Any differential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \quad (1)$$

in which $f(x,y)$ and $g(x,y)$ are homogeneous functions of x and y of the same degree is known as the homogeneous differential equation of first order.

Example of homogeneous functions:

$f_1(x,y) = ax + by$

$f_2(x,y) = ax^2 + bxy + cy^2$

$f_3(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$

$f_4(x,y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$

$\vdots$

are all homogeneous functions of x, y of degree 1, 2, 3, 4 respectively. Where as

$g_1(x,y) = ax + by + c$

$g_2(x,y) = ax^2 + bxy + cy^2 + d$

$g_3(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + e$

are not homogeneous functions of x and

Solve: $\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx} = \frac{x+y+a}{x+y+b}$ —— (1)

Solⁿ:-

Let $x+y = v$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

(1) $\Rightarrow$

$$\left(\frac{v-a}{v-b}\right)\left(\frac{dv}{dx} - 1\right) = \frac{v+a}{v+b}$$

$\Rightarrow \frac{dv}{dx} - 1 = \frac{v-b}{v-a}\,\frac{v+a}{v+b}$

$\Rightarrow \frac{dv}{dx} = 1 + \frac{(v-b)(v+a)}{(v-a)(v+b)}$

$$= \frac{v^2 - av + bv - ab + v^2 - bv + av - ab}{v^2 - av + bv - ab}$$

$$= \frac{2v^2 - 2ab}{v^2 + (b-a)v - ab} = \frac{2(v^2-ab)}{v^2+(b-a)v-ab}$$

$\Rightarrow \frac{v^2+(b-a)v-ab}{v^2-ab}\,dv = 2\,dx$ (variable separable)

$\Rightarrow \frac{(v^2-ab)+(b-a)v}{v^2-ab}\,dv = 2\,dx$

$\Rightarrow \int\left(1 + \frac{b-a}{2}\,\frac{2v}{v^2-ab}\right)dv = 2\int dx$

$\Rightarrow v + \frac{b-a}{2}\boxed{\ln(v^2-ab)} = 2x + c$

Ans.

Solve: $x\frac{dy}{dx} = y + x\sqrt{(x^2+y^2)}$

$\Rightarrow$

$$x\frac{dy}{dx} - y = x\sqrt{(x^2+y^2)}$$

$\Rightarrow$ $$\frac{x\,dy - y\,dx}{dx} = x\sqrt{(x^2+y^2)}$$

$\Rightarrow$ $$x\,dy - y\,dx = x\sqrt{(x^2+y^2)}\,dx \quad \text{---} \quad (1)$$

Let $x = r\cos\theta$
$y = r\sin\theta$

$\Rightarrow$ $x^2+y^2 = r^2$

Again $\frac{y}{x} = \tan\theta$

$$\frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\, d\theta$$

$\Rightarrow$ $x\,dy - y\,dx = x^2\sec^2\theta\, d\theta = r^2\cos^2\theta\sec^2\theta\, d\theta$

$\Rightarrow$ $x\,dy - y\,dx = r^2\, d\theta$

(1) $\Rightarrow$ $r^2\, d\theta = r\cos\theta\sqrt{(r^2)}\,dx$

$\Rightarrow$ $r^2\, d\theta = r\cos\theta\, r\, dx$

$\Rightarrow$ $$\frac{d\theta}{\cos\theta} = dx$$

$\Rightarrow$ $\sec\theta\, d\theta = dx$ (variable seperable)

integrating, we get

$$\ln(\sec\theta + \tan\theta) = x + \log c$$

$\Rightarrow$ $\ln(\sec\theta+\tan\theta) - \log c = x$

$\Rightarrow$ $\ln\left(\frac{\sec\theta+\tan\theta}{c}\right) = x$ $\Rightarrow$ $\frac{\sec\theta+\tan\theta}{c} = e^{x}$

Solve $\dfrac{x\,dx+y\,dy}{x\,dy-y\,dx}=\sqrt{\left(\dfrac{a^2-x^2-y^2}{x^2+y^2}\right)}$ ... ①

Solⁿ: Let $x=r\cos\theta$ — (2)

$y=r\sin\theta$ -- (3)

$\Rightarrow x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta$

$\Rightarrow x^2+y^2=r^2(\cos^2\theta+\sin^2\theta)$

$\Rightarrow x^2+y^2=r^2$ ... (4)

$\Rightarrow 2x\,dx+2y\,dy=2r\,dr$

$\Rightarrow x\,dx+y\,dy=r\,dr$ - (5)

Again, (3) ÷ (2) ⇒

$\dfrac{y}{x}=\tan\theta$

$\Rightarrow \dfrac{x\,dy-y\,dx}{x^2}=\sec^2\theta\,d\theta$

$\Rightarrow x\,dy-y\,dx=x^2\sec^2\theta\,d\theta=r^2\cos^2\theta\cdot\dfrac{1}{\cos^2\theta}\,d\theta$

$\Rightarrow x\,dy-y\,dx=r^2\,d\theta$ ... (6)

$\therefore$ (1) ⇒ $\dfrac{r\,dr}{r^2\,d\theta}=\sqrt{\left(\dfrac{a^2-r^2}{r^2}\right)}=\dfrac{\sqrt{a^2-r^2}}{r}$

$\Rightarrow \dfrac{dr}{\sqrt{a^2-r^2}}=d\theta$ (variable seperable)

Integrating, we get

$\sin^{-1}\left(\dfrac{r}{a}\right)=\theta+c \Rightarrow \dfrac{r}{a}=\sin(\theta+c)$

$\Rightarrow r=a\sin(\theta+c)\Rightarrow \sqrt{x^2+y^2}=a\sin\left(\tan^{-1}\left(\dfrac{y}{x}\right)+c\right)$

(4) $kx = e^{x/y}$ (5) $2xy = (x+y)^2 \log\left(\frac{k}{x+y}\right)$

(6) $y+\sqrt{y^2-x^2} = cx^3$ (7) $x \tan\left(\frac{y}{x}\right) = c$

(8) $cx^2 = y + \sqrt{x^2+y^2}$ (9) $\log y = c + \tan^{-1} \frac{y}{x}$

(10) $x + ye^{x/y} = c$ (11) $\tan\left(\frac{y}{2x}\right) = cx$

(12) $y = x \sinh(x+c)$ (13) $\log(cy) = \frac{-x^2}{2y^2}$

(14) $x = c \sin\left(\frac{y}{x}\right)$ (15) $ye^{y/x} = c.$

(16) $x+y = c(x^2+y^2)$

(17) $(2x+y)(2y-3x) = cx$

(18) $x^4+6x^2y^2+y^4 = c$

(19) $\log(3y^2+3xy-x^2) + \frac{1}{\sqrt{21}} \log\left[\frac{2\sqrt{3}y + x(\sqrt{3}-\sqrt{7})}{2\sqrt{3}y + x(\sqrt{3}+\sqrt{7})}\right] = c$

(20) $x = c \log\left(\frac{y}{x}\right)$

(iii) Equations reducible to homogeneous equations:

Equations of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \text{ where } \frac{a}{a'} \neq \frac{b}{b'}$$

can be solved by reducing it to homogeneous form using the transformations $\boxed{x=X+h,\ y=Y+k}$ where h and k are constants to be determined.

