Real Analysis II

MAT311

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Preface

This is a compilation of lecture notes with some books and my own thoughts. This document is not a holy text. So, if there is a mistake, solve it by your own judgement. Every “iff” from sheets are replaced as “if and only if”.

Contents

Part I

Class Note/Sheet(Before Pandemic)

Chapter 1

Metric Space

1.1 Euclidean Space

Euclidean space or Euclidean n-space, denoted by $\mathbb{R}^n$ consists of all ordered n-tuples of real numbers. Symbolically, $\mathbb{R}^n = \{x \mid x = (x_1, x_2, \ldots, x_n), x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
Here the element $x \in \mathbb{R}^n$ is called a point or a vector and $x_1, x_2, \ldots, x_n$ are called coordinates of x when $n > 1$.
If $x, y \in \mathbb{R}^n$ and if $\alpha \in \mathbb{R}^n$ then put,
$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ so that $x + y \in \mathbb{R}^n$ and $\alpha x \in \mathbb{R}^n$. This defines addition and scalar multiplication of vectors. These two operations satisfy the commutative, associative and distributive laws and make $\mathbb{R}^n$ into a vector space over the real field.

Theorem 1.1.1. $\mathbb{R}^n$ *with operations of addition and scalar multiplication defined previously is a vector space of dimension* n.

Definition 1 (Inner Product)**.** The inner product (or scalar product) of x and y in $\mathbb{R}^n$ is defined by $\langle x, y\rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$ and the *norm* or *length* of a vector $x \in \mathbb{R}^n$ is defined by $||x|| = \langle x, x\rangle^{1/2} = \sum_{i=1}^{n} (x_i^2)^2$ and the *distance* between two vectors x and y of $\mathbb{R}^n$ is the real number d(x, y) $= ||x - y|| = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}$

Definition 2. Let X be a metric space. All points and sets involved below are understood to be elements and subset of X.

1. A *neighborhood* of a point $p \in X$ is a set $N_\delta(p)$ containing all points q such that $d(p, q) < \delta$. The number δ is called the *radius* of $N_\delta(p)$. [Mathematically, $N_\delta(p) = \{q \mid d(p, q) < \delta\}$]
2. A point p is a *limit point* (accumulation point or cluster point) of the set E if every neighborhood $N_\delta(p)$ contains a point $q \neq p$ such that $q \in E$. [Mathematically, $(N_\delta(p) - \{p\}) \cap E \neq \varnothing$]
3. If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
4. E is *closed* if every limit point of E is a point of E.
5. A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$
   (i) E is *open* if every point of E is an interior point of E.
6. The *complement* of E, denoted by E^c is the set of all points $p \in X$ such that $p \notin E$
7. E is *perfect* if E is closed and every point of E is a limit point of E.
8. E is *bounded* if E is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

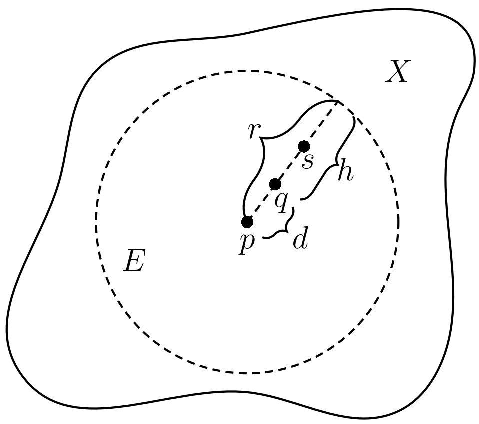
9. E is *dense* in X if every point of X is a limit point of E, or a point of E (or both).

Note. The segment $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

Note. In $\mathbb{R}^1$ neighborhoods are segments, whereas in $\mathbb{R}^2$ neighborhoods are interiors of circles and in $\mathbb{R}^3$ neighborhoods are interiors of spheres.

Theorem 1.1.2. *Every neighborhood is an open set.*

Proof. Consider the neighborhood $E = N_r(p) = \{q \in X \mid d(p, q) < r\}$ and let q be any point of E, where X is a metric space.



Then there is a positive real number h such that $d(p, q) = r - h$
Now for all points s such that $d(q, s) < h$,
We have then

$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r$	Here $d(p, q) < r$
so that $s \in E$	$\Rightarrow d(p, q) + h = r$
Therefore, $N_h(q) = \{s \in E \mid d(q, s) < h\} \subset E = N_r(p)$	$\Rightarrow d(p, q) = r - h$
Thus, q is an interior point of E.	$\because r - h = d(p, q) \geq 0$
Hence the theorem.	$\Rightarrow h \leq r$

□

Theorem 1.1.3. *If p is a limit point of a set E in a metric space X, then every neighborhood of p contains infinitely many points of E.*

Proof. Suppose there is a neighborhood N of $p \in X$ which contains only a finite number of points of E. Let $q_1, q_2, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p and put $r = \min_{1 \leq m \leq n} d(p, q_m)$.
[We use this notation to denote the smallest of the numbers $d(p, q_1), d(p, q_2), \ldots, d(p, q_n)$]
The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no part q of E such that $q \neq p$, so that p is not a limit point of E.

This contradiction establishes the theorem. □

Note. Here $r > 0 \Rightarrow r$ can be taken a large positive real number, however we please $\Rightarrow d(p, q_m), m = 1, 2, \ldots, n$ are bigger & bigger $\Rightarrow q_m$ are not close enough to $p \Rightarrow p$ is not a limit point of E.

Corollary 1.1.4. *A finite set has no limit points.*

Problem 1.1.1. Let us consider the following subsets of $\mathbb{R}^2$

1. The set of all complex Z such that $|Z| < 1$
2. The set of all complex Z such that $|Z| \leq 1$
3. A finite set

4. The set of all integers
5. The set consisting of the numbers $\frac{1}{n}(n = 1, 2, 3 \dots)$
6. The set of all complex numbers (that is, $\mathbb{R}^2$)
7. The segment (a, b)

If (4), (5) and (7) are regarded as subsets of $\mathbb{R}^1$, then identify whether the sets (1)-(7) are closed, open, perfect and bounded.

Theorem 1.1.5. *Let $\{E_\alpha\}$ be a collection of sets E_α, then $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha ({E_\alpha}^c)$*

Theorem 1.1.6. *A set E is open if and only if its complement is closed.*

Proof. First, suppose E^c is closed. Cause $x \in E$. Then $x \notin E^c$ and x is not a limit point E^c. Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E and E is open.

Next, suppose that E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, such that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed. □

Corollary 1.1.7. *A set F is closed if and only if its complement is open.*

Theorem 1.1.8.

1. *For any collection $\{G_\alpha\}$ is open sets, $\underset{\alpha}{\cup} G_\alpha$ is open.*
2. *For any collection $\{F_\alpha\}$ is closed sets, $\underset{\cap}{\alpha} F_\alpha$ is closed.*
3. *For any finite collection $G_1, G_2, \dots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.*
4. *For any finite collection $F_1, F_2, \dots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed.*

Note. Is the finiteness of the collection in parts (3) and (4) of the above theorem essential? Justify your answer.

Definition 3. If X is a metric space, if $E \subset X$ and if E' denotes the set of all limit points of E in X, then E' is called the derived set of E and $\bar{E} := E \cup E'$ is called the closure of E.

Example. $E = (0, 1) \cup \left\{e, \pi, \sqrt{7}, 11.5\right\}$, then $E' = [0, 1]$, $\bar{E} = E' \cup E = [0, 1] \cup \left\{e, \pi, \sqrt{7}, 11.5\right\}$

Theorem 1.1.9. *If X is a metric space and $E \subset X$, then*

(a) $\bar{E}$ is closed,

(b) $E = \bar{E}$ if and only if E is closed,

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

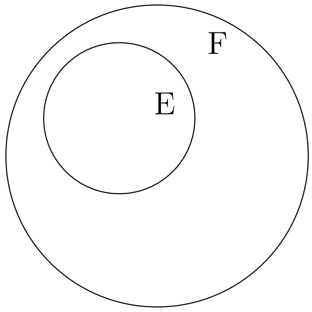
By 1 and 3, $\bar{E}$ is the smallest closed subset of X that contains E.

Proof.

(a) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E. Hence, p has a neighborhood which does not intersect E. The complement of $\bar{E}$ is therefore open. Hence, $\bar{E}$ is closed.

(b) If $E = \bar{E}$, (a) implies that E is closed. If E is closed, then $E' \subset E$(by definition (1) and (10)). Hence, $\bar{E} = E$.

(c) If E is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus, $F \supset \bar{E}$.



$F \supset F'$
$F \supset E$
$F \supset E'$
$F \supset \bar{E}$

□

1.2 Connected Set

Let A be a subset of metric space X. Two non-empty open sets U and V are said to *separate* A if they satisfy these condition

(i) $U \cap V \cap A = \varnothing$
(ii) $A \cap U \neq \varnothing$
(iii) $A \cap V \neq \varnothing$
(iv) $A \subset U \cup V$

We say that A is *disconnected (i.e., not connected)* if such set exist and if such sets do not exist, we say that A is *connected.*

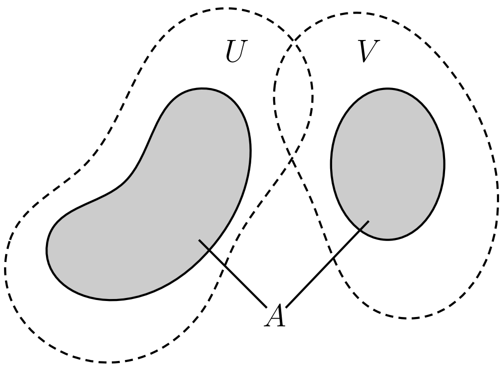


Figure 1.1: A is disconnected

Example.

(i) $\bar{\mathbb{Z}}$ is not connected

(ii) $S = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is connected

1.3 Compact Set

By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of an open subset of X such that $E \subset \bigcup_\alpha G_\alpha$.

A subset K of a metric space X is said to be *compact* if every open cover of K contains a finite subcover.

More explicitly, the requirement for completeness of $K \subset X$ is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha 1} \cup G_{\alpha 2} \cup \ldots \cup G_{\alpha n} \quad \text{i.e.,} k \subset \bigcup_{i=1}^{n} G_{\alpha i}$$

Example.

(i) $A = [1, 2]$ is *compact.*

(ii) $B = (0, 2)$ is *not compact.*

Theorem 1.3.1. *In a metric space, prove that closed subsets of a compact set is compact.*

1.4 Path-connected Sets

We say that map $\varphi : [a, b] \to M$ of an interval $[a, b]$ into a metric space M *continuous* if $t_\mu \to t$ implies $\varphi(t_\mu) \to \varphi(t)$ for every sequence t_μ in $[a, b]$ converging to some $t \in [a, b]$.

A *continuous path* joining two points x, y in a metric space M is a mapping $\varphi : [a, b] \to M$ such that $\varphi(a) = x$, $\varphi(b) = y$ and φ is continuous. Here x may or may not equal y and $b \geq a$.

A path φ is said to lie in a set A if $\varphi(t) \in A$ for all $t \in [a, b]$.

We say that a set A is path-connected if every two points in the set can be joined by a continuous path lying in the set.

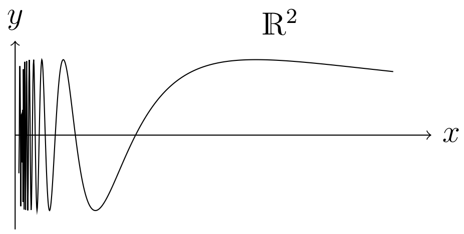


Figure 1.2: Not path-connected

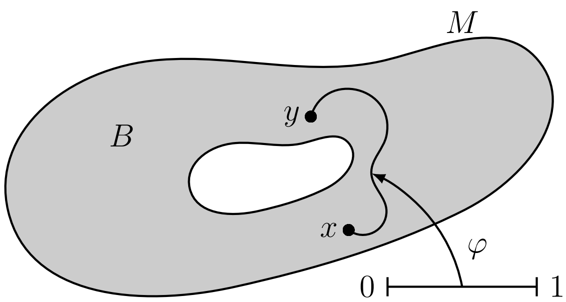


Figure 1.3: Path connected

- Figure 1.2: $A = \left\{(x, \sin\frac{1}{x}) \mid x > 0\right\} \cup \{(0, y) \mid y \subset [-1, 1]\} \subset \mathbb{R}^2$. A is not path connected.[1]
- Figure 1.3: A curve joining points x and y in A of a metric space M. Evidently, region A is path connected.

[1] This is also known as 'Topologist's sine curve'

Problem 1.4.1. Show that $B = [0, 1]$ is path connected.

Solution. Let $\varphi : B \to \mathbb{R}$ be a function defined by $\varphi(t) = (y - x)t + x$.
Here $\varphi(0) = x$, $\varphi(1) = y$, φ is continuous path (because φ is a linear polynomial in t) and φ lies in B.

Problem 1.4.2. Which of the following sets are path-connected?

(i) $[0, 3]$

(ii) $[1, 2] \cup [3, 4]$

(iii) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 2\}$

Problem 1.4.3. Let $\varphi : B = [0, 1] \to \mathbb{R}^2$ be a continuous path and $C = \varphi([0, 1])$. Show that C is path-connected.

Solution. This is intuitively clear, for we can use the path φ itself to join two points in C. Precisely, if $x = \varphi(a)$, $y = \varphi(b)$, where $0 \leq a \leq b \leq 1$, let $c : B \to \mathbb{R}^2$ be defined by $c(t) = \varphi(t)$. Thus, c is a path joining x to y and c lies in C.

Problem 1.4.4. Is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ connected?

Solution. No, for if $U = (1/2, \infty)$, $V = (-\infty, 1/4)$, then $\mathbb{Z} \subset U \subset V$, $\mathbb{Z} \cap U = \{1, 2, 3, \ldots\} \neq \varnothing$, $\mathbb{Z} \cap V = \{\ldots, -2, -1, 0\} \neq \varnothing$. Hence, $\mathbb{Z}$ is not disconnected(i.e., not connected).

Besides, $\mathbb{Z}$ is not path-connected.

Problem 1.4.5. Are $[0, 1] \cup [2, 3]$, $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\} \cup \{(x, 0) \mid 1 < x < 2\}$ connected?

Problem 1.4.6. Determine the compactness of

(i) finite set $A = \{x_1, x_2, \ldots, x_n\}$

(ii) $\mathbb{R}$

(iii) $B = [0, \infty)$

(iv) $C = (0, 1)$

Solution. 1. $A = \{x_1, x_2, \ldots, x_n\}$ − a finite subset of $\mathbb{R}$.
Let $\mathscr{G} = \{G_\alpha\}$ be any open cover of A, then each x_i is contained in some set $G_{\alpha i} \in \mathscr{G}$. Then $A \subset \bigcup_{i=1}^{n} G_{\alpha i} \Rightarrow \{G_{\alpha i}; i = 1, 2, \ldots, n\}$ is a finite sub-cover of $\mathscr{G}$. Since $\mathscr{G}$ is arbitrary so A is compact.

Theorem 1.4.1. *Path-connected sets are connected.*

Theorem 1.4.2 (Heine-Borel Theorem)**.** *A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 1.4.3 (Bolzano-Weirstrass Theorem)**.** *A subset of a metric space is compact if and only if it is sequentially compact.*

Problem 1.4.7. Show that $A = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is compact and connected.

Solution. To show that A is compact, we show it is closed and bounded. To show that it is closed consider $A^c = \mathbb{R}^n \backslash A = \{x \in \mathbb{R}^n \mid ||x|| > 1\} = B$. For $x \in B$, $N_\delta(x) \subset B$, with $\delta = ||x|| - 1$, so that B is open and hence A is closed. It is clear that A is bounded, since $A \subset N_2(0)$ and therefore A is compact.

