Chapter 1

Non-Linear Partial Differential Equation of First Order

Suppose the non-linear partial differential equation is

$$f(x, y, z, p, q) = 0$$

Type of solution of PDE of first order

1. *Complete solution/integral*: $z = (x + a)(y + b)$ is a complete solution/integral of $z = pq$.
2. *Particular solution:* $z = (x + 3)(y - 4)$ is a particular solution of $z = pq$.
3. *General solution:* The complete solution/integral $\Phi(x, y, z, a, b)$ of PDE $f(x, y, z, p, q) = 0$ is said to be general solution if we can write as $\Phi(x, y, z, a, b) = 0$
4. *Singular solution:* Let $z = ax + by + ab$ is a complete solution of $z = xp + qy + pq$.
   partial differential with respect to a and b gives

$$\begin{aligned} \frac{\partial z}{\partial a} &= x + b \\ \Rightarrow 0 &= x + b \\ \Rightarrow b &= -x \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial b} &= 0 + y + a \\ \Rightarrow a &= -y \end{aligned}$$

   Putting in complete solution $z = -yx - \cancel{xy} + \cancel{xy}$
   $z = -xy$ is a singular solution.

Type - I: First standard form $f(p, q) = 0$ which does not contain x, y, z explicitly.
Working rule:

1 Write

$$f(p, q) = 0 \tag{1.1}$$

Assume the solution of (1.1) is

$$z = ax + by + c \tag{1.2}$$

2 Differentiate (1.2) with respect to x and y partially, we get

$$\begin{aligned} \frac{\partial z}{\partial x} &= a \quad \Rightarrow \quad p = a \\ \frac{\partial z}{\partial y} &= b \quad \Rightarrow \quad q = b \end{aligned}$$

3 Putting the values of p, q in equation (1.1), we get

$$\begin{aligned} f(a, b) &= 0 \\ \Rightarrow b &= \Phi(a) \end{aligned}$$

4 Putting the values in (1.2)

$$z = ax + \Phi(a)y + c$$

This is the required complete solution.

Problem 1.0.1. Find the complete solution of PDE $p^2 + q^2 = m^2$

Solution. The given equation

$$p^2 + q^2 = m^2 \tag{1.3}$$

is the first standard form.
Suppose the solution is

$$z = ax + by + c \tag{1.4}$$

partial Differentiate with respect to x and y the equation (1.4) gives,

$$\frac{\partial z}{\partial x} = a \quad \Rightarrow \ p = a$$
$$\frac{\partial z}{\partial y} = b \quad \Rightarrow \ q = b$$

put in (1.3), gives

$$a^2 + b^2 = m^2$$
$$\Rightarrow b = \pm\sqrt{m^2 - a^2}$$

Hence from (1.4), we get

$$z = ax \pm \sqrt{m^2 - a^2}y + c \qquad \text{(contains 2 arbitrary constants)}$$

This is the required complete solution.

Problem 1.0.2. Find the complete solution of $p^2 + q^2 = npq$

Solution. The is

$$p^2 + q^2 = m^2 \tag{1.5}$$

first standard form.
Suppose the solution of (1.5) is

$$z = ax + by + c \tag{1.6}$$

partial Differentiate with respect to x and y the equation (1.6) gives,

$$\frac{\partial z}{\partial x} = a \quad \Rightarrow \ p = a$$
$$\frac{\partial z}{\partial y} = b \quad \Rightarrow \ q = b$$

put in (1.5), gives

$$a^2 + b^2 = nab$$
$$\Rightarrow b^2 - nab + a^2 = 0$$
$$\Rightarrow b = \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}$$

Hence from (1.6), we get

$$z = ax + \left[\frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}\right] y + c$$

This is the required complete solution.

