General Topology

MAT411

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Preface

This is a compilation of lecture notes with some books and my own thoughts. This document is not a holy text. So, if there is a mistake, solve it by your own judgement. Currently, the following topics are not included

- Fourier Transform and Applications
- Boundary value problems
- Eigenfunctions
- Green's functions
- Strum-Liouville problems
- Laplace Equation

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Books Recommended:

- Simmons, G.F.: Introduction to Topology and Modern Analysis
- Gal, S.: Point Set Topology
- Lipschutz, S.: General Topology
- Kelley, J.L.: General Topology
- Hockling and Young: Topology

Part I

Class Note/Sheet

Chapter 1

Topological Space

This chapter opens with the definition of a topology and then we study a number of ways of constructing a topology on a set so as to make it into a topological space. We also consider some of the elementary concepts associated with topological spaces. Open and closed sets, neighborhoods, limit points, interior, exterior and boundary of a set are introduced as natural generalization of corresponding ideas for the real line and Euclidean space.
Topology, like other branches of pure mathematics such as group theory, is an axiomatic subject. We start with a set of axioms and we use these axioms to prove propositions and theorems.

Definition 1.1. Let X be a non-empty set. A class τ of subsets of X is said to be a ***topology*** on X iff τ satisfies the following axioms:

(i) X and the empty set $\varnothing$ belongs to τ.

(ii) The union of any collection of elements of τ belongs to τ.

(iii) The intersection of any finite subcollection of elements of τ belongs to τ.

If τ is a topology on X, then the pair (X, τ) is called a ***topological space***. It is customary to denote this topological space simply by X if no confusion will arise.

Definition 1.2. Let (X, τ) be a topological space. Then members of τ are called open sets in X.

Obviously, we have,

(i) $\varnothing$ and X are open sets in X with respect to any topology τ on X.

(ii) Union of arbitrary number of open sets is an open set.

(iii) Intersection of finite number topological space is an open set.

Example 1.1. Consider the following collections of subsets of $X = \{a, b, c, d, e\}$:

$$\tau_1 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}\}$$
$$\tau_2 = \{X, \varnothing, \{b, c\}, \{b, c, e\}, \{b, c, d, e\}\}$$
$$\tau_3 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$$

Observe that τ_1 and τ_2 are topologies on X since they satisfy all the three axioms of the definition of topology whereas τ_3 is not, because the intersection

$$\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\}$$

does not belong to τ_3; i.e., τ_3 does not satisfy the condition (iii) of the definition of topology.

From this example we see that a set X may have many topologies but not every collection is a topology.

Example 1.2. Let X be a non-empty set and τ be the class of all subsets of X. Then obviously τ is a topology on X. This topology is known as ***discrete topology*** and the pair (X, τ) is called a ***discrete topological space***.

Example 1.3. For any non-empty set X, the class consisting of only X and $\varnothing$, i.e.; $\tau = \{X, \varnothing\}$ is a topology on X. This topology is called ***indiscrete topology*** and the pair (X, τ) is called an ***indiscrete topological space.***

The topologies in the Example 1.2 and Example 1.3 are known as ***trivial topologies***. All other topologies are known as ***non-trivial topologies***.

Remark. If $|X| = 1$, then discrete and indiscrete topologies on X coincide, otherwise they are different.

Example 1.4. Let $\mathcal{U}$ denotes the class of all open subsets of real numbers ($\mathbb{R}$). Then $\mathcal{U}$ is a topology on $\mathbb{R}$. For, $\varnothing$ and $\mathbb{R}$ are both open, arbitrary union of open sets is open and finite intersection of open sets is open. We call this topology a ***usual topology*** on $\mathbb{R}$.
This topology is also known as ***standard topology*** or ***Euclidean topology*** on $\mathbb{R}$.

Similarly, the class of all open sets in the plane $\mathbb{R}^2$ is a topology and also called usual topology/Euclidean topology on $\mathbb{R}^2$.

Example 1.5. Let (S, d) be a metric space and τ be the collection of all open sets in S. Then τ is a topology on S and called ***metric topology*** on S induced by the metric d.

Example 1.6. Let X be a non-empty set and τ be a collection of all those subsets U of X such that $X - U$ is finite or all of X. Then this collection forms a topology on X. This topology is known as the ***co-finite topology*** or the ***finite complement topology*** on X.
To prove that τ is a topology on X, we see that

(i) Both X, $\varnothing$ are in τ since $X - X = \varnothing$ is finite and $X - \varnothing = X$ is all of X.

(ii) If $\{U_\alpha\}$ is any collection of elements of τ, then, for each $\alpha \in \Omega$, $X - U_\alpha$ is finite and so $X - \cup U_\alpha = \cap(X - U_\alpha)$ is finite [by De Morgan's Law]. Hence, $\cup U_\alpha \in \tau$

(iii) If $\{U_1, U_2, \ldots, U_n)$ is a finite collection of elements of τ then $X - \bigcap_{k=1}^{n} U_k = \bigcup_{k=1}^{n} (X - U_k)$ is finite since each set $X - U_k$ is finite. Hence, $\bigcap_{k=1}^{n} U_k \in \tau$.

Remark. If X is finite set, then co-finite topology on X coincides with the discrete topology on X.

Example 1.7. Let X be any uncountable set and τ be a collection of all those subsets U of X such that $X - U$ is countable or all of X. This collection forms a topology on X and we call it **co-countable topology**.

To prove that τ is a topology on X, we see that

(i) $X, \varnothing \in \tau$, because $X - X = \varnothing$ which is countable and $X - \varnothing = X$ which is all of X.

(ii) If $\{U_\alpha\}$ is any collection of elements of τ, then for each $\alpha \in \Omega$, $X - U_\alpha$ is countable and so $X - \cup U_\alpha = \cap(X - U_\alpha)$ [by De Morgan's Law] is countable. Hence, $\cup U_\alpha \in \tau$.

(iii) If $\{U_1, U_2, \ldots, U_n\}$ is a finite collection of elements of τ, then $X - \bigcap_{k=1}^{n} U_k = \bigcup_{k=1}^{n} (X - U_k)$ which countable since each set $X - U_k$ is countable. Hence, $\bigcap_{k=1}^{n} U_k \in \tau$.

Remark. If X is a countable set, the co-countable topology on X coincides with the discrete topology on X.

Example 1.8. Let $\mathbb{N}$ be the set of natural numbers and let τ consists of $\varnothing$ and each subset S of $\mathbb{N}$ such that $\mathbb{N} - S$ is finite. Then τ is a topology on X.
By hypothesis, $\varnothing \in \tau$. Also $\mathbb{N} - \mathbb{N} = \varnothing$, a finite set, so $\mathbb{N} \in \tau$.
Thus, $\mathbb{N}, \varnothing \in \tau$. For each natural number n, define the set S_n, as follows:

$$S_n = \{1\} \cup \{n+1\} \cup \{n+2\} \cdots = 1 \cup \bigcup_{m=n+1}^{\infty} \{m\}$$

Then clearly each S_n is infinite and $\mathbb{N} - S_n$ is finite; so $S_n \in \tau$. Now, $\mathbb{N} - \cup S_n = \cap(\mathbb{N} - S_n)$ is finite since each $(\mathbb{N} - S)$ is finite. Hence, $\cup S_n \in \tau$.
Also $\mathbb{N} - \bigcap_1^n S_k = \bigcup_1^n (\mathbb{N} - S_k)$ is finite, so $\cap S_n \in \tau$.
Thus τ is a topology on $\mathbb{N}$.

Observe that the infinite intersection

$$\bigcap_{n=1}^{\infty} S_n = \{1\}$$

is finite whose compliment is not finite; so, it does not belong to τ. Thus, the infinite intersection of open sets may not be open.

Remark.

- In a topological space (X, τ), each member of τ is a subset of X, but each subset of X is not a member of τ. For this consider the topology $\tau = \{X, \varnothing, \{a\}, \{a, b\}\}$ on $X = \{a, b, c\}$. Then $\{b, c\} \subset X$ but $\{b, c\} \notin \tau$.

- Intersection of finite number of members of τ is a member of τ but intersection of any number members of τ need not be member of τ. For this, consider the usual topological space $(\mathbb{R}, \tau_u)$ and $(-n, n) \in \tau_u$, for each $n \in \mathbb{N}$. But

$$\bigcap_{n \in \mathbb{N}} (-n, n) = \{0\}$$

which is not a member of τ.

The set of all topologies on X

Given any nonempty set X, there always exists a topology on X viz. the discrete topology or the indiscrete topology. Hence, every non-empty set can be considered as a topological space.

The collection $\mathcal{F}$ of all topologies defined on a non-empty set X is surely non-empty and is partially ordered set (poset in short) under the partial ordering relation $\leq$ defined by $\tau_1 \leq \tau_2$ iff $\tau_1 \subseteq \tau_2$, for $\tau_1, \tau_2 \in \mathcal{F}$. The poset $(\mathcal{F}, \leq)$ is a bounded poset with indiscrete topology as the smallest element and discrete topology as the greatest element.

If $\tau_1 \subseteq \tau_2$, we say that τ_1 is ***coarser*** (or ***weaker***) than τ_2, or equivalently, we say that τ_2 is ***finer*** (or ***stronger***) than τ_1.
Whenever either $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$, we say that τ_1 and τ_2 are ***comparable***, otherwise they are ***non-comparable***.

