Chapter 1

Continuity

1.1 Limit

Definition 1 (Limit)**.** Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y (i.e., $f : E \subset X \to Y$), and p is a limit point of E. We write $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$ if there is a point $q \in Y$ with following property:
For every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_x(f(x), p) < \delta$[2]

Example. $E = (0, 2) \subset X = \mathbb{R}^1$, $Y = \mathbb{R}^1$; $f(x) = \frac{x^2-1}{x-1}$; $p = 1$ is a limit point of E,
Then $\lim_{x \to p} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

Theorem 1.1.1 (Sequential Criteron of Limits). Let x, y, E, f and p as in the above definition. Then $\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\langle p_n \rangle$ in E such that $p_n \neq p$, $\lim_{n \to \infty} p_n = p$

1.2 Continuity

Definition 2 (Continuity)**.** Suppose X and Y are metric spaces, $E \subset X$, $p \in E$ and f maps $E \to Y (f : E \to Y)$. Then f is said to be continuous at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{d_y(f(x), p)} < \epsilon \text{ for all points } x \in E \text{ for which } \mathrm{d_x(x, p)} < \delta$$

Theorem 1.2.1. Let $f : E \subset X \to Y$ be a mapping. Then the following assertions are equivalent:

(i) f is continuous on E.

(ii) For each convergent sequence $x_n \to x_0$, we have $f(x_n) \to f(x_0)$

(iii) For each open set U i Y, $f^{-1}(U) \subset E$ is open relative to E; that is, $f^{-1}(U) = E \cap V$ for some open set V.

(iv) For each closed set $F \in Y$, $f^{-1}(F) \subset E$ is closed relative to E; that is $f^{-1}(F) = E \cap G$ for some closed set G.

Theorem 1.2.2. Suppose $f : X \to Y$ is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

[2]The δ may depend on $f(x)$, p, and ϵ i.e., $\delta = \delta(p, f(x), \epsilon)$

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$, since f is continuous, by previous theorem each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices say $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that

$$X \subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \cdots \cup f^{-1}(V_{\alpha_n}) \tag{1.1}$$

since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, then (1.1) implies that $f(X) \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n}$ This completes the proof. □

note to self :: There may be some page left.