Chapter 1

Non-Linear Programming

1.1 Preliminary Concepts

Let a model be

$$\begin{aligned}
\text{maximize} \quad & Z = f(x_1, x_2, \dots, x_n) \\
\text{subject to} \quad & g^1(x_1, x_2, \dots, x_n) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_1 \\
& g^2(x_1, x_2, \dots, x_n) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_2 \\
& \qquad \vdots \qquad\qquad\qquad \vdots \\
& g^m(x_1, x_2, \dots, x_n) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_m \\
& \qquad\qquad x_i \geq 0 \qquad i = 1, 2, \dots, n
\end{aligned}$$

If f or g^i or both of them have one or more non-linear expression (i.e., have a variable that has degree 2 or above) then this type of problems are called non-linear programming problem.
Matrix form

$$\begin{aligned}
\text{maximize} \quad & Z = f(\underline{x}) \\
\text{subject to} \quad & g^i(\underline{x}) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad b_i \qquad i = 1, 2, \dots, m \\
& \Rightarrow h^i(\underline{x}) \quad \{\leq, \text{ or } = \text{ or } \geq\} \quad 0 \qquad \text{where } h^i(\underline{x}) = g^i(\underline{x}) - b_i
\end{aligned}$$

1.1.1 Principal minor

Let $Q_{n\times n}$ a matrix. It's k-th order ($k \leq n$) principal minor is a matrix that is obtained from Q by removing $(n-k)$ corresponding rows and column.
Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

its principal minors are

$$\begin{aligned}
\text{Order 1}: & \begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 5 \end{pmatrix}, \begin{pmatrix} 9 \end{pmatrix} \\
\text{Order 2}: & \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \\
\text{Order 3}: & \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\end{aligned}$$

Leading principal minor: Remove last $n-k$ corresponding row and column.
Principal determinant:

$$\begin{aligned}
\text{Order 1}: & \begin{vmatrix} 1 \end{vmatrix}, \begin{vmatrix} 5 \end{vmatrix}, \begin{vmatrix} 9 \end{vmatrix} \\
\text{Order 2}: & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \\
\text{Order 3}: & \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}
\end{aligned}$$

1.1.2 Quadratic form

$Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x_i x_j$
e.g., $Q(x) = x_1^2 + 7x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$ is a quadratic form because every term has degree of two.

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 7 & 6 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 & 7x_2 & x_1 + 6x_2 + 2x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 7 & 6 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 7x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$$

So, $Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x_i x_j = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ (here $\mathbf{A}$ can be symmetric) where $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Again,

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 7 & 3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 7x_2^2 + 2x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$$

The quadratic form $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ is said to be

1. Positive definite: if $Q(x) > 0 \, \forall \underline{x} \neq 0$
2. Positive semi-definite: if $Q(x) \geq 0$ for all x such that there exists at least one $x \neq 0$ satisfying $Q(x) = 0$
3. Negative definite: $-Q(x) > 0$ or $Q(x) < 0$
4. Negative semi-definite: $-Q(x) \geq 0$ or $Q(x) \leq 0$
5. Indefinite: if quadratic form does not fall into above categories.

Example.

- Positive definite: $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2$
- Positive semi-definite: $Q(x) = (x_1 - x_2)^2 + 2x_3^2 \; x_1 = x_2$ and $x_3 = 0$
- Negative definite: $Q(x) = -x_1^2 - 3x_2^2$
- Indefinite: $Q(x) = x_1^2 - 3x_2^2$

The necessary and sufficient condition:

1. Positive definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if leading principal determinant > 0
2. Positive semi-definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if leading principal determinant ≥ 0
3. Negative definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if sign of $k-$th leading principal determinant $= (-1)^k$
4. Negative semi-definite: $Q(x) = \mathbf{X}^T \mathbf{A}\, \mathbf{X}$ if sign of $k-$th leading principal determinant $= (-1)^k$ or zero
5. Indefinite: there must be two opposite sign in diagonal.

