Chapter 1

Transportation Problem

The transportation model is a special class of the linear programming problem. It deals with the situation in which a commodity is shipped from *sources* (e.g., factories) to *destinations* (e.g., warehouses). The objective is to determine the amounts shipped from each sources to each destination that minimize the total shipping cost while satisfying both the supply limits and the demand requirements. The model assumes that the shipping cost on a given route is directly proportional to the number of units shipped on that route.

Suppose that there are m sources and n destinations. Let a_i be the number of supply units available at source $i\,(i = 1, 2, \ldots, m)$ and let b_j be the number of demand units required at destination $j\,(j = 1, 2, \ldots, n)$. Let c_{ij} represent the unit transportation cost for transporting the units from source i to destination j. Then, if x_{ij} $(x_{ij} > 0)$ is the number of units shipped from source i to destination j, the problem is to determine the Transportation schedule so as to minimize the total transportation cost satisfying the supply and demand conditions.

Mathematically, the problem may be stated as follows:

$$
\begin{aligned}
\text{minimize} \quad & Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to} \quad & x_{i1} + x_{i2} + \cdots + x_{in} = a_i; \quad i = 1, 2, \ldots, m \quad \Rightarrow \sum_{j=1}^{n} x_{ij} = a_i \\
& x_{1j} + x_{2j} + \cdots + x_{mj} = b_i; \quad j = 1, 2, \ldots, n \quad \Rightarrow \sum_{i=1}^{m} x_{ij} = b_j \\
& x_{ij} \geq 0; \quad \text{for all } i \text{ and } j
\end{aligned}
$$

For a feasible solution to exist, it is necessary that total supply equals total requirement i.e.,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

This restriction causes one of the constraints to be redundant (and hence it can be deleted) so that the problem will have $(m + n - 1)$ independent constraints and $(m \times n)$ unknowns.

Note. The standard transportation problem has $(m+n)$ constraint, mn variables. In general, the number of basic variables in a basic feasible solution is given by the number of constraints. Nut in the TP, the number of variables that can take positive values is limited $(m + n - 1)$, since

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \qquad \text{and} \qquad \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$$

Also, we have that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ and hence, the transportation model have only $(m + n - 1)$ independent constraints.

1.1 Starting Basic Feasible Solution

To solve a standard TP, we shall use the following methods (for starting basic feasible solution):

1. North-west corner rule (NCR)
2. Least-cost rule (LCR)
3. Vogel's approximation rule/method (VAM)

Problem 1.1.1. Consider a transportation problem with 3 warehouses and 4 markets. The warehouse capacities are $a_1 = 3,\ a_2 = 7,\ a_3 = 5$. The market demands are $b_1 = 4,\ b_2 = 3,\ b_4 = 4$. The unit cost of shipping is given by the following table:

Markets

Warehouse	M_1	M_2	M_3	M_4	Supply
W_1	2	2	2	1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
Demand	4	3	4	4	15

Supply=Demand
$\therefore$ Standard TP

Solution. In this transportation problem,

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 15$$

So, it is a standard TP.
The transportation table is as follows:

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	$\cancel{3}$
W_2	1 $\boxed{10}$	3 $\boxed{8}$	3 $\boxed{5}$	$\boxed{4}$	$\cancel{7}$ $\cancel{6}$ $\cancel{3}$
W_3	$\boxed{7}$	$\boxed{6}$	1 $\boxed{6}$	4 $\boxed{8}$	$\cancel{5}$ $\cancel{4}$
Demand	$\cancel{4}$ $\cancel{1}$	$\cancel{3}$	$\cancel{4}$ $\cancel{1}$	4	

Here, we shall refrain from writing the variable names x_{ij}.
A basic feasible solution to this problem will have at most $(3+4-1) = 6$ positive variables.

1.1.1 North-west corner Rule

This rule generates a feasible solution with no more than $(m+n-1)$ basic variables.