Type - II: Second standard form

$$f(z, p, q) = 0 \tag{1.7}$$

which does not contain x, y explicitly.
Working rule:

1 Suppose

$$z = f(x + ay) \tag{1.8}$$

is a solution of (1.7)

2 Let $u = x + ay$ so that[1]

$$\begin{aligned} & z = f(u) \\ \Rightarrow & \frac{\partial z}{\partial x} = \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \cdot \frac{\partial u}{\partial x} \\ \Rightarrow & p = \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \cdot 1 \\ \Rightarrow & p = \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \end{aligned} \qquad \left[\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a\right]$$

and

$$\begin{aligned} \Rightarrow & q = \frac{\partial z}{\partial y} = \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \cdot \frac{\partial u}{\partial y} \\ \Rightarrow & q = a \cdot \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \\ \Rightarrow & q = a\ \frac{\mathrm{d}\, z}{\mathrm{d}\, u} \end{aligned}$$

3 Put in (1.7)

$$f\left(z,\ \frac{\mathrm{d}\, z}{\mathrm{d}\, u},\ a\,\frac{\mathrm{d}\, z}{\mathrm{d}\, u}\right) = 0$$

4 Solve the above equation for z. Which will be the required complete solution.

Problem 1.0.3. Solve $z = p^2 + q^2$ (Find the complete solution)

Solution. Given

$$z = p^2 + q^2 \tag{1.9}$$

This is second standard form.
We assume the solution

$$z = f(x + ay)$$

and $u = x + ay$, $p = \frac{\mathrm{d}\, z}{\mathrm{d}\, u}$, $q = a\,\frac{\mathrm{d}\, z}{\mathrm{d}\, u}$ Put in (1.9)

$$\begin{aligned} & z = \left(\frac{\mathrm{d}\, z}{\mathrm{d}\, u}\right)^2 + a^2\left(\frac{\mathrm{d}\, z}{\mathrm{d}\, u}\right)^2 \\ \Rightarrow & z = \left(\frac{\mathrm{d}\, z}{\mathrm{d}\, u}\right)^2\left(1 + a^2\right) \\ \Rightarrow & \sqrt{z} = \frac{\mathrm{d}\, z}{\mathrm{d}\, u}\sqrt{1 + a^2} \\ \Rightarrow & \frac{\mathrm{d}\, z}{\sqrt{z}} = \frac{\mathrm{d}\, u}{\sqrt{1 + a^2}} \end{aligned}$$

integrating we get,

$$\begin{aligned} \Rightarrow & 2\sqrt{z} = \frac{u}{\sqrt{1 + a^2}} + b \\ \Rightarrow & 2\sqrt{z} = \frac{x + ay}{\sqrt{1 + a^2}} \end{aligned}$$

This is the required solution.

Problem 1.0.4. Solve $p(1 + q^2 = q(z - a))$

[1]Note to self: check other resources to confirm

Solution. Given
$$p(1+q^2=q(z-a)) \tag{1.10}$$
This is second standard form.
We assume the solution
$$z=f(x+ay)$$
and $u=x+ay$, $p=\dfrac{\mathrm{d}\,z}{\mathrm{d}\,u}$, $q=a\dfrac{\mathrm{d}\,z}{\mathrm{d}\,u}$
Put in (1.10)
$$\begin{aligned}
\frac{\cancel{\mathrm{d}\,z}}{\cancel{\mathrm{d}\,u}}\left[1+a^2\left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2\right] &= b\frac{\cancel{\mathrm{d}\,z}}{\cancel{\mathrm{d}\,u}}(z-a)\\
\Rightarrow \left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2 &= \frac{b(z-a)-1}{b^2}\\
\Rightarrow \frac{b\,\mathrm{d}\,z}{\sqrt{b(z-a)}} &= \mathrm{d}\,u
\end{aligned}$$
integrating we get,
$$\Rightarrow b\int\frac{\mathrm{d}\,z}{\sqrt{b(z-a)}}=u+c$$
put $b(z-a)=t,\ b\,\mathrm{d}\,z=\mathrm{d}\,t$
$$\begin{aligned}
&\Rightarrow \int\frac{\mathrm{d}\,t}{\sqrt{t}}=u+c\\
&\Rightarrow 2\sqrt{t}=u+c\\
&\Rightarrow 2\sqrt{b(z-a)}=x+by+c
\end{aligned}$$
This is the required solution.

Problem 1.0.5. Solve $x^2p^2+y^2q^2=z^2$ (which is reducible to 2nd...)

