Real Analysis

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Contents

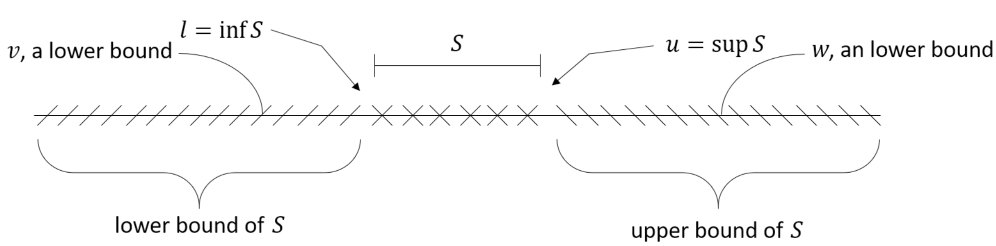
Chapter 1

Sets

1.1 Sets

Definition 1.1.1 (Supremum)**.** Let $S \subset \mathbb{R}$. A number $u \in \mathbb{R}$ is said to an upper bound of S, if $s \leq u$ for all $s \in S$. If S is bounded above, then the upper bound is said to be a supremum or a least upper bound (l.u.b) of S, written as $\sup S = u$, if no number less than u is an upper bound of S.

Definition 1.1.2 (Infimum)**.** Let $S \subset \mathbb{R}$. A number $l \in \mathbb{R}$ is said to an lower bound of S, if $l \leq s$ for all $s \in S$. If S is bounded below, then the lower bound l is said to be a infimum or a greatest lower bound (g.l.b) of S, written as $\inf S = l$, if no number greater than l is an lower bound of S.



Example 1.1.1. $S = (0, 1) \subset \mathbb{R}$, here -1 is a lower bound of S and 2 is an upper bound of S. clearly, $\sup S = 1$ and $\inf S = 0$.

Example 1.1.2. $S = [0, \infty)$ $\inf S = 0$ and $\sup S = \infty$ i.e., $\sup S$ does not exist.

Example 1.1.3. $S = (-\infty, 2]$ $\inf S = \infty$ i.e., $\inf S$ does not exist and $\sup S = 2$

Example 1.1.4. $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$ inf $S = 0$, sup $S = 1$.

Theorem 1.1.1. A number u is a supremum of a nonempty subset S of $\mathbb{R}$ if and only if u satisfies the two conditions:

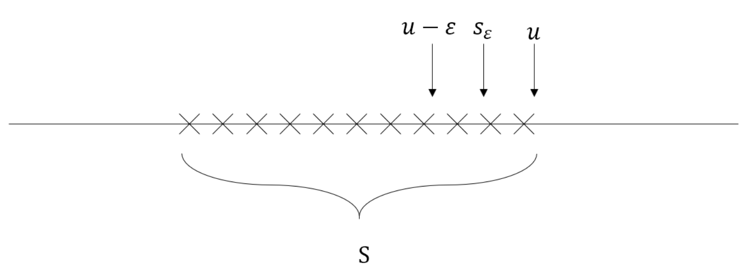
1. $s \leq u$ for all $s \in S$
2. if $v < u$, then there exists an $s' \in S$ such that $v < s'$

Supremum Characterization

Theorem 1.1.2. An upper bound u of a non-empty set S in $\mathbb{R}$ is the supremum of S if and only if for each $\varepsilon > 0$, there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$

Proof. Suppose that u is an upper bound of S and satisfies the stated condition. If $v < u$ and we take $\varepsilon := u - v$, then $\varepsilon > 0$, and the stated condition implies there exits a number $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$ i.e. $v < s_\varepsilon$ which implies that v is not an upper bound of S. Since v is an arbitrary number less than u, we conclude that $u =$sup S.

Conversely, suppose that $u =$sup S and let $\varepsilon > 0$. Since $u - \varepsilon < u$, then $u - \varepsilon$ is not an upper bound of S. Therefore, some elements s_ε of S must be greater than $u - \varepsilon$; that is, $u - \varepsilon < s_\varepsilon$ □



$\boxed{\text{Obtain the } \textit{infimum characterization.}}$

Example 1.1.5. Let $S = (a, b), a, b \in \mathbb{R}$. Show that sup $S = b$ and inf $S = a$.

