Chapter 1

Laguerre Polynomial

We define the standard solution of Laguerre's differential equation $x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}+ny=0$ as that for which $c_0=1$ and call it the Laguerre polynomial of order n and is denoted by $L_n(x)$.

$$\therefore\ L_n(x)=\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!\,(r!)^2}x^r$$

1.1 Generating Function of Laguerre Polynomial

Problem 1.1.1. Prove that

$$\frac{1}{(1-t)}e^{\frac{-tx}{1-t}}=\sum_{n=0}^{\infty}t^n\,L_n(x)$$

Proof. From exponential series we have

$$e_x=1\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^r}{r!}+\ldots \tag{1.1}$$

$$\begin{aligned}
\therefore\ \frac{1}{(1-t)}e^{\frac{-tx}{1-t}} &= \frac{1}{1-t}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{-tx}{1-t}\right)^r \qquad \text{[using 1.1]}\\
&=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!}\frac{t^r\,x^r}{(1-t)^{r+1}}\\
&=\sum_{r=0}^{\infty}(-1)^r\frac{t^r\,x^r}{r!}(1-t)^{-(r+1)}\\
&=\sum_{r=0}^{\infty}(-1)^r\frac{t^r\,x^r}{r!}\left[1+(r+1)^r+\frac{(r+1)(r+2)}{2!}t^2+\frac{(r+1)(r+2)(r+3)}{3!}t^3+\ldots\right]\\
&=\sum_{r=0}^{\infty}\left[(-1)^t\frac{t^r\,x^r}{r!}\sum_{s=0}^{\infty}\frac{(r+s)!}{r!\,s!}t^s\right] \qquad \text{[using binomial theorem]}\\
&=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r\frac{(r+s)!}{(r!)^2\,s!}x^r\,t^{r+s}
\end{aligned}$$

Let r be fixed. The coefficient of t^n can be obtained by setting $r+s=n$ i.e., $s=n-r$. Hence, for a fixed value of r the coefficient of t^n is given by

$$(-1)^r\frac{n!}{(r!)^2(n-r)!}x^r$$

Therefore, the total coefficient of t^n is obtained by summing over all allowed values of r.

Since $s=n-r$ and $s\geq 0$.

$\therefore\ n-r\geq 0$ or, $r\leq n$.

Hence, the coefficient of t^n is

$$\sum_{r=0}^{\infty}(-1)^r\frac{n!}{(r!)^2\,(n-r)!}x^r=L_n(x)$$

Thus

$$\frac{1}{(1-t)}e^{\frac{-tx}{1-t}}=\sum_{n=0}^{\infty}t^n\,L_n(x)$$

□

1.2 Rodrigue's Formula for Laguerre Polynomial

Problem 1.2.1. Prove that

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^n e^{-x}\right)$$

Proof. Right-hand side

$$\begin{aligned}
&\frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^n e^{-x}\right)\\
=&\frac{e^x}{n!}\left[x^n(-1)^n e^{-n} + n\cdot nx^{n-1}(-1)^{n-1}e^{-x} + \frac{n(n-1)}{2!}n(n-1)x^{n-2}(-1)^{n-2}e^{-x} + \cdots + n!e^{-x}\right]\\
=&\frac{e^x\cdot e^{-x}}{n!}\left[(-1)^n x^n + \frac{n(n!)}{1!(n-1)!}x^{n-1} + (-1)^{n-2}\frac{n(n-1)}{2!}\cdot\frac{n!}{(n-2)!}x^{n-2} + \cdots + n!\right]\\
=&(-1)^n\cdot\frac{n!}{(n!)^2}x^n + (-1)^{n-1}\frac{n!}{1!\{(n-1)!\}^2}x^{n-1} + (-1)^{n-2}\frac{n!}{2!\{(n-2)!\}^2}x^{n-2} + \dots\frac{n!}{n!}\\
=&\sum_{r=0}^{n}(-1)^r\frac{n!\,x^r}{\{r!\}^2\,(n-r)!}\\
=&L_n(x)
\end{aligned}$$

Hence

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^n e^{-x}\right)$$

□

1.3 Orthogonality Property of Laguerre Polynomials

Problem 1.3.1. Prove that

$$\int_0^\infty e^{-x}L_n(x)L_m(x)\,\mathrm{d}\,x = \begin{cases}0 & \text{if } m\neq n\\ 1 & \text{if } m=n\end{cases}$$

or, Prove that

$$\int_0^\infty e^{-x}L_n(x)L_m(x)\,\mathrm{d}\,x = \delta_{mn}$$

or, Show that Laguerre polynomials are orthogonal over $(0,\infty)$ with respect to the weighted function e^{-x}.

