Mathematical Method

MAT331

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Preface

This is a compilation of lecture notes with some books and my own thoughts. If there are any mistake/typing error or, for any query mail me at mehedi12@student.sust.edu. Currently, the following topics are not included

- Fourier Transform and Applications
- Boundary value problems
- Eigenfunctions
- Green's functions
- Strum-Liouville problems
- Laplace Equation

Contents

Syllabus

- Fourier series,
- Fourier integral,
- Fourier transform and their applications.
- Laplace transform and its properties with their applications in differential and integral equations.
- Harmonic functions:
  - Laplace equation in different coordinates and its applications.
- Concepts of singularities and series solutions;
- Legendre polynomials,
- Hermite polynomials,
- Laguerre functions,
- Bessel function and their properties.
- Boundary value problems involving second order ordinary differential equations;
- eigenfunction, expansions and Green's functions.
- Ideas about Strum-Liouville problems.
- Linear integral equations:
  - Elementary' ideas

Part I

Class Note/Sheet

Chapter 1

The Laplace Transform

1.1 Definition of The Laplace Transform

Let $F(t)$ be a function of t specified for $t > 0$. The *Laplace transform* of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,\mathrm{d}\,t \tag{1.1}$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex.

The Laplace transform of $F(t)$ is said to *exist* if the integer (1.1) *converges* for some values of s; otherwise it does not exist.

1.2 Laplace Transforms of Some Elementary Functions

$\mathbf{F(t)}$	$\mathcal{L}\{\mathbf{F(t)}\} = \mathbf{f(s)}$	
1	$\dfrac{1}{s}$	$s > 0$
t	$\dfrac{1}{s^2}$	$s > 0$
$t^n \quad n = 0, 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$\sin at$	$\dfrac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\dfrac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$s > \lvert a\rvert$
$\cosh at$	$\dfrac{s}{s^2-a^2}$	$s > \lvert a\rvert$

1.3 Some Important Properties of Laplace Transforms

1. First translation or shifting property.

   Theorem 1.3.1. If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$.

   Example. Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$, we have

$$\mathcal{L}\left\{e^{-t}\cos 2t\right\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

2. Second translation or shifting property.

   Theorem 1.3.2. If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$ then $\mathcal{L}\{G(t)\} = e^{-as} f(s)$.

   Example. Since $\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$, the Laplace transform of the function $G(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$ is $\frac{6e^{-2s}}{s^4}$.

3. Change of scale property.

   Theorem 1.3.3. If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

   Example. Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, we have $\mathcal{L}\{\sin 3t\} = \frac{1}{3}\frac{1}{(s/3)^2+1} = \frac{3}{s^2+9}$.

4. Laplace transform of derivatives.

   Theorem 1.3.4. If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s\,f(s) - F(0)$.
   If $F(t)$ is continuous for $0 \leqq t \leq N$ and of exponential order for $t > N$ while $F'(t)$ is sectionally continuous for $0 \leqq t \leqq N$.

   Example. If $F(t) = \cos 3t$, then $\mathcal{L}\{F'(t)\} = \frac{s}{s^2+9}$ and we have

   $$\mathcal{L}\{F'(t)\} = \mathcal{L}\{-3\sin 3t\} = s\left(\frac{s}{s^2+9}\right) - 1 = \frac{-9}{s^2+9}$$

   This method is useful in finding Laplace transforms without integration.

   Theorem 1.3.5. If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F''(t)\} = s^2\,f(s) - s\,F(0)$.
   If $F(t)$ and $F'(t)$ is continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ while $F''(t)$ is sectionally continuous for $0 \leq t \leqq N$.

5. Laplace transform of integrals.

   Theorem 1.3.6. If $\mathcal{L}\{F(t)\} = f(s)$ then

   $$\mathcal{L}\left\{\int_0^t F(u)\,\mathrm{d}\,u\right\} = \frac{f(s)}{s}$$

   Example. Since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, we have

   $$\mathcal{L}\left\{\int_0^t \sin 2u\,\mathrm{d}\,u\right\} = \frac{2}{s(s^2+4)}$$

   as can be verified directly.

6. Multiplication by t^n.

   Theorem 1.3.7. If $\mathcal{L}\{F(t)\} = f(s)$ then

   $$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\,s^n} f(s) = (-1)^n f^{(n)}(s)$$

   Example. Since $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$, we have

   $$\mathcal{L}\left\{te^{2t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$
   $$\mathcal{L}\left\{t^2e^{2t}\right\} = \frac{\mathrm{d}^2}{\mathrm{d}\,s^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^2}$$

1.4 Solved Problems

1.4.1 Laplace Transforms of Some Elementary Functions

Problem 1.4.1. Prove that

(a) $\mathcal{L}\{1\} = \frac{1}{s}$, $s > 0$

(b) $\mathcal{L}\{t\} = \frac{1}{s^2}$, $s > 0$

(c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-1}$, $s > a$

Solution. (a)

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \int_0^P e^{-st}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^P \\
&= \lim_{P\to\infty} \frac{1-e^{-sP}}{s} \\
&= \frac{1}{s} \quad \text{if } s > 0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty e^{-st}(t)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \int_0^P te^{-st}(t)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right|_0^P \\
&= \lim_{P\to\infty} \left(\frac{1}{s^2} - \frac{e^{-st}}{s^2} - \frac{Pe^{-sP}}{s}\right) \\
&= \frac{1}{s^2} \quad \text{if } s > 0
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st}\left(e^{at}\right)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \int_0^P e^{-(s-a)t}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^P \\
&= \lim_{P\to\infty} \frac{1-e^{-(s-a)P}}{s-a} \\
&= \frac{1}{s-a} \quad \text{if } s > a
\end{aligned}$$

Problem 1.4.2. Prove that

(a) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$

(b) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$

Solution.

(a)

$$\begin{aligned}
\mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st}\sin at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\int_0^P e^{-st}\sin at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\right|_0^P \\
&= \lim_{P\to\infty}\left\{\frac{a}{s^2+a^2} - \frac{e^{-sP}(a\sin aP + a\cos aP)}{s^2+a^2}\right\} \\
&= \frac{a}{s^2+a^2} \qquad \text{if } s>0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st}\cos at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\int_0^P e^{-st}\cos at \,\mathrm{d}\,t \\
&= \lim_{P\to\infty} \left.\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right|_0^P \\
&= \lim_{P\to\infty}\left\{\frac{s}{s^2+a^2} - \frac{e^{-sP}(a\cos aP - a\sin aP)}{s^2+a^2}\right\} \\
&= \frac{s}{s^2+a^2} \qquad \text{if } s>0
\end{aligned}$$

We have used here the results

$$\begin{aligned}
\int e^{\alpha t}\sin\beta t \,\mathrm{d}\,t &= \frac{e^{at}(\alpha\sin\beta t - \beta\cos\beta t)}{\alpha^2+\beta^2} \\
\int e^{\alpha t}\cos\beta t \,\mathrm{d}\,t &= \frac{e^{at}(\alpha\cos\beta t + \beta\sin\beta t)}{\alpha^2+\beta^2}
\end{aligned}$$

1.4.2 Translation and Change of Scale Properties

Problem 1.4.3. Prove the first translation or shifting property: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$.

Solution. We have, $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,\mathrm{d}\,t$
$= f(s)$

Then

$$\begin{aligned}
\mathcal{L}\left\{e^{at}F(t)\right\} &= \int_0^\infty e^{-st}\left\{e^{at}F(t)\right\}\,\mathrm{d}\,t \\
&= \int_0^\infty e^{-(s-a)t}F(t)\,\mathrm{d}\,t \\
&= f(s-a)
\end{aligned}$$

Problem 1.4.4. Find

(a) $\mathcal{L}\{t^2e^{3t}\}$

(b) $\mathcal{L}\{e^{-2t}\sin 4t\}$

(c) $\mathcal{L}\{e^{4t}\cosh 5t\}$

(d) $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

Solution.

(a) $\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$. Then $\mathcal{L}\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}$.

(b) $\mathcal{L}\{\sin 4t\} = \frac{4}{s^2+16}$. Then $\mathcal{L}\{e^{-2t}\sin 4t\} = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+20}$.

(c) $\mathcal{L}\{\cosh 5t\} = \frac{s}{s^2-25}$. Then $\mathcal{L}\{e^{4t}\cosh 5t\} = \frac{s-4}{(s-4)^2-25} = \frac{s-4}{s^2-8s-9}$.

Another method.

$$\begin{aligned}
\mathcal{L}\left\{e^{4t}\cosh 5t\right\} &= \mathcal{L}\left\{e^{4t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\right\} \\
&= \frac{1}{2}\mathcal{L}\left\{e^{9t}+e^{-t}\right\} \\
&= \frac{1}{2}\left\{\frac{1}{s-9}+\frac{1}{s+1}\right\} \\
&= \frac{s-4}{s^2-8s-9}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{L}\{3\cos 6t - 5\sin 6t\} &= 3\mathcal{L}\{\cos 6t\} - 5\mathcal{L}\{\sin 6t\} \\
&= 3\left(\frac{s}{s^2+36}\right) - 5\left(\frac{6}{s^2+36}\right) \\
&= \frac{3s-30}{s^2+36}
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{L}\left\{e^{-2t}(3\cos 6t - 5\sin 6t)\right\} &= \frac{3(s+2)-30}{(s+2)^2+36} \\
&= \frac{3s-24}{s^2+4s+40}
\end{aligned}$$

Problem 1.4.5. Prove the second translation or shifting property:
If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then $\mathcal{L}\{G(t)\} = e^{-as}f(s)$.

Solution.

$$\begin{aligned}
\mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st}G(t)\,\mathrm{d}\,t \\
&= \int_0^a e^{-st}G(t)\,\mathrm{d}\,t + \int_a^\infty e^{-st}G(t)\,\mathrm{d}\,t \\
&= \int_0^a e^{-st}(0)\,\mathrm{d}\,t + \int_a^\infty e^{-st}F(t-a)\,\mathrm{d}\,t \\
&= \int_a^\infty e^{-st}F(t-a)\,\mathrm{d}\,t \\
&= \int_a^\infty e^{-s(u+a)}F(u)\,\mathrm{d}\,u \\
&= e^{-as}\int_a^\infty e^{-su}F(u)\,\mathrm{d}\,u \\
&= e^{-as}f(s)
\end{aligned}$$

Where we have used the substitution $t = u + a$.

Problem 1.4.6. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$.

Solution.

Method 1.

$$\begin{aligned}
\mathcal{L}\{F(t)\} &= \int_0^{\frac{2\pi}{3}} e^{-st}(0)\,\mathrm{d}\,t + \int_{\frac{2\pi}{3}}^{\infty} e^{-st}\cos\left(t-\frac{2\pi}{3}\right)\,\mathrm{d}\,t \\
&= \int_0^{\infty} e^{-s\left(u+\frac{2\pi}{3}\right)}\cos u\,\mathrm{d}\,u \\
&= e^{-\frac{2\pi}{3}}\int_0^{\infty} e^{-su}\cos u\,\mathrm{d}\,u \\
&= \frac{se^{-\frac{2\pi}{3}}}{s^2+1}
\end{aligned}$$

Method 2. Since $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, it follows from second translation property, with $a = \frac{2\pi}{3}$, that

$$\mathcal{L}\{F(t)\} = \frac{se^{-\frac{2\pi}{3}}}{s^2+1}$$

1.4.3 Laplace Transform of Derivatives

Problem 1.4.7. Prove Theorem 1.3.4: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F'(t)\} = s\,f(s) - F(0)$.

Solution. Using integration by parts, we have

$$\begin{aligned}
\mathcal{L}\{F'(t)\} &= \int_0^{\infty} e^{-st}F'(t)\,\mathrm{d}\,t - \lim_{P\to\infty}\int_0^{P} e^{-st}F'(t)\,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\left\{ e^{-st}F(t)\Big|_0^P + s\int_0^P e^{-st}F(t)\,\mathrm{d}\,t \right\} \\
&= \lim_{P\to\infty}\left\{ e^{-sP}F(P) - F(0) + s\int_0^P e^{-st}F(t)\,\mathrm{d}\,t \right\} \\
&= s\int_0^{\infty} e^{-st}F(t)\,\mathrm{d}\,t - F(0) \\
&= s\,f(s) - F(0)
\end{aligned}$$

Using the fact that $F(t)$ is of exponential order γ as $t \to \infty$, so that $\lim_{P\to\infty} e^{-sP}F(P) = 0$ for $s > \gamma$.

Problem 1.4.8. Prove Theorem 1.3.5: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2\,f(s) - s\,F(0) - F'(0)$.

Solution. By Problem 1.4.3,

$$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0) = s\,g(s) - G(0)$$

Let $G(t) = F'(t)$. Then

$$\begin{aligned}
\mathcal{L}\{F''(t)\} &= s\mathcal{L}\{F'(t)\} - F'(0) \\
&= s\left[s\mathcal{L}\{F(t)\} - F(0)\right] - F'(0) \\
&= s^2\mathcal{L}\{F(t)\} - s\,F(0) - F'(0) \\
&= s^2 f(s) - s\,F(0) - F'(0)
\end{aligned}$$

The generalization to higher derivatives can be proved by using mathematical induction.

1.4.4 Multiplication By Powers of t

Problem 1.4.9. Prove Theorem 1.3.7: If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\,s^n} f(s) = (-1)^n f^{(n)}(s) \qquad \text{where } n = 1, 2, 3, \ldots$$

Solution. We have,

$$f(s) = \int_0^\infty e^{-st} F(t)\,\mathrm{d}\,t$$

Then by Leibniz's rule for differentiating under the integral sign,

$$\begin{aligned}\frac{\mathrm{d}\,f}{\mathrm{d}\,s} = f'(s) = \frac{\mathrm{d}}{\mathrm{d}\,s}\int_0^\infty e^{-st} F(t)\,\mathrm{d}\,t &= \int_0^\infty \frac{\partial}{\partial s} e^{-st} F(t)\,\mathrm{d}\,t \\ &= \int_0^\infty -te^{-st} F(t)\,\mathrm{d}\,t \\ &= -\int_0^\infty e^{-st}\left\{tF(t)\right\}\,\mathrm{d}\,t \\ &= -\mathcal{L}\left\{tF(t)\right\}\end{aligned}$$

Thus,

$$\mathcal{L}\left\{tF(t)\right\} = -\frac{\mathrm{d}\,f}{\mathrm{d}\,s} = -f'(s) \tag{1.2}$$

which proves the theorem for $n = 1$.

To establish the theorem in general, we use mathematical induction. Assume the theorem is true for $n = k$, i.e., assume

$$\int_0^\infty e^{-st}\left\{t^k F(t)\right\}\,\mathrm{d}\,t = (-1)^k f^{(k)}(s) \tag{1.3}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\int_0^\infty e^{-st}\left\{t^k F(t)\right\}\,\mathrm{d}\,t = (-1)^k f^{(k+1)}(s)$$

or by Leibniz's rule,

$$-\int_0^\infty e^{-st}\left\{t^{k+1} F(t)\right\}\,\mathrm{d}\,t = (-1)^k f^{(k+1)}(s)$$

i.e.,

$$\int_0^\infty e^{-st}\left\{t^{k+1} F(t)\right\}\,\mathrm{d}\,t = (-1)^{k+1} f^{(k+1)}(s) \tag{1.4}$$

It follows that if (1.3) is true, i.e., if the theorem holds for $n = k$, then (1.4) is true, i.e., the theorem holds for $n = k + 1$. But by (1.2) the theorem is true for $n = 1$. Hence, it is true for $n = 1 + 1 = 2$ and $n = 2 + 1 = 3$, etc., and thus for all positive integer values of n.

Problem 1.4.10. Find

(a) $\mathcal{L}\left\{t\sin at\right\}$

(b) $\mathcal{L}\left\{t^2\cos at\right\}$

Solution.

(a) Since $\mathcal{L}\left\{\sin at\right\} = \frac{a}{s^2+a^2}$, we have by multiplication by the powers of t

$$\mathcal{L}\left\{t\sin at\right\} = -\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

Another method
Since $\mathcal{L}\left\{\cos at\right\} = \int_0^\infty e^{-st}\cos at\,\mathrm{d}\,t = \dfrac{s}{s^2+a^2}$
We have by differentiating with respect to the parameter a [using Leibniz's rule],

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,a}\int_0^\infty e^{-st}\cos at\,\mathrm{d}\,t &= \int_0^\infty e^{-st}\left\{-t\sin at\right\}\,\mathrm{d}\,t = -\mathcal{L}\left\{t\sin at\right\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\,a}\left(\frac{s}{s^2+a^2}\right) = -\frac{2as}{(s^2+a^2)^2}\end{aligned}$$

from which

$$\mathcal{L}\left\{t\sin at\right\} = \frac{2as}{(s^2+a^2)^2}$$

Note that the result is equivalent to $\frac{\mathrm{d}}{\mathrm{d}\,a}\mathcal{L}\left\{\cos at\right\} = \mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}\,a}\cos at\right\}$.

(b) Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, we have by multiplication by the powers of t

$$\mathcal{L}\left\{t^2 \cos at\right\} = -\frac{\mathrm{d}^2}{\mathrm{d}\, s^2}\left(\frac{s}{s^2+a^2}\right) = \frac{2s^3-6a^2s}{(s^2+a^2)^3}$$

We can also use the second method of part (a) by writing

$$\mathcal{L}\left\{t^2 \cos at\right\} = \mathcal{L}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}\, a^2}\cos at\right\} = -\frac{\mathrm{d}^2}{\mathrm{d}\, a^2}\mathcal{L}\{\cos at\}$$

which gives the same result.

