Chapter 1

Charpit's Method

1.1 Derivation

Let us suppose the partial differential equation of first order is given by

$$f(x, y, z, p, q) = 0 \tag{1.1}$$

Also, we have

$$\mathrm{d}\,z = \frac{\partial\,z}{\partial\,x}\,\mathrm{d}\,x + \frac{\partial\,z}{\partial\,y}\,\mathrm{d}\,y$$
$$\Rightarrow \quad \mathrm{d}\,z = p\,\mathrm{d}\,x + q\,\mathrm{d}\,y \tag{1.2}$$

We assume that a relation

$$F(x, y, z, p, q) = 0 \tag{1.3}$$

exists such that after solving (1.1) and (1.3) simultaneously for p and q and putting these values of p, q in (1.2), (1.2) becomes integrable.
Thus, z, p, q may be expressed as functions of x and y.

Since these values identically satisfy (1.1) and (1.3) both, their differentiating coefficient with respect to x and y vanish.
We know that,

$$\begin{aligned}
p &= \frac{\partial\,z}{\partial\,x} \\
q &= \frac{\partial\,z}{\partial\,y} \\
\frac{\partial\,p}{\partial\,x} &= \frac{\partial^2\,z}{\partial\,x^2} = r \\
\frac{\partial\,q}{\partial\,y} &= \frac{\partial^2\,z}{\partial\,y^2} = t \\
\frac{\partial\,p}{\partial\,y} &= \frac{\partial^2\,z}{\partial\,y\,\partial\,x} = \frac{\partial^2\,z}{\partial\,x\,\partial\,y} = \frac{\partial\,q}{\partial\,x} = s
\end{aligned}$$

Now differentiating (1.1) with respect to x, we get

$$\frac{\partial\,f}{\partial\,x} + \frac{\partial\,f}{\partial\,z}\cdot\frac{\partial\,z}{\partial\,x} + \frac{\partial\,f}{\partial\,p}\cdot\frac{\partial\,p}{\partial\,x} + \frac{\partial\,f}{\partial\,q}\,\frac{\partial\,q}{\partial\,x} = 0$$
$$f_x + f_z\,p + f_p\,r + f_q\,s = 0 \tag{1.4}$$

Similarly differentiating (1.3) with respect to x, we get

$$F_x + F_z\,p + F_p\,r + F_q\,s = 0 \tag{1.5}$$

multiplying (1.4) by F_p and (1.5) by f_p we have

$$f_x\,F_p + f_z\,F_p\,p + f_p\,F_p\,r + f_q\,F_p\,s = 0 \tag{1.6}$$
$$f_p\,F_x + f_p\,F_z\,p + f_p\,F_p\,r + f_p\,F_q\,s = 0 \tag{1.7}$$

Subtracting (1.7) from (1.6), i.e., eliminating r we get,

$$(f_x F_p - f_p F_x) + (f_z F_p - f_p F_z)p + (f_q F_p - f_p F_q)s = 0 \tag{1.8}$$

Again, differentiating (1.1) and (1.3) with respect to y, we get

$$f_y + f_z q + f_p s + f_q t = 0 \tag{1.9}$$
$$F_y + F_z q + F_p s + F_q t = 0 \tag{1.10}$$

multiplying (1.9) by F_q and (1.10) by f_q we have

$$f_y F_q + f_z F_q q + f_p F_q s + f_q F_q t = 0 \tag{1.11}$$
$$f_q F_y + f_q F_z q + f_q F_p s + f_q F_q t = 0 \tag{1.12}$$

Subtracting (1.12) from (1.11), i.e., eliminating t we get,

$$(f_y F_q - f_q F_y) + (f_z F_p - f_q F_z)q + (f_p F_q - f_q F_p)s = 0 \tag{1.13}$$

