





Real Analysis II

MAT311

Prof. Md Shah Noor

Shahjalal University of Science and Technology

Edited by Mehedi Hasan







Preface

This is a compilation of lecture notes with some books and my own thoughts. This document is not a holy text. So, if there is a mistake, solve it by your own judgement.

Contents

1	No	${f otes}$	T	
1	Met	cric Space	2	
	1.1	Euclidean Space	2	
	1.2	Connected Set	5	
	1.3	Compact Set	5	
	1.4	Path-connected Sets	6	
2	Continuity			
	2.1	Limit	9	
	2.2	Continuity		
3	Sequences in Metric spaces			
	3.1	Sequences of real numbers	1	
	3.2	Convergent Sequence		
	3.3	Sequences of points or vectors in Metric Spaces	1	
	3.4	Convergent sequence in a Metric Space	1	
	3.5	Convergent sequence in normed space \mathbb{R}^n	1	
	3.6	Convergent sequence in arbitrary normed space V	1	
	3.7	Cauchy sequence	2	
	3.8	Complete Metric Space	2	
	3.9	Bounded Sequence	3	

Part I

Notes

Chapter 1

Metric Space

1.1 Euclidean Space

Euclidean space or Euclidean n-space, denoted by \mathbb{R}^n consists of all ordered n-tuples of real numbers. Symbolically, $\mathbb{R}^n = \{x \mid x = (x_1, x_2, \dots, x_n), x_1, x_2, \dots, x_n \in \mathbb{R}\}$

Here the element $x \in \mathbb{R}^n$ is called a point or a vector and x_1, x_2, \ldots, x_n are called coordinates of x when n > 1.

If $x, y \in \mathbb{R}^n$ and if $\alpha \in \mathbb{R}^n$ then put,

 $x+y=(x_1+y_1,x_2+y_2,\ldots,x_n+y_n)$ and $\alpha x=(\alpha x_1,\alpha x_2,\ldots,\alpha x_n)$ so that $x+y\in\mathbb{R}^n$ and $\alpha x\in\mathbb{R}^n$. This defines addition and scalar multiplication of vectors. This two operations satisfy the commutative, associative and distributive laws and make \mathbb{R}^n into a vector space over the real field.

Theorem 1.1.1. \mathbb{R}^n with operations of addition and scalar multiplication defined previously is a vector space of dimension n.

Definition 1.1.1 (Inner Product). The inner product (or scalar product) of x and y in \mathbb{R}^n is defined by $\langle x, y \rangle = x \cdot y = \sum_{i=1}^n x_i y_i$ and the *norm* or *length* of a vector $x \in \mathbb{R}^n$ is defined by $||x|| = \langle x, x \rangle^{1/2} = \sum_{i=1}^n \left(x_i^2\right)^2$ and the *distance* between two vectors x and y of \mathbb{R}^n is the real number $d(x,y) = ||x-y|| = \left\{\sum_{i=1}^n (x_i - y_i)^2\right\}^{1/2}$

Definition 1.1.2. Let X be a metric space. All points and sets involved below are understood to be elements and subset of X.

- 1. A neighborhood of a point $p \in X$ is a set $N_{\delta}(p)$ containing all points q such that $d(p,q) < \delta$. The number δ is called the radius of $N_{\delta}(p)$. [Mathematically, $N_{\delta}(p) = \{q \mid d(p,q) < \delta\}$]
- 2. A point p is a *limit point* (accumulation point or cluster point) of the set E if every neighborhood $N_{\delta}(p)$ contains a point $q \neq p$ such that $q \in E$. [Mathematically, $(N_{\delta}(p) \{p\}) \cap E \neq \Phi$]
- 3. If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
- 4. E is closed if every limit point of E is a point of E.
- 5. A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$
 - (i) E is open if every point of E is an interior point of E.
- 6. The complement of E, denoted by E^c is the set of all points $p \in X$ such that $p \notin E$

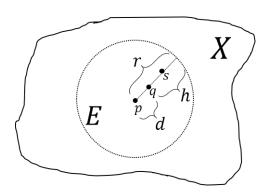
- 7. E is perfect if E is closed and every point of E is a limit point of E.
- 8. E is bounded if E is a real number M and a point $q \in X$ such that d(p,q) < M for all $p \in E$.
- 9. E is dense in X if every point of X is a limit point of E, or a point of E (or both).