1.4.5 Evaluation of Integral

Problem 1.4.11. Evaluate

(a) $\displaystyle\int_0^\infty te^{-2t}\cos t\,\mathrm{d}\,t$,

(b) $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,\mathrm{d}\,t$

Solution. (a) By multiplication by the powers of t,

$$\begin{aligned}\mathcal{L}\{t\cos t\} &= \int_0^\infty te^{-st}\cos t\,\mathrm{d}\,t \\ &= -\frac{\mathrm{d}}{\mathrm{d}\,s}\mathcal{L}\{\cos t\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\frac{s}{s^2+1}\right) \\ &= \frac{s^2-1}{(s^2+1)^2}\end{aligned}$$

Then letting $s=2$, we find

$$\int_0^\infty te^{-2t}\cos t\,\mathrm{d}\,t = \frac{3}{25}$$

(b) If $F(t) = e^{-t}-e^{-3t}$, then

$$f(s) = \mathcal{L}\{F(t)\} = \frac{1}{s+1} - \frac{1}{s+3}$$

Thus by division by powers of t,

$$\begin{aligned}\mathcal{L}\left\{\frac{e^{-t}-e^{-3t}}{t}\right\} &= \int_0^\infty \left\{\frac{1}{u+1}-\frac{1}{u+3}\right\}\mathrm{d}\,u \\ \Rightarrow \int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-3t}}{t}\right)\mathrm{d}\,t &= \ln\left(\frac{s+3}{s+1}\right)\end{aligned}$$

Taking the limit as $s\to 0^+$, we find

$$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,\mathrm{d}\,t = \ln 3$$

Chapter 2

Inverse Laplace Transform

2.1 Definition of Inverse Laplace Transform

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e., if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace Transform of $f(s)$, and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator.

Example. Since $\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$ we can write

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

2.2 Some Inverse Laplace Transforms

Here is a table of some inverse Laplace transforms

$\mathbf{f(s)}$	$\boldsymbol{\mathcal{L}^{-1}}\{\mathbf{f(s)}\} = \mathbf{F(t)}$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s^{n+1}}, n = 0, 1, 2, \ldots$	$\frac{t^n}{n!}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
$\frac{s}{s^2-a^2}$	$\cosh at$

2.3 Properties

1. Linearity property

   Theorem 2.3.1. If c_1 and c_2 are any constants while $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$$\begin{aligned}\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} &= c_1\mathcal{L}^{-1}\{f_1(s)\} + c_2\mathcal{L}^{-1}\{f_2(s)\} \\ &= c_1 F_1(s) + c_2 F_2(t)\end{aligned}$$

This result easily extended to more than two functions.

Example.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}\right\} &= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}+5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\\ &= 4e^{2t}-3\cos 4t+\frac{5}{2}\sin 2t\end{aligned}$$

Because of this property we can say that $\mathcal{L}^{-1}$ is a *linear operator* or that it has the *linearity property.*

2. First translation or shifting property

   Theorem 2.3.2. If $\mathcal{L}^{-1}\{f(s)\}=F(t)$, then

   $$\mathcal{L}^{-1}\{f(s-a)\}=e^{at}F(t)$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}=\frac{1}{2}\sin 2t$, we have

   $$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}=\frac{1}{2}e^t\sin 2t$$

3. Second translation or shifting property

   Theorem 2.3.3. If $\mathcal{L}^{-1}\{f(s)\}=F(t)$, then

   $$\mathcal{L}^{-1}\left\{e^{-as}f(s)\right\}=\begin{cases}F(t-a) & t>a\\ 0 & t<a\end{cases}$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$, we have

   $$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/3}}{s^2+1}\right\}=\begin{cases}\sin(t-\pi/3) & \text{if } t>\pi/3\\ 0 & \text{if } t<\pi/3\end{cases}$$

4. Change of scale property

   Theorem 2.3.4. If $\mathcal{L}^{-1}\{f(s)\}=F(t)$ then,

   $$\mathcal{L}^{-1}\{f(ks)\}=\frac{1}{k}F\left(\frac{t}{k}\right)$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}=\cos 4t$, we have

   $$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\}=\frac{1}{2}\cos\frac{4t}{2}=\frac{1}{2}\cos 2t$$

   as is verified directly.

5. Inverse Laplace transform of derivatives

   Theorem 2.3.5. If $\mathcal{L}^{-1}\{f(s)\}=F(t)$ then,

   $$\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\}=\mathcal{L}^{-1}\left\{\frac{\mathrm{d}^n}{\mathrm{d}s^n}f(s)\right\}=(-1)^n t^n F(t)$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$ and $\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right)=\frac{-2s}{(s^2+1)^2}$, we have

   $$\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}=-t\sin t \qquad \text{or} \qquad \mathcal{L}^{-1}\left\{\frac{s}{(s^1+1)^2}\right\}=\frac{1}{2}t\sin t$$

6. Inverse Laplace transform of integrals

   Theorem 2.3.6. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

$$\mathcal{L}^{-1}\left\{\int_s^\infty f(u)\,\mathrm{d}\,u\right\} = \frac{F(t)}{t}$$

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$, we have

$$\mathcal{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)\mathrm{d}\,u\right\} = \mathcal{L}^{-1}\left\{\ln\left(1 + \frac{1}{s}\right)\right\} = \frac{1 - e^{-t}}{t}$$

7. Multiplication by s^n

   Theorem 2.3.7. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t)$$

   Thus multiplication by s has the effect of differentiating $F(t)$.
   If $F(0) \neq 0$, then

$$\mathcal{L}^{-1}\{sf(s) - F(0)\} = F'(t)$$

   or,

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t) + F(0)\delta(t)$$

   where $\delta(t)$ is the Dirac delta function or unit impulse function.

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $\sin 0 = 0$, then

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{\mathrm{d}}{\mathrm{d}\,t}(\sin t) = \cos t$$

   Generalizations to $\mathcal{L}^{-1}\{s^n f(s)\}$, $n = 2, 3, \ldots$ are possible

8. Division by s

   Theorem 2.3.8. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then,

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,\mathrm{d}\,u$$

   Thus division by s (or multiplication by $1/s$) has the effect of integrating $F(t)$ from 0 to t.

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u\,\mathrm{d}\,u = \frac{1}{4}(1 - \cos 2t)$$

   Generalizations to $\mathcal{L}^{-1}\{f(s)/s^n\}$, $n = 2, 3, \ldots$ are possible

9. The convolution property

   Theorem 2.3.9. If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$ then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G.$$

   We call $F * G$ the convolution or faulting of F and G and the theorem is called the convolution theorem or property.

   Example. Since $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \int_0^t e^u e^{2(t-u)}\,\mathrm{d}\,u = e^{2t} - e^t$$

Problem 2.3.1. Prove $\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = (-1)^n t^n F(t)$, $n = 1, 2, 3, \ldots$

Proof. Since $\mathcal{L}\left\{t^n F(t)\right\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\, s^n} f(s) = (-1)^n f^{(n)}(s)$ we have

$$\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = (-1)^n t^n F(t)$$

□

Problem 2.3.2. Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

Solution. We have

$$\frac{\mathrm{d}}{\mathrm{d}\, s}\left\{\frac{1}{s^2+a^2}\right\} = \frac{-2s}{(s^2+a^2)^2}$$

Thus

$$\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{1}{s^2+a^2}\right)$$

Then since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$, we have by property of inverse Laplace transform of derivatives

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{1}{s^2+a^2}\right)\right\}\\ &= \frac{1}{2}t\left(\frac{\sin at}{a}\right)\\ &= \frac{t\sin at}{2a}\end{aligned}$$

Another Method.

Differentiating with respect to the parameter a, we find

$$\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{s}{s^2+a^2}\right) = \frac{-2as}{(s^2+a^2)^2}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{s}{s^2+a^2}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\, s}\left\{\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}\right\} = -2a\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

i.e.,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}\, a}(\cos at) = -\frac{1}{2a}(-t\sin at) = \frac{t\sin at}{2a}$$

Problem 2.3.3. Find $\mathcal{L}^{-1}\left\{\ln\left(1+\dfrac{1}{s^2}\right)\right\}$.

Solution. Let $f(s) = \ln\left(1+\dfrac{1}{s^2}\right) = \mathcal{L}\left\{F(t)\right\}$.
Then $f'(s) = \dfrac{-2}{s(s^2+1)} = -2\left\{\dfrac{1}{s} - \dfrac{s}{s^2+1}\right\}$.
Thus, since $\mathcal{L}^{-1}\left\{f'(s)\right\} = -2(1-\cos t) = -tF(t)$, $F(t) = \frac{2(1-\cos t)}{t}$.

2.4 The Convolution Theorem

The convolution theorem can be used to solved integral and integral-differential equations.

Theorem 2.4.1. If $\mathcal{L}^{-1}\left\{f(s)\right\} = F(t)$ and $\mathcal{L}^{-1}\left\{g(s)\right\} = G(t)$ then

$$\mathcal{L}^{-1}\left\{f(s)g(s)\right\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\, u = F * G.$$

We call $F * G$ the convolution or faulting of F and G and the theorem is called the convolution theorem. [Here, $*$ (asterisk) denotes convolution in this context, not standard multiplication.]

The formulation is especially useful for implementing a numerical convolution on a computer. The standard convolution algorithm has quadratic computational complexity. With the help of convolution theorem and the fast Fourier transform the complexity of the convolution can be reduced from $O(n^2)$ to $O(n \log n)$.

Problem 2.4.1. Prove the convolution theorem:
If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$ then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G.$$

Proof. The required result follows if we can prove that

$$\mathcal{L}\left\{\int_0^t F(u)G(t-u)\,\mathrm{d}\,u\right\} = f(s)g(s) \tag{2.1}$$

Where,

$f(s) = \mathcal{L}\{F(t)\}$ and
$g(s) = \mathcal{L}\{G(t)\}$

To show this we note the left side of (2.1) is

$$\begin{aligned}&\int_{t=0}^{\infty} e^{-st}\left\{\int_{u=0}^{t} F(u)G(t-u)\,\mathrm{d}\,u\right\}\mathrm{d}\,t\\ &= \int_{t=0}^{\infty}\int_{u=0}^{\infty} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t\\ &= \lim_{M\to\infty} s_M\end{aligned}$$

where,

$$s_M = \int_{t=0}^{M}\int_{u=0}^{t} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t \tag{2.2}$$

The region in the tu plane over which the integration (2.2) is performed is shown shaded in figure 2.1.

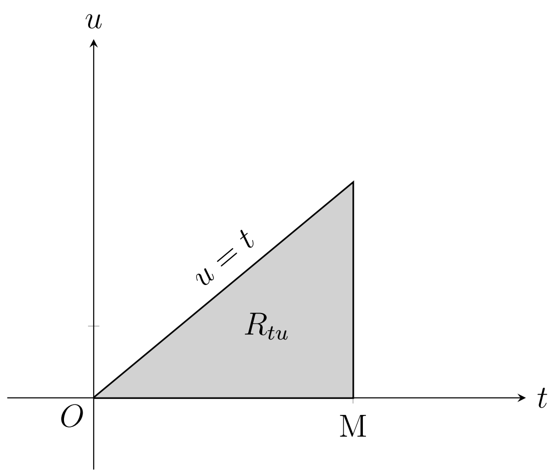

Figure 2.1:

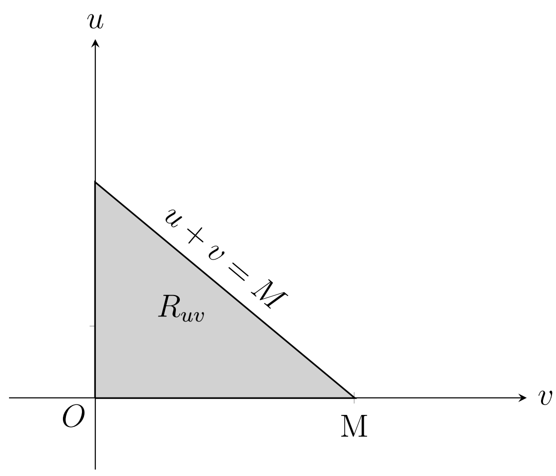

Figure 2.2:

Let, $t - u = v$ or $t = u + v$, the shaded region R_{tu} of the tu plane is transformed into the shaded region R_{uv} of the uv plane shown in figure 2.2. Then by a theorem on transformation on multiple integral,
We have

$$\begin{aligned}s_M &= \iint\limits_{R_{tu}} e^{-st}F(u)G(t-u)\,\mathrm{d}\,u\,\mathrm{d}\,t\\ &= \iint\limits_{R_{uv}} e^{-s(u+v)}F(u)G(v)\left|\frac{\partial(u,t)}{\partial(u,v)}\right|\mathrm{d}\,u\,\mathrm{d}\,v \qquad (2.3)\end{aligned}$$

where the Jacobian of the transformation is

$$J = \frac{\partial(u,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Thus, the right side of (2.3) is,

$$s_M = \int_{v=0}^{M} \int_{u=0}^{M} e^{-s(u+v)} F(u)G(v)\,\mathrm{d}\,u\,\mathrm{d}\,v \tag{2.4}$$

Let us define a function

$$k(u,v) = \begin{cases} e^{-s(u+v)} F(u)G(v) & \text{if } u+v \le M \\ 0 & \text{if } u+v > M \end{cases} \tag{2.5}$$

This function is defined over the square of figure 2.3 but as indicated in (2.5), is zero over

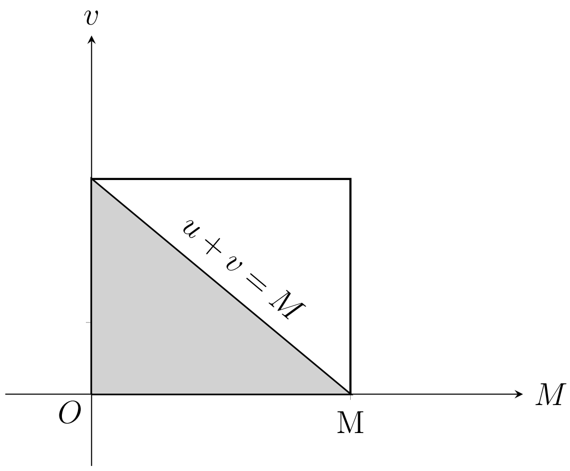

Figure 2.3:

the unshaded portion of the square. In terms of this new function we can write (2.4) as,

$$s_M = \int_{v=0}^{M} \int_{u=0}^{M} k(u,v)\,\mathrm{d}\,u\,\mathrm{d}\,v$$

Then,

$$\begin{aligned}
\lim_{M\to\infty} s_M &= \int_0^\infty \int_0^\infty k(u,v)\,\mathrm{d}\,u\,\mathrm{d}\,v \\
&= \int_0^\infty \int_0^\infty e^{-s(u+v)} F(u)G(v)\,\mathrm{d}\,u\,\mathrm{d}\,v \\
&= \left\{ \int_0^\infty e^{-su} F(u)\,\mathrm{d}\,u \right\} \left\{ \int_0^\infty e^{-sv} G(v)\,\mathrm{d}\,v \right\} \\
&= f(s)g(s)
\end{aligned}$$

Which establishes the theorem.

We call $\int_0^t F(u)G(t-u)\,\mathrm{d}\,u = F * G$ the convolution integral or convolution of F and G. □

Problem 2.4.2. Prove that $F * G = G * F$.