To show that A is connected, we show that A is path-connected. Let $x, y \in A$. Then the straight line joining x, y is the required path. Explicitly, we use $\varphi : [0,1] \to \mathbb{R}^n$, $\varphi(t) = (1-t)x + ty$. One sees that $\varphi(t) \in A$, since

$$\begin{aligned} ||\varphi(t)|| &\leq (1-t)\,||x|| + t\,||y|| \\ &\leq (1-t) + t = 1 \quad \text{by triangle inequality.} \end{aligned}$$

Theorem 1.4.4. *Closed subsets of a compact set is compact.*

Proof. Suppose $F \subset K \subset M$, F is closed subset and K is compact in the metric space M. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open Ω of K. Since K is compact, there is a finite sub-collection Φ of Ω which covers K, and hence F. If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F. We have thus shown that a finite sub-collection of $\{V_\alpha\}$ covers F. Hence, the theorem. $\square$

Chapter 2

Continuity

2.1 Limit

Definition 4 (Limit)**.** Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y (i.e., $f : E \subset X \to Y$), and p is a limit point of E. We write $f(x) \to q$ as $x \to p$ or $\lim_{x\to p} f(x) = q$ if there is a point $q \in Y$ with following property:
For every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_x(f(x), p) < \delta$[2]

Example. $E = (0, 2) \subset X = \mathbb{R}^1$, $Y = \mathbb{R}^1$; $f(x) = \frac{x^2-1}{x-1}$; $p = 1$ is a limit point of E,
Then $\lim_{x\to p} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = 2$

Theorem 2.1.1 (Sequential Criteron of Limits)**.** *Let x, y, E, f and p as in the above definition. Then* $\lim_{x\to p} f(x) = q$ *if and only if* $\lim_{n\to\infty} f(p_n) = q$ *for every sequence $\langle p_n\rangle$ in E such that $p_n \neq p$,* $\lim_{n\to\infty} p_n = p$

2.2 Continuity

Definition 5 (Continuity)**.** Suppose X and Y are metric spaces, $E \subset X$, $p \in E$ and f maps $E \to Y(f : E \to Y)$. Then f is said to be continuous at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that
$\mathrm{d_y(f(x), p)} < \epsilon$ for all points $x \in E$ for which $\mathrm{d_x(x, p)} < \delta$

Theorem 2.2.1. *Let $f : E \subset X \to Y$ be a mapping. Then the following assertions are equivalent:*

(i) f is continuous on E.

(ii) For each convergent sequence $x_n \to x_0$, we have $f(x_n) \to f(x_0)$

(iii) For each open set U i Y, $f^{-1}(U) \subset E$ is open relative to E; that is, $f^{-1}(U) = E \cap V$ for some open set V.

(iv) For each closed set $F \in Y$, $f^{-1}(F) \subset E$ is closed relative to E; that is $f^{-1}(F) = E \cap G$ for some closed set G.

Theorem 2.2.2. *Suppose $f : X \to Y$ is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$, since f is continuous, by previous theorem each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices say $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that

$$X \subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \cdots \cup f^{-1}(V_{\alpha_n}) \tag{2.1}$$

[2]The δ may depend on $f(x)$, p, and ϵ i.e., $\delta = \delta(p, f(x), \epsilon)$

since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, then (2.1) implies that $f(X) \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n}$ This completes the proof. $\square$

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Chapter 3

Sequences in Metric spaces

3.1 Sequences of real numbers

A sequence of real numbers in $\mathbb{R}$ is simply a function $f : \mathbb{N} \to \mathbb{R}$ which is usually defined by $f(n) = x_n$ and arranged in a particular order such as $x_1, x_2, \ldots, x_n, \ldots$.

For example, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be represented as $x_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$

3.2 Convergent Sequence

A sequence x_n in $\mathbb{R}$ is said to be *converged* to a *limit* $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N such that $|x_n - x| < \epsilon$ whenever $n \geq N$. In this case we write $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$

Note. $N := N(\epsilon)$, often smaller ϵ may require larger N.

3.3 Sequences of points or vectors in Metric Spaces

A sequence of points in a metric space $M := (M, d)$ is a function $f : \mathbb{N} \to M$, usually defined by $f(n) = x_k$ and arranged in a definite order such as $x_1, x_2, \ldots, x_n, \ldots$.

3.4 Convergent sequence in a Metric Space

A sequence x_k in a metric space (M, d) converges to $x \in M$ if for every given $\epsilon > 0$ there is a natural number N such as $n \geq N$ implies $d(x_k, x_n) < \epsilon$

3.5 Convergent sequence in normed space $\mathbb{R}^n$

A sequence v_k of vectors in $\mathbb{R}$ converges to the vector $v \subset \mathbb{R}^n$ if for every given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(v_k, v) = ||v_k - v|| < \epsilon$ whenever $k \geq N$.

3.6 Convergent sequence in arbitrary normed space V

$v_k \in V \to v \in V$ if $||v_k - v|| \to 0$ as $k \to \infty$.
If $v, v_k \in \mathbb{R}^n$, we write $v = (v^1, v^2, \ldots, v^n)$, $v_k = (v_k^1, v_k^2, \ldots, v_k^n)$

Theorem 3.6.1. *$v_k \to v$ in $\mathbb{R}^n$ if and only if each sequence of coordinates converges to the corresponding coordinate of v as a sequence in $\mathbb{R}$. That is,*
$\lim_{k\to\infty} v_k = v$ in $\mathbb{R}^n$ if and only if $\lim_{k\to\infty} v_k^i = v$ in $\mathbb{R}$ for each $i = 1, 2, \ldots, n$
or,

$$\lim_{k\to\infty}(v_k^1, \ldots, v_k^n) = \left(\lim_{k\to\infty} v_k^1, \ldots, \lim_{k\to\infty} v_k^n\right)$$

Example. Test the convergence of the sequences in $\mathbb{R}^2$:

(i) $v_k = (1/k,\ 1/k^2)$

(ii) $v_k = ((\sin n)^n/n,\ 1/n^2)$

Solution.

(i) Here the component sequences $1/k$ and $1/k^2$ each converges to 0. Hence, the vectors $v_k \to 0$, $0 = (0,0) \in \mathbb{R}^2$

(ii) Use Sandwich theorem ($v_n \to (0,0)$).
Here, $\left|\frac{(\sin n)^n}{n}\right| = \frac{|\sin n|^n}{n} \le \frac{1}{n} \Rightarrow -\frac{1}{n} \le \frac{(\sin n)^n}{n} \le \frac{1}{n}$
Hence by sandwich theorem, $\lim_{n\to\infty} -\frac{(\sin n)^n}{n} = 0 = \lim_{n\to\infty} \frac{(\sin n)^n}{n}$, therefore,
$\lim_{n\to\infty} \frac{(\sin n)^n}{n} = 0$.
Again $\lim_{n\to\infty} \frac{1}{n^2} = 0$
Therefore, $v_n \to (0,0)$

Theorem 3.6.2. *A set $A \subset M$ is closed $\Leftrightarrow$ for every sequence $x_k \in A$ converges to a point $x \in A$.*

Example. Let $x_n \in \mathbb{R}^m$ be a convergent sequence with $||x_n|| \le 1$ for all n. Show that the limit x also satisfies $||x|| \le 1$. If $||x_n|| < 1$, then must we have $||x|| < 1$?

Solution. The unit ball $B = \{y \in \mathbb{R}^m \mid ||y|| \le 1\}$ is closed. Let $x_n \in B$ and $x_n \to x \Rightarrow x \in B$ as B is closed, by theorem 3.6.2. This is not true if $\le$ is replaced by $<$; for example, on $\mathbb{R}$, consider $x_n = 1 - \frac{1}{n}$.

3.7 Cauchy sequence

Let (M, d) be a metric space. A *Cauchy sequence* is a sequence $x_k \in M$ such that for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $m, n \ge N$ implies $d(x_m, x_n) < \epsilon$.

3.8 Complete Metric Space

The metric space M is called *complete* if and only if every Cauchy sequence in M converges to a point in M.

In normed space, such as $\mathbb{R}^n$ a sequence v_k is Cauchy sequence if for every $\epsilon > 0$ there is an N such that $||v_k - v_j|| < \epsilon$ whenever $j, k \ge N$.

3.9 Bounded Sequence

A sequence x_k in a normed space is bounded if there is a number $M' > 0$ such that $||x_k|| \le M$ for every k.

In a metric space we require that there be a point x_0 such that $d(x_k, x_0) \le M'$ for all k.

Part II

Class Note/Sheet(After Pandemic)

Chapter 4

Metric Spaces

Definition 6 (Group)**.** A group G is a non-empty set of elements for which a binary operation $*$ is defined. This operation satisfies the following axioms:

(i) *Closure:* If $a, b \in G$ implies that $a * b \in G$

(ii) *Associativity:* If $a, b, c \in G$ implies that $(a * b) * c = a * (b * c)$

(iii) *Identity:* There exists a unique element $e \in G$(called the identity element) such that $a * e = e * a = a$ for all $a \in G$.

(iv) *Inverse:* For every $a \in G$ there exists an element $a' \in G$(called the inverse of a) such that $a * a' = a' * a = e$.

Note. When the binary operation is addition, G is called an additive group and when the binary operation is multiplication, G is called a multiplicative group.

Definition 7. A group G is called Abelian (or commutative) if for every $a, b \in G$, $a * b = b * a$.

Example. The set of all integers i.e., $\{0, \mp 1, \pm 2, \pm 3, \dots\}$ is a group with respect to the binary operation of addition.

Example. The set $\{\pm 1, \pm i\}$ where $i = \sqrt{-1}$ is a group with respect to the binary operation of multiplication.

Definition 8 (Ring)**.** An additive Abelian (or commutative) group $(G, +)$ with the following properties is said to be a ring:

(i) The group G is closed with respect to the binary operation of multiplication. i.e., for $a, b \in G \Rightarrow a \cdot b \in G$

(ii) Multiplication is associative, i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c$ for all $a, b, c \in G$.

(iii) Multiplication is distributive with respect to addition on both left and the right, that is $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in G$.

Example. The set $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ is a ring under binary operations of ordinary addition and multiplication.

Example. Consider the set $\bar{Z} = \{0, 1, 2, 3, 4, 5\}$, $\bar{Z}$ is a ring under the binary operation of addition and multiplication modulo 6.

Definition 9 (Field)**.** A field F is a commutative ring with unit element in which every non-zero element has a multiplicative inverse.

Example. Examples of fields are the ring of rational numbers, the ring of real numbers and the ring of complex numbers.

4.1 Metric Space

Definition 10. Euclidean space (or Euclidean $n-$space) denoted $\mathbb{R}^n$, consists of all ordered $n-$tuples of real numbers. Symbolically, $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}$.

Thus, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$($n$ times) is the Cartesian product of $\mathbb{R}$ with itself n times.

Example. The real line $\mathbb{R}$, two-dimensional plane $\mathbb{R}^2$, three-dimensional space $\mathbb{R}^3$ are examples of Euclidean spaces.

Definition 11 (Metric space)**.** A metric space (M, d) is a set M and a function $d : M \times M \to \mathbb{R}$ such that

(i) *Positivity:* $d(x, y) \geq 0$ for all $x, y \in M$

(ii) *Non degeneracy (identity of indiscernibles):* $d(x, y) = 0$ if and only if $x = y$

(iii) *Symmetry:* $d(x, y) = d(y, x)$ for every $x, y \in M$

(iv) *Triangle inequality:* $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$

Thus, a metric space M is a set equipped with a function $d : M \times M \to \mathbb{R}$ that gives a reasonable way of measuring the distance between two elements of M.

Example. The real line $\mathbb{R}$ is a metric space with the metric defined by $d(x, y) = |x - y|$. Similarly, the complex plane $\mathbb{C}$ and the Euclidean space $\mathbb{R}^n$ are metric spaces together with the metric $d(z, w) = |z - w|$ and the standard metric respectively.

Definition 12 (Discrete metric)**.** Let M be any set and let $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Then d is a discrete metric on M.

Definition 13 (Bounded metric)**.** If d is a metric on a set M and $\rho(x, y)$ is defined by $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$, then ρ is a metric called bounded metric. Observe that $\rho(x, y) < 1$ for all $x, y \in M$ i.e., ρ is bounded by 1.

Note. The distance function d on $\mathbb{R}^n$ is given by $d(x, y) = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}$.

4.2 Vector Space

A vector space over an arbitrary field F is a non-empty set V, whose elements are called vectors for which two operations are prescribed. The first operation, called *vector addition*, assigns to each pair of vectors u and v a vector denoted by $u + v$, called their sum. The second operation, called scalar multiplication assigns to each vector u in V and each scalar $\alpha \in F$ a vector denoted by αu which is in V.

Definition 14. A vector space (or a linear space) V is a set of elements called vectors, with given operations of vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$ such that:

A(i) *Commutativity:* $u + v = v + u$ for every $u, v \in V$.

A(ii) *Associativity:* $(u + v) + w = u + (v + w)$

A(iii) *Zero vector:* There is a zero vector 0 such that $u + 0 = u$ for every $u \in V$.

A(iv) *Negatives:* For each $u \in V$ there is a vector $-u$ such that $u + (-u) = 0$.

M(i) *Distributivity:* For $\alpha \in F$ and $u, v \in V$, $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$

M(ii) *Distributivity:* For any $\alpha, \beta \in F$ and $u \in V$, $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$

M(iii) *Associativity:* For any $\alpha, \beta \in F$ and $u \in V$, $(\alpha\beta) \cdot u = \alpha(\beta \cdot u)$

M(iv) *Multiplicative unity* For each $u \in V$ there is a unit scalar $e \in F$ such that $eu = u$.

If the field $F = \mathbb{R}$, then the linear space V is called a real linear space, similarly if $F = \mathbb{C}$, then the linear space is called a complex linear space. The subset S of a vector space V is called a subspace of V if S itself is a vector space.

4.3 Normed Linear Space (NLS)

A normed linear space $(V, ||\cdot||)$ is a vector space V and a function $||\cdot|| : V \to \mathbb{R}$ called a norm such that

(i) *Positivity:* $||u|| \geq 0$ for all $u \in V$.

(ii) *Non degeneracy:* $||u|| = 0$ if and only if $u = 0$.

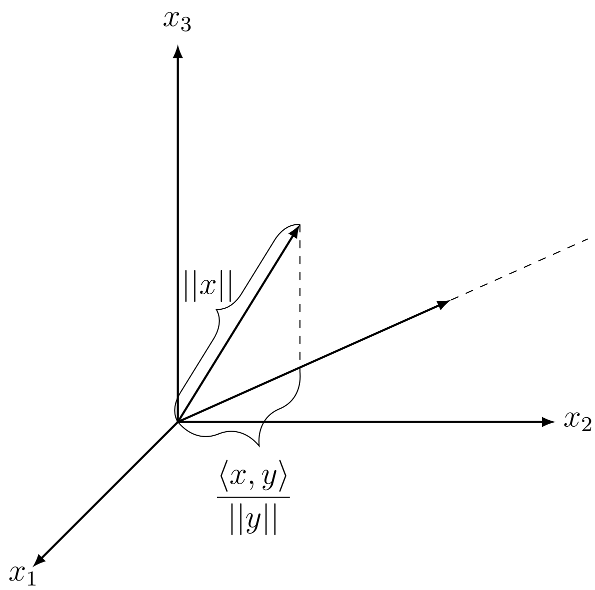
(iii) *Multiplicativity:* $||\alpha u|| = |\alpha| \ ||u||$ for every $u \in V$ and every scalar α.

(iv) *Triangle inequality:* $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in V$.

Definition 15. The norm or length of a vector x in $\mathbb{R}^n$ is defined by $||x|| = \{\sum_{i=1}^n x_i^2\}^{1/2}$, where $x = (x_1, x_2, \ldots, x_n)$. The distance between two vectors x and y in $\mathbb{R}^n$ is the real number

$$d(x, y) = ||x - y|| = \left\{\sum_{i=1}^n (x_i - y_i)^2\right\}^{1/2}$$

The inner product of x and y in $\mathbb{R}^n$ is defined by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Thus, $||x||^2 = \langle x, x\rangle$.
In $\mathbb{R}^3$, we are also familiar with $\langle x, y\rangle = ||x|| \ ||y|| \cos\theta$ where θ is the angle between x and y.