If $\tau_1, \tau_2 \in \mathcal{F}$, then $\tau_1 \cap \tau_2 \in \mathcal{F}$. Actually, arbitrary intersection of elements of $\mathcal{F}$ is also an element of $\mathcal{F}$. But the union of elements of $\mathcal{F}$ may not be a topology on X.

Theorem 1.1. The intersection of arbitrary family of topologies is again a topology.

Proof. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a family of topologies on a set X and let $\Im = \bigcap_{\alpha\in\Omega}\tau_\alpha$. Then for each $\alpha; X, \varnothing \in \tau_\alpha$ and so $X, \varnothing \in \bigcap_{\alpha\in\Omega}\tau_\alpha$
Let $\{U_\lambda\}_{\lambda\in\Omega}$ be a collection of elements of $\Im = \bigcap_{\alpha\in\Omega}\tau_\alpha$. Then for each α and for all $\lambda, U_\lambda \in \tau_\alpha$ and so $\cup U_\lambda \in \tau_\alpha$ for each α since τ_α is a topology. Therefore, $\cup U_\lambda \in \Im$.
Let $\{u_1, u_2, \ldots, u_n\}$ be a finite collection of elements of $\Im$. Then for all $k, U_k, \in \tau_\alpha$ for each α and hence $\bigcap_{k=1}^n U_k \in \tau_\alpha$ since each τ_α is a topology. Consequently, $\bigcap_{k=1}^n U_k \in \Im$.
Thus $\Im = \cap\tau_\alpha$ is a topology on X. □

Remark. The union of topologies needs not to be a topology.
For example, let $X = \{a, b, c\}$. Consider the two topologies on X:

$$\tau_1 = \{X, \varnothing, \{a\}\} \text{ and } \tau_2 = \{X, \varnothing, \{b\}\}$$

Then their union

$$\tau_1 \cup \tau_2 = \{X, \varnothing, \{a\}, \{b\}\}$$

is not a topology on X because $\{a\} \cup \{b\} \notin \tau_1 \cup \tau_2$ which violates the definition of topology.

1.1 Neighborhood System

Definition 1.3. Let (X, τ) be a topological space, and x be a point of X. A ***neighbourhood*** of x is any subset X which contains an open set containing the point x. That is, if G is an open set containing x, then G is a neighbourhood of x. Also, every superset of G is a neighbourhood of x.
Thus, a subset N of X is a neighborhood of x if there exists an open set U containing x such that $x \in U \subseteq N$.
The class of all neighborhoods of $p \in X$, denoted by N_p, is called the ***neighborhood system*** of p.

Example 1.9. Let $X = \{a, b, c, d, e\}$ and $T = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$ be a topology on X. Find the neighbourhood system of

(i) the point e,

(ii) the point c.

Solution.

(i) The open sets containing e are $\{a, b, e\}$ and X. The superset of $\{a, b, e\}$ are $\{a, b, e\}, \{a, b, c, e\}, \{a, b, d, e$ and X. The superset of X is X itself. Therefore, neighborhood system of e is

$$N_e = \{\{a, b, e\}, \{a, b, c, e\}, \{a, b, d, e\}, X\}$$

(ii) The open sets containing c are $\{a, c, d\}, \{a, b, c, d\}$ and X. The super sets of these open sets are $\{a, c, d\}, \{a, b, c, d\}, \{a, c, d, e\}$, and X. Hence, neighborhood system of c is

$$N_c = \{\{a, c, d\}, \{a, b, c, d\}, \{a, c, d, e\}, X\}$$

In this example we see that $-$
In N_e, the elements $\{a, b, e\}$ and X are open but the elements $\{a, b, c, e\}$ and $\{a, b, d, e\}$ are not open but they are all neighbourhood of e.
Similarly, in N_c, the element $\{a, c, d, e\}$ is not open in X.
Therefore, neighbourhood of a point may not be open.

Remark.

- In topological space (X, τ), any open set G in τ is a neighbourhood of each of its points.

- If N is a neighbourhood of a point $x \in X$, then any superset of N is also a neighbourhood of x.
- Each point $x \in X$ is contained in some neighbourhood.

Lemma 1.1. A subset V of a topological space X is open if and only if V is a neighborhood of each point belonging to V.

Proof. Suppose V is an open in X. Then, each point $p \in V$ belongs to the open set V contained in V. Hence, V is a neighborhood of each point belonging to V.
Conversely, suppose that V is a neighborhood of each its points. Then, by definition of neighborhood, given any point $p \in V$, there is an open set G_p, such that $p \in G_p \subseteq V$. Then clearly $\cup \{G_p : p \in V\} = V$and V is open since it is the union of open sets. □

1.2 The Subspace Topology

Definition 1.4. Let X be a topological space with a topology τ and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $G \cap A$ for $G \in \tau$.
i.e., $\tau_A = \{U \subseteq A : U = G \cap A, G \in \tau\}$.
Then τ_A is a topology, called the ***subspace topology*** on A or the ***relative topology*** on A. The pair (A, τ_A) is called the ***subspace*** of the topological space (X, τ).

To prove that τ_A is a topology on X, we see that $-$

(i) Both A and $\varnothing$ are in τ_A, because $\varnothing = \varnothing \cap A$ and $A = X \cap A$, where $\varnothing, X \in \tau$.

(ii) Let $\{U_\alpha\}_{\alpha\in\Omega}$ be an arbitrary collection of elements of τ_A Then for each $\alpha, U_\alpha = G_\alpha \cap A$ where $G_\alpha \in \tau$. Now, $\bigcup_{\alpha\in\Omega} U_\alpha = U_{\alpha\in\Omega}(G_\alpha \cap A) = (\bigcup_{\alpha\in\Omega} G_\alpha) \cap A$ and $\bigcup_{\alpha\in\Omega} G_\alpha \in \tau$.
So $\bigcup_{\alpha\in\Omega} U_\alpha \in \tau_A$

(iii) If $\{U_1, U_2, \ldots, U_n\}$ is a finite collection of elements of τ_A then for each k, $U_k = G_k \cap A$ where $G_k \in \tau$. Then $\bigcap_{k=1}^n U_k = \bigcap_{k=1}^n (G_k \cap A) = (\bigcap_{k=1}^n G_k) \cap A$ and $\bigcap_{k=1}^n G_k \in \tau$.
Hence, $\bigcap_{k=1}^n U_k \in \tau_A$

Example 1.10. Consider the following topology

$$\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

on $X = \{a, b, c, d.e\}$.
Let $A = \{a, c, e\}$. To find the relative topology on A. We see that $X \cap A = A$ and $\varnothing \cap A = \varnothing$, $X, \varnothing$ are open in τ, so that $A, \varnothing \in \tau_A$. Also, $\{a\} \cap A = \{a\}$, $\{a, b\} \cap A = \{a\}, \{a, c, d\} \cap A = \{a, c\}, \{a, b, c, d\} \cap A = \{a, c\}$ and $\{a, b, e\} \cap A = \{a, e\}$. Hence,

$$\tau_A = \{A, \varnothing, \{a\}, \{a,c\}, \{a,e\}\}.$$

Clearly it is a topology on A.
We observe that $\{a, c\}$ and $\{a, e\}$ are not open in τ but they are relatively open in A.

Example 1.11. Let (X, τ) be any topological space and $A = \{a\}$ for some $a \in X$. Then the relative topology τ_A on A is the indiscrete topology on A as $\tau_A = \{\varnothing, \{a\}\}$.

Example 1.12. Let $(\mathbb{R}, \tau_u)$ be a usual topological space and $N \subset \mathbb{R}$. Then the relative topology τ^* on $\mathbb{N}$ is a discrete topology on $\mathbb{N}$ as for any $n \in \mathbb{N}$, $\{n\} = \left(n - \frac{1}{2}, n + \frac{1}{2}\right) \cap \mathbb{N} \in \tau^*$
Similarly, the relative topology of τ_u on $\mathbb{Z}$ is a discrete topology,

Remark. Let (Y, τ^*) be a subspace of a topological space (X, τ). Then for each subset open in (Y, τ^*) to be open in (X, τ) it is necessary and sufficient that Y is open in X.

1.3 Closed sets

Definition 1.5. A subset A of a topological space X is said to be closed if its compliment is open, that is, if $X - A \in \tau$

Example 1.13. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. Find the closed sets of X.

Proof. We know that a set is closed if its compliment is open.
Hence, the compliment of each set in τ are

$$X, \varnothing, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}$$

These are the closed sets of X. □

Theorem 1.2. Let X be a topological space. Then the following conditions holds:

a) $\varnothing$ and X are closed.

b) arbitrary intersections of closed sets are closed.

c) finite unions of closed sets are closed.

Proof.

a) Since $\varnothing$ and X are the compliments of the open sets X and $\varnothing$ respectively, so they are closed sets.

b) Let $\{F_\alpha\}_{\alpha \in I}$ be a family of closed sets. Then by De Morgan's law,

$$X - \bigcap_{\alpha \in I} F_\alpha = \bigcup_{\alpha \in I} (X - F_\alpha)$$

Since the sets $X - F_\alpha$ are open, the union of arbitrary open sets is open. Thus $\bigcup_{\alpha \in I}(X - F_\alpha)$ is open; i.e., $X - \bigcap_{\alpha \in I} F_\alpha$ is open and therefore $\bigcap_{\alpha \in I} F_\alpha$ is closed.

c) Let F_i is closed for each $i = 1, 2, \ldots, n$. Then

$$X - \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (X - F_\alpha)$$

Since the sets $X - F_\alpha$ are open, the finite intersection of open sets is open. Thus, $\bigcap_{i=1}^{n}(X - F_\alpha)$ is open, i.e., $X - \bigcup_{i=1}^{n} F_i$ is open and therefore $\bigcup_{i=1}^{n} F$ is closed.