Proof. Letting $t - u = v$ or $u = t - v$ we have

$$\begin{aligned}
F * G &= \int_0^t F(u)G(t-u)\,\mathrm{d}\,u \\
&= \int_0^t F(t-v)G(v)\,\mathrm{d}\,v \\
&= \int_0^t G(v)F(t-v)\,\mathrm{d}\,v \\
&= G * F
\end{aligned}$$

This shows that convolution of F and G obeys the commutative law of algebra. It also obeys the associative law and distributive law. □

Problem 2.4.3. Evaluate each of the following by the use of the convolution theorem

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{\left(s^2+a^2\right)^2}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2\left(s+1\right)^2}\right\}$

Solution. (a) We can write

$$\frac{s}{\left(s^2+a^2\right)^2}=\frac{s}{s^2+a^2}\times\frac{1}{s^2+a^2}$$

Now,

$$\frac{s}{s^2+a^2}=\cos at \qquad \text{and}$$
$$\frac{1}{s^2+a^2}=\frac{\sin at}{a}$$

By the convolution theorem we get,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\} &= \int_0^t \cos au\,\frac{\sin a(t-u)}{a}\,\mathrm{d}\,u \\
&= \frac{1}{a}\int_0^t (\cos^2 au)(\sin at\cos au-\cos at\sin au)\,\mathrm{d}\,u \\
&= \frac{1}{a}\sin at\int_0^t \cos^2 au\,\mathrm{d}\,u-\frac{1}{a}\cos at\int_0^t \sin au\cos au\,\mathrm{d}\,u \\
&= \frac{1}{a}\sin at\int_0^t \frac{1+\cos 2au}{2}\,\mathrm{d}\,u-\frac{1}{a}\cos at\int_0^t \frac{\sin 2au}{2}\,\mathrm{d}\,u \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right)-\frac{1}{a}\cos at\left(\frac{1-\cos 2at}{4a}\right) \\
&= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin at\cos at}{2a}\right)-\frac{1}{a}\cos at\left(\frac{\sin^2 at}{2a}\right) \\
&= \frac{2\sin at}{2a}
\end{aligned}$$

(b) We have,

$$\frac{1}{s^2}=t \qquad \text{and}$$
$$\frac{1}{(s+1)^2}=te^{-t}$$

By the convolution theorem we get,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s+1\right)^2}\right\} &= \int_0^t ue^{-u}(t-u)\,\mathrm{d}\,u \\
&= \int_0^t \left(ut-u^2\right)e^{-u}\,\mathrm{d}\,u \\
&= \left(ut-u^2\right)\left(-e^{-u}\right)-(t-2u)\left(e^{-u}\right)+(-2)\left(-e^{-u}\right)\Big|_0^t \\
&= te^{-t}+2e^{-t}+t-2
\end{aligned}$$

Chapter 3

Applications Of Laplace Transform

3.1 Applications To Differential Equations

3.1.1 Ordinary Differential Equations With Constant Coefficients

The Laplace transform is useful in solving linear ordinary differential equations with constant coefficients. For example, suppose we wish to solve the second order linear differential equation

$$\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}Y}{\mathrm{d}t} + \beta Y = F(t) \qquad \text{or} \qquad Y'' + \alpha Y' + \beta Y = F(t) \tag{3.1}$$

where α and β are constants, subject to the initial or boundary conditions

$$Y(0) = A, \quad Y'(0) = B \tag{3.2}$$

where A and B are given constants. On taking the Laplace transform of both sides of (3.1) and using (3.2), we obtain an algebraic equation for determination of $\mathcal{L}\{Y(t)\} = y(s)$. The required solution is then obtained by finding the inverse Laplace transform of $y(s)$. The method is easily extended to higher order differential equations.

Problem 3.1.1. Solve $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$.

Solution. Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$\begin{aligned}
&\mathcal{L}\{Y''\} + \mathcal{L}\{Y\} = \mathcal{L}\{t\} \\
\Rightarrow &s^2y - sY(0) - Y'(0) + y = \frac{1}{s^2} \\
\Rightarrow &s^2y - s - 2 + y = \frac{1}{s^2}
\end{aligned}$$

Then

$$\begin{aligned}
y = \mathcal{L}\{Y\} &= \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1} \\
&= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1} \\
&= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}
\end{aligned}$$

and

$$Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\} = t + \cos t - 3\sin t$$

Check: $Y = t + \cos t - 3\sin t$, $Y' = 1 - \sin t - 3\cos t$, $Y'' = -\cos t + 3\sin t$. Then $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$ and the function obtained is the required solution.

Problem 3.1.2. Solve $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$.

Solution. We have,

$$\begin{aligned}
&\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\left\{e^{2t}\right\} \\
\Rightarrow&\{s^2y - sY(0) - Y'(0)\} - 3\{sy - Y(0)\} + 2y = \frac{4}{s-2} \\
\Rightarrow&\{s^2y + 3s - 5\} - 3\{sy + 3\} + 2y = \frac{4}{s-2} \\
\Rightarrow&(s^{-}3s + 2)y + 3s - 14 = \frac{4}{s-2} \\
\Rightarrow&y = \frac{4}{(s^2 - 3s + 2)(s-2)} + \frac{14 - 3s}{s^2 - 3s + 12} \\
\Rightarrow&y = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} \\
\Rightarrow&y = \frac{-7}{s-1}\frac{4}{s-2} + \frac{4}{(s-2)^2}
\end{aligned}$$

Thus,

$$Y = \mathcal{L}\left\{\frac{-7}{s-1}\frac{4}{s-2} + \frac{4}{(s-2)^2}\right\} = -7e^t + 4e^{2t} + 4te^{2t}$$

which can be verified as the solution.

3.1.2 Ordinary Differential Equations With Variable Coefficients

The Laplace transform can also be used in solving some ordinary differential equations in which the coefficients are variable. A particular differential equation where the method proves useful is one in which the terms have the form

$$t^m Y^{(n)}(t)$$

the Laplace transform of which is

$$(-1)^m \frac{\mathrm{d}^m}{\mathrm{d}\,s^m}\mathcal{L}\left\{Y^{(n)}(t)\right\}$$

Problem 3.1.3. Solve $tY'' + Y' + 4tY = 0$, $Y(0) = 3$, $Y'(0) = 0$.

Solution. We have,

$$\mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} + \mathcal{L}\{4tY\} = 0$$

or,

$$-\frac{\mathrm{d}}{\mathrm{d}\,s}\left\{s^2y - sY(0) - Y'(0)\right\} + \{sy - Y(0)\} - 4\frac{\mathrm{d}\,y}{\mathrm{d}\,s} = 0$$

i.e.,

$$(s^2 + 4)\frac{\mathrm{d}\,y}{\mathrm{d}\,s} + sy = 0$$

Then

$$\frac{\mathrm{d}\,y}{y} + \frac{s\,\mathrm{d}\,s}{s^2 + 4} = 0$$

and integrating

$$\ln y + \frac{1}{2}\ln(s^2 + 4) = c_1 \quad \text{or,} \quad y = \frac{c}{\sqrt{s^2 + 4}}$$

Inverting, we find

$$Y = cJ_0(2t)$$

To determine c we note that $Y(0) = cJ_0(0) = c = 3$. Thus,

$$Y = 3J_0(2t)$$

Problem 3.1.4. Solve $tY'' + 2Y' + tY = 0$, $Y(0+) = 1$, $Y'(\pi) = 0$.

Solution. Let $Y(0+) = c$. Then taking the Laplace transform of each term

$$-\frac{\mathrm{d}}{\mathrm{d}\,s}\{s^2y - sY(0+) - Y'(0+)\} + 2\{sy - Y(0+)\} - \frac{\mathrm{d}}{\mathrm{d}\,s}y = 0$$

or

$$-s^2y' - 2sy + 1 + 2sy - 2 - y' = 0$$

i.e.,

$$-(s^2+1)y' - 1 = 0 \quad \text{or} \quad y' = \frac{-1}{s^2+1}$$

Integrating

$$y = -\tan^{-1} s + A$$

Since $y \to 0$ as $s \to \infty$, we must have $A = \pi/2$. Thus,

$$y = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1}\frac{1}{s}$$

Then,

$$Y = \mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} = \frac{\sin t}{t}$$

3.1.3 Partial Differential Equations

Problem 3.1.5. Given the function $U(x,t)$ defined for $a \le x \le b$, $t > 0$. Find

(a) $\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} = \int_0^\infty e^{-st}\frac{\partial U}{\partial t}\,\mathrm{d}\,t$

(b) $\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial U}{\partial x}\,\mathrm{d}\,t$

assuming suitable restrictions on $U = U(x,t)$.

Solution.

(a) Integrating by parts, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} &= \int_0^\infty e^{-st}\frac{\partial U}{\partial t}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\int_0^P e^{-st}\frac{\partial U}{\partial t}\,\mathrm{d}\,t \\
&= \lim_{P\to\infty}\left\{e^{-st}U(x,t)\Big|_0^P + s\int_0^P e^{-st}U(x,t)\,\mathrm{d}\,t\right\} \\
&= s\int_0^\infty e^{-st}U(x,t)\,\mathrm{d}\,t - U(x,0) \\
&= su(x,s) - U(x,0) \\
&= su - U(x,0)
\end{aligned}$$

where $u = u(x,s) = \mathcal{L}\{U(x,t)\}$.

We have assumed that $U(x,t)$ satisfies the restrictions of sectionally continuous in finite interval, when regraded as a function of t.

(b) We have, using Leibniz's rule for differentiating under the integral sign,

$$\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial U}{\partial x}\,\mathrm{d}\,t = \frac{\mathrm{d}}{\mathrm{d}\,x}\int_0^\infty e^{-st}U\,\mathrm{d}\,t = \frac{\mathrm{d}\,u}{\mathrm{d}\,x}$$

Problem 3.1.6. Referring to problem 3.1.5, show that

(a) $\mathcal{L}\left\{\dfrac{\partial^2 U}{\partial t^2}\right\} = s^2 u(x,s) - sU(x,0) - U_t(x,0)$

(b) $\mathcal{L}\left\{\dfrac{\partial^2 U}{\partial x^2}\right\} = \dfrac{\mathrm{d}^2 u}{\mathrm{d}\, x^2}$

where $U_t(x,0) = \left.\dfrac{\partial U}{\partial t}\right|_{t=0}$ and $u = u(x,s) = \mathcal{L}\{U(x,t)\}$.

Solution. Let $V = \partial U/\partial t$. Then as in part (a) of Problem 3.1.5, we have

$$\begin{aligned}\mathcal{L}\left\{\frac{\partial^2 U}{\partial t^2}\right\} = \mathcal{L}\left\{\frac{\partial V}{\partial t}\right\} &= s\mathcal{L}\{V\} - V(x,0)\\ &= s\left[s\mathcal{L}\{U\} - U(x,0)\right] - U_t(x,0)\\ &= s^2 u - sU(x,0) - U_t(x,0)\end{aligned}$$

Problem 3.1.7. Find the solution of

$$\frac{\partial\, U}{\partial\, x} = 2\frac{\partial U}{\partial t} + U, \qquad U(x,0) = 6e^{-3x}$$

which is bounded for $x > 0$, $t > 0$.

Solution. Taking the Laplace transform of the given partial differential equation with respect to t and using Problem 3.1.5, we find

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, x} = 2\{su - U(x,0)\} + u$$

or,

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, x} - (2s+1)u = -12e^{-3x} \tag{3.3}$$

from the given boundary conditions. Note that the Laplace transformation has transformed the partial differential equation into an ordinary differential equation (3.3).

To solve (3.3) multiply both sides by the integrating factor $e^{\int -(2s+1)\,\mathrm{d}\, x} = e^{-(2s+1)x}$. Then (3.3) can be written

$$\frac{\mathrm{d}}{\mathrm{d}\, x}\left\{ue^{-(2s+1)x}\right\} = -12e^{-(2x+4)x}$$

Integration yields

$$ue^{-(2s+1)x} = \frac{6}{s+2}e^{-(2s+4)x} + c \qquad \text{or,} \qquad u = \frac{6}{s+2}e^{-3x} + ce^{(2s+1)x}$$

Now since $U(x,t)$ must be bounded as $x \to \infty$, we must have $u(x,s)$ also bounded as $x \to \infty$ and it follows that we must choose $c = 0$. Then

$$u = \frac{6}{s+2}e^{-3x}$$

and so, on taking the inverse, we find

$$U(x,t) = 6e^{-2t-3x}$$

This is easily checked as the required solution.

Problem 3.1.8. Solve $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}$, $U(x,0) = 3\sin 2\pi x$, $U(0,t) = 0$, $U(1,t) = 0$ where $0 < x < 1$, $t > 0$.

Solution. Taking the Laplace transform of the partial differential equation using Problem 3.1.5 and 3.1.6, we find

$$su - U(x,0) = \frac{\mathrm{d}^2 u}{\mathrm{d}\,x^2} \qquad \text{or} \qquad \frac{\mathrm{d}^2 u}{\mathrm{d}\,x^2} - su = -3\sin 2\pi x \tag{3.4}$$

where $u = u(x,s) = \mathcal{L}\{U(x,t)\}$. The general solution of (3.4) in

$$u = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{3}{s+4\pi^2}\sin 2\pi x \tag{3.5}$$

Taking the Laplace transform of those boundary conditions which involve t, we have

$$\mathcal{L}\{U(0,t)\} = u(0,s) = 0 \qquad \text{and} \qquad \mathcal{L}\{U(1,t)\} = u(1,s) = 0 \tag{3.6}$$

Using the first condition $[u(0,s) = 0]$ of (3.6) in (3.5), we have

$$c_1 + c_2 = 0 \tag{3.7}$$

Using the second condition $[u(1,s) = 0]$ of (3.6) in (3.5), we have

$$c_1 e^{\sqrt{s}} + c_2 e^{-\sqrt{s}} = 0 \tag{3.8}$$

From (3.7) and (3.8) we find $c_1 = 0$, $c_2 = 0$ and so (3.5) becomes

$$u = \frac{3}{s+4\pi^2}\sin 2\pi x \tag{3.9}$$

from which we obtain on inversion

$$U(x,t) = 3e^{-4\pi^2 t}\sin 2\pi x \tag{3.10}$$

This problem has an interesting physical interpretation. If we consider a solid bounded by the infinite plane faces $x = 0$ and $x = 1$, the equation

$$\frac{\partial U}{\partial t} = k\,\frac{\partial^2 U}{\partial x^2}$$

is the *equation for heat conduction* in this solid where $U = U(x,t)$ is the *temperature* at any plane face x at any time t and k is a constant called the *diffusivity*, which depends on the material of the solid. The boundary conditions $U(0,t) = 0$ and $U(1,t) = 0$ indicate that the temperatures at $x = 0$ and $x = 1$ are kept at temperature zero, while $U(x,0) = 3\sin 2\pi x$ represents the initial temperature everywhere in $0 < x < 1$. The result (3.10) then is the temperature everywhere in the solid at time $t > 0$.

3.2 Applications To Integral Equations

3.2.1 Integral Equations

An *integral equation* is an equation having the form

$$Y(t) = F(t) + \int_a^b K(u,t)Y(u)\,\mathrm{d}\,u$$

where $F(t)$ and $K(u,t)$ are known, a and b are either given constants or functions of t, and the function $Y(t)$ which appears under the integral sign is to be determined.

The function $K(u,t)$ is often called the *kernel* of the integral equation. If a and b are constants, the equation is often called a *Fredholm integral equation*. If a is a constant while $b = t$, it is called a *Volterra integral equation*.

It is possible to convert a linear differential equation into an integral equation.

Problem 3.2.1. Convert the differential equation

$$Y''(t) + 3Y'(t) + 2Y(t) = 4\sin t, \quad Y(0) = 1, \quad Y'(0) = -2$$

into an integral equation.

Solution. Integrating both sides of the given differential equation, we have

$$\int_0^t \{Y''(u) - 3Y'(u) + 2Y(u)\} \, \mathrm{d}\, u = \int_0^t 4 \sin u \, \mathrm{d}\, u$$

$$\Rightarrow Y'(t) - Y'(0) - 3Y(t) + 3Y(0) + 2\int_0^t Y(u) \, \mathrm{d}\, u = 4 - 4\cos t$$

This becomes, using $Y'(0) = -2$ and $Y(0) = 1$

$$\Rightarrow Y'(t) - 3Y(t) + 2\int_0^t Y(u) \, \mathrm{d}\, u = -1 - 4\cos t$$

Integrating again from 0 to t as before, we find

$$\Rightarrow Y(t) - Y(0) - 3\int_0^t Y(u) \, \mathrm{d}\, u + 2\int_0^t (t-u)Y(u) \, \mathrm{d}\, u = -t - 4\sin t$$

$$\Rightarrow Y(t) + \int_0^t \{2(t-u) - 3\} \, Y(u) \, \mathrm{d}\, u = 1 - t - 4\sin t$$

3.2.2 Integral Equations Of Convolution Type

A special integral equation of importance in applications is

$$Y(t) = F(t) + \int_0^t K(t-u)Y(u) \, \mathrm{d}\, u$$

This equation is of *convolution type* and can be written as

$$Y(t) = F(t) + K(t) * Y(t)$$

Taking the Laplace transform of both sides, assuming $\mathcal{L}\{F(t)\} = f(s)$ and $\mathcal{L}\{K(t)\} = k(s)$ both exist, we find

$$y(s) = f(s) + k(s)y(s) \qquad \text{or} \qquad y(s) = \frac{f(s)}{1 - k(s)}$$

The required solution may then be found by inversion.

Problem 3.2.2. Solve the integral equation $Y(t) = t^2 + \int_0^t Y(u) \sin(t-u) \, \mathrm{d}\, u$.

Solution. The integral equation can be written

$$Y(t) = t^2 + Y(t) * \sin t$$

Then taking the Laplace transform and using the convolution theorem, we find, if $y = \mathcal{L}\{Y\}$

$$y = \frac{2}{s^3} + \frac{y}{s^2 + 1}$$

solving,

$$\Rightarrow y = \frac{2(s^2+1)}{s^5}$$

$$\Rightarrow y = \frac{2}{s^3} + \frac{2}{s^5}$$

and so

$$\Rightarrow Y = 2\left(\frac{t^2}{2!}\right) + 2\left(\frac{t^4}{4!}\right) = t^2 + \frac{1}{12}t^4$$

This can be checked by direct substitution in the integral equation.

3.2.3 Applications of Integral Equation

A large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. Mathematical physics models, such as

- Diffraction problems
- Scattering in quantum mechanics
- Conformal mapping
- Water waves

Also contributed to the creation of integral equations.

Chapter 4

Power Series Solutions to the Legendre Equation

4.1 The Legendre equation

The equation

$$\left(1-x^2\right)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{4.1}$$

where α is any real constant, is called Legendre's equation. When $\alpha \in \mathbb{Z}^+$, the equation has polynomial solutions called Legendre polynomials. In fact, these are the same polynomial that encountered earlier in connection with the Gram-Schmidt process.
The equation (4.1) can be rewritten as

$$\left[\left(x^2-1\right)y'\right]' = \alpha(\alpha+1)y,$$

which has the form $T(y) = \lambda y$, where $T(f) = (pf')'t'$, with $p(x) = x^2 - 1$ and $\lambda = \alpha(\alpha+1)$.