Example. $Q(x) = 2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3$
Here, $A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

The leading principal determinants of A: $\begin{vmatrix} 2 \end{vmatrix} = 2$, $\begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} = 4$, $\begin{vmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 12$

So $Q(x)$ is positive definite.

1.1.3 Hessian matrix:

Let $f(\underline{x}) = f(x_1, x_2, \ldots, x_n)$ be a function that is continuous has double derivative. Then Hessian matrix of $f(\underline{x})$ is

$$H(\underline{x}) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

$$f(x_1, x_2, x_3) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_3} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2\,\partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_3\,\partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{pmatrix}$$

Hessian matrix is a symmetric matrix $[\because f_{ij} = f_{ji}]$.

Problem 1.1.1. Find the Hessian matrix of $f(x, y) = x^3 - 2xy - y^6$ at the point $(1, 2)$

Solution.

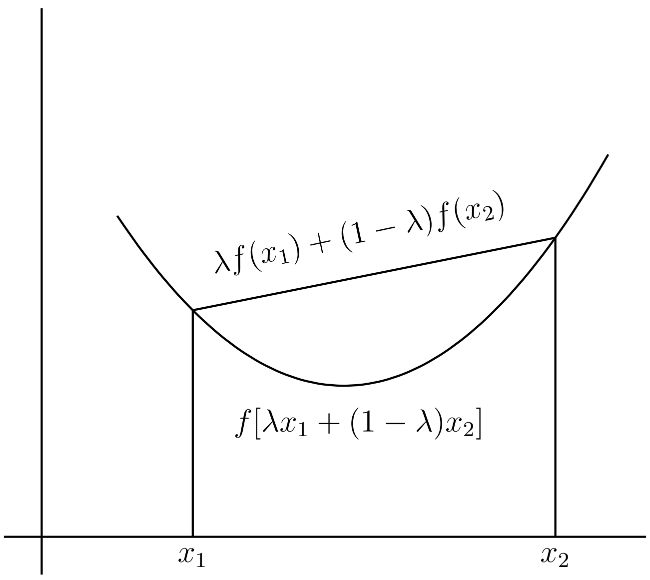
$$\frac{\partial f}{\partial x} = 3x^2 - 2y \qquad \frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial f}{\partial x\,\partial y} = -2 \qquad \frac{\partial f}{\partial y\,\partial x} = -2 \qquad \frac{\partial^2 f}{\partial y^2} = -30y^4$$

$$H = \begin{pmatrix} 6x & -2 \\ -2 & -30y^4 \end{pmatrix}$$

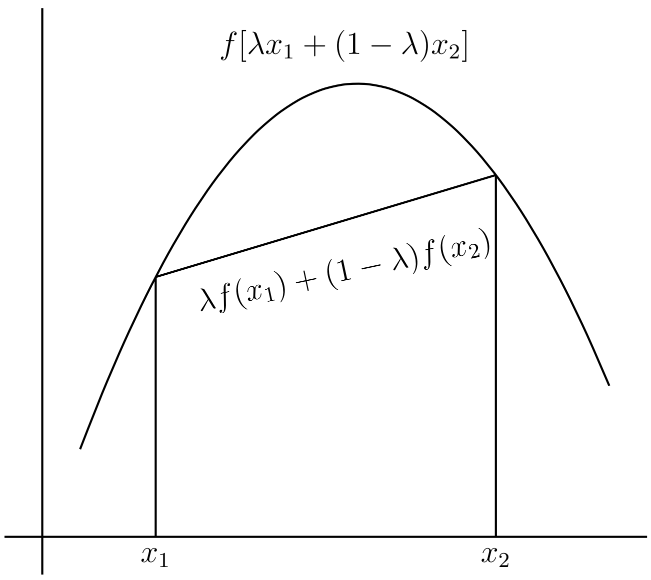
$$H_{(1,2)} = \begin{pmatrix} 6 & -2 \\ -2 & -480 \end{pmatrix}$$

1.1.4 Convex and concave function:

If $f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ where x_1, x_2 are points and $0 \leq \lambda \leq 1$, then $f(x)$ is a convex function. $f(x)$ is strictly convex if $f[\lambda x_1 + (1-\lambda)x_2] < \lambda f(x_1) + (1-\lambda) f(x_2)$ where $0 < \lambda < 1$.