□

Theorem 1.3. Let X be a topological space and $A \subseteq X$. If $F \subseteq A$, then F is relativity closed iff there is closed subset F^* of X such that $F = F^* \cap A$.

Proof. Let F is relatively closed; then $A - F$ is relatively open. Then there is an open set U in X such that

$$A - F = U \cap A$$

and

$$F = (X - U) \cap A$$

If we let $X - U = F^*$, then F^* is closed in X and $F = F^* \cap A$.
Conversely, let $F = F^* \cap A$ and F^* is closed in X. Then $X - F^*$ is open in X and so $(X - F^* \cap A)$ is open in A, by definition of the subspace topology. But $(X - F^* \cap A = A - F)$. Hence, $A - F$ is open in A and so F is closed in A. □

1.4 Limit points/Accumulation points

Definition 1.6. Let X be a topological space and $A \subseteq X$. A point $p \in X$ is said to be an ***accumulation point*** (or, ***limit point***, or, ***cluster point***) of A if for every open set G containing p contains a point of A other than p i.e., $\{G - \{p\}\} \cap A \neq \varnothing$.

The set of all limit points of A, denoted by A', is called the ***derived set*** of A.

Example 1.14. Let the class

$$\tau = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

be a topology on $X = \{a, b, c, d, e\}$ and $A = \{a, b, c\} \subseteq X$. Then $a \in X$ is not a limit point of A, because the open set $\{a\}$ contains no other point of A.
The set $\{c, d\}$ is an open set containing c, but it contains no other point of A. So c is not a limit point of A.
The point $b \in X$ is a limit point of A, because the only open set containing b are X and $\{b, c, d, e\}$ and both contain another point of A, namely c; i.e., $\{\{b, c, d, e\} - \{b\}\} \cap A = \{c\} \neq \varnothing$. So b is a limit point of A.
The point d is a limit point of A even though d is not in A because every open set containing d contains a point of A.
Similarly, the point e is a limit point of A even though c is not in A. Thus, the derived set of A is $A' = \{b, d, e\}$.

Example 1.15. Let (X, τ) be a discrete topological space and A be a subset of X. Then A has no limit points, since for each $x \in A$, $\{x\}$ is an open set containing no point of A other than x.

Example 1.16. Let (X, τ) be an indiscrete topological space and A be a subset of X with at least two elements. Then it is readily seen that every point of X is a limit of X.

The next theorem provides a useful way of testing whether a set is closed or not.

Theorem 1.4. Let (X, τ) be a topological space and $A \subseteq X$. Then A is closed iff A contains all its limit points ($A' \subseteq A$).

Proof. First suppose that A is closed in X. Let $p \in A'$. Then p is a limit point of A. If $p \notin A$, then $p \in X - A$ and $X - A$ is open. Then clearly $\{A - \{p\}\} \cap \{X - A\} = \varnothing$ which shows that p is not a limit point of A, a contradiction. Therefore, $p \in A$. Thus, $A' \subseteq A$, i,e., A contains all its limit points.
Conversely, let $A' \subseteq A$, i.e., A contains all its limit points. To show A is closed, we show that $X - A$ is open in X. So, let $y \in X - A$. Then y is not a limit point of A and so there is an open set U_y such that $U_y \cap A = \varnothing$. Then $U_y \subseteq X - A$ and therefore $X - A = \bigcup_{y \in X - A} U_y$. So x is a union of open sets and hence $X - A$ is open. Thus, A is closed. □

Example 1.17. As applications of Theorem 1.4, we have the following:

(i) The set $[a, b)$ is not closed in $\mathbb{R}$, since b is a limit point of $[a, b)$ and $b \notin [a, b)$.

(ii) The set $[a, b]$ is closed in $\mathbb{R}$ because all the limit points of $[a, b]$ are in $[a, b]$.

(iii) (a, b) is not closed in $\mathbb{R}$, because it does not contain the limits a and b.

(iv) $[a, \infty)$ is a closed subset of $\mathbb{R}$ because all the limit points of (a, ∞) are in $[a, \infty)$.

Theorem 1.5. Let A be a subset of a topological space (X, τ) and A' be the set of all limit points of A. Then $A \cup A'$ is closed.

Proof. It suffices to show that the set $A \cup A'$ contains all of its limit points or equivalently that no element of $X - (A \cup A')$ is a limit point of $A \cup A'$.
Let $p \in X - (A \cup A')$. Then $p \notin A$ and p is not a limit point of A. So, there is an open set U containing p such that $U \cap A = \varnothing$. We claim that $U \cap A' = \varnothing$. For; if $x \in U$, then $U \cap A = \varnothing$ implies x is not a limit point of A, i.e., $x \notin A'$. Thus, $U \cap A' = \varnothing$. Therefore, $U \cap (A \cup A') = \varnothing$ and $p \in U$. This implies p is not a limit point of $A \cup A'$. Thus, no element of $X - (A \cup A')$ is a limit point of $A \cup A'$. Hence, $A \cup A'$ is a closed set. $\square$

Problem 1.1. Let A and B are subsets of a topological space (X, τ). If $A \subseteq B$ then show that $A' \subseteq B'$.

Solution. Let $p \in A'$. Then by definition of limit point, there is an open set G containing p such that $\{G - \{p\}\} \cap A \neq \varnothing$. But $A \subseteq B$; hence $\{G - \{p\}\} \cap B \neq \varnothing$ and so $p \in B'$. Thus, $A' \subseteq B'$.

Problem 1.2. Let A and B are subsets of a topological space (X, τ). Then prove that $(A \cup B)' = A' \cup B'$.

Solution. We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$ and so $A' \subseteq (A \cup B)'$ and $B' \subset (A \cup B)'$; hence $A' \cup B' \subseteq (A \cup B)'$.
To show $(A \cup B)' \subseteq A' \cup B'$, we proceed by the method of contradiction. So, assume that $(A \cup B)' \nsubseteq A' \cup B'$. Then there is an element $p \in (A \cup B)'$ such that $p \notin (A' \cup B)$. This implies p is neither a limit point of A nor of B. Hence there are open sets G and H such that

$$p \in G \text{ and } G \cap A \subseteq \{p\}\,; \qquad \text{and} \qquad p \in H \text{ and } H \cap B \subseteq \{p\}$$

But $p \in G \cap H$ and $G \cap H$ is open. Now

$$(G \cap H) \cap (A \cup B) = (G \cap H \cap A) \cup (G \cap H \cap B) \subseteq \{p\}$$

Thus p is not a limit point of $A \cup B$, i.e., $p \notin (A \cup B)'$ which is a contradiction. Hence, $(A \cup B)' \subseteq A' \cup B'$.

Definition 1.7. Let (X, τ) be a topological space. A sequence $\langle x_n \rangle$ on X is said to converge to a point $x \in X$ if for every open set U containing x, there is an $N \in \mathbb{N}$ such that $X_n \in U$ for all $n > N$.

An important thing is that a sequence in a topological space may have more than one limit. Here are some examples where it happens.

Example 1.18. Let $X = \{1, 2, 3\}$ and $\tau = \{X, \varnothing, \{1, 2\}\}$ be a topology on X. Let $\langle x_n \rangle$ be a constant sequence such that $x_n = 1$ for every n. There are two open sets containing $1 : \{1, 2\}$ and X, every term of $\langle x_n \rangle$ is in the open set, thus, $\langle xn \rangle$ converges to 1. Also, the open sets containing 2 contains all the terms of $\langle x_n \rangle$, so $\langle x_n \rangle$ also converges to 2. It is easy to see that the open set containing 3 also contains all terms of $\langle x_n \rangle$. Thus, the set of limits of $\langle x_n \rangle$ is $\{1, 2, 3\}$. Therefore, the limit of the sequence on (X, τ) is not unique.

Example 1.19. Let X be any nonempty set and let $\tau = \{X, \varnothing\}$ be the indiscrete topology on X. Then every sequence $\langle x_n \rangle$ in X converges to every point of X. For; let $x \in X$ be any point. The only open set containing x is X and so, for all N and for any $n > N$, $X_n \in X$.

1.5 Adherent point

Definition 1.8. A point $p \in X$ is called an ***adherent point*** of $A \subseteq X$ iff every open set G containing p contains a point of A, i.e., for any open set G with $p \in G$ implies $G \cap A \neq \varnothing$. Thus, a point $p \in X$ is an ***adherent point*** of A if $p \in \bar{A}$.

From the definition, it is clear that an adherent point is either a point of A or a limit point of A. But every adherent point is not a limit point.