Note. The nonzero solutions of (4.1) are eigenfunctions of T corresponding to the eigenvalue $\alpha(\alpha+1)$.

Since $p(1) = p(1) = 0$, T is symmetric with respect to the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}\,x.$$

Thus, eigenfunctions belonging to distinct eigenvalues are orthogonal.

4.2 Power series solution for the Legendre equation

The Legendre equation can be put in the form

$$y'' + p(x)y' + q(x)y = 0,$$

where

$$p(x) = \frac{2x}{1-x^2} \text{ and } q(x) = \frac{\alpha(\alpha+1)}{1-x^2} \text{ if } x^2 \neq 1$$

Since $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$, both $p(x)$ and $q(x)$ have power series expansions in the open interval $(-1,1)$.
Thus, seek a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,\ x \in (-1,1).$$

Differentiating term by term, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^n - 1 \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Thus,

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n = \sum_{n=0}^{\infty} 2n a_n x^n$$

and

$$\begin{aligned}(1-x^2)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x_n - \sum_{n=0}^{\infty} n(n-1)a_n x^n \\ &= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] xn\end{aligned}$$

Substituting in (4.1), we obtain

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha+1)a_n = 0, n \geq 0,$$

which leads to a recurrence relation

$$a_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+1)(n+2)}a_n$$

Thus, we obtain

$$\begin{aligned} a2 &= -\frac{\alpha(\alpha+1)}{1\cdot 2}a_0 \\ a_4 &= -\frac{(\alpha-2)(\alpha+3)}{3\cdot 4} = (-1)^2\frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}a_0 \\ &\vdots \\ a_{2n} &= (-1)^n\frac{\alpha(\alpha-2)\ldots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\ldots(\alpha+2n-1)}{(2n)!}a_0\end{aligned}$$

Similarly, we can compute $a_3, a_5, a_7, \ldots,$ in terms of a_1 and obtain

$$\begin{aligned} a_3 &= \frac{(\alpha-1)(\alpha+2)}{2\cdot 3}a_1 \\ a_5 &= -\frac{(\alpha-3)(\alpha+4)}{4\cdot 5}a_3 = (-1)^2\frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}a_1 \\ &\vdots \\ a_{2n+1} &= (-1)^n\frac{(\alpha-1)(\alpha-3)\ldots(\alpha-2n+1)(\alpha+2)(\alpha+4)\ldots(\alpha+2n)}{(2n+1)!}a_1\end{aligned}$$

Therefore, the series for $y(x)$ can be written as

$$y(x) = a_0y_1(x) + a_1y_2(x),$$

where,

$$\begin{aligned} y1(x) &= 1 + \sum_{n=0}^{\infty}(-1)^n\frac{\alpha(\alpha-2)\ldots(\alpha-2n+2)\cdot(\alpha+1)(\alpha+3)\ldots(\alpha+2n-1)}{(2n)!}x^{2n}, \text{ and} \\ y2(x) &= x + \sum_{n=0}^{\infty}(-1)^n\frac{(\alpha-1)(\alpha-3)\ldots(\alpha-2n+1)\cdot(\alpha+2)(\alpha+4)\ldots(\alpha+2n)}{(2n+1)!}x^{2n+1}\end{aligned}$$

Note. The ratio test shows that $y_1(x)$ and $y_2(x)$ converges for $|x| < 1$. These solutions $y_1(x)$ and $y_2(x)$ satisfy the initial conditions

$$y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0, y_2'(0) = 1.$$

Since $y_1(x)$ and $y2(x)$ are independent, the general solution of the Legendre equation over $(-1, 1)$ is

$$y(x) = a_0y_1(x) + a_1y_2(x)$$

with arbitrary constants a_0 and a_1.

4.2.1 Observations

Case I.
When $\alpha = 0$ or $\alpha = 2m$, we note that

$$\alpha(\alpha-2)\ldots(\alpha-2n+2) = 2m(2m-2)\ldots(2m-2n+2) = \frac{2^n\, m!}{(m-n)!}$$

and

$$(\alpha+1)(\alpha+3)\ldots(\alpha+2n-1) = (2m+1)(2m+3)\ldots(2m+2n-1) = \frac{(2m+2n)!\, m!}{2^n(2m)!(m+n)!}$$

Then, in this case, $y_1(x)$ becomes

$$y_1(x) = 1 + \frac{(m!)^2}{(2m)!}\sum_{k=0}^{m}(-1)^k \frac{(2m+2k)!}{(m-k)!\,(m+k)!\,(2k)!}x^{2k}$$

which is a polynomial of degree $2m$. In particular, for $\alpha = 0, 2, 4 (m = 0, 1, 2)$, the corresponding polynomials are

$$y_1(x) = 1,\ 1-3x^2, 1-10x^2+\frac{35}{3}x^4$$

Note that the series $y_2(x)$ is not a polynomial when α is even because the coefficients of x^{2n+1} is never zero.
Case II.
When $\alpha = 2m+1$, $y_2(x)$ becomes a polynomial and $y_1(x)$ is not a polynomial.
In this case,

$$y_2(x) = x + \frac{(m!)^2}{(2m+1)!}\sum_{k=0}^{m}(-1)^k \frac{(2m+2k+1)!}{(m-k)!\,(m+k)!\,(2k+1)!}x^{2k+1}$$

For example, when $\alpha = 1, 3, 5 (m = 0, 1, 2)$, the corresponding polynomials are

$$y_2(x) = x, x-\frac{5}{3}x^3, x-\frac{14}{3}x^3+\frac{21}{5}x^5$$

4.3 The Legendre polynomial

Let,

$$P_n(x) = \frac{1}{2^n}\sum_{r=0}^{[n/2]}\frac{(-1)^r(2n-2r)!}{r!(n-r)!(n-2r)!}x^{n-2r} \tag{4.2}$$

where $[n/2]$ denotes the greatest integer $\leq n/2$.

- When n is even, it is a constant multiple of the polynomial $y1(x)$.
- When n is odd, it is a constant multiple of the polynomial $y2(x)$.

The first five Legendre polynomials are

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}\left(3x^2-1\right),\ P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4-30x^2+3\right),\ P_5(x) = \frac{1}{8}\left(63x^5-70x^3+15x\right)$$

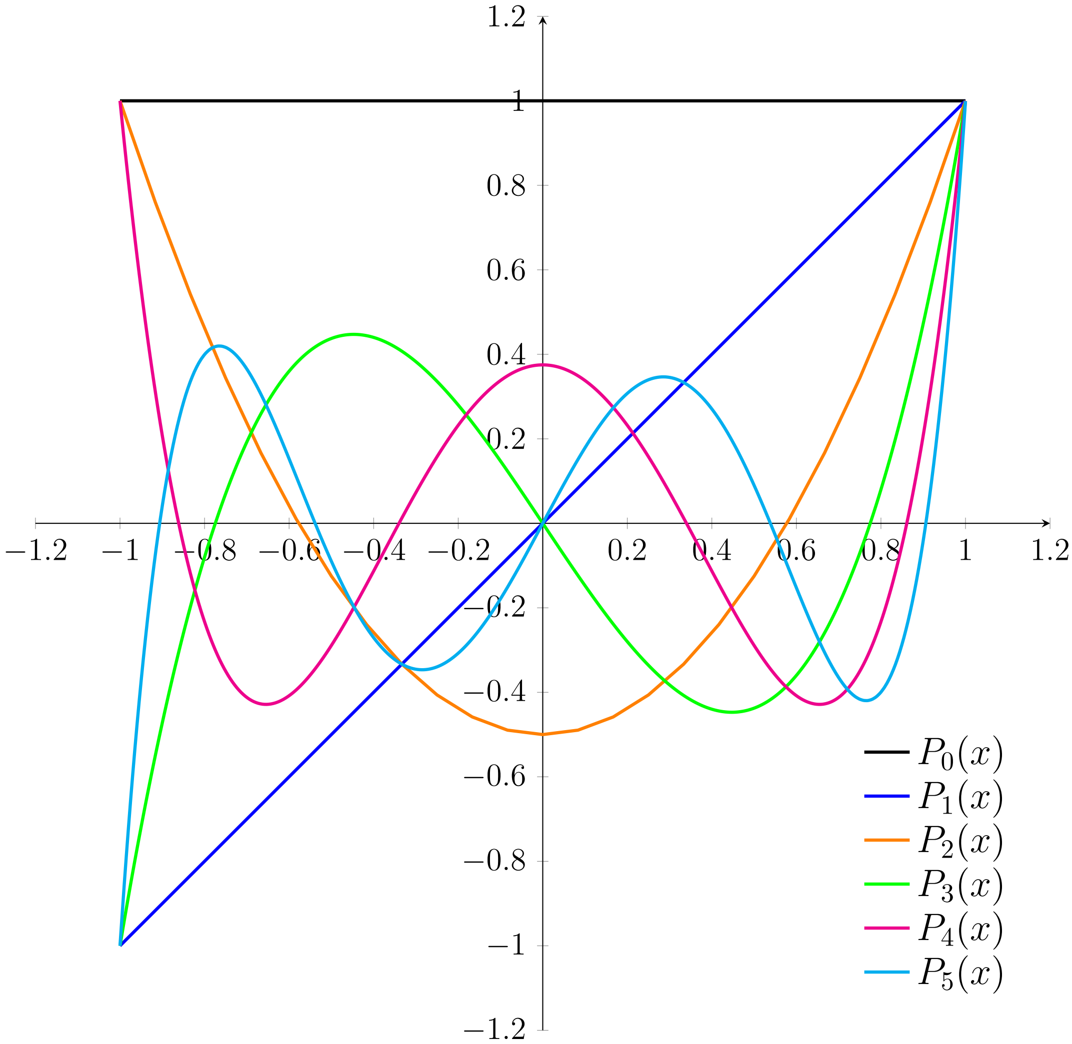

Figure 4.1: Legendre polynomial $(P_0(x), P_1(x), P_2(x), P_3(x), P_4(x), P_5(x))$ over the interval $[-1, 1]$

4.4 Rodrigues's formula for the Legendre polynomials

Note that

$$\frac{(2n-2r)!}{(n-2r)!}x^{n-2r} = \frac{\mathrm{d}^n}{\mathrm{d}\,x^n}x^{2n-2r} \quad \text{and} \quad \frac{1}{r!\,(n-r)!} = \frac{1}{n!}\binom{n}{r}$$

Thus,$P_n(x)$ in (4.2) can be expressed as

$$P_n(x) = \frac{1}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\sum_{r=0}^{[n/2]}(-1)^r\binom{n}{r}x^{2n-2r}$$

When $[n/2] < r \leq n$, the term x^{2n-2r} has degree less than n, so its nth derivative is zero. This gives

$$P_n(x) = \frac{1}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}x^{2n-2r} = \frac{1}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n$$

which is known as Rodrigues' formula.

4.5 Properties of the Legendre polynomials $P_n(x)$

- For each $n \geq 0$, $P_n(1) = 1$. Moreover, $P_n(x)$ is the only polynomial which satisfies the Legendre equation

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$$

  and $Pn(1) = 1$.

- For each $n \geq 0$, $P_n(-x) = (-1)^n P_n(x)$.

- $$\int_{-1}^{1} P_n(x)P_m(x) = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

- If $f(x)$ is a polynomial of degree n, we have

$$f(x) = \sum_{k=0}^{n} c_k P_k(x), \text{ where}$$

$$c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x)P_k(x) \,\mathrm{d}\, x$$

- It follows from the orthogonality relation that

$$\int_{-1}^{1} g(x)P_n(x) \,\mathrm{d}\, x = 0$$

  for every polynomial $g(x)$ with $\deg(g(x)) < n$.

Chapter 5

Legendre Function

5.1 Legendre Function

The differential equation $(1-x)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} - 2x \frac{\mathrm{d}\, y}{\mathrm{d}\, x} + n(n+1)y = 0$ is known as Legendre's differential equation; where n is a constant (real number). But in most applications only integral values of n are required.

Any solution of the Legendre's equation is called a Legendre function.

5.2 Rodrigues' Formula of Legendre Polynomial

We have obtained the Legendre polynomials as solutions of the Legendre's equation. There is another way of obtaining $P_n(x)$, which may be deduced directly from Legendre's differential equation without solving it. According to this formula $P_n(x)$ is given by,

$$P_n(x) = \frac{1}{2^n\, n!} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \left(x^2 - 1\right)^n$$

This is Rodrigues' formula.

Proof. Let,
$y = (x^2 - 1)^n$
$\therefore y_1 = 2nx\,(x^2-1)^{n-1}$

$$\begin{aligned}
\therefore & y_1 = 2nx \left(x^2 - 1\right)^{n-1} \\
\Rightarrow & y_1 \left(x^2 - 1\right) = 2nxy \\
\Rightarrow & y_2 \left(x^2 - 1\right) + 2xy_1 = 2nxy_1 + 2ny \\
\Rightarrow & y_2 \left(x^2 - 1\right) + 2(n-1)xy_1 - 2nx = 0
\end{aligned}$$

Now differentiating n times with respect to x, we get,

$$\Rightarrow y_{n+2}\left(x^2-1\right) + ny_{n+2-1}\cdot 2x + {}^nC_2 y_{n+2-2}\cdot 2 - 2(n-1)xy_{n+1} - 2n(n-1)y_n - 2ny_n = 0$$

$$\text{i.e.,} \qquad y_{n+2}\left(x^2-1\right) + 2xy_{n+1} - n(n+1)y_n = 0 \tag{5.1}$$

Put $\quad y_n = Z$

$\therefore y_{n+1} = \dfrac{\mathrm{d}\, Z}{\mathrm{d}\, x}, \qquad y_{n+2} = \dfrac{\mathrm{d}^2\, Z}{\mathrm{d}\, x^2}$

Substituting these values in (5.1) we get,

$$\begin{aligned}
&\left(x^2-1\right) \frac{\mathrm{d}^2 Z}{\mathrm{d}\, x^2} + 2x \frac{\mathrm{d}\, Z}{\mathrm{d}\, x} - n(n+1)y_n = 0 \\
\Rightarrow &\left(x^2-1\right) \frac{\mathrm{d}^2 Z}{\mathrm{d}\, x^2} - 2x \frac{\mathrm{d}\, Z}{\mathrm{d}\, x} + n(n+1)y_n = 0
\end{aligned}$$

This is a Legendre's differential equation of order n.
But since $Z = y_n = \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left\{(x^2-1)^n\right\}$, Z is a polynomial of degree n and since Legendre's equation has one and

only one distinct series solution of the form $P_n(x)$, it follows that $P_n(x)$ is a multiple of Z.
Hence,

$$P_n(x) = c \cdot Z = c\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left\{\left(x^2-1\right)^n\right\} \qquad [c \text{ is a constant}]$$

$$\text{or, } \sum_{r=0}^{N}(-1)^r \frac{1}{2^n\,r!}\frac{(2n-2r)!}{(n-r)!(n-2r)!}x^{n-2r} = c\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[x^{2n} - nx^{2(n-1)} + \frac{n(n-1)}{2!}x^{2(n-2)} + \dots\right]$$
$$= c\left[\frac{(2n)!}{n!}x^n - \frac{n(2n-2)!}{(n-2)!}x^{n-2} + \dots\right]$$

Equating the coefficient of x^n on both sides,

$$\frac{2n!}{2^n\,n!\,n!} = c\frac{(2n)!}{n!} \qquad [\text{Putting } r=0]$$

$$\text{ie.,} \qquad c = \frac{1}{2^n\,n!}$$

$$\therefore P_n(x) = c\cdot Z = \frac{2}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n$$

□

5.3 Generating Function for $P_n(x)$

Legendre polynomial $P_n(x)$ is the coefficient if h^n in $(1-2xh+h^2)^{-\frac{1}{2}}$ that is

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)h^n$$

Proof. The function $(1-2xh+h^2)^{-\frac{1}{2}}$ can be replaced by using binomial theorem as

$$\begin{aligned}\left(1-2xh+h^2\right)^{-\frac{1}{2}} &= \{1-h(2x-h)\}^{-\frac{1}{2}}\\ &= 1+\frac{1}{2}h(2x-h) + \frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{2!}h^2(2x-h)^2 + \frac{\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}h^3(2x-h)^3+\dots\\ &= 1+\frac{1}{2}h(2x-h)+\frac{3}{4\cdot 2}h^2\left(4x^2-4xh+h^2\right)+\frac{15}{8\cdot 6}h^3\left(8x^3-12x^2h+6xh^2-h^3\right)+\dots\\ &= 1+xh-\frac{h^2}{2}+\frac{3}{2}x^2h^2-\frac{3}{2}xh^3+\frac{3}{8}h^4+\frac{5}{2}x^3h^3+\dots\\ &= 1+xh+\left(\frac{3}{2}x^2-\frac{1}{2}\right)h^2+\left(\frac{5}{2}x^3-\frac{3}{2}x\right)h^3+\dots\end{aligned} \tag{5.2}$$

Again, we have,

$$P_n(x) = \frac{1}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n$$

Another form to find the value of c - Rajput. P-654
Integral form - Rajput. P-654