Solution. Observe that b is an upper bound of S because $s \leq b$ for all $s \in S$. Now given $\varepsilon > 0$, $b - \varepsilon < b$. Clearly $b - \varepsilon \in S$, so there exits an $s_\varepsilon \in S$ such that $b - \varepsilon < s_\varepsilon$. Hence by supremum characterization we have $b =$sup S.
Similarly, by using infimum characterization $a =$inf S.

Definition 1.1.3 (Maximum of a Set)**.** Let $S \subset \mathbb{R}$. Then an element $m \in S$ is called the maximum(or the greatest or the largest) element of S if $s \leq m$ for all $s \in S$.

Definition 1.1.4 (Minimum of a Set)**.** Let $S \subset \mathbb{R}$. Then an element $l \in S$ is called the minimum(or the least or the smallest) element of S if $l \leq s$ for all $s \in S$.

Example 1.1.6. $S = [1, 2)$. $\min S = 1$, $\max S$ does not exist.

Example 1.1.7. The sets $\mathbb{Z}$ and $\mathbb{Q}$ have neither maximum nor minimum. The set $\mathbb{N}$ has no maximum but has a minimum. i.e. $\min \mathbb{N} = 1$.

Triangle Inequality $\big||x| - |y|\big| \leq |x - y| \leq |x + y| \leq |x| + |y|$

1.1.1 The Completeness Property of the set of Real Number $\mathbb{R}$

Here we shall present a special property of $\mathbb{R}$ that is often called the "Completeness property", since it guarantees the *existence of elements in* $\mathbb{R}$ under certain hypothesis. We know that $\sqrt{2}$ is not a rational number, i.e. $\sqrt{2} \notin \mathbb{Q}$. This observation shows the necessity of an additional property to characterize the real number system. This additional property, the Completeness property, is an essential feature of $\mathbb{R}$.

There are several different versions of the Completeness property. Such as,

(i) The supremum Property (or The Infimum Property)

(ii) Dedikind's Axiom.

(iii) Cauchy Completeness Property.

The supremum property of the real number system $\mathbb{R}$:
Every non empty set of real numbers that has an upper bound has a supremum is $\mathbb{R}$.

The analogues infimum property of the real number system $\mathbb{R}$ states that, "Every non empty set of real numbers that has a lower bound has an infimum in $\mathbb{R}$."

One important consequence of the supremum property is that the subset $\mathbb{N}$ of natural numbers is not bounded above in $\mathbb{R}$. This means that given any real number x there exists a natural number n (depending on x) such that $x < n$.

Archimedean Property: If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

Proof. If the conclusion fails, then x is an upper bound of $\mathbb{N}$. Therefore by the supremum property, the non empty set $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Since $u - 1 < u$, it follows from the previous supremum characterization theorem that there exists $m \in \mathbb{N}$ such that $u - 1 < m$. But then $u < m + 1$ and since $m+ \in \mathbb{N}$, this contradicts the assumption that u is an upper bound of $\mathbb{N}$. □

Corollary 1.1.3. Let y and z be positive real numbers. Then

(i) There exists $n \in \mathbb{N}$ such that $z < ny$

(ii) There exists $n \in \mathbb{N}$ such that $o < \dfrac{1}{n} < y$

(iii) There exists $n \in \mathbb{N}$ such that $n - 1 \leq z < n$

Homework: Let S be a non empty subset of $\mathbb{R}$ that is bounded above and let $a \in \mathbb{R}$. Define the set $a + s = \{a + x : x \in S\}$. Show that sup $(a + S) = a+$ sup S.

1.1.2 Open and Closed Sets in $\mathbb{R}$

Definition 1.1.5 (Neighborhood)**.** Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε-neighborhood of a is the set $N_\varepsilon = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.

$$|x - a| < \varepsilon \quad \Rightarrow a - \varepsilon < x < a + \varepsilon$$

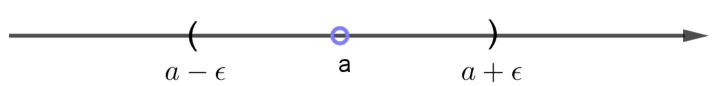


Figure 1.1: A $\varepsilon-$neighborhood of a

Definition 1.1.6 (Open Set)**.** Let $S \subset \mathbb{R}$ and $x \in S$, then S is an open set if $N_\varepsilon(x) \subset S$. Generally, open set is denoted by G.

