Chapter 1

Separation Axioms

Definition 1 (Quasi T_0-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a quasi T_0-space, if for every two distinct fuzzy points x_a and x_b with same support point x, there exists $U \in Q_\delta(x_a)$ such that $x_b \not\propto U$ or, there exists $V \in Q_\delta(x_b)$ such that $x_a \not\propto V$.

Definition 2 (Sub T_0-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a sub T_0-space, if for every two distinct $x, y \in X$, there exists $a \in [0,1]$ such that either $\exists U \in Q_\delta(x_a)$ with $y_a \not\propto U$ or, $\exists V \in Q_\delta(y_a)$ with $x_a \not\propto V$.

Definition 3 (T_0-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_0-space, if for every two distinct fuzzy points x_a and y_b, $\exists U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, $\exists V \in Q_\delta(y_b)$ with $x_a \not\propto V$.

Definition 4 (T_1-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_1-space, if for every two distinct fuzzy points x_a and y_b such that $x_a \not\leq y_b$ then there exists $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ and, $\exists V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.

Definition 5 (T_2-space)**.** Let, $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, $\langle \mathcal{F}(X), \delta \rangle$ is called a T_2-space, if for every two distinct fuzzy points x_a and y_b (i.e., $x_a \neq y_b$) then there exists $U \in Q_\delta(x_a)$ and, $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

Theorem 1.0.1. Quasi T_0 property is heriditary.
or, Every subspace of a Quasi T_0 space is Quasi T_0 space.

Proof. Suppose, $\langle X, \delta \rangle$ be a fuzzy topological space which is Quasi T_0-space. Let $\langle Y, \mu \rangle$ be the subspace of $\langle X, \delta \rangle$. We have to prove that, $\langle Y, \mu \rangle$ be a Q-T_0-space.

Now, since, $Y \subseteq X$ so every $V \in \mu$, $V = U_{\restriction Y}$ for some $U \in \delta$. Let y_a and y_b be two distinct fuzzy points in Y such that, $y_a \neq y_b$. Then as $Y \subseteq X$, we have y_a and y_b are in X with $y_a \neq y_b$.

Again, since $\langle X, \delta \rangle$ is a Quasi T_0-space there exist $U \in Q_\delta(y_a)$ such that $y_b \not\propto U$ or, there exist $V \in Q_\delta(y_b)$ such that $y_a \not\propto V$. This implies, there is $U_{\restriction y} \in Q_{\delta \restriction y}(y_a)$ such that $y_b \not\propto U_{\restriction Y}$ or there is $V_{\restriction Y} \in Q_{\delta \restriction}(y_b)$ such that $y_a \not\propto V_{\restriction Y}$.

Thus, by definition of a Q-T_0-space $\langle Y, \mu \rangle$ is a Q-T_0- space. $\square$

Theorem 1.0.2. Every subspace of a T_0-space is T_0-space.

Proof. Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $\langle Y, \mu \rangle$ be a subspace of $\langle X, \delta \rangle$. Let x_a and y_b be two distinct points in Y. Then since, $Y \subseteq X$, we have, x_a and y_b in X with $x_a \neq y_b$. Now since $\langle X, \delta \rangle$ is a fuzzy T_0-space. We have either there is $U \in Q_\delta(x_a)$ such that $y_b \not\propto U$ or, there is $V \in Q_\delta(y_b)$ such that $x_a \not\propto V$.
Now, $U_{\restriction Y} \in Q_{\delta \restriction Y}(x_a)$ such that $y_b \not\propto U_{\restriction Y}$ as x_a, $y_b \in Y$ and $V_{\restriction Y} \in Q_{\delta \restriction Y}(y_b)$ such that $x_a \not\propto V_{\restriction Y}$.
Thus, $\langle Y, \mu \rangle$ is a T_0-space. $\square$

Theorem 1.0.3. A fuzzy topological space$\langle \mathcal{F}(X), \delta \rangle$ is a quasi-T_0-space iff for every $x \in X$ and $a \in [0,1]$ there exists $B \in \delta$ such that $B(x) = a$.

