Chapter 1

The Topology of Euclidean Space

We want to study the basic properties of $\mathbb{R}^n$ which are important for the notion of a continuous function. We will study open sets, which generalize open intervals on $\mathbb{R}$, and closed sets, which generalize closed intervals. The study of open and closed sets constitutes the begging of topology.

1.1 Open Sets

Definition 1. Let (M, d) be a metric space. For each fixed $x \in M$ and $\varepsilon > 0$, the set $D(x, \varepsilon) = \{y \in M \mid d(x, y) < \varepsilon\}$ is called the ε-disk about x (also called the ε-neighbourhood or ε-ball about x).

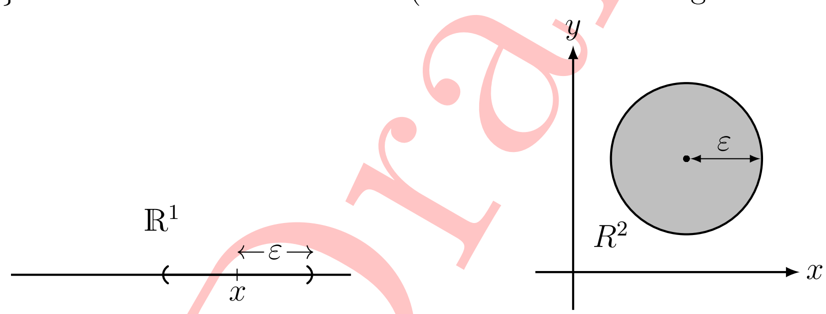


Figure 1.1: The ε-disk

A set $A \subset M$ is said to be open if for each $x \in A$, there exists an $\varepsilon > 0$ such that $D(x, \varepsilon) \subset A$. A neighborhood of a point in M is an open set containing that point.

Note. The empty set $\varnothing$ and the whole space M are open. It is important to realize that the ε required in the definition of an open set may depend on x. For example, the unit square in $\mathbb{R}^2$ not including the "boundary" is open, bur the ε-neighborhood get smaller as we approach the boundary. However, the ε-neighborhood cannot be zero for any x.

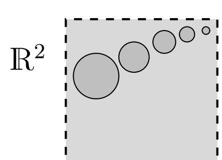


Figure 1.2: An open set

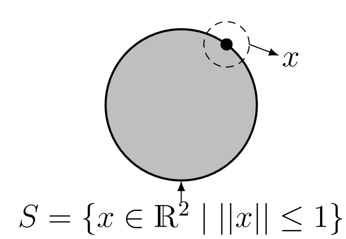
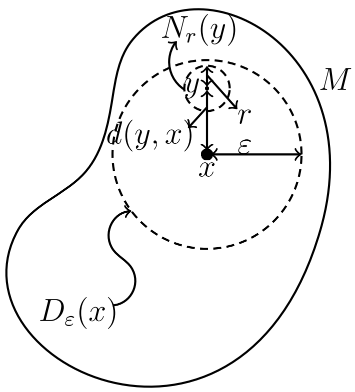


Figure 1.3: A non open set

Theorem 1.1.1. *In a metric space M, each ε-disk $D(x, \varepsilon)$ is open.*

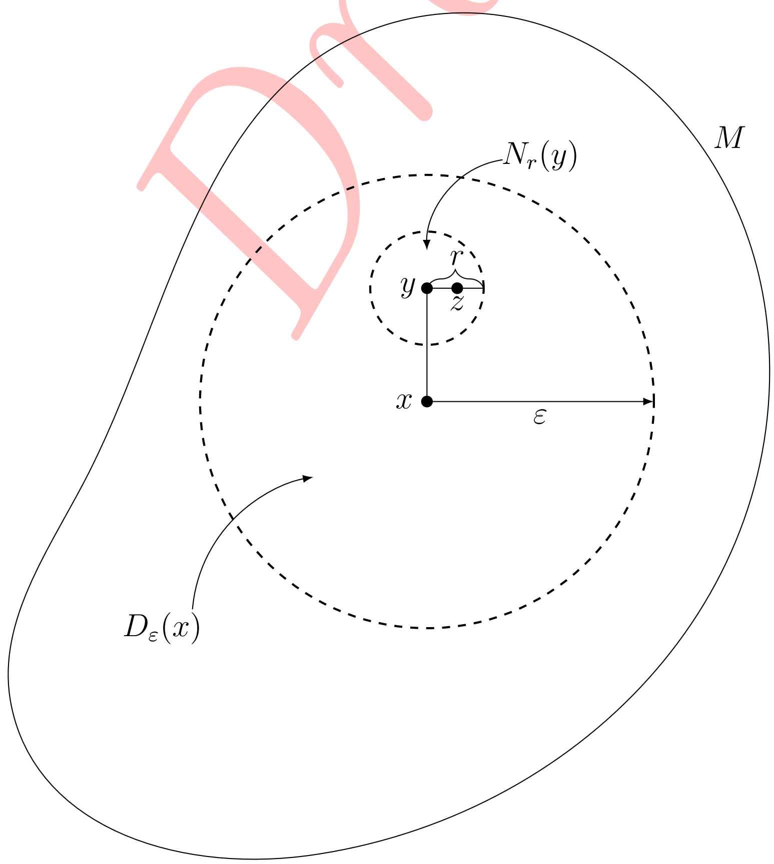
Proof. Assume $D_\varepsilon(x) \equiv D(x, \varepsilon)$



