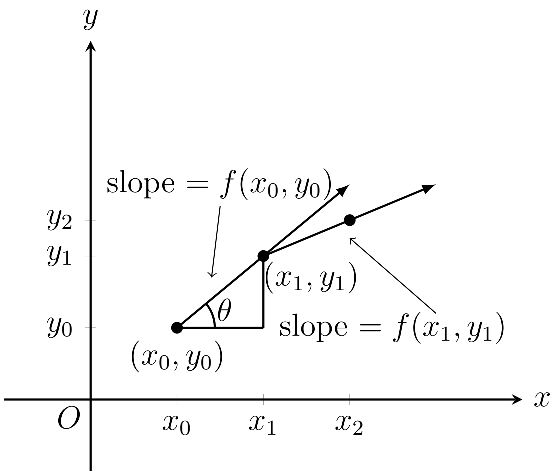
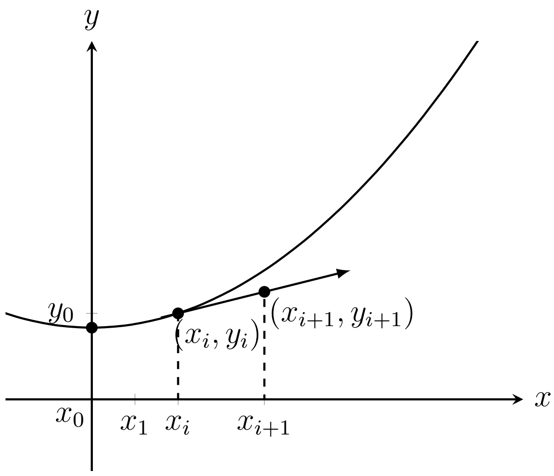
0.1 Euler Method

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x,y); \quad y(x_0) = y_0$$





$$\begin{aligned} \tan\ \theta &= \frac{y_1 - y_0}{x_1 - x_0} \\ \Rightarrow \tan\ \theta &= \frac{y_1 - y_0}{h} \\ \Rightarrow f(x_0, y_0) &= \frac{y_1 - y_0}{h} \\ \Rightarrow y_1 &= y_0 + hf(x_0, y_0) \end{aligned}$$

Similarly,

$$y_2 = y_1 + hf(x_1, y_1)$$

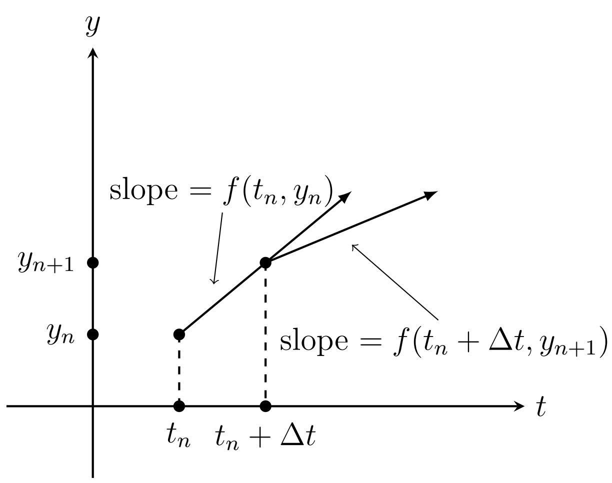
Generally,

$$y_{i+1} = y_i + hf(x_i, y_i)$$

which is Euler's method.

0.2 Modified Euler Method

$$\begin{aligned} \frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= f(x,y); \qquad y(x_0) = y_0 \\ \text{or, } \frac{\mathrm{d}\,y}{\mathrm{d}\,t} &= f(t,y); \qquad y(t_0) = y_0 \end{aligned}$$



$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_n + \Delta t, y_{n+1})\right]$$

Predictor-Corrector Method:

$$\begin{aligned} y_{n+1}^{p} &= y_n + \Delta t f(t_n, y_n) \\ \therefore \quad y_{n+1} &= y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_n + \Delta t, y_{n+1}^{p})\right] \end{aligned}$$

which is called Modified Euler method.
This is also called 2nd-order Runge-Kutta method.

0.3 Runge-Kutta Method (RK2) (Two stage)

$$\begin{aligned} k_1 &= \Delta t f(t_n, y_n) \\ k_2 &= \Delta t f(t_n + \Delta t, y_n + k_1) \\ \therefore \quad y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2) \end{aligned}$$

Example. Modified Euler Method:

First approximation:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

where

$$y_1 = y_0 + h f(x_0, y_0)$$

Now,

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &\vdots \end{aligned}$$

Let us consider $y_1^{(n+1)} = y_1^{(n)}$, then $y_1 = y_1^{(n+1)}$.

Second approximation: $x_1 = x_0 + h$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$$

where

$$y_2 = y_1 + h f(x_1, y_1)$$

Now,

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\ y_2^{(3)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] \\ &\vdots \end{aligned}$$

Generally,

$$y_{i+1}^{(n+1)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(n)})\right]$$

where

$$y_{i+1}^{(n)} = y_i + h f(x_i, y_i)$$

Problem 0.1. $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = x + y$, with $y(0) = 1$ for $x = 0.1$ taking $h = 0.05$

Solution. 1st Iteration

1st approximation:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] \tag{1}$$

where,

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.05(0 + 1) \\ &= 1.05 \end{aligned}$$

Now from (1)

$$\begin{aligned} y_1^{(1)} &= 1 + \frac{0.05}{2}\left[0 + 1 + 0.05 + 1.05\right] \\ &= 1.0525 \end{aligned}$$

Again,

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] && = 1 + \frac{0.05}{2}\left[0 + 1 + 0.05 + 1.0525\right] \\ &= 1.0526 \end{aligned}$$

[Note: $f(x_1, y_1^{(1)}) = x_1 + y_1^{(1)} = 0.05 + 1.0525$]

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] && = 1 + \frac{0.05}{2}\left[0 + 1 + 0.05 + 1.0526\right] \\ &= 1.0526 \end{aligned}$$

$f(x_1, y_1^{(2)}) = x_1 + y_1^{(2)} = 0.05 + 1.0526$
So $y = 1.0526$ at $x = 0.05$

2nd Iteration: $x_0 = 0.05$, $y_0 = 1.0526$, $h = 0.05$, $x_1 = x_0 + h = 0.05 + 0.05 = 0.10$

$$\therefore \quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] \tag{2}$$

where,

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1.0526 + 0.05f(0.05, 1.0526) \\ &= 1.1077 \end{aligned}$$

Now from (2)

$$\begin{aligned} y_1^{(1)} &= 1.0526 + \frac{0.05}{2}\left[f(0.05, 1.0526) + f(0.1, 1.1077)\right] \\ &= 1.0526 + \frac{0.05}{2}\left[0.05 + 1.0526 + 0.1 + 1.1077\right] \\ &= 1.1103 \end{aligned}$$

Again,

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] && = 1.0526 + \frac{0.05}{2}\left[f(0.05, 1.0526) + f(0.1, 1.1103)\right] \\ &= 1.0526 + \frac{0.05}{2}\left[0.05 + 1.0526 + 0.1 + 1.1103\right] \\ &= 1.1104 \end{aligned}$$

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] && = 1.0526 + \frac{0.05}{2}\left[f(0.05, 1.0526) + f(0.1, 1.1104)\right] \\ &= 1.0526 + \frac{0.05}{2}\left[0.05 + 1.0526 + 0.1 + 1.1104\right] \\ &= 1.1104 \end{aligned}$$

So $y = 1.1104$ at $x = 0.1$