When we put $x=X+h$, $y=Y+k$, then, $dx = dX$, $dy = dY$

Hence $\frac{dy}{dx} = \frac{dY}{dX}$

$\therefore$ The given equation reduces to

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

ie, $$\frac{dy}{dx} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$$

We choose h and k such that

$$ah+bk+c = 0 \quad \text{and}$$

By solving these equations for h and k

we get $h = \frac{bc'-b'c}{ab'-a'b}$

$$k = \frac{a'c-ac'}{ab'-a'b}$$

$\therefore$ when $\frac{a}{a'} \neq \frac{b}{b'}$ ie, when $ab'-a'b \neq 0$, for these values of h and k, the given equation reduces to

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$$

This is homogeneous and can be solved by putting $Y=VX$ and in the solution replace $X=x-h$ and $Y=y-k$, to get the solution of the given equation.

Worked Examples:

Solve the following equations:

(1) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solution : Put $x=X+h$ and $y=Y+k$

$$\therefore \frac{dY}{dX} = \frac{X+2Y+h+2k-3}{2X+Y+2h+k-3}$$

Put $h+2k-3 = 0$

$2h+k-3 = 0$

Solving these two equations simultaneously we get $h=1$, $k=1$

$$\therefore \frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

Put $Y=VX$ $\therefore \frac{dY}{dX} = V + X\frac{dV}{dX}$

$$\therefore V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX}$$

ie, $$V + X\frac{dV}{dX} = \frac{1+2V}{2+V}$$

$$\therefore X\frac{dV}{dX} = \frac{1+2V}{2+V} - V$$

$$= \frac{1+2V-2V-V^2}{2+V}$$

$$= \frac{1-V^2}{2+V}$$

$(2+V)dV \quad dX$

Integrating

$$2\int \frac{1}{1-V^2}dV+\int \frac{V}{1-V^2}dV = \int \frac{dX}{X} + \text{constant}$$

ie, $2.\frac{1}{2}\log\left(\frac{1+V}{1-V}\right)-\frac{1}{2}\log(1-V^2) = \log X + \log c$

ie, $\log\left(\frac{1+V}{1-V}\right)-\frac{1}{2}\log(1-V^2) = \log cx$

ie, $\left(\frac{1+V}{1-V}\right) = cX\sqrt{1-V^2}$

ie, $\frac{1+\frac{Y}{X}}{1-\frac{Y}{X}} = cX\sqrt{1-\frac{Y^2}{X^2}}$

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constant

ie, $\frac{X+Y}{X-Y} = c\sqrt{X^2-Y^2}$

ie, $\frac{x-1+y-1}{x-1-y+1} = c\sqrt{(x-1)^2-(y-1)^2}$

ie, $\frac{x+y-2}{x-y} = c\sqrt{(x-1+y-1)(x-1-y+1)}$

ie, $\frac{x+y-2}{x-y} = c\sqrt{(x+y-2)(x-y)}$

Squaring $\frac{(x+y-2)^2}{(x-y)^2} = c^2(x+y-2)(x-y)$

ie, $x+y-2 = c^2(x-y)^3$

ie, $x+y-2 = k(x-y)^3$ is the solution

(2) $(3x+2y-7)dx+(2x-3y+5)dy=0$

Solution : $\frac{dy}{dx} = -\frac{(3x+2y-7)}{(2x-3y+5)}$

Put $x = X+h$ and $y = Y+k$

$\therefore \frac{dY}{dX} = -\frac{(3X+2Y+3h+2k-7)}{2X-3Y+2h-3k+5}$

Choose $3h+2k-7 = 0$ and

$\underline{2h-3k+5 = 0}$

Solve these equations for h and k simultaneously. We get

$h = \frac{11}{13}$ and $k = \frac{29}{13}$

$\therefore \frac{dY}{dX} = -\frac{(3X+2Y)}{2X-3Y}$

Put $Y=VX$ $\therefore \frac{dY}{dX} = V+X\frac{dV}{dX}$

$\therefore V+X\frac{dV}{dX} = -\frac{(3X+2VX)}{2X-3VX}$

ie, $V+X\frac{dV}{dX} = -\frac{(3+2V)}{2-3V}$

$\therefore x\frac{dV}{dX} = -\frac{(3+2V)}{2-3V} - V$

$= \frac{-3-2V-2V+3V^2}{2-3V}$

$= \frac{-3-4V+3V^2}{2-3V}$

$\therefore \frac{(2-3V)\,dV}{3V^2-4V-3} = \frac{dX}{X}$

Integrating

$\int \frac{(2-3V)\,dV}{3V^2-4V-3} = \int \frac{dX}{X} + \text{constant}$

ie, $-\frac{1}{2}\int \frac{6V-4}{3V^2-4V-3}dV = \log X + \log C$

ie, $-\frac{1}{2}\log(3V^2-4V-3) = \log CX$

ie, $\frac{1}{\sqrt{3V^2-4V-3}} = CX$

ie, $\frac{X}{\sqrt{3Y^2-4XY-3X^2}} = CX$

or $3Y^2-4XY-3X^2 = k$

ie, $3\left(y-\frac{29}{13}\right)^2 - 4\left(x-\frac{11}{13}\right)\left(y-\frac{29}{13}\right) - 3\left(x-\frac{11}{13}\right)^2 = k$

(3) $\frac{dy}{dx} = \frac{2x+y+6}{y-x-3}$ when $y=0$, $x=0$

Solution : Put $x = X+h$, and $y = Y+k$

$\therefore \frac{dY}{dX} = \frac{2X+Y+2h+k+6}{Y-X+k-h-3}$

Put $2h+k+6 = 0$

and $-h+k-3 = 0$

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Solving we get h= −3 and k=0

$$\therefore \frac{dY}{dX} = \frac{2X+Y}{Y-X}$$

Put Y=VX

$$\therefore V+X\frac{dV}{dX} = \frac{2X+VX}{VX-X} = \frac{2+V}{V-1}$$

$$\therefore X\frac{dV}{dX} = \frac{2+V}{V-1} - V = \frac{2+V-V^2+V}{V-1}$$

$$\text{ie, } \frac{V-1}{2+2V-V^2}\, dV = \frac{dX}{X}$$

$$\text{ie, } -\frac{1}{2}\int \frac{2-2V}{2+2V-V^2}\, dV = \int \frac{dX}{X} + \text{constant}$$

$$\text{ie, } -\frac{1}{2}\log(2+2V-V^2) = \log X + \log C$$

$$\text{ie, } \frac{1}{\sqrt{2+2V-V^2}} = CX$$

$$\text{ie, } \frac{1}{\sqrt{2+\frac{2Y}{X} - \frac{Y^2}{X^2}}} = CX$$

$$\text{ie, } \frac{X}{\sqrt{2X^2+2XY-Y^2}} = CX$$

$$\therefore 2X^2+2XY-Y^2 = \frac{1}{C^2}$$

$$\therefore 2(x+3)^2 + 2(x+3)y - y^2 = \frac{1}{C^2}$$

$$\text{ie, } 2x^2+12x+18+2xy+6y-y^2 = \frac{1}{C^2}$$

$$\text{ie, } 2x^2+2xy-y^2+12x+6y+18 = \frac{1}{C^2}$$

When y=0, x=0, we get $\frac{1}{C^2} = 18$

∴ The required solution is

$$2x^2+2xy-y^2+12x+6y = 0$$

(4) $\frac{dy}{dx} = \frac{x-y+1}{x+2y-3}$

Solution : Put x = X+h, y = Y+k

$$\therefore \frac{dY}{dX} = \frac{X-Y+h-k+1}{X+2Y+h+2k-3}$$

Put $h-k+1 = 0$

$h+2k-3 = 0$

Solving, we get $h = \frac{1}{3}$, $k = \frac{4}{3}$

$$\therefore \frac{dY}{dX} = \frac{X-Y}{X+2Y}$$

Put Y=VX $\therefore \frac{dY}{dX} = V+X\frac{dV}{dX}$

$$\therefore V+X\frac{dV}{dX} = \frac{X-VX}{X+2VX} = \frac{1-V}{1+2V}$$

$$\therefore X\frac{dV}{dX} = \frac{1-V}{1+2V} - V$$

$$= \frac{1-V-V-2V^2}{1+2V} = \frac{1-2V-2V^2}{1+2V}$$

$$\therefore \frac{(1+2V)\, dV}{1-2V-2V^2} = \frac{dX}{X}$$

$$-\frac{1}{2}\int \frac{-2-4V}{1-2V-2V^2}\, dV = \int \frac{dX}{X} + \text{constant}$$