Solution. Given
$$(xp)^2+(yp)^2=z^2 \tag{1.11}$$
Put $X=\ln x;\quad Y=\ln y$
$$\frac{\partial\,X}{\partial\,x}=\frac{1}{x}\qquad \frac{\partial\,Y}{\partial\,y}=\frac{1}{y}$$
$$\begin{aligned}
p &= \frac{\partial\,z}{\partial\,x}\\
\Rightarrow p &= \frac{\partial\,z}{\partial\,X}\frac{\partial\,X}{\partial\,x}\\
\Rightarrow p &= \frac{1}{x}\frac{\partial\,z}{\partial\,X}\\
\Rightarrow xp &= \frac{\partial\,z}{\partial\,X}\\
\Rightarrow xp &= P
\end{aligned}$$
$$\begin{aligned}
q &= \frac{\partial\,z}{\partial\,y}\\
\Rightarrow q &= \frac{\partial\,z}{\partial\,Y}\frac{\partial\,Y}{\partial\,y}\\
\Rightarrow q &= \frac{1}{y}\frac{\partial\,z}{\partial\,Y}\\
\Rightarrow yq &= \frac{\partial\,z}{\partial\,Y}\\
\Rightarrow yp &= Q
\end{aligned}$$

From (1.11) we get,

$$P^2 + Q^2 = z^2 \tag{1.12}$$

This is 2nd standard form.
Assume the solution is

$$z = f(X + aY)$$

and $u = X + aY$, $P = \dfrac{\mathrm{d}\,z}{\mathrm{d}\,u}$, $Q = a\dfrac{\mathrm{d}\,z}{\mathrm{d}\,u}$
Put in (1.12)

$$\begin{aligned}
\left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2 + a^2\left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2 &= z^2 \\
\Rightarrow \left(\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\right)^2 (1 + a^2) &= z^2 \\
\Rightarrow \frac{\mathrm{d}\,z}{\mathrm{d}\,u} &= \frac{z}{\sqrt{1+a^2}} \\
\Rightarrow \frac{\mathrm{d}\,z}{z} &= \frac{\mathrm{d}\,u}{\sqrt{1+a^2}}
\end{aligned}$$

Integrating,

$$\begin{aligned}
\ln z &= \frac{u}{\sqrt{1+a^2}} + b \\
\Rightarrow \ln z &= \frac{\ln x + a\ln y}{\sqrt{1+a^2}} + b
\end{aligned}$$

This is the required complete solution.

Problem 1.0.6. Solve $xp^2 + yq^2 = z^2$

Solution. Given

$$(\sqrt{x}p)^2 + (\sqrt{y}p)^2 = z^2 \tag{1.13}$$

Put $X = 2x^{\frac{1}{2}}$; $\quad Y = 2y^{\frac{1}{2}}$

$$\frac{\partial\,X}{\partial\,x} = 2\cdot\frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{1}{\sqrt{x}} \qquad \frac{\partial\,Y}{\partial\,y} = \frac{1}{\sqrt{y}}$$

Now,

$$\begin{aligned}
p &= \frac{\partial\,z}{\partial\,x} \\
\Rightarrow p &= \frac{\partial\,z}{\partial\,X}\frac{\partial\,X}{\partial\,x} \\
\Rightarrow p &= \frac{1}{\sqrt{x}}\frac{\partial\,z}{\partial\,X} \\
\Rightarrow \sqrt{x}p &= \frac{\partial\,z}{\partial\,X} \\
\Rightarrow \sqrt{x}p &= P
\end{aligned}$$

$$\begin{aligned}
q &= \frac{\partial\,z}{\partial\,y} \\
\Rightarrow q &= \frac{\partial\,z}{\partial\,Y}\frac{\partial\,Y}{\partial\,y} \\
\Rightarrow q &= \frac{1}{\sqrt{y}}\frac{\partial\,z}{\partial\,Y} \\
\Rightarrow \sqrt{y}q &= \frac{\partial\,z}{\partial\,Y} \\
\Rightarrow \sqrt{y}p &= Q
\end{aligned}$$

continue...

Problem 1.0.7. Solve $\frac{p^2}{x^2} + \frac{q^2}{y^2} = z^2$

Solution. Given

$$\left(\frac{p}{x}\right)^2 + \left(\frac{q}{y}\right)^2 = z^2 \tag{1.14}$$

Put $X = \frac{1}{2}x^2$ $Y = \frac{1}{2}y^2$
continue...