1.1.3 Properties of Open Set

(a) The union of an arbitrary collection of open sets in $\mathbb{R}$ is open.

(b) The intersection of any finite collection of open sets on $\mathbb{R}$ is open.

Proof. (a) Let $\{G_\lambda : \lambda \in \Lambda\}$ be a family of sets in $\mathbb{R}$ that are open, and let G be their union; by the definition of union, x must belong to G_{λ_0} for some $\lambda_0 \in \Lambda$. Since G_{λ_0} is open, there exists a neighborhood V of x such that $V \subseteq G$. But $G_{\lambda_0} \subseteq G$, so that $V \subseteq G$. Since x is an arbitrary element of G, we conclude that G is open in $\mathbb{R}$.

[1] $\{G_\lambda : \lambda \in \Lambda\}$ is an arbitrary collection of open set.

$$G = \cup G_\lambda \quad x \in G \Rightarrow N_\varepsilon(x) \subset G$$

$$\begin{aligned} & x \in G \\ \Rightarrow & x \in \cup G_\lambda \\ \Rightarrow & x \in \cup G_{\lambda_0} \qquad \lambda_0 \in \Lambda \\ \Rightarrow & N_\varepsilon(x) \subset G_{\lambda_0} \subset G \end{aligned}$$

G is open if $x \in G \Rightarrow N_\varepsilon(x) \subset G$

(b) Suppose G_1 and G_2 are open and let $G := G_1 \cap G_2$. To show that G is open, we consider any $x \in G$; then $x \in G_1$ and $x \in G_2$. Since G_1 is open, there exits $\varepsilon_1 > 0$ such that $(x - \varepsilon_1, x + \varepsilon_1)$ is contained in G_1. Simillarly, since G_2 is open, there exits $\varepsilon_2 > 0$ such that $(x - \varepsilon_2, x + \varepsilon_2)$ is contained in G_2. If we now take ε to be smaller of ε_1 and ε_2, then the $\varepsilon-$neighborhood $U := (x - \varepsilon, x + \varepsilon)$ satisfies both $U \subseteq G_1$ and $U \subseteq G_2$. Thus, $x \in U \subseteq G$. Since x is an arbitrary element of G, we conclude that G is open in $\mathbb{R}$.

[2]

$$\begin{aligned} G &= \cap_{\lambda=1}^{N} G_\lambda \\ &= G_1 \cap G_2 \cap G_2 \cap \cdots \cap G_n \\ G &= G_1 \cap G_2 \end{aligned}$$

$$\begin{aligned} & x \in G \\ \Rightarrow & x \in G_1 \quad \text{and} \quad x \in G_2 \\ \Rightarrow & N_{\varepsilon_1}(x) \subset G_1 \quad \text{and} \quad N_{\varepsilon_2}(x) \subset G_2 \end{aligned}$$

[1] This is from class
[2] This is from class

$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

$$\begin{aligned} &N_\varepsilon(x) \subset G_1 \quad \text{and} \quad N_\varepsilon(x) \subset G_2 \\ \Rightarrow &N_{\varepsilon_1}(x) \subset G_1 \cap G_2 = G \end{aligned}$$

□

Definition 1.1.7 (Closed Set)**.** A set $F \subset \mathbb{R}$ is called closed set if F^c is open set. Generally, close set is denoted by F.

1.1.4 Properties of Closed Set

(a) The intersection of an arbitrary collection of closed sets in $\mathbb{R}$ is closed.

(b) The union of any finite collection of closed sets in $\mathbb{R}$ is closed.

Proof. (a) If $\{F_\lambda : \lambda \in \Lambda\}$ is a family of closed sets in $\mathbb{R}$ and $F := \cap_{\lambda\in\Lambda} F_\lambda$, then $C(F) = \cup_{\lambda\in\Lambda}$ is the union of open sets. Hence, $C(F)$ is open by properties of open set(a) and consequently, F is closed.

