Shahjalal University of Science and Technology
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Department of Mathematics
(MAT)

<u>ASSIGNMENT</u>

Group 10: Eigen Value Problem and Green's Functions

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1 Preliminary Concepts

Boundary Value Problem: A boundary value problem is a system of differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.

$$y'' + p(x)y' + q(x)y = r(x), \quad y(x_1) = y_1,\, y(x_2) = y_2 \tag{1}$$

This is a boundary value problem with boundary conditions $y(x_1) = y_1$, $y(x_2) = y_2$.

There are four kinds of boundary value problem.

Dirichlet or First kind	$y(x_1) = y_1,\, y(x_2) = y_2$
Neumann or Second kind	$y'(x_1) = y_1,\, y'(x_2) = y_2$
Robin or Third kind or mixed	$\alpha_1 y(x_1) + \alpha_2 y'(x_1) = y_1,\, \beta_1 y(x_2) + \beta_2 y'(x_2) = y_2$
Periodic	$y(x_1) = y(x_2),\, y'(x_1) = y'(x_2)$

By comparing with the above table we can say that equation (1) is boundary value problem of first kind or Dirichlet boundary value problem.
Solution of differential equation can vary massively depending on boundary conditions. This is illustrated in the following example.

Example 1.1. Let us consider a differential equation

$$y'' + y = 0 \tag{2}$$

This is a second-order linear differential equation. The general solution of this differential equation is $A\sin(x) + B\cos(x)$.

Now, if the boundary conditions are $y(0) = 1\, y\left(\frac{\pi}{2}\right) = 1$ then equation (2) has a unique solution. Both A and B is 1 so the particular solution becomes $\sin(x) + \cos(x)$.

Now, if we change the boundary conditions to $y(0) = 1\, y(\pi) = 1$ then equation (2) has no solutions. Because B takes the values 1 and -1 which is not possible.

Again, if we change the boundary conditions to $y(0) = 1\, y(2\pi) = 1$ then equation (2) has infinite numbers of solutions.

From this example we can say that small changes to boundary condition can dramatically change boundary value problem.

Problem 1.1. Solve this problem

$$y'' + 9y = 0$$

with boundary conditions $y(0) = 2$ and $y(2\pi) = 4$.

Solution. Given that,

$$y'' + 9y = 0$$

Here,

$$\begin{aligned} r^2 + 9 &= 0 \\ \Rightarrow r^2 &= -9 \\ \therefore r &= \pm 3i \end{aligned}$$

The roots are complex conjugate. So, the general solution to the differential equation is,

$$y(x) = A\cos(3x) + B\sin(3x)$$

Left boundary condition

$$\begin{aligned} y(0) &= A\cos(0) + B\sin(0) \\ \Rightarrow 2 &= A \\ \therefore A &= 2 \end{aligned}$$

Right boundary condition

$$\begin{aligned} y(2\pi) &= A\cos(6\pi) + B\sin(6\pi) \\ \Rightarrow 4 &= A \end{aligned}$$

On one hand, $A = 2$ and by the second equation $A = 4$. This is impossible and this boundary value problem has no solution.

2 Eigenvalue Problem

Consider the system

$$Ax = \lambda x$$

where for certain values of λ, called eigenvalues, there are non-zero solutions called eigenvectors. The values of λ where we get non-trivial solutions. The nontrivial solutions themselves are called *eigenfunctions*. This type of differential equations are *eigenvalue problem*.

Example 2.1. Let us consider a differential equation

$$y'' + \lambda y = 0, \quad y(0) = 0, \, y(\pi) = 0 \tag{3}$$

From equation (3) we can see there are three cases arises. So we need to consider them separately.