Note. The segment $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

Note. In \mathbb{R}^1 neighborhoods are segments, whereas in \mathbb{R}^2 neighborhoods are interiors of circles and in \mathbb{R}^3 neighborhoods are interiors of spheres.

Theorem 1.1.2. Every neighborhood is an open set.

Proof. Consider the neighborhood $E = N_r(p) = \{q \in X \mid d(p,q) < r\}$ and let q be any point of E, where X is a metric space.



Then there is a positive real number h such that d(p,q) = r - hNow for all points s such that d(q,s) < h,

We have then

we have then
$$d(p,s) \le d(p,q) + d(q,s) < r - h + h = r$$
 so that $s \in E$ $\Rightarrow d(p,q) + h = r$ $\Rightarrow d(p,q) + h = r$ Therefore, $N_h(q) = \{s \in E \mid d(q,s) < h\} \subset E = N_r(p)$ $\Rightarrow d(p,q) + h = r$ $\Rightarrow d(p,q) = r - h$ $\therefore r - h = d(p,q) \ge 0$ $\Rightarrow h \le r$

Theorem 1.1.3. If p is a limit point of a set E in a metric space X, then every neighborhood of p contains infinitely many points of E.

Proof. Suppose there is a neighborhood N of $p \in X$ which contains only a finite number of points of E. Let q_1, q_2, \ldots, q_n be those points of $N \cap E$, which are distinct from p and put $r = \min_{1 \le m \le n} d(p, q_m)$. [We use this notation to denote the smallest of the numbers $d(p, q_1), d(p, q_2), \ldots, d(p, q_n)$] The minimum of a finite set of positive numbers is clearly positive, so that r > 0.

The neighborhood $N_r(p)$ contains no part q of E such that $q \neq p$, so that p is not a limit point of E.

This contradiction establishes the theorem.

Note. Here $r > 0 \Rightarrow r$ can be taken a large positive real number, however we please $\Rightarrow d(p, q_m), m = 1, 2, \ldots, n$ are bigger & bigger $\Rightarrow q_m$ are not close enough to $p \Rightarrow p$ is not a limit point of E.

Corollary 1.1.4. A finite set has no limit points.

Problem 1.1.1. Let us consider the following subsets of \mathbb{R}^2

1. The set of all complex Z such that |Z| < 1

- 2. The set of all complex Z such that |Z| < 1
- 3. A finite set
- 4. The set of all integers
- 5. The set consisting of the numbers $\frac{1}{n}(n=1,2,3...)$
- 6. The set of all complex numbers (that is, \mathbb{R}^2)
- 7. The segment (a, b)

If (4), (5) and (7) are regarded as subsets of \mathbb{R}^1 , then identify whether the sets (1)-(7) are closed, open, perfect and bounded.

Theorem 1.1.5. Let
$$\{E_{\alpha}\}$$
 be a collection of sets E_{α} , then $(\bigcup_{\alpha} E_{\alpha})^{c} = \bigcap_{\alpha} (E_{\alpha}^{c})^{c}$

Theorem 1.1.6. A set E is open if and only if its complement is closed.

Proof. First, suppose E^c is closed. Cause $x \in E$. Then $x \notin E^c$ and x is not a limit point E^c . Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E and E is open.

Next, suppose that E is open. Let x be a limit point of E^c . Then every neighborhood of x contains a point of E^c , such that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

Corollary 1.1.7. A set F is closed if and only if its complement is open.

Theorem 1.1.8. 1. For any collection $\{G_{\alpha}\}$ is open sets, $\bigcup_{\alpha} G_{\alpha}$ is open.

- 2. For any collection $\{F_{\alpha}\}$ is closed sets, $\underset{\sim}{\alpha}F_{\alpha}$ is closed.
- 3. For any finite collection G_1, G_2, \ldots, G_n of open sets, $\bigcap_{i=1}^n G_i$ is open.
- 4. For any finite collection F_1, F_2, \ldots, F_n of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Note. Is the finiteness of the collection in parts (3) and (4) of the above theorem essential? Justify your answer.

