$a_0 = 0, \qquad b_0 = Z_0 = 0.5, \qquad c_0 = y_0 = 0$$

and

$$f_0(x) = 0.5x, \qquad 0 \le x \le 1$$

With k = 1,

$$a_1 = \frac{Z_2 - X_1}{2h_1} = \frac{2.5 - 0.5}{2(1)} = 1, \qquad b_1 = Z_1 = 0.5, \qquad c_1 = y_1 = 0.5$$

and

$$f_1(x) = (x-1)^2 + 0.5(x-1) + 0.5, \qquad 1 \le x \le 2$$

With k = 2,

$$a_2 = \frac{Z_3 - Z_2}{2h_2} = \frac{-3.5 - 2.5}{2(1)} = 3, \qquad b_2 = Z_2 = 2.5, \qquad c_2 = y_2 = 2$$

and

$$f_2(x) = -3(x-2)^2 + 2.5(x-2) + 2, \qquad 2 \le x \le 3$$

The quadratic spline function is

$$S(x) = \begin{cases} 0.5x, & 0 \le x \le 1 \\ (x-1)^2 + 0.5(x-1) + 0.5, & 1 \le x \le 2 \\ -3(x-2)^2 + 2.5(x-2) + 2, & 2 \le x \le 3 \end{cases}$$

Problem 1.2.3. Consider the points

x	y
0	0.0
1	0.5
2	2.0
3	1.5

(a) Find the natural cubic spline which fits the given data.

(b) Find the clamped cubic spline with conditions $S^{'}(0) = 1$ and $S^{'}(3) = -1$.

(c) Find the extrapolated cubic spline.

Solution. The governing recurrence is

$$h_k M_k + 2(h_k + h_{k+1})M_{k+1} + h_{k+1}M_{k+2} = 6\left[\frac{\Delta y_{k+1}}{h_{k+1}} - \frac{\Delta y_k}{h_k}\right], \qquad k = 0, 1, 2$$

First, compute the quantities

$$h_0 = h_1 = h_2 = 1$$

and

$$\frac{\Delta y_0}{h_0} = \frac{0.5 - 0}{1} = 0.5, \quad \frac{\Delta y_1}{h_1} = \frac{2 - 0.5}{1} = 1.5, \quad \frac{\Delta y_2}{h_2} = \frac{1.5 - 2.0}{1} = -1.5$$

(a) Using natural cubic spline
Here the end conditions are

$$M_0 = M_3 = 0$$

The equations corresponding to $k = 0, 1$ are

$$M_0 + 4M_1 + M_2 = 6(1.5 - 0.5) = 6$$

or

$$4M_1 + M_2 = 6 \tag{1.11}$$

and

$$M_1 + 4M_2 + M_3 = 6(-0.5 - 1.5) = -12$$

or

$$M_1 + 4M_2 = -12 \tag{1.12}$$

Solving (1.11) and (1.12),

$$M_1 = 2.4, \qquad M_2 = -3.6$$

With $k = 0$,

$$\begin{aligned}
a_0 &= \frac{M_1 - M_0}{6} = \frac{2.4}{6} = 0.4 \\
b_0 &= \frac{M_0}{2} = 0 \\
c_0 &= \frac{\Delta y_0}{h_0} - \frac{h_0}{6}(M_1 + 2M_0) = 0.5 - \frac{2.4}{6} = 0.1 \\
d_0 &= y_0 = 0
\end{aligned}$$

and

$$f_0(x) = 0.4x^3 + 0.1x, \qquad 0 \le x \le 1$$

With $k = 1$,

$$\begin{aligned}
a_1 &= \frac{M_2 - M_1}{6} = \frac{-6}{6} = -1 \\
b_1 &= \frac{M_1}{2} = \frac{2.4}{2} = 1.2 \\
c_1 &= \frac{\Delta y_1}{h_1} - \frac{h_1}{6}(M_2 + 2M_1) = 1.5 - \frac{1.2}{6} = 1.3 \\
d_1 &= y_1 = 0.5
\end{aligned}$$