Case 1: λ is positive, i.e., $\lambda > 0$

Here λ is positive. So we assume $\lambda = \mu^2$ and rewriting equation (3) we get,

$$y'' + \mu^2 y = 0$$

The characteristic equation is $r^2 + \mu^2 = 0$.
So,

$$\begin{aligned} r^2 + \mu^2 &= 0 \\ \Rightarrow r &= \sqrt{-\mu^2} \\ \therefore r &= \pm i\mu \end{aligned}$$

Here the roots are complex conjugate. So, the general solution is,

$$y(x) = A\cos(\mu x) + B\sin(\mu x)$$

Here $\mu \neq 0$ because $\lambda > 0$. From the first boundary condition we get $A = 0$. From the second boundary condition we get, $B\sin(\mu x) = 0$.
For non-trivial solution B must be not equal to zero. So,

$$\begin{aligned} \sin(\mu x) &= 0 \\ \therefore \mu &= 1, 2, 3, \dots \end{aligned}$$

So, for $\mu = 1, 2, 3, \dots$ we get eigenvalues $\lambda = 1, 4, 9, \dots, n^2$. Here B is an arbitrary function and let $B = 1$. Thus, the eigenfunctions are

$$y_1(x) = \sin(x),\ y_2(x) = \sin(2x), \dots, y_n(x) = \sin(nx)$$

Case 2: λ is negative, i.e., $\lambda < 0$

Here λ is negative. So we assume $\lambda = -\mu^2$ and rewriting equation (3) we get,

$$y'' - \mu^2 y = 0$$

The characteristic equation is $r^2 - \mu^2 = 0$.
So,

$$\begin{aligned} r^2 - \mu^2 &= 0 \\ \Rightarrow r &= \sqrt{\mu^2} \\ \therefore r &= \pm\mu \end{aligned}$$

Here the roots are real and distinct. So, the general solution in hyperbolic form is,

$$y(x) = A\cosh(\mu x) + B\sinh(\mu x)$$

The first boundary condition gives $A = 0$ because $\cosh(0) = 1$ and $\sinh(0) = 0$. And the second boundary condition gives

$$B\sinh(\mu x) = 0$$

Here $\mu \neq 0$ so, $\sinh(\mu x) \neq 0$, therefore, $B = 0$.
So for $\lambda < 0$ only trivial solution is possible thus no eigenvalues.

Case 3: λ is zero, i.e., $\lambda = 0$

Here λ is zero. So by rewriting equation (3) we get,

$$y'' = 0$$

The general solution by integrating twice is,

$$y(x) = Ax + B$$

Using boundary condition we get $A = 0$ and $B = 0$.
So for $\lambda = 0$ only trivial solution is possible thus no eigenvalues.

So, we only get real eigenvalues and eigenfunctions when $\lambda > 0$. There may be complex eigenvalues.

Consider a basic problem

$$y'' + \lambda y = 0, \qquad y(0) = 0,\ y(L) = 0$$

Here the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad y_n(x) = \sin\left(\frac{n\pi x}{L} \quad \text{for } n = 1, 2, 3 \dots\right)$$

This problem is known as the classical *Euler Buckling* problem.

Note. The purpose of the eigenvalue problems is to find out non-trivial solution.

Problem 2.1. Find the eigenvalues and the corresponding eigenfunctions of

$$y'' + \lambda y = 0, \qquad y(0) = 0,\ y'(\pi) = 0$$

Solution. The cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$ will be considered separately.

Case 1: If $\lambda = 0$, we have $y'' = 0$

Which has the solution

$$y(x) = Ax + B \tag{4}$$

Applying the boundary conditions we have

$$\begin{aligned} y(0) &= A \times 0 + B \\ \Rightarrow 0 &= B \\ \therefore B &= 0 \end{aligned}$$

Differentiating both sides of equation (4) with respect to x we get,

$$y'(x) = A$$

So,

$$\begin{aligned} y'(\pi) &= A \\ \therefore A &= 0 \end{aligned}$$

So we obtain $A = 0$ and $B = 0$. Hence, from equation (4) we have $y = 0$, which is not an eigenfunction.