$$\text{ie, } -\frac{1}{2}\log(1-2V-2V^2) = \log X + \log C$$

$$\text{ie, } \frac{1}{\sqrt{1-2V-2V^2}} = CX$$

$$\text{ie, } \frac{X}{\sqrt{X^2-2XY-2Y^2}} = CX$$

$$\text{or } X^2-2XY-2Y^2 = \frac{1}{C^2}$$

$$\text{ie, } \left(x-\frac{1}{3}\right)^2 - 2\left(x-\frac{1}{3}\right)\left(y-\frac{4}{3}\right) - 2\left(y-\frac{4}{3}\right)^2 = k$$

Exercise 5.4

Solve the following Equations :

(1) $\frac{dy}{dx} = \frac{x-y+1}{x+y-2}$

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(3) $\frac{(3x+5y+6)}{(x+7y+2)}\frac{dy}{dx}=1$ (4) $\frac{dy}{dx}=\frac{x-y+1}{x+2y-3}$

(5) $\frac{dy}{dx}+\frac{3x+2y-5}{2x+3y-5}=0$

(6) $(x+y+1)dx+(3x+4y+4)dy=0$

(7) $(6x+2y-10)\frac{dy}{dx}=(2x+9y-20)$

(8) $2(x-3y+1)\frac{dy}{dx}=(4x-2y+1)$

(9) $(2x+3y-5)\frac{dy}{dx}+(3x+2y-5)=0$

(10) $(3x-7y-30\frac{dy}{dx}=3y-7x+7$

(11) $\frac{dy}{dx}=\frac{y-x+1}{y+x+5}$ (12) $(2x+y+3)dy=(x+2y+3)dx$

(13) $(2x+3y-6)\frac{dy}{dx}+2x+3y+1=0$

(14) $(3y-7x-3)dx+(7y-3x-7)dy=0$

(15) $(2x+3y-8)dx=(x+2y-3)dy$

Answers Ex. 5.4

1. $(y-\frac{3}{2})^2+2(x-\frac{1}{2})(y-\frac{3}{2})-(x-\frac{1}{2})^2=c$
2. $4(y-\frac{1}{8})^2-4(x+\frac{5}{4})(y-\frac{1}{8})-(x+\frac{5}{4})^2=c$
3. $[(x+2)^2+4(x+2)y-5y^2]\left[\frac{5y-5x-10}{5y+x+2}\right]^{5/3}=c$
4. $\left(x-\frac{1}{3}\right)^2-2\left(x-\frac{1}{3}\right)\left(y-\frac{4}{3}\right)-2\left(y-\frac{4}{3}\right)^2=c$
5. $3(x-1)^2+4(x-1)(y-1)+3(y-1)^2=c$
6. $\log\left(\frac{x+2y+2}{c}\right)=\frac{x}{2(x+2y+2)}$
7. $x+2y-5=c(y-2x)^2$
8. $2\left(x+\frac{1}{10}\right)^2-2\left(x+\frac{1}{10}\right)\left(y-\frac{3}{10}\right)+3\left(y-\frac{3}{10}\right)^2=c$
9. $3\left(x-\frac{13}{5}\right)^2+4\left(x-\frac{13}{5}\right)\left(y+\frac{7}{5}\right)+3\left(y+\frac{7}{5}\right)^2=c$
10. $(x-1)^6=c(x+y-1)^5(y-x+1)$
11. $\log\left[\frac{c}{\sqrt{(x+2)^2+(y+3)^2}}\right]=\tan^{-1}\left(\frac{y+3}{x+2}\right)$
12. $x+y+2=c(x-y)^3$
13. $x+y-\frac{29}{12}=c.$
14. $(x-y+1)^2=cx^4(x+y-1)^5$
15. $(y+2)^2-(x-7)(y+2)-(x-7)^2=\left[\frac{2y-x+11+\sqrt{5}(x-7)}{2y-x+11-\sqrt{5}(x-7)}\right]^{2/\sqrt{5}}$

(iv) Linear Equations :-

An equation of the form $\frac{dy}{dx}+Py=Q$ where P and Q are functions of x alone is called a Linear differential equation of first order.

To solve $\frac{dy}{dx}+Py=Q$ ....(1)

Multiply equation (1) by $e^{\int Pdx}$ then we get

$$\frac{dy}{dx}e^{\int Pdx}+Py\,e^{\int Pdx}=Q\,e^{\int Pdx}$$

The LHS can be written as

$$\frac{d}{dx}(ye^{\int Pdx})$$

$$\therefore \frac{d}{dx}(ye^{\int Pdx})=Q\,e^{\int Pdx}$$

Integrating, we get

$$ye^{\int Pdx}=\int [Qe^{\int Pdx}]\,dx+c$$

Which is the required solution

$e^{\int Pdx}$ is called the **Integrating Factor (I.F.).**

Working Rule :

(i) Find I.F. $=e^{\int Pdx}$

(ii) The solution is

$$y(\text{I.F.})=\int Q\,(\text{I.F.})\,dx+c.$$

NOTE:

The equation $\frac{dx}{dy}+Px=Q$ where P and Q are functions of y alone, is also a linear differential equation and I.F. in this case is $e^{\int Pdy}$ and the solution is $x(\text{I.F.})=\int Q(\text{I.F.})dy+c$.

Worked Examples:

Solve the following equations:

(1) $\frac{dy}{dx}+\frac{3x^2y}{1+x^3}=\frac{\sin^2x}{1+x^3}$

Solution : $IF = e^{\int \frac{3x^2}{1+x^3}dx} = e^{\log(1+x^3)}$

$= (1+x^3)$

∴ The solution is

$y(I.F.) = \int Q\,(I.F.)\,dx + c.$

ie, $y(1+x^3) = \int \frac{\sin^2x}{(1+x^3)}(1+x^3)\,dx + c,$

ie, $y(1+x^3) = \int \sin^2 x\,dx + c = \int \frac{1}{2}(1-\cos 2x)dx + c$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

∴ $y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + c$ is the required solution.

(2) $(x^2-a^2)\frac{dy}{dx} + xy = (x+a)\sqrt{x^2-a^2}$

Solution : Divide throughout by (x^2-a^2)

∴ $\frac{dy}{dx} + \frac{x}{x^2-a^2}y = \frac{(x+a)}{\sqrt{x^2-a^2}}$

$I.F. = e^{\int \frac{x}{x^2-a^2}dx} = e^{\frac{1}{2}\log(x^2-a^2)}$

$= \sqrt{x^2-a^2}.$

∴ The solution is

$y\sqrt{x^2-a^2} = \int \frac{x+a}{\sqrt{x^2-a^2}}\sqrt{x^2-a^2}\,dx + c$

ie, $y\sqrt{x^2-a^2} = \int (x+a)dx + c$

ie, $y\sqrt{x^2-a^2} = \frac{x^2}{2} + ax + c$

(3) $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$

Solution : The given equation can be written as

$(1+x)\frac{dy}{dx} = y - y^2$

∴ $\frac{dy}{dx} = \frac{y-y^2}{1+x}$ ∴ $\frac{dx}{dy} = \frac{1+x}{y-y^2}$

ie, $\frac{dx}{dy} = \frac{1}{y-y^2} + \frac{1}{y-y^2}x.$

ie, $\frac{dx}{dy} - \frac{1}{y-y^2}x = \frac{1}{y-y^2}$

∴ $I.F. = e^{-\int \frac{1}{y(1-y)}dy}$

$= e^{-\int\left[\frac{1}{y}+\frac{1}{1-y}\right]dy}$

$= e^{-[\log y - \log(1-y)]}$

$= e^{-\log\left(\frac{y}{1-y}\right)}$

$= \frac{1-y}{y}.$

∴ The solution is

$x\,(I.F.) = \int Q\,(I.F.)dx + c.$

ie, $\frac{x(1-y)}{y} = \int \frac{1}{y(1-y)}\frac{(1-y)}{y}dy + c$

ie, $\frac{x(1-y)}{y} = \int \frac{1}{y^2}dy + c = \frac{-1}{y} + c$

∴ $x(1-y) = -1 + cy$

or $cy = x(1-y) + 1$ is the required solution.