Example 1.20. Consider the topology

$$\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{a,b,e\}\}$$

on $X = \{a,b,c,d,e\}$. Then the closed subsets of X are X, $\varnothing$, $\{c,d\}$, $\{c,d,e\}$, $\{b,c,d,e\}$.
If we consider $A = \{a,c,d\}$, then $A' = \{b,c,d,e\}$. Here the point a is not a limit point of A but it is an adherent point of A, because the open sets containing a are $G_1 = \{a\}$, $G_2 = \{a,b\}$, $G_3 = \{a,b,e\}$ and $G_4 = X$ and any of the case, $G_i \cap A \neq \varnothing$.

1.6 Closure of a Set

Definition 1.9. Let A be a subset of a topological space X. Then the ***closure*** of A, denoted by Cl (A) or $\bar{A}$, is the smallest closed set containing A. Mathematically;

$$\bar{A} = \cap \{F : F \text{ is closed and } A \subseteq F\}$$

From this definition, it is obvious that

1. $\bar{A}$ is closed.
2. $A^{\circ} \subseteq A \subseteq \bar{A}$
3. If A is closed, then $A = \bar{A}$.

To find the closure of a particular set. We shall find all the closed sets containing that set and then select the smallest.

Example 1.21. Let $X = \{a,b,c,d,e\}$ and $t = \{X, \varnothing, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$ Show that $\overline{\{b\}} = \{b,e\}$, $\overline{\{a,c\}} = X$ and $\overline{\{b,d\}} = \{b,c,d,e\}$

Proof. The closed sets are X, $\varnothing$, $\{b,c,d,e\}$, $\{a,b,e\}$, $\{b,e\}$ and $\{a\}$. So, the smallest closed set containing $\{b\}$ is $\{b,e\}$; i.e., $\overline{\{b\}} = \{b,e\}$. Similarly, $\overline{\{a,c\}} = X$ and $\overline{\{b,d\}} = \{b,c,d,e\}$. □

Problem 1.3. Prove that $\bar{A} = A \cup A'$

Proof. Since $\bar{A}$ is the smallest closed set containing A and $A \cup A'$ is a closed set containing A [by Theorem 1.5], so $\bar{A} \subseteq A \cup A'$. Again, since $A \subseteq \bar{A}$ and $\bar{A}$ is closed, so $A' \subseteq (\bar{A})' \subseteq \bar{A}$ and hence $A \cup A' \subseteq \bar{A}$. Thus, $\bar{A} = A \cup A'$. □

Problem 1.4. If $A \subseteq B$ then prove that $\bar{A} \subseteq \bar{B}$.

Proof. We know that if $A \subseteq B$ then $A' \subseteq B'$ and hence $A \cup A' \subseteq B \cup B'$; i.e.; $\bar{A} \subseteq \bar{B}$. □

1.7 Dense Set

Definition 1.10. Let A be a nonempty subset of a topological space (X, τ). Then A is said to be dense in X (or everywhere dense in X) iff for every nonempty open set U of X, $U \cap A \neq \varnothing$.

In the Example 1.18, we have seen that $\{a,c\}$ is dense in X.

Theorem 1.6. Let A be a subset of a topological space (X, τ). Then A is dense in X iff $\bar{A} = X$.

Proof. First suppose that A is dense in X. Then for every nonempty open set U of X, $U \cap A \neq \varnothing$. If $A = X$, then clearly A is dense in X. If $A \neq X$, let $x \in X - A$. Then for any open set U containing x, $U \cap A \neq \varnothing$. Since $x \in A$, so x is a limit point of A and hence $(X - A) \subseteq A'$. Then $\bar{A} = A \cup A' = X$.

Conversely, assume that $\bar{A} = X$. Then every point of $X - A$ is a limit point of A. Suppose A is not dense in X. Then there is an open set U of X such that $U \cap A = \varnothing$. Then for any $x \in U$, $x \notin A$ and so $x \in X - A$. Clearly, x is not a limit point of A, since $U \cap A = \varnothing$. This is a contradiction. So, our supposition is false and hence $U \cap A \neq \varnothing$. Therefore, A is dense in X. □

Problem 1.5. For any subset A of a topological space (X, τ), determine all the dense subsets of X when

(i) X is discrete

(ii) X is indescrete.

Solution.

(i) In a discrete space X, every subset of X is closed. So, for any $A \subseteq X$, $\bar{A} = A$ and hence X is the only closed subset of X for which $\bar{X} = X$. Thus, the only dense subset of X is X itself.

(ii) In an indiscrete space X, the only closed subsets of X are $\varnothing$ and X. Let A be any non-empty subset of X. Since X is the only closed set containing A, so $\bar{A} = X$. Therefore, every non-empty subset A of X is dense in X.

1.8 Interior, Exterior and Boundary of a Set

Definition 1.11 (Interior Point). Let X be a topological space and $A \subseteq X$. Then a point $p \in A$ is called an interior point of A if there is an open set G containing p such that $p \in G \subseteq A$.

Definition 1.12. The ***interior*** of a set A is the set of all interior points of A. The interior of A is denoted $int(A)$, or A°. The interior of a set has the following properties.

- A° is an open subset of A.
- A° is the union of all open sets contained in A.
- A° is the largest open set contained in A.
- A set A is open if and only if $A = A^{\circ}$.
- $(A^{\circ})^{\circ} = A^{\circ}$ If A is a subset of S, then A° is a subset of S°.

Sometimes the second or the third property above is taken as the definition of the interior of a set.

Example 1.22. Consider the set $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$.
If we choose the set $A = \{a, c\} \subset X$, then $a \in A$ is an interior point of A if we let $G = \{a\} \in \tau$ since $a \in G \subset \{a, c\} = A$.
However, the point $c \in A$ is not an interior point with respect to this topology τ. The only open set that contains c is X and $c \in X \nsubseteq \{a, c\} = A$. Therefore, $A^{\circ} = \{a\}$.

Definition 1.13 (Exterior Point). Let A be a subset of a topological space X. Then a point $p \in X$ is called an exterior point of A if p is the interior point of $X - A$.

The exterior of A is the set of all exterior points of A and we denote it by $ext(A)$. Clearly an exterior point of a set A is neither a point of A nor a limit point of A. It is the interior point of A^c.

Definition 1.14 (Boundary point). Let A be a subset of a topological space X. Then a point $p \in X$ is called a ***boundary point (frontier point)*** of A if for any open set G containing p, $G \cap A \neq \varnothing$ and $G \cap (X - A) \neq \varnothing$.

From the above definition, we see that the boundary of A is the set of those points of X which are neither exterior points of A nor interior points of A. We denote it by $bd(A)$ or δA. Also, the set $\bar{A} - A^{\circ}$ is called the boundary of A.

Example 1.23. Consider the topology $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$ on $X = \{a, b, c, d, e\}$. Let $A = \{a, b, c\}$. Find the interior points, exterior points and boundary points of A.

Solution. Here the open subsets in A are, $\varnothing$, $\{a\}$, $\{a, b\}$.Clearly, their union is $\{a, b\}$ and so $A^{\circ} = \{a, b\}$.
$A^c = \{d, e\}$. We see that there are no open sets containing d or e lies in A^c. So $ext(A) = int(A^c) = \varnothing$. The boundary of A is neither exterior nor interior, so clearly $bd(A) = \{c, d, e\}$.

From the above example, we see that

$$X = int(A) \cup bd(A) \cup ext(A)$$

Or

$$X = int(A) \cup bd(A) \cup int(A^c)$$

Theorem 1.7. Let A be a subset of a topological space X. Then the closure of A is the union of interior and boundary of A, i.e., $\bar{A} = int(A) \cup bd(A)$.

Proof. Since $X = int(A) \cup bd(A) \cup ext(A)$, so

$$(int(A) \cup bd(A))^c = ext(A) \tag{1.1}$$

It suffices to show that $(\bar{A})^c \subseteq ext(A)$.
Let $p \in ext(A)$. Then there exist an open set G containing p such that $G \subseteq A^c$ which implies $G \cap A = \varnothing$. Then p is not a limit point of A. Also, $p \notin A$. Hence, $p \notin A \cup A' = \bar{A}$; i.e.; $p \in (\bar{A})^c$. Thus,

$$ext(A) \subseteq (\bar{A})^c \tag{1.2}$$

Now let $p \in (\bar{A})^c = (A \cup A')^c$. Then $p \notin A'$ and so there exist an open set G containing p such that $\{G - \{p\}\} \cap A = \varnothing$. Also $p \notin A$, so $G \cap A = \varnothing$ and $p \in G \subset A^c$. Thus, p is an interior point of A^c, i.e., $p \in ext(A)$. Thus,

$$(\bar{A})^c \subseteq ext(A) \tag{1.3}$$

From (1.2) and (1.3), we get $(\bar{A})^c = ext(A)$.
Hence,

$$\bar{A} = (ext(A))^c = int(A) \cup bd(A) \qquad \text{by (1.1)}$$

□

Theorem 1.8. Let A be a subset of a topological space X. Prove that $\bar{A}$ is a closed set.

Proof. We know that

$$X = int(A) \cup bd(A) \cup int(A^c)$$

Then $int(A) \cup bd(A) = X - int(A^c)$. Since $\bar{A} = int(A) \cup bd(A)$ and $int(A^c)$ is open, so $\bar{A} = X - int(A^c)$ is closed. □

Problem 1.6. Let $f : X \to Y$ be a function from a non-empty set X into a topological space (Y, τ^*). Define $\tau = \{f^{\leftarrow}(G) : G \in \tau^*\}$. Show that, τ is a topology on X.

