Chapter 1

Applications Of Laplace Transform

1.1 Applications To Differential Equations

1.1.1 Ordinary Differential Equations With Constant Coefficients

The Laplace transform is useful in solving linear ordinary differential equations with constant coefficients. For example, suppose we wish to solve the second order linear differential equation

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + \alpha \frac{\mathrm{d}Y}{\mathrm{d}t} + \beta Y = F(t) \qquad \text{or} \qquad Y'' + \alpha Y' + \beta Y = F(t) \tag{1.1}$$

where α and β are constants, subject to the initial or boundary conditions

$$Y(0) = A, \quad Y'(0) = B \tag{1.2}$$

where A and B are given constants. On taking the Laplace transform of both sides of (1.1) and using (1.2), we obtain an algebraic equation for determination of $\mathcal{L}\{Y(t)\} = y(s)$. The required solution is then obtained by finding the inverse Laplace transform of $y(s)$. The method is easily extended to higher order differential equations.

Problem 1.1.1. Solve $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$.

Solution. Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$\begin{aligned}
&\mathcal{L}\{Y''\} + \mathcal{L}\{Y\} = \mathcal{L}\{t\} \\
\Rightarrow &s^2 y - sY(0) - Y'(0) + y = \frac{1}{s^2} \\
\Rightarrow &s^2 y - s - 2 + y = \frac{1}{s^2}
\end{aligned}$$

Then

$$\begin{aligned}
y = \mathcal{L}\{Y\} &= \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1} \\
&= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1} \\
&= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}
\end{aligned}$$

and

$$Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\} = t + \cos t - 3\sin t$$

Check: $Y = t + \cos t - 3\sin t$, $Y' = 1 - \sin t - 3\cos t$, $Y'' = -\cos t + 3\sin t$. Then $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$ and the function obtained is the required solution.

Problem 1.1.2. Solve $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$.

Solution. We have,

$$\begin{aligned}
&\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\left\{e^{2t}\right\} \\
\Rightarrow&\{s^2y - sY(0) - Y'(0)\} - 3\{sy - Y(0)\} + 2y = \frac{4}{s-2} \\
\Rightarrow&\{s^2y + 3s - 5\} - 3\{sy + 3\} + 2y = \frac{4}{s-2} \\
\Rightarrow&(s^{-}3s + 2)y + 3s - 14 = \frac{4}{s-2} \\
\Rightarrow&y = \frac{4}{(s^2 - 3s + 2)(s-2)} + \frac{14 - 3s}{s^2 - 3s + 12} \\
\Rightarrow&y = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} \\
\Rightarrow&y = \frac{-7}{s-1}\frac{4}{s-2} + \frac{4}{(s-2)^2}
\end{aligned}$$

Thus,

$$Y = \mathcal{L}\left\{\frac{-7}{s-1}\frac{4}{s-2} + \frac{4}{(s-2)^2}\right\} = -7e^t + 4e^{2t} + 4te^{2t}$$

which can be verified as the solution.

1.1.2 Ordinary Differential Equations With Variable Coefficients

The Laplace transform can also be used in solving some ordinary differential equations in which the coefficients are variable. A particular differential equation where the method proves useful is one in which the terms have the form

$$t^m Y^{(n)}(t)$$

the Laplace transform of which is

$$(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}\, s^m}\mathcal{L}\left\{Y^{(n)}(t)\right\}$$

Problem 1.1.3. Solve $tY'' + Y' + 4tY = 0$, $Y(0) = 3$, $Y'(0) = 0$.