Theorem 4.3.1. *For vectors in $\mathbb{R}^n$, we have*

1. *Properties of the inner product:*

   (i) Positivity: $\langle x, x\rangle \geq 0$

   (ii) Non degeneracy: $\langle x, x\rangle = 0$ *if and only if* $x = 0$

   (iii) Distributivity: $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(iv) Multiplicativity: $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ *for* $\alpha \in \mathbb{R}$

(v) Symmetry: $\langle x, y\rangle = \langle y, x\rangle$

2. *Properties of the norm:*

(i) $||x|| \geq 0$

(ii) $||x|| = 0$ *if and only if* $x = 0$.

(iii) $||\alpha x|| = |\alpha|\ ||x||$ *for* $\alpha \in \mathbb{R}$.

(iv) $||x + y|| \leq ||x|| + ||y||$

3. *Properties of the distance:*

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ *if and only if* $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$

4. *The Cauchy Schwarz inequality:*
$|\langle x, y\rangle| \leq ||x||\ ||y||$ *(Also, named Cauchy-Bunyakovskii-Schwarz inequality).*

4.3.1 Examples of normed linear space (NLS)

Example. The real line $\mathbb{R}$ is a NLS with the norm $||x|| = |x|$. Similarly, the set of complex numbers $\mathbb{C}$ is a NLS with $||z|| = |z|$.

Example (Taxicab norm)**.** Consider the space $\mathbb{R}^2$, but instead of the usual norm on it, set $||(x, y)||_1 = |x| + |y|$. Then $||\cdot||_1$ is a norm on $\mathbb{R}^2$, called the taxicab norm. If $P = (x, y)$ and $Q = (a, b)$, then $d_1(P, Q) = ||P - Q||_1 = |x - a| + |y - b|$. This is the sum of the vertical and horizontal separations. You must travel this distance to get from P to Q if you always travel parallel to the axes (stay on the streets in a taxicab).

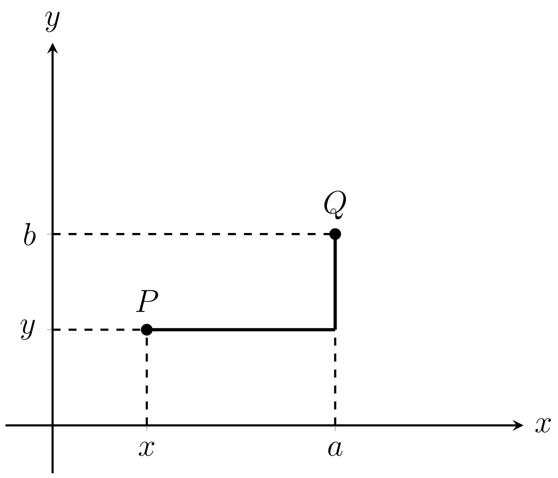


Figure 4.1: The taxicab metric

Example (Supremum norm)**.** Let M = all real-valued functions on the interval $[0, 1]$ that are bounded. That is, let $M = \{f : [0, 1] \to \mathbb{R} \mid$ there is a number B with $|f(x)| \leq B$ for every $x \in [0, 1]\}$. For each f in M, $f([0, 1])$ is a bounded subset of $\mathbb{R}$, and so $\{|f(x) \mid x \in [0, 1]|\}$ is also. It then has a fine least upper bound and $||f||_\infty = \sup\{|x| \mid x \in [0, 1]\}$ defines a function $||\cdot||_\infty : M \to \mathbb{R}$. The set M is a vector space and $||\cdot||_\infty$ is a norm on it, called supremum norm.

The metric in the space M of all bounded functions on $[0,1]$ is thus defined by $d(f,g) = ||f-g||_\infty = \sup\{|f(x)-g(x)| \mid 0 \le x \le 1\}$. Thus, the metric given by the sup norm is the largest vertical separation between the graphs:

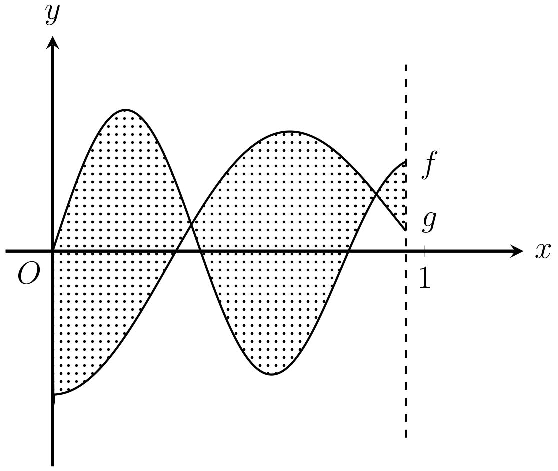


Figure 4.2: The sup distance between function is the largest distance between their graphs.

Proposition 4.3.2. *If $(V, ||\cdot||)$ is a normal vector space and $d(u,v)$ is defined by $d(u,v) = ||u-v||$, then d is a metric in V.*

4.4 Inner Product Space

A vector space V over an arbitrary field F is called an inner product space if there is a function $\langle \cdot,\cdot \rangle : V \times V \to F$ that associates a scalar $\langle u,v \rangle \in F$ with each pair of vectors u and v in V in such a way that the following axioms are satisfied for all vectors u, v and w in V and all scalars $\alpha, \beta \in F$

(i) *Positivity:* $\langle u,u \rangle \ge 0$

(ii) *Non degeneracy:* $\langle u,u \rangle = 0$ if and only if $u = 0$

(iii) *Hermitian symmetry:* $\langle u,v \rangle = \overline{\langle v,u \rangle}$

(iv) *Distributivity:* $\langle u+v,w \rangle = \langle u,w \rangle + \langle v,w \rangle$

(v) *Multiplicativity:* $\langle \alpha u,v \rangle = \alpha \langle u,v \rangle$

Note. The function $\langle \cdot,\cdot \rangle : V \times V \to F$ is called the inner product on V and $(V, \langle \cdot,\cdot \rangle)$ is called the inner product space.

Note. If $F = \mathbb{R}$ (real field), then the inner product space $(V, \langle \cdot,\cdot \rangle)$ is called a real inner product space. In this case the Hermitian symmetry $\langle u,v \rangle = \overline{\langle v,u \rangle}$ becomes simply symmetry $\langle u,v \rangle = \langle v,u \rangle$, and the second distributive property $\langle u,v+w \rangle = \langle u,v \rangle + \langle u,w \rangle$ holds by the properties (iii) and (iv).

Similarly, if $F = \mathbb{C}$, the inner product space $(V, \langle \cdot,\cdot \rangle)$ is called a complex inner product space or UNITARY space. With the help of (iii) and (v) we have $\langle \alpha u,v \rangle = \bar{\alpha} \langle u,v \rangle$ if $\alpha \in \mathbb{C}$.

(v) implies that $\langle 0,y \rangle = 0$ for all $y \in V$.

By (i), we may define $||u||$, the norm of the vector $x \in V$ to be the non-negative square roots of $\langle u,u \rangle$. Thus, $||u||^2 = \langle u,u \rangle$. The properties (i) to (v) excluding (ii) imply that $|\langle x,y \rangle| \le ||x||\ ||y||$ for all $x, y \in V$.

4.4.1 The Cauchy-Schwarz Inequality

If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, then $|\langle x, y\rangle| \leq ||x||\ \ ||y||$ for all x and $y \in V$. The equality holds if and only if x and y are linearly dependent.

Proof. Method 1: If either x and y is 0, then $\langle x, y\rangle = 0$, and so the inequality holds. Therefore, we can assume $x \neq 0$, and $y \neq 0$. Then $\langle x, x\rangle > 0$ and $\langle y, y\rangle > 0$. Then for any α and β in $\mathbb{C}$, we have

$$\begin{aligned}
0 \leq ||\alpha x + \beta y||^2 &= \langle \alpha x + \beta y, \alpha x + \beta y\rangle \ \text{ where } \alpha \text{ and } \beta \text{ are not both zero}\\
&= \alpha\bar{\alpha}\langle x, x\rangle + \alpha\bar{\beta}\langle x, y\rangle + \bar{\alpha}\beta\langle y, x\rangle + \beta\bar{\beta}\langle y, y\rangle\\
&= |\alpha|^2 ||x||^2 + \alpha\bar{\beta}\langle x, y\rangle + \overline{\alpha\bar{\beta}\langle x, y\rangle} + |\beta|^2 ||y||^2\\
&= |\alpha|^2 ||x||^2 + 2\mathrm{Re}\left\{\alpha\bar{\beta}\langle x, y\rangle\right\} + |\beta|^2 ||y||^2\\
&\leq |\alpha|^2 ||x||^2 + 2|\alpha||\beta||\langle x, y\rangle| + |\beta|^2 ||y||^2 \qquad [\text{As } \mathrm{Re}(z) \leq |z| \text{ and } \left|\bar{\beta}\right| = |\beta|]
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow |\alpha|^2 a + 2|\alpha||\beta| b + \left|\beta^2\right| c \geq 0 \qquad \text{Where } a = ||x||^2,\ b = |\langle x, y\rangle|,\ \text{and } c = ||y||^2\\
&\Rightarrow \left|\frac{\alpha}{\beta}\right|^2 a + 2\left|\frac{\alpha}{\beta}\right| b + c \geq 0 \qquad \text{If } \beta \neq 0\\
&\Rightarrow ax^2 + 2bx + c \geq 0 \qquad \text{Where } \left|\frac{\alpha}{\beta}\right| = x,\ \text{a real variable}\\
&\Rightarrow 0 \leq a\left(x^2 + 2\cdot x\cdot\frac{b}{a} + \frac{b^2}{a^2}\right) + c - \frac{b^2}{a}\\
&\Rightarrow 0 = a\left(x + \frac{b}{a}\right)^2 + \frac{ca - b^2}{a} \qquad (4.1)
\end{aligned}$$

Inequality (4.1) holds if and only if $\frac{ca-b^2}{a} \geq 0$ since $\left(x + \frac{b}{a}\right)^2 \geq 0$

$$\begin{aligned}
&\Rightarrow b^2 \leq ac\\
&\Rightarrow |\langle x, y\rangle| \leq ||x||\ \ ||y||
\end{aligned}$$

For equality, there must be a value of x of which $ax^2 + 2bx + c = 0$, which is possible if and only if $\alpha x + \beta y = 0$ where not bot of α and β are zero, which implies that x and y are linearly dependent.[1] □

Method 2: Let $a = ||x||^2$, $b = |\langle x, y\rangle|$, and $c = ||y||^2$.
There is a complex number α such that $|\alpha| = 1$ and $\alpha\langle y, x\rangle = b$.
For any real r, we then have

$$\begin{aligned}
0 \leq \langle x - r\alpha y, x - r\alpha y\rangle &= \langle x, x\rangle - r\alpha\langle y, x\rangle - r\bar{\alpha}\langle x, y\rangle + r^2\langle y, y\rangle\\
&= cr^2 - 2br + a \qquad \overline{\alpha\langle y, x\rangle} = \bar{\beta}\\
\text{i.e., } f(r) &= cr^2 - 2br + a \geq 0 \qquad \bar{\alpha}\langle x, y\rangle = b \text{ as } b \text{ is real}
\end{aligned}$$

Here $\frac{\mathrm{d}f}{\mathrm{d}r} = 2cr - 2b$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} = 2c > 0$.
Since, $\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} > 0$ so the quadratic expression $f(r)$ has a minimum which occurs when $\frac{\mathrm{d}f}{\mathrm{d}r} = 0$ i.e., $r = \frac{b}{c}$.
Therefore, we insert the value of r and obtain,

$$\begin{aligned}
&c\cdot\frac{b^2}{c^2} - 2b\cdot\frac{b}{c} + a \geq 0\\
\Rightarrow\ &\frac{b^2}{c} \leq a\\
\Rightarrow\ &b^2 \leq ac\\
\Rightarrow\ &|\langle x, y\rangle| \leq ||x||\ \ ||y||
\end{aligned}$$

[1] If $\alpha \neq 0$, $x = \frac{-\beta}{\alpha}y$ and if $\beta \neq 0$, $y = \frac{-\alpha}{\beta}x$

The second part is followed if and only if $x - r\alpha y = 0$, so x and y are linearly dependent. $\square$

Note. The above inequality also variously known as the Schwarz, the Cauchy-Schwarz or the Cauchy-Buniakowsky-Schwarz inequality.

Remark. A consequence of this remark is that the linear function $f(x) = \langle x, y\rangle$ $[f : V \to F$ (here field $F = \mathbb{C}$)] is bounded by $||y||$, and from this it follows that $\langle x, y\rangle$ is a continuous function from $V \times V$ to $\mathbb{C}$.

Theorem 4.4.1. *If $(V, \langle\cdot,\cdot\rangle)$ is an inner product space and $||\cdot||$ is defined for $v \in V$ by $||v|| = \sqrt{\langle v, v\rangle}$ then $||\cdot||$ is a norm on V.*

Proof. Hints for triangle inequality,

$$\begin{aligned} ||v+w||^2 &= \langle v+w, v+w\rangle \\ &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= ||v||^2 + 2\langle v, w\rangle + ||w||^2 \\ &\leq ||v||^2 + 2\,||v||\,||w|| + ||w||^2 \\ &= (||v|| + ||w||)^2 \qquad \text{and so } ||v+w|| \leq ||v|| + ||w|| \end{aligned}$$

$\square$

Chapter 5

Sequence in Metric Space

5.1 Sequence of Real Numbers[2]

A sequence of real numbers in $\mathbb{R}$ is simply a function $f : \mathbb{N} \to \mathbb{R}$ which us usually defined by $f(n) = x_n$ and arranged in a particular order such as $x_1, x_2, x_3, \ldots, x_n, \ldots$.

For example, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ can be represented as $x_n = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$.

5.2 Convergent Sequence

A sequence x_n in $\mathbb{R}$ is said to converge to a limit $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N such that $|x_n - x| < \epsilon$, whenever $n \geq N$.

In this case we write $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

Note. $N := N(\epsilon)$, often smaller ϵ may require larger N.

5.3 Sequence of points or Vectors in Metric Spaces

A sequence of points in a metric space $M := (M, d)$ is a function $f : \mathbb{N} \to M$, usually defined by $f(n) = x_k$ and arranged in a definite order such as $x_1, x_2, x_3, \ldots, x_n, \ldots$.

5.4 Convergent Sequence in a Metric Space

A sequence x_k in a metric space (M, d) converges to $x \in M$ if for every given $\epsilon > 0$ there is a natural number N such that $n \geq N$ implies $d(x_k)$.

5.5 Convergent Sequence in Normed Space $\mathbb{R}^n$

A sequence v_k of vector converges to the vector $v \in \mathbb{R}^n$ if for every given $\varepsilon > 0$, there exists such that $d(v_k, v) = ||v_k - v|| < \varepsilon$ whenever $k \geq N$.

5.6 Convergent Sequence in Arbitrary Normed Space V

$v_k \in V \to v$, $||v_k - v|| \to 0$ as $k \to \infty$.

If $v, v_k \in \mathbb{R}^n$, we write $v = (v^1, v^2, \ldots, v^n)$, $v_k = (v_k^1, v_k^2, \ldots, v_k^n)$

[2]Marsden. P.36

Theorem 5.6.1. *$v_k \to v$ in $\mathbb{R}^n$ if and only if each sequence of coordinates converges to the corresponding coordinate of v as a sequence in $\mathbb{R}$. That is,*

$$\lim_{k\to\infty} v_k = v \text{ in } \mathbb{R}^n \text{ if and only if } \lim_{k\to\infty} v^i = v \text{ in } \mathbb{R} \text{ for each } i = 1, 2, \ldots, n$$

or,

$$\lim_{k\to\infty} \left(v_k^1, v_k^2, \ldots, v_k^n\right) = \left(\lim_{k\to\infty} v_k^1, \lim_{k\to\infty} v_k^2, \ldots, \lim_{k\to\infty} v_k^n\right)$$

Problem 5.6.1. Test the convergence of the sequences in $\mathbb{R}^2$

1. $v_k = ({}^1/_2, {}^1/_{k^2})$
2. $v_n = \left({}^{(\sin n)^n}/_n, {}^1/_{n^2}\right)$

Solution.