(b) Suppose $F_1, F_2, \ldots, F_n$ are closed in $\mathbb{R}$ and let $F := F_1 \cup F_2 \cup \cdots \cup F_n$. By the De Morgan identity of the complement[1] of F is given by

$$C(F) = C(F_1) \cap \cdots \cap C(F_n)$$

Since each set $C(F_i)$ is open, it follows properties of open set(b) that $C(F)$ is open. Hence F is closed.

□

[1] $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

Chapter 2

Series and Series of Function

2.1 Sequence

Definition 2.1.1 (Sequence)**.** A sequence of real numbers (or a sequence in $\mathbb{R}$)is a function defined on the set $\mathbb{N} = 1, 2, \ldots$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

Since $\mathbb{N}$ is the domain of a sequence $x(n) = x_n$ so a sequence is an ordered set and is denoted by $\langle x_n \rangle$ or (x_n).

Remark. A sequence is a set of real numbers with an order.

Example 2.1.1. 1. $\langle x_n \rangle = \left\langle \dfrac{1}{n} \right\rangle = \langle 1, 1/2, 1/3, \ldots, 1/n, \ldots \rangle$

2. $\langle x_n \rangle = \langle n \rangle$

3. $\langle x_n \rangle = \langle (-1)^n \rangle$

2.2 Series

Definition 2.2.1 (Series)**.** Sum of the terms of an infinite sequence is called series. Given a series $\displaystyle\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \ldots$, let s_n denote its $n-$th partial sum:

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + x_3 + \cdots + x_n.$$

If the sequence s_n is convergent, i.e. if x is a real number such that $\lim(s_n) = x$, then the series $\sum x_n$ is called *convergent* and we write

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k = x_1 + x_2 + x_3 + \cdots = x$$

The number x is called the sum of the series. Otherwise, the series is *divergent.*

Note that $\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k$

Theorem 2.2.1. The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

if $|r| \geq 1$, the series is divergent.

Example 2.2.1. Test whether the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ is convergent or divergent.

Solution. $\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 4(4/3)^{n-1}$ is a geometric series with $a = 4$ and $r = 4/3 > 1$. So, the series is divergent.

Theorem 2.2.2. The $p-$ series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$. When $p = 1$ the series is called the *harmonic* series.

Theorem 2.2.3. If the series $\sum_{n=1}^{\infty} x_n$ is convergent, them $\lim(x_n) = 0$. But the converse is not true in general, e.g, harmonic series.

Theorem 2.2.4 (The Test for Divergent). If $x_n \to \infty$ or $x_n \nrightarrow 0$ then the series $\sum_{n=1}^{\infty} x_n$ is divergent.

Example 2.2.2. Test whether the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is convergent or divergent.

Solution. Here $x_n = \frac{n^2}{5n^2+4} \to 1/5 \neq 0$ so the test for divergence implies that the series is divergent.

Theorem 2.2.5. The sum, difference, and scalar multiple of two convergent series are convergent.

Example 2.2.3. Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$

Solution. Here the second series is geometric series with $a = 1/2$ and $r = 1/2$, so $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = 1$. The first series is $\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3 \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = \lim_{n\to\infty} (1 - \frac{1}{n+1}) = 3$. Therefore the sum of the given series is $3 + 1 = 4$.

Theorem 2.2.6 (Cauchy Criterion for Series b). A series $\sum x_n$ in $\mathbb{R}$ is convergent iff for each $\epsilon > 0$ there is a natural number $K := K(\epsilon)$ such that if $m > n \geq K$, then

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \epsilon$$

Definition 2.2.2. We say that a series $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent in $\mathbb{R}$. A series is conditionally convergent if it is convergent but not absolutely convergent.

Theorem 2.2.7. If a series is absolutely convergent, then it is convergent.

2.2.1 Tests for Absolute Convergence

Theorem 2.2.8 (Comparison Test b). Let x_n and y_n be real sequences such that for some natural number K,

$$0 \leq x_n \leq y_n \qquad \text{for } x \geq K$$

Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$ and the divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

Example 2.2.4. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

Solution. Note that $\dfrac{5}{2n^2+4n+3} < \dfrac{5}{2n^2}$ by the $p-$ series $\sum \dfrac{1}{n^2}$ converges and hence by the comparison test the given series is convergent.

