Chapter 1

Metric Space

1.1 Euclidean Space

Euclidean space or Euclidean n-space, denoted by $\mathbb{R}^n$ consists of all ordered n-tuples of real numbers. Symbolically, $\mathbb{R}^n = \{x \mid x = (x_1, x_2, \ldots, x_n), x_1, x_2, \ldots, x_n \in \mathbb{R}\}$
Here the element $x \in \mathbb{R}^n$ is called a point or a vector and $x_1, x_2, \ldots, x_n$ are called coordinates of x when $n > 1$.
If $x, y \in \mathbb{R}^n$ and if $\alpha \in \mathbb{R}^n$ then put,
$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ so that $x + y \in \mathbb{R}^n$ and $\alpha x \in \mathbb{R}^n$. This defines addition and scalar multiplication of vectors. These two operations satisfy the commutative, associative and distributive laws and make $\mathbb{R}^n$ into a vector space over the real field.

Theorem 1.1.1. $\mathbb{R}^n$ *with operations of addition and scalar multiplication defined previously is a vector space of dimension* n.

Definition 1 (Inner Product)**.** The inner product (or scalar product) of x and y in $\mathbb{R}^n$ is defined by $\langle x, y\rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$ and the *norm* or *length* of a vector $x \in \mathbb{R}^n$ is defined by $||x|| = \langle x, x\rangle^{1/2} = \sum_{i=1}^{n} (x_i^2)^2$ and the *distance* between two vectors x and y of $\mathbb{R}^n$ is the real number d(x, y) $= ||x - y|| = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}$

Definition 2. Let X be a metric space. All points and sets involved below are understood to be elements and subset of X.

1. A *neighborhood* of a point $p \in X$ is a set $N_\delta(p)$ containing all points q such that $d(p, q) < \delta$. The number δ is called the *radius* of $N_\delta(p)$. [Mathematically, $N_\delta(p) = \{q \mid d(p, q) < \delta\}$]

2. A point p is a *limit point* (accumulation point or cluster point) of the set E if every neighborhood $N_\delta(p)$ contains a point $q \neq p$ such that $q \in E$. [Mathematically, $(N_\delta(p) - \{p\}) \cap E \neq \varnothing$]

3. If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.

4. E is *closed* if every limit point of E is a point of E.

5. A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$

   (i) E is *open* if every point of E is an interior point of E.

6. The *complement* of E, denoted by E^c is the set of all points $p \in X$ such that $p \notin E$

7. E is *perfect* if E is closed and every point of E is a limit point of E.

8. E is *bounded* if E is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

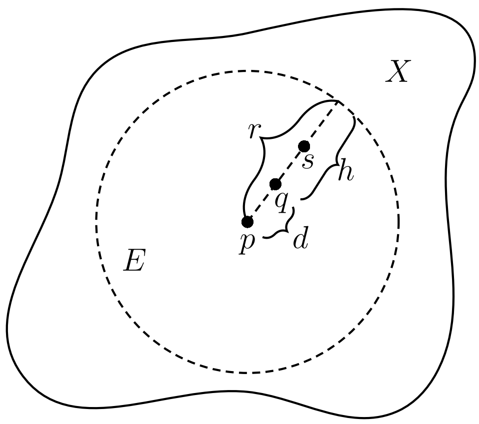
9. E is *dense* in X if every point of X is a limit point of E, or a point of E (or both).

Note. The segment $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

Note. In $\mathbb{R}^1$ neighborhoods are segments, whereas in $\mathbb{R}^2$ neighborhoods are interiors of circles and in $\mathbb{R}^3$ neighborhoods are interiors of spheres.

Theorem 1.1.2. *Every neighborhood is an open set.*

Proof. Consider the neighborhood $E = N_r(p) = \{q \in X \mid d(p, q) < r\}$ and let q be any point of E, where X is a metric space.



