Chapter 1

Graphical Theory For Solution of ODE

1.1 Integral Curves or Solution Curves

Let

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y) \tag{1.1}$$

be a first order ordinary differential equation, then the graph of the explicit solution of (1.1) in the xy plane are called integral curves or solution curves.

1.2 Line Element

Let

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f(x, y) \tag{1.2}$$

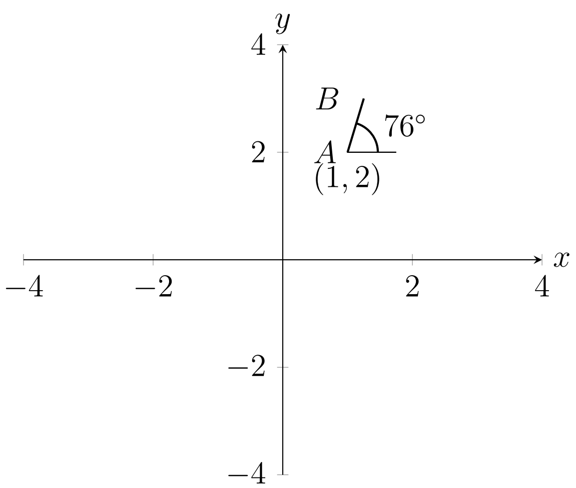
be a first order ordinary differential equation, then a short segment of the tangent line through the point (a, b) and with the slope $f(a, b)$ is called the line element.

1.2.1 Example of Line Element

Suppose

$$y' = 2x + y \tag{1.3}$$

be an ordinary differential equation.
Here, $f(x, y) = 2x + y$
The slope of (1.3) at $(1, 2)$ is 4.
Thus, through $(1, 2)$ we can draw short line AB with an inclination $\approx 76°$.



Here, AB is line Element at $(1, 2)$.

1.2.2 Line Element Configuration

If we draw a large number of line element for a large number of points then we obtain a configuration called line element configuration.

1.3 Direction Field

The totality of the line element together with the corresponding directions constitute a field which is called direction field of the differential equation.

1.4 Graphical Method

A procedure which yields the line element configuration of the direction field of a differential equation is called the graphical method. It provides approximate graphs of solution curves.

Problem 1.1. Construct a line element configuration (direction field) of the differential equation $y' = \frac{y}{x}$ and sketch the several integral curves.

Solution. We have

$$y' = \frac{y}{x} \tag{1.4}$$

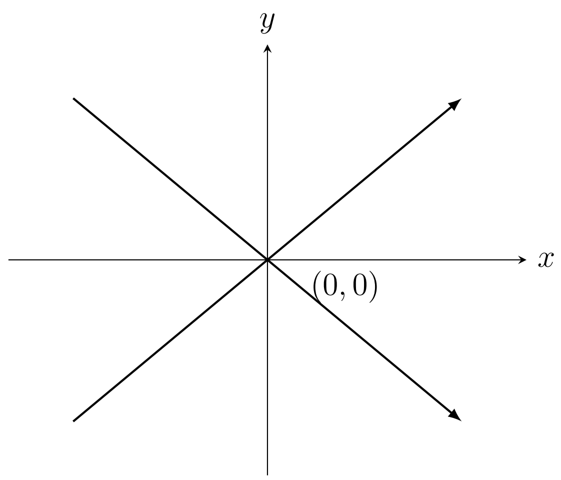
Here, the slope of the differential equation is

$$m = \tan\theta = \frac{y}{x}$$

The slope of the approximate integral curves of (1.4) are calculated at some selected points are given below.

x	-1	1	-2	2	1	-1	2	3	3	-2	-3	-3
y	-1	1	-2	2	-1	1	-2	3	-3	2	3	-3
m	1	1	1	1	-1	-1	-1	1	-1	-1	-1	-1
θ	$45°$	$45°$	$45°$	$45°$	$-45°$	$-45°$	$-45°$	$45°$	$-45°$	$-45°$	$-45°$	$-45°$

Now construct the line element at the selected points and sketch several smooth curves.



We observe that the integral curves represent a family of straight line passing through the origin.

Problem 1.2. Construct a line element configuration (direction field) of the differential equation $y' = -\frac{x}{y}$ and sketch several integral curves.

