Chapter 1

Gauss-Seidel and SOR Method for Systems of Linear Equations

1.1 Gauss-Seidel Method

$$x_i^{(k)} = \frac{-\sum_{j=1}^{i-1}\left(a_{ij}x_j^{(k)}\right) - \sum_{j=i+1}^{n}\left(a_{ij}x_j^{(k-)}\right) + b_i}{a_{ij}}$$

for each $i = 1, 2, 3, \ldots, n$ is called the Gauss-Seidel iterative technique.

Example. The linear system

$$\begin{aligned}
10x_1 - x_2 + 2x_3 \phantom{{}- x_4} &= 6\\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25\\
2x_1 - x_2 + 10x_3 - x_4 &= -11\\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}$$

can be written

$$\begin{aligned}
x_1^{(k)} &= \phantom{-\frac{1}{5}x_1^{(k)} +{}} \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5}\\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11}\\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10}\\
x_4^{(k)} &= -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} + \frac{15}{8}
\end{aligned}$$

Letting $x^{(0)} = (0, 0, 0, 0)^t$, we generate the iterates in the table below.

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.0300	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.0370	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	-0.9873	-1.0140	-1.0025	-1.0003	-1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

Since

$$\frac{\left\|x^{(5)} - x^{(4)}\right\|_\infty}{\left\|x^{(5)}\right\|_\infty} = \frac{0.0008}{0.2000} = 4 \times 10^{-4}$$

$x^{(5)}$ is accepted as a reasonable approximation to the solution.

Comment: Jacobi method for this example require twice the iterations for the same degree of accuracy.

So the Gauss-Seidel method is superior to the Jacobi method. This is generally but not always true. There are linear systems for which Jacobi method is convergent but not Gauss-Seidel and others for which Gauss-Seidel method converges and the Jacobi method does not.

Definition 1. Suppose $\tilde{x} \in \mathbb{R}^n$ is an approximation to the solution of the linear system defined by $Ax = b$. The residual vector for $\tilde{x}$ with respect to this system is $r = b - A\tilde{x}$.

1.2 Successive Over Relaxation

Gauss-Seidel procedure can be modified as follows

$$x_i^{(k)} = x_i^{(k-1)} + \omega \frac{r_{ii}^{(k)}}{a_{ii}} \tag{1.1}$$

for certain choices of positive ω reduces the norm of the residual vector and leads to significantly faster convergence. Where,

$$r_{ii}^{(k)} = b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} - a_{ii}x_i^{(k-1)}$$

Methods involving equation (1.1) are called relaxation methods. For choices of ω with $0 < \omega < 1$, the procedures are called under-relaxation methods and can be used to obtain convergence of some systems that are not convergent by the Gauss-Seidel method. For choices of ω with $\omega > 1$, the procedures are called over-relaxation methods, which are used to accelerate the convergence for systems that are convergent by the Gauss-Seidel technique.

These methods are called Successive Over-Relaxation (SOR) method and are particularly useful for solving the linear systems that occur in the numerical solution of certain partial-differential equations.

Note.

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right]$$

Example. The linear system $Ax = b$ given by

$$\begin{aligned} 4x_1 + 3x_2 \qquad\quad &= 24 \\ 3x_1 + 4x_2 - \quad x_3 &= 30 \\ -4x_2 + 4x_3 &= -24 \end{aligned}$$

has the solution $(3, 4, -5)^t$.

We want to solve the above system by Gauss-Seidel and SOR (with $\omega = 1.25$) method using $x^{(0)} = (1, 1, 1)^t$ for both methods.

Gauss-Seidel method:

$$\begin{aligned} x_1^{(k)} &= \qquad\quad - 0.75x_2^{(k-1)} \qquad\qquad\qquad + 6 \\ x_2^{(k)} &= -0.75x_1^{(k)} \qquad\qquad + 0.25x_3^{(k-1)} + 7.5 \\ x_3^{(k)} &= \qquad\quad 0.25x_2^{(k)} \qquad\qquad\qquad - 6 \end{aligned}$$

SOR method with $\omega = 1.25$:

$$\begin{aligned} x_1^{(k)} &= -0.25x_1^{(k-1)} - 0.9375x_2^{(k-1)} \qquad\qquad\qquad + 7.5 \\ x_2^{(k)} &= -0.9375x_1^{(k)} - 0.25x_2^{(k-1)} + 0.3125x_3^{(k-1)} + 9.375 \\ x_3^{(k)} &= \qquad\qquad 0.3125x_2^{(k)} \qquad - 0.25x_3^{(k-1)} - 7.5 \end{aligned}$$

The first 7 iterations are listed in the tables below. To obtain 7 digit accuracy Gauss-Seidel needs 34 and SOR required 14 iterations.