and

$$f_1(x) = -(x-1)^3 + 1.2(x-1)^2 + 1.3(x-1) + 0.5 \qquad 0 \le x \le 1$$

With $k = 2$,

$$\begin{aligned}
a_2 &= \frac{M_3 - M_2}{6} = \frac{3.6}{6} = 0.6 \\
b_2 &= \frac{M_2}{2} = \frac{-3.6}{2} = -1.8 \\
c_2 &= \frac{\Delta y_2}{h_2} - \frac{h_2}{6}(M_3 + 2M_2) = -0.5 - \frac{-7.2}{6} = 0.7 \\
d_2 &= y_2 = 2
\end{aligned}$$

and

$$f_2(x) = 0.6(x-2)^3 - 1.8(x-2)^2 + 0.7(x-2) + 2, \qquad 0 \le x \le 1$$

The natural cubic spline function is

$$S(x) = \begin{cases} 0.4x^3 + 0.1x, & 0 \le x \le 1 \\ -(x-1)^3 + 1.2(x-1)^2 + 1.3(x-1) + 0.5, & 1 \le x \le 2 \\ 0.6(x-2)^3 - 1.8(x-2)^2 + 0.7(x-2) + 2, & 2 \le x \le 3 \end{cases}$$

(b) With clamped spline condition
The first derivative boundary conditions are:

$$S^{'}(0) = 0.2 \qquad \text{and} \qquad S^{'}(3) = -1$$

The equations involving M's are

$$\begin{aligned}
&\text{At left end:} & 2M_0 + M_1 &= 6(0.5 - 0.2) = 1.8 \\
&\text{For } k = 0: & M_0 + 4M_1 + M_2 &= 6(1.5 - 0.5) = 6 \\
&\text{For } k = 1: & M_1 + 4M_2 + M_3 &= 6(-0.5 - 1.5) = -12 \\
&\text{At right end:} & M_2 + 2M_3 &= 6(-1 + 0.5) = -3
\end{aligned}$$

Solution of the above equations are

$$M_0 = -0.36, \qquad M_1 = 2.52, \qquad M_2 = -3.72, \qquad M_3 = 0.36$$

Corresponding spline coefficients are

$$\begin{aligned}
k=0: &\qquad a_0 = 0.48, \quad b_0 = -0.18, \quad c_0 = 0.2, \quad d_0 = 0 \\
k=1: &\qquad a_1 = -1.04, \quad b_1 = 1.26, \quad c_1 = 1.28, \quad d_1 = 0.5 \\
k=2: &\qquad a_2 = 0.68, \quad b_2 = -1.86, \quad c_2 = 0.68, \quad d_2 = 2
\end{aligned}$$

Thus the clamped cubic spline function is

$$S(x) = \begin{cases} 0.48x^3 - 0.18x^2 + 0.2x, & 0 \le x \le 1 \\ -1.04(x-1)^3 + 1.26(x-1)^2 + 1.28(x-1) + 0.5, & 1 \le x \le 2 \\ 0.68(x-2)^3 - 1.8(x-2)^2 + 0.68(x-2) + 2, & 2 \le x \le 3 \end{cases}$$

(c) With extrapolated boundary condition
The equations involving M's are

$$\begin{aligned}
\text{At left end:} &\qquad M_1 - M_0 = M_2 - M_1 \\
\text{or} &\qquad M_0 - 2M_1 + M_2 = 0 \\
\text{For } k=0: &\qquad M_0 + 4M_1 + M_2 = 6(1.5 - 0.5) = 6 \\
\text{For } k=1: &\qquad M_1 + 4M_2 + M_3 = 6(-0.5 - 1.5) = -1.2 \\
\text{At right end:} &\qquad M_3 - M_2 = M_2 - M_1 \\
\text{or} &\qquad M_1 - 2M_2 + M_3 = 0
\end{aligned}$$