Here x_{11} is selected as the 1st basic variable and is assigned a value as much as possible consistent with the supply and demand restriction.
Set $x_{11} = \min\{3, 4\} = 3$. We set the variable $x_{12} = x_{13} = x_{14} = 0$. The remaining demand of the market M_1 is 1 unit. The next variable x_{21} and $x_{21} = \min\{7, 1\} = 1$.
Similarly, $x_{22} = \min\{6, 3\} = 3$
In the same manner, finally we obtain the table-2.

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	3
W_2	1 $\boxed{10}$	3 $\boxed{8}$	3 $\boxed{5}$	$\boxed{4}$	7
W_3	$\boxed{7}$	$\boxed{6}$	1 $\boxed{6}$	4 $\boxed{8}$	5
Demand	4	3	4	4	

Table 1.1: tab-2

Since we have six positive variables in the solution, so for the initial basic feasible solution, the total cost is given by

$$Z = 3 \cdot 2 + 1 \cdot 10 + 3 \cdot 8 + 3 \cdot 5 + 1 \cdot 6 + 4 \cdot 8 = 93$$

1.1.2 Least Cost Rule (LCR)

In this rule, the variable with the least shipping cost will be chosen as the basic variable.

According to this rule, in our present problem x_{14} must be chosen as the 1st basic variable, and $x_{14} = \min\{3, 4\} = 3$.

	M_1	M_2	M_3	M_4	Supply
W_1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	3 $\boxed{1}$	$\cancel{3}$
W_2	2 $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	1 $\boxed{4}$	$\cancel{7}\ \cancel{6}\ \cancel{2}$
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	$\cancel{5}\ \cancel{2}$
Demand	$\cancel{4}\ \cancel{2}$	$\cancel{3}$	$\cancel{4}$	$\cancel{4}\ \cancel{1}$	

Now, out of the remaining unassigned cells, the variable x_{24} has the least cost and $x_{24} = \min\{7, 1\} = 1$. In this way, finally, we obtain the following table which gives the initial basic feasible solution.

	M_1	M_2	M_3	M_4	Supply
W_1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	3 $\boxed{1}$	3
W_2	2 $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	1 $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

The total cost of transportation for the above basic feasible solution is

$$Z = 3 \cdot 1 + 2 \cdot 10 + 4 \cdot 5 + 1 \cdot 4 + 2 \cdot 7 + 3 \cdot 6 = 79$$

1.1.3 Vogel's Approximation Method (VAM)

In Vogel's approximation method, we compute a penalty for each row and column. Vogel defined the penalty as the absolute difference between the smallest and the next smallest cost in a row or a column. In two or more cells lie for the minimum cost, then the penalty is set to zero.

	M_1	M_2	M_3	M_4	Supply	Penalty 1st	2nd	3rd	4th
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	$\cancel{3}$	$(2-1)=1$			
W_2	$\boxed{10}$	$\boxed{8}$	3 $\boxed{5}$	4 $\boxed{4}$	$\cancel{7}\ \cancel{3}$	1	1	3	3
W_3	1 $\boxed{7}$	3 $\boxed{6}$	1 $\boxed{6}$	$\boxed{8}$	$\cancel{5}\ \cancel{4}\ \cancel{3}$	0	0	0	0
Demand	$\cancel{4}\ \cancel{1}$	3	$\cancel{4}\ \cancel{1}$	$\cancel{4}$					
1st Penalty	$7-2=5$	4	3	3					
2nd Penalty	3	2	1	4					
3rd Penalty	3	2	1						
4th Penalty		2	1						

Now, the row or column with the largest penalty is identified and the variable that has the smallest cost in that row or column is selected as the basic variable.

In this problem, column 1 has the largest penalty and so, the variable x_{11} will be selected as the 1st basic variable. We set $x + 11 = \min\{3, 4\} = 3$.

The 2nd set of penalties are shown in table. Applying the same procedure, the variable x_{24} is selected as basic variable and $x_{24} = \min\{7, 4\} = 4$.

