Chapter 1

Navier-Stokes Equation

Problem 1.1. Derive the Navier-Stokes equation in tensor form.

Solution. Let $\vec{u}$ be the fluid velocity at time t at the position vector $\vec{x}$, so that $\vec{u}$ is a function of t and $\vec{x}$. The components of $\vec{u}$ are $u_i = (u_1, u_2, u_3)$, so that each component is a function of t and x:

$$u(t, x_1, x_2, x_3, \dots) \quad \text{etc.}$$

Consider a small volume v in which the velocity components do not vary significantly. The total momentum in this volume is given by

$$\int_V \rho \,\mathrm{d}\, v \cdot \vec{u} \tag{1.1}$$

It can be shown that the rate of change of this quantity

$$\int \rho \frac{D}{Dt}\vec{u}(t,\vec{x}) \,\mathrm{d}\, v = \int \rho \,\mathrm{d}\, v \{\frac{\partial u}{\partial t} + (\vec{u} \cdot \vec{\nabla})\}$$

where $\vec{u} \cdot \vec{\nabla} = (u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3})$. Which is simply the sum of the products of mass and acceleration for all the elements of the material volume V, can be rewritten as,

$$\int_V \rho \,\mathrm{d}\, v \cdot \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + (\vec{u} \cdot \vec{\nabla}) u_i$$

A portion of fluid is acted on, in general by both volume and surface forces.

We denote the vector resultant of the volume forces per unit mass of fluid, by $\vec{F}$, so that the total volume force on the selected portion of fluid is

$$\int F_i \rho \,\mathrm{d}\, v$$

The $i-$th component of the surface on contact force exerted across a surface element of area $\mathrm{d}\, s$ and normal $\vec{n}$ may be represented as $\sigma_{ij} n_j \,\mathrm{d}\, s$, where σ_{ij} is the stress tensor and the total surface force exerted on the selected portion of fluid by

$$\int \sigma_{ij} n_j \,\mathrm{d}\, s = \int \frac{\partial \sigma_{ij}}{\partial x_j} \,\mathrm{d}\, v$$

$$\text{(Total force = Body force + surface force)}$$

$$\Rightarrow \rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{1.2}$$

This is the equation of the motion for a fluid where the stress tensor σ_{ij} can be written as follows

$$\sigma_{ij} = -P\delta_{ij} + 2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij})$$

substituting this into (1.2), the equation of motion we get,

$$\rho \frac{Du_i}{Dt} = \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} 2\mu(e_{ij} - \frac{1}{3}\Delta\delta_{ij})$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{and} \quad \Delta = e_{ij}$$

$$
\begin{aligned}
\therefore \frac{\partial\left(e_{ij}-\frac{1}{3}\Delta\delta_{ij}\right)}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}+\frac{\partial^2 u_j}{\partial x_j\,\partial x_i}\right)-\left(\frac{1}{3}\cdot\frac{\partial\nabla}{\partial x_j}\delta_{ij}\right)\\
&= \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}+\frac{\partial^2 u_j}{\partial x_j\,\partial x_i}\right)-\left(\frac{1}{3}\cdot\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_j}\right)\delta_{ij}\right)\\
&= \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}\right)+\frac{1}{2}\frac{\partial^2 u_j}{\partial x_j\,\partial x_i}-\frac{1}{3}\frac{\partial^2 u_i}{\partial x_j\,\partial x_i}\\
&= \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}\right)+\frac{1}{6}\frac{\partial}{\partial x_j}(\frac{\partial u_i}{\partial x_i})\\
&= \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}\right)+\frac{1}{6}\frac{\partial\,\nabla}{\partial x_j}\\
&= \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}\right)+\frac{1}{6}\frac{\partial}{\partial x_j}(\vec{\nabla}\cdot\vec{u})
\end{aligned}
$$

For incompressible fluid, $\vec{\nabla}\cdot\vec{u}=0$,

$$
\begin{aligned}
\frac{\partial}{\partial x_j}\left(e_{ij}-\frac{1}{3}\Delta\delta_{ij}\right) &= \frac{1}{2}\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}\\
&= \frac{1}{2}\frac{\partial^2 u_i}{\partial x_j^2}\\
&= \frac{1}{2}\nabla^2 u
\end{aligned}
$$

$$
\therefore\ \rho\frac{Du_i}{Dt} = \rho F_i - \frac{\partial P}{\partial x_i} + 2\mu\cdot\frac{1}{2}\nabla^2 u_i
$$

$$
\begin{aligned}
\therefore\ \rho\frac{Du_i}{Dt} &= \rho F_i - \frac{\partial P}{\partial x_i} + \mu\nabla^2 u_i\\
\frac{Du_i}{Dx_i} &= 0
\end{aligned}
\tag{1.3}
$$

$\therefore$ (1.3) is the Navier-Stokes equation in tensor form.
We may rewrite (1.3) in Lamb vector form,

$$
\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}
$$

$$
\begin{aligned}
\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\cdot\vec{u} &= \vec{F} - \frac{1}{\rho}\left(\frac{\partial P}{\partial x}\vec{i}+\frac{\partial P}{\partial y}\vec{j}+\frac{\partial P}{\partial z}\vec{k}\right)+\nu\nabla^2\vec{u}\\
\Rightarrow\ \frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\cdot\vec{u} &= \vec{F} - \frac{1}{\rho}\vec{\nabla}P + \nu\nabla^2\vec{u}\\
\vec{\nabla}\cdot\vec{u} &= 0
\end{aligned}
$$