Case 2: If $\lambda < 0$ (i.e., λ is negative), the equation $y'' + \lambda y = 0$ can be written as

$$y'' - \left(\sqrt{-\lambda}\right)^2 y = 0$$

which has the solution

$$y(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \tag{5}$$

where $-\lambda$ and $\sqrt{-\lambda}$ are positive.

Applying the boundary conditions we have

$$y(0) = Ae^0 + Be^0$$
$$\therefore A + B = 0 \tag{6}$$

Differentiating both sides of equation (5), with respect to x we get,

$$y'(x) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$$

So,

$$y'(\pi) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi}$$
$$\therefore A\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} = 0 \tag{7}$$

Solving equation (6) and (7) we get $A = 0$ and $B = 0$. Hence, from equation (5) we have $y = 0$. Which is not an eigenfunction.

Case 3: If $\lambda > 0$ (i.e., λ is positive), then $y'' + \lambda y = 0$ can be written as

$$y'' + \left(\sqrt{\lambda}\right)^2 = 0$$

Whose solution is

$$y(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x) \tag{8}$$

Applying the given boundary conditions we have

$$\begin{aligned} y(0) &= A\cos(0) + B\sin(0) \\ \Rightarrow 0 &= A + 0 \\ \therefore A &= 0 \end{aligned}$$

Differentiating both sides of equation (8), with respect to x we get,

$$y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\, x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\, x)$$

So,

$$\begin{aligned} y'(\pi) &= -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi) \\ \Rightarrow 0 &= B\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi) \\ \therefore B\sqrt{\lambda}\cos(\sqrt{\lambda}\,\pi) &= 0 \end{aligned}$$

Now for $\theta > 0$, $\cos\theta = 0$ if and only if θ is a positive odd multiple of $\frac{\pi}{2}$.
That is for $\theta = \dfrac{(2n-1)\pi}{2} = \left(n - \dfrac{1}{2}\right)\pi$ where $n = 1, 2, 3, \ldots$. Therefore, to satisfy the boundary conditions, we must have $A = 0$ either $B = 0$ or $\cos\sqrt{\lambda}\pi = 0$.
This last equation is equivalent to

$$\begin{aligned} \sqrt{\lambda}\,\pi &= \left(n - \frac{1}{2}\right)\pi \\ \therefore \sqrt{\lambda} &= \left(n - \frac{1}{2}\right) \end{aligned}$$

The choice $B = 0$ results in the trivial solution $y = 0$ which is not an eigenfunction.
While, the choice $\sqrt{\lambda} = \left(n - \frac{1}{2}\right)$ results in the nontrivial solution $y_n = B_n \sin\left(n - \frac{1}{2}\right)x$.

Collecting all three cases, we conclude that the eigenvalues are $\lambda_n = (n - \frac{1}{2})^2$ and the corresponding eigenfunctions are

$$y_n = B_n \sin\left(n - \frac{1}{2}\right)x \qquad \text{where, } n = 1, 2, 3, \ldots$$

2.1 Strum-Liouville Problems

In this section we consider eigenvalue problems of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0, \quad B_1(y) = 0,\ B_2(y) = 0 \tag{9}$$

where,

$$\begin{aligned} B_1(y) &= \alpha y(a) + \beta y'(a) \quad \text{and} \\ B_2(y) &= \rho y(b) + \delta y'(b) \end{aligned}$$

Here, α, β, ρ and δ are real numbers with

$$\alpha^2 + \beta^2 > 0 \quad \text{and} \quad \rho^2 + \delta^2$$

P_0, P_1, P_2 and R are continuous and P_0 and R are positive on $[a, b]$.
We say that λ is an eigenvalue of equation (9) if it has a non-trivial solution y. In this case, y is an eigenfunction associated with λ or a $\lambda-$eigenfunction.

Note. Solving the eigenvalue problems means finding all eigenvalues and associated eigenfunctions of the equation (9).