Similarly, we take the 3rd penalty and so on. Finally, the is as follows:

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	3 $\boxed{5}$	4 $\boxed{4}$	7
W_3	1 $\boxed{7}$	3 $\boxed{6}$	1 $\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

The total cost of shipping is

$$Z = 3 \cdot 2 + 3 \cdot 5 + 4 \cdot 4 + 1 \cdot 7 + 3 \cdot 6 + 1 \cdot 6 = 68$$

Note. From the starting basic solution, we observe that the best starting solution is obtained by VAM having the least transportation cost of 68.

Remark. Vogel's approximation method (VAM) yields a very good initial solution, which, sometimes may be the optimal solution.

1.2 Improving the initial basic feasible solution

1.2.1 U-V method or MODI (Modified Distribution Method)

For any basic feasible solution find numbers u_i for warehouses i and market j such that

$$u_i + v_j = c_{ij} \qquad \text{for every basic } x_{ij}$$

These number can be positive, negative or zero. Then $\bar{c}_{ij} = c_{ij} - (u_i + v_j)$ for all non-basic variables x_{ij}

If all $\bar{c}_{ij}$ are non-negative (cost T.P.) then the current basic feasible solution is optimal. If not, there exists a non-basic variable x_{pq} such that $\bar{c}_{pq} = \min\{\bar{c}_{ij} < 0\}$ and x_{pq} is made as basic variable to improve the objective function.

	M_1	M_2	M_3	M_4	Supply
W_1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	3 $\boxed{1}$	3
W_2	2 $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	1 $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Table 1.2: Tab-1

To apply U-V method, we have to compute the numbers u_1, u_2, u_3, v_1, v_2, v_3, v_4 from the initial basic feasible solution table of LCR.

$$\begin{aligned}
u_1 + v_4 &= 1 \\
u_2 + v_1 &= 10 \\
u_2 + v_3 &= 5 \\
u_2 + v_4 &= 4 \\
u_3 + v_1 &= 7 \\
u_3 + v_2 &= 6
\end{aligned}$$

Let us suppose that

$$\begin{aligned}
u_1 &= 0 && \Rightarrow v_4 = 1 \\
u_2 &= 3 && \Rightarrow v_1 = 7 \\
u_2 &= 3 && \Rightarrow v_3 = 2 \\
u_2 &= 0 && \Rightarrow v_2 = 6
\end{aligned}$$

Now,

$$\begin{aligned}
\bar{c}_{11} &= c_{11} - (u_1 + v_1) = 2 - 7 = -5 \\
\bar{c}_{12} &= c_{12} - (u_1 + v_2) = 2 - (0 + 6) = -4 \\
\bar{c}_{13} &= c_{13} - (u_1 + v_3) = 2 - (0 + 2) = 0 \\
\bar{c}_{22} &= c_{22} - (u_2 + v_2) = -1 \\
\bar{c}_{33} &= c_{33} - (u_3 + v_3) = -4 \\
\bar{c}_{34} &= c_{34} - (u_3 + v_4) = 7
\end{aligned}$$

Since $\min\{\bar{c}_{ij} < 0\} = -5$. So, the non-basic variable x_{11} is introduced into the basis.

To determine the maximum increase in x_{11}. We assign an unknown non-negative value θ. We add or subtract θ from the basic variables so that the row sums and column sums are equal to the corresponding supply and demand respectively, which is shown in the table 2.

	M_1	M_2	M_3	M_4	Supply
W_1	θ $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$3-\theta$ $\boxed{1}$	3
W_2	$2-\theta$ $\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	$1+\theta$ $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Now θ is increased as long as the solution remains non-negative. In this problem, the maximum value of θ is 2, and the basic variable x_{21} is removed from the basis, i.e., x_{21} is replaced by x_{11}. The new basic feasible solution is given by the following table.