Equation (1.8) and (1.13) contains s and for elimination of s we add (1.8) and (1.13), we get

$$\begin{aligned}(f_x F_p - f_p F_x) + (f_y F_q - f_q F_y) + (f_z F_p - f_p F_z)p + (f_z F_q - f_q F_z)q = 0\\ \Rightarrow (f_x + p f_z)F_p + (f_y + q f_z)F_q + (-p f_p - q f_q)F_z + (-f_p)F_x + (-f_q)F_y = 0\end{aligned}$$

This is a linear equation of order one with x, y, z, p, q as independent variables and F is dependent variable. Therefore, as in Lagrange's method, the auxiliary equations are

$$\frac{\mathrm{d}\,p}{f_x + p f_z} = \frac{\mathrm{d}\,q}{f_y + q f_z} = \frac{\mathrm{d}\,z}{-p f_p - q f_q} = \frac{\mathrm{d}\,x}{-f_p} = \frac{\mathrm{d}\,y}{-f_q} = \frac{\mathrm{d}\,F}{0} \tag{1.14}$$

Any integral of (1.14) will satisfy (1.8) and (1.13). The simplest relation involving at least one of p and q may be taken as $F = 0$. Now from equation (1.1) and (1.3) that is $F = 0$ and $f = 0$ the values of p and q should be found in terms of x and y and should be substituted in (1.2) which on integration gives the solution.

Generally, Charpit's auxiliary equations are written as

$$\frac{\mathrm{d}\,p}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,z}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,x}{-\frac{\partial f}{\partial p}} = \frac{\mathrm{d}\,y}{-\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,F}{0}$$

Either this form or the form given in (1.14) should be memorized.

1.2 Charpit's Method

Working rule:

1. Let us suppose we have non-linear partial differential equation

$$f(x, y, z, p, q) = 0 \tag{1.15}$$

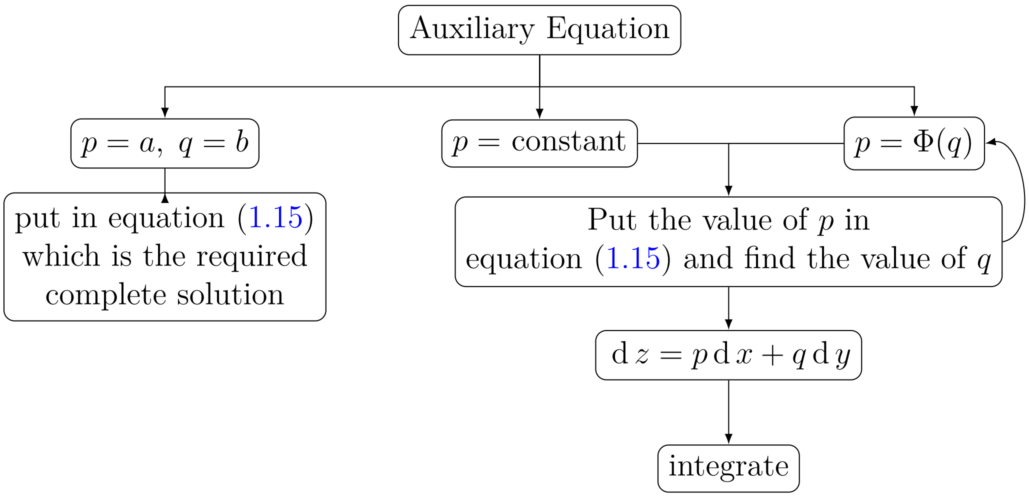
2. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial p}$, $\frac{\partial f}{\partial q}$

3. Write the Charpit's auxiliary equations

$$\frac{\mathrm{d}\,p}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,z}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,x}{-\frac{\partial f}{\partial p}} = \frac{\mathrm{d}\,y}{-\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,F}{0}$$

4. Find the values of p and q such that p and q are independent to each other.

5. Since $\mathrm{d}\,z = p\,\mathrm{d}\,x + q\,\mathrm{d}\,y$
   putting the values of p and q and integrate which gives the required complete solution.