Solution of the above equations are

$$M_0 = 4, \qquad M_1 = 1, \qquad M_2 = -2, \qquad M_3 = -5$$

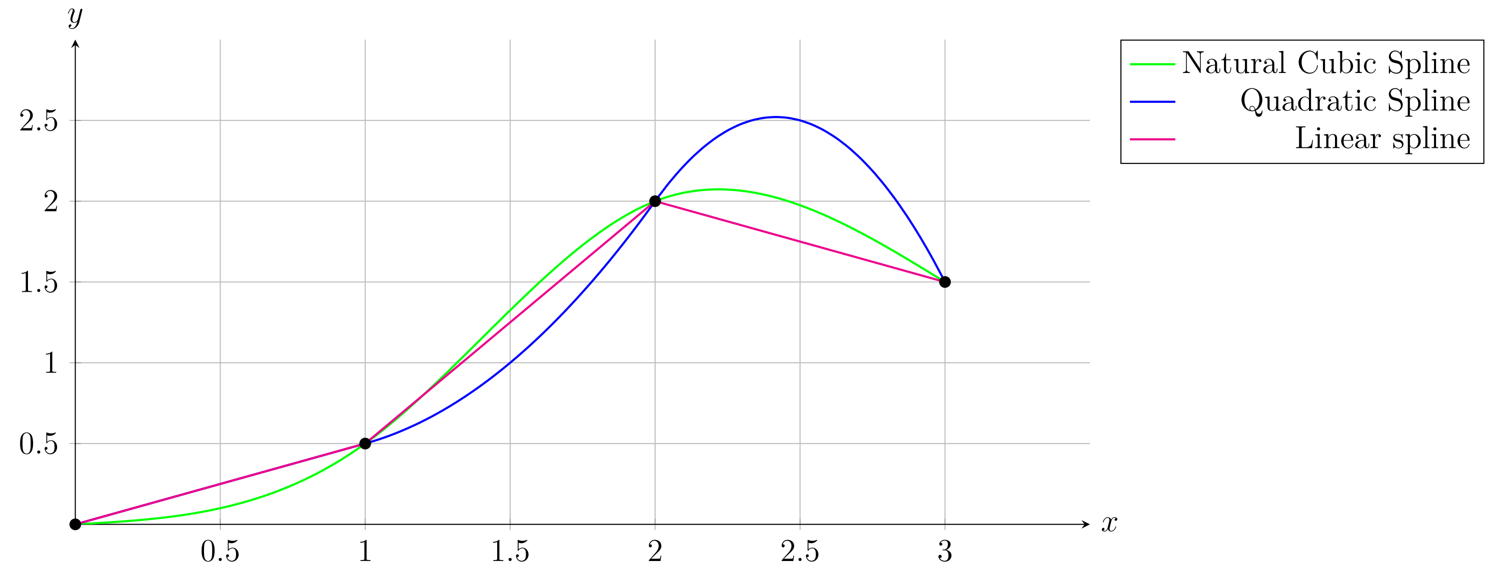
Corresponding spline coefficients are

$$\begin{aligned}
k=0: &\qquad a_0 = -0.5, \quad b_0 = 2, \quad c_0 = -1, \quad d_0 = 0 \\
k=1: &\qquad a_1 = -0.5, \quad b_1 = 0.5, \quad c_1 = 1.5, \quad d_1 = 0.5 \\
k=2: &\qquad a_2 = -0.5, \quad b_2 = -1, \quad c_2 = 1, \quad d_2 = 2
\end{aligned}$$

Thus the cubic spline function with extrapolated boundary condition is

$$S(x) = \begin{cases} -0.5x^3 + 2x^2 - x, & 0 \le x \le 1 \\ -0.5(x-1)^3 + 0.5(x-1)^2 + 1.5(x-1) + 0.5, & 1 \le x \le 2 \\ -0.5(x-2)^3 - (x-2)^2 + (x-2) + 2, & 2 \le x \le 3 \end{cases}$$

Comparison of the three types of spline curves are shown below:



