Chapter 1

Compactness

Definition 1 (Cover and $C-$compactness)**.** Let $\langle \mathcal{F}(X), \delta\rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then, $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a cover of A if $A \subseteq \vee\mathcal{A}$.

- $\langle \mathcal{F}(X), \delta\rangle$ is called $C-$compact if every open cover of $\langle \mathcal{F}(X), \delta\rangle$ has a finite subcover.
- $\mathcal{A}$ is called an open cover of A, if $\mathcal{A} \subseteq \delta$ and if $\mathcal{A}$ is a cover of A.
- $\mathcal{B} \subseteq \mathcal{A}$ is called a subcover if $\mathcal{B}$ is still a cover of A.

In particularly, $\mathcal{A}$ is a cover of $\langle \mathcal{F}(X), \delta\rangle$ if $\mathcal{A}$ is a cover of $\underline{1}$.

Definition 2 ($\alpha-$cover and $\alpha-$compactness)**.** Let $\langle \mathcal{F}(X), \delta\rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an $\alpha-$cover, if for very $x \in X \exists A \in \mathcal{A} \ni A(x) > \alpha$.

- ft is called an $\alpha-$compact, if for every open $\alpha-$cover of $\langle \mathcal{F}(X), \delta\rangle$ has a finite sub$-\alpha-$cover where $\alpha \in [0, 1)$.

Definition 3 (Strong Compact)**.** A fuzzy topological space $\langle \mathcal{F}(X), \delta\rangle$ is called strongly compact if it is $\alpha-$compact for every $\alpha \in [0, 1)$.

Definition 4 (α^*-cover and α^*-compactness)**.** Let $\langle \mathcal{F}(X), \delta\rangle$ be a fuzzy topological space and $\alpha \in [0, 1)$. Then a family $\mathcal{A} \subseteq \mathcal{F}(X)$ is called an α^*-cover, if for every $x \in X$, there exists $A \in \mathcal{A}$ such that, $A(x) \geq \alpha$.

- For $\alpha \in [0, 1)$, $\langle \mathcal{F}(X), \delta\rangle$ is called an α^*-compact, if for every open α^*-cover of $\langle \mathcal{F}(X), \delta\rangle$ has a finite sub α^*-cover.

Example. Given $X = \{a, b, c\}$, $\mathcal{A} = \{A, B, C\}$, $\alpha \in [0, 1)$, $\delta = \{\underline{0}, \underline{1}, A, B, C\}$ where,

$$\begin{aligned} A &: a \mapsto 0.2,\ b \mapsto 0.4,\ c \mapsto 0.6; \\ B &: a \mapsto 0.4,\ b \mapsto 0.6,\ c \mapsto 0.8; \\ C &: a \mapsto 0.6,\ b \mapsto 0.8,\ c \mapsto 0.9; \end{aligned}$$

Check whether $\mathcal{A}$ is $\alpha-$compact or, α^*-compact corresponding to the given value of α.

Solution.

1. Let $\alpha = 0.7$
   $a \in X : \alpha = 0.7 > A(a),\ B(a), C(a)$.
   Hence, for $\alpha = 0.7$, $\mathcal{A}$ is not an $\alpha-$cover.

2. Let $\alpha = 0.3$
   $a \in X : \alpha = 0.3 < C(a) = 0.6,\ B(a) = 0.4$
   $b \in X : \alpha = 0.3 < A(b) = 0.4,\ B(b) = 0.6,\ C(b) = 0.8$
   $c \in X : \alpha = 0.3 < A(c) = 0.6,\ B(c) = 0.8,\ C(c) = 0.9$
   $\therefore\ \mathcal{A}$ is an $\alpha-$compact space for $a = 0.3$.

3. Let $\alpha = 0.6$
   For, $a \in X : \alpha = 0.6 = C(a)$
   For, $b \in X : \alpha = 0.6 = B(b),\ \alpha = 0.6 < C(b) = 0.8$
   For, $c \in X : \alpha = 0.6 = A(c) = 0.6,\ \alpha = 0.6 < B(c) = 0.8,\ C(c) = 0.9$
   $\therefore\ \mathcal{A}$ is an α^*-compact space for $a = 0.6$.

Definition 5 ($Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\mathcal{A} \subseteq \mathcal{F}(X)$ is called a $Q-$cover of A if for every $x \in Supp(A)$, there exists $U \in \mathcal{A}$ such that $x_{A(x)} \propto U$.

Definition 6 ($Q-$compact)**.** A fuzzy set A is called $Q-$compact if every open $Q-$cover of A has a finite sub $Q-$cover. A fuzzy topological space $\langle \mathcal{F}(X), \delta \rangle$ is called $Q-$compact if $\underline{1}$ is $Q-$compact.