Solution. Since τ^* is a topology on Y, so $\varnothing, Y \in \tau^*$. Again, since $X = f^{\leftarrow}(Y)$ and $\varnothing = f^{\leftarrow}(\varnothing)$, by definition of τ, $\varnothing, X \in \tau$. Let $\{U_i\}$ be a class of sets in τ. Then for each i, there exist G_i such that $U_i = f^{\leftarrow}(G_i)$. Now,

$$\cup U_i = \cup f^{\leftarrow}(G_i) = f^{\leftarrow}(\cup G_i) \in \tau \quad \cup G_i \in t^*$$

Finally, for any two elements $U_1, U_2 \in \tau$.

$$U_1 \cap U_2 = f^{\leftarrow}(G_1) \cap f^{\leftarrow}(G_2) = f^{\leftarrow}(G_1 \cap G_2) \in \tau \text{ as } G_1 \cap G_2 \in \tau^*$$

Thus t is a topology on X.

Problem 1.7. Let τ be a topology on a set X consisting of four subsets, i.e.,

$$\tau = \{X, \varnothing, A, B\}$$

where A and B are non-empty distinct proper subsets of X. What conditions A and B must satisfy?

Solution. Since $\tau = \{X, \varnothing, A, B\}$ is a topology on X; So $A \cap B \in \tau$ and $A \cup B \in \tau$.
Case I. If $A \cap B = \varnothing$, then $A \cup B$ cannot be A or B; but it must be an element of τ; hence $A \cup B = X$. Thus, $\{A, B\}$ is a partition of X.
Case II. If $A \cap B \neq \varnothing$, then either $A \cap B = A$ or $A \cap B = B$; i.e., either $A \subseteq B$ or $B \subseteq A$. Then the members of τ are totally ordered set; i.e., $\varnothing \subseteq A \subseteq B \subseteq X$ or $\varnothing \subseteq B \subseteq A \subseteq X$.

Problem 1.8. List all topologies on $X = \{a, b, c\}$ which consist of exactly four elements.

Solution. Each topology τ on X with four members is of the form $\tau = \{X, \varnothing, A, B\}$ where either $\{A, B\}$ is a partition of X or the members of τ is totally ordered.
Case I. $\{A, B\}$ is a partition of X.
In this case the topologies are

$$\tau_1 = \{X, \varnothing, \{a\}, b, c\}, \tau_2 = \{X, \varnothing, \{b\}, \{a, c\}\} \text{ and } \tau_3 = \{X, \varnothing, \{c\}, \{a, b\}\}$$

Case II. The members of τ are totally ordered.
The topologies in this case are the following:

$$\begin{aligned} \tau_4 &= \{X, \varnothing, \{a\}, \{a, b\}\}, & \tau_5 &= \{X, \varnothing, \{a\}, \{a, c\}\}, \\ \tau_6 &= \{X, \varnothing, \{b\}, \{a, b\}\}, & \tau_7 &= \{X, \varnothing, \{b\}, \{b, c\}\}, \\ \tau_8 &= \{X, \varnothing, \{c\}, \{a, c\}\}, & \tau_9 &= \{X, \varnothing, \{c\}, \{b, c\}\} \end{aligned}$$

1.9 Exercise

1. Let $p \in X$. Define $\tau = \{\varnothing\} \cup \{B \subseteq X : p \in B\}$. Then show that τ is a topology on X.

2. Let $p \in X$. Define $\tau = \{X\} \cup \{B \subseteq X : p \notin B\}$. Then show that τ is a topology on X.

3. Let $\tau = \{\varnothing\} \cup \{A \subseteq \mathbb{R} | \forall p \in A \exists a, b \in \mathbb{R}\}$ such that $p \in [a, b) \subseteq A$. Then show that τ is a topology on $\mathbb{R}$.

4. Let $\tau = \{\varnothing\} \cup \{A_n | n = 1, 2, \dots\}$ where $A_n = \{n, n+1, n+2, \dots\}$. Then prove that τ is a topology on $\mathbb{N}$.

Chapter 2

Base for a Topology

Definition 2.1. Let (X, τ) be a topological space. Then a class $\mathcal{B} \subseteq \tau$ is a **base** for the topology τ on X if every member of τ can be written as a union of members of $mathcalB$. Equivalently, $U \in \tau$ can be written as $U = \cup_{B \in \mathcal{B}} B$.

Chapter 3

Continuity

The central concept in topology is continuity, defined for functions between sets equipped with a notion of nearness (topological spaces) which is preserved by a continuous function. Topology is one kind of geometry in which the important properties of a figure are those are preserved under continuous motions.

Definition 3.1. Let X and Y be two topological spaces and $f : X \to Y$ be a mapping. Then f is said to be continuous at p in X if given any open set V containing $f(p)$ there exist an open set U containing p such that $f(U) \subseteq V$.

If f is continuous for each $p \in X$, then f is said to be continuous on X.

Example 3.1. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$ be a topology on X. Define $f : X \to Y$ by $f(a) = b$, $f(b) = d$, $f(c) = b$, $f(d) = c$. Discuss/examine/check the continuity at c and d.

Solution. Continuity of f at c:
We see that at c, $f(c) = b$ and the open sets containing $f(c)$ are $X, \{b\}, \{a, b\}, \{b, c, d\}$. If we take $V = \{a, b\}$, then

$$f^{-1}(V) = f^{-1}(\{a, b\}) = \{a, c\}$$

The open sets containing c are X and $\{b, c, d\}$. Now, $f(X) = \{b, c, d\}$, and $f(\{b, c, d\}) = \{b, c, d\}$. But none of them contained in $V = \{a, b\}$. Hence, f is not continuous at c. **Continuity of f at d:** Here $f(d) = c$ and the open sets containing $f(d)$ are X and $\{b, c, d\}$. Also, the open sets containing d are X and $\{b, c, d\}$.

If we take $V = X$, then we get an open set $\{b, c, d\}$ containing d with $f(\{b, c, d\}) = \{b, c, d\} \subseteq V$. And if we take $V = \{b, c, d\}$, we get the open set $\{b, c, d\}$ containing d with $f(\{b, c, d\}) = \{b, c, d\} \subseteq V$. Therefore, f is continuous at d.

Example 3.2. If a singleton set $\{p\}$ is an open in a topological space (X, τ) then any function $f : X \to Y$, is continuous at $p \in X$.

Proof. Suppose Hbe a open set containing $f(p)$. But

$$f(p) \in H \quad \text{implies} \quad p \in f^{\leftarrow}(H) \quad \text{implies} \quad \{p\} \subseteq f^{\leftarrow}(H)$$

This implies $f(\{P\}) \subseteq H$. Hence, f is continuous at p. □

From this example we can say that any function defined on a discrete space is continuous.

Theorem 3.1. A function $f : X \to Y$ is continuous iff for each open subset V in Y, $fa(V)$ is open in X.

Proof. First suppose f is continuous on X and let V be any open subset of Y. Let $U = f^{\leftarrow}(V)$. Choose any point $p \in U$. Then $f(p) \in V$. Since f is continuous at p, there exist an open set W_p

containing p such that $f(W_p) \subseteq V$. Then $p \in W_p \subseteq f^{\leftarrow}(V) = U$. Hence, U is a neighborhood of p. Since p is arbitrary, so U is a neighborhood of each point of U. Therefore, $U = f^{\leftarrow}(V)$ is open.

Conversely, let for each open subset V of Y, $f^{\leftarrow}(V)$ is open in X. Let $U = f^{\leftarrow}(V)$. Then $f(U) = f(f^{\leftarrow}(V)) \subseteq V$.
Hence, by definition, f is continuous. $\square$

Example 3.3. Let $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ be given by $f(x) = x$ for all $x \in \mathbb{R}$; that is, f is an identity function. Then for any open set V in $\mathbb{R}$, $f^{\leftarrow}(V) = V$ and so $f^{\leftarrow}(V)$ is open. Hence, f is continuous.

Example 3.4. Let $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ be given by $f(x) = c$ for all $x \in \mathbb{R}$; that is, f is a constant function. Then for any open set V in $\mathbb{R}$, clearly $f^{\leftarrow}(V) = \mathbb{R}$ if $c \in V$ and $f^{\leftarrow}(V) = \varnothing$ if $c \notin V$. In both cases $f^{\leftarrow}(V)$ is open. Hence, f is continuous.

Example 3.5. Let (X, τ) and (Y, τ^*) be two topologies defined by $X = \{a, b, c\}$ $\tau = \{X, \varnothing, \{a\}, \{b, c\}\}$ and $Y = \{p, q, r\}$ $\tau^* = \{Y, \varnothing, \{r\}, \{p, q\}\}$. Define $f : X \to Y$ by $f(a) = p$, $f(b) = q$, $f(c) = r$. The f is not continuous, because if we take the open set $V = \{r\}$ in Y, the $f^{\leftarrow}(V) = \{c\}$ which is not open in X.

Theorem 3.2. A function $f : ((X, \tau)) \to (Y, \tau^*)$ is continuous iff for each member of a base $\mathcal{B}$ for Y, $f^{\leftarrow} = (B)$ is open in X.

Proof. Let f be continuous and $B \in \mathcal{B}$. Then B is open in Y since it is a member of base $\mathcal{B}$, and hence $f^{\leftarrow}(B)$ is open in X.