Definition 1.1.3. If X is a metric space, if $E \subset X$ and if E' denotes the set of all limit points of E in X, then E' is called the derived set of E and $\bar{E} := E \cup E'$ is called the closure of E.

Example.
$$E = (0,1) \cup \{e, \pi, \sqrt{7}, 11.5\}$$
, then $E' = [0,1]$, $\bar{E} = E' \cup E = [0,1] \cup \{e, \pi, \sqrt{7}, 11.5\}$

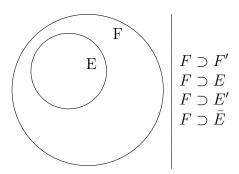
Theorem 1.1.9. If X is a metric space and $E \subset X$, then

- (a) \bar{E} is closed,
- (b) $E = \bar{E}$ if and only if E is closed,
- (c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

By 1 and 3, \bar{E} is smallest closed subset of X that contains E.

Proof. (a) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. The complement of \bar{E} is therefore open. Hence \bar{E} is closed.

- (b) If $E = \bar{E}$, (a) implies that E is closed. If E is closed, then $E' \subset E$ (by definition (1) and (10)). Hence $\bar{E} = E$.
 - (c) If E is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.



1.2 Connected Set

Let A be a subset on metric space X. Two non-empty open sets U and V are said to separate A if the satisfy these condition

- 1. $U \cap V \cap A = \Phi$
- 2. $A \cap U \neq \Phi$
- 3. $A \cap V \neq \Phi$
- 4. $A \subset U \cup V$

We say that A is disconnected (i.e., not connected) if such set exist and if such sets do not exist, we say that A is connected.

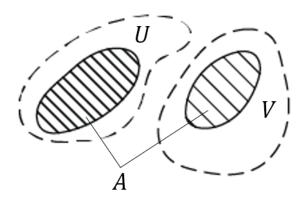


Figure 1.1: A is disconnected

Example.

- 1. $\bar{\mathbb{Z}}$ is not connected
- 2. $S = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \le 1\}$ is connected

1.3 Compact Set

By an open cover of a set E in a metric space X we mean a collection $\{G_{\alpha}\}$ of an open subset of X such that $E \subset \bigcup G_{\alpha}$.

A subset K of a metric space X is said to be *compact* if every open cover of K contains a finite subcover.

More explicitly, the requirement for completeness of $K \subset X$ is that if $\{G_{\alpha}\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$$
 i.e., $k \subset \bigcup_{i=1}^n G_{\alpha_i}$

Example.

- 1. A = [1, 2] is *compact*.
- 2. B = (0, 2) is not compact.

Theorem 1.3.1. In a metric space, prove that closed subsets of a compact set is compact.

1.4 Path-connected Sets

We say that map $\varphi : [a, b] \to M$ of an interval [a, b] into a metric space M continuous if $t_{\mu} \to t$ implies $\varphi(t_{\mu}) \to \varphi(t)$ for every sequence t_{μ} in [a, b] converging to some $t \in [a, b]$.

A continuous path joining two points x, y in a metric space M is a mapping $\varphi : [a, b] \to M$ such that $\varphi(a) = x$, $\varphi(b) = y$ and φ is continuous. Here x may or may not equal y and $b \ge a$.

A path φ is said to lie in a set A if $\varphi(t) \in A$ for all $t \in [a, b]$.

We say that a set A is path-connected if every two points in the set can be joined by a continuous path lying in the set.

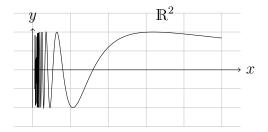


Figure 1.2: Not path connected

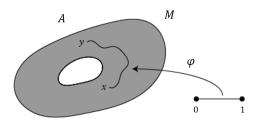


Figure 1.3: Path connected

• Figure 1.2: $A = \left\{ (x, \sin \frac{1}{x}) \mid x > 0 \right\} \cup \left\{ (0, y) \mid y \subset [-1, 1] \right\} \subset \mathbb{R}^2$. A is not path connected.