Problem 2.2. Solve the eigenvalue problem

$$y'' + 3y' + 2y + \lambda y = 0, \qquad y(0) = 0, y(1) = 0$$

Solution. Given,

$$y'' + 3y' + 2y + \lambda y = 0 \tag{10}$$

The characteristic equation of equation (10) is,

$$r^2 + 3r + 2 + \lambda = 0$$

Here the roots are

$$r_1 = \frac{-3 + \sqrt{1 - 4\lambda}}{2} \qquad \text{and} \qquad r_1 = \frac{-3 - \sqrt{1 - 4\lambda}}{2}$$

If $\lambda < \frac{1}{4}$ then r_1 and r_2 are real and distinct roots. So, the general solution is

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}$$

Using boundary condition we get

$$\begin{aligned} A + B &= 0 \\ Ae^{r_1} + Be^{r_2} &= 0 \end{aligned}$$

Solving this system of equation we see that this system has only trivial solution. Therefore, there is no eigenvalue for $\lambda < \frac{1}{4}$.

If $\lambda = \frac{1}{4}$ then r_1 and r_2 are real, and they are $r_1 = r_2 = -\frac{3}{2}$. So, the general solution is

$$y(x) = Ae^{\frac{-3x}{2}} + Bxe^{\frac{-3x}{2}}$$

From the first boundary condition we get $A = 0$. Using $A = 0$ we get,

$$y(x) = Bxe^{\frac{-3x}{2}}$$

Now to satisfy second boundary condition B must be equal to zero. So, we see that this system has only trivial solution. Therefore, there is no eigenvalue for $\lambda = \frac{1}{4}$.

If $\lambda > \frac{1}{4}$ then r_1 and r_2 are complex conjugate roots, and they are

$$r_1 = -\frac{3}{2} + i\omega \quad \text{and } r_2 = -\frac{3}{2} + i\omega$$

where

$$\begin{aligned} \omega &= \frac{\sqrt{4\lambda - 1}}{2} \\ \lambda &= \frac{1 + 4\omega^2}{4} \end{aligned}$$

So, the general solution is

$$y(x) = Ae^{-\frac{3x}{2}}\cos(\omega x) + Be^{-\frac{3x}{2}}\sin(\omega x)$$

From the first boundary condition we get $A = 0$. Using $A = 0$ we get,

$$y(x) = Be^{-\frac{3x}{2}}\sin(\omega x)$$

Now the second boundary condition holds if and only if $\omega = n\pi$, where n is a positive integer. So, the eigenvalues are

$$\lambda_n = \frac{1 + 4n^2\pi^2}{4}$$

and the associated eigenfunctions,

$$y_n = e^{\frac{-3x}{2}}\sin(n\pi x), \qquad \text{where, } n = 1, 2, 3, \ldots$$

3 Green's Function

The general form of the second order differential equation,

$$\begin{aligned} P_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} + P_1(x)\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + P_0(x)y &= f(x) \\ \left[P_2(x)\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + P_1(x)\frac{\mathrm{d}}{\mathrm{d}\,x} + P_0(x)\right]y &= f(x) \\ \therefore\ L[y] &= f(x) \end{aligned}$$

$\therefore$ The non-homogeneous differential equation is

$$L[y] = f(x), \qquad a \leq x \leq b$$

where L is a differential operator and $y(x)$ satisfies boundary conditions at $x = a$ and $x = b$.
The solution is given by

$$y = L^{-1}[f]$$

The inverse of a differential operator in an integral operator, which is in the form,

$$y(x) = \int_a^b G(x,\zeta)f(\zeta)\,\mathrm{d}\,\zeta$$

The function $G(x,\zeta)$ is referred to as the kernel of the integral operator and is called the *Green's function.*

The solution of the boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(p(x)\frac{\mathrm{d}\,y(x)}{\mathrm{d}\,x}\right) + q(x)y(x) = f(x), \qquad a < x < b, \qquad y(a) = 0,\ y(b = 0)$$

takes the form

$$y(x) = \int_a^b G(x,\zeta)f(\zeta)\,\mathrm{d}\,\zeta$$

where the Green's function is the piece-wise defined function

$$G(x,\zeta) = \begin{cases} \dfrac{y_1(\zeta)y_2(x)}{pW}, & a \leq \zeta \leq x \\ \dfrac{y_1(x)y_2(\zeta)}{pW}, & x \leq \zeta \leq b \end{cases}$$

where W is Wronskinan of the system, and $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous problem satisfying $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$.