Proof. Suppose, $\langle \mathcal{F}(X), \delta \rangle$ be a quasi T_0-space. If $a = 0$, then it suffices to take $B = \underline{0}$. If $0 < a < 1$, we take a strictly monotonic increasing sequence of positive real numbers converging to a. Let $\Delta_n = (a_n, a_{n+1}]$, $n = 1, 2, 3, \ldots$.

Since $\langle \mathcal{F}(X), \delta\rangle$ be a quasi T_0-space, then for any $x \in X$ and $\Delta = (a_1, a_2)$ with $0 \le a_1 < a_2 < 1$, there exists $B \in \delta$ such that $B(x) \in \Delta$.
From this property, we can say that, $\exists B_n \in \delta$ such that $B_n(x) \in \Delta_n$, for each n

$$\therefore\ B = \bigvee_{n=1}^{\infty} B_n \in \delta \qquad \text{and} \qquad B(x) = a.$$

Conversely, suppose x_a and x_b are two fuzzy points with $b < a$ where $a, b \in [0, 1]$. Then by hypothesis, there is an open set B such that $B(x) = 1 - b > 1 - a$.
This implies, B is an open Q-nbd of x_a but not quasi-conincident with x_b [since, B is a nbd of x_{1-a}]. Hence, $\langle \mathcal{F}(X), \delta\rangle$ is a quasi T_0-space. □

Theorem 1.0.4. A fuzzy topological space$\langle \mathcal{F}(X), \delta\rangle$ is T_1-space iff for every $x \in X$ and each $a \in [0, 1]$ there exists $B \in \delta$ such that $B(x) = 1 - a$ and $B(y) = 1$ for $y \neq x$.
Or, $\langle \mathcal{F}(X), \delta\rangle$ is a T_1-space $\Leftrightarrow$ every fuzzy point in $\langle X, \delta\rangle$ is closed.

Proof. Suppose $\langle \mathcal{F}(X), \delta\rangle$ be a T_1-space. If $a = 0$ then it suffices to take $B = \underline{1}$.
Suppose, $a > 0$ and x_a is a fuzzy point. Since, every fuzzy point in a T_1-space is closed, so, x_a is a closed set.
$\therefore$ We have, $B = 1 - x_a \in \delta$ and hence $B(x) = 1 - a$ and $B(y) = 1$. if $y \neq x$.
Conversely, let x_a be a fuzzy point. Then by hypothesis there exists $B \in \delta$ such that $B(x) = 1-a$ and $B(y) = 1$ with $y \neq x$. This implies, $B = 1 - x_a$ and hence $B^c = x_a$ which is closed. Thus, $B \in \delta$. Hence, $\langle \mathcal{F}(X), \delta\rangle$ is a T_1-space. □

Definition 6 (Purely T_2-space)**.** $\langle \mathcal{F}(X), \delta\rangle$ is called purely T_2-space if for every two zero-meet fuzzy points x_a and y_b, $\exists U \in Q_\delta(x_a)$ and $V \in Q_\delta(y_b)$ such that $U \wedge V = \underline{0}$.

Theorem 1.0.5. For a fuzzy topological space$\langle \mathcal{F}(X), \delta\rangle$ the following statements are equivalent

1. $\langle X, \delta\rangle$ is a fuzzy T_0-space.
2. For $x, y \in X$, $x \neq y$, $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.

Proof. (1) $\Rightarrow$ (2), Suppose $\langle X, \delta\rangle$ is a fuzzy T_0-space. Thus, we have $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$. □

Theorem 1.0.6 (X). A fuzzy topological space $\langle \mathcal{F}(X), \delta\rangle$ is called a fuzzy T_0 space iff

(i) $\forall\, x, y \in X$, $\exists U \in \delta$ such that $U(x) = 1$, $U(y) = 0$ or $U(y) = 1$, $U(x) = 0$.

(ii) For all $\forall x, y \in X$, $x \neq y$, $\overline{x_1(y)} \cap \overline{y_1(x)} < 1$

(iii) $\forall x, y \in X$, $x \neq y$, such that $U(x) < U(y)$ or $U(y) < U(x)$

Theorem 1.0.7. For a fuzzy topological space $\langle X, \delta\rangle$ the following statements are equivalent

(i) For all $x,\ y \in X$, $x \neq y$, $\bar{x}_1(y) \cap \bar{y}_1(x) < 1$,

(ii) For all $x,\ y \in X$, $x \neq y$, $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.