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	1	5.250000	3.1406250	3.0878906	3.0549316	3.0343323	3.0214577	3.0134110
$x_2^{(k)}$	1	3.812500	3.8828125	3.9267578	3.9542236	3.9713898	3.9821186	3.9888241
$x_3^{(k)}$	1	-5.046875	-5.0292969	-5.0183105	-5.0114441	-5.0071526	-5.0044703	-5.0027940

Table 1.1: Gauss-Seidel

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	1	6.312500	2.6223145	3.1333027	2.9570512	3.0037211	2.9963276	3.0000498
$x_2^{(k)}$	1	3.5195313	3.9585266	4.0102646	4.0074838	4.0029250	4.0009262	4.0002586
$x_3^{(k)}$	1	-6.6501465	-4.6004238	-5.0966863	-4.9734897	-5.0057135	-4.9982822	-5.0003486

Table 1.2: SOR method with $\omega = 1.25$

Note. There is no any general answer to know perfect choice of the values of ω for solving a linear system of equation.

For certain situations we can follow the following theorems:

Theorem 1.2.1 (Kahan). If $a_{ii} \neq 0$ for each $i = 1, 2, \dots, n$, then $\rho(T_\omega) \geq |\omega - 1|$. This implies that the SOR method can converge only if $0 < \omega < 2$.

Theorem 1.2.2 (Ostrowski-Reich). If A is a positive definite matrix and $0 < \omega < 2$, then the SOR method converges for any choice of initial approximate vector $x^{(0)}$.

Theorem 1.2.3. If A is positive definite and tridiagonal, then $\rho(T_g) = [\rho(T_j)]^2 < 1$, and the optimal choice of ω for the SOR method is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_j)]^2}}$$

with this choice of ω we have $\rho(T_\omega) = 1 - \omega$.

Problem 1.1 (H.W.)**.** Find the first four iterations by Jacobi and Gauss-Seidel method using $\mathbf{x}^{(0)} = 0$, and check the relations

$$\frac{\left\| x^{(4)} - x^{(3)} \right\|_\infty}{\left\| x^{(4)} \right\|_\infty}$$

for the all systems.

1. $$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned}$$

2. $$\begin{aligned} 10x_1 - x_2 &= 9 \\ -x_1 + 10x_2 - 2x_3 &= 7 \\ - 2x_2 + 10x_3 &= 6 \end{aligned}$$

3. $$\begin{aligned} 10x_1 + 5x_2 &= 6 \\ 5x_1 + 10x_2 - 4x_3 &= 25 \\ - 4x_2 + 8x_3 - x_4 &= -11 \\ - x_3 + 5x_4 &= -11 \end{aligned}$$

4. $$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2 \\ x_1 + 4x_2 - x_3 - x_4 &= -1 \\ -x_1 - x_2 + 5x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 + 3x_4 &= 1 \end{aligned}$$

1.2.1 The SOR Method

Problem 1.2. Consider a linear system $Ax = b$, where

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

(i) Check that the SOR method with $\omega = 1.25$ of the relaxation parameter can be used to solve this system.

(ii) Compute the first four iteration by the SOR method starting at the point $x^{(0)} = (0, 0, 0)^t$

Solution.

(i) Let us verify the sufficient condition for using the SOR method. We have to check if matrix A is symmetric, positive definite.
(spd): A is symmetric, so let us check positive definiteness:

$$\det(3) = 3 > 0, \quad \det \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = 8 > 0, \quad \det(A) = 20 > 0$$

All the leading principal minors are positive and so the matrix A is positive definite. We know that for spd matrices the SOR method converges for values of the relaxation ω from the interval $0 < \omega < 2$.

Conclusion: The SOR method with the value $\omega = 1.25$ can be used to solve this system.