(4) $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

Solution : Divide the given equation by $y(\log y)^2$

∴ $\frac{dy}{dx}\cdot\frac{1}{y(\log y)^2} + \frac{1}{x}\cdot\frac{1}{\log y} = \frac{1}{x^2}$

Put $\frac{1}{\log y} = v$

∴ $\frac{-1}{(\log y)^2}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$

∴ The equation becomes

$$-\frac{dv}{dx}+\frac{1}{x}v=\frac{1}{x^2}$$

ie, $$\frac{dv}{dx}-\frac{1}{x}v=\frac{-1}{x^2}$$

$$\therefore \text{ I.F.} = e^{\int \frac{-1}{x}dx} = e^{-\log x}=\frac{1}{x}$$

∴ The solution is

$$v\left(\frac{1}{x}\right)=\int \frac{-1}{x^2}\cdot\frac{1}{x}\,dx+c$$

ie, $$\frac{v}{x}=-\int x^{-3}\,dx+c=-\frac{x^{-2}}{-2}+c$$

∴ $\frac{v}{x}=\frac{1}{2x^2}+c$ is the required solution.

(5) $y^2+\left(x-\frac{1}{y}\right)\frac{dy}{dx}=0$

Solution : $\left(x-\frac{1}{y}\right)\frac{dy}{dx}=-y^2$

$$\therefore \frac{dy}{dx}=\frac{-y^2}{x-\frac{1}{y}}$$

$$\therefore \frac{dx}{dy}=\frac{x-\frac{1}{y}}{-y^2}$$

$$=\frac{-1}{y^2}x+\frac{1}{y^3}$$

$$\therefore \frac{dx}{dy}+\frac{1}{y^2}x=\frac{1}{y^3}$$

$$\text{I.F.} = e^{\int \frac{1}{y^2}dy}=e^{\left(\frac{-1}{y}\right)}$$

∴ The solution is

$$xe^{\frac{-1}{y}}=\int \frac{1}{y^3}e^{\frac{-1}{y}}\,dy+c$$

Put $\frac{-1}{y}=t$ ∴ $\frac{1}{y^2}dy=dt$

$$\therefore xe^{\frac{-1}{y}}=\int(-t)e^t\,dt+c$$

$$=-[te^t-e^t]+c$$

$$=-e^{\frac{-1}{y}}\left(\frac{-1}{y}-1\right)+c$$

$$\therefore xe^{\frac{-1}{y}}=e^{\frac{-1}{y}}\left(\frac{1}{y}+1\right)+c$$

∴ $x=\frac{1}{y}+1+ce^{\frac{1}{y}}$ is the required solution

(6) $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

Solution : $(x-e^{\tan^{-1}y})\frac{dy}{dx}=-(1+y^2)$

$$\therefore \frac{dy}{dx}=\frac{-(1+y^2)}{x-e^{\tan^{-1}y}}$$

$$\therefore \frac{dx}{dy}=\frac{e^{\tan^{-1}y}-x}{1+y^2}$$

ie, $$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\therefore \text{ I.F.} = e^{\int\frac{1}{1+y^2}}=e^{\tan^{-1}y}$$

∴ The solution is

$$xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}\cdot e^{\tan^{-1}y}\,dy+c$$

$$=\int\frac{e^{2\tan^{-1}y}}{1+y^2}\,dy+c$$

Put $\tan^{-1}y=t$

$$\therefore \frac{1}{1+y^2}dy=dt$$

$$\therefore xe^{\tan^{-1}y}=\int e^{2t}\,dt+c$$

$$=\frac{1}{2}e^{2t}+c$$

ie, $$xe^{\tan^{-1}y}=\frac{1}{2}e^{2\tan^{-1}y}+c$$

(7) $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Solution : Divide throughout by $\cos^2 y$

$\therefore \quad \sec^2 y \frac{dy}{dx} + \frac{x.2 \sin y \cos y}{\cos^2 y} = x^3$

ie, $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$

Put $\tan y = v \quad \therefore \sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \quad \frac{dv}{dx} + 2xv = x^3$

I.F. $= e^{\int 2x dx} = e^{x^2}$

$\therefore$ The solution is

$v e^{x^2} = \int x^3 e^{x^2} dx + c$

Put $x^2 = t \quad \therefore 2xdx = dt$

$v e^{x^2} = \frac{1}{2} \int t e^t dt + c$

$= \frac{1}{2}\left[t e^t - \int e^t dt \right] + c$

$= \frac{1}{2}[t e^t - e^t] + c$

$\therefore \quad v e^{x^2} = \frac{1}{2}[x^2 e^{x^2} - e^{x^2}] + c$

ie, $\tan y = \frac{1}{2}(x^2 - 1) + c e^{-x^2}$

(8) $(1+y+x^2y)dx + (x+x^3)dy = 0$

Solution : $\frac{dy}{dx} = \frac{-(1+y+x^2y)}{x+x^3}$

$= \frac{-[1+y(1+x^2)]}{x(1+x^2)}$

$= \frac{-1}{x(1+x^2)} - \frac{y}{x}$

$\therefore \quad \frac{dy}{dx} + \frac{y}{x} = \frac{-1}{x(1+x^2)}$

$\int \frac{1}{x} dx$

$\therefore$ The solution is

$yx = \int \frac{-1}{x(1+x^2)} x \, dx + c$

ie, $xy = -\int \frac{1}{1+x^2} dx + c$

ie, $xy = -\tan^{-1} x + c$

or $xy + \tan^{-1} x = c$

Exercise 5.5

Solve the following equations:

(1) $\cos^3 x \frac{dy}{dx} + y \cos x = \sin x$

(2) $\frac{dy}{dx} + 2y \tan x = \sin x$ given that $y=0$ when $x = \frac{\pi}{3}$

(3) $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$ given $y=0$ when $x=0$

(4) $dx + xdy = e^{-y} \sec^2 y \, dy$ (5) $\cos x \, dy = (\sin x - y)dx$

(6) $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

(7) $x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$

(8) $\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{1}{(1+x^2)^2}$ given that $y=0$ when $x=1$

(9) $\cos^2 x \frac{dy}{dx} + y = \tan x$ (10) $x\frac{dy}{dx} - y = x^5 \log x$

(11) $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{\sin 2x}{\log x}$ (12) $2x\frac{dy}{dx} + y = 2x^3$

(13) $x \log x \frac{dy}{dx} + y = 2 \log x$ (14) $\sec x \frac{dy}{dx} = y + \sin x$

(15) $(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x$ (16) $\frac{dy}{dx} + \frac{2xy}{(1-x^2)} = \frac{x\sqrt{1-x^2}}{(1-x^2)}$