¹This is also known as 'Topologist's sine curve'

• Figure 1.3: A curve joining points x and y in A of a metric space M. Evidently, region A is path connected.

Problem 1.4.1. Show that B = [0, 1] is path connected.

Solution. Let $\varphi: B \to \mathbb{R}$ be a function defined by $\varphi(t) = (y - x)t + x$.

Here $\varphi(0) = x$, $\varphi(1) = y$, φ is continuous path (because φ is a linear polynomial in t) and φ lies in B.

Problem 1.4.2. Which of the following sets are path-connected?

- 1. [0,3]
- $[1,2] \cup [3,4]$
- 3. $\{(x,y) \in \mathbb{R}^2 \mid 0 < x \le 2\}$

Problem 1.4.3. Let $\varphi: B = [0,1] \to \mathbb{R}^2$ be a continuous path and $C = \varphi([0,1])$. Show that C is path-connected.

Solution. This is intuitively clear, for we can use the path φ itself to join two points in C. Precisely, if $x = \varphi(a)$, $y = \varphi(b)$, where $0 \le a \le b \le 1$, let $c : B \to \mathbb{R}^2$ be defined by $c(t) = \varphi(t)$. Thus c is a path joining x to y and c lies in C.

Problem 1.4.4. Is $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ connected?

Solution. No, for if $U = (1/2, \infty)$, $V = (-\infty, 1/4)$, then $\mathbb{Z} \subset U \subset V$, $\mathbb{Z} \cap U = \{1, 2, 3, \dots\} \neq \Phi$, $\mathbb{Z} \cap V = \{\dots, -2, -1, 0\} \neq \Phi$. Hence \mathbb{Z} is not disconnected (i.e., not connected).

Besides, Z is not path-connected.

Problem 1.4.5. Are $[0,1] \cup [2,3]$, $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\} \cup \{(x,0) \mid 1 < x < 2\}$ connected?

Problem 1.4.6. Determine the compactness of

- 1. finite set $A = \{x_1, x_2, \dots, x_n\}$
- 2. R
- 3. $B = [0, \infty)$
- 4. C = (0, 1)

Solution. 1. $A = \{x_1, x_2, \dots, x_n\}$ – a finite subset of \mathbb{R} .

Let $\mathscr{G} = \{G_{\alpha}\}$ be any open cover of A, then each x_i is contained in some set $G_{\alpha i} \in \mathscr{G}$. Then $A \subset \bigcup_{i=1}^n G_{\alpha i} \Rightarrow \{G_{\alpha i}; i=1,2,\ldots,n\}$ is a finite sub-cover of \mathscr{G} . Since \mathscr{G} is arbitrary so A is compact.

Theorem 1.4.1. Path-connected sets are connected.

Theorem 1.4.2 (Heine-Borel Theorem). A set $A \subset \mathbb{R}^n$ is compact iff it is closed and bounded.

Theorem 1.4.3 (Bolzano-Weirstrass Theorem). A subset of a metric space is compact iff it is sequentially compact.

Problem 1.4.7. Show that $A = \{x \in \mathbb{R}^n \mid ||x|| \le 1\}$ is compact and connected.

Solution. To show that A is compact, we show it is closed and bounded. To show that it is closed consider $A^c = \mathbb{R}^n \setminus A = \{x \in \mathbb{R}^n \mid ||x|| > 1\} = B$. For $x \in B$, $N_{\delta}(x) \subset B$, with $\delta = ||x|| - 1$, so that B is open and hence A is closed. It is clear that A is bounded, since $A \subset N_2(0)$ and therefore A is compact.

To show that A is connected, we show that A is path-connected. Let $x, y \in A$. Then the straight line joining x, y is the required path. Explicitly, we use $\varphi : [0,1] \to \mathbb{R}^n$, $\varphi(t) = (1-t)x + ty$. One sees that $\varphi(t) \in A$, since

$$||\varphi(t)|| \le (1-t)||x|| + t||y||$$

 $\le (1-t) + t = 1$ by triangle inequality.

Theorem 1.4.4. Closed subsets of a compact set is compact.