Convex function



Concave function

If $f[\lambda x_1 + (1-\lambda)x_2] \geq \lambda f(x_1) + (1-\lambda) f(x_2)$ where x_1, x_2 are points and $0 \leq \lambda \leq 1$, then $f(x)$ is a concave function. $f(x)$ is strictly concave if $f[\lambda x_1 + (1-\lambda)x_2] > \lambda f(x_1) + (1-\lambda) f(x_2)$ where $0 < \lambda < 1$.
Tests for a function to be convex/concave: $H(x)$ is the Hessian matrix of $f(x)$

- Convex: if $H(x)$ is positive semi-definite
- Strictly convex: if $H(x)$ is positive definite
- Concave: if $H(x)$ is negative semi-definite
- Strictly concave: if $H(x)$ is negative definite

Problem 1.1.2. Test the convexity of the function $f(\underline{x}) = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 6x_1 - 4x_2 - 2x_3$ at the point (x_1, x_2, x_3).

Solution. The given function is $f(\underline{x}) = 3x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 6x_1 - 4x_2 - 2x_3$

$$\begin{array}{l|l|l}
\dfrac{\partial f}{\partial x_1} = 6x_1 - 2x_2 - 2x_3 - 6 & \dfrac{\partial^2 f}{\partial x_1^2} = 6 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} = -2 = \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} \\
\dfrac{\partial f}{\partial x_2} = 4x_2 - 2x_1 + 2x_3 - 4 & \dfrac{\partial^2 f}{\partial x_2^2} = 4 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_3} = -2 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_1} \\
\dfrac{\partial f}{\partial x_3} = 3x_3 - 2x_1 + 2x_2 - 2 & \dfrac{\partial^2 f}{\partial x_3^2} = 2 & \dfrac{\partial^2 f}{\partial x_2\,\partial x_3} = 2 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_2}
\end{array}$$

Then the Hessian matrix of the function $f(x)$ is

$$H(\underline{x}) = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$

Since,

(i) $H(\underline{x})$ is symmetric

(ii) All diagonal elements are positive

(iii) The leading principal minor determinant $\begin{vmatrix} 6 \end{vmatrix} > 0$, $\begin{vmatrix} 6 & -2 \\ -2 & 4 \end{vmatrix} = 20 > 0$, $\begin{vmatrix} H(\underline{x}) \end{vmatrix} = 16 > 0$

So, $H(\underline{x})$ is positive definite for all values of (x_1, x_2, x_3) which implies that $f(\underline{x})$ is strictly convex function.

Problem 1.1.3. Test the convexity of the function $f(\underline{x}) = -x_1^2 - 3x_2^2 - 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$.

Solution. The given function is $f(\underline{x}) = -x_1^2 - 3x_2^2 - 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$

$$\begin{array}{l|l|l}
\dfrac{\partial f}{\partial x_1} = -2x_1 + 4x_2 + 2x_3 & \dfrac{\partial^2 f}{\partial x_1^2} = -2 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} = 4 = \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} \\
\dfrac{\partial f}{\partial x_2} = -6x_2 + 4x_1 + 4x_3 & \dfrac{\partial^2 f}{\partial x_2^2} = -6 & \dfrac{\partial^2 f}{\partial x_1\,\partial x_3} = 2 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_1} \\
\dfrac{\partial f}{\partial x_3} = -4x_3 + 2x_1 + 4x_2 & \dfrac{\partial^2 f}{\partial x_3^2} = -4 & \dfrac{\partial^2 f}{\partial x_2\,\partial x_3} = 4 = \dfrac{\partial^2 f}{\partial x_3\,\partial x_2}
\end{array}$$