1. Here the component sequences $\frac{1}{k}$ and $\frac{1}{k^2}$ each converge to 0. Hence, the vector $v_k \to 0,\ 0 = (0,0) \in \mathbb{R}^2$.
2. Use sandwich theorem ($v_n \to (0,0)$)
   Here,
   $$\left|\frac{(\sin n)^n}{n}\right| = \frac{|\sin n|^n}{n} \le \frac{1}{n} \Rightarrow -\frac{1}{n} \le \frac{(\sin n)^n}{n} \le \frac{1}{n}$$
   Hence, by sandwich theorem,
   $$\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$$
   Therefore
   $$\lim_{n\to\infty} \frac{(\sin n)^n}{n} = 0$$
   Again
   $$\lim_{n\to\infty} \frac{1}{n^2} = 0$$
   Therefore, $v_n \to (0,0)$

Theorem 5.6.2. *A set $A \subset M$ is closed $\Leftrightarrow$ for every sequence $x_k \in A$ converges to a point $x \in A$.*

Problem 5.6.2. Let $x_n \in \mathbb{R}^m$ be a convergent sequence with $||x_n|| \le 1$ for all n. Show that x also satisfies $||x|| \le 1$. If $||x_n|| < 1$, then must we have $||x|| < 1$?

Solution. The unit ball $B = \{y \in \mathbb{R}^m \mid ||y|| \le 1\}$ is closed. Let $x_n \in B$, and $x_n \to x \Rightarrow x \in B$ as B is closed, by the above theorem. This is not true if $\le$ is replaced by $<$; for example, on $\mathbb{R}$ consider $x_n = 1 - {}^1/_n$.

5.7 Cauchy Sequence

Let (M, d) be a metric space. A *Cauchy sequence* is a sequence $x_k \in M$ such that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that in $n \ge N$ implies $d(x_m, x_n) < \varepsilon$.

5.8 Complete Metric Space

The metric space M is called *complete* if and only if every Cauchy sequence in M converges to a point in M.

In Normed space, such as $\mathbb{R}^n$, a sequence v_k is Cauchy sequence if for every $\varepsilon > 0$ there us an N such that $||v_k - v_j|| < \varepsilon$ whenever $j, k \ge N$.

5.9 Bounded Sequence

A sequence x_k in a normed space is *bounded* if there is a number $M' > 0$ such that $||x_k|| \leq M$ for every k.

In a metric space we require that there be a point x_c such that $d(x_k, x_c) \leq M'$ for every k.

Theorem 5.9.1. *A convergent sequence in a normed or metric space is bounded.*

Theorem 5.9.2.

(i) Every convergent sequence in a metric space is a Cauchy sequence.

(ii) A Cauchy sequence in a metric space is bounded.

(iii) If a subsequence of a Cauchy sequence converges to x, then the sequence converges to x.

Theorem 5.9.3. *A sequence $x_k \in \mathbb{R}^n$ converges to a point in $\mathbb{R}^n$ if and only if it is a Cauchy sequence.*

Problem 5.9.1 (2.8.8 - P.125, Marsden)**.** Let (M, d) be a complete metric space and $B \subset M$ a closed subset. Show that B is complete as well.

Problem 5.9.2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(\sin n)^n}{n^2}, \frac{1}{n^2} \right)$ converges.

Solution. The first component series $\sum_{n=1}^{\infty} \frac{(\sin n)^n}{n^2}$ is absolutely convergent and hence convergent. For absolutely convergence, $\sum_{n=1}^{\infty} \left| \frac{(\sin n)^n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$, by comparison theorem/test. Since $\sum \frac{1}{n^2}$ is convergent, so $\sum \left| \frac{(\sin n)^n}{n^2} \right|$ is convergent and hence $\sum_{n=1}^{\infty} \frac{(\sin n)^n}{n^2}$ is convergent.
The second component series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, according to p-series test.
Therefore, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(\sin n)^n}{n^2}, \frac{1}{n^2} \right)$ is convergent series in $\mathbb{R}^2$.

5.10 Series of Real Numbers and Vectors

Definition 16. Let V be a normed space. A series $\sum_{k=1}^{\infty} x_k$, where $x_k \in V$, is said to converge to $x \in V$ if the sequence of partial sums $s_k = \sum_{i=1}^{k} x_i$ converges to $x \in V$, and if so we write $\sum_{k=1}^{\infty} x_k = x$ or simply $\sum x_k = x$.

Theorem 5.10.1. *$\sum x_k = x$ is equivalent to corresponding component series converging to components of x.*

5.11 Cauchy Criterion for Series of Vectors

Let V be a complete normed space (such as $\mathbb{R}^n$). A series $\sum x_k$ in V converges if and only if for every $\varepsilon > 0$, there is an N such that $k \geq N$ implies

$$||x_k + x_{k+1} + \cdots + x_{k+p}|| < \varepsilon \qquad \text{for } p = 0, 1, 2, \ldots$$

5.12 Absolutely Convergent Series

A series $\sum x_k$ is said to be *absolutely convergent* if and only if the real series $\sum ||x_k||$ converges.

5.13 Conditionally Convergent Series

A series that is converged but not absolute convergent is said to be conditionally convergent.

Example.

1. If a series of non-negative real numbers is convergent, then it is obviously absolutely convergent.
2. The series $\sum \frac{(-1)^n}{n^3}$ is *absolutely convergent* because $\sum \left|\frac{(-1)^n}{n^3}\right| = \sum \frac{1}{n^3}$ is convergent.
3. The series $\sum \frac{(-1)^{n-1}}{n}$ is convergent (by Leibniz alternating test) but not absolutely convergent because the harmonic series $\sum \left|\frac{(-1)^{n-1}}{n}\right| = \sum \frac{1}{n}$ is divergent. So, $\sum \frac{(-1)^{n-1}}{n}$ is *conditionally convergent.*

Theorem 5.13.1. *In a complete normed space, if $\sum x_k$ converges absolutely, then $\sum x_k$ converges.*

5.13.1 P-series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

5.14 Geometric Series

The series $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Problem 5.14.1. Let $x_n = \left(\frac{1}{n^2}, \frac{1}{n}\right)$. Does $\sum x_n$ converge?

Solution. No, because the harmonic series $\sum \frac{1}{n}$ diverges even though the $p = 2$ series $\sum \frac{1}{n^2}$ converges.

Problem 5.14.2. Let $||x_n|| \leq \frac{1}{2^n}$; prove that $\sum x_n$ converges and $||\sum_{n=0}^{\infty} x_n|| \leq 2$.

Solution.

$$\sum_{n=0}^{\infty} ||x_n|| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - {}^1/_2} = 2 \qquad \text{(Geometric series } \sum \frac{1}{2^n} \text{ is convergent)}$$

By comparison theorem with the convergent geometric series $\sum {}^1/_{2^n}$, the series $\sum x_n$ is absolutely convergent and hence is convergent.
Again the partial sums satisfy

$$||s_n|| = \left|\left|\sum_{k=0}^{n} x_k\right|\right| \leq \sum_{k=0}^{n} ||x_k|| \leq \sum_{k=0}^{n} \frac{1}{2^n} = 2$$

Let $B = \{y \in \mathbb{R}^n \mid ||y|| \leq 2\}$. Clearly B is closed. If $s_n \in B$ and $s_n \to s$, then $s \in B$ as B is closed. Hence, $||s|| \leq 2$.

Problem 5.14.3. Test for convergence: $\sum_{n=1}^{\infty} \frac{n}{3^n}$

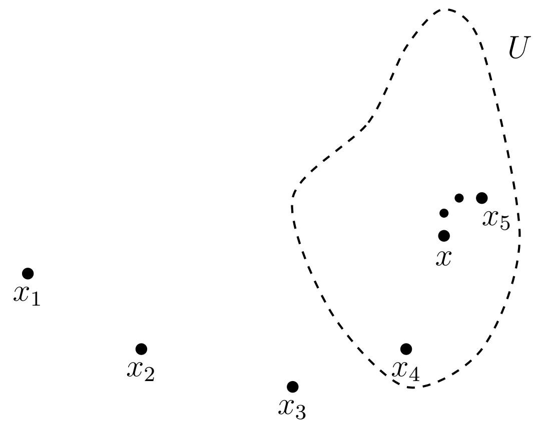
Solution. The ratio test is applicable: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{3 \cdot 3^n} \cdot \frac{3^n}{n} = \frac{1}{3} \cdot \frac{n+1}{n} \to \frac{1}{3}$ and so the series converges.

Problem 5.14.4. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges.

Solution. Observe that $\frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{1}{2^n}$, and so by comparison with divergent series $\frac{1}{2} \sum \frac{1}{n}$, we get divergence.

5.15 Sequence in Metric Space

Definition 17. Let (M,d) be a metric space, and $\langle x_n \rangle$ a sequence of points in M. We say that $\langle x_n \rangle$ converges to a point $x \in M$, written $\lim_{k\to\infty} x_k = x$ or $x_k \to x$ as $k \to \infty$.



Provided that for every open set U containing x, there us an integer N such that $x_k \in U$ whenever $k \geq N$.
This definition coincides with the usual $\varepsilon - \delta$ definition as the next theorem shows.

Proposition 5.15.1. *A sequence $\langle x_k \rangle$ in M converges to $x \in M$ if and on;y if for every $\varepsilon > 0$ there is an N such that $k \geq N$ implies $d(x, x_k) < \varepsilon$.*

Thus, a sequence $\langle v_k \rangle$ of points in $\mathbb{R}^n$ converges to $v \in \mathbb{R}^n$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(v, v_k) = ||v_k - v|| < \varepsilon$ whenever $k \geq N$.

Definition 18. Let (M,d) be a metric space. A Cauchy sequence is a sequence $\langle x_k \rangle$ in M such that for all $\varepsilon > 0$, there is an N such that $k, l \geq N$ implies $d(x_k, x_l) < \varepsilon$. The space M is called *complete* if and only if every Cauchy sequence in M converges to a point in M.

In a normed space, such as $\mathbb{R}^n$, a sequence v_k is a Cauchy sequence if for every $\varepsilon > 0$ there is an N such that $||v_k - v_j|| < \varepsilon$ whenever $k, j \geq N$.

Definition 19. A sequence $\langle x_k \rangle$ in a normed space is bounded if there is a number M such that $||x_k|| \leq M \forall k$. In a metric space, we require that there be a point x_0 such that $d(x_k, x_0) \leq M$ for all k.

Theorem 5.15.2. *(i) Every convergent sequence in a metric space is a Cauchy sequence.*

(ii) A Cauchy sequence in a metric space is bounded.

x If a subsequence of a Cauchy sequence converges to x then the sequence converges to x.

Proof. *H.W.* □

Example. $\mathbb{R}$ is a complete metric space. An example of an incomplete metric space is the set of rational numbers with $d(x,y) = |x - y|$.
Another example is $\mathbb{R} \mid \{0\}$ with the same metric.

Theorem 5.15.3 (Completeness of the metric space $\mathbb{R}^n$)**.** *A sequence $\langle x_k \rangle$ in $\mathbb{R}^n$ converges to a point in $\mathbb{R}^n$ if and only if it is a Cauchy sequence.*

Proof. If x_k converges to x, then for $\varepsilon > 0$, choose N so that $k \geq N$ implies $||x_k - x|| < {}^{\varepsilon}/_2$. Then, for $k, l \geq N$, $||x_k - x_l|| = ||(x_k - x) + (x - x_l)|| \leq ||x_k - x|| + ||x - x_l|| < {}^{\varepsilon}/_2 + {}^{\varepsilon}/_2 = \varepsilon$, by the triangle inequality. Thus, $\langle x_k \rangle$ is a Cauchy sequence.

Conversely, suppose $\langle x_k \rangle$ is a Cauchy sequence. Since $|x_k^i - x_l^i| \leq ||x_k - x_l||$, the components are also Cauchy sequence on the real line. By the completeness of $\mathbb{R}$, x_k^i converges to, say, x^i.
Therefore, $\langle x_k \rangle$ converges to $x = (x^1, x^2, \ldots, x^n)$. □

5.16 Contraction Mapping

A function $\varphi : (M, d) \to (M, d)$ is called a contraction mapping if there exists a number $k(0 < k < 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq kd(x, y) \qquad \text{for all } x, y \in M$$

A point x_k is said to be a fixed point of φ if $\varphi(x_k) = x_k$.

5.17 Contraction Mapping Principle (Banach Fixed Point Theorem)

Let φ be a contraction mapping on a complete metric space M. Then there is a unique fixed point for φ. In fact, if x_0 is any point in M, and we define $x_1 = \varphi(x_0)$, $x_2 = \varphi(x_2), \ldots, x_{n+1} = \varphi(x_n), \ldots$, then $\lim_{n\to\infty} x_n = x_*$.

Intuitively, φ is shrinking distances, and so as φ iterates, points bunch up.

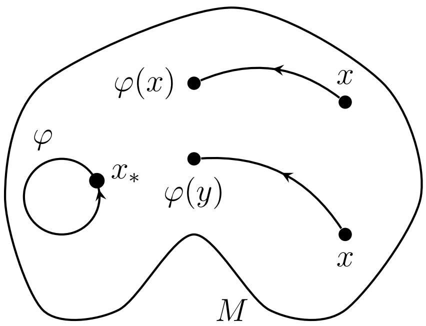


Figure 5.1: A contraction shrinks distances between points

Proof. First we show the existence of a fixed point, then its uniqueness. Let $x_0 \in M$ and $x_1, x_2, x_3, \ldots$ be as in the theorem. If $x_1 = x_0$, $\varphi(x_0) = x_0$ and so x_0 is fixed. If not, then $d(x_1, x_0)$ is not 0, and we start by showing that the points $\{x_n\}$ form a Cauchy sequence in M.
To show this, we write

$$d(x_2, x_1) = d\left(\varphi(x_1), \varphi(x_0)\right) \leq k\, d(x_1, x_0)$$
$$d(x_3, x_2) = d\left(\varphi(x_2), \varphi(x_1)\right) \leq k\, d(x_2, x_1) \leq k^2\, d(x_1, x_0);$$

inductively, $d(x_{n+1}, x_n) \leq k^n\, d(x_1, x_0)$.
Also,

$$d\left(x_{n+p}, x_n\right) \leq d\left(x_{n+p}, x_{n+p-1}\right) + d\left(x_{n+p-1}, x_{n+p-2}\right) + \cdots + d\left(x_{n+1}, x_n\right)$$

by the triangle inequality, and so

$$d\left(x_{n+p}, x_n\right) \leq \left(k^{n+p-1} + k^{n+p-2} + \cdots + k^n\right) D(x_1, x_0)$$

But the geometric series $\sum_{i=0}^{\infty} k^i$ converges, since $0 \leq k < 1$, and so it satisfies the Cauchy criterion for the series: given $\varepsilon > 0$, there is an N such that $k^{n+p-1} + \cdots + k^n < \frac{\varepsilon}{d(x_1,x_0)}$ if $n \geq N$ and p is arbitrary. Hence, $d\left(x_{n+p}, x_n\right) < \varepsilon$ if $n \geq N$ with p arbitrary, and so $\{x_n\}$ is a Cauchy sequence.