Proof. Suppose $F \subset K \subset M$, F is closed subset and K is compact in the metric space M. Let $\{V_{\alpha}\}$ be an open cover of F. If F^c is adjoined to $\{V_{\alpha}\}$, we obtain an open Ω of K. Since K is compact, there is a finite sub-collection Φ of Ω which covers K, and hence F. If F^c is a member of Φ , we may remove it from Φ and still retain an open cover of F. We have thus shown that a finite sub-collection of $\{V_{\alpha}\}$ covers F. Hence the theorem.

Chapter 2

Continuity

2.1 Limit

Definition 2.1.1 (Limit). Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y (i.e., $f: E \subset X \to Y$), and p is a limit point of E. We write $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$ if there is a point $q \in Y$ with following property:

For every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_x(f(x), p) < \delta^2$

Example.
$$E = (0, 2) \subset X = \mathbb{R}^1$$
, $Y = \mathbb{R}^1$; $f(x) = \frac{x^2 - 1}{x - 1}$; $p = 1$ is a limit point of E , Then $\lim_{x \to p} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$

Theorem 2.1.1 (Sequential Criteron of Limits). Let x, y, E, f and p as in the above definition. Then $\lim_{x\to p} f(x) = q$ iff $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\langle p_n \rangle$ in E such that $p_n \neq p$, $\lim_{n\to\infty} p_n = p$

2.2 Continuity

Definition 2.2.1 (Continuity). Suppose X and Y are metric spaces, $E \subset X$, $p \in E$ and f maps $E \to Y(f:E\to Y)$. Then f is said to be continuous at p if for every $\epsilon>0$ there exists a $\delta>0$ such that

 $d_{y}(f(x), p) < \epsilon$ for all points $x \in E$ for which $d_{x}(x, p) < \delta$

Theorem 2.2.1. Let $f: E \subset X \to Y$ be a mapping. Then the following assertions are equivalent:

- (i) f is continuous on E.
- (ii) For each convergent sequence $x_n \to x_0$, we have $f(x_n) \to f(x_0)$
- (iii) For each open set U i Y, $f^{-1}(U) \subset E$ is open relative to E; that is, $f^{-1}(U) = E \cap V$ for some open set V.
- (iv) For each closed set $F \in Y$, $f^{-1}(F) \subset E$ is closed relative to E; that is $f^{-1}(F) = E \cap G$ for some closed set G.

Theorem 2.2.2. Suppose $f: X \to Y$ is a continuous mapping of a compact metric space X into a metric space Y. Then f(X) is compact.

The δ may depend on f(x), p, and ϵ i.e., $\delta = \delta(p, f(x), \epsilon)$

Proof. Let $\{V_{\alpha}\}$ be an open cover of f(X), since f is continuous, by previous theorem each of the sets $f^{-1}(V_{\alpha})$ is open. Since X is compact, there are finitely many indices say $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that

$$X \subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \dots \cup f^{-1}(V_{\alpha_n})$$
 (2.1)

since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, then (2.1) implies that $f(X) \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n}$ This completes the proof.

note to self:: There may be some page left.

Chapter 3

Sequences in Metric spaces

3.1 Sequences of real numbers

A sequence of real numbers in \mathbb{R} is simply a function $f: \mathbb{N} \to \mathbb{R}$ which is usually defined by $f(n) = x_n$ and arranged in a particular order such as $x_1, x_2, \ldots, x_n, \ldots$

For example, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ can be represented as $x_n = \frac{1}{n}$ for $n = 1, 2, 3, \dots$

3.2 Convergent Sequence

A sequence x_n in \mathbb{R} is said to be *converge* to a *limit* $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N such that $|x_n - x| < \epsilon$ whenever $n \ge N$. In this case we write $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$ Note. $N := N(\epsilon)$, often smaller ϵ may require larger N.