Consider $r > 0$ such that $r + d(y, x) < \varepsilon$.
Now let $z \in N_r(y)$ then $d(y, z) < r$
So,

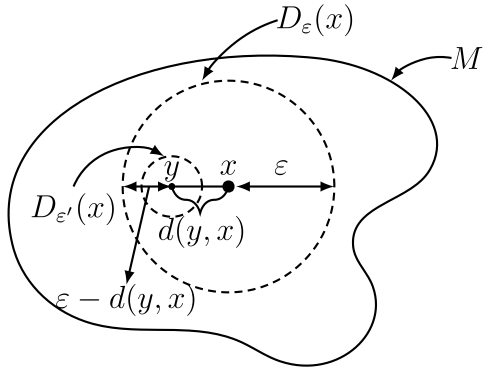
$$\begin{aligned} d(x, z) &\le d(x, y) + d(y, z) \\ &< r + d(x, y) < \varepsilon \\ \Rightarrow z \in D_\varepsilon(x) \end{aligned}$$

Hence $N_r(y) \subset D_\varepsilon(x)$ showing that $D_\varepsilon(x)$ is open in M. $\square$



Theorem 1.1.2. *In a metric space M, each ε-disk (or ε-neighborhood or neighborhood of $x \in M$) $D_\varepsilon(x)$ is open.*

Proof. Choose $y \in D_\varepsilon(x)$. We must produce an ε' such that $D_{\varepsilon'}(y) \subset D_\varepsilon(x)$.



The figure suggests that we try $\varepsilon' = \varepsilon - d(x,y)$, which is strictly positive, since $d(x,y) < \varepsilon$. With this choice (which depends on y), we shall show that $D_{\varepsilon'}(y) \subset D_\varepsilon(x)$. Let $z \in D_{\varepsilon'}(y)$, so that $d(z,y) < \varepsilon'$. We need to prove that $d(z,x) < \varepsilon$. But, by the triangle inequality, $d(z,x) \le d(z,y) + d(y,x) < \varepsilon' + d(y,x) = \varepsilon$. □

Theorem 1.1.3. *In a metric space (M,d),*

(i) both Φ and X are open

(ii) any union of open sets is open

(iii) any intersection of finite number of open sets is open.

To appreciate the difference between assertions ((ii)) and ((iii)), note that the intersection of an arbitrary family of open sets need not be open. For example, in $\mathbb{R}^1$, a single point (which is not an open set) is the intersection of the collection of all open intervals containing it. [$G_k = \left\{\left(-\frac{1}{k}, \frac{1}{k}\right) : k \in \mathbb{N}\right\}$ Take $\bigcap_1^\infty G_k = \{0\}$, a single point set which is not an open set]

Note. A set with a specified collection of subsets (called, bu=y definition, open sets) obeying the rules in Theorem (1.1.3) above, and containing the empty set and the whole space is called a *TOPOLOGICAL SPACE.*

Proof.

(i) Since there are no points in Φ, each point in Φ is the center of an ε-disk contained in Φ. For any $x \in M$, every ε-neighbourhood $D_\varepsilon(x)$ is contained in M.

(ii) Consider a family of open sets $\{G_\alpha : \alpha \in \mathbb{N}\}$ with $\bigcup_{\alpha=1}^\infty G_\alpha = A$. Let $x \in A$, then $x \in G_\alpha$ for same $\alpha \in \mathbb{N}$. Hence, since G_α is open, $D_\varepsilon(x) \subset G_\alpha \subset A$ for some $\varepsilon > 0$, proving that A is open.