By completeness of M, $\lim_{n\to\infty} x_n$ exists in M. Call this limit x_*; i.e., $x_* = \lim_{n\to\infty} x_n$.
We now show that φ is (uniformly) continuous. Given $\varepsilon > 0$, let $\delta = {}^{\epsilon}/_{k}$. Then $d(x, y) < \delta \Rightarrow d(\varphi(x), \varphi(y)) \leq k\, d(x, y) < k\, \delta = \varepsilon$.
Consider, $x_{n+1} = \varphi(x_n)$; $x_{n+1} \to x_*$, and by the continuity of φ, $\varphi(x_n) \to \varphi(x_*)$. Thus, $x_* = \varphi(x_*)$, so x_* is fixed.

Finally, we prove the uniqueness of the fixed point x_*. Let y_* be another point, i.e., $\varphi(y_*) = y_*$. Then

$$d\left(x_*, y_*\right) = d\left(\varphi(x_*), \varphi(y_*)\right) \leq k\, d\left(x_*, y_*\right) \qquad \text{i.e., } (1-k)d\left(x_*, y_*\right) \leq 0$$

By $k < 1$, and so $(1 - k) > 0$, implying $d(x_*, y_*) = 0$, i.e., $x_* = y_*$, and thus the fixed point is unique. $\square$

Chapter 6

The Topology of Euclidean Space

We want to study the basic properties of $\mathbb{R}^n$ which are important for the notion of a continuous function. We will study open sets, which generalize open intervals on $\mathbb{R}$, and closed sets, which generalize closed intervals. The study of open and closed sets constitutes the begging of topology.

6.1 Open Sets

Definition 20. Let (M, d) be a metric space. For each fixed $x \in M$ and $\varepsilon > 0$, the set $D(x, \varepsilon) = \{y \in M \mid d(x, y) < \varepsilon\}$ is called the ε-disk about x (also called the ε-neighbourhood or ε-ball about x).

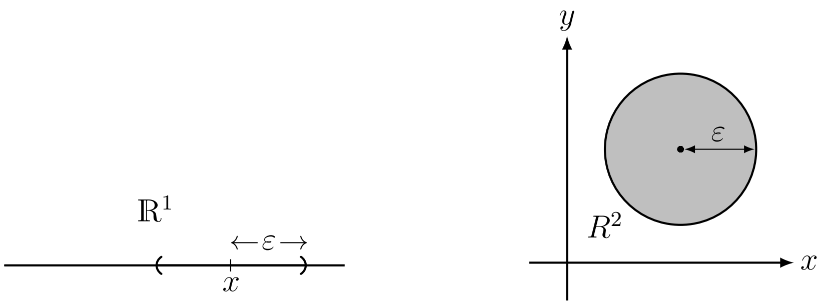


Figure 6.1: The ε-disk

A set $A \subset M$ is said to be open if for each $x \in A$, there exists an $\varepsilon > 0$ such that $D(x, \varepsilon) \subset A$. A neighborhood of a point in M is an open set containing that point.

Note. The empty set $\varnothing$ and the whole space M are open. It is important to realize that the ε required in the definition of an open set may depend on x. For example, the unit square in $\mathbb{R}^2$ not including the "boundary" is open, bur the ε-neighborhood get smaller as we approach the boundary. However, the ε-neighborhood cannot be zero for any x.

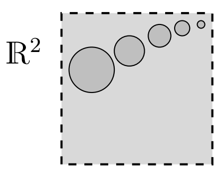
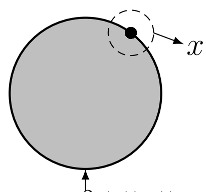


Figure 6.2: An open set

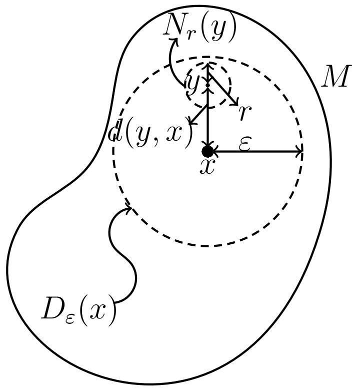


$S = \{x \in \mathbb{R}^2 \mid ||x|| \leq 1\}$

Figure 6.3: A non open set

Theorem 6.1.1. *In a metric space M, each ε-disk $D(x, \varepsilon)$ is open.*

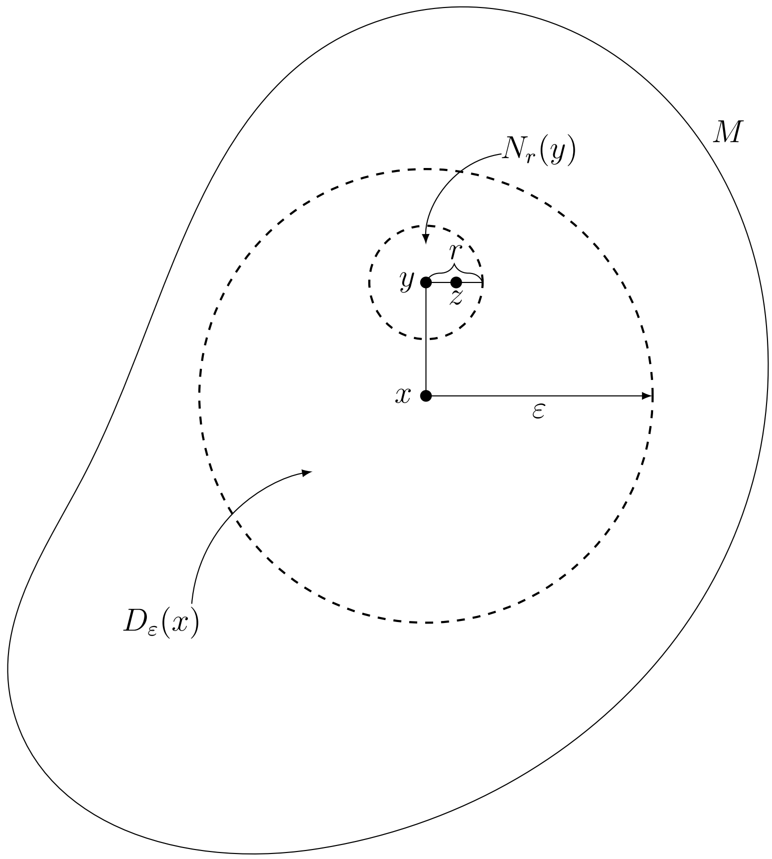
Proof. Assume $D_\varepsilon(x) \equiv D(x, \varepsilon)$



Consider $r > 0$ such that $r + d(y, x) < \varepsilon$.
Now let $z \in N_r(y)$ then $d(y, z) < r$
So,

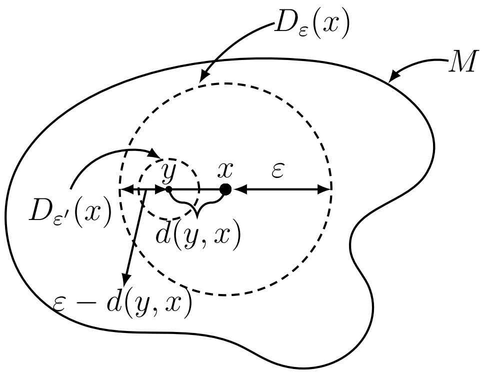
$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< r + d(x, y) < \varepsilon \\ \Rightarrow z \in D_\varepsilon(x) \end{aligned}$$

Hence $N_r(y) \subset D_\varepsilon(x)$ showing that $D_\varepsilon(x)$ is open in M. □



Theorem 6.1.2. *In a metric space M, each ε-disk (or ε-neighborhood or neighborhood of $x \in M$) $D_\varepsilon(x)$ is open.*

Proof. Choose $y \in D_\varepsilon(x)$. We must produce an ε' such that $D_{\varepsilon'}(y) \subset D_\varepsilon(x)$.



The figure suggests that we try $\varepsilon' = \varepsilon - d(x,y)$, which is strictly positive, since $d(x,y) < \varepsilon$. With this choice (which depends on y), we shall show that $D_{\varepsilon'}(y) \subset D_\varepsilon(x)$. Let $z \in D_{\varepsilon'}(y)$, so that $d(z,y) < \varepsilon'$. We need to prove that $d(z,x) < \varepsilon$. But, by the triangle inequality, $d(z,x) \leq d(z,y) + d(y,x) < \varepsilon' + d(y,x) = \varepsilon$. □

Theorem 6.1.3. *In a metric space (M,d),*

(i) both $\varnothing$ and X are open

(ii) any union of open sets is open

(iii) any intersection of finite number of open sets is open.

To appreciate the difference between assertions ((ii)) and ((iii)), note that the intersection of an arbitrary family of open sets need not be open. For example, in $\mathbb{R}^1$, a single point (which is not an open set) is the intersection of the collection of all open intervals containing it. [$G_k = \left\{\left(-\frac{1}{k}, \frac{1}{k}\right) : k \in \mathbb{N}\right\}$ Take $\bigcap\limits_1^\infty G_k = \{0\}$, a single point set which is not an open set]

Note. A set with a specified collection of subsets (called, by definition, open sets) obeying the rules in Theorem (6.1.3) above, and containing the empty set and the whole space is called a *TOPOLOGICAL SPACE.*

Proof.

(i) Since there are no points in $\varnothing$, each point in $\varnothing$ is the center of an ε-disk contained in $\varnothing$. For any $x \in M$, every ε-neighbourhood $D_\varepsilon(x)$ is contained in M.

(ii) Consider a family of open sets $\{G_\alpha : \alpha \in \mathbb{N}\}$ with $\bigcup_{\alpha=1}^\infty G_\alpha = A$. Let $x \in A$, then $x \in G_\alpha$ for same $\alpha \in \mathbb{N}$. Hence, since G_α is open, $D_\varepsilon(x) \subset G_\alpha \subset A$ for some $\varepsilon > 0$, proving that A is open.

(iii) It satisfies to prove that the intersection of two open sets is open, since we can use induction to get the general result by writing $G_1 \cap G_2 \cap \cdots \cap G_n = (G_1 \cap G_2 \cap \cdots \cap G_{n-1}) \cap G_n$.

Let A and B be open and $C = A \cap B$; if $C = \varnothing$, C is open, by (i). Therefore, suppose $x \in C$. Since A and B are open, there exist $\varepsilon, \varepsilon' > 0$ such that $D_\varepsilon(x) \subset A$ and $D_{\varepsilon'}(x) \subset B$.
Let ε'' be the smaller of ε and ε'. Then $D_{\varepsilon''}(x) \subset D_\varepsilon(x)$ and so $D_{\varepsilon''}(x) \subset A$; and similarly, $D_{\varepsilon''}(x) \subset B$, and so $D_{\varepsilon''}(x) \subset A \cap B = C$, as required.

□

Problem 6.1.1. Let $A \subset \mathbb{R}^n$ be open and $B \subset \mathbb{R}^n$. Define $A+b = \{x + y \in \mathbb{R}^n \mid x \in A \text{ and } y \in B\}$. Prove that $A + B$ is open.

Proof. Let $w \in A + B$. There are points $x \in A$ and $y \in B$ with $w = x + y$. Since A is open, there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \subset A$. We claim that $D_\varepsilon(w) \subset A + B$. Suppose $z \in D_\varepsilon(w)$, then $d(w, z) = ||w - z|| < \varepsilon$. But $\varepsilon > ||z - w|| = ||z - (x + y)|| = ||(z - y) - x|| = d(x, z - y)$, so $z - y \in D_\varepsilon(x) \subset A$. Since $y \in B$, this forces $z = (z - y) + y$ to be in $A + B$. Thus, $D_\varepsilon(w) \subset A + B$ and hence $A + B$ is an open set. □

6.2 Interior of a Set

Definition 21. Let M be a metric space and $A \subset M$. A point $x \in A$ is called an interior point of A if there is an open set U such that $x \in U \subset A$. The interior of A is the collection of all interior points of A and is denoted $\text{int}(A)$. The set might be empty.

Equivalently, x is an interior point of A if there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \subset A$.

Example.

1. The interior of a single point in $\mathbb{R}^n$ is empty.
2. The interior of the unit disk in $\mathbb{R}^2$, including its boundary, is the unit disk without its boundary.

The interior of A also can be described as the union of all open subsets of A. Thus, by theorem (6.1.3), $\text{int}(A)$ is open. Hence, $\text{int}(A)$ is the largest open subset of A. If there are no open subsets of A, then $\text{int}(A) = \varnothing$. Also, it is evident that A is open if and only if $\text{int}(A) = A$.

Problem 6.2.1. Let $S = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$. Find $\text{int}(S)$.

Solution. To determine the interior points, we locate points about with it is possible to draw an ε-disk entirely contained in S. Notice that there are points (x, y) where $0 < x < 1$. Thus, $\text{int}(S) = \{(x, y) \mid 0 < x < 1\}$.

6.3 Closed Sets

Definition 22. A set B in a metric space M is said to be closed if its complement (that is, the set $M \setminus B$) is open.

For example, a single point in $\mathbb{R}^n$ is a closed set. The set in $\mathbb{R}^2$ containing of the unit disk with its boundary is closed. Roughly speaking, a set is closed if it contains its "boundary points".

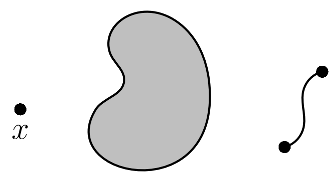


Figure 6.4: Closed sets

It is possible to have a set that is neither open nor closed. For example, in $\mathbb{R}^1$, a half-open interval $(0, 1]$ is neither open nor closed. Thus, even if we know that A is not open, we *cannot* conclude that it is closed or not closed.

Theorem 6.3.1. *In a metric space* (M, d)

(i) The whole space M *and the empty set* $\varnothing$ *are closed.*

(ii) The union of a finite number of closed subsets is closed.

(iii) The intersection of an arbitrary collection of closed subsets is closed.

Proof. Use defn and theorem (6.1.3) $\square$

Note. Let, $I_n = [-n, n]$, then $\bigcup_{n=1}^{\infty}[-n, n] = [-1, 1] \cup [-2, 2] \cup [-3, 3] \cup \cdots = (-\infty, \infty) = \mathbb{R}$,open. This suggests that union of arbitrary collection of closed sets is not closed.

6.4 Accumulation Point

Definition 23. A point x in a metric space M is said to be an accumulation point of a set $A \subset M$ if every open set U containing x contains same points of A other than x.

some

Equivalently, $x \in M$ is an accumulation point of $A \subset M$ if for every $\varepsilon > 0$, the ε-disk $D_\varepsilon(x)$ contains same points of y of A with $y \neq x$. For example, in $\mathbb{R}^1$, a set consisting of a single point has no accumulation points and the open interval $(0, 1)$ has all points of $[0, 1]$ as accumulation points.

Theorem 6.4.1. *The set* $A \subset M$ *is closed if and only if the accumulation point of* A *belong to* A.

Proof. First, suppose A is closed. Then $M \setminus A$ is open. Thus, if $x \in M \setminus A$, there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \subset M \setminus A$; i.e., $D_\varepsilon(x) \cap A = \varnothing$. Thus, x is not an accumulation point, and so A contains all its accumulation points.

Conversely, suppose A contains all its accumulation points. Let $x \in M \setminus A$. Since x is not an accumulation point and $x \notin A$, there is an $\varepsilon > 0$ such that $D_\varepsilon(x) \cap A = \varnothing$; i.e., $D_\varepsilon(x) \subset M \setminus A$.

Hence, $M \setminus A$ is open, and so A is closed. $\square$

Problem 6.4.1. Let $S = \{x \in R \mid x \in [0, 1]\}$ and x is rational. Find the accumulation points of S.

Solution. The set of accumulation points consists of all points in $[0, 1]$. Indeed, let $y \in [0, 1]$ and $D_\varepsilon(y) = (y - \varepsilon, y + \varepsilon)$ be a neighborhood of y. We can find rational points in $(0, 1)$ arbitrary close to y (other than y) and in particular in $D_\varepsilon(y)$. Hence, y is an accumulation point. Any point $y \notin [0, 1]$ is not an accumulation point, because y has an ε-disk containing it that does not meet $[0, 1]$.