Then the Hessian matrix of the function $f(x)$ is

$$H(\underline{x}) = \begin{pmatrix} -2 & 4 & 2 \\ 4 & -6 & 4 \\ 2 & 4 & -4 \end{pmatrix}$$

Since,

(i) $H(\underline{x})$ is symmetric

(ii) All diagonal elements are negative

(iii) The leading principal minor determinant $\begin{vmatrix} -2 \end{vmatrix} < 0$, $\begin{vmatrix} -2 & 4 \\ 4 & -6 \end{vmatrix} = -4 < 0$, $\begin{vmatrix} H(\underline{x}) \end{vmatrix} = 136 > 0$

So, $H(\underline{x})$ is an indefinite matrix which implies that $f(\underline{x})$ is neither convex nor concave function.

1.2 Unconstrained Optimization:

To find stationary point: $\nabla f(\underline{x}) = 0$. Let $\underline{x}_0$ is a stationary point.

(i) If $H(\underline{x}_0$ is positive definite then x_0 is a minimum point.

(ii) If $H(\underline{x}_0$ is negative definite then x_0 is a maximum point.

- If $H(\underline{x}_0)$ is indefinite then $x = x_0$ is a point of inflection (saddle point).

Problem 1.2.1. Determine the local maximum or minimum (if any) of the function $f(\underline{x}) = x_1^2 + 2x_2^2 + x_3^2 + x_1x_2 - 2x_3 - 7x_1 + 12$.

Solution. The given function is $f(\underline{x}) = x_1^2 + 2x_2^2 + x_3^2 + x_1x_2 - 2x_3 - 7x_1 + 12$.
The necessary condition to obtain the maximum or minimum is $\nabla f(\underline{x}) = 0$.
Now,

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= 0 \qquad \Rightarrow 2x_1 + x_2 - 7 = 0 \\ \frac{\partial f}{\partial x_2} &= 0 \qquad \Rightarrow 4x_2 + x_1 = 0 \\ \frac{\partial f}{\partial x_3} &= 0 \qquad \Rightarrow 2x_3 - 2 = 0 \end{aligned}$$

Solving these three equation we get $x_1 = 4$, $x_2 = -1$, $x_3 = 1$ i.e., $\underline{x}_0 = (x_1^0,\, x_2^0,\, x_3^0) = (4,\, -1, 1)$.
For the sufficient condition, let us find the Hessian matrix $H(\underline{x}^0)$.

$$H(\underline{x}^0) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Since,

(i) $H(\underline{x}_0)$ is symmetric.

(ii) All diagonal elements are positive.

(iii) The leading principal minor determinants are $\begin{vmatrix} 2 \end{vmatrix} > 0$, $\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7 > 0$, $\left|H(\underline{x})\right| = 14 > 0$

Thus, $H(\underline{x}_0)$ is positive definite, so the function $f(\underline{x})$ is strictly convex. Hence, $f(\underline{x})$ is minimum at $x^0 = (4, -1, 1)$.

1.3 Constrained Problems

1.3.1 Lagrangian Method

We can only use this method if the constraints are equation i.e., of $=$ type.
Consider the problem

$$\begin{gathered} \text{optimize } f(\underline{x}) \\ \text{subject to } \underline{g}(\underline{x}) = 0 \end{gathered}$$

where $\underline{x} = (x_1,\, x_2, \ldots, x_n)$ and $\underline{g} = (g_1,\, g_2, \ldots, g_m)^T$. The functions $f(\underline{x})$ and $g_i(\underline{x})$, $i = 1,\, 2 \ldots, m$ are assumed to be twice differentiable.
Let $L(\underline{x},\, \underline{\lambda}) = f(\underline{x}) - \underline{\lambda}\underline{g}(\underline{x})$. Here $L =$ Lagrangian function, $\underline{\lambda} =$ Lagrangian multipliers.
The necessary conditions for determining the stationary points of $f(\underline{x})$ are subject to $\underline{g}(\underline{x}) = 0$ is given by

$$\frac{\partial L}{\partial \underline{x}} = 0 \qquad \frac{\partial L}{\partial \underline{\lambda}} = 0$$