Conversely, let V be any open set in Y. We show that $f^{\leftarrow}(V)$ is open in X. Since $\mathcal{B}$ is a base for Y, every open set in Y is the union of members of $\mathcal{B}$ and so $V = \cup \{B : B \in \mathcal{B}\}$. Then

$$f^{\leftarrow}(V) = f^{\leftarrow}(\cup \{B : B \in \mathcal{B}\}) = \cup \{f^{\leftarrow}(B) : B \in \mathcal{B}\}$$

But, by the hypothesis $f^{\leftarrow}(B)$ is open in X and their union is also open in X. Hence, $f^{\leftarrow}(V)$ is open. Thus, f is continuous. $\square$

Theorem 3.3. Let $f : ((X, \tau)) \to (Y, \tau^*)$ and $\mathcal{A}$ be a subbase for the topology τ^* on Y. Then f is continuous iff the inverse of each member of the subbase $\mathcal{A}$ is an open subset of X.

Proof. Let $f : ((X, \tau)) \to (Y, \tau^*)$ be continuous and $\mathcal{A}$ be a subbase to τ^*. Then each element $\mathcal{S}$ of $\mathcal{A}$ is open in Y and so $f^{\leftarrow}(\mathcal{S})$ is open in X, f being continuous.

Conversely, suppose for any $\mathcal{S} \in \mathcal{A}$, $f^{\leftarrow}(\mathcal{S})$ is open in X. We show that f is continuous; i.e., $G \in \tau^*$ implies $f^{\leftarrow}(G) \in \tau$. Let $G \in \tau^*$. Then by definition of subbase,

$$G = \cup(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n), \quad \text{where } \mathcal{S}_i \in \mathcal{A}$$

Hence

$$\begin{aligned} f^{\leftarrow}(G) &= f^{\leftarrow}(\cup(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n)) \\ &= \cup f^{\leftarrow}(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n) \\ &= \cup[f^{\leftarrow}(\mathcal{S}_1) \cap f^{\leftarrow}(\mathcal{S}_2) \cap \cdots \cap f^{\leftarrow}(\mathcal{S}_n)] \end{aligned}$$

But, by hypothesis, $\mathcal{S}_i \in \mathcal{A}$ implies $f^{\leftarrow}(\mathcal{S}_i)$ is open in X and hence $f^{\leftarrow}(G)$ is open in X since the union of finite intersection of open sets is open. Therefore, f is continuous. $\square$

Theorem 3.4. A function $f : X \to Y$ is continuous iff for any subset of Y, $f^{\leftarrow}(B^{\circ}) \subseteq f^{\leftarrow}(B)^{\circ}$.

Proof. Suppose f is continuous on X and let B be any subset of Y. Then B° is open in Y and so $f^{\leftarrow}(B^\circ)$ is open in X. Now, we have, $B^\circ \subseteq B$ and so $f^{\leftarrow}(B^\circ) \subseteq f^{\leftarrow}(B)$, and then $[f^{\leftarrow}(B^\circ)]^\circ \subseteq [f^{\leftarrow}(B^\circ)]^\circ = f^{\leftarrow}(B)^\circ$.
But $f^{\leftarrow}(B^\circ)$ is open, so $[f^{\leftarrow}(B^\circ)]^\circ = f^{\leftarrow}(B^\circ)$ and hence $f^{\leftarrow}(B^\circ) \subseteq f^{\leftarrow}(B)^\circ$.

Conversely, let for any subset B of Y, $f^{\leftarrow}(B^\circ) \subseteq [f^{\leftarrow}(B)]^\circ$. Let V be any open set in Y. Then $f^{\leftarrow}(V^\circ) \subseteq [f^{\leftarrow}(B)]^\circ$. But V is open, so $V^\circ = V$. Hence $f^{\leftarrow}(V^\circ) = f^{\leftarrow}(V) \subseteq [f^{\leftarrow}(V)]^\circ$.
Bur it is always the case that $[f^{\leftarrow}(V)]^\circ \subseteq f^{\leftarrow}(V)$. Hence $f^{\leftarrow}(V) = [f^{\leftarrow}(V)]^\circ$ which is open in X. Therefore, f is continuous on X. $\square$

Theorem 3.5. A function $f : X \to Y$ is continuous iff for each closed subset F of Y, $f^{\leftarrow}(F)$ is a closed subset in X.

Proof. Suppose f is continuous on X and let F be any closed subset of Y. Let $V = Y - F$. Then V is open in Y and hence $f^{\leftarrow}(V)$ is open in X. Now,

$$X - f^{\leftarrow}(F) = f^{\leftarrow}(Y - F) = f^{\leftarrow}(V)$$

which is open in X. Hence, $f^{\leftarrow}(F)$ is closed in X.

Conversely, let V be any open set in Y. Then $Y - V$ is closed and hence $f^{\leftarrow}(Y - V)$ is closed ain X. But,

$$f^{\leftarrow}(Y - V) = X - f^{\leftarrow}(V)$$

which is closed in X. Hence, $f^{\leftarrow}(V)$ is open in X. Thus, f is continuous on X. $\square$

Theorem 3.6. A function $f : X \to Y$ is continuous iff for any subset A of X, $f(\overline{A}) \subseteq \bar{(}f(A))$.

Proof. Suppose f is continuous on X and let A be any subset of X. Then $f(A)$ is a subset of Y and $\overline{f(A)}$ is closed in Y; hence $f^{\leftarrow}(\overline{f(A)})$ is closed in X. Now, we have

$$f(A) \subseteq \overline{f(A)}$$

and so

$$f^{\leftarrow}(f(A)) \subseteq f^{\leftarrow}(\overline{f(A)})$$

But $A \subseteq f^{\leftarrow}(f(A))$; hence $A \subseteq f^{\leftarrow}(\overline{f(A)})$. Since $f^{\leftarrow}(\overline{f(A)})$ is closed and $\overline{A}$ is the smallest closed set containing, it follows that

$$\overline{A} \subseteq f^{\leftarrow}(\overline{f(A)})$$

and so

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Conversely, let for any subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$. Let F be any closed set in Y. Then $f^{\leftarrow}(F)$ is subset of X. We claim that $f^{\leftarrow}(F)$ is closed in X. Since $fa(F)$ is subset of X, so

$$f\overline{(f^{\leftarrow}(F))} \subseteq \overline{f(f^{\leftarrow}(F))} = \overline{F} = F$$

$\therefore \overline{(f^{\leftarrow}(F))} \subseteq f^{\leftarrow}(F)$.
But it is always the case that $f^{\leftarrow}(F) \subseteq \overline{(f^{\leftarrow}(F))}$. Hence $f^{\leftarrow}(F) = \overline{(f^{\leftarrow}(F))}$ i.e., $f^{\leftarrow}(F)$ is closed and therefore f is continuous on X. $\square$

3.1 Sequential Continuity

Definition 3.2. A function $f : X \to Y$ is said to be sequentially continuous at a point $p \in X$ iff for every sequence $\langle a_n \rangle$ converging to p, the sequence $f(a_n)$ converges to $f(p)$; i.e., iff $a_n \to p$ implies $f(a_n) \to f(p)$.

Continuity and sequential continuity at a point are related as follows:

Theorem 3.7. If a function $f : X \to Y$ is continuous at $p \in X$, then it is sequentially continuous at p.

Proof. Let the sequence $\langle a_n \rangle$ in X converges to p. Let M be the neighborhood of $f(p)$. Then f being continuous at p implies $f^{\leftarrow}(M)$ is open in X containing p. Let $N = f^{\leftarrow}(M)$. Then, since $\langle a_n \rangle$ converges to p, so $a_n \in N$ for almost all $n \in \mathbb{N}$. This implies $f(a_n) \in f(N) = f(f^{\leftarrow}(M)) = M$ for almost all $n \in \mathbb{N}$. So, the sequence $\langle f(a_n) \rangle$ converges to $f(p)$. Hence, f is sequentially continuous at p. $\square$

3.2 Open and Closed functions

A function $f : X \to Y$ is called an **open function** if the image of every open set is open.
Similarly, a function $f : X \to Y$ is called a **closed function** if the image of every closed set is closed.
In general, functions which are not open need not be closed and vice versa.

Example 3.6. Let $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ be given by $f(x) = c$ for all $x \in \mathbb{R}$. Then f is continuous (see ex 3.3). Let V be a open set and H be a closed set in R. Then,

$$f(v) = \{c\} \text{ and } f(H) = \{c\} \text{ for all } x \in V \text{ and for all } x \in H$$

Since $\{c\}$ is finite, it is closed but not open. Therefore f is a closed map and continuous but it is not open.

Example 3.7. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$, $Y = \{p, q, r\}$ and $\tau^* = \{\varnothing, \{p\}, \{p, r\}, Y\}$.

1. Define $f : X \to Y$ by $f(a) = p$, $f(b) = q$, $f(c) = r$. Then f is an open map but it is not continuous.

2. Define $g : X \to Y$ by $g(x) = q$ for all $x \in X$. Then g is a closed map and it is continuous but not open.

3. Define $h : X \to Y$ by $h(x) = p$ for all $x \in X$. Then h is an open map and it is not continuous and not open.

3.3 Homeomorphism

Between any two topological spaces (X, τ) and (Y, τ^*), there are many functions $f : X \to Y$. We choose to discuss continuous, or open or closed functions rather than arbitrary functions since these functions preserves some aspects of the structure of the spaces (X, τ) and (Y, τ^*).
If the function $f : X \to Y$ defines a one to one correspondence between the open sets in X and the open sets in Y, then the spaces (X, τ) and (Y, τ^*) are identical from the topological point of view.