Putting $n = 0, 1, 2, 3, \dots$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2} 2x = x$$

$$\begin{aligned} P_2(x) &= \frac{1}{4 \cdot 2} \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} \left(x^2 - 1\right)^2 \\ &= \frac{1}{8} \frac{\mathrm{d}\, \left\{2\left(x^2 - 1\right) 2x\right\}}{\mathrm{d}\,x} \\ &= \frac{1}{2} \frac{\mathrm{d}\, \left(x^3 - x\right)}{\mathrm{d}\,x} \\ &= \frac{1}{2}\left(3x^2 - 1\right) \end{aligned}$$

$$\begin{aligned} P_3(x) &= \frac{1}{8 \cdot 6} \frac{\mathrm{d}^3}{\mathrm{d}\,x^3} \left(x^2 - 1\right)^3 \\ &= \frac{1}{8 \cdot 6} \frac{\mathrm{d}^3}{\mathrm{d}\,x^3} [x^6 - 3x^4 + 3x^2 - 1] \\ &= \frac{1}{8 \cdot 6} \frac{\mathrm{d}^2}{\mathrm{d}\,x^2} \left(6x^5 - 12x^3 + 6x\right) \\ &= \frac{1}{8 \cdot 6} \left(6 \cdot 5 \cdot 4x^3 - 12 \cdot 3 \cdot 2x\right) \\ &= \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) \end{aligned}$$

So from (5.2) we get,

$$\begin{aligned} \left(1 - 2xh + h^2\right)^{-\frac{1}{2}} &= P_0(x) + P_1(x)h + P_2(x)h^2 + P_3(x)h^3 + \dots \\ &= \sum_{n=0}^{\infty} P_n(x)h^n \end{aligned}$$

Thus by expanding $(1 - 2xh + h^2)^{-\frac{1}{2}}$, we can obtain the Legendre's polynomials of different order as the coefficient of corresponding power of h.
This is why $(1 - 2xh + h^2)^{-\frac{1}{2}}$ is known as the generating function if $P_n(x)$. □

5.4 Recurrence Relation for $P_n(x)$

5.4.1 First Relation $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$

We know,

$$\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)h^n$$

Now differentiating with respect to h we have

$$\begin{aligned} &(x - h)\left(1 - 2xh + h^2\right)^{-\frac{3}{2}} = h \sum_{n=0}^{\infty} P_n(x)h^{n-1} \\ \Rightarrow &(x - h)\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = h\left(1 - 2xh + h^2\right) \sum_{n=0}^{\infty} P_n(x)h^{n-1} \\ \Rightarrow &(x - h) \sum P_n(x)h^n = \sum \left[nP_n(x)h^{n-1} - 2hxP_n(x)h^n + nP_nh^{n+1}\right] \\ \Rightarrow &x \sum P_n(x)h^n - \sum P_n(x)h^{n+1} = \sum \left[nP_n(x)h^{n-1} \dots\right] \end{aligned}$$

Now equating the coefficient of h^n from both sides,

$$\begin{aligned} xP_n(x) - P_{n-1}(x) &= (n+1)P_{n+1}(X) - 2xnP_n(x) + (n-1)P_{n-1}(x) \\ \Rightarrow &(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \end{aligned}$$

Replacing n by $n-1$, we get

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0$$

The other relations are

- $P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x) = P_{n-1}(x)$
- $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$
- $P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x)$
- $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$
- $(x^2-1)\,P_n'(x) = n\left\{xP_n(x) - P_{n-1}(x)\right\}$
- $(x^2-1)\,P_n'(x) = (n+1)\left\{P_{n+1}(x) - xP_n(x)\right\}$

5.5 Orthogonal Properties of Legendre Polynomial

Problem 5.5.1. Prove that

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\,x = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

Proof. Since $P_n(x)$ is a solution of the Legendre's differential equation, we have

$$\begin{aligned} &(1-x)^2 \frac{\mathrm{d}^2}{\mathrm{d}\,x^2}(P_n(x)) - 2x\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x)) + n(n+1)P_n(x) = 0 \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\,x}\left\{\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x))\right\} + n(n+1)P_n(x) = 0 \\ &\Rightarrow \int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}\,x}\left\{\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x))\right\}P_m(x)\,\mathrm{d}\,x + n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \\ &\Rightarrow \left[P_m(x)\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}(P_n(x))\right]_{-1}^{1} - \int_{-1}^{1}P_m(x)\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}\,x}\left(P_n(x)\right)\,\mathrm{d}\,x + n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \\ &\Rightarrow -\int_{-1}^{1}\left(1-x^2\right)P_m'(x)P_n'(x)\,\mathrm{d}\,x + n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \end{aligned} \tag{5.3}$$

Interchanging m and n in (5.3), we get

$$\Rightarrow -\int_{-1}^{1}\left(1-x^2\right)P_n'(x)P_m'(x)\,\mathrm{d}\,x + m(m+1)\int_{-1}^{1}P_n(x)P_m(x)\,\mathrm{d}\,x = 0 \tag{5.4}$$

Subtracting (5.4) from (5.3),

$$\begin{aligned} &\Rightarrow (n-m)(m+n+1)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \\ &\Rightarrow (n-m)\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0 \quad \Big|\, m+n+1 \neq 0 \\ &\Rightarrow \int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}\,x = 0, \text{ if } m \neq n \end{aligned}$$

Again, we have

$$P_n(x) = \frac{1}{2^n\, n!}\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(x^2-1\right)^n$$

$$P_m(x) = \frac{1}{2^m\, m!}\frac{\mathrm{d}^m}{\mathrm{d}\, x^m}\left(x^2-1\right)^m$$

$$\begin{aligned}
&\therefore \int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\,x\\
&= \frac{1}{2^{m+n}\, m!\, n!}\int_{-1}^{1}\frac{\mathrm{d}^m}{\mathrm{d}\, x^m}\left(x^2-1\right)^m \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(x^2-1\right)^n\,\mathrm{d}\,x\\
&= \frac{1}{2^{m+n}\, m!\, n!}\left\{\left[\frac{\mathrm{d}^m}{\mathrm{d}\, x^m}\left(x^2-1\right)^m \frac{\mathrm{d}^{n-1}}{\mathrm{d}\, x^{n-1}}\left(x^2-1\right)^n\right]_{-1}^{1} - \int_{-1}^{1}\frac{\mathrm{d}^{m+1}}{\mathrm{d}\, x^{m+1}}(x^2-1)^m \frac{\mathrm{d}^{n-1}}{\mathrm{d}\, x^{n-1}}\left(x^2-1\right)^n\,\mathrm{d}\,x\right\}\\
&= -\frac{1}{2^{m+n}\, m!\, n!}\int_{-1}^{1}\frac{\mathrm{d}^{m+1}}{\mathrm{d}\, x^{m+1}}\left(x^2-1\right)^m \frac{\mathrm{d}^{n-1}}{\mathrm{d}\, x^{n-1}}\left(x^2-1\right)^n\,\mathrm{d}\,x
\end{aligned}$$

Continuing this process m times, we get

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\,x = \frac{(-1)^m}{2^{m+n}\, m!\, n!}\int_{-1}^{1}\frac{\mathrm{d}^{m+m}}{\mathrm{d}\, x^{m+m}}\left(x^2-1\right)^m \frac{\mathrm{d}^{n-m}}{\mathrm{d}\, x^{n-m}}\left(x^2-1\right)^n\,\mathrm{d}\,x$$

If $m = n$,

$$\begin{aligned}
\int_{-1}^{1}\{P_n(x)\}^2\,\mathrm{d}\,x &= \frac{(-1)^n}{2^{2n}\,(n!)^2}\int_{-1}^{1}\left\{\frac{\mathrm{d}^{2n}}{\mathrm{d}\, x^{2n}}\left(x^2-1\right)^n\right\}(x^2-1)^n\,\mathrm{d}\,x\\
&= (-1)^n\frac{1}{2^{2n}\,(n!)^2}\int_{-1}^{1}(2n)!\left(x^2-1\right)^n\left(x^2-1\right)^n\,\mathrm{d}\,x\\
&= \frac{2(-1)^{2n}(2n)!}{2^{2n}\,(n!)^2}\int_0^1\left(1-x^2\right)^n\,\mathrm{d}\,x\\
&= \frac{2(2n)!}{2^{2n}\,(n!)^2}\int_0^{\frac{\pi}{2}}\cos^{2n+1}\theta\,\mathrm{d}\,\theta\\
&= \frac{2(2n)!}{2^{2n}\,(n!)^2}\frac{2n(2n-2)(2n-4)\ldots 6\cdot 4\cdot 2}{(2n+1)(2n-1)(2n-3)\ldots 5\cdot 3\cdot 1}\\
&= \frac{2(2n)!}{2^{2n}\,(n!)^2}\frac{\{2^n\, n!\}^2}{(2n+1)!}\\
&= \frac{2}{2n+1}
\end{aligned}$$

□

Problem 5.5.2. Show that $P_n(-x) = (-1)^n P_n(x)$

Solution. we have,

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)h^n \tag{5.5}$$

Now replacing x by $-x$ and h by $-h$ in (8.8) we get,

$$\left(1+2xh+h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(-x)h^n \tag{5.6}$$

$$\left(1+2xh+h^2\right)^{-\frac{1}{2}} = (-1)^n\sum_{n=0}^{\infty} P_n(x)h^n \tag{5.7}$$

$\frac{\mathrm{d}^{2n}}{\mathrm{d}\, x^{2n}}\left(x^2-1\right)^n = (2n)!$
$(2n+1)! = (2n+1)(2n)(2n-1)(2n-2)\ldots$
$\{2n(2n-2)(2n-4)\ldots\}^2$
$= [2\{n(n-1)(n-2)\ldots\}]^2$
$= \{2^n n!\}^2$
If $I_n = \int_0^{\pi/2}\cos^n\,\mathrm{d}\,x$, then $I_n = \frac{n-1}{n}I_{n-2}$

From (5.6) and (5.7) we get

$$\sum P_n(-x)h^n = (-1)^n \sum P_n(x)h^n$$

Equating the coefficients of h^n we get,

$$P_N = n(x) = (-1)^n P_n(x)$$

Problem 5.5.3. Prove that

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x = \frac{2n}{4n^2-1}$$

Proof. We have the recurrence relation

$$\begin{aligned} nP_n(x) &= (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \\ \Rightarrow (2n-1)xP_{n-1}(x) &= nP_n(x) + (n-1)P_{n-2}(x) \end{aligned}$$

Multiplying both sides of the above equation by $P_n(x)$ and then integrating from -1 to 1 we get,

$$(2n-1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x = n\int_{-1}^{1} [P_n(x)]^2\,\mathrm{d}\,x + (n-1)\int_{-1}^{1} P_n(x)P_{n-2}(x)\,\mathrm{d}\,x \tag{5.8}$$

From the orthogonal property, we have,

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}\,x = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

$$\begin{aligned} \therefore \int_{-1}^{1} P_n(x)P_{n-2}(x)\,\mathrm{d}\,x &= 0 \quad \text{since} n \neq n-2 \\ \text{And } \int_{-1}^{1} [P_n(x)]^2\,\mathrm{d}\,x &= \frac{2}{2n+1} \end{aligned}$$

So from (5.8),

$$\begin{aligned} (2n-1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x &= \frac{2n}{2n+1} \\ \therefore \int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathrm{d}\,x &= \frac{2n}{4n^2-1} \end{aligned}$$

□

Problem 5.5.4. Prove that

$$P_n(1) = 1$$

Proof. If $x = 1$ then,

$$\begin{aligned} \left(1-2xh+h^2\right)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} P_n(x)h^n \\ \Rightarrow (1-h)^{-1} &= \sum_{n=0}^{\infty} h_n P_n(1) \\ \Rightarrow (1-h)^{-1} &= 1 + hP_1(x) + h^2P_2(1) + \cdots + h^nP_n(1) + \ldots \\ \Rightarrow 1 + h + h^2 + h^3 + \cdots + h^n + \cdots &= 1 + hP_1(x) + h^2P_2(1) + \cdots + h^nP_n(1) + \ldots \end{aligned}$$

Equating the coefficients of h^n from both sides,

$$P_n(1) = 1$$

□

Problem 5.5.5. Show that

$$\int_{-1}^{1} P_0(x)P_1(x)\,\mathrm{d}\,x = 0$$

Solution.

$$P_n(x) = \frac{1}{2^n\,n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n$$

$$\therefore P_0(x) = 1, \quad P_1(x) = \frac{1}{2\cdot 1}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left(x^2-1\right) = x$$

$$\int_{-1}^{1} P_0(x)P_1(x)\,\mathrm{d}\,x = \int_{-1}^{1} x\,\mathrm{d}\,x = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

Problem 5.5.6. Compute

$$\int_{-1}^{1} [P_2(x)]^{\frac{1}{2}}\,\mathrm{d}\,x$$

Solution.

$$P_2(x) = \frac{1}{2^2\,2!}\,\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}\left(x^2-1\right) = \frac{1}{2}\left(3x^2-1\right)$$

$$\begin{aligned}
\therefore \int_{-1}^{1}\left\{\frac{1}{2}\left(3x^2-1\right)\right\}^{\frac{1}{2}}\mathrm{d}\,x &= \frac{1}{\sqrt{2}}\int_{-1}^{1}\sqrt{\left\{(\sqrt{3}x)^2-1\right\}}\,\mathrm{d}\,x \\
&= \frac{1}{\sqrt{2}\sqrt{3}}\left[\frac{\sqrt{3}x\sqrt{3x^2-1}}{2} + \frac{1}{2}\log\left(3x^2+\sqrt{3x^2-1}\right)\right]_{-1}^{1} \\
&= \frac{1}{\sqrt{6}}\frac{\sqrt{3}\sqrt{3-1}}{2} + \frac{1}{2}\log\left(\sqrt{3}+\sqrt{3-1}\right) + \frac{\sqrt{3}\sqrt{3-1}}{2} - \frac{1}{2}\log\left(-\sqrt{3}+\sqrt{3-1}\right) \\
&= \frac{1}{\sqrt{6}}\left\{\frac{\sqrt{6}}{2} + \frac{1}{2}\log\dots\right\}
\end{aligned}$$

Remark.

$$\begin{aligned}
&\left(1-2xh+h^2\right)^{-\frac{1}{2}} = \sum h^n P_n(x) \\
\Rightarrow &\left\{1-h\left(2x-h\right)\right\}^{-\frac{1}{2}} = \sum h^n P_n(x) \\
\Rightarrow &1 + hx + h^2\frac{3x^2-1}{2} + h^3\frac{5x^3-3x}{2} + \dots = P_0(x) + hP_1(x) + h^2P_2(x) + \dots
\end{aligned}$$

Equating,

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2-1}{2} \text{ and so on}$$

Chapter 6

Bessel's Equation and Bessel's Function

6.1 Bessel's Equation and Bessel's Function

The differential equation

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + \left(x^2 - n^2\right) y = 0$$

where n is a positive constant (not necessarily an integer) is known as the Bessel's equation.

Since it is a linear differential equation of second order, it must have two linearly independent solutions.