(17) $(1+y^2)dx = (\tan^{-1}y - x)\,dy$ (18) $(2x-10y^3)\frac{dy}{dx} + y = 0$

(19) $\frac{dy}{dx} - e^{x-y}(e^x - e^y) = 0$ (20) $\frac{dy}{dx} + \frac{1}{x} = e^y x^{-2}$

(21) $\frac{dy}{dx} + 3x^2 y = x^5$ (22) $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

(25) $\frac{dy}{dx}+\frac{xy}{1-x^2}=x\sqrt{y}$ (26) $2xydy-(x^2+y^2+1)\,dx=0$

(27) $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$ (28) $x\frac{dy}{dx}+y\log y=xye^x$

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(29) $y(2xy+e^x)\,dx-e^x\,dy=0$ (30) $\frac{dy}{dx}=(\sin x-\sin y)\frac{\cos x}{\cos y}$

(31) $\frac{dy}{dx}+\frac{1}{x}\tan y=\frac{1}{x^2}\tan y\sin y$

(32) $\frac{dy}{dx}=x^3y^3-xy$ (33) $\frac{dy}{dx}=1-x(y-x)-x^3(y-x)^3$

(34) $(y\log x-1)\,ydx=xdy$ (35) $(y\log y-1)y\,dx+x\,dy=0$

Answers 5.5

(1) $ye^{\tan x}=c+e^{\tan x}(\tan x-1)$ (2) $y\sec^2x=\sec x-2$

(3) $3y(1+x^2)=4x^3$ (4) $xe^y=\tan y+c$

(5) $x+(y-1)(\sec x+\tan x)=c$ (6) $y(x-1)=x^4-x^3+cx^2$

(7) $y+ax=cx\sqrt{1-x^2}$ (8) $y(1+x^2)=\tan^{-1}x-\frac{\pi}{4}$

(9) $y=\tan x-1+ce^{-\tan x}$ (10) $\frac{y}{x}=\frac{x^4}{4}\log x-\frac{x^4}{16}+c$

(11) $2y\log x+\cos 2x=c$ (12) $7y=2x^3+c\sqrt{x}$

(13) $y\log x=(\log x)^2+c$ (14) $y+\sin x+1=ce^{\sin x}$

(15) $y+1-\tan^{-1}x=ce^{-\tan^{-1}x}$ (16) $\frac{y}{1-x^2}=\frac{1}{\sqrt{1-x^2}}+c$

(17) $xe^{\tan^{-1}y}=e^{\tan^{-1}y}(\tan^{-1}y-1)+c$

(18) $xy^2=2y^5+c$ (19) $e^y=e^x-1+ce^{(-e^x)}$

(20) $e^{-y}=\frac{1}{2x}+cx$ (21) $y=\frac{1}{3}(x^3-1)+ce^{-x^3}$

(22) $y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=x+\frac{2}{3}x\sqrt{x}+c$ (23) $1+x(y^2-2+ce^{\frac{-y^2}{2}})=0$

(24) $xy^4+4y=4cx$ (25) $3\sqrt{y}=(x^2-1)+c\sqrt[4]{1-x^2}$

(26) $y^2=x^2-1+cx$ (27) $\sin y=e^x(1+x)+c(1+x)$

(28) $x\log y=e^x(x-1)+c$ (29) $e^x=y(c-x^2)$

(30) $\sin y=\sin x-1+ce^{-\sin x}$ (31) $2x\operatorname{cosec} y=1+cx^2$

(32) $\frac{1}{y^2}=(x^2+1)+ce^{x^2}$ (33) $\frac{1}{(y-x)^2}=-(x^2+1)+ce^{x^2}$

(34) $\frac{1}{y}=-\log x+1+cx$ (35) $x\log y=x+c$

(v) Equations reducible to linear form : Bernoulli's Equation:

An equation of the form

$$\frac{dy}{dx}+Py=Qy^n \qquad ...(1)$$

where P and Q are functions of x alone, is called Bernoulli's equation which can be solved by reducing it to linear form.

Divide the given equation by y^n

$$\therefore y^{-n}\frac{dy}{dx}+Py^{1-n}=Q \qquad ...(2)$$

Put $y^{1-n}=v$ $\quad\therefore (1-n)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$

$\therefore$ The equation (2) becomes

$$\frac{1}{(1-n)}\frac{dv}{dx}+Pv=Q.$$

ie, $\frac{dv}{dx}+(1-n)Pv=(1-n)Q.$

This is linear in v and x which can be solved by finding the I.F.

$$=e^{\int(1-n)P\,dx}$$

The solution is $v\,(\text{I.F.})=\int(1-n)Q\,(\text{I.F.})\,dx+c.$

By replacing v by y^{1-n} we get the required solution.

NOTE: $\frac{dx}{dy}+Px=Qx^n$ where P and Q are functions of y alone, is also a Bernoulli's equation. For solving divide throughout by x^n.

$$x^{-n}\frac{dx}{dy}+Px^{1-n}=Q$$

Put $x^{1-n}=v$ $\quad\therefore (1-n)x^{-n}\frac{dx}{dy}=\frac{dv}{dy}$

$$\therefore \frac{1}{(1-n)}\frac{dv}{dy}+Pv=Q$$

or $\frac{dv}{dy}+(1-n)Pv=(1-n)Q$

This is linear in v and y which can be solved by finding I.F

$$= e^{\int (1-n)P\,dy}$$

Solution is v (IF) $= \int (1-n)\, Q\, (\text{I.F})\,dy + c$

By replacing v by x^{1-n} we get the required solution.

Worked Examples

Solve the following Equations

(1) $\frac{dy}{dx} - \frac{2}{x} y = \frac{y^2}{x^3}$

Solution : Divide by y^2

$$\therefore\ y^{-2}\frac{dy}{dx} - \frac{2}{x}y^{-1} = \frac{1}{x^3}$$

Put $y^{-1} = v$ $\therefore (-1)y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

The above equation becomes

$$-\frac{dv}{dx} - \frac{2}{x}v = \frac{1}{x^3}$$

or $\frac{dv}{dx} + \frac{2}{x}v = \frac{-1}{x^3}$. This is linear

$$\text{I.F} = e^{\int \frac{2}{x}dx} = e^{2\log x} = x^2$$

$\therefore$ Solution is $v\,x^2 = \int \frac{-1}{x^3}\, x^2\, dx + \text{constant}$

$$\text{ie, } vx^2 = -\int \frac{1}{x}dx + \text{constant}$$

$$\text{ie, } vx^2 = -\log x + \log c$$

$$\text{ie, } vx^2 = \log\frac{c}{x}$$

Replacing v by y^{-1} we get

$$\frac{x^2}{y} = \log\frac{c}{x} \quad \text{or} \quad x^2 = y\log\frac{c}{x}.$$

