1 Assignment - I

1. Define groupoid, semi group, monoid and group. Show the following by giving examples:
   (a) groupoid but not semi-group
   (b) semi group bun not monoid,
   (c) monoid but not group
2. Prove that a group of order 3 is Abelian.
3. What is the order of an element of a group? Prove that order of an element divides the order of the group.
4. Write down the composition table for permutation on $S = \{1, 2, 3\}$. Hence, find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.
5. Prove that any permutation of a finite set containing at least two elements can be written as the product of transpositions. Show that product of transpositions of a permutation is not unique.
6. Define cyclic group. Find all generators of the cyclic group $(\mathbb{Z}_9; +)$.
7. Prove that a non-empty subset H of a group G is a subgroup iff
   (a) for any $a, b \in H$, $ab \in H$
   (b) for $x \in H$, $x^{-1} \in H$
8. Find all the normal subgroups of the symmetric group S_3.

Problem 1.1. Define groupoid, semigroup, monoid and group. Show the following by giving examples:

1. groupoid but not semigroup,
2. semigroup but not monoid,
3. monoid but not group

Solution.
Groupoid: A non-empty set of elements G is said to form a groupoid if in G is defined a binary operation called the product denoted by $*$ such that $a * b \in G$ for all $a, b \in G$.

Here the binary operation $*$ defined on the set G does not need to be associative, i.e., $(a * b) * c \neq a * (b * c)$ for all $a, b, c \in G$, so we can say that the groupoid $(G, *)$ is a set on which is defined a non-associative binary operation which is closed on G.

Semigroup: Let S be a non-empty set. S is said to be a semigroup, if on S is defined a binary operation '$\cdot$' such that

1. For all $a, b \in S$ we have $a \cdot b \in S$ (closure).
2. For all $a, b, c \in S$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associative law).

$(S, \cdot)$ is a semigroup.

Monoid: Let $(S, \cdot)$ be a semigroup. If S contains an element e such that $e \cdot s = s \cdot e = s$ for all $s \in S$ then S is a monoid.

Group: A non-empty set G is called a group under an operation $(a, b) \to ab$ defined on $G \times G$ iff the following properties hold.

1. $ab \in G$ for all $a, b \in G$ (closure property)
2. $(ab)c = a(bc)$ holds for all $a, b, c \in G$ (associative law)
3. There exists $e \in G$ such that $ea = ae = a$ holds for all $a \in G$ (existence of identity)
4. For every $a \in G$ there exists $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (existence of inverse).

<u>Groupoid but not semigroup:</u>
Let us consider the set of integers $\mathbb{Z}$ with an operation '$-$' on $\mathbb{Z}$ that is usual subtraction; $(\mathbb{Z}, -)$ is a groupoid but not semigroup. Because, for all $a, b \in \mathbb{Z}$ under subtraction $a - b \in \mathbb{Z}$, but it does not satisfy associative law, i.e., $(a - b) - c \neq a - (b - c)$.

<u>Semigroup but not monoid:</u>
Let $\mathbb{Z}^+ = 1, 2, \ldots, \infty$. $\mathbb{Z}^+$ is a semigroup under addition. Now, $(\mathbb{Z}^+, +)$ is only a semigroup and not a monoid, because there is not any identity element (i.e., 0) in the set $\mathbb{Z}^+$. For a semigroup to be monoid it has to have an identity element. So $(\mathbb{Z}^+, +)$ is only a semigroup and not a monoid.

<u>Monoid but not group:</u>