Type - III: Third standard form $f_1(x,p) = f_2(y,q)$
Working rule:

1 Let

$$\begin{aligned} f_1(x,p) &= f_2(y,q) = a \\ f_1(x,p) &= a \quad \text{and} \quad f_2(y,q) = a \\ \Rightarrow p &= \Phi_1(x,a) \quad \text{and} \quad q = \Phi_2(y,a) \end{aligned}$$

2 Solve

$$\begin{aligned} \mathrm{d}\, z &= p \,\mathrm{d}\, u + q \,\mathrm{d}\, u \\ &= \Phi_1(x,a) \,\mathrm{d}\, u + \Phi_2(y,a) \,\mathrm{d}\, u \end{aligned}$$

Integrate on both side

$$z = \int \Phi_1(x,a) \,\mathrm{d}\, u + \Phi_2(y,a) \,\mathrm{d}\, u + c$$

This is the required complete solution.

Problem 1.0.8. Solve $p^2 - q^2 = x - y$

Solution. Given,

$$p^2 - x = q^2 - y \tag{1.15}$$

This is 3rd standard form.
Let

$$\begin{aligned} p^2 - x &= q^2 - y = a \\ \Rightarrow p &= \sqrt{a+x} \quad \text{and} \quad q = \sqrt{a+y} \end{aligned}$$

Since,

$$\begin{aligned} \mathrm{d}\, z &= p \,\mathrm{d}\, x + q \,\mathrm{d}\, y \\ \mathrm{d}\, z &= \sqrt{a+x} \,\mathrm{d}\, x + \sqrt{a+y} \,\mathrm{d}\, y \end{aligned}$$

Integrating,

$$\begin{aligned} z &= \int \sqrt{a+x} \,\mathrm{d}\, x + \int \sqrt{a+y} \,\mathrm{d}\, y \\ z &= \frac{2}{3}(a+x)^{\frac{3}{2}} + \frac{2}{3}(a+y)^{\frac{3}{2}} + c \end{aligned}$$

This is the required complete solution.

Problem 1.0.9. Solve $yp + xq + pq = 0$

Solution. Given,

$$\begin{aligned} yp + xq &= -pq \\ \Rightarrow \frac{yp}{-pq} + \frac{xq}{-pq} &= 1 \\ \Rightarrow \frac{x}{p} &= 1 + \frac{y}{q} \end{aligned}$$

This is 3rd standard form.
Let

$$\frac{x}{p} = 1 + \frac{y}{q} = a$$
$$\Rightarrow p = \frac{x}{a} \quad \text{and} \quad q = \frac{y}{a-1}$$

Since,

$$\mathrm{d}\,z = p\,\mathrm{d}\,x + q\,\mathrm{d}\,y$$
$$\mathrm{d}\,z = \frac{x}{a}\,\mathrm{d}\,x + \frac{y}{a-1}\,\mathrm{d}\,y$$

Integrating,

$$z = \frac{1}{2a}x^2 + \frac{1}{2(a-1)}y^2 + c$$

This is the required complete solution.

Problem 1.0.10. Solve $z^2(p^2 + q^2) = x^2 + y^2$

Solution. Given,

$$(zp)^2 + (zq)^2 = x^2 + y^2 \tag{1.16}$$

Taking $Z = \frac{z^2}{2}$ and $\frac{\partial Z}{\partial z} = z$
Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}$$
$$p = \frac{1}{z}P$$

and,

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y}$$
$$zq = Q \qquad \left[\text{Where } P = \frac{\partial Z}{\partial x},\ Q = \frac{\partial Z}{\partial y}\right]$$

From (1.16)

$$P^2 + Q^2 = x^2 + y^2$$
$$\Rightarrow P^2 - x^2 = y^2 - Q^2$$

This is 3rd standard form.
Let

$$P^2 - x^2 = y^2 - Q^2 = a^2$$
$$\Rightarrow P^2 = a^2 + x^2 \quad \text{and} \quad Q^2 = y^2 - a^2$$

Since,

$$\mathrm{d}\,Z = P\,\mathrm{d}\,x + Q\,\mathrm{d}\,y$$
$$\Rightarrow \mathrm{d}\,Z = (a^2 + x^2)\,\mathrm{d}\,x + (y^2 - a^2)\,\mathrm{d}\,y$$

Integrating,

$$Z = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log(x + \sqrt{x^2+a^2})\right] + \frac{1}{2}\left[y\sqrt{y^2-a^2} - a^2\log(y + \sqrt{y^2-a^2})\right] + c$$
$$\Rightarrow \frac{z^2}{2} = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log(x + \sqrt{x^2+a^2})\right] + \frac{1}{2}\left[y\sqrt{y^2-a^2} - a^2\log(y + \sqrt{y^2-a^2})\right] + c$$
$$\Rightarrow z^2 = \left[x\sqrt{x^2+a^2} + a^2\log(x + \sqrt{x^2+a^2})\right] + \left[y\sqrt{y^2-a^2} - a^2\log(y + \sqrt{y^2-a^2})\right] + c$$

This is the required complete solution.