To establish the sufficient condition:
Define

$$H^B = \left(\begin{array}{c|c} O & P \\ \hline P^T & Q \end{array}\right)_{(m+n)+(m+n)}$$

where

$$P = \begin{pmatrix} \nabla g_1(\underline{x}) \\ \nabla g_2(\underline{x}) \\ \vdots \\ \nabla g_m(\underline{x}) \end{pmatrix}_{m \times n} \qquad \text{and } Q = \left(\frac{\partial^2 L(\underline{x},\, \underline{\lambda})}{\partial x_i\, \partial x_j}\right)_{n \times n} \quad \text{for all } i \text{ and } j$$

$O =$ Null matrix whose order is adjusted to make H^B a square matrix $\quad H^B =$ Bordered Hessian matrix

Given stationary point $(\underline{x}^0, \underline{\lambda}^0)$ for the Lagrangian function $L(\underline{x},\, \underline{\lambda})$ and bordered Hessian matrix H^B evaluated at $(\underline{x}^0, \underline{\lambda}^0)$, then $\underline{x}^0$ is

1. a *maximum* point, if starting with the principal minor determinants of order $(2m + 1)$ the last $(n - m)$ principal minor determinants of H^B form an alternating sign pattern starting with $(-1)^{m+1}$.

2. a *minimum* point, if starting with the principal minor determinants of order $(2m+1)$ the last $(n-m)$ principal minor determinants of H^B have the sign of $(-1)^m$.

Problem 1.3.1. Solve the following problem by using Lagrangian method.

$$\begin{aligned} \text{minimize} \quad & f(\underline{x}) = x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} \quad & x_1 + x_2 + 3x_3 = 2 \\ & 5x_1 + 2x_2 + x_3 = 5 \end{aligned}$$

Solution. Suppose that

$$\begin{aligned} f(x_1,\, x_2,\, x_3) &= x_1^2 + x_2^2 + x_3^2 \\ g_1(x_1,\, x_2,\, x_3) &= x_1 + x_2 + 3x_3 - 2 = 0 \\ g_2(x_1,\, x_2,\, x_3) &= 5x_1 + 2x_2 + x_3 - 5 = 0 \end{aligned}$$

$$L(\underline{x}, \underline{\lambda}) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

where

$$\underline{x} = (x_1,\, x_2,\, x_3) \quad \text{and} \quad \underline{\lambda} = (\lambda_1,\, \lambda_2)$$

The necessary condition:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda = 0 \tag{1.1}$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda = 0 \tag{1.2}$$
$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - 2\lambda = 0 \tag{1.3}$$
$$\frac{\partial L}{\partial \lambda_1} = -x_1 - x_2 - 3x_3 + 2 = 0 \tag{1.4}$$
$$\frac{\partial L}{\partial \lambda_2} = -5x_1 - 2x_2 - x_3 + 5 = 0 \tag{1.5}$$

From (1.1) and (1.2)

$$2(x_1 - x_2) = 3\lambda_2 \quad \Rightarrow \lambda_2 = \frac{2}{3}(x_1 - x_2) \quad \text{and} \quad \lambda_1 = \frac{1}{3}(-4x_1 + 10x_2) \tag{1.6}$$

From (1.3) and (1.6)

$$2x_3 - 3 \times \frac{1}{3}(-4x_1 + 10x_2) - \frac{2}{3}(x_1 - x_2) = 0$$
$$\Rightarrow 5x_1 - 14x_2 + 3x_3 = 0 \tag{1.7}$$