Remark. We use Green's function when the solution of differential equation is trivial.

Problem 3.1. Solve the boundary value problem

$$y'' = x^2, \qquad y(0) = 0,\, y(1) = 0$$

using the boundary value Green's function.

Solution. We first solve the homogeneous equation $y'' = 0$.
After two integrations, we have,

$$y(x) = Ax + B$$

for A and B constants to be determined.
We need one solution satisfying $y_1(0) = 0$. Thus,

$$0 = y_1(0) = B$$

So, we can pick $y_1(x) = x$, since A is arbitrary.
The other solution has to satisfy $y_2(1) = 0$. So,

$$\begin{aligned} 0 &= y_2(1) = A + B \\ \Rightarrow & A + B = 0 \\ \therefore & B = -A \end{aligned}$$

Again, A is arbitrary, and we will choose $A = -1$. Thus,

$$\begin{aligned} y_2(x) &= Ax + B \\ \Rightarrow y_2(x) &= Ax - A \\ \Rightarrow y_2(x) &= A(x-1) \\ \Rightarrow y_2(x) &= (-1)(x-1) \\ \Rightarrow y_2(x) &= -x + 1 \\ \therefore y_2(x) &= 1 - x \end{aligned}$$

For this problem $p(x) = 1$. Thus, for $y_1(x) = x$ and $y_2(x) = 1 - x$ we get,

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x & 1-x \\ 1 & -1 \end{vmatrix} \\ &= (-x - 1 + x) \\ &= -1 \\ \therefore\ p(x)W(x) &= -1 \end{aligned}$$

Here $p(x)W(x)$ is constant, as it should be.
Now,

$$G(x,\zeta) = \begin{cases} -\zeta(1-x), & 0 \le \zeta \le x, \\ -x(1-\zeta), & x \le \zeta \le 1. \end{cases}$$

Finally,

$$
\begin{aligned}
y(x) &= \int_0^1 G(x,\zeta) f(\zeta)\,\mathrm{d}\,\zeta \\
&= \int_0^1 G(x,\zeta)\zeta^2\,\mathrm{d}\,\zeta \\
&= -\int_0^x \zeta(1-x)\zeta^2\,\mathrm{d}\,\zeta - \int_x^1 x(1-\zeta)\zeta^2\,\mathrm{d}\,\zeta \\
&= -(1-x)\int_0^x \zeta^3\,\mathrm{d}\,\zeta - x\int_x^1 (\zeta^2-\zeta^3)\,\mathrm{d}\,\zeta \\
&= -(1-x)\left[\frac{\zeta^4}{4}\right]_0^x - x\left[\frac{\zeta^3}{3}-\frac{\zeta^4}{4}\right]_x^1 \\
&= -\frac{1}{4}(1-x)x^4 - \frac{1}{12}4x + \frac{1}{12}3x + \frac{1}{12}x\left(4x^3-3x^4\right) \\
&= -\frac{1}{4}x^4 + \frac{1}{4}x^5 - \frac{1}{3}x + \frac{1}{4}x + \frac{1}{3}x^4 - \frac{1}{4}x^5 \\
&= \frac{1}{12}x^4 - \frac{1}{12}x \\
\therefore\ y(x) &= \frac{1}{12}\left(x^4 - x\right)
\end{aligned}
$$

Checking the answer, we can easily verify that $y'' = x^2$, $y(0) = 0$ and $y(1) = 0$.