Problem 1.0.11. Solve $p^2 - q^2 = z(x-y)$

Solution. Given,

$$\left(\frac{p}{\sqrt{z}}\right)^2 - \left(\frac{q}{\sqrt{z}}\right)^2 = x - y \tag{1.17}$$

Taking $Z = 2\sqrt{z}$ and $\frac{\partial Z}{\partial z} = \frac{1}{\sqrt{z}}$
Now,

$$\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} \\ \Rightarrow\ p &= \sqrt{z}\frac{\partial Z}{\partial x} \\ \Rightarrow\ \frac{p}{\sqrt{z}} &= P \end{aligned}$$

and,

$$\begin{aligned} q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y} \\ &= \sqrt{z}\frac{\partial Z}{\partial y} \\ \Rightarrow\ \frac{q}{\sqrt{z}} &= Q \qquad \left[\text{Where } P = \frac{\partial Z}{\partial x},\ Q = \frac{\partial Z}{\partial y}\right] \end{aligned}$$

From (1.17)

$$\begin{aligned} P^2 - Q^2 &= x - y \\ \Rightarrow P^2 - x &= Q^2 - y \end{aligned}$$

This is 3rd standard form.
Let

$$\begin{aligned} P^2 - x &= Q^2 - y = a \\ \Rightarrow P^2 &= x + a \quad \text{and} \quad Q^2 = a + y \\ \Rightarrow P &= \sqrt{x+a} \quad \text{and} \quad Q = \sqrt{a+y} \end{aligned}$$

Since,

$$\begin{aligned} \mathrm{d}Z &= P\,\mathrm{d}x + Q\,\mathrm{d}y \\ \Rightarrow\ \mathrm{d}Z &= \sqrt{x+a}\,\mathrm{d}x + \sqrt{y+a}\,\mathrm{d}y \end{aligned}$$

Integrating,

$$\begin{aligned} Z &= \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y+a)^{\frac{3}{2}} + c \\ \Rightarrow\ 2\sqrt{z} &= \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y+a)^{\frac{3}{2}} + c \end{aligned}$$

This is the required complete solution.

Problem 1.0.12. Solve $z(p^2 - q^2) = x - y$

Solution. Given,

$$\left(\sqrt{z}p\right)^2 - \left(\sqrt{z}q\right)^2 = x - y \tag{1.18}$$

Put $Z = \frac{2}{3}z^{\frac{3}{2}}$ and $\frac{\partial Z}{\partial z} = \sqrt{z}$

Problem 1.0.13. Solve $p^2 + q^2 = z^2(x^2 + y^2)$

Solution. Given,

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2 + y^2 \tag{1.19}$$

Put $Z = \log z$ and $\frac{\partial Z}{\partial z} = \frac{1}{z}$

Type - IV: Forth standard form

$$z = px + qy + f(p, q)$$

The complete solution is

$$z = ax + by + f(a, b) \tag{1.20}$$

Singular solution
Partial differentiate (1.20) with respect to a and b

$$0 = x + 0 + f'(a, b) \tag{1.21}$$
$$0 = 0 + y + f'(a, b) \tag{1.22}$$

Solve (1.21) and (1.22) find the value of a, b put in (1.20), which gives the required singular solution.

Problem 1.0.14. Find the complete and singular solution of

$$z = px + qy + p^2 + q^2$$

Solution. Given,

$$z = px + qy + p^2 + q^2 \tag{1.23}$$

This is fourth standard form.
The complete solution is

$$z = ax + by + a^2 + b^2 \tag{1.24}$$

Singular solution
Partial differentiate (1.24) with respect to a and b

$$0 = x + 0 + 2a + 0 \qquad \Rightarrow \; a = \frac{-x}{2}$$
$$0 = 0 + y + 0 + 2b \qquad \Rightarrow \; b = \frac{-y}{2}$$

Putting in (1.24)

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{y^2}{4} + \frac{x^2}{4}$$
$$z = -\frac{x^2}{4} - \frac{y^2}{4}$$

$x^2 + y^2 + 4z = 0$ is singular solution.