Then there is a positive real number h such that $d(p, q) = r - h$
Now for all points s such that $d(q, s) < h$,
We have then

$d(p, s) \le d(p, q) + d(q, s) < r - h + h = r$	Here $d(p, q) < r$
so that $s \in E$	$\Rightarrow d(p, q) + h = r$
Therefore, $N_h(q) = \{s \in E \mid d(q, s) < h\} \subset E = N_r(p)$	$\Rightarrow d(p, q) = r - h$
Thus, q is an interior point of E.	$\because r - h = d(p, q) \ge 0$
Hence the theorem.	$\Rightarrow h \le r$

□

Theorem 1.1.3. *If p is a limit point of a set E in a metric space X, then every neighborhood of p contains infinitely many points of E.*

Proof. Suppose there is a neighborhood N of $p \in X$ which contains only a finite number of points of E. Let $q_1, q_2, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p and put $r = \min_{1 \le m \le n} d(p, q_m)$.
[We use this notation to denote the smallest of the numbers $d(p, q_1), d(p, q_2), \ldots, d(p, q_n)$]
The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no part q of E such that $q \ne p$, so that p is not a limit point of E.
This contradiction establishes the theorem. □

Note. Here $r > 0 \Rightarrow r$ can be taken a large positive real number, however we please $\Rightarrow d(p, q_m), m = 1, 2, \ldots, n$ are bigger & bigger $\Rightarrow q_m$ are not close enough to $p \Rightarrow p$ is not a limit point of E.

Corollary 1.1.4. *A finite set has no limit points.*

Problem 1.1.1. Let us consider the following subsets of $\mathbb{R}^2$

1. The set of all complex Z such that $|Z| < 1$

2. The set of all complex Z such that $|Z| \le 1$

3. A finite set

4. The set of all integers

5. The set consisting of the numbers $\frac{1}{n}(n = 1, 2, 3\dots)$

6. The set of all complex numbers (that is, $\mathbb{R}^2$)

7. The segment (a, b)

If (4), (5) and (7) are regarded as subsets of $\mathbb{R}^1$, then identify whether the sets (1)-(7) are closed, open, perfect and bounded.

Theorem 1.1.5. *Let $\{E_\alpha\}$ be a collection of sets E_α, then $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha ({E_\alpha}^c)$*

Theorem 1.1.6. *A set E is open if and only if its complement is closed.*

Proof. First, suppose E^c is closed. Cause $x \in E$. Then $x \notin E^c$ and x is not a limit point E^c. Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E and E is open.

Next, suppose that E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, such that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed. □

Corollary 1.1.7. *A set F is closed if and only if its complement is open.*

Theorem 1.1.8.

1. *For any collection $\{G_\alpha\}$ is open sets, $\underset{\alpha}{\cup} G_\alpha$ is open.*

2. *For any collection $\{F_\alpha\}$ is closed sets, $\underset{\cap}{\alpha} F_\alpha$ is closed.*

3. *For any finite collection $G_1, G_2, \dots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.*

4. *For any finite collection $F_1, F_2, \dots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed.*

Note. Is the finiteness of the collection in parts (3) and (4) of the above theorem essential? Justify your answer.

Definition 3. If X is a metric space, if $E \subset X$ and if E' denotes the set of all limit points of E in X, then E' is called the derived set of E and $\bar{E} := E \cup E'$ is called the closure of E.

Example. $E = (0, 1) \cup \left\{e, \pi, \sqrt{7}, 11.5\right\}$, then $E' = [0, 1]$, $\bar{E} = E' \cup E = [0, 1] \cup \left\{e, \pi, \sqrt{7}, 11.5\right\}$

Theorem 1.1.9. *If X is a metric space and $E \subset X$, then*

(a) $\bar{E}$ is closed,

(b) $E = \bar{E}$ if and only if E is closed,

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

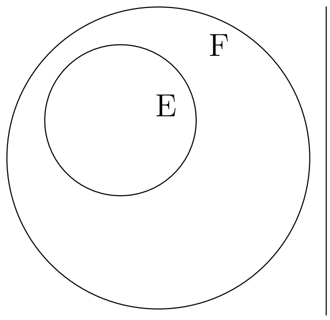
By 1 and 3, $\bar{E}$ is the smallest closed subset of X that contains E.

Proof.

(a) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E. Hence, p has a neighborhood which does not intersect E. The complement of $\bar{E}$ is therefore open. Hence, $\bar{E}$ is closed.

(b) If $E = \bar{E}$, (a) implies that E is closed. If E is closed, then $E' \subset E$(by definition (1) and (10)). Hence, $\bar{E} = E$.

(c) If E is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus, $F \supset \bar{E}$.