(2) $\frac{dy}{dx} + 2xy + xy^4 = 0$

Solution : $\frac{dy}{dx} + 2xy = -xy^4$

Dividing by y^4, we get

$\frac{dy}{dx} + 2xy + xy^4 = 0$

$\Rightarrow y^{-4}\frac{dy}{dx} + 2xy^{-3} = -x$

Put, $y^{-3} = v$ $\therefore 3y^{-4}\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dv}{dx} - 6xv = 3x,$

$$y^{-4}\frac{dy}{dx} + 2xy^{-3} = -x$$

Put $y^{-3} = v$ $\therefore -3y^{-4}\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \frac{-1}{3}\frac{dv}{dx} + 2xv = -x$$

or $\frac{dv}{dx} - 6xv = 3x$. This is linear

$$\text{I.F.} = e^{\int -6x\,dv} = e^{-6\frac{x^2}{2}} = e^{-3x^2}$$

$\therefore$ Solution is $ve^{-3x^2} = \int e^{-3x^2}(3x)dx + c$

Put $x^2 = t$ $\therefore$ $2x\,dx = dt$

$$\therefore ve^{-3x^2} = \int e^{-3t}\,\frac{3}{2}dt + c$$

$$\text{ie, } ve^{-3x^2} = \frac{3}{2}\frac{e^{-3t}}{-3} + c$$

$$ve^{-3x^2} = \frac{-1}{2}e^{-3x^2} + c.$$

Replacing v by y^{-3}

$$y^{-3}e^{-3x^2} = \frac{-1}{2}e^{-3x^2} + c$$

$$\text{or} \quad \frac{1}{y^3} = \frac{-1}{2} + ce^{3x^2}$$

(3) $\frac{dy}{dx} - y\tan x = -y^2 \sec x$

Solution : Divide by y^2

$$\therefore y^{-2}\frac{dy}{dx} - y^{-1}\tan x = -\sec x.$$

Put $y^{-1} = v$ $\therefore (-1)y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore -\frac{dv}{dx} - v\tan x = -\sec x$$

ie, $\frac{dv}{dx} + (\tan x)\,v = \sec x$. This is linear

$$\therefore \text{I.F} = e^{\int \tan x\,dx} = e^{\log\sec x} = \sec x$$

$\therefore$ Solution is $v\sec x = \int \sec x \sec x\,dx + c$

ie, $v \sec x = \int \sec^2 x \, dx + c$

ie, $v \sec x = \tan x + c$

$\therefore \; y^{-1} \sec x = \tan x + c$

$\therefore \quad \sec x = y (\tan x + c)$

(4) $\frac{dy}{dx} = \frac{1}{x^2y^3+xy}$

Solution : The given equation is written as

$$\frac{dx}{dy} = x^2y^3+xy$$

ie, $\frac{dx}{dy} - yx = y^3 x^2$

Divide by x^2

$\therefore x^{-2} \frac{dx}{dy} - yx^{-1} = y^3$

Put $x^{-1} = v$. $\quad \therefore (-1)x^{-2} \frac{dx}{dy} = \frac{dv}{dy}$

$\therefore -\frac{dv}{dy} - yv = y^3$

ie, $\frac{dv}{dy} + yv = -y^3$. This is linear.

$\therefore$ I.F $= e^{\int y \, dy} = e^{\left(\frac{y^2}{2}\right)}$

$\therefore$ Solution is $v\left(e^{\frac{y^2}{2}}\right) = \int -y^3 . e^{\frac{y^2}{2}} dy + c.$

Put $\frac{y^2}{2} = t \qquad \therefore y \, dy = dt$

$$v e^{\frac{y^2}{2}} = \int -2t \, e^t \, dt + c.$$

$$= -2[t e^t - e^t] + c$$

$$= -2e^{\frac{y^2}{2}} \left(\frac{y^2}{2} - 1\right) + c$$

$$v e^{\frac{y^2}{2}} = -e^{\frac{y^2}{2}} (y^2-2) + c$$

ie, $x^{-1} e^{\frac{y^2}{2}} = -e^{\frac{y^2}{2}} (y^2-2) + c$

or $\quad \frac{1}{x} = 2 - y^2 + c e^{\frac{-y^2}{2}}$

(5) $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$

Solution : Put $x+y=z$

$\therefore \; 1 + \frac{dy}{dx} = \frac{dz}{dx} \qquad \therefore \; \frac{dy}{dx} = \frac{dz}{dx} - 1$

$\therefore \; \frac{dz}{dx} - 1 + xz = x^3z^3 - 1$

ie, $\frac{dz}{dx} + xz = x^3z^3$ This is Bernoulli's equation.

Divide by z^3

$\therefore \; z^{-3} \frac{dz}{dx} + xz^{-2} = x^3$

Put $z^{-2} = v \qquad \therefore (-2) z^{-3} \frac{dz}{dx} = \frac{dv}{dx}$

$\therefore \; \frac{-1}{2} \frac{dv}{dx} + xv = x^3$

ie, $\frac{dv}{dx} - 2xv = -2x^3$. This is linear

$\therefore$ I.F. $= e^{\int -2x \, dx} = e^{-x^2}$

$\therefore$ Solution is $ve^{-x^2} = \int -2x^3 . e^{-x^2} dx + c$

Put $x^2 = t \qquad \therefore 2x \, dx = dt$

$\therefore \quad ve^{-x^2} = \int -t \, e^{-t} dt + c$

$$= -[\, t(-e^{-t}) - \int -e^{-t} dt \,] + c$$

$$= te^{-t} + e^{-t} + c$$

ie, $ve^{-x^2} = e^{-x^2}(x^2+1) + c$

ie, $z^{-2} e^{-x^2} = e^{-x^2}(x^2+1) + c$

ie, $\frac{1}{(x+y)^2} = (x^2+1) + ce^{x^2}$

Exercise 5.6

Solve the following equations:

(1) $\frac{dy}{dx} = x^3y^3 - xy \qquad$ (2) $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

(3) $2\frac{dy}{dx}=\frac{y}{x}+\frac{y^2}{x^2}$ (4) $3x(1-x^2)y^2\frac{dy}{dx}+(2x^2-1)y^3=ax^2$

(5) $\frac{dy}{dx}=1-x(y-x)-x^3(y-x)^3$ (6) $(y\log x-1)\,y\,dx = x\,dy$

(7) $y(2y+e^x)\,dx - e^x\,dy=0$ (8) $x\frac{dy}{dx}+y=y^2\log x$

(9) $\frac{1}{y}\frac{dy}{dx}+\frac{x}{1-x^2}=xy^{\frac{-1}{2}}$ (10) $\frac{dy}{dx}+\frac{y}{x}=x^2y^6$

(11) $\frac{dy}{dx}-2y\tan x=y^2\tan^2 x$ (12) $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$

(13) $\frac{dy}{dx}+2y\tan x=y^2$

(14) $\frac{dy}{dx}+y\cos x=y^n\sin 2x$ where $n\neq 1$

(15) $(xy^2-e^{1/x^3})\,dx-x^2y\,dy=0$

(16) $2\sin x\frac{dy}{dx}-y\cos x=xy^3e^x$ (17) $\frac{dy}{dx}+\frac{2}{x}y=\frac{y^3}{x^3}$

(18) $y\,dx-x\,dy+3x^2y^2e^{x^3}\,dx=0$ (19) $(1-x^2)\frac{dy}{dx}+xy=xy^2$

(20) $x\frac{dy}{dx}+y\log y=xye^x$

Answers Ex 5.6

1. $(x^2+1)+\frac{1}{y^2}=ce^{x^2}$
2. $y^2(2x+c)=e^{x^2}$
3. $x=y(1+c\sqrt{x})$
4. $y^3=\frac{x(1+x)}{a}\log\left(\frac{1+x}{1-x}\right)-\frac{x}{2}$
5. $\frac{1}{(y-x)^2}+(x^2+1)=ce^{x^2}$
6. $\frac{1}{xy}=(1-\log x)\,e^{\log x}+c$
7. $e^x=y(c-2x)$
8. $xy\,[(1-\log x)e^{\log x}+c]=1$
9. $\sqrt{y}=-\frac{1}{3}(1-x^2)+c(1-x^2)^{1/4}$
10. $y^5(5x^3+cx^5)=2$
11. $\frac{\sec^2 x}{y}=-\frac{\tan^3 x}{3}+c$
12. $2x=y(1+2cx^2)$
13. $\frac{\cos^2 x}{y}=\frac{-1}{2}[x+\frac{\sin 2x}{2}]+c$
14. $y^{1-n}=2[\sin x-\frac{1}{n-1}]-ce^{(n-1)\sin x}$.