Problem 1.2.1. Solve $z = px + qy + pq$

Solution.

$$z = px + qy + pq \tag{1.16}$$

Given,

$$f(x, y, z, p, q) = z - px - qy - pq$$

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = -q, \quad \frac{\partial f}{\partial z} = 1, \quad \frac{\partial f}{\partial p} = -x - q, \quad \frac{\partial f}{\partial q} = -y - p$$

The Charpit's auxiliary equation,

$$\frac{\mathrm{d}\,p}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,z}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,x}{-\frac{\partial f}{\partial p}} = \frac{\mathrm{d}\,y}{-\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,F}{0}$$

$$\Rightarrow \frac{\mathrm{d}\,p}{-p + p} = \frac{\mathrm{d}\,q}{-q + q} = \frac{\mathrm{d}\,z}{-p(-x - q) - q(-y - p)} = \frac{\mathrm{d}\,x}{-(x - q)} = \frac{\mathrm{d}\,y}{-(y - p)} = \frac{\mathrm{d}\,F}{0}$$

From first two fractions or ratios, we get

$$\mathrm{d}\,p = 0 \qquad \text{and} \qquad \mathrm{d}\,q = 0$$

Integrating we get,

$$p = a \qquad \text{and} \qquad q = b$$

putting the values of p and q in (1.16)

$$z = ax + by + ab$$

This is the required complete solution.

Problem 1.2.2. Solve $pxy + pq + qy = yz$

Solution.

$$pxy + pq + qy = yz \tag{1.17}$$

Given,

$$f = pxy + pq + qy - yz$$

$$\frac{\partial f}{\partial x} = py, \quad \frac{\partial f}{\partial y} = pz + q - z, \quad \frac{\partial f}{\partial z} = -y, \quad \frac{\partial f}{\partial p} = xy + q, \quad \frac{\partial f}{\partial q} = p + y$$

The Charpit's auxiliary equation,

$$\frac{\mathrm{d}\,p}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\mathrm{d}\,z}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,x}{-\frac{\partial f}{\partial p}} = \frac{\mathrm{d}\,y}{-\frac{\partial f}{\partial q}} = \frac{\mathrm{d}\,F}{0}$$

$$\Rightarrow \frac{\mathrm{d}\,p}{py + p(-y)} = \frac{\mathrm{d}\,q}{(px + q - z) + q(-y)} = \frac{\mathrm{d}\,z}{-p(xy + q) - q(p + y)} = \frac{\mathrm{d}\,x}{-(xy + q)} = \frac{\mathrm{d}\,y}{-(p + y)} = \frac{\mathrm{d}\,F}{0}$$

$$\Rightarrow \mathrm{d}\,p = 0$$

Integrating we get, $p =$ constant $= a$ putting the values of p in (1.17)

$$
\begin{aligned}
& axy + aq + qy = yz \\
\Rightarrow\ & q(a+y) = yz - axy \\
\Rightarrow\ & q = \frac{y(z-ax)}{a+y}
\end{aligned}
$$

Since,

$$
\begin{aligned}
& \mathrm{d}\,z = p\,\mathrm{d}\,x + q\,\mathrm{d}\,y \\
\Rightarrow\ & \mathrm{d}\,z = a\,\mathrm{d}\,x + \frac{y(z-ax)}{a+y}\,\mathrm{d}\,y \\
\Rightarrow\ & \mathrm{d}\,z - a\,\mathrm{d}\,x = \frac{y(z-ax)}{a+y}\,\mathrm{d}\,y \\
\Rightarrow\ & \frac{\mathrm{d}\,(z-ax)}{z-ax} = \frac{y}{a+y}\,\mathrm{d}\,y \\
\Rightarrow\ & \frac{\mathrm{d}\,(z-ax)}{z-ax} = \left(1 - \frac{a}{a+y}\right)\,\mathrm{d}\,y
\end{aligned}
$$

Integrating we get,

$$\ln(x - ax) = y - a\ln(a+y) + c$$

This is the required complete solution.