	M_1	M_2	M_3	M_4	Supply
W_1	2 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	1 $\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	4 $\boxed{5}$	3 $\boxed{4}$	7
W_3	2 $\boxed{7}$	3 $\boxed{6}$	$\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Now, we proceed as before to find the relative cost coefficients of non-basic variables.

$$
\begin{aligned}
u_1 + v_1 &= 2 \\
u_1 + v_4 &= 1 \\
u_2 + v_3 &= 5 \\
u_2 + v_4 &= 4 \\
u_3 + v_1 &= 7 \\
u_3 + v_2 &= 6
\end{aligned}
$$

Let

$$
\begin{aligned}
u_1 = 0 \quad & \Rightarrow v_1 = 2 \\
& \Rightarrow v_4 = 1 \\
u_2 = 3 \quad & \Rightarrow v_3 = 2 \\
u_2 = 5 \quad & \Rightarrow v_2 = 1
\end{aligned}
$$

Now,

$$
\begin{aligned}
\bar{c}_{12} &= 2 - (0 + 1) = 1 \\
\bar{c}_{13} &= 2 - (0 + 2) = 0 \\
\bar{c}_{21} &= 10 - (3 + 2) = 5 \\
\bar{c}_{22} &= 4 \\
\bar{c}_{33} &= -1 \\
\bar{c}_{34} &= 2
\end{aligned}
$$

Since the relative cost-coefficients of non-basic variable is x_{33} is -1. Hence, x_{33} is introduced as a basic variable at a non-negative θ. This produces the following change of the variables:

	M_1	M_2	M_3	M_4	Supply
W_1	$2+\theta$ $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$1-\theta$ $\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	$4-\theta$ $\boxed{5}$	$3+\theta$ $\boxed{4}$	7
W_3	$2-\theta$ $\boxed{7}$	3 $\boxed{6}$	θ $\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Since the maximum value of θ is 1 and x_{33} replace x_{14}.
The new basic feasible solution is given by the following table.

	M_1	M_2	M_3	M_4	Supply
W_1	3 $\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{1}$	3
W_2	$\boxed{10}$	$\boxed{8}$	3 $\boxed{5}$	4 $\boxed{4}$	7
W_3	$\boxed{7}$	3 $\boxed{6}$	1 $\boxed{6}$	$\boxed{8}$	5
Demand	4	3	4	4	

Now proceeding as above, we get

$$\begin{aligned}
u_1 + v_1 &= 2 \\
u_2 + v_3 &= 5 \\
u_2 + v_4 &= 4 \\
u_3 + v_1 &= 7 \\
u_3 + v_2 &= 6 \\
u_3 + v_3 &= 6
\end{aligned}$$

Let

$$\begin{aligned}
u_1 = 0 \qquad &\Rightarrow v_1 = 2 \\
u_2 = 4 &\Rightarrow v_2 = 1 \\
&\Rightarrow v_3 = 1 \\
u_3 = 5 \qquad &\Rightarrow v_4 = 0
\end{aligned}$$

Now,

$$\begin{aligned}
\bar{c}_{12} &= 1 \\
\bar{c}_{13} &= 1 \\
\bar{c}_{14} &= 1 \\
\bar{c}_{21} &= 4 \\
\bar{c}_{22} &= 3 \\
\bar{c}_{34} &= 3
\end{aligned}$$

Since all $\bar{c}_{ij} > 0$ for all non-basic variables. Thus, the above table represents unique optimal solution. The optimal shipping schedule is to ship

3 units from warehouse w_1 to market M_1
3 units from warehouse w_2 to market M_3
4 units from warehouse w_2 to market M_4
1 units from warehouse w_3 to market M_1
3 units from warehouse w_3 to market M_2
1 units from warehouse w_3 to market M_3

The least cost of shipping is

$$Z = 3 \cdot 2 + 3 \cdot 5 + 4 \cdot 4 + 1 \cdot 7 + 3 \cdot 6 + 1 \cdot 6 = 68 \text{ units}$$