17. $\frac{3x^2}{y^2}=(1+3cx^6)$
18. $x=y(e^{x^3}+c)$
19. $\sqrt{x^2-1}=y\sqrt{x^2-1}+cy$
20. $x\log y=e^x(x^2-2x+2)+c$

(vi) Exact Equations:

A differential equation is said to be **exact** if it can be derived from its solution by direct differentiation.

ie, an equation of the form $Mdx+Ndy=0$ where M and N are functions of x and y is said to be exact if there exists a function u of x and y such that $Mdx+Ndy=du$ and hence its solution is $u(x,y)=c$.

Eg : $(x+y)dx+(x-y)dy=0$ is exact, since there exists a function of x and y

$$u(x,y)=\frac{x^2}{2}+xy-\frac{y^2}{2} \quad \text{such that}$$

$$du=\frac{2x}{2}dx+xdy+ydx-\frac{2y}{2}dy$$

$$=(x+y)dx+(x-y)dy$$

Hence the solution of the given equation is

$$\frac{x^2}{2}+xy-\frac{y^2}{2}=c.$$

// **Theorem :** The necessary and sufficient condition for the differential equation $Mdx+Ndy=0$ to be exact is that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Proof : Necessary part :

Let the equation $Mdx+Ndy=0$ ..(1) be exact.

$\therefore$ There exists a function $u(x,y)$ such that $du=Mdx+Ndy$ ...(2)

By differentiating $u(x,y)$ totally, we have

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy \quad ..(3)$$

Equating the coefficients of dx and dy in (2) and (3) we get

$$M=\frac{\partial u}{\partial x} \text{ and } N=\frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 y}{\partial y\partial x}$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x\partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence if equation (1) is exact, the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is satisfied.

Sufficiency part :

Let the condition $\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$ be satisfied.

Let f(x,y) be a function such that

$$\int M\,dx = f \qquad \dots (1)$$

ie, f(x,y) is obtained by integrating M dx partially w.r.t. x, treating y as a constant

$$\therefore \frac{\partial}{\partial x}\left[\int M dx\right] = \frac{\partial f}{\partial x}$$

$$\text{ie,} \quad M = \frac{\partial f}{\partial x} \qquad \dots (2)$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Integrating both sides treating y as a constant, we get

$$N = \frac{\partial f}{\partial y} + (\text{a constant})$$

$$= \frac{\partial f}{\partial y} + g(y) \qquad \dots (3) \text{ since y is a constant.}$$

$\therefore$ from (2) and (3)

$$M\,dx + N dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + g(y)\,dy$$

$$= df + g(y)dy.$$

$$= d[\,f + \int g(y)\,dy] \qquad \dots (4)$$

which is an exact differential.

Hence Mdx +Ndy =0 is an exact differential equation.

Working rule for the solution of Mdx +Ndy =0

(i) Verify the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(ii) Integrate M w.r.t. x treating y as constant.

(iii) Integrate w.r.t. y those terms in N which do not contain x

(iv) Equate the sum of the results of (ii) and (iii) to a constant c, which is the required solution.

In symbols, the solution is given by

$\int M dx + \int (\text{Terms in N not containing x})\, dy = c.$

(y is constant)

Worked Examples

Solve the following Equations:

(1) $x\,dx + y\,dy = \frac{a^2(xdy - ydx)}{x^2+y^2}$

Solution : The given equation can be written as

$$\left(x + \frac{a^2 y}{x^2+y^2}\right)dx + \left(y - \frac{a^2 x}{x^2+y^2}\right)dy = 0$$

$$\therefore M = x + \frac{a^2 y}{x^2+y^2}, \; N = y - \frac{a^2 x}{x^2+y^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{(x^2+y^2)a^2 - a^2y(2y)}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{-a^2(x^2+y^2)+a^2x(2x)}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given equation is exact. $\therefore$ The solution is given by

$\int M\,dx + \int (\text{Those terms in N not containing x})\, dy = c$

y is constant

$$\text{ie,} \int \left(x + \frac{a^2 y}{x^2+y^2}\right) dx + \int y\,dy = c.$$

$$\text{ie,} \frac{x^2}{2} + a^2 y \cdot \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} = c$$

$$\text{ie,} \quad \frac{x^2}{2} + a^2 \tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} = c$$

(2) $\cos x\,(\cos x - \sin\alpha \sin y)\,dx + \cos y\,(\cos y - \sin\alpha \sin x)\,dy = 0$

Solution : $M = \cos^2 x - \sin\alpha \cos x \sin y$; $N = \cos^2 y - \sin\alpha \cos y \sin x$

$$\therefore \frac{\partial M}{\partial y} = -\sin\alpha \cos x \cos y. ; \qquad \frac{\partial N}{\partial x} = -\sin\alpha \cos y \cos x.$$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and hence the equation is exact.

$\therefore$ The solution is given by

$\int M dx + \int$ (Those terms in N not containing x) dy = c

y is constant

ie, $\int (\cos^2 x - \sin\alpha \ \cos x \sin y)\, dx + \int \cos^2 y \, dy = c.$

ie, $\int \left(\dfrac{1+\cos 2x}{2}\right) dx - \sin\alpha \sin y \int \cos x \, dx + \int \dfrac{1+\cos 2y}{2} dy = c$

ie, $\dfrac{1}{2}x + \dfrac{\sin 2x}{4} - \sin\alpha \sin y \sin x + \dfrac{1}{2}y + \dfrac{\sin 2y}{4} = c.$

or $2(x+y) + \sin 2x + \sin 2y - 4 \sin\alpha \sin x \sin y = k$

(3) $(ax+hy+g)\, dx + (hx+by+f)\, dy = 0.$

Solution : $M = ax + hy + g$, $N = hx+by+f$

$\dfrac{\partial M}{\partial y} = h$, $\dfrac{\partial N}{\partial x} = h$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and hence the equation is exact.

$\therefore$ The solution is

$\int M\, dx + \int$ (Those terms in N not containing x) dy = constant

y is constant

ie, $\int (ax+hy+g)\, dx + \int (by+f) dy$ = constant

ie, $\dfrac{ax^2}{2} + hyx + gx + \dfrac{by^2}{2} + fy$ = constant

or $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

(4) $(e^y+1)\cos x \, dx + e^y \sin x \, dy = 0$

Solution : $M = (e^y+1)\cos x$, $N = e^y \sin x$

$\therefore \dfrac{\partial M}{\partial y} = e^y \cos x$, $\dfrac{\partial N}{\partial x} = e^y \cos x.$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and hence equation is exact.

$\therefore$ The solution is given by

$\int M dx + \int$ (Those terms in N not containing x) dy = constant.

ie, $\int (e^y+1) \cos x + 0$ = constant. since there is no term in N not containing x.

$\therefore \quad (e^y+1) \sin x = c.$

(5) $(x^2 - ay)\, dx - (ax - y^2)\, dy = 0.$

Solution : $M = x^2 - ay$, $N = -(ax - y^2)$

$\dfrac{\partial M}{\partial y} = -a$, $\dfrac{\partial N}{\partial x} = -a$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and hence equation is exact.

