0.1 Problems

Problem 0.1. Employ the method of isoclines to sketch several approximate integral curves of the following differential equations.

(a) $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = x^2 + y^2$

(b) $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = y^3 - x^2$

(c) $\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{y}{x^2}$

Solution (a)**.**

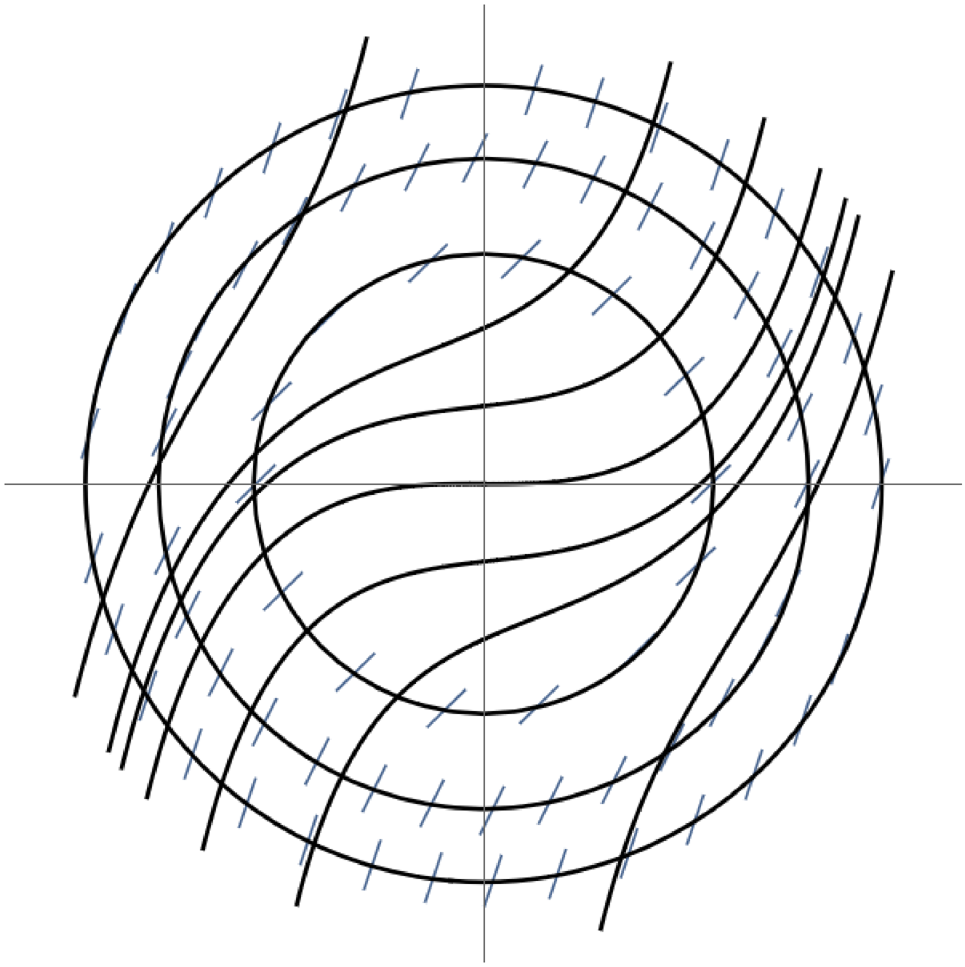
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = x^2 + y^2 \tag{1}$$

and the isocline of (1) is given by

$$\begin{aligned} \frac{\mathrm{d}\,y}{\mathrm{d}\,x} &= c \\ \Rightarrow x^2 + y^2 &= c \end{aligned} \tag{2}$$

Where c is the slope of the line element of the curves obtained for different values of c.
For different values of parameter c (2) represent a family of circle.
We construct the curve (2) for $c = 0,\ 0.5,\ 0.75,\ 1,\ 1.5,\ 2,\ 3$ etc.
On each of these circles we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $x^2 + y^2 = 0$,	$\theta = \tan^0 c = 0°$
When $c = 0.5$,	then $x^2 + y^2 = 0.5$,	$\theta = \tan^{0.5} c = 26.56°$
When $c = 0.75$,	then $x^2 + y^2 = 0.75$,	$\theta = \tan^{0.75} c = 36.86°$
When $c = 1$,	then $x^2 + y^2 = 1$,	$\theta = \tan^1 c = 45°$
When $c = 1.5$,	then $x^2 + y^2 = 1.5$,	$\theta = \tan^{1.5} c = 56.3°$
When $c = 2$,	then $x^2 + y^2 = 2$,	$\theta = \tan^2 c = 63.43°$
When $c = 3$,	then $x^2 + y^2 = 3$,	$\theta = \tan^3 c = 71.56°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1).

Solution (b)**.**

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = y^3 - x^2 \tag{3}$$

and the isocline of (3) is given by

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c$$
$$\Rightarrow y^3 - x^2 = c \tag{4}$$

Where c is the slope of the line element of the curves obtained for different values of c.
For different values of parameter c (4) represent a family of curves.
We construct the curve (4) for $c = 0, \pm 1, \pm 23$ etc.
On each of these circles we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$. When $c = 0$, then $y^3 = x^2$, $\theta = \tan^0 c = 0°$