Case 1 : *n is not an integer*

The complete solution of the Bessel's equation can be expressed as

$$y = AJ_n(x) + BJ_{-n}(x)$$

Where $J_n(x)$ is called the Bessel function of the first kind of order n and $J_{-n}(x)$ is called the Bessel function of first kind of order $-n$.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

and,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,(-n+r)!}$$

Case 2 : *n is an integer*

The complete solution of the Bessel's equation can be expressed as

$$y = AJ_n(x) + BY_n(x)$$

Where $Y_n(x)$ is called Bessel function of second kind of order n,

$$Y_n(x) = J_n(x) \int \frac{d x}{x \left(J_n(x)\right)^2}$$

Problem 6.1.1. Prove that $J_{-n}(x) = (-1)^n J_n(x)$

Proof. We have,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,(-n+r)!} \tag{6.1}$$

Let,

$$r - n = s$$
$$\Rightarrow r = n + s$$

From (6.1),

$$\begin{aligned} J_{-n}(x) &= \sum \frac{(-1)^{n+s} \left(\frac{x}{2}\right)^{-n+2(n+s)}}{(n+s)!\,(-n+n+s)!} \\ &= (-1)^n \sum \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{s!\,(n+s)!} \\ &= (-1)^n J_n(x) \end{aligned}$$

□

Problem 6.1.2. Prove the following

(i) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(ii) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

(iii) $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$

(iv) $J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x\right)$

Proof. We have

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

Putting $r = 0, 1, 2, \dots$

$$\begin{aligned} J_n(x) &= \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{\left(\frac{x}{2}\right)^2}{(n+1)!} + \frac{\left(\frac{x}{2}\right)^4}{2!\,(n+2)!} - \dots\right] \\ &= \frac{\left(\frac{x}{2}\right)^n}{n!} \left[1 - \frac{\left(\frac{x}{2}\right)^2}{n+1} + \frac{\left(\frac{x}{2}\right)^4}{2!\,(n+2)(n+1)} - \dots\right] \end{aligned} \tag{6.2}$$

(i) Putting $n = \frac{1}{2}$ in (6.2) we get,

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!} \left[1 - \frac{\left(\frac{x}{2}\right)^2}{\frac{1}{2}+1} + \frac{\left(\frac{x}{2}\right)^4}{2!\,\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+1\right)} - \dots\right] \\ &= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)!} \left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots\right] \\ &= \sqrt{\frac{x}{2}} \cdot \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right] \\ &= \sqrt{\frac{x}{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right] \\ &= \sqrt{\frac{2}{\pi x}} \sin x \end{aligned}$$

(ii) Putting $n = -\frac{1}{2}$ in (6.2) we get,

$$\begin{aligned} J_{-\frac{1}{2}}(x) &= \frac{\left(\frac{x}{2}\right)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} \left[1 - \frac{\left(\frac{x}{2}\right)^2}{-\frac{1}{2}+1} + \frac{\left(\frac{x}{2}\right)^4}{2!\,\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+1\right)} - \dots\right] \\ &= \frac{\left(\frac{x}{2}\right)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)!} \left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right] \\ &= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \cdot \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right] \\ &= \sqrt{\frac{2}{\pi x}} \cos x \end{aligned}$$

$\Gamma(n+1) = n!$
$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

(iii) From the recurrence relation we have,

$$\frac{2n}{x} J_n(x) = \{J_{n+1}(x) + J_{n-1}(x)\}$$

Putting $n = \frac{1}{2}$ we get

$$\begin{aligned} J_{\frac{1}{2}}(x) &= x\left\{J_{\frac{3}{2}}(x) + J_{-\frac{1}{2}}(x)\right\} \\ \Rightarrow J_{\frac{3}{2}}(x) &= \frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) \\ \Rightarrow J_{\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \end{aligned}$$

(iv) Again putting $n = -\frac{1}{2}$ in the recurrence relation we get

$$\begin{aligned} -J_{-\frac{1}{2}}(x) &= x\left\{J_{\frac{1}{2}}(x) + J_{-\frac{3}{2}}(x)\right\} \\ \Rightarrow J_{-\frac{3}{2}}(x) &= -\frac{1}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) \\ \Rightarrow J_{-\frac{3}{2}}(x) &= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right) \end{aligned}$$

□

Problem 6.1.3. Show that

$$J_n'(x) J_{-n}(x) - J_{-n}'(x) J_n(x) = \frac{2\sin n\pi}{\pi x}$$

Solution. The Bessel's differential equation is

$$\begin{aligned} & x^2 \frac{\mathrm{d}^2 y}{\mathrm{d} x^2} + x\frac{\mathrm{d} y}{\mathrm{d} x} + \left(x^2 - n^2\right) y = 0 \\ \Rightarrow & \frac{\mathrm{d}^2 y}{\mathrm{d} x^2} + \frac{1}{x}\frac{\mathrm{d} y}{\mathrm{d} x} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \end{aligned}$$

Since $J_n(x)$ and $J_{-n}(x)$ satisfies the Bessel's differential equation,

$$J_n''(x) + \frac{1}{x} J_n'(x) + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0 \tag{6.3}$$

And,

$$J_{-n}''(x) + \frac{1}{x} J_{-n}'(x) + \left(1 - \frac{n^2}{x^2}\right) J_{-n}(x) = 0 \tag{6.4}$$

Now (6.3)$\times J_{-n}(x) -$(6.4)$\times J_n x$,

$$J_n''(x) J_{-n}(x) - J_{-n}''(x) J_n(x) + \left[J_n'(x) J_{-n}(x) - J_{-n}'(x) J_n(x)\right] = 0 \tag{6.5}$$

Put $\quad z = J_n'(x) J_{-n}(x) - J_{-n}'(x) J_n(x)$

$\therefore\ z' = J_n''(x) J_{-n}(x) + J_n'(x) J_{-n}'(x) - J_{-n}''(x) J_n(x) - J_{-n}'(x) J_n'(x)$

From (6.5),

$$\begin{aligned} & z' + \frac{1}{x} z = 0 \\ \Rightarrow & \frac{z'}{z} + \frac{1}{x} = 0 \\ \Rightarrow & \log z + \log x = \log c \\ \Rightarrow & zx = c \\ \Rightarrow & J_n'(x) J_{-n}(x) - J_{-n}'(x) J_n(x) = \frac{c}{x} \qquad (6.6) \end{aligned}$$

But

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} \qquad \text{and,}$$
$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}$$
$$\therefore\ J_n^{'}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{(n+2r)}{2\cdot r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1}$$
$$\therefore\ J_{-n}^{'}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{(-n+2r)}{2\cdot r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r-1}$$

From (6.6),

$$\sum_{r=0}^{\infty}(-1)^r \frac{(n+2r)}{2\cdot r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1} \cdot \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}$$
$$- \sum_{r=0}^{\infty}(-1)^r \frac{(-n+2r)}{2\cdot r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r-1} \cdot \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = \frac{c}{x}$$
$$\Rightarrow \sum_{r=0}^{\infty}(-1)^{2r} \frac{(n+2r)x^{4r-1}}{2^{4r}(r!)^2(n+r)!(-n+r)!} - \sum_{r=0}^{\infty}(-1)^{2r} \frac{(-n+2r)x^{4r-1}}{2^{4r}(r!)^2(n+r)!(-n+r)!} = \frac{c}{x}$$

Equating the coefficient of $\frac{1}{x}$ from both sides,

$$\frac{1}{n!\,(-n)!}\{n-(-n)\} = c$$
$$\Rightarrow \frac{2n}{\Gamma(n+1)\,\Gamma(-n+1)} = c$$
$$\Rightarrow c = \frac{2}{\Gamma(n)\,\Gamma(1-n)}$$
$$\Rightarrow c = \frac{2\sin n\pi}{\pi}$$

$$\therefore\ J_n^{'}(x)J_{-n}(x) - J_{-n}^{'}(x)J_n(x) = \frac{2\sin n\pi}{\pi x}$$

6.2 Orthogonality of Bessel Functions

Problem 6.2.1. Show that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,\mathrm{d}\,x = 0$$

where α and β are different roots of $J_n(x) = 0$.

Proof. We have $u = J_n(\alpha x)$ and $v = J_n(\beta c)$ respectively be the solution of

$$x^2u'' + xu' + \left(\alpha^2x^2 - n^2\right)u = 0 \tag{6.7}$$
$$x^2v'' + xv' + \left(\alpha^2x^2 - n^2\right)v = 0 \tag{6.8}$$

Now (6.7) $\times \frac{v}{x}$ − (6.8) $\times \frac{u}{x}$,

$$x\left(u''v - uv''\right) + \left(u'v - uv'\right) + \left(\alpha^2 - \beta^2\right)xuv = 0$$
$$\Rightarrow \frac{\mathrm{d}\,\{x\left(u'v - uv'\right)\}}{\mathrm{d}\,x} = \left(\beta^2 - \alpha^2\right)xuv$$
$$\Rightarrow = \int_0^1 \left(\beta^2 - \alpha^2\right)xuv\,\mathrm{d}\,x = \left[x\left(u'v - uv'\right)\right]_0^1$$
$$\Rightarrow = \int_0^1 \left(\beta^2 - \alpha^2\right)xuv\,\mathrm{d}\,x = \left[\left(u'v - uv'\right)\right]_{x=1}$$
$$\Rightarrow = \int_0^1 (xuv)\,\mathrm{d}\,x = \frac{1}{\beta^2 - \alpha^2}\left[\left(u'v - uv'\right)\right]_{x=1} \tag{6.9}$$

But $u' = \alpha J_n'(\alpha x)$, $v' = \beta J_n'(\beta x)$
From (6.9)

$$\Rightarrow \int_0^1 (xuv)\,\mathrm{d}\,x = \frac{\alpha J_n'(\alpha x) J_n(\beta) - \beta J_n(\alpha) J_n'(\beta x)}{\beta^2 - \alpha^2} \tag{6.10}$$

If α and β are distinct roots of $J_n(x) = 0$ then $J_n(\alpha) = J_n(\beta) = 0$
From (6.10),

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,\mathrm{d}\,x = 0$$

Note. The Bessel's equation is

$$x^2 \frac{\mathrm{d}^2\,y}{\mathrm{d}\,x^2} + x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + \left(x^2 - n^2\right) y = 0$$

Let $x = \alpha r$, we get

$$r^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\,r} + r\frac{\mathrm{d}\,y}{\mathrm{d}\,r} + \left(\alpha^2 r^2 - n^2\right) y = 0$$

$$\Rightarrow x^2 \frac{\mathrm{d}^2\,y}{\mathrm{d}\,x^2} + x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} + \left(\alpha^2 x^2 - n^2\right) y = 0$$

$$\begin{aligned} &x = \alpha r \\ &\therefore \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\mathrm{d}\,y}{\mathrm{d}\,r} \cdot \frac{\mathrm{d}\,r}{\mathrm{d}\,x} \\ &\Rightarrow \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{1}{\alpha}\frac{\mathrm{d}\,y}{\mathrm{d}\,r} \\ &\therefore x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\alpha r}{\alpha}\frac{\mathrm{d}\,y}{\mathrm{d}\,r} \\ &\therefore x\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = r\frac{\mathrm{d}\,y}{\mathrm{d}\,r} \end{aligned}$$

□

Problem 6.2.2. Show that

$$\int_0^x x^n J_{n-1}(x)\,\mathrm{d}\,x = x^n J_n(x)$$

Proof. We have,

$$x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!} (x)^{2n+2r} \frac{1}{2^{n+2r}}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}\,x}\left(x^n J_n(x)\right) = x^n J_{n-1}(x) \tag{6.11}$$

Integrating (6.11) with respect to x from 0 to x we get,

$$\begin{aligned} \int_0^x x^n J_{n-1}(x)\,\mathrm{d}\,x &= \left[x^n J_n(x)\right]_0^x \\ &= x^n J_n(x) + \lim_{x\to 0} x^n J_n(x) \\ &= x^n J_n(x) + 0 \\ &= x^n J_n(x) \end{aligned}$$

□

Problem 6.2.3. Show that

$$\int_0^x x^{-n} J_{n+1}(x)\,\mathrm{d}\,x = \frac{1}{2^n\,n!} - x^{-n} J_n(x)$$

Proof. We have,

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,(n+r)!} \frac{1}{2^{n+2r}} (x)^{2r}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}\,x}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) \tag{6.12}$$

Integrating (6.12) with respect to x from 0 to x we get,

$$\begin{aligned} \int_0^x -x^{-n} J_{n+1}(x)\,\mathrm{d}\,x &= \left[-x^{-n} J_n(x)\right]_0^x \\ &= -x^{-n} J_n(x) + \lim_{x\to 0} x^{-n} J_n(x) \end{aligned} \tag{6.13}$$

Now,

$$\begin{aligned}
\lim_{x\to 0}\left(x^{-n}J_n(x)\right) &= \lim_{x\to 0}\frac{J_n(x)}{x^n} \\
&= \lim_{x\to 0}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+2r)!}\frac{1}{2^{n+2r}}x^{2n} \\
&= \lim_{x\to 0}\left[\frac{1}{2^n\,n!}-\frac{1}{2^{n+2}(n+2)!}x^2+\dots\right] \\
&= \frac{1}{2^n\,n!}
\end{aligned}$$

From (6.13)

$$\int_0^x -x^{-n}J_{n+1}(x)\,\mathrm{d}\,x = \frac{1}{2^n n!} - x^{-n}J_n(x)$$

□

6.3 Recurrence Relation

Problem 6.3.1. Prove the following recurrence formula for $J_n(x)$

(i) $\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^nJ_n(x)\right] = x^nJ_{n-1}x$

(ii) $\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}x$

(iii) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]$

(iv) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$

(v) $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

Proof.

(i) From the Bessel function of the first kind of order n
We have,

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

$$\therefore\ x^nJ_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{2n+2r}}{2^{n+2r}r!\,(n+r)!}$$

$$\begin{aligned}
\therefore\ \frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^nJ_n(x)\right] &= \sum_{r=0}^{\infty}(-1)^r\frac{1}{2^{n+2r}\,n!\,(n+r)!}\cdot 2(n+r)x^{2(n+r)-1} \\
&= \sum(-1)^r\frac{1}{r!\,(n+r-1)!}\frac{x^n - x^{n+2r-1}}{2^{n+2r-1}} \\
&= x^n\sum(-1)^r\frac{1}{r!\,(n-1+r)!}\left(\frac{x}{2}\right)^{(n-1)+2r} \\
&= x^nJ_{n-1}(x)
\end{aligned}$$

(ii)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left[\sum(-1)^r\frac{x^{2r}}{r!\,(n+r)!\,2^{n+2r}}\right] \\
&= \sum(-1)^r\frac{2r\cdot x^{2r-1}}{2^{n+2r}\,n!\,(n+r)!} \\
&= \sum(-1)^r\frac{x^{2r-1}}{r!\,(n+r-1)!2^{n-1+2r}} \\
&= -\sum(-1)^{r-1}\frac{1}{(r-1)!\,(n+r)!}\frac{x^{n+2r-1}\cdot x^{-n}}{2^{n-1+2r}} \\
&= -x^{-n}\sum_{r=0}^{\infty}(-1)^{r-1}\frac{1}{(r-1)!\,(n+r)!}\cdot\frac{x^{n+1+2(r-1)}}{x^{n+1+2(r-1)}}
\end{aligned}$$

When $r = 0$ $(r-1)! = (-1)! = \infty$
i.e., $\frac{1}{(r-1)!} = 0$
$\therefore$ When $r = 0$, the first term vanishes. So,

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] = -x^{-n}\sum_{r=1}^{\infty}(-1)^r\frac{1}{(r-1)!\,(n+1+r-1)!}\cdot\left(\frac{x}{2}\right)^{n+2r-1}$$

Putting $r-1=k$ i.e., $r=k+1$

$$\begin{aligned}\therefore\ \frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] &= -x^{-n}\sum_{k=0}^{\infty}(-1)^r\frac{1}{k!\,(n+1+k)!}\cdot\frac{x}{2}^{n+1+2k} \\ &= -x^{-n}J_{n+1}(x)\end{aligned}\qquad \left|\ \begin{array}{l}\text{When,}\\ \quad r=1,\ k=0\end{array}\right.$$

(iii) We have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^nJ_n(x)\right] = x^nJ_{n-1}(x)\\ &\Rightarrow x^nJ_n^{'}(x) + nx^{n-1}J_n(x) = x^nJ_{n-1}(x)\\ &\Rightarrow J_n^{'}(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)\end{aligned}\tag{6.14}$$

Also,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\,x}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)\\ &\Rightarrow x^{-n}J_n^{'}(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)\\ &\Rightarrow -J_n^{'}(x) + \frac{n}{x}J_n(x) = J_{n+1}(x)\end{aligned}\tag{6.15}$$

Adding (6.14) with (6.15) with we get,

$$\begin{aligned}&\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)\\ &\therefore J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]\end{aligned}$$

(iv) Subtracting (6.15) from (6.14) we get,

$$\begin{aligned}&2J_n^{'}(x) = J_{n-1}(x) - J_{n+1}(x)\\ &\therefore J_n^{'}(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\end{aligned}$$

(v) We have

$$\begin{aligned}&J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]\\ &\Rightarrow \frac{2n}{x}J_n(x) = \left[J_{n-1}(x) + J_{n+1}(x)\right]\end{aligned}\tag{6.16}$$

Again

$$\begin{aligned}&J_n^{'}(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\\ &\Rightarrow 2J_n^{'}(x) = J_{n-1}(x) - J_{n+1}(x)\end{aligned}\tag{6.17}$$

Subtracting (6.17) from (6.16) we get,

$$\begin{aligned}&\frac{2n}{x}J_n(x) - 2J_n^{'} = 2J_{n+1}(x)\\ &\therefore xJ_n^{'}(x) = nJ_n(x) - xJ_{n+1}(x)\end{aligned}$$

$\square$

Problem 6.3.2. Show that

$$xJ_n' = -nJ_n + xJ_{n-1}$$

Solution.

$$\begin{aligned}
J_n &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} \\
\therefore J_n' &= \sum_{r=0}^{\infty} \frac{(-1)^r(n+2r)}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1}\cdot\frac{1}{2} \\
\Rightarrow xJ_n' &= \sum \frac{(-1)^r(2n+2r-n)}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}\cdot\frac{1}{2} \\
\Rightarrow xJ_n' &= -n\sum \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} + \sum \frac{(-1)^r 2(n+r)}{r!\,(n+r)!}\cdot\frac{x^{n+2r}}{2} \\
\Rightarrow xJ_n' &= -nJ_n + x\sum \frac{(-1)^r}{r!\,(n+r-1)!}\cdot\frac{x^{n+2r-1}}{2} \\
\Rightarrow xJ_n' &= -nJ_n + xJ_{n-1}
\end{aligned}$$

6.4 Generating Function of the Bessel Function $J_n(x)$

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Proof. From the exponential series, we have

$$e^z = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \tag{6.18}$$

$$\begin{aligned}
e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} &= e^{\frac{1}{2}tx}\cdot e^{\frac{-x}{2t}} \\
&= \left[1 + \frac{tx}{2\cdot 1!} + \frac{t^2x^2}{2^2\cdot 2!} + \frac{t^3x^3}{2^3\cdot 3!} + \dots + \frac{t^nx^n}{2^n\cdot n!} + \frac{t^{n+1}x^{n+1}}{2^{n+1}\cdot(n+1)!} + \dots\right]\times \\
&\left[1 - \frac{x}{2t\cdot 1!} + \frac{x^2}{2^2\cdot t^2\cdot 2!} - \frac{x^3}{2^3\cdot t^3\cdot 3!} + \dots + (-1)^n\frac{x^n}{2^n\cdot t^n\cdot n!} + (-1)^{n+1}\frac{x^{n+1}}{2^{n+1}\cdot t^{n+1}\cdot(n+1)!} + \dots\right]
\end{aligned}$$