Theorem 2.2.9 (Limit Comparison Test b). Let x_n and y_n be positive real sequences and $L = \lim(x_n/y_n)$

1. If $L \neq 0$, then $\sum x_n$ is convergent iff $\sum y_n$ is convergent.

2. If $L = 0$ and $\sum y_n$ is convergent then $\sum x_n$ is convergent.

Example 2.2.5. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.

Solution. Note that the dominant part od the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5}$. This suggests taking $x_n = \dfrac{2n^2+3n}{\sqrt{5+n^5}}$, $y_n = \dfrac{2n^2}{\sqrt{n^5} = \frac{2}{n^{1/2}}}$

$\displaystyle\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \times \frac{n^{1/2}}{2} = 1$ Since $\sum y_n = 2\sum 1/n^{1/2}$ is divergent (p-series with $p = 1/2 < 1$)

Theorem 2.2.10 (Root test b). Let x_n be a sequence in $\mathbb{R}$ and

$$r := \lim(|x_n|^{1/n}).$$

Then $\sum x_n$ is absolutely convergent if $r < 1$ and divergent if $r > 1$.

Example 2.2.6. Test the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Solution. Root test with $x_n = \left(\dfrac{2n+3}{3n+2}\right)^n$ gives $\sqrt[n]{|x_n|} = \dfrac{2n+3}{3n+2} \to 2/3 < 1$. Thus, the given series is convergent by the root test.

Theorem 2.2.11 (Ratio test b). Let x_n be a sequence in $\mathbb{R}$ and

$$r := \lim(|x_{n+1}| \,/\, |x_n|)$$

Then $\sum x_n$ is absolutely convergent if $r < 1$ and divergent if $r > 1$

Example 2.2.7. Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Solution. Ratio test with $x_n = (-1)^n \frac{n^3}{3^n}$ gives: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{3}\frac{n+1}{n}^3\right| = \left|\frac{1}{3}(1+1/n)^3\right| \to \frac{1}{3} < 1$. Thus, by ratio test, the given series is absolutely convergent and therefore convergent.

Theorem 2.2.12 (Raabe's Test b)*.* Let x_n be a sequence of nonzero real numbers and

$$r := \lim(n(1 - \frac{|x_{n+1}|}{|x_n|})$$

Then $\sum x_n$ is absolutely convergent if $r > 1$ and is not absolutely convergent if $r < 1$.

Theorem 2.2.13 (Integral Test b)*.* Let f be a continuous, positive decreasing function on $[1, \infty)$ and $x_n = f(n)$. Then if $\int_1^{\infty} f(x)\mathrm{d}x$ is convergent (divergent), then $\sum x_n$ is convergent (divergent).

Example 2.2.8. Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

Solution. The function $f(x) = \frac{\ln x}{x}$ is positive and continuous for $x > 1$ because the logarithm function is continuous. But it is not clear whether or not f is decreasing, so $f'(x) = \frac{1 - \ln x}{x^2}$ Thus, $f'(x) < 0$ when $1 - \ln x < 0$ i.e., $x > e$. So f is decreasing when $x > e$. Here $\int_1^{\infty} \frac{\ln x}{x}\mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x}\mathrm{d}x = \infty$. Therefore, by the integral test the given series is divergent.

2.2.2 Test For Nonabsolute Convergence

Definition 2.2.3 (Alternating Series)**.** An alternating series is a series whose terms are alternately positive and negative. The $n-$th term of an alternating series is defined by $x_n = (-1)^{n-1}y_n$, $x_n = (-1)^n y_n$,$x_n = (-1)^{n+1}y_n$ where $n \in \mathbb{N}$ and $y_n > 0$

Theorem 2.2.14 (Alternating Series Test b)*.* Let x_n be a decreasing sequence of positive real numbers with $\lim(x_n) = 0$. Then the alternating series $\sum(-1)^{n+1}x_n$ is convergent.

Example 2.2.9. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$