Example. Consider, $X = \{a, b, c\}$, $\delta = \{\underline{0}, \underline{1}, U, V, W\}$ where

$$U : a \mapsto 0.3,\ b \mapsto 0.5,\ c \mapsto 0.7;$$
$$V : a \mapsto 0.4,\ b \mapsto 0.6,\ c \mapsto 0.8;$$
$$W : a \mapsto 0.6,\ b \mapsto 0.8,\ c \mapsto 0.9;$$

Consider $\mathcal{A} = \{U, V\} \subseteq \delta$ and let, $A : a \mapsto 0.1,\ b \mapsto 0.2,\ c \mapsto 0.3$. Then, find the $Q-$cover of A.

Solution. Here, $Supp(A) = \{a, b, c\}$
For, $x = a$, $a_{A(a)} = a_{0.1} = 0.1$
For, $x = b$, $b_{A(b)} = b_{0.2} = 0.2$
For, $x = c$, $c_{A(c)} = c_{0.3} = 0.3$
For $x = a$, we have $U_a, : 0.3 + 0.1 < 1$, $V_a = 0.4 + 0.1 < 1$. Hence $\mathcal{A}$ is not a $Q-$cover of A.

If $A : a \mapsto 0.7,\ b \mapsto 0.6,\ c \mapsto 0.5$.
Then, For $x = a$, $a_{A(a)} = a_{0.7} = 0.7$
For, $x = b$, $b_{A(b)} = b_{0.6} = 0.6$
For, $x = c$, $c_{A(c)} = c_{0.5} = 0.5$
For, $x = a, 0.3 + 0.7 \geq 1,\ 0.4 + 0.7 > 1$
For, $x = b, 0.5 + 0.6 > 1,\ 0.6 + 0.6 > 1$
For, $x = c, 0.7 + 0.5 > 1,\ 0.8 + 0.5 > 1$
Hence, for every $x \in Supp(A)$, $x_{A(x)} \propto U$.
$\therefore \mathcal{A}$ is a $Q-$cover of A.

Definition 7 ($\alpha - Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\Phi \subseteq \mathcal{F}(X)$ is called an $\alpha - Q-$cover of A, if for every $x_a \subseteq A$, there exists $U \in \Phi$ such that $x_a \propto U$. It is denoted by $\vee \Phi \hat{q} A(\alpha)$.

Definition 8 ($\bar{\alpha} - Q-$cover)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then a collection $\Phi \subseteq \mathcal{F}(X)$ is called an $\bar{\alpha} - Q-$cover of A, if there exists $\gamma \in B^*(\alpha)$ such that γ is a $\gamma - Q-$cover of A.

- $B(b) = \{a \in L : a \propto b\}$, where the binary relation $\propto$ is defined as, for $a, b \in L$, $a \propto b \Leftrightarrow$ for every subset $D \subseteq L$, $b \leq Sup\, D$ implies the existence of $d \in D$ with $a \leq$
- $B^*(b) = B(b) \cap M(L)$, where, $M(L) = (0, 1]$.

Definition 9. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. A is called $N-$compact if for every $\alpha \in (0, 1] - M([0, 1])$, every open $\alpha - Q-$cover of A has a finite subfamily which is an $\bar{\alpha} - Q-$cover of A. $\langle \mathcal{F}(X), \delta \rangle$ is called $N-$compact, if $\underline{1}$ is compact.

Theorem 1.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space, $A \in \mathcal{F}(X)$. Then A is $N-$compact iff the following conditionds hold:

(a) For every $\alpha \in (0, 1]$, every open $\alpha - Q-$cover of A has a finite sub $\alpha - Q-$cover.

(b) For every $\alpha \in (0, 1]$, every open $\alpha - Q-$cover of A which consists of just one subset is an $\bar{\alpha} - Q-$cover of A.

Proof. (a) Let, A be $N-$compact, $\alpha \in (0, 1]$ and φ is an open $\alpha - Q-$cover of A. By the definition of $N-$compact, φ has a finite subfamily ψ such that, ψ is an $\bar{\alpha} - Q-$cover of A. Hence, $\vee \psi \hat{q} A(\alpha)$ i.e., ψ is an $\alpha - Q-$cover of A.

(b) Suppose, $U \in \delta$ and $\varphi = \{U\}$ is an open $\alpha - Q-$cover of A. Then, by the $N-$compactness of A, φ has a subfamily ψ such that ψ is an $\bar{\alpha} - Q-$cover of A. But, clearly, $\varphi = \psi$. Hence, ψ is an open $\alpha - Q-$cover of A.