From (1.4) and (1.5)

$$3x_1 - 5x_3 = 1 \tag{1.8}$$

From (1.4) and (1.7)

$$4x_1 - 15x_2 = 2 \tag{1.9}$$

Solving (1.7), (1.8) and (1.9) we get

$$x_1 = \frac{37}{46}, \qquad x_2 = \frac{16}{46}, \qquad x_3 = \frac{13}{46}$$
$$\therefore \lambda_1 = \frac{4}{46}, \qquad \lambda_2 = \frac{14}{46}$$

So, the stationary point is given by

$$\underline{x}^0 = \left(\frac{37}{46},\, \frac{16}{46},\, \frac{13}{46}\right) \quad \underline{\lambda}^0 = \left(\frac{4}{46},\, \frac{14}{46}\right)$$

For the sufficient condition the Lagrangian function:

$$\begin{aligned} &\frac{\partial^2 L}{\partial x_1^2} = 2 && \frac{\partial^2 L}{\partial x_1\, \partial x_2} = 0 && \frac{\partial^2 L}{\partial x_1\, \partial x_3} = 0 \\ &\frac{\partial^2 L}{\partial x_2\, \partial x_1} = 0 && \frac{\partial^2 L}{\partial x_2^2} = 2 && \frac{\partial^2 L}{\partial x_2\, \partial x_3} = 0 \\ &\frac{\partial^2 L}{\partial x_3\, \partial x_1} = 0 && \frac{\partial^2 L}{\partial x_3\, \partial x_2} = 0 && \frac{\partial^2 L}{\partial x_3^2} = 2 \end{aligned}$$

$$\therefore Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\nabla g_1(\underline{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}$$
$$\nabla g_2(\underline{x}) = \begin{pmatrix} \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix} \qquad P^T = \begin{pmatrix} 1 & 5 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$\therefore \text{ Bordered Hessian matrix, } H^B(\underline{x}^0, \underline{\lambda}^0) = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ \hline 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{array}\right) \quad \text{taking } O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Here, $m = 2$, $n = 3$ $\therefore 2m + 1 = 5$. So, $\left|H^B(x^0,\, \lambda^0)\right| = 460 > 0$ i.e., having the sign $(-1)^2$. So this point is sufficient.

Problem 1.3.2. Consider the problem

$$\begin{aligned} &\text{minimize} \quad Z = x_1^2 + x_2^2 + x_3^2 \\ &\text{subject to} \quad 4x_1 + x_2^2 + 2x_3 - 14 = 0 \end{aligned}$$

1.3.2 Inequality Constraints [Karush-Kuhn-Tucker (KKT) conditions]

Problem 1.3.3. Use KKT conditions to solve

$$\begin{aligned} \text{maximize} \quad & f(\underline{x}) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & 2x_1 + 3x_2 \le 12 \\ & x_1,\, x_2,\, x_3 \ge 0 \end{aligned}$$

Solution. We have

$$\begin{aligned} f(\underline{x}) &= -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 \\ h^1(\underline{x}) &= x_1 + x_2 - 2 \\ h^2(\underline{x}) &= 2x_1 + 3x_2 - 12 \end{aligned}$$

The KKT conditions for maximization problem are

$$\begin{aligned} \nabla f(\underline{x}) - \underline{\lambda} \nabla \underline{h}(\underline{x}) &= 0 \\ \lambda_i h^i(\underline{x}) &= 0 \\ h^i(\underline{x}) &\le 0 \\ \underline{\lambda} &\ge 0 \end{aligned}$$

Applying these conditions we get,

$$-2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0 \tag{1.10}$$
$$-2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0 \tag{1.11}$$
$$-2x_3 = 0 \tag{1.12}$$
$$\lambda_1(x_1 + x_2 - 2) = 0 \tag{1.13}$$
$$\lambda_2(2x_1 + 3x_2 - 12) = 0 \tag{1.14}$$
$$x_1 + x_2 \le 2 \tag{1.15}$$
$$2x_1 + 3x_2 \le 12 \tag{1.16}$$
$$x_1,\, x_2,\, x_3 \ge 0 \;\; \lambda_1,\, \lambda_2 \ge 0 \tag{1.17}$$

Now these arise following four cases:

Case 1: If $\lambda_1 = 0$, $\lambda_2 = 0$, then from (1.10), (1.11) and (1.12) we get,

$$\begin{aligned} -2x_1 + 4 &= 0 \\ -2x_2 + 6 &= 0 \\ -2x_3 &= 0 \end{aligned}$$

Solving these equations we get $x_1 = 2$, $x_2 = 3$ and $x_3 = 0$. This solution violates the equation (1.15) and (1.16). So, it is rejected.