Solution. We have,

$$\mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} + \mathcal{L}\{4tY\} = 0$$

or,

$$-\frac{\mathrm{d}}{\mathrm{d}\, s}\left\{s^2y - sY(0) - Y'(0)\right\} + \{sy - Y(0)\} - 4\frac{\mathrm{d}\, y}{\mathrm{d}\, s} = 0$$

i.e.,

$$(s^2 + 4)\frac{\mathrm{d}\, y}{\mathrm{d}\, s} + sy = 0$$

Then

$$\frac{\mathrm{d}\, y}{y} + \frac{s\,\mathrm{d}\, s}{s^2 + 4} = 0$$

and integrating

$$\ln y + \frac{1}{2}\ln(s^2 + 4) = c_1 \quad \text{or,} \quad y = \frac{c}{\sqrt{s^2 + 4}}$$

Inverting, we find

$$Y = cJ_0(2t)$$

To determine c we note that $Y(0) = cJ_0(0) = c = 3$. Thus,

$$Y = 3J_0(2t)$$

Problem 1.1.4. Solve $tY'' + 2Y' + tY = 0$, $Y(0+) = 1$, $Y'(\pi) = 0$.

Solution. Let $Y(0+) = c$. Then taking the Laplace transform of each term

$$-\frac{\mathrm{d}}{\mathrm{d}\,s}\{s^2y - sY(0+) - Y'(0+)\} + 2\{sy - Y(0+)\} - \frac{\mathrm{d}}{\mathrm{d}\,s}y = 0$$

or

$$-s^2y' - 2sy + 1 + 2sy - 2 - y' = 0$$

i.e.,

$$-(s^2+1)y' - 1 = 0 \quad \text{or} \quad y' = \frac{-1}{s^2+1}$$

Integrating

$$y = -\tan^{-1} s + A$$

Since $y \to 0$ as $s \to \infty$, we must have $A = \pi/2$. Thus,

$$y = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1}\frac{1}{s}$$

Then,

$$Y = \mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} = \frac{\sin t}{t}$$

1.1.3 Partial Differential Equations

Problem 1.1.5. Given the function $U(x,t)$ defined for $a \le x \le b$, $t > 0$. Find

(a) $\mathcal{L}\left\{\dfrac{\partial U}{\partial t}\right\} = \displaystyle\int_0^\infty e^{-st}\frac{\partial U}{\partial t}\,\mathrm{d}\,t$

(b) $\mathcal{L}\left\{\dfrac{\partial U}{\partial x}\right\} = \displaystyle\int_0^\infty e^{-st}\frac{\partial U}{\partial x}\,\mathrm{d}\,t$

assuming suitable restrictions on $U = U(x,t)$.

Solution.

(a) Integrating by parts, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} &= \int_0^\infty e^{-st}\frac{\partial U}{\partial t}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\int_0^P e^{-st}\frac{\partial U}{\partial t}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\left\{e^{-st}U(x,t)\Big|_0^P + s\int_0^P e^{-st}U(x,t)\,\mathrm{d}\,t\right\} \\
&= s\int_0^\infty e^{-st}U(x,t)\,\mathrm{d}\,t - U(x,0) \\
&= su(x,s) - U(x,0) \\
&= su - U(x,0)
\end{aligned}$$

where $u = u(x,s) = \mathcal{L}\{U(x,t)\}$.

We have assumed that $U(x,t)$ satisfies the restrictions of sectionally continuous in finite interval, when regraded as a function of t.

(b) We have, using Leibniz's rule for differentiating under the integral sign,

$$\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial U}{\partial x}\,\mathrm{d}\,t = \frac{\mathrm{d}}{\mathrm{d}\,x}\int_0^\infty e^{-st}U\,\mathrm{d}\,t = \frac{\mathrm{d}\,u}{\mathrm{d}\,x}$$

Problem 1.1.6. Referring to problem 1.1.5, show that

(a) $\mathcal{L}\left\{\dfrac{\partial^2 U}{\partial t^2}\right\} = s^2 u(x,s) - sU(x,0) - U_t(x,0)$

(b) $\mathcal{L}\left\{\dfrac{\partial^2 U}{\partial x^2}\right\} = \dfrac{\mathrm{d}^2 u}{\mathrm{d}\, x^2}$

where $U_t(x,0) = \left.\dfrac{\partial U}{\partial t}\right|_{t=0}$ and $u = u(x,s) = \mathcal{L}\{U(x,t)\}$.