$c = 0$

x	0	± 0.35	± 1	± 1.84	2.83	± 3.95
y	0	0.5	1	1.5	2	2.5

When $c = 1$, then $y^3 = x^2 + 1$, $\theta = \tan^1 c = 45°$

$c = 1$

x	0	± 1.54	± 2.65	± 3.82
y	1	1.5	2	2.5

When $c = -1$, then $y^3 = x^2 - 1$, $\theta = \tan^{-1} c = -45°$

$c = -1$

x	0	± 0.94	± 1	± 1.06	± 1.4	± 2.03	± 3
y	-1	-0.5	0	0.5	1	1.5	2

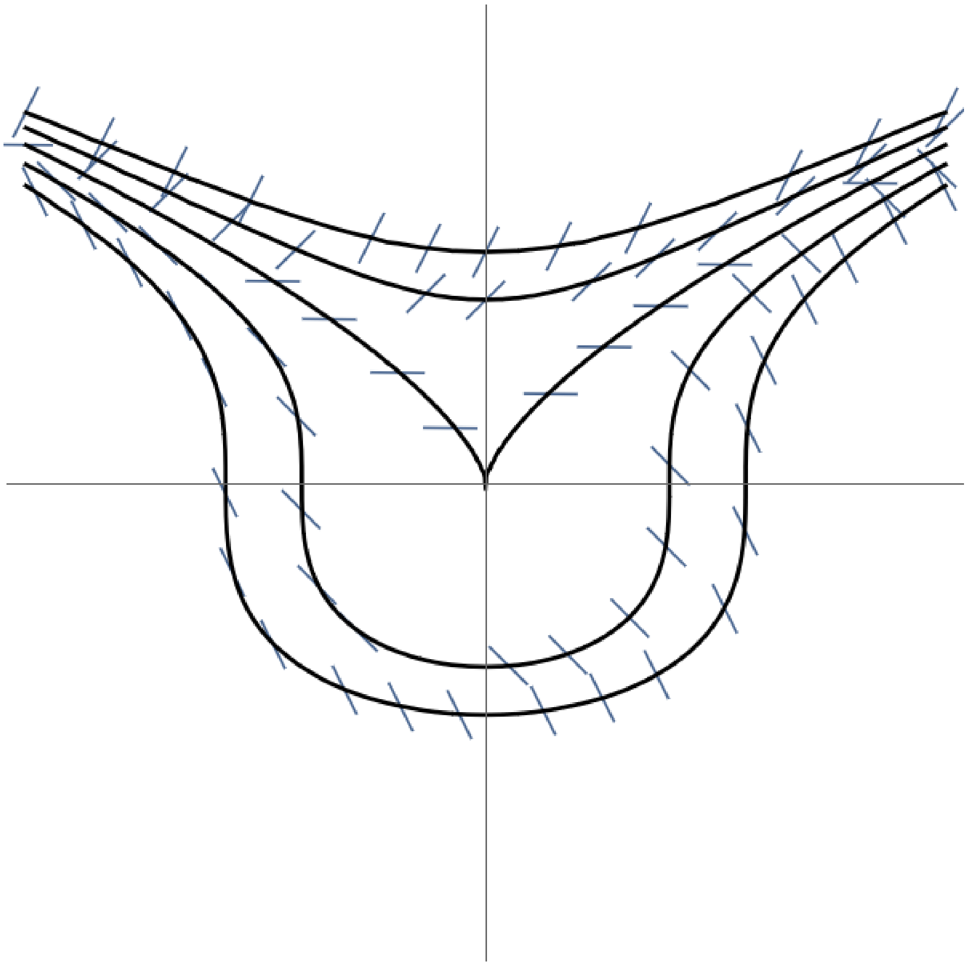
When $c = 2$, then $y^3 = x^2 + 2$, $\theta = \tan^2 c = 63.43°$

$c = 2$

x	0	± 0.5	± 1	± 1.5	± 2	± 2.44	± 3	± 3.69	± 4
y	1.25	1.31	1.44	1.61	1.81	2	2.22	2.5	2.62

When $c = -2$, then $y^3 = x^2 - 2$, $\theta = \tan^{-2} c = -63.43°$

$c = -2$	x	0	± 1	± 1.5	± 2	± 2.3	± 3.16	± 3.65	± 4
	y	-1.25	-1	0.62	1.25	1.5	2	2.25	2.41



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (3).