(ii) The iterations of the SOR method are easier to compute by elements than in the vector form.
Write the system as equations:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= -1 \\ -x_1 + 3x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -7 \end{aligned}$$

First we write down the equations for the Gauss-Seidel (GS) iterations:

$$\begin{aligned} x_1^{(k+1)} &= \frac{\left(-1 + x_2^{(k)} - x_3^{(k)}\right)}{3} \\ x_2^{(k+1)} &= \frac{\left(7 + x_1^{(k+1)} + x_3^{(k)}\right)}{3} \\ x_3^{(k+1)} &= \frac{\left(-7 - x_1^{(k+1)} + x_2^{(k+1)}\right)}{3} \end{aligned}$$

Now multiply the RHS by the parameter $\omega = 1.25$ and add to it the vector $x^{(k)}$ from the previous iteration multiplied by the factor of $(1 - \omega)$:

$$\begin{aligned} x_1^{(k+1)} &= (1-\omega)x_1^{(k)} + \frac{\omega\left(-1 + x_2^{(k)} - x_3^{(k)}\right)}{3} \\ x_2^{(k+1)} &= (1-\omega)x_2^{(k)} + \frac{\omega\left(7 + x_1^{(k+1)} + x_3^{(k)}\right)}{3} \\ x_3^{(k+1)} &= (1-\omega)x_3^{(k)} + \frac{\omega\left(-7 - x_1^{(k+1)} + x_2^{(k+1)}\right)}{3} \end{aligned}$$

For $k = 0, 1, 2, \ldots$ compute $x^{(k+1)}$ from these equations, starting by the first one.

Computation for $k = 0$:

$$\begin{aligned} x_1^{(1)} &= (1-\omega)x_1^{(0)} + \frac{\omega\left(-1 + x_2^{(0)} - x_3^{(0)}\right)}{3} \\ &= (1 - 1.25) \times 0 + \frac{1.25\,(-1 + 0 - 0)}{3} \\ &= -0.41667 \\ x_2^{(1)} &= (1-\omega)x_2^{(0)} + \frac{\omega\left(7 + x_1^{(1)} + x_3^{(0)}\right)}{3} = 2.7431 \\ x_3^{(1)} &= (1-\omega)x_3^{(0)} + \frac{\omega\left(-7 - x_1^{(1)} + x_2^{(1)}\right)}{3} = -1.6001 \end{aligned}$$

Similarly, the next three iterations are presented in the following table:

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
2	1.4972	2.1880	-2.2288
3	1.0494	1.8782	-2.0141
4	0.9128	2.0007	-1.9723

Note. 1. The spectral radius $\rho(A)$ of a matrix A is defined by $\rho(A) = \max|\lambda|$, λ is an eigen value of A.

2. The $n \times n$ matrix A is said to be strictly diagonally dominant when $|a_{ii}| > \sum_{j=1 j\neq i}^{n} |a_{ij}|$ holds for each $i = 1, 2, \ldots, n$.

3. A matrix A is positive definite if it is symmetric and if $x'Ax > 0$ for every $n-$dimensional column vector $x \neq 0$.
Here,

$$x'Ax = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Theorem 1.2.4. If A is an $n \times n$ positive definite matrix, then

(i) A is non-singular;

(ii) $a_{ii} > 0$ for each $i = 1, 2, \ldots, n$;

(iii) $\max\limits_{1\leq k,j\leq n} |a_{kj}| \leq \max\limits_{1\leq i\leq n} |a_{ii}|$;

(iv) $(a_{ij})^2 < a_{ii}a_{jj}$ for each $i \neq j$.

Theorem 1.2.5. A symmetric matrix A is positive definite iff each of its leading principal sub matrices has a positive determinant.

Example. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$\det A_1 = \det \begin{bmatrix} 2 \end{bmatrix} = 2, \qquad \det A_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 > 0, \qquad \det A_3 = \det A = 4 > 0$$

Problem 1.3 (H.W.)**.** Find the first 3 iterations of the SOR method with $\omega = 1.1$ for the following linear systems, using $x^{(0)} = 0$:

(i)
$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned}$$

(ii)
$$\begin{aligned} 10x_1 - x_2 &= 9 \\ -x_1 + 10x_2 - 2x_3 &= 7 \\ - 2x_2 + 10x_3 &= 6 \end{aligned}$$

(iii)
$$\begin{aligned} 10x_1 + 5x_2 &= 6 \\ 5x_1 + 10x_2 - 4x_3 &= 25 \\ - 4x_2 + 8x_3 - x_4 &= -11 \\ - x_3 + 5x_4 &= -11 \end{aligned}$$

(iv)
$$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2 \\ x_1 + 4x_2 - x_3 - x_4 &= -1 \\ -x_1 - x_2 + 5x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 + 3x_4 &= 1 \end{aligned}$$