6.5 Closure of a Set

Definition 24. Let (M, d) be a metric space, and $A \subset M$. The closure of A denoted $\mathrm{cl}(A)$, is defined to be the intersection of all closed sets containing A.

Since, the intersection of any family of closed sets is closed. $\mathrm{cl}(A)$ is closed; it is also clear that A is closed if and only if $\mathrm{cl}(A) = A$. For example, on $\mathbb{R}^1$, $\mathrm{cl}((0, 1)) = [0, 1]$. The connection between closure and accumulation points is the following:

Theorem 6.5.1. *For* $A \subset M$, $\mathrm{cl}(A)$ *consists of* A *plus the accumulation points of* A. *That is,* $\mathrm{cl}(A) = A \cup \{$ *accumulation points of* $A\}$.

Problem 6.5.1. Find the closure of $A = [0, 1) \cup \{2\}$ in $\mathbb{R}$.

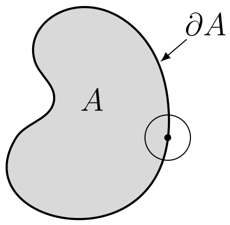
Solution. The accumulation point of A are $[0, 1]$, and so the closure is $[0, 1] \cup \{2\}$. This is clearly the smallest closed set we could find containing A.

6.6 Boundary of a Set

Definition 25. For a given set A in a metric space (M, d), the boundary is defined to be the set $\mathrm{bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(M \setminus A)$. Sometimes the notation $\delta A = \mathrm{bd}(A)$ is used.

Since the intersection of two closed sets is again a closed set, $\mathrm{bd}(A)$ is a closed set. Also note that $\mathrm{bd}(A) = \mathrm{bd}(M \setminus A)$.

Proposition 6.6.1. *Let $A \subset M$. Then $x \in bd(A)$ if and only if for every $\varepsilon > 0$, $D_\varepsilon(x)$ contains point of A and of $M \setminus A$ (these points might include the points x itself).*

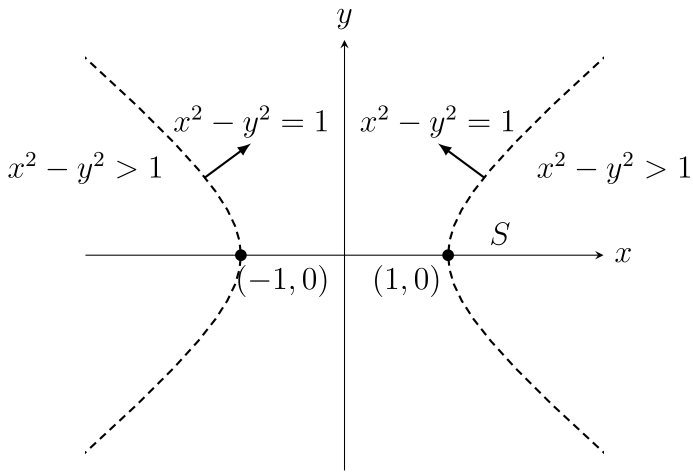


Problem 6.6.1. Let $A = \{x \in \mathbb{R} \mid x \in [0, 1] \text{ and } x \text{ is rational}\}$. Find $\mathrm{bd}(A)$.

Solution. $\mathrm{bd}(A) = [0, 1]$, since, for any $\varepsilon > 0$ and $x \in [0, 1]$, $D_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ contains both rational and irrational points. One can also verify that $\mathrm{bd}(A) = [0, 1]$ using the original definition of $\mathrm{bd}(A)$.

Problem 6.6.2. Let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 > 1\}$. Find $\mathrm{bd}(S)$.

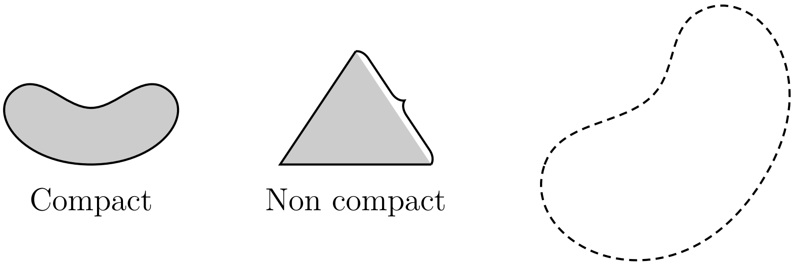
Solution. Clearly, $\mathrm{bd}(S)$ consists of the hyperbola $x^2 - y^2 = 1$.



Chapter 7

Compact and Connected Sets

Definition 26. Let M be a metric space. A subset $A \subset M$ is called *sequentially compact* if every sequence in A has a subsequence that converges to a point in A.



Definition 27 (Some useful definitions)**.** Let M be a metric space and $A \subset M$ a subset. A *cover of* A is collection $\{U_i\}$ of sets whose union contains A; it is an *open cover* if each U_i is open. A sub-cover of a given cover is a sub-collection of $\{U_i\}$ whose union also contains A; it is a *finite sub-cover* if the sub-collection contains only a finite number of sets.

Open covers are not necessarily countable collections of open sets. For example, the uncountable set of disks $\{D_\varepsilon((x,0))\} = \{D_1((x,0)) \mid x \in \mathbb{R}^1\}$ in $\mathbb{R}^2$ covers the real axis, and the sub-collection of all disks $D_1((n,0))$ centered at integer points on the real line forms a countable sub-cover. Note that the set of disks $D_1((2n,0))$ centered at even integer points on the real line does not form a sub-covering (why?).

Definition 28 (Compact set)**.** A subset A of a metric space M is called *compact* if every open cover of A has a finite sub-cover.

7.1 Bolzano-Weierstrass Theorem

Theorem 7.1.1 (Bolzano-Weierstrass Theorem)**.** *A subset of a metric space is compact if and only if it is sequentially compact.*

We will provide the proof of the theorem later on. Some simple observations will help give a feel for compactness and for the theorem.

First, a sequentially compact set must be closed:
Indeed, if $x_n \in A$ converges to $x \in M$, then by assumption there is a subsequence converging to a point $x_0 \in A$; by uniqueness of limits $x = x_0$, and so A is closed.

Secondly, a sequentially compact set A must be bounded:
For if not, there is a point $x_0 \in A$ and a sequence $x_n \in A$ with $d(x_n, x_0) \geq n$. Then x_n cannot have any convergent subsequence. To show directly that a compact set is bounded, use the fact that for any $x_0 \in A$, the open balls $D_n(x_0), n = 1, 2, \ldots$ cover A, so there is a finite sub-cover.

Definition 29 (Totally Bounded)**.** A set A in a metric space M is called totally bounded if for each $\varepsilon > 0$ there is a finite set $\{x_1, x_2, \ldots, x_N\}$ in M such that $A \subset \bigcup_{i=1}^{N} D(x_i, \varepsilon)$.

Note that, a totally bounded set is bounded:
If A is totally bounded, then for each $\varepsilon > 0$, there is a finite set $\{x_1, x_2, \ldots, x_N\}$ in a metric space M such that $A \subset \bigcup_{i=1}^{N} D(x_i, \varepsilon)$. Observe that $D(x_i, \varepsilon) \subset D(x_1, \varepsilon + d(x_i, x_1))$, so that if $R = \varepsilon + \max\{d(x_2, x_1), \ldots, d(x_N, x_1)\}$, then $A \subset D(x_1, R)$ and so a totally bounded set is bounded.

Example. The entire real line is *not* compact, for it is unbounded. Another reason is that $\{D(n, 1) = (n-1, n+1) \mid n = 0, \pm 1, \pm 2, \pm 3, \ldots\}$ is on open cover of $\mathbb{R}$ but does not have a finite sub-cover (why?).

Problem 7.1.1. Let $A = (0, 1]$. Find an open cover with no finite sub-cover.

Solution. Consider the open cover $\left\{(\frac{1}{n}, 2) \mid n = 1, 2, 3, \ldots\right\}$. Then we have $A = (0, 1] \subset (1, 2) \cup (\frac{1}{2}, 2) \cup (\frac{1}{3}, 2) \cup \cdots = (0, 2)$. Clearly, this open cover cannot have a finite sub-cover. This time compactness fails because A is not closed; the point 0 is "missing" from A.

Proof of Bolzano-Weierstrass Theorem
We begin with two lemmas:

Lemma 1. *A compact set $A \subset M$ is closed.*

Proof. We will show that $M \setminus A$ is open. Let $x \in M \setminus A$ and consider the following collection of open sets: $U_n = \{y \mid d(y, x) > {}^1/_n\}$. Since every $y \in M$ with $y \neq x$ has $d(y, x) > 0$, y lies in some U_n. Thus, the U_n cover A, and since A is compact, so there must be a finite sub-cover. One of these has the largest index, say, U_N. If $\varepsilon = \frac{1}{N}$, then, by conclusion(?/ contradiction), $D(x, \frac{1}{N}) \subset M \setminus A$, and so $M \setminus A$ is open. □

Lemma 2. *If M is a compact metric space and $B \subset M$ is closed, then B is compact.*

Proof. Let $\{U_i\}$ be an open covering of B and let $V = M \setminus B$, so that V is open. Thus, $\{U_i, V\}$ is an open cover of M. Therefore, M has a finite cover, say $\{U_1, U_2, \ldots, U_N, V\}$. Then $\{U_1, U_2, \ldots, U_N\}$ is a finite open cover of B. Hence, B is compact. □

Bolzano-Weierstrass Theorem Proof. Let A be compact. Assume that there exists a sequence $x_k \in A$ that has no convergent subsequences. In particular, this means that x_k has infinitely many distinct points, say $y_1, y_2, \ldots$. Since there are no convergent subsequences, there is some neighborhood U_k of y_k containing no other y_i. This is because if every neighborhood of y_k contained another y_i we could, by choosing the neighborhoods $D(y_k, {}^1/_m)$, $m = 1, 2, 3, \ldots$, select a subsequence converging to y_k. We claim that the set $\{y_1, y_2, y_3, \ldots\}$ is closed. Indeed, it has no accumulation points, by the assumption that there are no convergent subsequences. Applying lemma (2) to $\{y_1, y_2, y_3, \ldots\}$ as a subset of A, we find that $\{y_1, y_2, y_3, \ldots\}$ is compact. But $\{U_k\}$ is an open cover that has no finite sub-cover, a contradiction. Thus, x_k has a convergent subsequence. The limit lies in A, since A is closed, by lemma (1).

Conversely, suppose that A is sequentially compact. To prove that A is compact, let $\{U_i\}$ be an open cover of A. We need to prove that this has a finite sub-cover. To show this we proceed in several steps. □

Lemma 3. *There is an $r > 0$ such that for each $y \in A$, $D(y, r) \subset U_i$ for some U_i.*

Proof. If not, then for every integer n, there is some y_n such that $D(y_n, {}^1/_n)$ is not contained in any U_i. By hypothesis, y_n has a convergent subsequence, say $z_n \to z \in A$. Since the U_i cover A, $z \in Ui_0$. Choosing $\varepsilon > 0$ such that $D(z, \varepsilon) \subset U_{i_0}$[1], which is possible since U_{i_0} is open. Choose N large enough so that [2]$d(z_N, z) < {}^\varepsilon/_2$ and ${}^1/_N < {}^\varepsilon/_2$. Then $D(z_n, {}^1/_N) \subset U_{i_0}$, a contradiction. □

[1] $D_\varepsilon(z) = \{z^* \colon d(z, z^*) < \varepsilon\}$
[2] $D_{\varepsilon/_2(z) \subset D_\varepsilon(z)}$ when ${}^1/_N < {}^\varepsilon/_2$
$\Rightarrow D_{1/_N}(z_n) \subset D_{\varepsilon/_2}(z_n) \subset D_\varepsilon(z_n) \subset U$

Lemma 4. *A is totally bounded.*

Proof. If A is not totally bounded, then some $\varepsilon > 0$, we cannot cover A with finitely many disks. Choose $y_1 \in A$ and $y_2 \in A \setminus D(y_1, \varepsilon)$. By assumption, we can repeat; choose $y_n \in A \setminus [D(y_1, \varepsilon) \cup \cdots \cup D(y_{n-1}, \varepsilon)]$. This is a sequence with $d(y_n, y_m) \geq \varepsilon$ for all n and m, and so y_n has no convergent subsequence, a contradiction to the assumption that A is sequentially compact. □

Bolzano-Weierstrass Theorem Proof (continued). To complete our proof, let r be as in lemma (3). By lemma (4) we can write $A \subset D(y_1, r) \cup D(y_2, r) \cup \cdots \cup D(y_n, r)$ for finitely many y_i. By lemma (3), $D(y_i, r) \subset U_{i_j}, j = 1, 2, \ldots, n$ for some index j. Then $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ cover A. □

7.2 Heine-Borel Theorem

Theorem 7.2.1 (Heine-Borel Theorem)**.** *A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. (In fact, a compact set is closed and bounded in any metric space.)*

Proof. We have already proved that compact sets are closed and bounded. We must now show that a set $S \subset \mathbb{R}^n$ is compact if it is closed and bounded. In fact, we shall prove that a closed and bounded set A is sequentially compact.

Let $x_k = (x_k^1, x_k^2, \ldots, x_k^n) \in \mathbb{R}^n$ be a sequence. Since A is bounded x_k^1 has a convergent subsequence, say, $x_{f_1(k)}^1$. Then x_k^2 has a convergent subsequence, say, $x_{f_2(k)}^2$. Continuing, we get a further subsequence $x_{f_n(k)} = \left(x_{f_1(k)}^1, \ldots, x_{f_n(k)}^n\right)$, all of whose components converge. This, $x_{f_n(k)}$ converges in $\mathbb{R}^n$. The limit lies in A since A is closed. Thus, A is sequentially compact, and so is compact. □

Theorem 7.2.2 (Nested Set Property)**.** *Let F_k be a sequence of compact non-empty sets in a metric space M such that $F_{k+1} \subset F_k$ for all $k = 1, 2, 3, \ldots$. Then there is at least one point in $\bigcap_{k=1}^{\infty} F_k$.*

7.3 Path Connected Sets

Definition 30. We call a map $\varphi : [a, b] \to M$ of an interval $[a, b]$ into a metric space M *continuous* if $(t_k \to t)$ implies $(\varphi(t_k) \to \varphi(t))$ for every sequence t_k in $[a, b]$ converging to some $t \in [a, b]$. A *continuous path* joining two points x, y in a metric space M is a mapping $\varphi : [a, b] \to M$ such that $\varphi(a) = x$, $\varphi(b) = y$, and φ is continuous: the x may or may not be equal y, and $b \geq a$. A path φ is said to lie in a set A if $\varphi(t) \in A$ for all $t \in [a, b]$.

We say a set is *path-connected* if every two points in the set can be joined by a continuous path lying in the set.

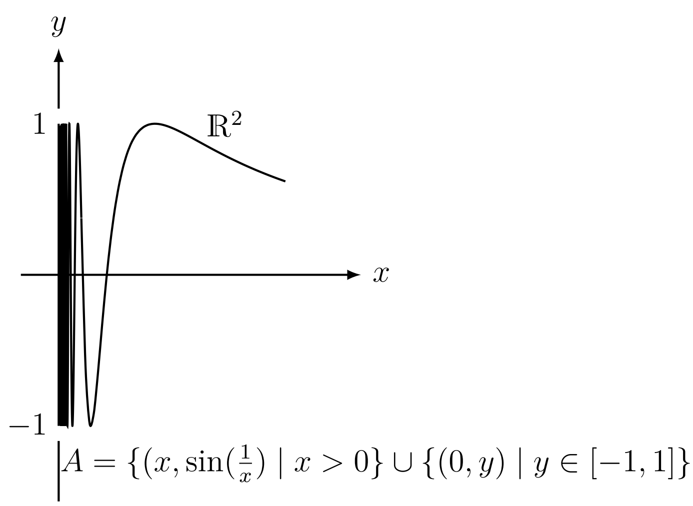


Figure 7.1: A is not path-connected

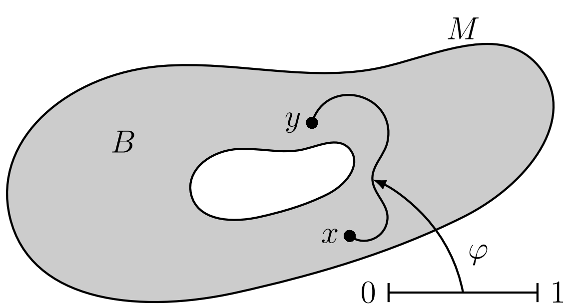


Figure 7.2: B is path-connected

Example. $[0,1] \subset \mathbb{R}^1$ is *path-connected*: To prove this, let $x, y \in [0,1]$ and define $\varphi : [0,1] \to \mathbb{R}$ by $\varphi(t) = (y-x)t + x$. This is a continuous path connecting x and y, and it lies in $[0,1]$.

