Chapter 1

Applications of Sylow Theorems

Some Examples on the Use of Sylows Theorems

Example. A group of order 40 must contain a normal subgroup of order 5.

Solution. Here $|G| = 40 = 2^3 \cdot 5$.
s_5 divides 8 and has the form $1 + 5k$. So, necessarily $s_5 = 1$.
Since there is only one subgroup of order 5, then this subgroup is normal in G.

Example. Show that no group of order 30 is simple.

Solution. To show: G has at least one non-trivial normal subgroup.
Here $|G| = 30 = 2 \cdot 3 \cdot 5$.
s_5, the number of Sylow 5-subgroups, divides 6 and has the form $(1 + 5k)$. Should $k = 0$. We are done (i .e., there exists unique subgroup of order 5 is then normal in G).
Should $k = 1$, the total number of Sylow 5-subgroups is then 6. Such subgroups can have only the identity element e in common; so they would account for $4 \times 6 = 24$ distinct elements of G, not counting e.
In this situation, s_3 (the number of Sylow 3-subgroups), being a divisor of 10 and of the form $1 + 3m$, can only be $= 1(m = 3$ is ruled out by counting elements of G, since G would have then at least $25 + 20 = 45$ elements).
So, the Sylow 3-subgroup is a normal subgroup.
So, it is proved that either G has a unique Sylow 5-subgroup or it has a unique Sylow 3-subgroup.
!n either case, we get a non-trivial normal subgroup of G.