Chapter 1

Hypergeometric Function

1.1 Introduction

The hypergeometric differential equation is an equation of the form

$$\left(x^2 - x\right) y'' + \Big[(1+\alpha+\beta)x - \gamma\Big] y' + \alpha\beta y = 0 \tag{1.1}$$

where the parameters α, β, γ are constant, and it is assumed that γ is not a negative integer.

Equation (1.1) can be written as

$$y'' + X_1 y' + X_2 y = 0 \tag{1.2}$$

where

$$X_1 = \frac{(1+\alpha+\beta)x - \gamma}{x(x-1)}, \qquad X_2 = \frac{\alpha\beta}{x(x-1)}$$

Equation (1.1) and (1.2) has singularities at $x = 0$, 1 and ∞.
For $x = 0$, the general solution of (1.1) is $y = Au + Bv$, where A, B are constant and

$$\begin{aligned} u &= 1 + \frac{\alpha\beta}{1\cdot\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \dots \\ &= \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{r!\,(\gamma)_r} x^r \\ &= F(\alpha,\beta;\gamma;x) \quad \text{or} \quad {}_2F_1(\alpha,\beta;\gamma;x) \end{aligned}$$

and

$$v = x^{1-\gamma} F(\alpha', \beta'; \gamma'; x) \qquad \left| \qquad \begin{aligned} \text{Where, } \alpha' &= 1 - \gamma + \alpha \\ \beta' &= 1 - \gamma + \beta \\ \gamma' &= 2 - \gamma \end{aligned} \right.$$

Similarly, *for* $x = 1$ *and* $x = \infty$, the solutions of (1.1) are

$$y = AF(\alpha,\beta;1+\alpha+\beta-\gamma;1-x) + B(1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-x)$$

and

$$y = Ax^{-\alpha} F\left(\alpha, \alpha-\beta+1; \alpha-\beta+1; \frac{1}{x}\right) + Bx^{-\beta} F\left(\beta, \beta-\gamma+1; \beta-\alpha+1; \frac{1}{x}\right)$$

respectively.

One of the solutions of the hypergeometric differential equation

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!} x^r$$

is known as hypergeometric function.

1.1.1 Pochhammer Symbol

The Pochhammer symbol is denoted and defined by

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+r-1), \qquad \text{with } (\alpha)_0 = 1$$

$(\alpha)_r$ can also be expressed as

$$(\alpha)_r = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}$$

1.2 Integral Formula for the Hypergeometric Function

Problem 1.2.1. If $|x| < 1$ and if $\gamma > \beta > 0$, prove that

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,\mathrm{d}\,t $$

Proof. By definition, we have

$$ F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!} x^r \tag{1.3} $$

where

$$ \begin{aligned} (\alpha)_r &= \alpha(\alpha+1)(\alpha+2)\dots(\alpha+r-1) \\ &= \frac{1\cdot 2\cdot 3\dots(\alpha-1)\alpha(\alpha+1)(\alpha+2)\dots(\alpha+r-1)}{1\cdot 2\cdot 3\dots(\alpha-1)} \\ &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \end{aligned} $$

$$ \begin{aligned} \therefore\ \frac{(\beta)_r}{(\gamma)_r} &= \frac{\Gamma(\beta+r)}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+r)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta+r)\Gamma(\gamma-\beta)}{\Gamma(\gamma+r)\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta+r)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma+r-\beta)} \cdot \frac{1}{\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta+r)\Gamma(\gamma-\beta)}{\Gamma(\beta+r+\gamma-\beta)} \cdot \frac{1}{\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\cdot\Gamma(\gamma-\beta)} \cdot B(\beta+r, \gamma-r) \qquad \text{where } \beta+r>0\, \gamma-\beta>0 \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\cdot\Gamma(\gamma-\beta)} \cdot \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1}\,\mathrm{d}\,t \qquad \text{since } \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,\mathrm{d}\,t \\ &= \frac{1}{\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)}} \cdot \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1}\,\mathrm{d}\,t \\ &= \frac{1}{B(\beta,\gamma-\beta)} \cdot \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1}\,\mathrm{d}\,t \end{aligned} $$