Case 2: If $\lambda_1 = 0$, $\lambda_2 \neq 0$, then from (1.14) we get

$$2x_1 + 3x_2 - 12 = 0 \tag{1.18}$$

and from (1.10) and (1.11) we get

$$\begin{aligned} -2x_1 + 4 - 2\lambda_2 &= 0 \\ -2x_2 + 6 - 3\lambda_2 &= 0 \end{aligned}$$

By manipulating these two equation we get

$$\begin{aligned} 6x_1 - 4x_2 &= 0 \\ \Rightarrow x_1 &= \frac{2}{3}x_2 \end{aligned} \tag{1.19}$$

From (1.18) and (1.19) we get $x_1 = \frac{24}{13}$, $x_2 = \frac{36}{13}$ and from (1.12) we get $x_3 = 0$. This solution violates the inequality (1.15). This is also rejected.

Case 3: If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, then from (1.13), (1.14) we get,

$$\begin{aligned} x_1 + x_2 - 2 &= 0 \\ -2x_1 + x_2 - 12 &= 0 \end{aligned}$$

Solving these equations we get $x_1 = -6$, $x_2 = 8$. This solution violates the inequality (1.17). So, this solution is also rejected.

Case 4: If $\lambda_1 \neq 0$, $\lambda_2 = 0$, then from (1.13) we get

$$x_1 + x_2 - 2 = 0$$

and from (1.10) and (1.11) we get

$$-2x_1 + 2x_2 - 2 = 0$$

By solving these two equation we get $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{2}$
Again from (1.10) we get $\lambda_1 = 3$ and (1.12) we get $x_3 = 0$.
Observe that the solution $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{2}$, $x_3 = 0$ and $\lambda_1 = 3$, $\lambda_2 = 0$ satisfies all the KKT conditions.

So the optimum solution of the given non-linear programming problem is

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{3}{2}, \quad x_3 = 0 \qquad \text{and} \qquad f(\underline{x})_{\max} = \frac{17}{2}$$

For sufficient condition the objective function and constraints must be concave functions.

$$f(\underline{x}) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

$$H(\underline{x}^0) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The leading principal determinants are -2, 4, -8. So $H(\underline{x}^0)$ is negative definite and hence the function is concave. Here the constraints are linear, so they are also concave. Hence, this point is sufficient.

Remark. Non negativity is not mandatory for non-linear problems.

Let a problem be

$$\begin{aligned} \max \text{ or } \min\ Z &= f(\underline{x}) \\ \text{subject to } g_i(\underline{x}) &\leq 0 \quad i = 1, 2, \ldots, r \\ g_i(\underline{x}) &\geq 0 \quad i = r+1, r+2, \ldots, p \\ g_i(\underline{x}) &= 0 \quad i = p+1, p+2, \ldots, m \end{aligned}$$

Sense of Optimization	Required Conditions		
	$f(\underline{x})$	$g_i(\underline{x})$	λ_i
Maximization	Concave	Convex	$\geq 0 \quad (1 \leq i \leq r)$
		Concave	$\leq 0 \quad (r+1 \leq i \leq p)$
		Linear	Unrestricted $(p+1 \leq i \leq m)$
Minimization	Convex	Convex	$\leq 0 \quad (1 \leq i \leq r)$
		Concave	$\geq 0 \quad (r+1 \leq i \leq p)$
		Linear	Unrestricted $(p+1 \leq i \leq m)$