Solution (c)**.**

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{y}{x^2} \tag{5}$$

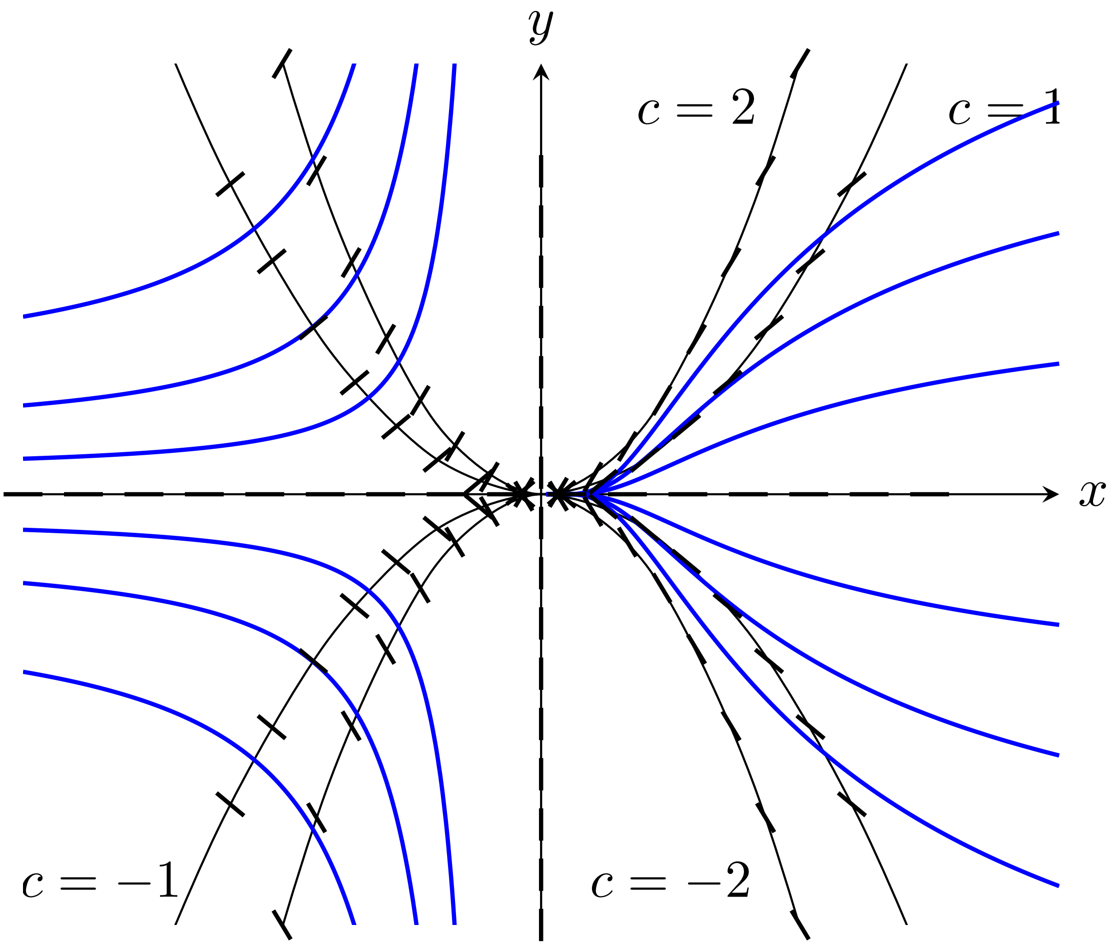
and the isocline of (5) is given by

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c$$
$$\Rightarrow \frac{y}{x^2} = c$$
$$\Rightarrow y = cx^2 \tag{6}$$

Where c is the slope of the line element of the curves obtained for different values of c.
For different values of parameter c (4) represent a family of parabola.
We construct the curve (4) for $c = 0$, ± 1, ± 2 etc.
On each of these circles we then construct a number of line elements having the approximate incli-

nations $\tan^{-1} c$.

When $c = 0$,	then $y = 0$,	$\theta = 0°$
When $c = 1$,	then $y = x^2$,	$\theta = 45°$
When $c = -1$,	then $y = x^2$,	$\theta = -45°$
When $c = 2$,	then $y = 2x^2$,	$\theta = 63.43°$
When $c = -2$,	then $y = -2x^2$,	$\theta = -63.43°$
When $c = \infty$,	then $x = 0$,	$\theta = 90°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (5).

Problem 0.2. Examine the nature and stability of the critical points of the following autonomous system and sketch the phase portrait of each cases.

(a)
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x + x^2 - 3xy$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = -2x + y + 3y^2$$

(b)
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x(4 - 2x - 4y)$$
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = y(x - 1)$$

(c) $\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} = y - x^2$

$\dfrac{\mathrm{d}\,y}{\mathrm{d}\,t} = 8x - y^2$

Solution (a)**.** We have

$$\left.\begin{aligned} \dot{x} &= x + x^2 - 3xy \\ \dot{y} &= -2x + y + 3y^2 \end{aligned}\right\} \tag{7}$$

The critical point of the system (7) are given by $\dot{x} = 0$ and $\dot{y} = 0$. i.e.,

$$x + x^2 - 3xy = 0 \tag{8}$$
$$-2x + y + 3y^2 = 0 \tag{9}$$

From (8) we get,

$$\begin{aligned} x = 0, \qquad 1 + x - 3y &= 0 \\ \Rightarrow\ x &= 3y - 1 \end{aligned}$$

Putting $x = 0$ in (9) we get,

$$\begin{aligned} & y + 3y^2 = 0 \\ \Rightarrow\ & y(1 + 3y) = 0 \\ \Rightarrow\ & y = 0, \quad y = -\frac{1}{3} \end{aligned}$$

Again put $x = 3y - 1$ in (9) we get,

$$\begin{aligned} & -6y + 2 + y + 3y^2 = 0 \\ \Rightarrow\ & 3y^2 - 5y + 2 = 0 \\ \Rightarrow\ & y = 1, \quad y = \frac{2}{3} \end{aligned}$$

Hence the critical points are $(0, 0)$, $(0, -\frac{1}{3})$, $(2, 1)$, $(1, \frac{2}{3})$.

(I) Investigation for critical point $(0, 0)$*:*
The corresponding linearized system of (7) is

$$\left.\begin{aligned} \dot{x} &= x \\ \dot{y} &= -2x + y \end{aligned}\right\} \tag{10}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \neq 0$

(ii) $\displaystyle\lim_{\substack{x \to 0 \\ y \to 0}} \frac{P(x, y)}{\sqrt{x^2 + y^2}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{Q(x, y)}{\sqrt{x^2 + y^2}} = 0$

Where $P(x, y) = x^2 - 3xy$; $Q(x, y) = 3y^2$

Hence the behavior of the paths of the system (7) near $(0, 0)$ would be similar to that of the paths of the related linearized system (10).
The characteristic equation of (10) is

$$\begin{aligned} & \lambda^2 - (1 + 1)\lambda + 1 - 0 = 0 \\ \Rightarrow\ & \lambda^2 - 2\lambda + 1 = 0 \\ \Rightarrow\ & (\lambda - 1)^2 = 0 \\ \Rightarrow\ & \lambda = 1, \ \ 1 \end{aligned}$$

The characteristic roots are real, equal and positive. Thus, the critical points $(0,0)$ is unstable node.