Solution. Let $V = \partial U/\partial t$. Then as in part (a) of Problem 1.1.5, we have

$$\begin{aligned}\mathcal{L}\left\{\frac{\partial^2 U}{\partial t^2}\right\} = \mathcal{L}\left\{\frac{\partial V}{\partial t}\right\} &= s\mathcal{L}\{V\} - V(x,0) \\ &= s\left[s\mathcal{L}\{U\} - U(x,0)\right] - U_t(x,0) \\ &= s^2 u - sU(x,0) - U_t(x,0)\end{aligned}$$

Problem 1.1.7. Find the solution of

$$\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U, \qquad U(x,0) = 6e^{-3x}$$

which is bounded for $x > 0$, $t > 0$.

Solution. Taking the Laplace transform of the given partial differential equation with respect to t and using Problem 1.1.5, we find

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, x} = 2\{su - U(x,0)\} + u$$

or,

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, x} - (2s+1)u = -12e^{-3x} \tag{1.3}$$

from the given boundary conditions. Note that the Laplace transformation has transformed the partial differential equation into an ordinary differential equation (1.3).

To solve (1.3) multiply both sides by the integrating factor $e^{\int -(2s+1)\,\mathrm{d}\, x} = e^{-(2s+1)x}$. Then (1.3) can be written

$$\frac{\mathrm{d}}{\mathrm{d}\, x}\left\{ue^{-(2s+1)x}\right\} = -12e^{-(2x+4)x}$$

Integration yields

$$ue^{-(2s+1)x} = \frac{6}{s+2}e^{-(2s+4)x} + c \qquad \text{or,} \qquad u = \frac{6}{s+2}e^{-3x} + ce^{(2s+1)x}$$

Now since $U(x,t)$ must be bounded as $x \to \infty$, we must have $u(x,s)$ also bounded as $x \to \infty$ and it follows that we must choose $c = 0$. Then

$$u = \frac{6}{s+2}e^{-3x}$$

and so, on taking the inverse, we find

$$U(x,t) = 6e^{-2t-3x}$$

This is easily checked as the required solution.

Problem 1.1.8. Solve $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}$, $U(x,0) = 3\sin 2\pi x$, $U(0,t) = 0$, $U(1,t) = 0$ where $0 < x < 1$, $t > 0$.

Solution. Taking the Laplace transform of the partial differential equation using Problem 1.1.5 and 1.1.6, we find

$$su - U(x,0) = \frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2} \qquad \text{or} \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2} - su = -3\sin 2\pi x \tag{1.4}$$

where $u = u(x,s) = \mathcal{L}\{U(x,t)\}$. The general solution of (1.4) in

$$u = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{3}{s + 4\pi^2}\sin 2\pi x \tag{1.5}$$

Taking the Laplace transform of those boundary conditions which involve t, we have

$$\mathcal{L}\{U(0,t)\} = u(0,s) = 0 \qquad \text{and} \qquad \mathcal{L}\{U(1,t)\} = u(1,s) = 0 \tag{1.6}$$

Using the first condition $[u(0,s) = 0]$ of (1.6) in (1.5), we have

$$c_1 + c_2 = 0 \tag{1.7}$$

Using the second condition $[u(1,s) = 0]$ of (1.6) in (1.5), we have

$$c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \tag{1.8}$$

From (1.7) and (1.8) we find $c_1 = 0$, $c_2 = 0$ and so (1.5) becomes

$$u = \frac{3}{s + 4\pi^2}\sin 2\pi x \tag{1.9}$$

from which we obtain on inversion

$$U(x,t) = 3e^{-4\pi^2 t}\sin 2\pi x \tag{1.10}$$

This problem has an interesting physical interpretation. If we consider a solid bounded by the infinite plane faces $x = 0$ and $x = 1$, the equation

$$\frac{\partial U}{\partial t} = k\,\frac{\partial^2 U}{\partial x^2}$$

is the *equation for heat conduction* in this solid where $U = U(x,t)$ is the *temperature* at any plane face x at any time t and k is a constant called the *diffusivity*, which depends on the material of the solid. The boundary conditions $U(0,t) = 0$ and $U(1,t) = 0$ indicate that the temperatures at $x = 0$ and $x = 1$ are kept at temperature zero, while $U(x,0) = 3\sin 2\pi x$ represents the initial temperature everywhere in $0 < x < 1$. The result (1.10) then is the temperature everywhere in the solid at time $t > 0$.