Solution. We have

$$y' = -\frac{x}{y} \tag{1.5}$$

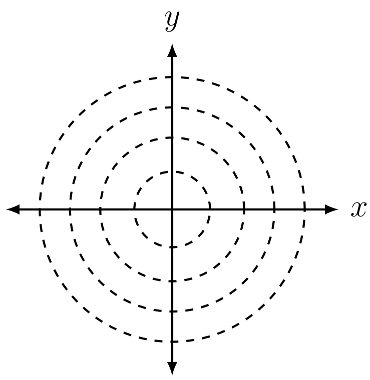
Here, the slope of the differential equation is

$$m = \tan\theta = -\frac{x}{y}$$

The slope of the approximate integral curves of (1.5) are calculated at some selected points are given below.

x	1	−1	−1	2	−2	−2	0	0	0	0	3	−3
y	1	1	−1	2	2	−2	1	2	−2	−1	3	3
m	−1	1	−1	−1	1	−1	0	0	0	0	−1	1
θ	$-45°$	$45°$	$-45°$	$-45°$	$45°$	$-45°$	$0°$	$0°$	$0°$	$0°$	$-45°$	$45°$

We now construct the line element at the selected points and sketch several integral curves.



We observe that the integral curves represent a family of circles which are centered as $(0, 0)$.

1.5 Method of Isoclines

Let us consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1.6}$$

A curve along which the slope $f(x, y)$ has a constant value c, is called a isocline of the differential equation (1.6) are curves $f(x, y) = c$ for different values of c.

Problem 1.3. Employ the method of isocline to sketch the several approximate integral curves of $y' = 3x - y$

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x - y \tag{1.7}$$

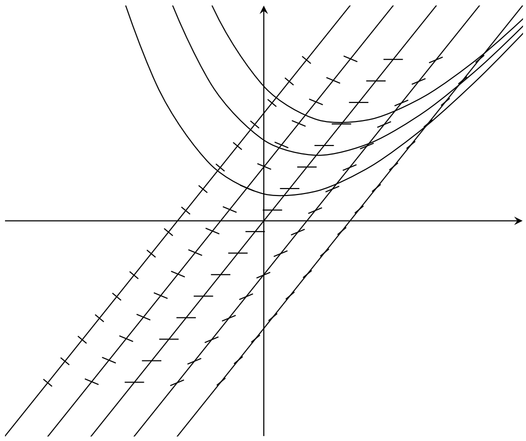
and the isocline of (1.7) is given by

$$3x - y = c$$
$$\Rightarrow y = 3x - c \tag{1.8}$$

For different values of c (1.8) represent a family of straight line.
We construct the line (1.8) for $c = 0, \pm1, \pm2, \pm3, \ldots$ etc.

On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = 3x$,	$\theta = \tan^{-1} c = 0°$
When $c = 1$,	then $y = 3x - 1$,	$\theta = \tan^{-1} c = 45°$
When $c = -1$,	then $y = 3x + 1$,	$\theta = \tan^{-1} c = -45°$
When $c = 2$,	then $y = 3x - 2$,	$\theta = \tan^{-1} c = 63.43°$
When $c = -2$,	then $y = 3x + 2$,	$\theta = \tan^{-1} c = -63.43°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.7)

Problem 1.4. Employ the method of isocline to sketch the several approximate integral curves of $y' = 2x + y$

Solution.

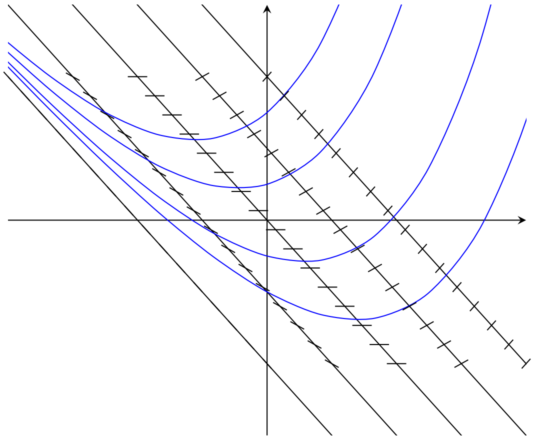
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 2x + y \tag{1.9}$$

The isocline of (1.9) is given by

$$2x + y = c \tag{1.10}$$

For different values of c (1.10) represent a family of straight line.
We construct the line (1.10) for $c = 0, \pm 1, \pm 2, \pm 3, \ldots$ etc.
On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = -2x$,	$\theta = 0°$
When $c = 1$,	then $2x + y = 1$,	$\theta = 45°$
When $c = -1$,	then $2x + y = -1$,	$\theta = -45°$
When $c = 2$,	then $2x + y = 2$,	$\theta = 63.43°$
When $c = -2$,	then $2x + y = -2$,	$\theta = -63.43°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.9)

Problem 1.5. Employ the method of isocline to sketch the several approximate integral curves of $y' = \frac{y-x}{y+x}$

Solution.