Conversely, suppose (a) and (b) holds.
Let $\alpha \in (0,1]$ and φ is an open $\alpha - Q-$cover of A.
By (a), φ has a finite sub $\alpha - Q-$cover ψ of A. Take $U = \vee\psi$. Then $\{U\}$ is an $\alpha - Q-$cover of A.
By (b), $\{U\}$ is also an $\bar{\alpha} - Q-$cover of A. By the definition of $\bar{\alpha} - Q-$cover, there exists $\gamma \in B^*(\alpha)$ such that x_γ is a quasi-coincident with U for every $x_\gamma \subseteq A$. Hence, $\gamma + U(x) > 1 \Rightarrow \gamma > 1 - U(x)$
i.e., $\gamma \leq (U(x))' \;\Rightarrow \gamma \not\leq (U\psi(x))' = \wedge\{(W(x))'|W \in \psi\}$
i.e., $W \in Q_\gamma(x_\gamma)$. So, ψ is an $\bar{\alpha} - Q-$cover of A. Hence, A is $N-$compact. □

Theorem 1.0.2. Continuous image of an $N-$compact space is $N-$compact.

Proof. Let $f^\rightarrow : \langle \mathcal{F}(X), \delta\rangle \rightarrow \langle \mathcal{F}(Y), \mu\rangle$ be a continuous fuzzy mapping and A be a $N-$compact fuzzy set in $\mathcal{F}(X)$. For $\alpha \in (0,1]$, let $\mathcal{A}$ be an open $\alpha - Q-$cover of $f^\rightarrow(A)$. Then for every $x_\alpha \leq A$, $f^\rightarrow(x_\alpha) = f(x)_\alpha \leq f^\rightarrow(A)$, there exists $U \in \mathcal{A}$ such that $f(x)_\alpha \propto U \;\Rightarrow f(x)_\alpha \not\propto U^c \;\Rightarrow \alpha \not\leq U^c(f(x)) \;\Rightarrow \alpha \not\leq f^\leftarrow(U^c)(x) = f^\leftarrow(U)^c(x)$. That is $x_a \propto f^\leftarrow(U)$. Since, $f^\rightarrow$ is continuous, $f^\leftarrow(U) \in \delta$ and hence $f^\leftarrow(U) \in Q(x_\alpha)$. Thus, $f^\leftarrow(A)$ is an open $\alpha - Q-$cover of A.

Since A is $N-$compact, $\mathcal{A}$ has a finite subfamily $\mathcal{A}_n = \{U_i : 1 \leq i \leq n\}$ such that $f^\leftarrow(\mathcal{A}_n)$ is an $\bar{\alpha} - Q-$cover of A.

Now, we show that, $\mathcal{A}_n$ is an $\bar{\alpha} - Q-$cover of $f^\rightarrow(A)$. Since, $f^\leftarrow(\mathcal{A})_n$ is an open $\bar{\alpha} - Q-$cover of A, there exists $\gamma \in \mathcal{B}(\alpha)$ such that $f^\leftarrow(\mathcal{A}_\backslash)$ is $\gamma - Q-$cover of A. This implies, $\gamma \sqsubseteq a$ and hence $\exists\lambda \in (0,1]$ such that $\gamma \sqsubseteq \lambda \sqsubseteq \alpha$. So, $\lambda \in \mathcal{B}(\alpha)$ and hence we have, $\lambda \leq f^\leftarrow(A)(y) = \vee\{A(x) : x \in X, f(x) = y\}$. Now, $\gamma \sqsubseteq \lambda$ implies, $\gamma \not\leq (f^\leftarrow(U_i))^c(x) = f^\leftarrow(U_i^c)(x) = U_i^x(f(x)) = U_i^c(y)$, for some $1 \leq i \leq n$ such that $x_\gamma \propto f^\leftarrow(U_i)$.
By $\gamma \sqsubseteq \lambda$ and hence $\gamma \leq \lambda$, we have $\lambda \not\leq U_i^c(y)$. Thus $y_\lambda \propto U_i$ for some $1 \leq i \leq n$. So, $\mathcal{A}_n$ is an open $\lambda - Q-$cover of $f^\rightarrow(A)$ and hence $\mathcal{A}_n$is an $\bar{\alpha} - Q-$cover of $f^\leftarrow(A)$.
Therefore, $f^\rightarrow(A)$ is an $N-$compact. □

Definition 10 (Net in X)**.** Let X be a non-empty ordinary set and D be a directed set then every mapping $S : D \rightarrow X$ is called a net in X and D is called the index set of S.

Theorem 1.0.3. Let $\langle \mathcal{F}(X), \delta\rangle$ be a fuzzy topological space. Let $A, B, C \in \mathcal{F}(X)$ such that A be a $N-$compact and B be closed. Then $A \wedge B$ is $N-$compact.

Proof. Let S be an $\alpha-$net in $A \wedge B$. Then S is also an $\alpha-$net in A. Since, A is $N-$compact, S has a cluster point x_α in A such that $ht(\alpha) = \alpha$. But, S is also a net in closed subset B, we have $x_\alpha \leq B$.
So, $x_a \leq A \wedge B$, i.e., x_a is a cluster point of δ in $A \wedge B$ such that $ht(\alpha) = \alpha$. Hence, $A \wedge B$ is $N-$compact. □