(iii) It satisfies to prove that the intersection of two open sets is open, since we can use induction to get the general result by writing $G_1 \cap G_2 \cap \cdots \cap G_n = (G_1 \cap G_2 \cap \cdots \cap G_{n-1}) \cap G_n$.

Let A and B be open and $C = A \cap B$; if $C = \varnothing$, C is open, by (i). Therefore, suppose $x \in C$. Since A and B are open, there exist $\varepsilon, \varepsilon' > 0$ such that $D_\varepsilon(x) \subset A$ and $D_{\varepsilon'}(x) \subset B$.
Let ε'' be the smaller of ε and ε'. Then $D_{\varepsilon''}(x) \subset D_\varepsilon(x)$ and so $D_{\varepsilon''}(x) \subset A$; and similarly, $D_{\varepsilon''}(x) \subset B$, and so $D_{\varepsilon''}(x) \subset A \cap B = C$, as required.

$\square$

Problem 1.1.1. Let $A \subset \mathbb{R}^n$ be open and $B \subset \mathbb{R}^n$. Define $A + b = \{x + y \in \mathbb{R}^n \mid x \in A \text{ and } y \in B\}$. Prove that $A + B$ is open.

Proof. Let $w \in A + B$. There are points $x \in A$ and $y \in B$ with $w = x + y$. Since A is open, there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \subset A$. We claim that $D_\varepsilon(w) \subset A + B$. Suppose $z \in D_\varepsilon(w)$, then $d(w, z) = ||w - z|| < \varepsilon$. But $\varepsilon > ||z - w|| = ||z - (x + y)|| = ||(z - y) - x|| = d(x, z - y)$, so $z - y \in D_\varepsilon(x) \subset A$. Since $y \in B$, this forces $z = (z - y) + y$ to be in $A + B$. Thus, $D_\varepsilon(w) \subset A + B$ and hence $A + B$ is an open set. $\square$

1.2 Interior of a Set

Definition 2. Let M be a metric space and $A \subset M$. A point $x \in A$ is called an interior point of A if there is an open set U such that $x \in U \subset A$. The interior of A is the collection of all interior points of A and is denoted $\operatorname{int}(A)$. The set might be empty.

Equivalently, x is an interior point of A if there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \subset A$.

Example.

1. The interior of a single point in $\mathbb{R}^n$ is empty.

2. The interior of the unit disk in $\mathbb{R}^2$, including its boundary, is the unit disk without its boundary.

The interior of A also can be described as the union of all open subsets of A. Thus, by theorem (1.1.3), $\operatorname{int}(A)$ is open. Hence, $\operatorname{int}(A)$ is the largest open subset of A. If there are no open subsets of A, then $\operatorname{int}(A) = \varnothing$. Also, it is evident that A is open if and only if $\operatorname{int}(A) = A$.

Problem 1.2.1. Let $S = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$. Find $\operatorname{int}(S)$.

Solution. To determine the interior points, we locate points about with it is possible to draw an ε-disk entirely contained in S. Notice that there are points (x, y) where $0 < x < 1$. Thus, $\operatorname{int}(S) = \{(x, y) \mid 0 < x < 1\}$.

1.3 Closed Sets

Definition 3. A set B in a metric space M is said to be closed if its complement (that is, the set $M \setminus B$) is open.

For example, a single point in $\mathbb{R}^n$ is a closed set. The set in $\mathbb{R}^2$ containing of the unit disk with its boundary is closed. Roughly speaking, a set is closed if it contains its "boundary points".

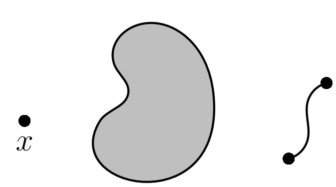


Figure 1.4: Closed sets

It is possible to have a set that is neither open nor closed. For example, in $\mathbb{R}^1$, a half-open interval $(0, 1]$ is neither open nor closed. Thus, even if we know that A is not open, we *cannot* conclude that it is closed or not closed.

Theorem 1.3.1. *In a metric space* (M, d)

(i) The whole space M *and the empty set* $\varnothing$ *are closed.*

(ii) The union of a finite number of closed subsets is closed.

(iii) The intersection of an arbitrary collection of closed subsets is closed.

Proof. Use defn and theorem (1.1.3) □

Note. Let, $I_n = [-n, n]$, then $\bigcup_{n=1}^{\infty}[-n, n] = [-1, 1] \cup [-2, 2] \cup [-3, 3] \cup \cdots = (-\infty, \infty) = \mathbb{R}$,open. This suggests that union of arbitrary collection of closed sets is not closed.