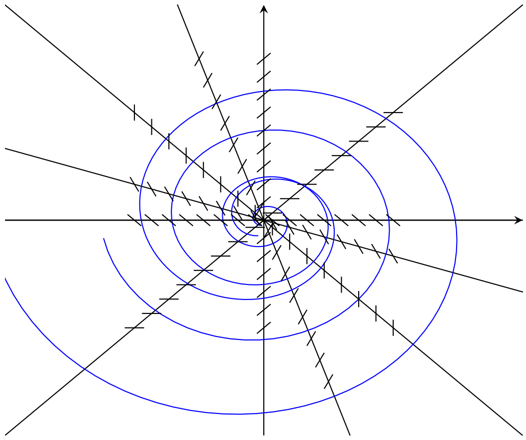
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{y-x}{y+x} \tag{1.11}$$

The isocline of (1.11) is given by

$$\begin{aligned}
&\frac{y-x}{y+x} = c \\
\Rightarrow\, & y - x = cy + cx \\
\Rightarrow\, & (1+c)x = (1-c)y \\
\Rightarrow\, & y = \frac{1+c}{1-c}x
\end{aligned} \tag{1.12}$$

For different values of c (1.12) represent a family of straight line passing through origin. We construct the line (1.12) for $c = 0, \pm 1, \pm 2, \pm 3, \ldots$ etc.
On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = x$,	$\theta = 0°$
When $c = 1$,	then $x = 0$,	$\theta = 45°$
When $c = -1$,	then $y = 0$,	$\theta = -45°$
When $c = 2$,	then $y = -3x$,	$\theta = 63.43°$
When $c = -2$,	then $y = -\frac{1}{3}x$,	$\theta = -63.43°$
When $c = \infty$,	then $y = -x$,	$\theta = 90°$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.11)

Problem 1.6. Employ the method of isocline to sketch the several approximate integral curves of $y' = \frac{3x-y}{x+y}$

Solution.

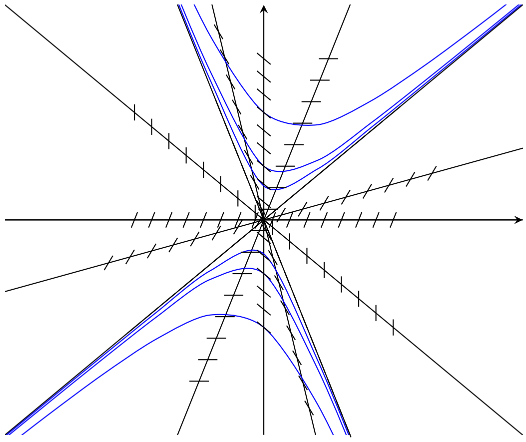
$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{3x-y}{x+y} \tag{1.13}$$

The isocline of (1.13) is given by

$$\begin{aligned}
&\frac{3x-y}{x+y} = c\\
&\Rightarrow 3x - y = cy + cx\\
&\Rightarrow (3-c)x = (c+1)y\\
&\Rightarrow y = \frac{3-c}{c+1}x
\end{aligned} \tag{1.14}$$

For different values of c (1.14) represent a family of straight line passing through origin.
We construct the line (1.14) for $c = 0, \pm 1, \pm 2, \pm 3, \ldots$ etc.
On each of these lines we then construct a number of line elements having the approximate inclinations $\tan^{-1} c$.

When $c = 0$,	then $y = 3x$,	$\theta = 0^\circ$
When $c = 1$,	then $y = x$,	$\theta = 45^\circ$
When $c = -1$,	then $x = 0$,	$\theta = -45^\circ$
When $c = 2$,	then $y = \dfrac{1}{3}x$,	$\theta = 63.43^\circ$
When $c = -2$,	then $y = -5x$,	$\theta = -63.43^\circ$
When $c = 3$,	then $y = 0$,	$\theta = 71.57^\circ$
When $c = -3$,	then $y = -3x$,	$\theta = -71.57^\circ$
When $c = \infty$,	then $y = -x$,	$\theta = 90^\circ$



Finally we draw several smooth curves. These smooth curves represent the approximate integral curves of (1.13)