(II) Investigation for critical point $(0,-\frac{1}{3})$*:*
For the critical point $(0,-\frac{1}{3})$ we make the transformation $x=\xi$, $y=\eta-\frac{1}{3}$. So that $\dot{x}=\dot{\xi}$, $\dot{y}=\dot{\eta}$. Which transforms the critical point $x=0$, $y=-\frac{1}{3}$ to $\xi=0$, $\eta=0$ in the $\xi\eta$ plane.
With this transformation, we get from (7)

$$\begin{aligned} \dot{\xi} &= \xi+\xi^2-3\xi\left(\eta-\frac{1}{3}\right) \\ \dot{\eta} &= -2\xi+\eta-\frac{1}{3}+3\left(\eta-\frac{1}{3}\right)^2 \\ \Rightarrow \left.\begin{aligned} \dot{\xi} &= 2\xi+\xi^2-3\xi\eta \\ \dot{\eta} &= -2\xi-\eta+3\eta^2 \end{aligned}\right\} \end{aligned} \tag{11}$$

The corresponding linearized system of (11) is

$$\left.\begin{aligned} \dot{\xi} &= 2\xi \\ \dot{\eta} &= -2\xi-\eta \end{aligned}\right\} \tag{12}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ -2 & -1 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{P_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = \lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{Q_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = 0$

Where $P_1(\xi,\eta)=\xi^2-3\xi\eta$; $Q_1(\xi,\eta)=3\eta^2$

Hence the behavior of the paths of the system (11) near $(0,0)$ would be similar to that of the paths of the related linearized system (10).
The characteristic equation of (12) is

$$\begin{aligned} &\lambda^2-(2-1)\lambda+(-2)-0=0 \\ \Rightarrow\ &\lambda^2-\lambda-2=0 \\ \Rightarrow\ &\lambda^2-2\lambda+\lambda-2=0 \\ \Rightarrow\ &\lambda=2,\quad -1 \end{aligned}$$

Since the characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (12) but also an unstable saddle point of the system (11). So the critical point $(0,-\frac{1}{3})$ is an unstable saddle point of the system (7).

(III) Investigation for critical point $(2,1)$*:*
For the critical point $(2,1)$ we make the transformation $x=\xi_1+2$, $y=\eta_1+1$. So that $\dot{x}=\dot{\xi}_1$, $\dot{y}=\dot{\eta}_1$. Which transforms the critical point $x=2$, $y=1$ to $\xi_1=0$, $\eta_1=0$ in the $\xi_1\eta_1$ plane.
With this transformation, we get from (7)

$$\begin{aligned} \dot{\xi}_1 &= \xi_1+2+(\xi_1+2)^2-3(\xi_1+2)(\eta_1+1) \\ \dot{\eta}_1 &= -2(\xi_1+2)+(\eta_1+1)+3(\eta_1+1)^2 \\ \Rightarrow \left.\begin{aligned} \dot{\xi}_1 &= 2\xi_1-6\eta_1+\xi_1^2-3\xi_1\eta_1 \\ \dot{\eta}_1 &= -2\xi_1+7\eta_1+3\eta_1^2 \end{aligned}\right\} \end{aligned} \tag{13}$$

The corresponding linearized system of (13) is

$$\left.\begin{aligned} \dot{\xi}_1 &= 2\xi_1-6\eta_1 \\ \dot{\eta}_1 &= -2\xi_1+7\eta_1 \end{aligned}\right\} \tag{14}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2 & -6 \\ -2 & 7 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{P_2(\xi_1, \eta_1)}{\sqrt{\xi_1^2 + \eta_1^2}} = \lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{Q_2(\xi_1, \eta_1)}{\sqrt{\xi_1^2 + \eta_1^2}} = 0$

Where $P_2(\xi_1, \eta_1) = \xi_1^2 - 3\xi_1\eta_1$; $Q_2(\xi_1, \eta_1) = 3\eta_1^2$

Hence the behavior of the paths of the system (13) near $(0,0)$ would be similar to that of the paths of the related linearized system (14).
The characteristic equation of (12) is

$$\begin{aligned} &\lambda^2 - (2+7)\lambda + 14 - 12 = 0 \\ \Rightarrow\ &\lambda^2 - 9\lambda + 2 = 0 \\ \Rightarrow\ &\lambda = \frac{3}{2} \pm \sqrt{\frac{73}{4}} \end{aligned}$$

Since the characteristic roots are real, unequal and of positive sign. Hence, not only $(0,0)$ is an unstable node of the system (14) but also an unstable node of the system (13). So the critical point $(2,1)$ is an unstable node of the system (7).

(IV) Investigation for critical point $(1, \frac{2}{3})$*:*
For the critical point $(1, \frac{2}{3})$ we make the transformation $x = \xi_2 + 1$, $y = \eta_2 + \frac{2}{3}$. So that $\dot{x} = \dot{\xi}_2$, $\dot{y} = \dot{\eta}_2$. Which transforms the critical point $x = 1$, $y = \frac{2}{3}$ to $\xi_2 = 0$, $\eta_2 = 0$ in the $\xi_2\eta_2$ plane.
With this transformation, we get from (7)

$$\begin{aligned} \dot{\xi}_2 &= (\xi_2 + 1) + (\xi_2 + 1)^2 - 3(\xi_2 + 1)\left(\eta_2 + \frac{2}{3}\right) \\ \dot{\eta}_2 &= -2(\xi_2 + 1) + (\eta_2 + \frac{2}{3}) + 3\left(\eta_2 + \frac{2}{3}\right)^2 \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} \dot{\xi}_2 &= \xi_2 - 3\eta_2 + \xi_2^2 - 3\xi_2\eta_2 \\ \dot{\eta}_2 &= -2\xi_2 + 5\eta_2 + 3\eta_2^2 \end{aligned}\right\} \tag{15}$$

The corresponding linearized system of (15) is

$$\left.\begin{aligned} \dot{\xi}_2 &= \xi_2 - 3\eta_2 \\ \dot{\eta}_2 &= -2\xi_2 + 5\eta_2 \end{aligned}\right\} \tag{16}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi_2 \to 0 \\ \eta_2 \to 0}} \dfrac{P_3(\xi_2, \eta_2)}{\sqrt{\xi_2^2 + \eta_2^2}} = \lim\limits_{\substack{\xi_2 \to 0 \\ \eta_2 \to 0}} \dfrac{Q_3(\xi_2, \eta_2)}{\sqrt{\xi_2^2 + \eta_2^2}} = 0$

Where $P_3(\xi_2, \eta_2) = \xi_2^2 - 3\xi_2\eta_1$ and $Q_3(\xi_2, \eta_2) = 3\eta_2^2$

Hence the behavior of the paths of the system (15) near $(0,0)$ would be similar to that of the paths of the related linearized system (16).
The characteristic equation is

$$\begin{aligned} &\lambda^2 - (1+5)\lambda + 5 - 6 = 0 \\ \Rightarrow\ &\lambda^2 - 6\lambda - 1 = 0 \\ \Rightarrow\ &\lambda = 3 \pm \sqrt{10} \end{aligned}$$

Since the characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (16) but also an unstable saddle point of the system (15). So the critical point $(1, \frac{2}{3})$ is an unstable saddle point of the system (7).