3.3 Sequences of points or vectors in Metric Spaces

A sequence of points in a metric space M := (M, d) is a function $f : \mathbb{N} \to M$, usually defined by $f(n) = x_k$ and arranged in a definite order such as $x_1, x_2, \ldots, x_n, \ldots$

3.4 Convergent sequence in a Metric Space

A sequence x_k in a metric space (M, d) converges to $x \in M$ if for every given $\epsilon > 0$ there is a natural number N such as $n \geq N$ implies $d(x_k, x_n) < \epsilon$

3.5 Convergent sequence in normed space \mathbb{R}^n

A sequence v_k of vectors in \mathbb{R} converges to the vector $v \subset \mathbb{R}^n$ if for every given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(v_k, v) = ||v_k - v|| < \epsilon$ whenever $k \geq N$.

3.6 Convergent sequence in arbitrary normed space V

$$v_k \in V \to v \in V \text{ if } ||v_k - v|| \to 0 \text{ as } k \to \infty.$$

If $v, v_k \in \mathbb{R}^n$, we write $v = (v^1, v^2, \dots, v^n), v_k = (v_k^1, v_k^2, \dots, v_k^n)$

Theorem 3.6.1. $v_k \to v$ in \mathbb{R}^n iff each sequence of coordinates converges to the corresponding coordinate of v as a sequence in \mathbb{R} . That is,

 $\lim_{\substack{k\to\infty\\\text{or,}}} v_k = v \text{ in } \mathbb{R}^n \text{ iff } \lim_{\substack{k\to\infty\\}} v_k^i = v \text{ in } \mathbb{R} \text{ for each } i=1,\,2,\,\ldots,n$

$$\lim_{k \to \infty} (v_k^1, \dots, v_k^n) = \left(\lim_{k \to \infty} v_k^1, \dots, \lim_{k \to \infty} v_k^n\right)$$

Example. Test the convergence of the sequences in \mathbb{R}^2 :

- (i) $v_k = (1/k, 1/k^2)$
- (ii) $v_k = ((\sin n)^n / n, 1/n^2)$

Solution.

- (i) Here the component sequences 1/k and $1/k^2$ each converges to 0. Hence the vectors $v_k \to 0$, $0 = (0,0) \in \mathbb{R}^2$
- (ii) Use Sandwich theorem $(v_n \to (0,0))$. Here, $\left| \frac{(\sin n)^n}{n} \right| = \frac{|\sin n|^n}{n} \le \frac{1}{n} \Rightarrow -\frac{1}{n} \le \frac{(\sin n)^n}{n} \le \frac{1}{n}$ Hence by sandwich theorem, $\lim_{n \to \infty} -\frac{(\sin n)^n}{n} = 0 = \lim_{n \to \infty} \frac{(\sin n)^n}{n}$, therefore, $\lim_{n \to \infty} \frac{(\sin n)^n}{n} = 0$. Again $\lim_{n \to \infty} \frac{1}{n^2} = 0$ Therefore, $v_n \to (0,0)$

Theorem 3.6.2. A set $A \subset M$ is closed \Leftrightarrow for every sequence $x_k \in A$ converges to a point $x \in A$.

Example. Let $x_n \in \mathbb{R}^m$ be a convergent sequence with $||x_n|| \le 1$ for all n. Show that the limit x also satisfies $||x|| \le 1$. If $||x_n|| < 1$, then must we have ||x|| < 1?

Solution. The unit ball $B = \{y \in \mathbb{R}^m \mid ||y|| \le 1\}$ is closed. Let $x_n \in B$ and $x_n \to x \Rightarrow x \in B$ as B is closed, by theorem 3.6.2. This is not true if \le is replaced by <; for example, on \mathbb{R} , consider $x_n = 1 - \frac{1}{n}$.

3.7 Cauchy sequence

Let (M, d) be a metric space. A Cauchy sequence is a sequence $x_k \in M$ such that for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon$.

3.8 Complete Metric Space

The metric space M is called *complete* iff every Cauchy sequence in M converges to a point in M.

In normed space, such as \mathbb{R}^n a sequence v_k is cauchy sequence if for every $\epsilon > 0$ there is an N such that $||v_k - v_j|| < \epsilon$ whenever $j, k \geq N$.

3.9 Bounded Sequence

A sequence x_k in a normed space is bounded if there is a number M' > 0 such that $||x_k|| \le M$ for every k.

In a metric space we require that there be a point x_0 such that $d(x_k, x_0) \leq M'$ for all k.