Chapter 1

Numerical Solution of Non-Linear Systems of Equations: Fixed Point Iteration

1.1 Fixed Points For Functions of Several Variables

A system of nonlinear equations has the form

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}\right\} \tag{1.1}$$

where each function f_i can be thought of as mapping a vector $x = (x_1, x_2, \ldots, x_n)^t$ of the $n-$dimensional space $\mathbb{R}^n$ into the real line $\mathbb{R}$.

This system of n nonlinear equations in n unknowns can alternatively be represented by defining a function F mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ by

$$F(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n))^t$$

If the vector notation is used to represent the variables $x_1, x_2, \ldots, x_n$, then the system (1.1) assumes the form

$$F(x) = 0$$

The functions $f_1, f_2, \ldots, f_n$ are called coordinate functions of F.

Definition 1. A function $\mathbf{G}$ from $D \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$ has a fixed point at $\mathbf{P} \in D$ if $G(\mathbf{P}) = \mathbf{P}$.

Theorem 1.1.1. Let f be a function from $D \subset \mathbb{R}$ into $\mathbb{R}$ and $\mathbf{x}_0 \in D$. Suppose that all the partial derivatives of f exist and constants $\delta > 0$ and $K > 0$ exist so that whenever $\|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in D$, we have

$$\left|\frac{\partial f(\mathbf{x})}{\partial x_j}\right| \leq K \qquad \text{for each } j = 1, 2, \ldots, n$$

Then f is continuous at $\mathbf{x}_0$.

Theorem 1.1.2. Let $D = \{(x_1, x_2, \ldots, x_n)^t : a_i \leq x_i \leq b_i \text{ for each } i = 1, 2, 3, \ldots, n\}$ for some collection of constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. Suppose $\mathbf{G}$ is a continuous function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with the property that $\mathbf{G}(\mathbf{x}) \in D$ whenever $\mathbf{x} \in D$. Then $\mathbf{G}$ has a fixed point in D.

Suppose, in addition, that G has continuous partial derivatives and a constant $k < 1$ exists with $\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \leq \frac{k}{n}$ whenever $\mathbf{x} \in D$ for each $j = 1, 2, \ldots, n$ and each component function g_i. Then the sequence $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ defined by an arbitrarily selected $\mathbf{x}^{(0)} \in D$ and generated by

$$\mathbf{x}^{(k)} = G\left(\mathbf{x}^{(k-1)}\right) \qquad \text{for each } k \geq 1$$

converges to the unique fixed point $\mathbf{p} \in D$ and

$$\left\|\mathbf{x}^{(k)} - \mathbf{p}\right\|_\infty \leq \frac{k^n}{1-k}\left\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right\|_\infty \tag{1.2}$$

Example. Consider the non linear system

$$
\begin{aligned}
&3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0 \\
&{x_1}^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0 \\
&e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0
\end{aligned}
$$

if the ith equation is solved for x_i, the system is changed into the fixed-point problem

$$
\left.
\begin{aligned}
x_1 &= \tfrac{1}{3}\cos(x_2 x_3) + \tfrac{1}{6} \\
x_2 &= \tfrac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06} - 0.1 \\
x_3 &= -\tfrac{1}{20} e^{-x_1 x_2} - \tfrac{10\pi - 3}{60}
\end{aligned}
\right\} \tag{1.3}
$$

Let $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}))^t$, where

$$
\begin{aligned}
g_1(x_1, x_2, x_3) &= \frac{1}{3}\cos(x_2 x_3) + \frac{1}{6} \\
g_2(x_1, x_2, x_3) &= \frac{1}{9}\left(x_1^2 + \sin x_3 + 1.06\right)^{\frac{1}{2}} - 0.1 \\
g_3(x_1, x_2, x_3) &= -\frac{1}{20} e^{-x_1 x_2} - \frac{10\pi - 3}{60}
\end{aligned}
$$

Now, we will show that $\mathbf{G}$ has a unique fixed point in $D = \{(x_1, x_2, x_3)^t : -1 \le x_i \le 1;\ \text{for each } i = 1, 2, 3\}$ by theorems 1.1.1 and 1.1.2.
For $\mathbf{x} = (x_1, x_2, x_3)^t$ in D.

$$
\begin{aligned}
|g_1(x_1, x_2, x_3)| &\le \frac{1}{3}|\cos(x_2 x_3)| + \frac{1}{6} \le 0.5 \\
|g_2(x_1, x_2, x_3)| &\le \left|\frac{1}{9}\left(x_1^2 + \sin x_3 + 1.06\right)^{\frac{1}{2}} - 0.1\right| \\
&\le \frac{1}{9}\sqrt{1 + \sin 1 + 1.06} - 0.1 \ < \ 0.09
\end{aligned}
$$

and

$$
\begin{aligned}
|g_3(x_1, x_2, x_3)| &= \frac{1}{20} e^{-x_1 x_2} + \frac{10\pi - 3}{60} \\
&\le \frac{1}{20} e + \frac{10\pi - 3}{60} \ < \ 0.61
\end{aligned}
$$