Finally, we get the critical point

(i) $(0,0)$ is unstable node of (7)

(ii) $(0,-\frac{1}{3})$ is unstable saddle point of (7)

(iii) $(2,1)$ is unstable node of (7)

(iv) $(1,\frac{2}{3})$ is unstable saddle point of (7)

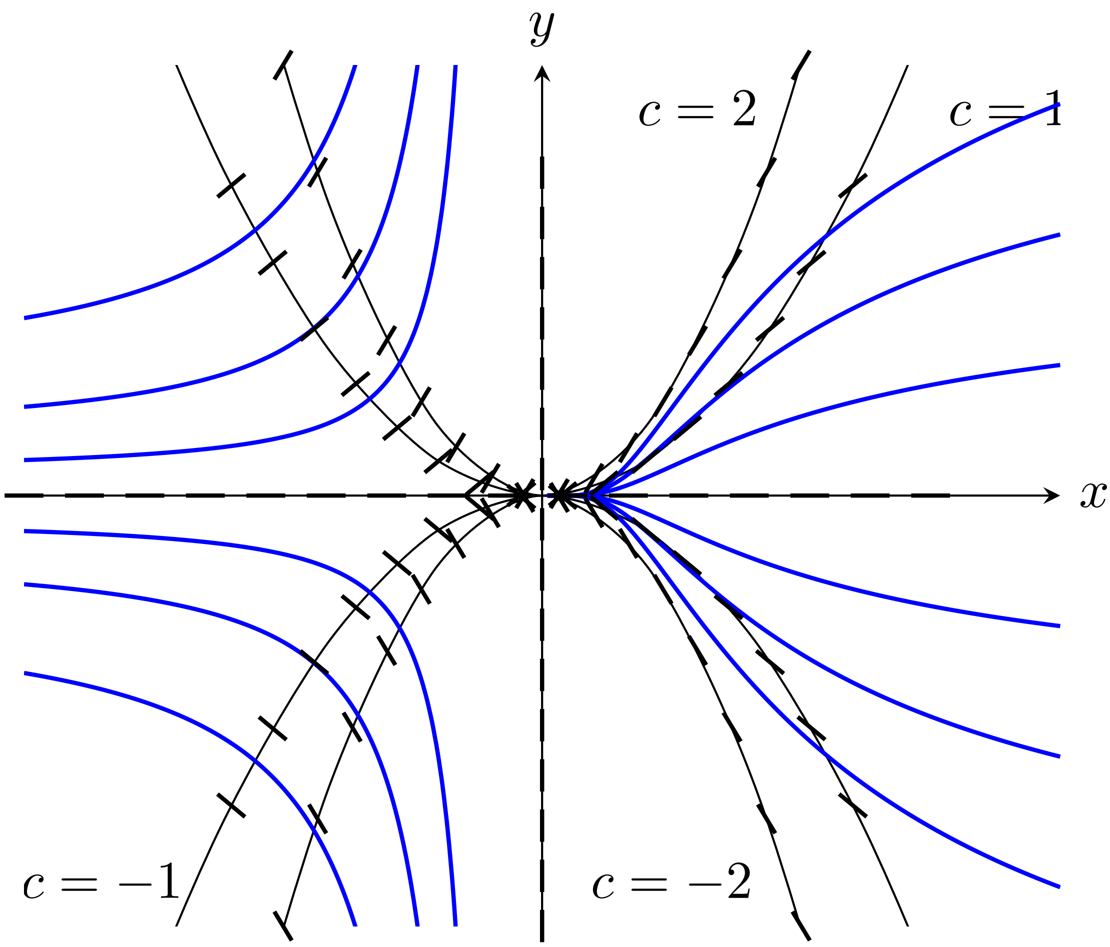
Phase Portrait: From (7) we get

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{-2x + y + 3y^2}{x + x^2 - 2xy}$$

The isocline of the above differential equation are given by

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c$$
$$\Rightarrow \frac{-2x + y + 3y^2}{x + x^2 - 2xy} = c$$

Where c is the slope of the line element of the canvas obtained for different values of c.
For $c = 0$, $-2x + y + 3y^2 = 0 \quad \Rightarrow \left(y + \frac{1}{6}\right)^2 = \frac{2}{3}\left(x + \frac{1}{24}\right)$ then $\theta = 0°$
For $c = \infty$, $x(1 + x - 2y) = 0 \quad \Rightarrow x = 0, \quad \frac{x}{-1} + \frac{y}{1/3} = 1$ then $\theta = 90°$



Finally we draw several smooth curves. These smooth curves complete the phase portrait of (7).

Solution (b)**.** We have

$$\left.\begin{aligned} \dot{x} &= x(4-2x-4y) \\ \dot{y} &= y(x-1) \end{aligned}\right\} \tag{17}$$

The critical point of the system (17) are given by $\dot{x}=0$ and $\dot{y}=0$. i.e.,

$$x(4-2x-4y)=0 \tag{18}$$
$$y(x-1)=0 \tag{19}$$

From (19) we get, $y=0$, $x=1$
Putting $y=0$ in (18) we get, $x=0$, $x=2$
Putting $x=1$ in (18) we get, $y=1/2$

Hence, the critical points are $(0,0)$, $(2,0)$, $(1,\frac{1}{2})$.