$F \supset F'$
$F \supset E$
$F \supset E'$
$F \supset \bar{E}$

□

1.2 Connected Set

Let A be a subset of metric space X. Two non-empty open sets U and V are said to *separate* A if they satisfy these condition

(i) $U \cap V \cap A = \varnothing$
(ii) $A \cap U \neq \varnothing$
(iii) $A \cap V \neq \varnothing$
(iv) $A \subset U \cup V$

We say that A is *disconnected (i.e., not connected)* if such set exist and if such sets do not exist, we say that A is *connected.*

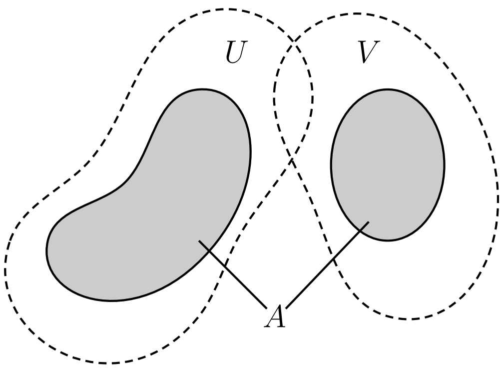


Figure 1.1: A is disconnected

Example.

(i) $\bar{\mathbb{Z}}$ is not connected

(ii) $S = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is connected

1.3 Compact Set

By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of an open subset of X such that $E \subset \bigcup_\alpha G_\alpha$.

A subset K of a metric space X is said to be *compact* if every open cover of K contains a finite subcover.

More explicitly, the requirement for completeness of $K \subset X$ is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha 1} \cup G_{\alpha 2} \cup \ldots \cup G_{\alpha n} \quad \text{i.e.,} k \subset \bigcup_{i=1}^{n} G_{\alpha i}$$

Example.

(i) $A = [1, 2]$ is *compact.*

(ii) $B = (0, 2)$ is *not compact.*

Theorem 1.3.1. *In a metric space, prove that closed subsets of a compact set is compact.*

1.4 Path-connected Sets

We say that map $\varphi : [a, b] \to M$ of an interval $[a, b]$ into a metric space M *continuous* if $t_\mu \to t$ implies $\varphi(t_\mu) \to \varphi(t)$ for every sequence t_μ in $[a, b]$ converging to some $t \in [a, b]$.

A *continuous path* joining two points x, y in a metric space M is a mapping $\varphi : [a, b] \to M$ such that $\varphi(a) = x$, $\varphi(b) = y$ and φ is continuous. Here x may or may not equal y and $b \geq a$.

A path φ is said to lie in a set A if $\varphi(t) \in A$ for all $t \in [a, b]$.

We say that a set A is path-connected if every two points in the set can be joined by a continuous path lying in the set.

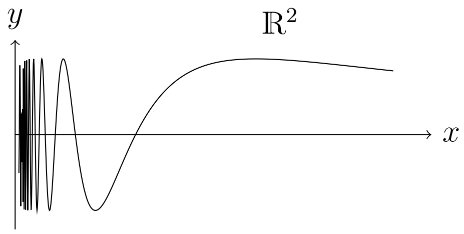


Figure 1.2: Not path-connected

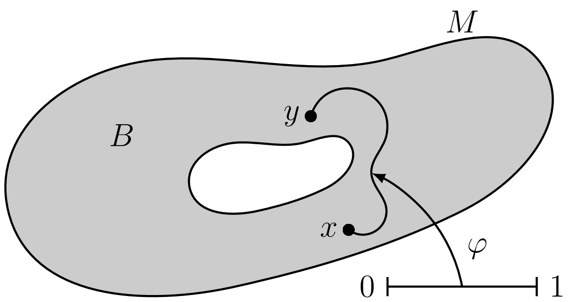


Figure 1.3: Path connected

- Figure 1.2: $A = \left\{(x, \sin \frac{1}{x}) \mid x > 0\right\} \cup \{(0, y) \mid y \subset [-1, 1]\} \subset \mathbb{R}^2$. A is not path connected.[1]
- Figure 1.3: A curve joining points x and y in A of a metric space M. Evidently, region A is path connected.

Problem 1.4.1. Show that $B = [0, 1]$ is path connected.