Thus from (1.3), we have

$$ \begin{aligned} F(\alpha, \beta; \gamma; x) &= \sum_{r=0}^{\infty} \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1} \times \frac{(\alpha)_r}{r!} \cdot x^r\,\mathrm{d}\,t \\ &= \sum_{r=0}^{\infty} \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \times \frac{(\alpha)_r}{r!} \cdot (xt)^r\,\mathrm{d}\,t \\ &= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \left\{ \sum_{r=0}^{\infty} \frac{(\alpha)_r (xt)^r}{r!} \right\} \mathrm{d}\,t \end{aligned} $$

Note. The general term in the expansion of $(1-xt)^{-\alpha}$ is[1]

$$\begin{aligned}(1-xt)^{-\alpha} &= \frac{(-\alpha)(-\alpha-1)\dots(-\alpha-r+1)}{r!}(-xt)^r \\ &= (-1)^r\frac{\alpha(\alpha+1)\dots(\alpha+r-1)}{r!}(-1)^r(xt)^r \\ &= \frac{\alpha(\alpha+1)\dots(\alpha+r-1)}{r!}x^r\,t^r \\ &= \frac{(\alpha)_r}{r!}\cdot x^r\cdot t^r\end{aligned}$$

$$\begin{aligned}\therefore\ F(\alpha,\beta;\gamma;x) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,\mathrm{d}\,t \\ \text{or, } F(\alpha,\beta;\gamma;x) &= \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,\mathrm{d}\,t\end{aligned}$$

Which is known as the integral formula for hypergeometric function and is valid if $|x|<1$ and $\gamma>\beta>0$. □

1.3 Gauss's Theorem

Theorem 1.3.1.

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

Proof. From the definition of integral formula for the hypergeometric function, we have,

$$F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,\mathrm{d}\,t \tag{1.4}$$

Putting $x=1$ in (1.4), we get,

$$\begin{aligned}F(\alpha,\beta;\gamma;1) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha}\,\mathrm{d}\,t \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{(\gamma-\beta-\alpha)-1}\,\mathrm{d}\,t \\ &= \frac{B(\beta,\gamma-\beta-\alpha)}{B(\beta,\gamma-\beta)} \\ &= \frac{\dfrac{\Gamma(\beta)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\beta+\gamma-\beta-\alpha)}}{\dfrac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)}} \\ &= \frac{\Gamma(\beta)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)}\times\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\end{aligned}$$

Hence $F(\alpha,\beta;\gamma;1) = \dfrac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$ □

[1]
$$\sum_{r=0}^{\infty}\frac{(\alpha)_r(xt)^r}{r!} = 1+\frac{\alpha(xt)}{1!}+\frac{\alpha(\alpha+1)}{2!}(xt)^2+\cdots = (1-xt)^{-\alpha}$$

1.4 Problems

Problem 1.4.1. Prove that

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

Proof. From Rodrigue's formula for Legendre polynomial, we have

$$\begin{aligned}
P_n(x) &= \frac{1}{2^n\, n!} \frac{\mathrm{d}^n}{\mathrm{d}\,x^n} \left(x^2-1\right)^n \\
&= \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\,x^n} \left[(x-1)^n \left\{\frac{1}{2}(x+1)\right\}^n\right] \\
&= \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\,x^n} \left[(1-x)^n \left\{1-\frac{1}{2}(1-x)\right\}^n\right] \\
&= \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\,x^n} \left[(1-x)^n \left\{1-n\frac{1}{2}(1-x)+\frac{n(n-1)}{2!}\cdot\frac{(1-x)^2}{4}-\frac{n(n-1)(n-2)}{3!}\cdot\frac{(1-x)^3}{8}+\dots\right\}\right] \\
&= \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\,x^n} \left[(1-x)^n - n\frac{1}{2}(1-x)^{n+1}+\frac{n(n-1)}{2!\,2^2}\cdot(1-x)^{n+2}-\frac{n(n-1)(n-2)}{3!\cdot 2^3}\cdot(1-x)^{n+3}+\dots\right] \\
&= \frac{(-1)^n}{n!} \left[(-1)^n\, n! - \frac{n}{2}(-1)^n\frac{(n+1)!}{1!}(1-x)+\frac{n(n-1)}{2!}(-1)^n\frac{(n+2)!}{2!}(1-x)^2-\dots\right] \\
&= 1+\frac{(-n)(n+1)}{1\cdot 1!}\left(\frac{1-x}{2}\right)+\frac{(-n)(-n+1)(n+1)(n+2)}{1\cdot 2\cdot 2!}\left(\frac{1-x}{2}\right)^2+\dots \\
&= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)
\end{aligned}$$

Hence $P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$. □