(I) Investigation for critical point $(0,0)$*:*
The corresponding linearized system of (17) is

$$\left.\begin{aligned} \dot{x} &= 4x \\ \dot{y} &= -y \end{aligned}\right\} \tag{20}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & -1 \end{vmatrix} \neq 0$

(ii) $\displaystyle \lim_{\substack{x\to 0\\ y\to 0}} \frac{P(x,y)}{\sqrt{x^2+y^2}} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{Q(x,y)}{\sqrt{x^2+y^2}} = 0$

Where $P(x,y)=-2x^2-4xy$ and $Q(x,y)=xy$

Hence the behavior of the paths of the system (17) near $(0,0)$ would be similar to that of the paths of the related linearized system (20).
The characteristic equation of (20) is

$$\begin{aligned} &\lambda^2-(4-1)\lambda-4-0=0 \\ \Rightarrow\ &\lambda^2-3\lambda-4=0 \\ \Rightarrow\ &\lambda^2-4\lambda+\lambda-4=0 \\ \Rightarrow\ &\lambda=4,\ \ -1 \end{aligned}$$

The characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (20) but also an unstable saddle point of the system (17).

(II) Investigation for critical point $(2,0)$*:*
For the critical point $(2,0)$ we make the transformation $x=\xi+2$, $y=\eta$. So that $\dot{x}=\dot{\xi}$, $\dot{y}=\dot{\eta}$. Which transforms the critical point $x=2$, $y=0$ to $\xi=0$, $\eta=0$ in the $\xi\eta$ plane.
With this transformation, we get from (17)

$$\begin{aligned} &\dot{\xi}=4(\xi+2)-2(\xi+2)^2-4(\xi+2)\eta \\ &\dot{\eta}=\eta(\xi+2)-\eta \end{aligned}$$
$$\Rightarrow \left.\begin{aligned} \dot{\xi} &= -4\xi-8\eta-2\xi^2-4\xi\eta \\ \dot{\eta} &= \eta+\xi\eta \end{aligned}\right\} \tag{21}$$

The corresponding linearized system of (11) is

$$\left.\begin{aligned} \dot{\xi} &= -4\xi - 8\eta \\ \dot{\eta} &= \eta \end{aligned}\right\} \tag{22}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -4 & -8 \\ 0 & 1 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi \to 0 \\ \eta \to 0}} \dfrac{P_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = \lim\limits_{\substack{\xi \to 0 \\ \eta \to 0}} \dfrac{Q_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = 0$

Where $P_1(\xi,\eta) = -2\xi^2 - 4\xi\eta$, and $Q_1(\xi,\eta) = \xi\eta$

Hence the behavior of the paths of the system (21) near $(0,0)$ would be similar to that of the paths of the related linearized system (20).
The characteristic equation of (12) is

$$\begin{aligned} &\lambda^2 - (-4+1)\lambda - 4 = 0 \\ \Rightarrow\ &\lambda^2 + 3\lambda - 4 = 0 \\ \Rightarrow\ &\lambda^2 + 4\lambda - \lambda - 4 = 0 \\ \Rightarrow\ &\lambda = -4,\ \ 1 \end{aligned}$$

Since the characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (22) but also an unstable saddle point of the system (21). So the critical point $(0, -\frac{1}{3})$ is an unstable saddle point of the system (17).

(III) Investigation for critical point $(1, 1/2)$*:*
For the critical point $(1, 1/2)$ we make the transformation $x = \xi_1 + 1$, $y = \eta_1 + \frac{1}{2}$. So that $\dot{x} = \dot{\xi}_1$, $\dot{y} = \dot{\eta}_1$. Which transforms the critical point $x = 1$, $y = \frac{1}{2}$ to $\xi_1 = 0$, $\eta_1 = 0$ in the $\xi_1\eta_1$ plane.
With this transformation, we get from (17)

$$\begin{aligned} &\dot{\xi} = 4(\xi_1+1) - 2(\xi_1+1)^2 - 4(\xi_1+1)\left(\eta_1 + \frac{1}{2}\right) \\ &\dot{\eta} = \left(\eta_1 + \frac{1}{2}\right)(\xi_1+1) - \left(\eta_1 + \frac{1}{2}\right) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} \dot{\xi} &= -2\xi_1 - 4\eta_1 - 2\xi_1^2 - 4\xi_1\eta_1 \\ \dot{\eta} &= \tfrac{1}{2}\xi_1 + \xi_1\eta_1 \end{aligned}\right\} \tag{23}$$

The corresponding linearized system of (23) is

$$\left.\begin{aligned} \dot{\xi}_1 &= 2\xi_1 - 4\eta_1 \\ \dot{\eta}_1 &= \tfrac{1}{2}\xi_1 \end{aligned}\right\} \tag{24}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -2 & -4 \\ \frac{1}{2} & 0 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{P_2(\xi_1,\eta_1)}{\sqrt{\xi_1^2+\eta_1^2}} = \lim\limits_{\substack{\xi_1 \to 0 \\ \eta_1 \to 0}} \dfrac{Q_2(\xi_1,\eta_1)}{\sqrt{\xi_1^2+\eta_1^2}} = 0$

Where $P_2(\xi_1,\eta_1) = -2\xi_1^2 - 4\xi_1\eta_1$ and $Q_2(\xi_1,\eta_1) = \xi_1\eta_1$

Hence the behavior of the paths of the system (23) near $(0,0)$ would be similar to that of the paths of the related linearized system (24).
The characteristic equation of (12) is

$$\lambda^2 + 2\lambda + 2 = 0$$
$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2}$$
$$\Rightarrow \lambda = -1 \pm i$$

Since the characteristic roots are conjugate complex with negative real parts. Hence, $(0,0)$ is an asymptotically stable spiral point of the system (24) and hence of (23). So the critical point $(2,1)$ is an asymptotically stable spiral point of (7).

