CHAPTER 1

Automorphism

Definition 1 (Automorphism). An automorphism of a group G is an isomorphism[1] of G onto itself.

Theorem 0.1. The set $Aut(G)$ of all automorphisms of a group G is a group under the operation of composition of mappings.

Proof. Here $Aut(G)$ is the set of all automorphisms of a group G and the operation is the composition of mappings.
Let $f, g \in Aut(G)$.
Then the composite map $g \circ f$ is bijective, because f and g are bijective.
Using the hypotheses that f and g are group homomorphisms, we can conclude that $g \circ f$ is also a group homomorphism, because

$$\begin{aligned}(g \circ f)(ab) &= g(f(ab)) \\ &= g(f(a)f(b)) \\ &= g(f(a))g(f(b)) \\ &= (g \circ f)(a)(g \circ f)(b)\end{aligned}$$

So, $g \circ f \in Aut(G)$.
This is the closure property.

The associative law holds for $Map(G)$, the set of all mappings of G into itself; so it holds in $Aut(G)$, because $Aut(G)$ is closed under composition of mappings.

Clearly, 1_G is the identity element of $Aut(G)$.

If $f \in Aut(G)$, the inverse mapping $f^{-1} : G \to G$ exists and is likewise bijective.
Let $f \in Aut(G)$ and $a, b, x, y \in G$ such that $f(a) = x$ and $f(b) - y$. Then we have $a = f^{-1}(x)$ and $b = f^{-1}(y)$.
Since f is a group homomorphism, we have $f(ab) = f(a)f(b) = xy$.
It gives, $f^{-1}(xy) = ab = f^{-1}(x)f^{-1}(y)$.
This implies that f^{-1} is also a group homomorphism.
Hence, $f^{-}1 \in Aut(G)$.
Therefore, $Aut(G)$ is a group under composition of mappings. □

[1]Homomorphism: Suppose G, G' are multiplicative groups. A mapping $f : G \to G'$ is called a group homomorphism iff $f(ab) = f(a)f(b)$ holds for all $a, b \in G$.
Isomorphism: A bijective group homomorphism is called an isomorphism.

1. Inner Automorphisms

For any fixed $a \in G$, we define a mapping $f_a : G \to G$ by setting $f_a(x) = axa^{-1}$, $f_a \in Aut(G)$ for every $a \in G$.

Proof. f_a is injective (by the cancellation law), for

$$f_a(x) = f_a(y) \;\Rightarrow\; axa^{-1} = aya^{-1} \;\Rightarrow\; x = y.$$

f_a is surjective, because

$$f_a(a^{-1}xa) = a(a^{-1}xa)a^{-1} = x.$$

f_a is group homomorphism, because for all $x, y \in G$, we have

$$f_a(xy) = a(xy)a^{-1} = (axa^{-1})(aya^{-1}) = f_a(x)f_a(y).$$

$\square$

Definition 2 (Inner Automorphism). For any fixed $a \in G$ the mapping $f_a : G \to G$ defined by $f_a(x) = axa^{-1}$ is called the inner automorphism determined by a.

Theorem 1.1. The set $Inn(G)$ of all inner automorphisms of a group G is a subgroup of $Aut(G)$.

Proof. The relation $f_a \circ f_b = f_{ab}$ is the key.
This is easily proved, for

$$\begin{aligned}(f_a \circ f_b)(x) &= f_a(f_b(x)) \\ &= f_a(bxb^{-1}) \\ &= a(bxb^{-1})a^{-1} \\ &= (ab)x(ab)^{-1} \\ &= f_{ab}(x) \qquad \text{holds for all } x \in G\end{aligned}$$

So, $Inn(G)$ Is closed under composition of mappings.
The identity mapping l_G belongs to $Inn(G)$, because $f_e = 1_G$.
The inverse of f_a, which is obviously an automorphism, is the inner automorphism determined by a^{-1}, because

$$f_a \circ f_{a^{-1}} = f_{aa^{-1}} = f_e = 1_G$$

and

$$f_{a^{-1}} \circ f_a = f_{a^{-1}a} = f_e = 1_G$$

So, $Inn(G)$ is a subgroup of $Aut(G)$. It remains to show that $Inn(G)$ is a normal subgroup of $Aut(G)$. For any $\sigma \in Aut(G)$, we have $\sigma \circ f_a \circ \sigma^{-1} = f_{\sigma(a)}$, because

$$\begin{aligned}
(\sigma \circ f_a \circ \sigma^{-1})(x) &= (\sigma \circ f_a)(\sigma^{-1}(x)) \\
&= \sigma(a\sigma^{-1}(x)a^{-1}) \\
&= \sigma(a)\sigma(\sigma^{-1}(x))\sigma(a^{-1}) \\
&= \sigma(a)x\sigma(a^{-1}) \\
&= \sigma(a)x(\sigma(a))^{-1} \\
&= f_{\sigma(a)}(x) \quad \in Inn(G)
\end{aligned}$$

So, $Inn(G)$ is a normal subgroup of $Aut(G)$ $\square$