In this product the coefficient of t^n is

$$\begin{aligned}
&\frac{x^n}{2^n\cdot n!} - \frac{x^{n+1}}{2^{n+1}\cdot(n+1)!}\cdot\frac{x}{2} + \frac{x^{n+2}}{2^{n+2}\cdot(n+2)!}\cdot\frac{x^2}{2^2\cdot 2!} - \frac{x^{n+3}}{2^{n+3}\cdot(n+3)!}\cdot\frac{x^3}{2^3\,3!} + \dots \\
=&\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{1!\,(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \frac{1}{3!\,(n+3)!}\left(\frac{x}{2}\right)^{n+6} + \dots \\
=&\sum_{m=0}^{\infty}(-1)^m\frac{1}{m!\,(n+m)!}\left(\frac{x}{2}\right)^{n+2m} \\
=&J_n(x)
\end{aligned}$$

Also in the product, the coefficient of t^{-n} is

$$\begin{aligned}
&(-1)^n\left[2^n\frac{x^n}{n!} - \frac{x^{n+1}}{2^{n+1}\cdot(n+1)!}\cdot\frac{x}{2} + \frac{x^{n+2}}{2^{n+2}\cdot(n+2)!}\cdot\frac{x^2}{2^2\cdot 2!} - \frac{x^{n+3}}{2^{n+3}\cdot(n+3)!}\cdot\frac{x^3}{2^3\,3!} + \dots\right] \\
=&(-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{1!\,(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \frac{1}{3!\,(n+3)!}\left(\frac{x}{2}\right)^{n+6} + \dots\right] \\
=&(-1)^nJ_n(x) \\
=&J_{-n}(x)
\end{aligned}$$

Thus all the integral powers of t both positive and negative occur in the product.
Hence, we have

$$\begin{aligned}
e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} &= J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \dots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \dots \\
&= \sum_{n=\infty}^{\infty} t^nJ_n(x)
\end{aligned}$$

For this reason $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$ is called the generating function of Bessel function. ☐

Chapter 7

Hermite Polynomial

The differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} - 2x\frac{\mathrm{d}\, y}{\mathrm{d}\, x} + 2ny = 0$ when n is a constant, is called Hermite's differential equation. The solutions of Hermite's equation is called the Hermite polynomial. Hermite polynomial of order n is denoted and defined by

$$H_n(x) = \sum_{r=0}^{N} (-1)^r \frac{n!}{r!\,(n-2r)!}(2x)^{n-2r},\ N = \begin{cases} \frac{n}{2}, & n \text{ is even} \\ \frac{n-1}{2}, & n \text{ is odd} \end{cases} \tag{7.1}$$

7.1 Relation Between Legendre and Hermite Polynomial

Problem 7.1.1. Prove that

$$P_n(x) = \frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2} H_n(xt)\,\mathrm{d}\, t$$

Proof. We have

$$\begin{aligned} H_n(x) &= \sum_{r=0}^{N} (-1)^r \frac{n!}{r!\,(n-2r)!}(2x)^{n-2r} \\ \Rightarrow H_n(xt) &= \sum_{r=0}^{N} (-1)^r \frac{n!}{r!\,(n-2r)!}(2xt)^{n-2r} \end{aligned}$$

Now,

$$\begin{aligned} &\frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2} H_n(xt)\,\mathrm{d}\, t \\ =&\frac{2}{\sqrt{\pi}\, n!}\int_0^\infty t^n e^{-t^2}\left[\sum_{r=0}^{N} (-1)^r \frac{n!}{r!\,(n-2r)!}(2xt)^{n-2r}\right]\mathrm{d}\, t \\ =&\sum_{r=0}^{N}\frac{(-1)^r 2^{n-2r+1} x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!}\int_0^\infty e^{-t^2} t^{2n-2r}\,\mathrm{d}\, t \end{aligned} \tag{7.2}$$

Now,

$$\begin{aligned} &\int_0^\infty e^{-t^2} t^{2n-2r-1+1}\,\mathrm{d}\, t \\ =&\int_0^\infty e^{-t^2} t^{2\left(n-r+\frac{1}{2}\right)-1}\,\mathrm{d}\, t \\ =&\frac{1}{2}\Gamma\left(n-r+\frac{1}{2}\right) && \left|\ 2\int_0^\infty e^{-t^2} t^{2n-1}\,\mathrm{d}\, t = \Gamma(n)\right. \\ =&\frac{1}{2}\cdot\frac{(2n-2r)!}{2^{2n-2r}(n-r)!}\sqrt{\pi} && \left|\ \Gamma\left(x+\frac{1}{2}\right) = \frac{(2x)!}{2^{2x}\,(x)!}\sqrt{\pi}\right. \end{aligned}$$

From (7.2),

$$\begin{aligned} &\sum_{r=0}^{N}\frac{(-1)^r 2^{n-2r+1} x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!}\frac{1}{2}\cdot\frac{(2n-2r)!}{2^{2n-2r}(n-r)!}\sqrt{\pi} \\ =&\sum_{r=0}^{N}(-1)^r\frac{(2n-2r)}{2^n\, r!\,(n-r)!\,(n-2r)!}x^{n-2r} \\ =&P_n(x) \end{aligned}$$

$$\therefore P_n(x) = \frac{2}{\sqrt{\pi}\, n!} \int_0^\infty t^n e^{-t^2} H_n(xt)\, \mathrm{d}\, t$$

□

7.2 Generating Function of Hermite Polynomial

Problem 7.2.1. Prove that

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

Proof.

$$\begin{aligned}
e^{2tx-t^2} &= e^{2tx} \cdot e^{-t^2} \\
&= \left\{1 + \frac{2tx}{1!} + \frac{2^2t^2x^2}{2!} + \dots + \frac{(2tx)^r}{r!} + \dots\right\} \times \left\{1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} \dots + \frac{(t^2)^s}{s!} + \dots\right\} \\
&= \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \sum_{s=0}^{\infty} \frac{(-1)^s\,(t^2)^s}{s!} \\
&= \sum_{r,s=0}^{\infty} (-1)^s \frac{(2x)^r \cdot t^{r+2s}}{r!\, s!}
\end{aligned}$$

Let $r + 2s = n$ so that $r = n - 2s$
so for a fixed value of s, the coefficient of t^n is given by

$$(-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\, s!}$$

The total value of t^n is obtained by summing over all allowed values of s and since $r = n - 2s$.
$\therefore\ n - 2s \geq 0$ or $s \leq \frac{n}{2}$
Thus if n is even, s goes from 0 to $\frac{n}{2}$ and if n is odd, s goes from 0 to $\frac{n-1}{2}$.
So coefficient of t^n

$$\begin{aligned}
t^n &= \sum_{s=0}^{\frac{n}{2}} \frac{(-1)^s(2x)^{n-2s}}{(n-2s)!\, s!} \\
&= \sum_{s=0}^{\frac{n}{2}} \frac{(-1)^s n!}{(n-2s)!\, s!} \cdot (2x)^{n-2s} \cdot \frac{1}{n!} \\
&= \frac{H_n(x)}{n!}
\end{aligned}$$

Since

$$\sum_{r=0}^{\frac{n}{2}} (-1)^r \frac{n!}{r!\,(n-2r)!} (2x)^{n-2r}$$

Hence

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

$$\text{Or, } e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

□

7.3 Hermite Polynomials of Different Forms

Theorem 7.3.1. Prove that

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \left(e^{-x^2}\right)$$

Proof. Using the generating function, we have

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \tag{7.3}$$

Or,

$$f(x,t) = e^{x^2-(t-x)^2} = \frac{H_0(x)}{0!}t^0 + \frac{H_1(x)}{1!}t^1 + \frac{H_2(x)}{2!}t^2 + \cdots + \frac{H_n(x)}{n!}t^n + \frac{H_{n+1}(x)}{(n+1)!}t^{n+1} + \ldots \tag{7.4}$$

Differentiating both sides of (7.4) partially with respect to t n times

$$e^{x^2} \cdot \frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\} = 0 + \frac{H_n(x)}{n!}n! + \frac{H_{n+1}(x)}{(n+1)!}(n+1)n(n-1)\ldots 2t \cdots + \ldots \tag{7.5}$$

Putting $t = 0$ in (7.5), we get

$$\begin{aligned} e^{x^2}\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} &= \frac{H_n(x)\, n!}{n!} + 0 \\ \Rightarrow e^{x^2}\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} &= H_n(x) \\ \Rightarrow H_n(x) = e^{x^2}&\left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} \end{aligned} \tag{7.6}$$

Putting $t - x = u$ so that $\frac{\partial}{\partial t} = \frac{\partial}{\partial u}$
But at $t = 0$ $-x = u$ i.e., $x = -u$
Therefore,

$$\begin{aligned} \left[\frac{\partial^n}{\partial t^n}\left\{e^{-(t-x)^2}\right\}\right]_{t=0} &= \frac{\partial^n}{\partial u^n}\left(e^{-u^2}\right) \\ &= (-1)^n \frac{\partial^n}{\partial\, x^n}\left(e^{-x^2}\right) \\ &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(e^{-x^2}\right) \end{aligned}$$

Thus from (7.6), we get

$$\begin{aligned} H_n(x) &= e^{x^2} \cdot (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(e^{-x^2}\right) \\ \Rightarrow H_n(x) &= (-1)^n \cdot e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left(e^{-x^2}\right) \end{aligned}$$

Which is also known as the *Rodrigue's formula for* $H_n(x)$. □

7.4 Orthogonality Properties of Hermite Polynomials

Problem 7.4.1. Prove that

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,\mathrm{d}\,x = \begin{cases} 0 & \text{if } m \neq n \\ 2^n \sqrt{\pi}\, n! & \text{if } m = n \end{cases}$$

Proof. From generating function of Hermite polynomial, we have

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{7.7}$$

and

$$e^{-s^2+2sx} = \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!} \tag{7.8}$$

Multiplying the corresponding sides of (8.2) and (8.3), we can write

$$e^{-t^2+2tx} \cdot e^{-s^2+2sx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \cdot \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$$
$$\Rightarrow e^{2tx-t^2+2sx-s^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{H_n(x)\, H_m(x)\, t^n\, s^m}{n!\, m!} \tag{7.9}$$

Multiplying both sides of (8.4) by e^{-x^2} and then integrating both sides with respect to x from $-\infty$ to ∞, we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \,\mathrm{d}\,x \right] \frac{t^n\, s^m}{n!\, m!}$$
$$= \int_{-\infty}^{\infty} e^{-x^2+2(t+s)x\left(t^2+s^2\right)} \,\mathrm{d}\,x$$
$$= \int_{-\infty}^{\infty} e^{-x^2+2(t+s)x\left(t^2+s^2\right)} \times e^{(t+s)^2-\left(t^2+s^2\right)} \,\mathrm{d}\,x$$
$$= e^{2st} \int_{-\infty}^{\infty} e^{-(x-(t+s))^2} \,\mathrm{d}\,x$$

Putting $x - (t+s) = y$ so that $\mathrm{d}\,x = \mathrm{d}\,y$

$$\text{Limits} \quad \left.\begin{matrix} x = \infty \\ y = \infty \end{matrix}\right\} \qquad \left.\begin{matrix} x = -\infty \\ y = -\infty \end{matrix}\right\}$$
$$= e^{2st} \cdot \int_{-\infty}^{\infty} e^{-y^2} \,\mathrm{d}\,y$$
$$= e^{2st} \cdot 2 \int_{0}^{\infty} e^{-y^2} \,\mathrm{d}\,y$$
$$= 2e^{2st} \cdot \frac{\sqrt{pi}}{2} \qquad \text{since} \int_0^{\infty} e^{-y^2} \,\mathrm{d}\,y = \frac{\sqrt{\pi}}{2}$$
$$= \sqrt{\pi} e^{2st}$$
$$= \sqrt{\pi} \left[1 + \frac{2st}{1!} + \frac{(2st)^2}{2!} + \frac{(2st)^3}{3!} + \dots \right]$$
$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2st)^n}{n!}$$
$$= \sqrt{\pi} \sum_{n=0}^{\infty} 2^n \frac{s^n t^n}{n!}$$

Thus coefficient of $t^n s^m$ in the expansion of $\int_{-\infty}^{\infty} e^{-x^2+2(t+s)x\left(t^2+s^2\right)} \,\mathrm{d}\,x$ is $\begin{cases} 0 & \text{if } m \neq n \\ \frac{2^n \sqrt{\pi}}{n!} & \text{if } m = n \end{cases}$

Hence

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \,\mathrm{d}\,x = \begin{cases} 0 & \text{if } m \neq n \\ 2^n \sqrt{\pi}\, n! & \text{if } m = n \end{cases}$$

□

Making use of the *kronecker delta* we have

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \,\mathrm{d}\,x = \sqrt{\pi} 2^n\, n!\, \delta_{mn} \qquad \text{since } \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$
$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} (H_n(x))^2 \,\mathrm{d}\,x = \sqrt{\pi} 2^n\, n!$$

7.5 Recurrence Relation of Hermite Polynomial

(i) $H_n'(x) = 2nH_{n-1}(x)$, $n \geq 1$; $H_0'(x) = 0$.

(ii) $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$, $n \geq 1$; $H_1(x) = 2xH_0(x)$.

(iii) $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$.

(iv) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$.

7.6 Integral Formula for Hermite Polynomial

Let us assume

$$y_n = \frac{1}{2\pi\, i} \oint \rho^{-n-1} e^{\left\{x^2-(\rho-x)^2\right\}}\,\mathrm{d}\,\rho \tag{7.10}$$

where the contour is taken around a circle having centre at origin.
If we differentiate (9.3) with respect to x we get,

$$\begin{aligned}
\frac{\mathrm{d}\,y_n}{\mathrm{d}\,x} &= \frac{1}{2\pi i} \oint 2\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}}\,\mathrm{d}\,\rho \\
\Rightarrow \frac{\mathrm{d}^2\,y_n}{\mathrm{d}\,x^2} &= \frac{1}{2\pi i} \oint 4\rho^{-n+1} e^{\left\{x^2-(\rho-x)^2\right\}}\,\mathrm{d}\,\rho
\end{aligned}$$

Thus

$$\begin{aligned}
& y_n^{''} - 2xy_n^{'} - 2ny_n \\
=& \frac{1}{2\pi i} \oint 4\rho^{-n+1} e^{\left\{x^2-(\rho-x)^2\right\}}\,\mathrm{d}\,\rho - \frac{2x}{2\pi i} \oint 2\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}}\,\mathrm{d}\,\rho + \frac{2n}{2\pi\, i} \oint \rho^{-n-1} e^{\left\{x^2-(\rho-x)^2\right\}}\,\mathrm{d}\,\rho \\
=& \frac{1}{2\pi i} \oint \left(4\rho^2 - 4x\rho + 2n\right) e^{\left\{x^2-(\rho-x)^2\right\}} \rho^{-n-1}\,\mathrm{d}\,\rho \\
=& \frac{-2}{2\pi i} \oint \frac{\mathrm{d}}{\mathrm{d}\,\rho} \left[\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}}\right]\,\mathrm{d}\,\rho
\end{aligned}$$

But

$$\oint \frac{\mathrm{d}}{\mathrm{d}\,\rho} \left[\rho^{-n} e^{\left\{x^2-(\rho-x)^2\right\}}\right]\,\mathrm{d}\,\rho = 0$$

$$\therefore y_n^{''} - 2xy_n^{'} - 2ny_n = 0$$

Which is Hermite equation and hence y_n given by (9.3) is also a solution of the Hermite equation.

So we may have $H_n(x) = cy_n(x)$, c is a constant but if we put $x = 0$ in (7.1) we have,

$$H_n(0) = (-1)^{\frac{n}{2}} \frac{n!}{\frac{n}{2}}$$

and form (9.3),

$$y_n(0) = \frac{1}{2\pi\, i} \oint \rho^{-n-1} e^{-(\rho-x)^2}\,\mathrm{d}\,\rho$$

But by contour integration we have,

$$\oint \rho^{-n-1} e^{-(\rho-x)^2}\,\mathrm{d}\,\rho = \frac{2\pi i (-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}$$

$$\therefore y_n(0) = \frac{(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}$$

Thus

$$\begin{aligned}
& H_n(0) = cy_n(0) \\
\Rightarrow & (-1)^{\frac{n}{2}} \frac{n!}{\frac{n}{2}} = \frac{c(-1)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} \Rightarrow \qquad c = n! \\
& \therefore H_n(x) = n!\, y_n(x) \\
\Rightarrow & H_n(x) = \frac{n!}{2\pi\, i} \oint \rho^{-n-1} e^{\left\{x^2-(\rho-x)^2\right\}}\,\mathrm{d}\,\rho
\end{aligned}$$

Which is the integral form of Hermite polynomial.