1.4 Accumulation Point

Definition 4. A point x in a metric space M is said to be an accumulation point of a set $A \subset M$ if every open set U containing x contains same points of A other than x.

Equivalently, $x \in M$ is an accumulation point of $A \subset M$ if for every $\varepsilon > 0$, the ε-disk $D_\varepsilon(x)$ contains same points of y of A with $y \neq x$. For example, in $\mathbb{R}^1$, a set consisting of a single point has no accumulation points and the open interval $(0, 1)$ has all points of $[0, 1]$ as accumulation points.

Theorem 1.4.1. *The set* $A \subset M$ *is closed if and only if the accumulation point of* A *belong to* A.

Proof. First, suppose A is closed. Then $M \setminus A$ is open. Thus, if $x \in M \setminus A$, there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \subset M \setminus A$; i.e., $D_\varepsilon(x) \cap A = \varnothing$. Thus, x is not an accumulation point, and so A contains all its accumulation points.

Conversely, suppose A contains all its accumulation points. Let $x \in M \setminus A$. Since x is not an accumulation point and $x \notin A$, there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \cap A = \varnothing$; i.e., $D_\varepsilon(x) \subset M \setminus A$.

Hence, $M \setminus A$ is open, and so A is closed. □

Problem 1.4.1. Let $S = \{x \in R \mid x \in [0, 1]\}$ and x is rational. Find the accumulation points of S.

Solution. The set of accumulation points consists of all points in $[0, 1]$. Indeed, let $y \in [0, 1]$ and $D_\varepsilon(y) = (y - \varepsilon, y + \varepsilon)$ be a neighborhood of y. We can find rational points in $(0, 1)$ arbitrary close to y (other than y) and in particular in $D_\varepsilon(y)$. Hence, y is an accumulation point. Any point $y \notin [0, 1]$ is not an accumulation point, because y has an ε-disk containing it that does not meet $[0, 1]$.

1.5 Closure of a Set

Definition 5. Let (M, d) be a metric space, and $A \subset M$. The closure of A denoted $\mathrm{cl}(A)$, is defined to be the intersection of all closed sets containing A.

Since, the intersection of any family of closed sets is closed. $\mathrm{cl}(A)$ is closed; it is also clear that A is closed if and only if $\mathrm{cl}(A) = A$. For example, on $\mathbb{R}^1$, $\mathrm{cl}((0, 1)) = [0, 1]$. The connection between closure and accumulation points is the following:

Theorem 1.5.1. *For $A \subset M$, $\text{cl}(A)$ consists of A plus the accumulation points of A. That is, $\text{cl}(A) = A \cup \{$ accumulation points of $A\}$.*

Problem 1.5.1. Find the closure of $A = [0, 1) \cup \{2\}$ in $\mathbb{R}$.

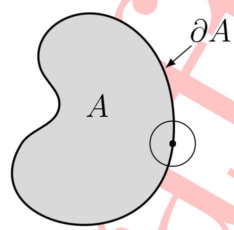
Solution. The accumulation point of A are $[0, 1]$, and so the closure is $[0, 1] \cup \{2\}$. This is clearly the smallest closed set we could find containing A.

1.6 Boundary of a Set

Definition 6. For a given set A in a metric space (M, d), the boundary is defined to be the set $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(M \setminus A)$. Sometimes the notation $\delta A = \text{bd}(A)$ is used.

Since the intersection of two closed sets is again a closed set, $\text{bd}(A)$ is a closed set. Also note that $\text{bd}(A) = \text{bd}(M \setminus A)$.

Proposition 1.6.1. *Let $A \subset M$. Then $x \in bd(A)$ if and only if for every $\varepsilon > 0$, $D_\varepsilon(x)$ contains point of A and of $M \setminus A$ (these points might include the points x itself).*



Problem 1.6.1. Let $A = \{x \in \mathbb{R} \mid x \in [0, 1]$ and x is rational$\}$. Find $\text{bd}(A)$.

Solution. $\text{bd}(A) = [0, 1]$, since, for any $\varepsilon > 0$ and $x \in [0, 1]$, $D_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ contains both rational and irrational points. One can also verify that $\text{bd}(A) = [0, 1]$ using the original definition of $\text{bd}(A)$.

Problem 1.6.2. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 > 1\}$. Find $\text{bd}(S)$.

Solution. Clearly, $\text{bd}(S)$ consists of the hyperbola $x^2 - y^2 = 1$.