3.1 Properties of Green's Function

1. Differential equation:

$$\frac{\partial}{\partial x}\left(p(x)\,\frac{\partial\, G(x,\zeta)}{\partial\, x}\right) + q(x)G(x,\zeta) = 0, \qquad x \neq \zeta$$

2. Boundary conditions: Whatever conditions $y_1(x)$ and $y_2(x)$ satisfy, $G(x,\zeta)$ will satisfy.

3. Symmetry or Reciprocity:

$$G(x,\zeta) = G(\zeta,x)$$

4. Continuity of G at $x=\zeta$:

$$G(\zeta^+,\zeta) = G(\zeta^-,\zeta)$$

5. Jump discontinuity of $\frac{\partial\, G}{\partial\, x}$ at $x = \zeta$:

$$\frac{\partial\, G(\zeta^+,\zeta)}{\partial\, x} - \frac{\partial\, G(\zeta^-,\zeta)}{\partial\, x} = \frac{1}{p(\zeta)}$$

We can construct Green's function using these properties. Here is an example.

Problem 3.2. Construct the Green's function for the problem

$$y'' + \omega^2 y = f(x), \qquad 0 < x < 1, \qquad y(0) = 0,\ y(1) = 0$$

with $\omega \neq 0$.

Solution. 1. Finding solutions to the homogeneous equation
Given that,

$$y'' + \omega^2 y = 0 \tag{11}$$

Here the characteristic equation is,

$$\begin{aligned} m^2 + \omega^2 &= 0 \\ \Rightarrow m^2 &= -\omega^2 \\ \therefore m &= -i\omega \end{aligned}$$

A general solution to the homogeneous equation is

$$y(x) = c_1 \sin(\omega x) + c_2 \cos(\omega x)$$

If we use boundary condition the solution is trivial. So we have to use Green's function. Thus, for $x \neq \zeta$,

$$G(x, \zeta) = c_1(\zeta) \sin(\omega x) + c_2(\zeta) \cos(\omega x) \tag{12}$$

2. Boundary conditions:
We have $G(0, \zeta) = 0$ for $0 \leq x \leq \zeta$. So,

$$\begin{aligned} G(0, \zeta) &= c_1(\zeta) \sin(0) + c_2(\zeta) \cos(\omega x) \\ &= c_2(\zeta) \cos(\omega x) \\ &= 0 \end{aligned}$$

So,

$$G(x, \zeta) = c_1(\zeta) \sin \omega x, \qquad 0 \leq x \leq \zeta$$

Again, we have $G(1, \zeta) = 0$ for $\zeta \leq x \leq 1$. So,

$$\begin{aligned} G(1, \zeta) &= c_1(\zeta) \sin(\omega) + c_2(\zeta) \cos(\omega) \\ \therefore c_1(\zeta) \sin(\omega) + c_2(\zeta) \cos(\omega) &= 0 \\ \Rightarrow c_2(\zeta) \cos(\omega) &= -c_1(\zeta) \sin(\omega) \\ \Rightarrow c_2(\zeta) &= -c_1(\zeta) \frac{\sin(\omega)}{\cos(\omega)} \\ \therefore c_2(\zeta) &= -c_1(\zeta) \tan(\omega) \end{aligned}$$

Putting these value in equation (12), we have,

$$\begin{aligned}
G(x,\zeta) &= c_1(\zeta)\sin(\omega x) + c_2(\zeta)\cos(\omega x) \\
&= c_1(\zeta)\sin(\omega x) - c_1(\zeta)\tan(\omega)\cos(\omega x) \\
&= c_1(\zeta)\sin(\omega x) - c_1(\zeta)\frac{\sin(\omega)}{\cos(\omega)}\cos(\omega x) \\
&= \frac{c_1(\zeta)\sin(\omega x)\cos(\omega) - c_1(\zeta)\sin(\omega)\cos(\omega x)}{\cos(\omega)} \\
&= \frac{c_1(\omega)}{\cos(\omega)}\left[\sin(\omega x)\cos(\omega) - \sin(\omega)\cos(\omega x)\right] \\
&= \frac{c_1(\omega)}{\cos(\omega)}\left[\sin(\omega x - \omega)\right] \\
&= \frac{c_1(\omega)}{\cos(\omega)}\cdot\sin\omega(x-1) \\
&= \frac{-c_1(\omega)}{\cos(\omega)}\cdot\sin\omega(1-x) \\
\therefore\ G(x,\zeta) &= \frac{-c_1(\omega)}{\cos(\omega)}\cdot\sin\omega(1-x) \qquad (13)
\end{aligned}$$