Definition 3.3. Let X and Y be topological spaces. A bijective function $f : X \to Y$ is said to be a homeoporphism if f is open and continuous, or equivalently, both f and $f^{\leftarrow}$ are continuous.
If there exists a homeoporphism between X and Y, we say that X and Y are **homeomorphic** spaces, or that they are topologically equivalent, and write $X \cong Y$.

Lemma 3.1. If $f : X \to Y$ is a homeoporphism, then so is the inverse map $f^{\leftarrow} : Y \to X$.

Lemma 3.2. If $f : X \to Y$ and $g : Y \to Z$ are homeoporphisms, then so is the composite map $gf : X \to Z$.

Example 3.8. For each space X the identity function $i_d : X \to X$, with $i_d(x) = x$ for all $x \in X$, is a homeoporphism.

Example 3.9. Any two open intervals of the real line are homeomorphic. For example, if $S = (-1, 1)$ and $T = (0, 5)$, then define $f : S \to T$ and $g : T \to S$ by $f(x) = \frac{5}{2}(x+1)$, $g(x) = \frac{2}{5}(x-1)$. These maps are continuous, being composites of addition and multiplication, and it is easy to verify that they are inverse to each other. So f and g are homeomorphisms, and $(-1, 1)$ and $(0, 5)$ are homeomorphic.

Example 3.10. The function $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{x}{1-x^2}$ is homeoporphism. To find the inverse of f, we rewrite the equation $\frac{x}{1-x^2} = y$ as $yx^2 + x - y = 0$ and solve for x as a function of $y \in \mathbb{R}$, namely

$$f^{-1}(y) = \frac{-1 + \sqrt{1 + 4y^2}}{2y} = \frac{2y}{1 + \sqrt{1 + 4y^2}}$$

It is well known that both f and f^{-1} are continuous, hence $\mathbb{R}$ is homeomorphic to any open interval (a, b).

If we define a continuous map $f : (-1, 1) \to \mathbb{R}$ by

$$f(x) = \tan(\frac{\pi}{2}x)$$

This is a bijection and has a continuous invgrse $g : \mathbb{R} \to (-1, 1)$ given by

$$g(x) = \frac{2}{\pi}\tan^{-1}(x)$$

Example 3.11. A solid square is homeomorphic to a solid disc.
We will illustrate this with the square $Q = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ and disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
Define $f : D \to Q$ by

$$f(x, y) = \frac{\sqrt{x^2 + y^2}}{\max(|x|, |y|)}(x, y)$$

if $(x, y) \neq (0, 0)$ and $f(0, 0) = (0, 0)$. Its inverse $g : Q \to D$ is given by

$$g(x, y) = \frac{\max(|x|, |y|)}{\sqrt{x^2 + y^2}}(x, y)$$

if $(x, y) \neq (0, 0)$ and $g(0, 0) = (0, 0)$.

The idea of these maps is that f pushes the disc out radially to form a square, and g contracts the square radially to form a disc. Using this idea, you can see that the preimage of an open subset

Insert fig

of Q under f will be open in D and similarly for g. So, they are continuous maps.

3.4 Topological Properties

A property P is said to be a topological property or a topological invariant if, whenever a topological space (X, τ) has the property P, then every space homeomorphic to (X, τ) also has the property P. Briefly, a property, that is preserved under a homeomorphism, is called a topological property or topological invariant.

Example 3.12. Let $X = (0, \infty)$. Define a function $f : X \to X$ by $f(x) = \frac{1}{x}$. Then f is a homeomorphism. Observe that the sequence

$$\langle a_n \rangle = 1, \frac{1}{2}, \frac{1}{3}, ...$$

correspond, under homeoporphism, to the sequence

$$\langle f(a_n) \rangle = 1, 2, 3, ...$$

We see that the sequence $\langle a_n \rangle$ is a Cauchy sequence but the sequence $\langle f(a_n) \rangle$ is not. Hence the property of being a Cauchy sequence is not topological.

Example 3.13. Being a finite topological space, having the discrete, trivial or cofinte topology, or being a Hausdorff space, are all examples of topological properties. So, if X is a Hausdorff space and $X \cong Y$ then Y is a Hausdorff space. Compactness and connectedness are also topological properties.

Problem 3.1. Show that an identity map on a topological space is continuous but the identity map in different topological spaces may not be continuous.

Solution. Let $f : (X, \tau) \to (X, \tau)$ defined by $f(x) = x$ for all $x \in X$; that is, f is an identity map. Then for any open set V in X, $fa(V) = V$ and so $f^{\leftarrow}(V)$ is open. Hence f is continuous.
To prove the 2nd part, let $\tau =$ co-finite topology on $\mathbb{R}$ and $\tau_u =$ usual topology on $\mathbb{R}$.
Let $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ be an identity map. Let $V = (0, 1) \in \tau_u$. Then $i^{\leftarrow}(0, 1) = (0, 1) \notin \tau$ because $\mathbb{R} - (0, 1)$ is not finite. Thus we can see that, though $V = (0, 1)$ is open in $(\mathbb{R}, \tau_u)$, $i^{\leftarrow}(0, 1)$ is not open in $(\mathbb{R}, \tau)$. Hence the identity map $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_u)$ is not continuous.

Again, let $i : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau)$ be an identity map. Let $G \in \tau$. Then $\mathbb{R} - G$ is finite. Hence $i^{\leftarrow}(\mathbb{R} - G) = \mathbb{R} - G$ is closed in $\mathbb{R}, \tau_u$. Hence G is open in $\mathbb{R}$. Thus i is continuous.
Therefore, the identity map on different topological spaces may not be continuous.

Chapter 4

Topological spaces and metric spaces

The topology on $\mathbb{R}^n$ is defined in terms of open balls, which in turn are defined in terms of distance between points. There are many other spaces whose topology can be defined in a similar way in terms of a suitable notion of distance between points in the space.

Definition 4.1. A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A metric space (X, d) is a set equipped with a metric d on X. We denote a metric space simply by X.

The **open ball** of radius r centered at x in a metric space (X, d) is defined as

$$B_r(x) = \{y \in X | d(x, y) < r\}$$

that is, the points in X within r in distance from x. It is also known as **open sphere**, or $r-$ **neighborhood** of x.

Definition 4.2. A subset U of a metric space (X, d) is open if for any $x \in U$ there is an $r > 0$ so that $B_r(x) \subseteq U$. We note the following properties of open subsets of metric spaces.

1. An open ball $B_r(x)$ is an open set in (X, d)
2. An arbitrary union of open subsets is open.
3. The finite intersection of open subsets is open.

Theorem 4.1. Let (X, d) be a metric space. For any $x \in X$ and $r > 0$, let

$$B_r(x_= \{y \in X | d(x, y) < r\})$$

Define $\tau_d = \{A \subseteq X : \forall X \in A \exists r > 0 \text{ such that } B_r \subseteq A\} \cup \varnothing$. Then τ_d is a topology on X.

Proof. (i) By definition, $\varnothing \in \tau_d$. Also, for any $x \in X$, there exist an $r > 0$ such that $B_r(x) \subseteq X$. Hence $X \in \tau_d$.

(ii) Let $A, B \in \tau_d$. If $A \cap B = \varnothing$, then clearly $A \cap B \in \tau_d$. If $A \cap B \neq \varnothing$, then for $x \in A \cap B$ we get $x \in A$ and $x \in B$. Now, $x \in A$ implies there exists $r_1 > 0$ such that $B_{r_1}(x) \subseteq A$ and $x \in B$ implies there exists $r_2 > 0$ such that $B_{r_2}(x) \subseteq B$. Let $r = \min(r_1, r_2)$. Then $B_r(x) \subseteq B_{r_1}(x)$ and $B_r(x) \subseteq B_{r_2}(x)$. Thus $B_r(x) \subseteq A \cap B$. Hence $A \cap B \in \tau_d$.

(iii) Let $\{A_\alpha\}$ be a family of members of τ_d. Let $x \in \cup_{\alpha\in\Omega}A_\alpha$. Then $x \in A_{\alpha_0}$ for some $\alpha_0 \in \tau_d$, there exist $r > 0$ such that $B_r(x) \subseteq A_{\alpha_0}$ and hence $B_r(x) \subseteq \cup_{\alpha\in\Omega}A_\alpha$. Therefore $\cup_{\alpha\in\Omega}A_\alpha \in \tau_d$.

From (i), (ii) and (iii) τ_d is a topology on X. $\square$

This topology τ_d is called the topology induced by the metric d on X.

Let (X, d) be a metric space and τ be the collection of all open sets in (X, d). Then τ is a topology on X, called a **metric topology** generated by (induced by) the metric d and the open balls of all points are a basis for this topology.

Example 4.1. Let d be a usual metric on the real line $\mathbb{R}$, i.e., $d(x, y) = |x - y|$, then the open balls in $\mathbb{R}$ are precisely the finite open intervals and these open intervals forms a topology on $\mathbb{R}$ called usual topology. Hence the usual metric on $\mathbb{R}$ induces the usual topology on $\mathbb{R}$. Similarly, the usual metric on the plane $\mathbb{R}^2$ induces the usual topology on $\mathbb{R}^2$.