Proof. From generating function of Laguerre polynomial we have,

$$\sum_{n=0}^{\infty} t^n\,L_n(x) = \frac{1}{(1-t)}e^{\frac{-tx}{1-t}}$$

and

$$\sum_{m=0}^{\infty} s^m\,L_m(x) = \frac{1}{(1-s)}e^{\frac{-sx}{1-s}}$$

$$\begin{aligned}
\therefore \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_n(x)\,L_m(x)t^n s^m &= \frac{1}{(1-t)(1-s)}e^{\frac{-tx}{1-t}}\cdot e^{\frac{-sx}{1-s}}\\
&= \frac{1}{(1-t)(1-s)}e^{-x\left\{\frac{t}{1-t}+\frac{s}{1-s}\right\}} \qquad (1.2)
\end{aligned}$$

Multiplying both sides of (1.2) by e^{-x} and then integrating both sides with respect to x from 0 to ∞, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\,\mathrm{d}\,x\right] t^n s^m \\
&= \frac{1}{(1-t)(1-s)} \int_0^{\infty} e^{-x\left\{1+\frac{t}{1-t}+\frac{s}{1-s}\right\}}\,\mathrm{d}\,x \\
&= \frac{1}{(1-t)(1-s)}\left[\frac{e^{-x\left\{1+\frac{t}{1-t}+\frac{s}{1-s}\right\}}}{-\left(1+\frac{t}{1-t}+\frac{s}{1-s}\right)}\right]_0^{\infty} \\
&= \frac{1}{(1-t)(1-s)}\cdot\frac{1}{1+\frac{t}{1-t}+\frac{s}{1-s}} \\
&= \frac{1}{(1-t)(1-s)+t(1-s)+s(1-t)} \\
&= \frac{1}{1-t-s+st+t-ts+s-st} \\
&= \frac{1}{1-st} \\
&= (1-st)^{-1} \\
&= 1+st+(st)^2+\cdots+(st)^n+\dots \\
&= \sum_{n=0}^{\infty} s^n t^n \qquad \text{using binomial theorem}
\end{aligned} \tag{1.3}$$

Now we see that the indices of t and s are always equal in each term on right hand side of (1.3). Hence when $m \neq n$, equating coefficient of $t^n s^m$ on both sides of (1.3) gives

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,\mathrm{d}\,x = 0 \qquad \text{if } m \neq n \tag{1.4}$$

Again equating coefficients of $t^n s^m$ on both sides of (1.3) gives

$$\int_0^{\infty} \big(L_n(x)\big)^2\,\mathrm{d}\,x = 1 \tag{1.5}$$

Hence

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,\mathrm{d}\,x = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{1.6}$$

Let

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{1.7}$$

Thus from (1.6) and (1.7), we have

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,\mathrm{d}\,x = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

□

1.4 Recurrence Formula for Laguerre Polynomial

(i) $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$

(ii) $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$

(iii) $L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)$

1.5 Problems on Laguerre Polynomial

Problem 1.5.1. Expand $x^3 + x^2 - 3x + 2$ in a series of Laguerre polynomial.

Solution. Let $f(x) = x^3 + x^2 - 3x + 2$. By definition of Laguerre polynomial, we know that $L_n(x)$ is a polynomial of degree n. Since $x^3 + x^2 - 3x + 2$ is a polynomial of degree 3, we may write

$$\begin{aligned}
&x^3 + x^2 - 3x + 2 \\
&= C_0 L_0(x) + C_1 L_1(x) + C_2 L_2(x) + C_3 L_3(x) \\
&= C_0 + C_1(1 - x) + C_2\left(1 - 2x + \frac{1}{2}x^2\right) + C_3\left(1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3\right) \\
&= C_0 + C_1 - C_1 x + C_2 - 2C_2 x + \frac{C_2}{2}x^2 + C_3 - 3C_3 x + \frac{3}{2}C_3 x^2 - \frac{1}{6}C_3 x^3 \\
&= (C_0 + C_1 + C_2 + C_3) - (C_1 + 2C_2 + 3C_3)x + \left(\frac{C_3}{2} + \frac{3}{2}C_3\right)x^2 - \frac{1}{6}C_3 x^3 \qquad (1.8)
\end{aligned}$$

Equating the coefficients of like powers of x from both sides of (1.8), we get

$$\begin{aligned}
C_0 + C_1 + C_2 + C_3 &= 2 \\
C_1 + 2C_2 + 3C_3 &= 3 \\
\frac{C_3}{2} + \frac{3}{2}C_3 &= 1 \\
-\frac{1}{6}C_3 &= 1
\end{aligned}$$

Solving these for C_0, C_1, C_2, and C_3, we get

$$C_3 = -6, \quad C_2 = 20, \quad C_1 = -19, \quad C_0 = -7$$

Thus $f(x) = x^3 + x^2 - 3x + 2 = 7L_0(x) - 19L_1(x) + 20L_2(x) - 6L_3(x)$.