Solution. Let $\varphi : B \to \mathbb{R}$ be a function defined by $\varphi(t) = (y - x)t + x$.
Here $\varphi(0) = x$, $\varphi(1) = y$, φ is continuous path (because φ is a linear polynomial in t) and φ lies in B.

[1] This is also known as 'Topologist's sine curve'

Problem 1.4.2. Which of the following sets are path-connected?

(i) $[0, 3]$

(ii) $[1, 2] \cup [3, 4]$

(iii) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 2\}$

Problem 1.4.3. Let $\varphi : B = [0, 1] \to \mathbb{R}^2$ be a continuous path and $C = \varphi([0, 1])$. Show that C is path-connected.

Solution. This is intuitively clear, for we can use the path φ itself to join two points in C. Precisely, if $x = \varphi(a)$, $y = \varphi(b)$, where $0 \leq a \leq b \leq 1$, let $c : B \to \mathbb{R}^2$ be defined by $c(t) = \varphi(t)$. Thus, c is a path joining x to y and c lies in C.

Problem 1.4.4. Is $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ connected?

Solution. No, for if $U = (1/2, \infty)$, $V = (-\infty, 1/4)$, then $\mathbb{Z} \subset U \subset V$, $\mathbb{Z} \cap U = \{1, 2, 3, \dots\} \neq \varnothing$, $\mathbb{Z} \cap V = \{\dots, -2, -1, 0\} \neq \varnothing$. Hence, $\mathbb{Z}$ is not disconnected(i.e., not connected).

Besides, $\mathbb{Z}$ is not path-connected.

Problem 1.4.5. Are $[0, 1] \cup [2, 3]$, $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\} \cup \{(x, 0) \mid 1 < x < 2\}$ connected?

Problem 1.4.6. Determine the compactness of

(i) finite set $A = \{x_1, x_2, \dots, x_n\}$

(ii) $\mathbb{R}$

(iii) $B = [0, \infty)$

(iv) $C = (0, 1)$

Solution. 1. $A = \{x_1, x_2, \dots, x_n\}$ – a finite subset of $\mathbb{R}$.
Let $\mathscr{G} = \{G_\alpha\}$ be any open cover of A, then each x_i is contained in some set $G_{\alpha i} \in \mathscr{G}$. Then $A \subset \bigcup_{i=1}^{n} G_{\alpha i} \Rightarrow \{G_{\alpha i}; i = 1, 2, \dots, n\}$ is a finite sub-cover of $\mathscr{G}$. Since $\mathscr{G}$ is arbitrary so A is compact.

Theorem 1.4.1. *Path-connected sets are connected.*

Theorem 1.4.2 (Heine-Borel Theorem)**.** *A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 1.4.3 (Bolzano-Weirstrass Theorem)**.** *A subset of a metric space is compact if and only if it is sequentially compact.*

Problem 1.4.7. Show that $A = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ is compact and connected.

Solution. To show that A is compact, we show it is closed and bounded. To show that it is closed consider $A^c = \mathbb{R}^n \backslash A = \{x \in \mathbb{R}^n \mid ||x|| > 1\} = B$. For $x \in B$, $N_\delta(x) \subset B$, with $\delta = ||x|| - 1$, so that B is open and hence A is closed. It is clear that A is bounded, since $A \subset N_2(0)$ and therefore A is compact.

To show that A is connected, we show that A is path-connected. Let $x, y \in A$. Then the straight line joining x, y is the required path. Explicitly, we use $\varphi : [0, 1] \to \mathbb{R}^n$, $\varphi(t) = (1 - t)x + ty$. One sees that $\varphi(t) \in A$, since

$$\begin{aligned} ||\varphi(t)|| \leq&\ (1 - t)\,||x|| + t\,||y|| \\ \leq&\ (1 - t) + t = 1 \quad \text{by triangle inequality.} \end{aligned}$$

Theorem 1.4.4. *Closed subsets of a compact set is compact.*

Proof. Suppose $F \subset K \subset M$, F is closed subset and K is compact in the metric space M. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open Ω of K. Since K is compact, there is a finite sub-collection Φ of Ω which covers K, and hence F. If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F. We have thus shown that a finite sub-collection of $\{V_\alpha\}$ covers F. Hence, the theorem. $\square$