Example (H.W.)**.** Which of the sets are path-connected?

(i) $[0,3]$

(ii) $[1,2] \cup [3,4]$

(iii) $\{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$

(iv) $\{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$

Example. Let $\varphi : B = [0,1] \to \mathbb{R}^2$ be a continuous path, and $C = \varphi([0,1])$. Show that C is path-connected.

Solution. This is intuitively clear, for we can i=use the path φ itself to join two points in C. Precisely, if $x = \varphi(a)$, $y = \varphi(b)$, where $0 \leq a \leq b \leq 1$, let $c : B \to \mathbb{R}^2$ be defined by $c(t) = \varphi(t)$. Then c is path joining x to y and c lies in C.

7.4 Connected Sets

Definition 31. Let A be a subset of a metric space M. Then A is said to be disconnected if there exists two open sets U and V such that

(i) $U \cap V \cap A = \varnothing$

(ii) $U \cap A \neq \varnothing$

(iii) $V \cap A \neq \varnothing$

(iv) $A \subset U \cup V$

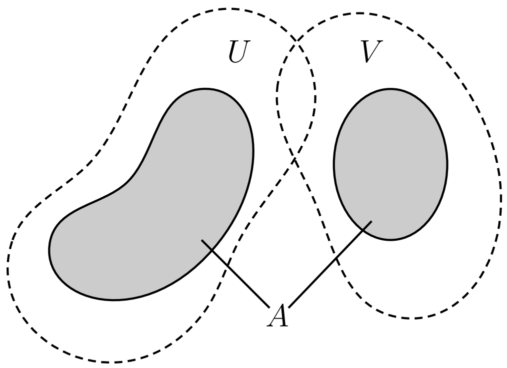


Figure 7.3: A is neither connected nor path-connected

Theorem 7.4.1. *Path-connected sets are connected.*

Problem 7.4.1. Is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\} \subset \mathbb{R}$ connected?

Solution. No, for if $U = (\frac{1}{2}, \infty)$ and $V = (-\infty, \frac{1}{4})$, then $\mathbb{Z} \subset U \cup V$, $\mathbb{Z} \cap U = \{1, 2, 3, \ldots\} \neq \varnothing$, $\mathbb{Z} \cap V = \{\ldots, -2, -1, 0\} \neq \varnothing$, and $\mathbb{Z} \cap U \cap V = \varnothing$. Hence, $\mathbb{Z}$ is disconnected(i.e., not connected).

Problem 7.4.2. Is $\{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is connected?

Solution. Yes, because $\{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is path-connected and hence is connected by theorem 7.4.1.

Example (H.W.)**.** Are $[0,1] \cup (2,3]$ and $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\} \cup \{(x,0) \mid 1 < x < 2\}$ connected? Prove or disprove.

Problem 7.4.3. Determine the compactness of

(i) finite set $A = \{x_1, x_2, \ldots, x_n\}$

(ii) $\mathbb{R}$

(iii) $B = [0, \infty) \to G_n = (-1, n) \Rightarrow B \subset \bigcup_1^\infty G_n \Rightarrow B \not\subset \bigcup_{i=1}^k G_{n_i}$

(iv) $C = (0,1)$

Solution.

(i) $A = \{x_1, x_2, \ldots, x_n\} -$ a finite subset of $\mathbb{R}$,

Let $\mathscr{G} = \{G_\alpha\}$ be any open cover of A, then each x_i is contained in some ..... Then $A \subset \bigcup_{i=1}^n G_{\alpha_i} \Rightarrow \{G_{\alpha_i} : i = 1, 2, \ldots, n\}$ is a finite sub-cover of $\mathscr{G}$. Since $\mathscr{G}$ is arbitrary, so A is compact.

Problem 7.4.4. Show that $A = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is compact and connected.

Solution. To show that A is compact, we show it is closed and bounded. To show that it is closed, consider $A^\complement = \mathbb{R}^n \setminus A = \{x \in \mathbb{R}^n \mid ||x|| > 1\} = B$. For $x \in B$, $||x|| = 1$, $N_\delta(x) \subset B$, with $\delta = ||x|| - 1$, so that B is open and hence A is closed. It is clear that A is bounded, since $A \subset N_2(0)$ and therefore A is compact.

To show that A is connected, we show that A is path-connected. Let $x, y \in A$. Then the straight line joining x, y is the required path. Explicitly, we use $\varphi : [0,1] \to \mathbb{R}^n, \varphi(t) = (1-t)x + ty$. One sees that $\varphi(t) \in A$, since $||\varphi(t)|| \leq (1-t)\,||x|| + t\,||y|| \leq (1-t) + t = 1$, by triangle inequality.

Chapter 8

Continuous Mappings on Metric Spaces

Definition 32. Let (M, d) and (N, ρ) be two metric spaces, $A \subset M$, and $f : A \to N$ be a mapping. Suppose that x_0 is an accumulation point of A. We say that $b \in N$ is *the limit of f at x_0,* written $\lim_{x \to x_0} f(x) = b$, if given any $\epsilon > 0$ there exists $\delta > 0$ (possibly depending on f, x_0, and ϵ) such that for all $x \in A$ satisfying $x \neq x_0$ and $d(x_0, x) < \delta$, we have $\rho\left(f(x), b\right) < \epsilon$.

Intuitively, this says that as x approached x_0, $f(x)$ approaches b. We also write $f(x) \longrightarrow b$ as $x \longrightarrow x_0$.

Definition 33. Let (M, d) and (N, ρ) be two metric spaces and $a \subset M$ and $f : A \to N$ be a mapping. We say that f is continuous at x_0 in its domain if and only if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in A$, $d(x, x_0) < \delta$ implies $\rho\left(f(x), f(x_0)\right) < \epsilon$.

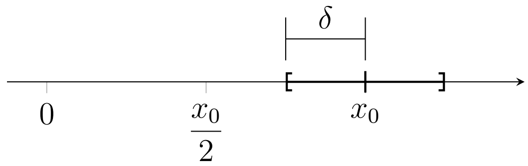
Note. A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x_0 \in A$ if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in A$ with $||x - x_0|| < \delta$, we have $||f(x) - f(x_0)|| < \epsilon$.

Example (HW)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function $x \mapsto x$. Show that f is continuous.

Example. Let $f : (0, \infty) \to \mathbb{R}$, $x \mapsto \frac{1}{x}$, show that f is continuous.

Solution. Fix $x_0 \in (0, \infty)$, that is, fix $x_0 > 0$. To determine how to choose δ, we examine the expression

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x_0 - x|}{|x\,x_0|}$$



If $|x - x_0| < \delta$, then we would get

$$|f(x) - f(x_0)| < \frac{\delta}{|x\,x_0|} = \frac{\delta}{x\,x_0}$$

If $\delta < \frac{x_0}{2}$, then $x > \frac{x_0}{2}$ and so $\frac{\delta}{x\,x_0} < \frac{2\delta}{x_0^2}$. Thus, given $\epsilon > 0$, choose $\delta = \min\left(\frac{x_0}{2}, \frac{\epsilon x_0^2}{2}\right)$. Then $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$, and so f is continuous.

Theorem 8.0.1. *Suppose that (M, d) and (N, p) are two metric spaces, $f : M \to N$ is continuous and $K \subset M$ is connected. Then $f(K)$ is connected. Similarly, if K is path-connected, so is $f(K)$.*

Proof. Suppose $f(k)$ is not connected. By definition, we can write $f(K) \subset U \cup V$, when $U \cap V \cap f(K) = \varnothing$, $U \cap f(K) \neq \varnothing$, $V \cap f(K) \neq \varnothing$, and U, V are open sets. Now, $f^{-1}(U) = U' \cap K$ for some open set U', and similarly, $f^{-1}(V) = V' \cap K$ for some open set V'. From the conditions on U, V, we see that $U' \cap V' \cap K = \varnothing$, $K \subset U' \cup V'$, $U' \cap K \neq \varnothing$, and $V' \cap K \neq \varnothing$. Thus, K is not connected, which proves the first assertion. $\square$

8.1 The Boundedness of Continuous Functions on Compact Sets

Theorem 8.1.1 (Maximum-Minimum Theorem (Boundedness Theorem)). *Let (M, d) be a metric space, let $A \subset M$, and let $f : A \to \mathbb{R}$ be continuous. Let $K \subset A$ be a compact set. Then f is bounded on K; that is, $B = \{f(x) \mid x \in K\} \subset \mathbb{R}$ is a bounded set. Furthermore, there exists points $x_0, x_1 \in K$ such that $f(x_0) = \inf(B)$ and $f(x_1) = \sup(B)$. We call $\sup(B)$ the (absolute) maximum of f on K and $\inf B$ the (absolute) minimum of f on K.*

Proof. First, B is bounded, for $B = f(K)$ is compact, since the continuous image of a compact set is compact. Therefore, it is closed and bounded, by the definition of compactness. Second, we want to produce an x_1 such that $x_1 \in K$ and $f(x_1) = \sup B$. Now, since B is closed, $\sup B \in B = f(k)$. Thus, $\sup B = f(x_1)$ for some $x_1 \in K$. The case of $\inf B$ is similar. □

To appreciate the result, let us consider what can happen on a non-compact set:

Note. First, *a continuous function need not be bounded.* Consider the function $f(x) = \frac{1}{x}$ on $(0, 1)$. As x gets closer to 0, the function becomes arbitrarily large, but f is nevertheless continuous.

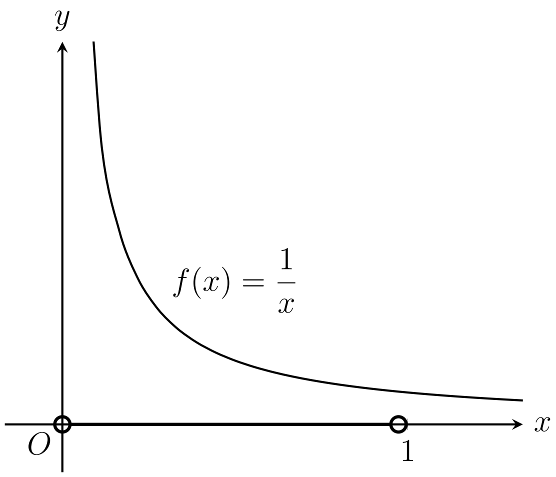


Figure 8.1: An unbounded continuous function

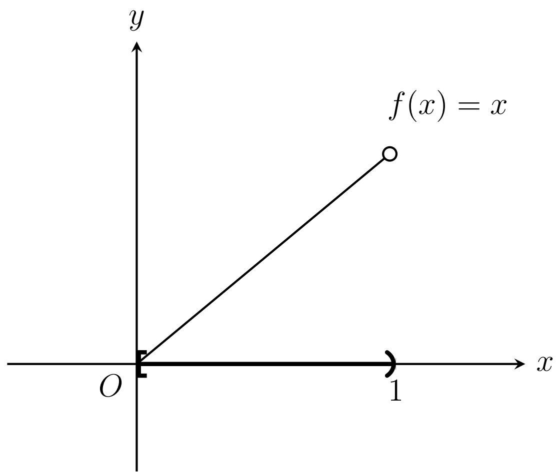


Figure 8.2: A function with no maximum

Note. Second, *if a function is bounded and continuous, it might not assume its maximum at any points of its domain.*
Let $f(x) = x$ on $[0, 1)$. This function never attain a maximum value, because even though there are an infinite number of points as near to 1 as we please, there is no point x for which $f(x) = 1$.

Problem 8.1.1. Give an example of an unbounded discontinuous function on a compact set.

Solution. Let $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ if $x > 0$ and $f(0) = 0$. Clearly, this function exhibits the same unboundedness property as does $\frac{1}{x}$ on $(0, 1]$

Problem 8.1.2. Verify the Maximum-Minimum theorem for $f(x) = \frac{x}{x^2+1}$ on $[0, 1]$

Solution. $f(0) = 0$, $f(1) = \frac{1}{2}$. We shall verify explicitly that, the maximum is at $x = 1$, and the minimum is at $x = 0$. First, as $0 \le x \le 1$, so $\frac{x}{x^2+1} \ge 0$, since $x \ge$, $x^2 + 1 \ge 1$, so that $f(x) \ge f(0)$ for $0 \le x \le 1$. Thus, 0 is the minimum. Next, note that $0 \le (x - 1)^2 = x^2 - 2x + 1$, so that $x^2 + 1 \ge 2x$, and hence for $x \ne 0$, $\frac{x}{x^2+1} \le \frac{x}{2x} = \frac{1}{2}$ so that $f(x) \le f(1) = \frac{1}{2}$ and thus $x = 1$ is the maximum point.

Problem 8.1.3. Verify the Maximum-Minimum theorem for $f(x) = x^3 - x$ on $[-1, 1]$

8.2 The Intermediate Value Theorem (IMVT)

From the context of elementary calculus, it states that a continuous function on an interval assumes all values between any two given elements of its range (Fig 8.3 below). This theorem is not true for the case of a discontinuous function (Fig 8.4 below). Also, this is not true when the function is continuous the domain is not connected (Fig 8.5 below). Therefore, the crucial assumptions for this theorem to hold for a function f be *continuous* and the domain of definition be *connected.*

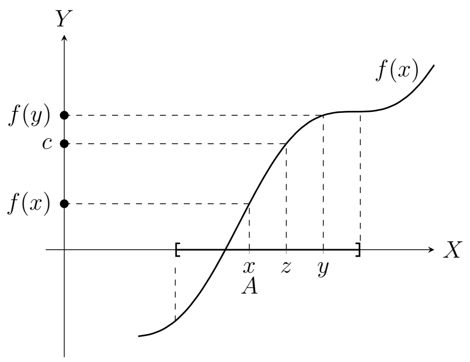


Figure 8.3: IMVT

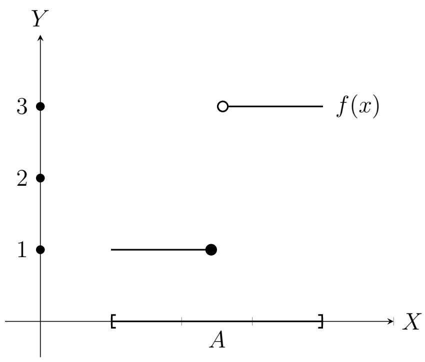


Figure 8.4: IMVT

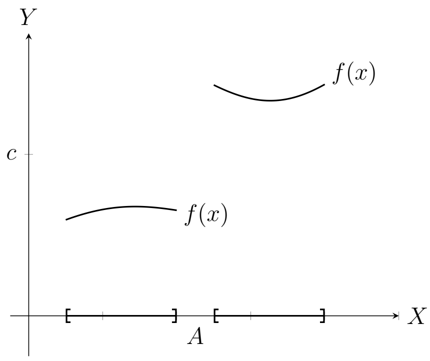


Figure 8.5: Continuous function with disconnected domain

8.2.1 Intermediate Value Theorem

Suppose M is a metric space, $A \subset M$, and $f : A \to \mathbb{R}$ is continuous. Suppose that $K \subset A$ is connected and $x, y \in K$. For every number $c \in \mathbb{R}$ such that $f(x) < c < f(y)$, there exists a point $z \in K$ such that $f(z) = c$.