Since the coefficient is arbitrary at this point, we can write the result as,

$$G(x,\zeta) = d_1\zeta\sin\omega(1-x), \qquad \zeta \le x \le 1$$

$\therefore\ y_2(x) = \sin\omega(1-x)$ satisfies the second boundary condition.

3. Symmetry or Reciprocity:
   We now impose that $G(x,\zeta) = G(\zeta,x)$. To this point we have that,

$$G(x,\zeta) = \begin{cases} c_1(\zeta)\sin\omega x, & 0 \le x \le \zeta \\ d_1(\zeta)\sin\omega(1-x), & \zeta \le x \le 1 \end{cases}$$

   We can make the branches symmetric by picking the right forms for $c_1(\omega)$ and $d_1(\omega)$. We choose $c_1(\omega) = C\sin\omega(1-\zeta)$ and $d_1(\omega) = C\sin\omega$.
   Then,

$$G(x,\zeta) = \begin{cases} C\sin\omega(1-\zeta)\sin\omega x, & 0 \le x \le \zeta \\ C\sin\omega(1-x)\sin\omega\zeta, & \zeta \le x \le 1 \end{cases} \qquad (14)$$

   Now the Green's function is symmetric and C is constant.

4. Continuity of $G(x,\zeta)$:
   We already have continuity by virtue of the symmetric imposed in the last step.

5. Jump discontinuity in $\frac{\partial}{\partial x}G(x,\zeta)$:
   Using the jump discontinuity in the derivative:

$$\frac{\partial G(\zeta^+,\zeta)}{\partial x} - \frac{\partial G(\zeta^-,\zeta)}{\partial x} = \frac{1}{p(\zeta)}$$

For this problem $p(x) = 1$. Inserting the Green's function, we have,

$$
\begin{aligned}
1 &= \frac{\partial G(\zeta^+, \zeta)}{\partial x} - \frac{\partial G(\zeta^-, \zeta)}{\partial x} \\
&= \frac{\partial}{\partial x}\left[C \sin\omega(1-x)\sin\omega\zeta\right]_{x=\zeta} - \frac{\partial}{\partial x}\left[C \sin\omega(1-\zeta)\sin\omega x\right]_{x=\zeta} \\
&= -\omega C \cos\omega(1-\zeta)\sin\omega\zeta - \omega C \sin\omega(1-\zeta)\cos\omega\zeta \\
&= -\omega C \left[\cos\omega(1-\zeta)\sin\omega\zeta + \sin\omega(1-\zeta)\cos\omega\zeta\right] \\
&= -\omega C \sin\omega(\zeta + 1 - \zeta) \\
&= -\omega C \sin\omega \\
\therefore\ -\omega C \sin\omega &= 1 \\
\therefore\ C &= \frac{-1}{\omega \sin\omega}
\end{aligned}
$$

Finally, we have the Green's function:
From the equation (14), we get,

$$
G(x,\zeta) = \begin{cases} \dfrac{-1}{\omega\sin\omega} \cdot \sin\omega(1-\zeta)\sin\omega x, & 0 \le x \le \zeta \\ \dfrac{-1}{\omega\sin\omega} \cdot \sin\omega(1-x)\sin\omega\zeta, & \zeta \le x \le 1 \end{cases}
$$

$$
\therefore\ G(x,\zeta) = \begin{cases} -\dfrac{\sin\omega(1-\zeta)\sin\omega x}{\omega\sin\omega}, & 0 \le x \le \zeta \\ -\dfrac{\sin\omega(1-x)\sin\omega\zeta}{\omega\sin\omega}, & \zeta \le x \le 1 \end{cases}
$$

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