Chapter 8

Laguerre Polynomial

We define the standard solution of Laguerre's differential equation $x\,\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2}+(1-x)\,\frac{\mathrm{d}\,y}{\mathrm{d}\,x}+ny=0$ as that for which $c_0=1$ and call it the Laguerre polynomial of order n and is denoted by $L_n(x)$.

$$\therefore\ L_n(x)=\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!\,(r!)^2}x^r$$

8.1 Generating Function of Laguerre Polynomial

Problem 8.1.1. Prove that

$$\frac{1}{(1-t)}e^{\frac{-tx}{1-t}}=\sum_{n=0}^{\infty}t^n\,L_n(x)$$

Proof. From exponential series we have

$$e_x=1\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^r}{r!}+\ldots \tag{8.1}$$

$$\begin{aligned}
\therefore\ \frac{1}{(1-t)}e^{\frac{-tx}{1-t}} &= \frac{1}{1-t}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{-tx}{1-t}\right)^r \qquad\qquad \text{[using 8.1]}\\
&=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!}\frac{t^r\,x^r}{(1-t)^{r+1}}\\
&=\sum_{r=0}^{\infty}(-1)^r\frac{t^r\,x^r}{r!}(1-t)^{-(r+1)}\\
&=\sum_{r=0}^{\infty}(-1)^r\frac{t^r\,x^r}{r!}\left[1+(r+1)^r+\frac{(r+1)(r+2)}{2!}t^2+\frac{(r+1)(r+2)(r+3)}{3!}t^3+\ldots\right]\\
&=\sum_{r=0}^{\infty}\left[(-1)^t\frac{t^r\,x^r}{r!}\sum_{s=0}^{\infty}\frac{(r+s)!}{r!\,s!}t^s\right] \qquad\qquad \text{[using binomial theorem]}\\
&=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r\frac{(r+s)!}{(r!)^2\,s!}x^r\,t^{r+s}
\end{aligned}$$

Let r be fixed. The coefficient of t^n can be obtained by setting $r+s=n$ i.e., $s=n-r$. Hence, for a fixed value of r the coefficient of t^n is given by

$$(-1)^r\frac{n!}{(r!)^2(n-r)!}x^r$$

Therefore, the total coefficient of t^n is obtained by summing over all allowed values of r.

Since $s=n-r$ and $s\geq 0$.

$\therefore\ n-r\geq 0$ or, $r\leq n$.

Hence, the coefficient of t^n is

$$\sum_{r=0}^{\infty}(-1)^r\frac{n!}{(r!)^2\,(n-r)!}x^r=L_n(x)$$

Thus

$$\frac{1}{(1-t)}e^{\frac{-tx}{1-t}}=\sum_{n=0}^{\infty}t^n\,L_n(x)$$

□

8.2 Rodrigue's Formula for Laguerre Polynomial

Problem 8.2.1. Prove that

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^n e^{-x}\right)$$

Proof. Right-hand side

$$\begin{aligned}
&\frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^n e^{-x}\right)\\
=&\frac{e^x}{n!}\left[x^n(-1)^n e^{-n} + n\cdot nx^{n-1}(-1)^{n-1}e^{-x} + \frac{n(n-1)}{2!}n(n-1)x^{n-2}(-1)^{n-2}e^{-x} + \cdots + n!e^{-x}\right]\\
=&\frac{e^x\cdot e^{-x}}{n!}\left[(-1)^n x^n + \frac{n(n!)}{1!(n-1)!}x^{n-1} + (-1)^{n-2}\frac{n(n-1)}{2!}\cdot\frac{n!}{(n-2)!}x^{n-2} + \cdots + n!\right]\\
=&(-1)^n\cdot\frac{n!}{(n!)^2}x^n + (-1)^{n-1}\frac{n!}{1!\{(n-1)!\}^2}x^{n-1} + (-1)^{n-2}\frac{n!}{2!\{(n-2)!\}^2}x^{n-2} + \ldots\frac{n!}{n!}\\
=&\sum_{r=0}^{n}(-1)^r\frac{n!\,x^r}{\{r!\}^2\,(n-r)!}\\
=&L_n(x)
\end{aligned}$$

Hence

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^n e^{-x}\right)$$

□

8.3 Orthogonality Property of Laguerre Polynomials

Problem 8.3.1. Prove that

$$\int_0^\infty e^{-x}L_n(x)L_m(x)\,\mathrm{d}\,x = \begin{cases}0 & \text{if } m\neq n\\ 1 & \text{if } m=n\end{cases}$$

or, Prove that

$$\int_0^\infty e^{-x}L_n(x)L_m(x)\,\mathrm{d}\,x = \delta_{mn}$$

or, Show that Laguerre polynomials are orthogonal over $(0,\infty)$ with respect to the weighted function e^{-x}.

Proof. From generating function of Laguerre polynomial we have,

$$\sum_{n=0}^{\infty} t^n\,L_n(x) = \frac{1}{(1-t)}e^{\frac{-tx}{1-t}}$$

and

$$\sum_{m=0}^{\infty} s^m\,L_m(x) = \frac{1}{(1-s)}e^{\frac{-sx}{1-s}}$$

$$\begin{aligned}
\therefore \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_n(x)\,L_m(x)t^n s^m &= \frac{1}{(1-t)(1-s)}e^{\frac{-tx}{1-t}}\cdot e^{\frac{-sx}{1-s}}\\
&= \frac{1}{(1-t)(1-s)}e^{-x\left\{\frac{t}{1-t}+\frac{s}{1-s}\right\}} \qquad (8.2)
\end{aligned}$$

Multiplying both sides of (8.2) by e^{-x} and then integrating both sides with respect to x from 0 to ∞, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\,\mathrm{d}\,x\right] t^n s^m \\
&= \frac{1}{(1-t)(1-s)}\int_0^{\infty} e^{-x\left\{1+\frac{t}{1-t}+\frac{s}{1-s}\right\}}\,\mathrm{d}\,x \\
&= \frac{1}{(1-t)(1-s)}\left[\frac{e^{-x\left\{1+\frac{t}{1-t}+\frac{s}{1-s}\right\}}}{-\left(1+\frac{t}{1-t}+\frac{s}{1-s}\right)}\right]_0^{\infty} \\
&= \frac{1}{(1-t)(1-s)}\cdot\frac{1}{1+\frac{t}{1-t}+\frac{s}{1-s}} \\
&= \frac{1}{(1-t)(1-s)+t(1-s)+s(1-t)} \\
&= \frac{1}{1-t-s+st+t-ts+s-st} \\
&= \frac{1}{1-st} \\
&= (1-st)^{-1} \\
&= 1+st+(st)^2+\dots+(st)^n+\dots \\
&= \sum_{n=0}^{\infty} s^n t^n \qquad \text{using binomial theorem}
\end{aligned} \tag{8.3}$$

Now we see that the indices of t and s are always equal in each term on right hand side of (8.3). Hence when $m \neq n$, equating coefficient of $t^n s^m$ on both sides of (8.3) gives

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,\mathrm{d}\,x = 0 \qquad \text{if } m \neq n \tag{8.4}$$

Again equating coefficients of $t^n s^m$ on both sides of (8.3) gives

$$\int_0^{\infty}\left(L_n(x)\right)^2\mathrm{d}\,x = 1 \tag{8.5}$$

Hence

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,\mathrm{d}\,x = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{8.6}$$

Let

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{8.7}$$

Thus from (8.6) and (8.7), we have

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,\mathrm{d}\,x = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

□

8.4 Recurrence Formula for Laguerre Polynomial

(i) $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$

(ii) $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$

(iii) $L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)$

8.5 Problems on Laguerre Polynomial

Problem 8.5.1. Expand $x^3 + x^2 - 3x + 2$ in a series of Laguerre polynomial.

Solution. Let $f(x) = x^3 + x^2 - 3x + 2$. By definition of Laguerre polynomial, we know that $L_n(x)$ is a polynomial of degree n. Since $x^3 + x^2 - 3x + 2$ is a polynomial of degree 3, we may write

$$\begin{aligned}
&x^3 + x^2 - 3x + 2 \\
&= C_0L_0(x) + C_1L_1(x) + C_2L_2(x) + C_3L_3(x) \\
&= C_0 + C_1(1-x) + C_2\left(1 - 2x + \frac{1}{2}x^2\right) + C_3\left(1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3\right) \\
&= C_0 + C_1 - C_1x + C_2 - 2C_2x + \frac{C_2}{2}x^2 + C_3 - 3C_3x + \frac{3}{2}C_3x^2 - \frac{1}{6}C_3x^3 \\
&= (C_0 + C_1 + C_2 + C_3) - (C_1 + 2C_2 + 3C_3)x + \left(\frac{C_3}{2} + \frac{3}{2}C_3\right)x^2 - \frac{1}{6}C_3x^3 \qquad (8.8)
\end{aligned}$$

Equating the coefficients of like powers of x from both sides of (8.8), we get

$$\begin{aligned}
C_0 + C_1 + C_2 + C_3 &= 2 \\
C_1 + 2C_2 + 3C_3 &= 3 \\
\frac{C_3}{2} + \frac{3}{2}C_3 &= 1 \\
-\frac{1}{6}C_3 &= 1
\end{aligned}$$

Solving these for C_0, C_1, C_2, and C_3, we get

$$C_3 = -6, \quad C_2 = 20, \quad C_1 = -19, \quad C_0 = -7$$

Thus $f(x) = x^3 + x^2 - 3x + 2 = 7L_0(x) - 19L_1(x) + 20L_2(x) - 6L_3(x)$.

Chapter 9

Hypergeometric Function

9.1 Introduction

The hypergeometric differential equation is an equation of the form

$$\left(x^2 - x\right) y'' + \left[(1+\alpha+\beta)x - \gamma\right] y' + \alpha\beta y = 0 \tag{9.1}$$

where the parameters α, β, γ are constant, and it is assumed that γ is not a negative integer.

Equation (9.1) can be written as

$$y'' + X_1 y' + X_2 y = 0 \tag{9.2}$$

where

$$X_1 = \frac{(1+\alpha+\beta)x - \gamma}{x(x-1)}, \qquad X_2 = \frac{\alpha\beta}{x(x-1)}$$

Equation (9.1) and (9.2) has singularities at $x = 0$, 1 and ∞.

For $x = 0$, the general solution of (9.1) is $y = Au + Bv$, where A, B are constant and

$$\begin{aligned} u &= 1 + \frac{\alpha\beta}{1\cdot\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \dots \\ &= \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{r!\,(\gamma)_r}x^r \\ &= F(\alpha,\beta;\gamma;x) \quad \text{or} \quad {}_2F_1(\alpha,\beta;\gamma;x) \end{aligned}$$

and

$$v = x^{1-\gamma}F(\alpha',\beta';\gamma';x) \qquad \text{Where, } \alpha' = 1-\gamma+\alpha,\ \beta' = 1-\gamma+\beta,\ \gamma' = 2-\gamma$$

Similarly, *for* $x = 1$ *and* $x = \infty$, the solutions of (9.1) are

$$y = AF(\alpha,\beta;1+\alpha+\beta-\gamma;1-x) + B(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-x)$$

and

$$y = Ax^{-\alpha}F\left(\alpha,\alpha-\beta+1;\alpha-\beta+1;\frac{1}{x}\right) + Bx^{-\beta}F\left(\beta,\beta-\gamma+1;\beta-\alpha+1;\frac{1}{x}\right)$$

respectively.

One of the solutions of the hypergeometric differential equation

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\,r!}x^r$$

is known as hypergeometric function.

9.1.1 Pochhammer Symbol

The Pochhammer symbol is denoted and defined by

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+r-1), \qquad \text{with } (\alpha)_0 = 1$$

$(\alpha)_r$ can also be expressed as

$$(\alpha)_r = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}$$

9.2 Integral Formula for the Hypergeometric Function

Problem 9.2.1. If $|x| < 1$ and if $\gamma > \beta > 0$, prove that

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, \mathrm{d}\, t $$

Proof. By definition, we have

$$ F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!} x^r \tag{9.3} $$

where

$$ \begin{aligned} (\alpha)_r &= \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1) \\ &= \frac{1\cdot 2\cdot 3\ldots(\alpha-1)\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1)}{1\cdot 2\cdot 3\ldots(\alpha-1)} \\ &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \end{aligned} $$

$$ \begin{aligned} \therefore\ \frac{(\beta)_r}{(\gamma)_r} &= \frac{\Gamma(\beta+r)}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+r)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta+r)\Gamma(\gamma-\beta)}{\Gamma(\gamma+r)\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta+r)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma+r-\beta)} \cdot \frac{1}{\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta+r)\Gamma(\gamma-\beta)}{\Gamma(\beta+r+\gamma-\beta)} \cdot \frac{1}{\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\cdot\Gamma(\gamma-\beta)} \cdot B(\beta+r, \gamma-r) \qquad \text{where } \beta+r>0\, \gamma-\beta>0 \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\cdot\Gamma(\gamma-\beta)} \cdot \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1}\, \mathrm{d}\, t \qquad \text{since } \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, \mathrm{d}\, t \\ &= \frac{1}{\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)}} \cdot \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1}\, \mathrm{d}\, t \\ &= \frac{1}{B(\beta, \gamma-\beta)} \cdot \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1}\, \mathrm{d}\, t \end{aligned} $$

Thus from (9.3), we have

$$ \begin{aligned} F(\alpha, \beta; \gamma; x) &= \sum_{r=0}^{\infty} \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta+r-1}(1-t)^{\gamma-\beta-1} \times \frac{(\alpha)_r}{r!} \cdot x^r\, \mathrm{d}\, t \\ &= \sum_{r=0}^{\infty} \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \times \frac{(\alpha)_r}{r!} \cdot (xt)^r\, \mathrm{d}\, t \\ &= \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \left\{ \sum_{r=0}^{\infty} \frac{(\alpha)_r (xt)^r}{r!} \right\} \mathrm{d}\, t \end{aligned} $$

Note. The general term in the expansion of $(1-xt)^{-\alpha}$ is[1]

$$\begin{aligned}(1-xt)^{-\alpha} &= \frac{(-\alpha)(-\alpha-1)\dots(-\alpha-r+1)}{r!}(-xt)^r \\ &= (-1)^r\frac{\alpha(\alpha+1)\dots(\alpha+r-1)}{r!}(-1)^r(xt)^r \\ &= \frac{\alpha(\alpha+1)\dots(\alpha+r-1)}{r!}x^r\,t^r \\ &= \frac{(\alpha)_r}{r!}\cdot x^r\cdot t^r\end{aligned}$$

$$\therefore\ F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,\mathrm{d}\,t$$

$$\text{or, } F(\alpha,\beta;\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,\mathrm{d}\,t$$

Which is known as the integral formula for hypergeometric function and is valid if $|x|<1$ and $\gamma>\beta>0$. □

9.3 Gauss's Theorem

Theorem 9.3.1.

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

Proof. From the definition of integral formula for the hypergeometric function, we have,

$$F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,\mathrm{d}\,t \tag{9.4}$$

Putting $x=1$ in (9.4), we get,

$$\begin{aligned}F(\alpha,\beta;\gamma;1) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha}\,\mathrm{d}\,t \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{(\gamma-\beta-\alpha)-1}\,\mathrm{d}\,t \\ &= \frac{B(\beta,\gamma-\beta-\alpha)}{B(\beta,\gamma-\beta)} \\ &= \frac{\dfrac{\Gamma(\beta)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\beta+\gamma-\beta-\alpha)}}{\dfrac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)}} \\ &= \frac{\Gamma(\beta)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)}\times\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\end{aligned}$$

Hence $F(\alpha,\beta;\gamma;1) = \dfrac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$ □

[1]

$$\sum_{r=0}^{\infty}\frac{(\alpha)_r(xt)^r}{r!} = 1+\frac{\alpha(xt)}{1!}+\frac{\alpha(\alpha+1)}{2!}(xt)^2+\cdots = (1-xt)^{-\alpha}$$

9.4 Problems

Problem 9.4.1. Prove that

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$$

Proof. From Rodrigue's formula for Legendre polynomial, we have

$$\begin{aligned}
P_n(x) &= \frac{1}{2^n\, n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left(x^2-1\right)^n \\
&= \frac{1}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[(x-1)^n\left\{\frac{1}{2}(x+1)\right\}^n\right] \\
&= \frac{(-1)^n}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[(1-x)^n\left\{1-\frac{1}{2}(1-x)\right\}^n\right] \\
&= \frac{(-1)^n}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[(1-x)^n\left\{1-n\frac{1}{2}(1-x)+\frac{n(n-1)}{2!}\cdot\frac{(1-x)^2}{4}-\frac{n(n-1)(n-2)}{3!}\cdot\frac{(1-x)^3}{8}+\dots\right\}\right] \\
&= \frac{(-1)^n}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[(1-x)^n-n\frac{1}{2}(1-x)^{n+1}+\frac{n(n-1)}{2!\,2^2}\cdot(1-x)^{n+2}-\frac{n(n-1)(n-2)}{3!\cdot 2^3}\cdot(1-x)^{n+3}+\dots\right] \\
&= \frac{(-1)^n}{n!}\left[(-1)^n\,n!-\frac{n}{2}(-1)^n\frac{(n+1)!}{1!}(1-x)+\frac{n(n-1)}{2!}(-1)^n\frac{(n+2)!}{2!}(1-x)^2-\dots\right] \\
&= 1+\frac{(-n)(n+1)}{1\cdot 1!}\left(\frac{1-x}{2}\right)+\frac{(-n)(-n+1)(n+1)(n+2)}{1\cdot 2\cdot 2!}\left(\frac{1-x}{2}\right)^2+\dots \\
&= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)
\end{aligned}$$

Hence $P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right)$. □