1.3 EXERCISES

1. Determine whether this function is a first degree spline:

$$f(x) = \begin{cases} x, & -1 \leq x \leq 1 \\ 1 - 2(x-1), & 1 \leq x \leq 2 \\ -1 + 3(x-2), & 2 \leq x \leq 3 \end{cases}$$

2. Is $f(x) = |x|$ a first degree spline? Why or why not?

3. Are these functions quadratic splines? Explain why or why not.

   (a)
$$f(x) = \begin{cases} 0.1x^2, & 0 \leq x \leq 1 \\ 9.3x^2 - 18.4x + 9.2, & 1 \leq x \leq 1.3 \end{cases}$$

   (b)
$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ x, & x > 0 \end{cases}$$

4. Find first-degree and quadratic splines for the following data:

x	y
-1.0	2
0.0	1
0.5	0
1.0	1
2.0	2

5. Prove that the derivative of a quadratic spline is a first degree spline.

6. Show that the indefinite integral of a first-degree spline is a second-degree spline.

7. Determine whether $f(x)$ is a cubic spline with knots -1, 0, 1 and 2:

$$f(x) = \begin{cases} 1 + 2(x+1) + (x+1)^3, & -1 \le x \le 0 \\ 4 + 5x + 3x^3, & 0 \le x \le 1 \\ 11 + 1(x-1) + 3(x-1)^2 + (x-1)^3, & 1 \le x \le 2 \end{cases}$$

8. A natural cubic spline S on $[0, 2]$ is defined by

$$S(x) = \begin{cases} 1 + 2x - x^3, 0 \le x \le 1 \\ 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & 1 \le x \le 2 \end{cases}$$

Find b, c, and d.

9. A natural cubic spline for a function $f(x)$ on $[-1, 2]$ is defined by

$$f(x) = \begin{cases} A(x+1)^3 + B(x+1)^2 - 5(x+1) + 5, & -1 \le x \le 0 \\ x^3 + 3x^2 - 2x + 1, & 0 \le x \le 1 \\ a(x+1)^3 + b(x+1)^2 + c(x+1) + d, & 1 \le x \le 2 \end{cases}$$

Find the values of A, B, a, b, c and d. Hence, estimate the values of $f(-0.5)$ and $f(1.5)$.

10. A clamped cubic spline for a function $f(x)$ is defined on $[1, 3]$ by

$$f(x) = \begin{cases} 3(x-1) + 2(x-1)^2 - (x-1)^3, & 0 \le x \le 1 \\ a + b(x-2) + c(x-2)^2 + d(x-2)^3, & 1 \le x \le 2 \end{cases}$$

Given $f'(1) = f'(3)$, find a, b, c, and d.

11. Find the natural cubic splines satisfying the following data points:

    (a) $(0, 1)$, $(1, 1)$ and $(2, 5)$

    (b) $(-1, 1)$, $(0, 2)$ and $(1, -1)$

12. Find the natural cubic spline which fits the following data:

x	y
1	1
2	5
3	11
4	8

and hence find the values of $y(1.5)$.

13. Find the natural cubic spline which fits the following data:

x	$f(x)$
1	6
2	-3
3	6
4	2
5	-6

Find $f(x)$ at $x = 1.3$.

14. Consider the points

x	y
-1	9
0	26
3	56
4	29

(a) Find the natural cubic spline which fits this data and hence estimate the value of $y(1)$.

(b) Find the clamped cubic spline with conditions $S'(-1) = 1$ and $S'(4) = -1$.

(c) Find the extrapolated cubic spline.

15. Consider the points $(0, 1)$, $(1, 4)$, $(2, 0)$ and $(3, -2)$. Find

(a) the natural cubic spline.

(b) the clamped cubic spline with conditions $S'(0) = 2$ and $S'(3) = 2$.

(c) the extrapolated cubic spline.

(d) the parabolically terminated cubic spline.

(e) the curvature adjusted cubic spline with the second derivative boundary conditions $S''(0) = -1.5$ and $S''(3) = 3$.