Finally, we get the critical point

(i) $(0,0)$ is an unstable saddle point of (17)

(ii) $(2,0)$ is an unstable saddle point of (17)

(iii) $(1,1/2)$ is an asymptotically stable spiral point of (17)

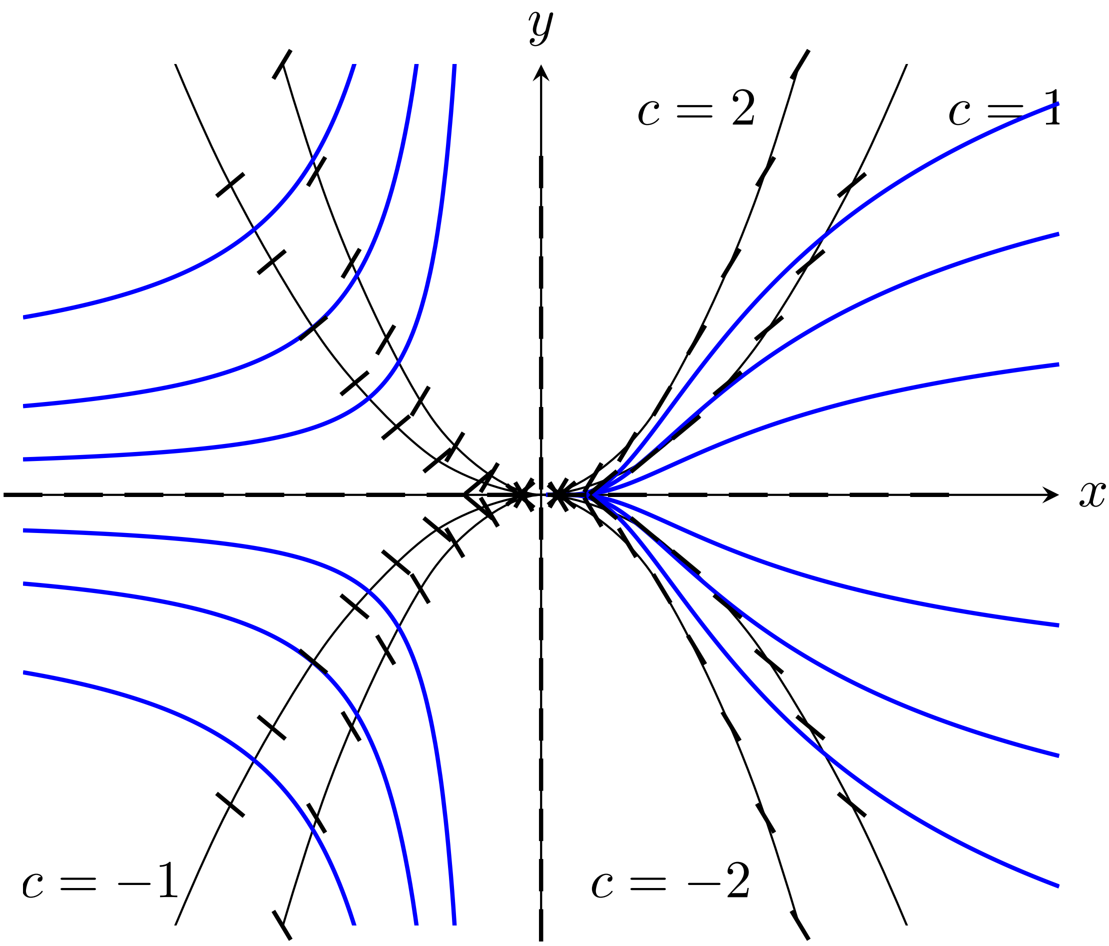
Phase Portrait: From (17) we get

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = \frac{y(x-1)}{x(4-2x-4y)}$$

The isocline of the above differential equation are given by

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = c$$
$$\Rightarrow \frac{y(x-1)}{x(4-2x-4y)} = c$$

Where c is the slope of the line element of the canvas obtained for different values of c.
When $c = 0$, then $y = 0,\ x = 1$ and $\theta = 0°$ When $c = \infty$, then $x = 0 \quad 2x + 4y = 4 \Rightarrow \frac{x}{2} + \frac{y}{1}$ and $\theta = 90°$



Finally we draw several smooth curves. These smooth curves complete the phase portrait of (17).

Solution (c)**.** We have

$$\left.\begin{aligned} \dot{x} &= y - x^2 \\ \dot{y} &= 8x - y^2 \end{aligned}\right\} \tag{25}$$

The critical point of the system (25) are given by $\dot{x} = 0$ and $\dot{y} = 0$. i.e.,

$$y - x^2 = 0 \tag{26}$$
$$8x - y^2 = 0 \tag{27}$$

Substituting (26) in (**??**) we get

$$\begin{aligned} &8x - x^4 = 0 \\ \Rightarrow &x(8 - x^3) = 0 \\ \Rightarrow &x = 0, \quad x = 2 \end{aligned}$$

Putting $x = 2$ in (26) we get, $y = 0$, $y = 4$

Hence, the critical points are $(0, 0)$, $(2, 4)$.

(I) Investigation for critical point $(0, 0)$*:*
The corresponding linearized system of (25) is

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= 8x \end{aligned}\right\} \tag{28}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 8 & 0 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{P(x,y)}{\sqrt{x^2+y^2}} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{Q(x,y)}{\sqrt{x^2+y^2}} = 0$

Where $P(x,y) = -x^2$ and $Q(x,y) = -y^2$

Hence the behavior of the paths of the system (25) near $(0,0)$ would be similar to that of the paths of the related linearized system (28).
The characteristic equation of (28) is

$$\lambda^2 - 8 = 0$$
$$\Rightarrow \lambda = \pm 2\sqrt{2}$$

The characteristic roots are real, unequal and of opposite sign. Hence, not only $(0,0)$ is an unstable saddle point of the system (28) but also an unstable saddle point of the system (25).