Proposition 4.1. The collection of all open balls $B_r(x)$ for $r > 0$ and $x \in X$ forms a base for a topology on X.

Proof. First a preliminary observation: For a point $y \in B_r(X)$ the ball $B_s(y)$ is contained in $B_r(x)$ if $s \leq r - d(x, y)$, since for $z \in B_s(y)$, we have $d(z, y) < s$ and hence

$$d(z, x) \leq d(z, y) + d(y, z) < s + d(x, y) \leq r$$

Now to show the condition to have a basis is satisfied, suppose $y \in B_{r_1}(x_1) \cap B_{r_2}(x_2)$. Then the observation in the preceding paragraph implies that $B_s(y) \subseteq B_{r_1}(x_1) \cap B_{r_2}(x_2)$, for any $s \leq \min\{r_1 - d(x_1, y), r_2 - d(x_2, y)\}$. Therefore the collection of all open balls $B_r(x)$ is a base for a topology on X. $\square$

A topological space (X, τ) together with a metric d that induces the topology τ is called a **metric topological space** or a **metric space** and it is denoted by (X, d).

Definition 4.3. A topological space (X, τ) is said to be **metrizable** if there is a metric d on X which induces the topology τ.

Example 4.2. $(\mathbb{R}, \tau_u)$ is a metrizable space.

Example 4.3. Discrete topological space is a metrizable space.

Example 4.4. Let $X = \{x, y\}$ and τ be the indiscrete topology. Then τ is not metrizable. Indeed, assume that τ is a metric topology for some metric d. Let $r = d(x, y)$. Then $B_r(x) = \{x\}$ is an open set. But $\{x\}$ is not an element of τ. A contradiction.

Example 4.5. Let X be an arbitrary set and let τ be a discrete topology on X. Let d be a metric on X defined by

$$d(x, y) = \begin{cases} 0, & \text{for } x = y \\ 1, & \text{for } x \neq y \end{cases}$$

Then $B_{\frac{1}{2}}(x) = \{x\}$; so, singleton subsets are open and hence d induces the discrete topology on X. Thus, we find a trivial metric d on X which induces the given topology τ. Accordingly, (X, τ) is metrizable.

4.0.1 Distance between Sets, Diameters

Let d be a metric on a set X. The **distance** between two non-empty sets A and B is denoted and defined by

$$d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$$

The distance between a point $p \in X$ and a non-empty subset B of X is denoted and defined by

$$d(p, B) = \inf\{d(p, b) : b \in B\}$$

The **diameter** of a non-empty subset E of X is denoted and defined by

$$d(E) = \sup\{d(a, b) : a, b \in E\}$$

If the diameter of a non-empty subset E of X is finite, i.e., $d(E) < \infty$, then E is said to be bounded. If $d(E) = \infty$, then E is said to be unbounded. Clearly a set has diameter 0 iff it is a singleton set.

Example 4.6. Let d be a trivial metric on X defined by

$$d(x, y) = \begin{cases} 0, & \text{for } x = y \\ 1, & \text{for } x \neq y \end{cases}$$

Then for any $p \in X$ and $A, B \subseteq X$.

$$d(p, A) = \begin{cases} 0, & \text{for } p \in A \\ 1, & \text{for } p \notin A \end{cases} \qquad d(A, B) = \begin{cases} 0, & \text{if } A \cap B = \varnothing \\ 1, & \text{if } A \cap B \neq \varnothing \end{cases}$$

Theorem 4.2. Let d be a metric on a set X. For any nonempty subset E of X, $d(\bar{E}) = d(E)$.

Proof. We know that $E \subset \bar{E}$. Now, if $d(\bar{E})$ is infinite, then there is nothing to prove. So, let $d(E) = r$ which is finite. If $d(\bar{E}) = r'$, then $r' \geq r$. Suppose $r' \neq r$ and let $r' - r = s > 0$. Then any point $x_0 \in \bar{E} - E$ must be a limit point of E and any open sphere centered at x_0 contains some points of E. But the open sphere $N_{\frac{s}{2}}(x_0)$ does not contain any point of E.

Hence our assumption that $r' > r$ is wrong and therefore $r' = r$, i.e., $d(\bar{E}) = d(E)$. □

Theorem 4.3. For any non-empty set A of a metric space X, the closure $\bar{A}$ of A is the set of points whose distance from A is 0.

This theorem can be stated as:
Let A be a non-empty subset of a metric space X. Then $d(x, A) = 0$ iff $x \in \bar{A}$.

Proof. Let $d(x, A) = 0$. Then every open sphere with center at x contains at least one point of A and therefore every open set G containing x also contains at least one point of A. Hence, x is a limit point of A and so $x \in \bar{A}$.

Conversely, let $x \in \bar{A}$. Suppose that $d(x, A) \neq 0$ and $d(x, A) = \varepsilon > 0$. Then the open sphere $S_{\frac{\varepsilon}{2}}(x)$ with center x contains no points of A and so x is an exterior point of A; i.e., $x \notin \bar{A}$, a contradiction. Hence, $\bar{A} = \{x : d(x, A) = 0\}$. □

Theorem 4.4. Let A and B be closed disjoint subset of a metric space X. Then there exist disjoint open subsets G and H in X such that $A \subseteq G$ and $B \subseteq H$.

Proof. If either A or B is empty, say $A = \varnothing$, the $\varnothing$ and X are disjoint open sets such that $A \subseteq \varnothing$ and $B \subseteq X$. Hence, we may assume that A and B are non empty.
Let $a \in A$. THen since A and B are disjoint, $a \notin B$ and so $d(a, B) > 0$. Similarly, if $b \in B$, then $d(b, A) > 0$. Set

$$S_a = S_{\frac{\delta}{3}}(a) \quad \text{and} \quad S_b = S_{\frac{\delta}{3}}(b)$$

Clearly, $a \in S_a$ and $b \in S_b$.

Let $G = \{S_a : a \in A\}$ and $H = \{s_b : b \in B\}$. Then clearly G and H are open because they are the union of open spheres and $A \subseteq G$ and $B \subseteq H$. We now have to show that $G \cap H = \varnothing$.
Suppose $G \cap H \neq \varnothing$ and let $p \in G \cap H$. Then $p \in G$ and $p \in H$ implies $p \in S_{a_0}$ and $p \in S_{b_0}$ for some $a_0 \in A$ and $b_0 \in B$ respectively. Let $d(a_0, b_0) = \varepsilon > 0$. Then $d(a_0, B) < \varepsilon$ and $d(b_0, A) < \varepsilon$. But $d(a_0, p) < \frac{\delta}{3}$ and $d(b_0, p) < \frac{\delta}{3}$. Therefore, by triangle inequality,

$$\varepsilon = d(a_0, b_0) \leq d(a_0, p) + d(p, b_0) < \frac{\delta}{3} + \frac{\delta}{3} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

which is impossible. Hence, G and H are disjoint. □

4.1 Euclidean $n-$dimensional space

In $\mathbb{R}^n$ space, the function d defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ with $x_i, y_i \in \mathbb{R}$ is a metric called the **Euclidean** metric on $\mathbb{R}^n$ and the space $(\mathbb{R}^n, d)$ is known as **Euclidean $n-$dimensional space**.

Theorem 4.5. Euclidean $n-$dimensional space is a metric space.

4.2 Hilbert Space

Hilbert Space is an immediate generalization of Euclidean $n-$dimensional space $\mathbb{R}^n$ arises when we replace $n-$tuples $x = (x_1, x_2, \ldots, x_n)$ with sequences $x = \langle x_1, x_2, \ldots \rangle$.
Let ℓ_2 denote the set of all sequences of real numbers such that

$$\sum_1^\infty (x_k)^2 < \infty$$

i.e., such that the series $x_1^2 + x_2^2 + \ldots$ converges and define

$$d(x, y) = \sqrt{\sum_1^\infty (x_k - y_k)^2}$$

Then d is a metric on ℓ_2. The resulting metric space ℓ_2 is usually called ℓ_2 space or **Hilbert** space, named after one of the most important and influential mathematician of his time, David Hilbert (1862-1943).

Theorem 4.6. Hilbert space or ℓ_2 space is a metric space.

4.3 Normed Space

A **norm** on a linear space is a function that gives a notion of the ‘length’ of a vector. The formal definition of a norm on a linear space is given below:
A norm on a linear space X is a function $||\cdot|| : X \to \mathbb{R}$ with the following properties:

1. $||x|| \geq 0$, for all $x \in X$ and $||x|| = 0$ implies $x = 0$
2. $||\lambda x|| = |\lambda| \; ||x||$, for all $x \in X$ and $\lambda \in \mathbb{R}$
3. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$

A linear space X together with a norm is called a **normed linear space**.
A normed linear space X is metric space with the metric

$$d(x, y) = ||x - y||$$

And it is known as induced metric on X.
The set of real numbers $\mathbb{R}$ with the absolute value norm $||x|| = |x|$ is a one-dimensional real normed linear space. More generally, $\mathbb{R}^n$, where $n = 1, 2, \dots,$ is an $n-$dimension linear space. We define **Euclidean norm** of a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}$ by

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

A normed linear space X that is complete (every Cauchy sequence on X converges in X) with respect to the metric d is called a **Banach space**.

Chapter 5

Separation Axioms