1.2 Applications To Integral Equations

1.2.1 Integral Equations

An *integral equation* is an equation having the form

$$Y(t) = F(t) + \int_a^b K(u,t)Y(u)\,\mathrm{d}\,u$$

where $F(t)$ and $K(u,t)$ are known, a and b are either given constants or functions of t, and the function $Y(t)$ which appears under the integral sign is to be determined.

The function $K(u,t)$ is often called the *kernel* of the integral equation. If a and b are constants, the equation is often called a *Fredholm integral equation*. If a is a constant while $b = t$, it is called a *Volterra integral equation*.

It is possible to convert a linear differential equation into an integral equation.

Problem 1.2.1. Convert the differential equation

$$Y''(t) + 3Y'(t) + 2Y(t) = 4\sin t, \quad Y(0) = 1, \quad Y'(0) = -2$$

into an integral equation.

Solution. Integrating both sides of the given differential equation, we have

$$\int_0^t \{Y''(u) - 3Y'(u) + 2Y(u)\}\, \mathrm{d}\, u = \int_0^t 4\sin u \,\mathrm{d}\, u$$
$$\Rightarrow Y'(t) - Y'(0) - 3Y(t) + 3Y(0) + 2\int_0^t Y(u)\,\mathrm{d}\, u = 4 - 4\cos t$$

This becomes, using $Y'(0) = -2$ and $Y(0) = 1$

$$\Rightarrow Y'(t) - 3Y(t) + 2\int_0^t Y(u)\,\mathrm{d}\, u = -1 - 4\cos t$$

Integrating again from 0 to t as before, we find

$$\Rightarrow Y(t) - Y(0) - 3\int_0^t Y(u)\,\mathrm{d}\, u + 2\int_0^t (t-u)Y(u)\,\mathrm{d}\, u = -t - 4\sin t$$
$$\Rightarrow Y(t) + \int_0^t \{2(t-u) - 3\}\, Y(u)\,\mathrm{d}\, u = 1 - t - 4\sin t$$

1.2.2 Integral Equations Of Convolution Type

A special integral equation of importance in applications is

$$Y(t) = F(t) + \int_0^t K(t-u)Y(u)\,\mathrm{d}\, u$$

This equation is of *convolution type* and can be written as

$$Y(t) = F(t) + K(t) * Y(t)$$

Taking the Laplace transform of both sides, assuming $\mathcal{L}\{F(t)\} = f(s)$ and $\mathcal{L}\{K(t)\} = k(s)$ both exist, we find

$$y(s) = f(s) + k(s)y(s) \qquad \text{or} \qquad y(s) = \frac{f(s)}{1-k(s)}$$

The required solution may then be found by inversion.

Problem 1.2.2. Solve the integral equation $Y(t) = t^2 + \int_0^t Y(u)\sin(t-u)\,\mathrm{d}\, u$.

Solution. The integral equation can be written

$$Y(t) = t^2 + Y(t) * \sin t$$

Then taking the Laplace transform and using the convolution theorem, we find, if $y = \mathcal{L}\{Y\}$

$$y = \frac{2}{s^3} + \frac{y}{s^2+1}$$

solving,

$$\Rightarrow y = \frac{2(s^2+1)}{s^5}$$
$$\Rightarrow y = \frac{2}{s^3} + \frac{2}{s^5}$$

and so

$$\Rightarrow Y = 2\left(\frac{t^2}{2!}\right) + 2\left(\frac{t^4}{4!}\right) = t^2 + \frac{1}{12}t^4$$

This can be checked by direct substitution in the integral equation.

1.2.3 Applications of Integral Equation

A large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. Mathematical physics models, such as

- Diffraction problems
- Scattering in quantum mechanics
- Conformal mapping
- Water waves

Also contributed to the creation of integral equations.