So, $-1 \le g_i(x_1, x_2, x_3) \le 1$ for each $i = 1, 2, 3$. Thus, $\mathbf{G}(\mathbf{x}) \in D$ whenever $\mathbf{x} \in D$.
To find the bounds for partial derivatives on D, we have

$$
\left|\frac{\partial g_1}{\partial x_1}\right| = 0, \quad \left|\frac{\partial g_2}{\partial x_2}\right| = 0, \ \text{and} \quad \left|\frac{\partial g_3}{\partial x_3}\right| = 0
$$

as well as

$$
\begin{aligned}
\left|\frac{\partial g_1}{\partial x_2}\right| &\le \frac{1}{3}|x_3|\,|\sin x_2 x_3| \le \frac{1}{3}\sin 1 < 0.281 \\
\left|\frac{\partial g_1}{\partial x_3}\right| &\le \frac{1}{3}|x_2|\,|\sin x_2 x_3| \le \frac{1}{3}\sin 1 < 0.281 \\
\left|\frac{\partial g_2}{\partial x_1}\right| &= \frac{|x_1|}{9\sqrt{{x_1}^2 + \sin x_3 + 1.06}} < \frac{1}{9\sqrt{0.218}} < 0.238 \\
\left|\frac{\partial g_2}{\partial x_3}\right| &= \frac{|\cos x_3|}{18\sqrt{{x_1}^2 + \sin x_3 + 1.06}} < \frac{1}{18\sqrt{0.218}} < 0.119 \\
\left|\frac{\partial g_3}{\partial x_1}\right| &= \frac{|x_2|}{20} e^{-x_1 x_2} \le \frac{1}{20} e < 0.14
\end{aligned}
$$

and

$$\left|\frac{\partial g_3}{\partial x_2}\right| = \frac{|x_1|}{20}e^{-x_1x_2} \leq \frac{1}{20}e < 0.14$$

Since the partial derivatives of g_1, g_2 and g_3 are bounded on D, theorem 1.1.1 implies that these functions are continuous on D. Consequently, G is continuous on D. Moreover, for every $\mathbf{x} \in D$

$$\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \leq 0.281 \qquad \text{for each } i = 1,2,3 \text{ and } j = 1,2,3$$

and the condition $\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \leq \frac{K}{n}$ holds with $K = 3 \times 0.281 = 0.843$.

To approximate the fixed point $\mathbf{p}$, we chose $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$, the sequence of vectors generated by

$$\begin{aligned}
x_1^{(k)} &= \frac{1}{3}\cos x_2^{(k-1)}x_3^{(k-1)} + \frac{1}{6} \\
x_2^{(k)} &= \frac{1}{9}\sqrt{\left(x_1^{(k-1)}\right)^2 + \sin x_3^{(k-1)} + 1.06} - 0.1 \\
x_3^{(k)} &= -\frac{1}{20}e^{-x_1^{(k-1)}x_2^{(k-1)}} - \frac{10\pi - 3}{60}
\end{aligned}$$

converges to the unique solution of the given non-linear system.

The results are listed in the table below. The sequences are generated until $\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty < 10^{-5}$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty$
0	0.10000000	0.10000000	-0.10000000	
1	0.49998333	0.00944115	-0.52310127	0.423
2	0.49999593	0.00002557	-0.52336331	9.4×10^{-3}
3	0.50000000	0.00001234	-0.52359814	2.3×10^{-4}
4	0.50000000	0.00000003	-0.52359847	1.2×10^{-5}
5	0.50000000	0.00000002	-0.52359877	3.1×10^{-7}

using the error bound formula $\left\|\mathbf{x}^{(k)} - \mathbf{p}\right\|_\infty \leq \frac{K^n}{1-K}\left\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right\|_\infty$ with $k = 0.843$ gives

$$\left\|\mathbf{x}^{(5)} - \mathbf{p}\right\|_\infty \leq \frac{(0.843)^n}{1 - 0.843}(0.423) < 1.15$$

which does not indicate the true accuracy of $\mathbf{x}^{(5)}$ because of the inaccurate initial approximation. The actual solution is

$$\mathbf{p} = \left(0.5, 0, \frac{-\pi}{6}\right)^t \approx (0.5, 0, -0.5235987757)^t$$

So the true error is $\left\|\mathbf{x}^{(5)} - \mathbf{p}\right\|_\infty \leq 2 \times 10^{-8}$.

To accelerate the convergence of the fixed-point iteration, we can use the Gauss-Seidel method, we have

$$\begin{aligned}
x_1^{(k)} &= \frac{1}{3}\cos x_2^{(k-1)}x_3^{(k-1)} + \frac{1}{6} \\
x_2^{(k)} &= \frac{1}{9}\left(\left(x_1^{(k)}\right)^2 + \sin x_3^{(k-1)} + 1.06\right)^{\frac{1}{2}} - 0.1 \\
x_3^{(k)} &= -\frac{1}{20}e^{-x_1^{(k)}x_2^{(k)}} - \frac{10\pi - 3}{60}
\end{aligned}$$

The results are represented in the table below, with the approximation $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty$
0	0.10000000	0.10000000	-0.10000000	
1	0.49998333	0.02222979	-0.52304613	0.423
2	0.49997747	0.00002815	-0.52359807	2.2×10^{-2}
3	0.50000000	0.00000004	-0.52359877	2.8×10^{-5}
4	0.50000000	0.00000000	-0.52359877	1.2×10^{-8}

Comment: The iterate $\mathbf{x}^{(4)}$ is accurate within 10^{-7} in the l_∞ norm; so the convergence is accelerated with Gauss-Seidel method.