Proof. Suppose no such z exists. Let $U = (-\infty, c)$ and $V = (c, \infty)$. Clearly, U and V are open sets. Since f is continuous, we have $f^{-1}(U) = U_0 \cap K$ an open set U_0, and similarly, $f^{-1}(V) = V_0 \cap K$ an open set V_0 (by the following theorem). By the definition if U and V, we have $U_0 \cap V_0 \cap K = \varnothing$, and by the assumption that $\{z \in K \mid f(z) = c\} = \varnothing$. We have $U_0 \cap V_0 \supset K$. Also, $U_0 \cap K \neq \varnothing$, since $x \in U$; and $V_0 \cap K \neq \varnothing$, since $y \in V$. Hence, K is not connected, a contradiction. □

Theorem 8.2.1. *Let $f : A \subset M \to N$ be a mapping. Then the following assumptions are equivalent:*

(i) f is continuous on A

(ii) For each convergent sequence $x_k \to x_0$ in A, we have $f(x_k) \to f(x_0)$

(iii) For each open set U in N, $f^{-1}(U) \subset A$ is open relative to A; i.e., $f^{-1}(U) = U_0 \cap A$ for some open set U_0

(iv) For each closed set $F \subset N$, $f^{-1}(F) \subset A$ is closed relative to A; i.e., $f^{-1}(F) = F_0 \cap A$ for some closed set F_0

Problem 8.2.1. Let $f(x)$ be a cubic polynomial. Show that f has a (real) root x_0 (i.e., $f(x_0) = 0$).

Solution. We can write $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Suppose that $a > 0$. For x large and positive, ax^3 is large (and positive) and will be bigger than the other terms, so that $f(x) > 0$ if x is large. To see it exactly, note that $ax^3 + bx^2 + cx + d = ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right)$ and the factor in parentheses tends to 1 as $x \to \infty$. Similarly, $f(x) < 0$ if x is large and negative. Hence, we can apply the Intermediate value with $K = \mathbb{R}$ to conclude the existence of a point x_0 where $f(x_0) = 0$.

Problem 8.2.2. Let $f : [1,2] \to [0,3]$ be a continuous function satisfying $f(1) = 0$ and $f(2) = 3$. Show that f has a fixed point. That is, show that there us a point $x_0 \in [1,2]$ such that $f(x_0) = x_0$.

Solution. Let $g(x) = f(x) - x$. Then g is continuous, $g(1) = f(1) - 1 = -1$ and $g(2) = f(2) - 2 = 3 - 2 = 1$. Hence, by the Intermediate value theorem g must vanish at some $x_0 \in [1,2]$, and this x_0 is a fixed point for $f(x)$.

8.3 Uniform Continuity

Definition 34. Let (M,d) and (N,ρ) be metric spaces, $A \subset M$, $f : A \to N$, and $B \subset A$. We say that f is *uniformly continuous on the set* B if for every $\epsilon > 0$ there is a $\delta > 0$ such that $x, y \in B$ and $d(x,y) < \delta$ imply $\rho(f(x), f(y)) < \epsilon$.

The definition is similar to that of continuity, except that here we are required to chose δ to work for all x, y once ϵ is given. For continuity, we were required only to choose a δ once we were given $\epsilon > 0$ and a particular x_0. Clearly, if f is uniformly continuous, then f is continuous.

For example, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then f is certainly continuous, but it is not uniformly continuous. Indeed, for $\epsilon > 0$ and $x_0 > 0$ given, the $\delta > 0$ we need is at least as small as $\epsilon/(x_0)$ (WHY?), and so if we choose x_0 large, δ must get smaller; i.e., no single δ will do for all $x > 0$. The phenomenon cannot happen on compact sets, as the next theorem shows:

Theorem 8.3.1 (Uniform Continuity Theorem)**.** *Let $f : A \subset M \to N$ be continuous and let $K \subset A$ be compact set. Then f is uniformly continuous.*

Proof. Given $\epsilon > 0$ and $x \in K$, choose δ_x such that $d(x,y) < \delta_x$ implies $\rho(f(x), f(y)) < \frac{\epsilon}{2}$. The sets $D(x, \frac{\delta_x}{2})$ cover K and are open. Therefore, there is a fine covering, say, $D(x_1, \frac{\delta_{x_1}}{2}), D(x_2, \frac{\delta_{x_2}}{2}), \ldots, D(x_N, \frac{\delta_{x_N}}{2})$. Let $\delta =$ minimum $\frac{\delta_{x_1}}{2}, \frac{\delta_{x_2}}{2}, \ldots, \frac{\delta_{x_N}}{2}$. If $d(x,y) < \delta$, then there is an x_i such that $d(x_i, y) \leq d(x, x_i) + d(x_i, y)$. Thus, by the choice of δ, $\rho(f(x), f(y)) \leq \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Problem 8.3.1. Let $f : (0,1] \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $[a,1]$ for $a > 0$.

Solution. Since $[a,1]$ is a compact subset of $(0,1]$ where $a > 0$ and f is continuous on $(0,1]$, and therefore, by the uniform continuity theorem, we conclude that f is uniformly continuous in $[a,1]$.

Problem 8.3.2. Let $f : (a,b) \to \mathbb{R}$ be differentiable and suppose that there is a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a,b)$. Here a and b may be $\pm\infty$. Show that f is uniformly continuous on (a,b).

Solution. The definition of uniform continuity asks us to estimate the difference $|f(x) - f(y)|$ in terms of $|x - y|$. This suggests using the mean value theorem.

Indeed, $f(x) - f(y) = f'(x_0)(x - y)$ for some x_0 between x and y. Hence, $|f(x) - f(y)| \leq M\,|x - y|$; a mapping with this property is called *Lipschitz.*

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$. Then $|x - y|$ implies $|f(x) - f(y)| < M \cdot \frac{\epsilon}{M} = \epsilon$. Hence, f is uniformly continuous on (a,b).

Problem 8.3.3. Show that $\sin x : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

Part III

Questions

Questions from Previous Years

2014-2015 (2017)

1. Marks : $4 + 4 + 6 = 14$

   (a) Define metric space with example. Let d be a metric on a set M and let ρ be defined by

   $$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

   show that ρ is bounded metric on M.

   (b) Define a normed liner space with example. Prove that an Euclidean space is a normed linear space.

   (c) What is meant by an inner product space? State and prove Cauchy-Schwarz inequality.

2. Marks : $4 + 3 + 7 = 14$

   (a) For a metric space X, define

   i. neighborhood of a point of X,

   ii. limit point of a subset of X,

   iii. closed subset of X,

   iv. interior point of a subset X.

   (b) Prove that every neighborhood is an open set.

   (c) Let X be a metric space. Prove that the intersection of finite collection of open sets is open. Also give an example to show that intersection of any collection of open set may not be open.

3. Marks : $5 + 4 + 5 = 14$

   (a) Define compact set with example. Show that the set of real numbers is not a compact set.

   (b) Explain what is meant by connected set. Is the set of integers a connected set? Support your answer.

   (c) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub-collection of $\{K_\alpha\}$ is non-empty, then prove that $\cap K_\alpha$ is non-empty.

4. Marks : $5 + 4 + 5 = 14$

   (a) Define continuous function in a metric space. Prove that a function f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open X for every open set V in Y.

   (b) Prove that every continuous image of a connected subset of a metric space is connected.

   (c) Show that the set $\{x \in \mathbb{R}^n : ||x|| = 1\}$ is compact and connected.

5. Marks : $8 + 6 = 14$

   (a) State and prove the Maximum and Minimum theorem. Hence, verify it for
   $$f(x) = \frac{x}{1+x^2}$$
   on $[0, 1]$.

   (b) State Heine-Borel Theorem. Prove that every compact subset of a metric space is closed.

6. Marks : $5 + 5 + 4 = 14$

   (a) What is the Intermediate Value Theorem? Can you prove it? How?

   (b) Show that there exists a real root of a cube polynomial.

   (c) Let $f : (a, b) \to \mathbb{R}$ be differentiable, and suppose that there is a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$. Show that f is uniformly continuous on (a, b).

7. Marks : $5 + 4 + 5 = 14$

   (a) Define pointwise convergent and uniform convergence of a sequence of functions in metric spaces. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by
   $$f_n(x) = \frac{\sin x}{n}$$
   Show that $f_n \to f = 0$ uniformly as $n \to \infty$.

   (b) Let $f_n(x) = x^n,\ 0 \leq x \leq 1$. Does f_n converge uniformly? What is the situation for $0 \leq x < 1$?

   (c) Suppose $f_n(x) = \frac{x^n}{1+x^n}$ for $x \in [0, 2]$. Show that $\langle f_n \rangle$ converges pointwise on $[0, 2]$ but that the convergence is not uniformly.

8. Marks : $7 + 7 = 14$

   (a) State the Contraction Mapping Principle. Consider the integral equation $f(x) = a + \int_0^x f(y) x e^{-xy}\, \mathrm{d}\, y$. Check directly on which intervals $[0, r]$ we get a contraction.

   (b) Let Ω be the set of all invertible linear operator on $\mathbb{R}^n$. Then prove that Ω is an open subset of $L(R^n)$, the set of all linear transformation on $\mathbb{R}^n$ and the mapping $A \to A^{-1}$ is continuous on Ω.

2015-2016 (2018)

1. Marks : $5 + 5 + 4 = 14$

   (a) Define Euclidean space. Prove that Euclidean space is a metric space.

   (b) What do you mean by supremum norm? If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space and $||\cdot||$ is defined for $v \in V$ by $||v|| = \sqrt{\langle v, v \rangle}$, then prove that $||\cdot||$ is a norm on V.

   (c) Let $C([0, 1])$ be the vector space of all continuous functions defined on $[0, 1]$. For $f, g \in C([0, 1])$ define $\langle f, g \rangle = \int_0^1 f(x) g(x)\, \mathrm{d}\, x$. Show that $\langle f, g \rangle$ is an inner product and hence $C([0, 1])$ is an inner product space.

2. Marks : $3 + 4 + 7 = 14$

   (a) Define interior, boundary, and closure of a set with examples.

   (b) Let $A \subset \mathbb{R}^n$ be open and $B \subset \mathbb{R}^n$, define $A+B = \{x + y \in \mathbb{R}^n \mid x \in A \text{ and } y \in B\}$, prove that $A + B$ is open.

   (c) In a metric space, prove that the union of finite number of closed subsets is closed, and the intersection of arbitrary collection of closed subsets is closed. What do you say about the union of an infinite collection of closed sets?

3. Marks : $6 + 8 = 14$

   (a) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub-collection of $\{K_\alpha\}$ is non-empty, then prove that $\bigcap_\alpha K_\alpha$ is non-empty.

   (b) State and prove the Heine-Borel theorem.

4. Marks : $4 + 4 + 6 = 14$

   (a) Define Cauchy sequence in a metric space. Prove that every convergent sequence in a metric space is a Cauchy sequence.

   (b) Define diameter of a subset of a metric space. Prove that diam$\bar{E}$ =diam$E$ where $\bar{E}$ is the closure of a subset E of a metric space.

   (c) Define complete metric space. Prove that $\mathbb{R}^k$ is complete.

5. Marks : $7 + 7 = 14$

   (a) Define pointwise convergence and uniform convergence of sequence of functions in a metric space. Distinguish between these two sorts of convergences. Show that the sequence of functions

$$f_k(x) = \begin{cases} 0 & x \geq {}^1/_k \\ -kx + 1 & 0 \leq x < {}^1/_k \end{cases}$$

   converges pointwise on $[0, 1]$.

   (b) Can we differentiate and integrate an infinite series of function? When and how? Examine the uniform convergence of the series

$$\sum_{n=0}^{\infty} \frac{(\sin nx)^2}{\pi^2}$$

   on $\mathbb{R}$.

6. Marks : $7 + 7 = 14$

   (a) Prove that a mapping $f : X \to Y$ of metric spaces is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

   (b) Define uniformly continuous mapping on a metric space. Prove that every continuous mapping on compact metric spaces is uniformly continuous.

7. Marks : $7 + 4 + 3 = 14$

   (a) State and prove the Maximum-Minimum Theorem. Hence, justify it for $f(x) = x^3 - x$ on $[-1, 1]$.

   (b) Let $f : [1, 2] \to [0, 3]$ be continuous such that $f(1) = 0$ and $f(2) = 3$, show that f has a fixed point.

(c) Let $f : (0, 1] \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $[a, 1]$ for $a > 0$.

8. Marks : $7 + 7 = 14$

   (a) State and prove the Contraction Mapping Principle.

   (b) Find a function f such that $f'(x) = xf(x)$ for x near 0 and $f(0) = 3$.

2016-2017 (2019)

1. Marks : $4 + 4 + 6 = 14$

   (a) Define metric space with an example. Let (M, d) be a metric space. Construct a function $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Then show that (M, ρ) is a metric space and $\rho(x, y)$ is bounded by 1.

   (b) What is meant by a normed space? If $(V, ||\cdot||)$ is a normed space, then prove that it is a metric space defined by the function $d(x, y) = ||x - y||$.

   (c) State and prove Cauchy Schwarz Inequality. Put the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}\,x$ on the space of all continuous functions $C([0, 1])$. Then verify the stated inequality with $f(x) = x$ and $g(x) = x^2$ on $[0, 1]$.

2. Marks : $5 + 6 + 3 = 14$

   (a) Define neighbourhood of a point in a metric space with example. Prove that neighbourhood of a point in a metric space is an open set.

   (b) Prove that intersection of a finite number of open subsets of a metric space is open. How is about the intersection of an arbitrary family of open sets? Justify your answer.

   (c) Find the interior, exterior, and boundary of $S = (x, y) : x^2 + y^2 \leq 1$.

3. Marks : $5 + 4 + 5 = 14$

   (a) Let $K \subset Y \subset X$. Then prove that K is compact relative to X if and only if K is compact relative to Y.

   (b) Prove that every infinite subset E of a compact space K has a limit point in K.

   (c) Prove that every $k-$cell is compact.

4. Marks : $7 + 7 = 14$

   (a) Define compact set, connected set, and path-connected set with example. Show that $A = x \in \mathbb{R}^n : ||x|| \leq 2$ is compact and connected.

   (b) State and prove the Heine-Borel Theorem.

5. Marks : $7 + 7 = 14$

   (a) Define continuous function on a metric space. Prove that a mapping $f : X \to Y$, where X and Y are metric space, is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

   (b) Define uniformly continuous function on a metric space. Prove that every continuous mapping from a compact metric space into a metric space is uniformly continuous.

6. Marks : $5 + 4 + 5 = 14$

   (a) State and prove the Intermediate-Value Theorem. Show that f is uniformly continuous on (a, b).

(b) Show that the equation $ax^3 + bx^2 + cx + d = 0$, $(a \neq 0)$ has at least a real solution.

(c) Let $f : (a, b) \to \mathbb{R}$ be differentiable, and suppose that there is a constant $M > 0$ such that $||f'(x)|| \leq M$ for all $x \in (a, b)$. Here, a or b may be $\pm\infty$, and f' stands for the derivative of f.

7. Marks : $8 + 6 = 14$

(a) Distinguish between pointwise convergence and uniform convergence for sequence of functions. Show that the sequence $f_n(x) = \frac{x^n}{(1+x^n)}$, $x \in [0, 2]$ converges pointwise on $[0, 2]$ but that the convergence is not uniform.

(b) Prove that every Cauchy sequence in a compact metric space is convergent.

8. Marks : $7 + 7 = 14$

(a) State and prove the Inverse Function Theorem.

(b) Define contraction of a metric space. If X is a complete metric space, and φ is a contraction on X, then prove that there is a unique $x \in X$ such that $\varphi(x) = x$.