Proof. (i) $\rightarrow$ (ii)
Suppose $\langle X, \delta\rangle$ be a fuzzy T_0 space. Thus, we have, $\bar{x}_1(y) \cap \bar{y}_1(x) < 1$. This implies that either $\bar{x}_1(y) < 1$ or $\bar{y}_1(x) < 1$. Consider $\bar{x}_1(y) < 1$. Hence, $1 - \bar{x}_1(y) > 0$ and $1 - \bar{x}_1(x) = 0$.
Let, $1 - \bar{x}_1 = U$, then $U \in \delta$ and $U(x) = 0$, $U(y) > 0$.
Similarly, from $\bar{y}_1(x) < 1$ we can show that there exists $V \in \delta$ such that $V(y) = 0$, $V(x) > 0$.

(ii) $\rightarrow$ (i)
From (ii), we have for any two points $x, y \in X$ with $x \neq y$ $\exists U \in \delta$ such that $U(x) > 0$, $U(y) = 0$ or $U(y) > 0$, $U(x) = 0$.
Then

$$\begin{aligned} 1 - U(x) < 1 &\qquad \text{and} \qquad 1 - U(y) = 1 \\ \text{or,}\ 1 - U(x) = 1 &\qquad \text{and} \qquad 1 - U(y) < 1 \end{aligned}$$

Since the complement of U is closed, we see that $\bar{x}_1(y) < 1$ or $\bar{y}_1(x) < 1$. This implies $\bar{x}_1(y) \cap \bar{y}_1(x) < 1$. □

Theorem 1.0.8. For any fuzzy topological space $\langle X, \delta\rangle$ the following are equivalent:

(i) For all $x,\ y \in X$, $x \neq y$, $\exists U, V \in \delta$ such that $U(x) = 1 = V(y)$ and $U \subset V^c$

(ii) If $x \in X$, then for each $y \in X$, $y \neq x$, $\exists U \in \delta$ such that $U(x) = 1$ and $\bar{U}(y) = 0$

Proof. Let $\langle X, \delta\rangle$ be a fuzzy topological space. Let $x, y \in X$ with $x \neq y$. Then by (i) there exist $U, V \in \delta$ such that

$$\begin{aligned} & U(x) = 1 = V(y) \qquad \text{and} \qquad U \subset V^c = 1 - V \\ \Rightarrow & \bar{U} \subset \overline{1 - V} \subset 1 - V \end{aligned}$$

So,

$$\bar{U}(y) \subset (1 - V)(y) = 1 - V(y) = 0 \quad \text{i.e., } \bar{V}(y) = 0$$

(ii) $\rightarrow$ (i)
Let $\langle X, \delta\rangle$ be a fuzzy topological space. Let $x, y \in X$ with $x \neq y$. By (ii) there exist $U \in \delta$ with $U(x) = 1$ and $\bar{U}(y) = 0$ or, there exist $V \in \delta$ with $V(y) = 1$ and $\bar{V}(x) = 0$.
Let $V = 1 - \bar{U}$. Then,

$$\begin{aligned} V(y) &= (1 - \bar{U})(y) \\ &= 1 - \bar{U}(y) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Again, $U \subset \bar{U} = 1 - V = V^c$. Hence $U \subset V^c$, $U(x) = V(y) = 1$ and it is also clear that $U,\ V \in \delta$. □

Definition 7 (Regular Space)**.** A fuzzy topological space $\langle X, \delta\rangle$ is said to be fuzzy regular iff for each $x \in X$ and each closed fuzzy set U with $U(x) = 0$ there exists $V, W \in \delta$ such that $V(x) = 1$, $U \subset W$ and $V \subseteq 1 - W$.

Theorem 1.0.9. Let $\langle X, \delta\rangle$ be a fuzzy topological space. Then the following are equivalent:

(i) $\langle X, \delta\rangle$ is a fuzzy regular space.

(ii) For each $x \in X$, $U \in \delta$ with $U(x) = 1$, $\exists V \in \delta$ with $V(x) = 1$ and $V \subset \bar{V} \subset U$.