(II) Investigation for critical point $(2,4)$*:*
For the critical point $(2,4)$ we make the transformation $x = \xi + 2$, $y = \eta + 4$. So that $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{\eta}$. Which transforms the critical point $x = 2$, $y = 4$ to $\xi = 0$, $\eta = 0$ in the $\xi\eta$ plane.
With this transformation, we get from (25)

$$\begin{aligned} \dot{\xi} &= 4 + \eta - (\xi+2)^2 \\ \dot{\eta} &= 8(\xi+2) - (\eta+4)^2 \end{aligned}$$
$$\Rightarrow \left.\begin{aligned} \dot{\xi} &= -4\xi + \eta - \xi^2 \\ \dot{\eta} &= 8\xi - 8\eta - \eta^2 \end{aligned}\right\} \tag{29}$$

The corresponding linearized system of (29) is

$$\left.\begin{aligned} \dot{\xi} &= -4\xi + \eta \\ \dot{\eta} &= 8\xi - 8\eta \end{aligned}\right\} \tag{30}$$

Here we observe that

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ 8 & -8 \end{vmatrix} \neq 0$

(ii) $\lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{P_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = \lim\limits_{\substack{\xi\to 0\\ \eta\to 0}} \dfrac{Q_1(\xi,\eta)}{\sqrt{\xi^2+\eta^2}} = 0$

Where $P_1(\xi,\eta) = -\xi^2$, and $Q_1(\xi,\eta) = -\eta^2$

Hence the behavior of the paths of the system (30) near $(0,0)$ would be similar to that of the paths of the related linearized system (29).
The characteristic equation of (30) is

$$\lambda^2 - (-4-8)\lambda + 32 - 8 = 0$$
$$\Rightarrow \lambda^2 + 12\lambda + 24 = 0$$
$$\Rightarrow \lambda = -6 \pm \sqrt{12}$$

Since the characteristic roots are real, unequal and of negative sign. Hence, not only $(0,0)$ is an asymptotically stable node of the system (30) but also an asymptotically stable node of the system (29). So the critical point $(2,4)$ is an asymptotically stable node of the system (25).

Finally, we get the critical point

(i) $(0,0)$ is an unstable saddle point of (17)

(ii) $(2,4)$ is an asymptotically stable node of (25)

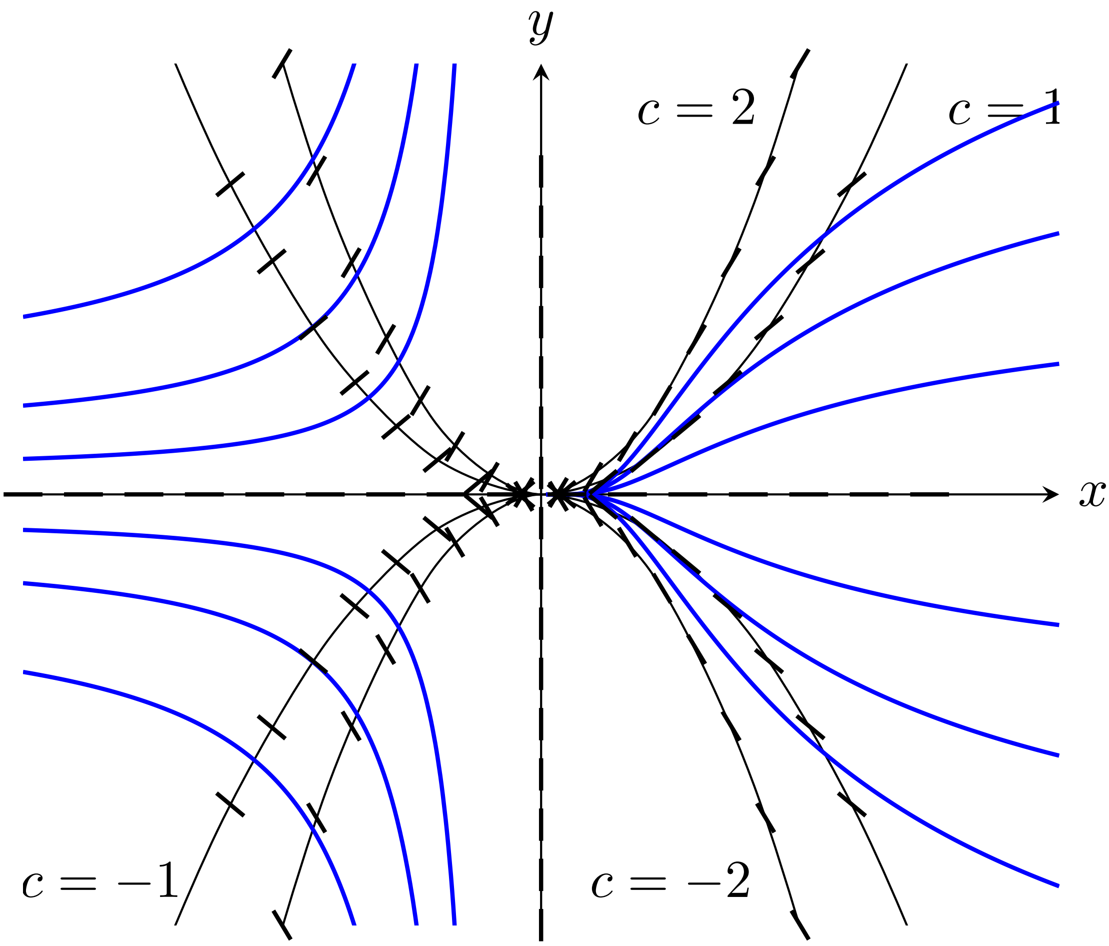
Phase Portrait: From (25) we get

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{8x - y^2}{y - x^2}$$

The isocline of the above differential equation are given by

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = c$$
$$\Rightarrow \frac{8x - y^2}{y - x^2} = c$$

Where c is the slope of the line element of the canvas obtained for different values of c. When $c = 0$, then $y^2 = 8x$ and $\theta = 0°$ When $c = \infty$, then $y = x^2$ and $\theta = 90°$



Finally we draw several smooth curves. These smooth curves complete the phase portrait of (25).