$\therefore$ The solution is

$\int (x^2 - ay) dx + \int y^2 \, dy$ = constant.

y is constant.

ie, $\dfrac{x^3}{3} - ay\, x + \dfrac{y^3}{3}$ = constant

ie, $x^3 - 3axy + y^3 = c.$

(6) $(1+e^{x/y})\, dx + e^{x/y}(1 - x/y)\, dy = 0.$

Solution : $M = 1 + e^{x/y}$, $N = e^{x/y}(1 - x/y)$

$\therefore \dfrac{\partial M}{\partial y} = e^{x/y}\left(\dfrac{-x}{y^2}\right),$

$\dfrac{\partial N}{\partial x} = e^{x/y}\left(\dfrac{-1}{y}\right) + \left(1 - \dfrac{x}{y}\right) e^{x/y}\left(\dfrac{1}{y}\right)$

ie, $\dfrac{\partial M}{\partial y} = \dfrac{-x}{y^2} e^{x/y}$, $\dfrac{\partial N}{\partial x} = -\dfrac{1}{y} e^{x/y} + \dfrac{1}{y} e^{x/y} - \dfrac{x}{y^2} e^{x/y} = -\dfrac{x}{y^2} e^{x/y}$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ equation is exact

$\therefore$ The solution is

$\int (1+e^{x/y}) dx + 0$ = constant since N has no term not containing x

y is constant

$\therefore \quad x + \dfrac{e^{x/y}}{1/y} = c$

ie, $x + y\, e^{x/y} = c.$

(7) $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$

Solution : The equation can be written as

$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0$

$\therefore$ M =y cos x +siny +y, N = sin x + x cosy +x

$\therefore \frac{\partial M}{\partial y} = \cos x + \cos y + 1$, $\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence equation is exact.

$\therefore$ The solution is

$\int (y \cos x + \sin y + y)dx + 0 = \text{constant.}$

y is constant

ie, y sin x + x sin y +xy = c.

(8) $\left[y \left(1+\frac{1}{x}\right) + \cos y \right] dx + [\,x + \log x - x \sin y]\, dy = 0$

Solution : $M = y\left(1+\frac{1}{x}\right) \cos y$, $N = x + \log x - x \sin y$

$\therefore \quad \frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y \qquad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence equation is exact.

$\therefore$ The solution is

$\int [\,y\,(1+\frac{1}{x}) + \cos y]\, dx + 0 = \text{constant}$

y is constant

ie y (x + logx) + x cosy =c.

(9) $(y^2 e^{xy^2} + 4x^3)\, dx + (2xye^{xy^2} - 3y^2)dy = 0.$

Solution : $M = y^2 e^{xy^2} + 4x^3$, $N = 2xye^{xy^2} - 3y^2$

$\therefore \frac{\partial M}{\partial y} = y^2 e^{xy^2}\ 2xy + e^{xy^2}\ 2y$

$\frac{\partial N}{\partial x} = 2xye^{xy^2}\ y^2 + e^{xy^2} . 2y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence equation is exact.

$\therefore$ The solution is

$\int (y^2 e^{xy^2} + 4x^3)dx + \int -3y^2 dy = \text{constant.}$

y is constant.

ie, $y^2 \frac{e^{xy^2}}{y^2} + \frac{4x^4}{4} - 3\frac{y^3}{3} = \text{constant}$

ie, $e^{xy^2} + x^4 - y^3 = c.$

(10) $(12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\, dx$
$+ (2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})dy = 0$

Solution ;

$M = 12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y$

$\therefore \frac{\partial M}{\partial y} = 12x^2 + 4xy - 12y^2 + 2e^{2x} - e^y$

$N = 2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x}$

$\frac{\partial N}{\partial x} = 4xy + 12x^2 - 12y^2 - e^y + 2e^{2x}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence equation is exact.

$\therefore$ The solution is

$\int (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)dx + \int 3y^2 dy = \text{constant}$

y is constant

ie, $4x^3y + x^2y^2 + x^4 - 4xy^3 + ye^{2x} - xe^y + y^3 = c.$

Exercise 5.7

Solve the following equations:

(1) $\left(x - \frac{y}{x^2+y^2}\right) dx + \left(y + \frac{x}{x^2+y^2}\right) dy = 0.$

(2) $(x^2 - 2xy - y^2)dx - (x+y)^2 dy = 0$

(3) $[\cos x \tan y + \cos(x+y)]\, dx + [\,\sin x \sec^2 y + \cos(x+y)]dy = 0$

(4) $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)dy = 0$

(5) $[2\cos 2x + \sin(x+2y)]\, dx + [\cos y + 2\sin(x+2y)\,]dy = 0$

(6) $3y\,(x^2-1)dx + (x^3 + 8y - 3x)dy = 0$. given that y=1 when x=0

(7) $x\,(x^2+3y^2)dx + y\,(y^2+3x^2)dy = 0$

(8) $x\,(x^2+y^2-a^2x)dx + y\,(x^2+y^2-b^2y)dy = 0$

(9) $\frac{1}{x}[\,x\,(y - \sin y) + y]\, dx + [x\,(1-\cos y) + \log x]dy = 0$

(10) $(x^2 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$

(11) $(x+2y-3)\, dy - (2x-y+1)\, dx = 0.$

(12) $(2xy-\tan y)dx + (x^2-x\sec^2 y)dy=0$

(13) $(\sin x \cos y + e^{2x})\, dx + (\cos x \sin y + \tan y)dy=0$

(14) $(a^2-2xy-y^2)dx-(x+y)^2dy=0$

(15) $(\cos 2y-3x^2y^2)dx + (\cos 2y-2x \sin 2y-2x^3y)dy=0.$

Answers 5.7

1) $\frac{x^2}{2}+\frac{y^2}{2}=c+\tan^{-1}\frac{x}{y}$ 2) $x^3-3x^2y-3xy^2-y^3=c$

3) $\sin x \tan y + \sin(x+y)=c$ 4) $x^3-6x^2y-6xy^2+y^3=c$

5) $\sin 2x - \cos(x+2y) + \sin y=c$ 6) $x^3y-3xy+4y^2=4$

7) $x^4+6x^2y^2+y^4=c$ 8) $\frac{x^4}{4}+\frac{x^2y^2}{2}-\frac{a^2x^3}{3}+\frac{y^4}{4}-\frac{b^2y^3}{3}=c.$

9) $x(y-\sin y)+y\log x=c$ 10) $\frac{x^3}{3}-x^2y^2+xy^4+\cos y=c$

11) $y^2+xy-x^2-x-3y=c$ 12) $x^2y-x\tan y=c$

13) $-\cos x \cos y+\frac{e^{2x}}{2}+\log \sec y=c$ 14) $a^2x-x^2y-xy^2-\frac{y^3}{3}=c$

15) $x\cos 2y-x^3y^2+\frac{\sin 2y}{2}=c$

(vii) Equations reducible to exact form with standard integrating factors:

Integrating factors : By integrating factors, we mean those functions of x and y which when multiplied with the differential equation will make it exact. Hence some equations which are not exact, can be made exact by multiplying the equation by integrating factors. Then the equations can be solved either by rearranging the terms so that we get exact differentials or by the method of solving exact differential equation.

(a) Integrating factors by inspection :

We consider some equations which can be made exact by means of integrating factors which can be determined by inspection. The following exact differentials are useful:

(1) $d(xy) = xdy + ydx.$ (2) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(3) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$ (4) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}$

(7) $d\left(\log\frac{y}{x}\right)=\frac{xdy-ydx}{xy}$ (8) $d\left(\frac{e^x}{y}\right)=\frac{e^x(ydx-dy)}{y^2}$

(9) $d[\log(x^2+y^2)]=\frac{2xdx+2ydy}{x^2+y^2}$

(10) $d\left(\frac{-1}{xy}\right)=\frac{ydx+xdy}{(xy)^2}$

Worked Examples

Solve the following equations:

(1) $(y^2e^x+2xy)dx-x^2dy=0$

Solution : $y^2e^x\,dx+2xydx-x^2dy=0$

Dividing by y^2 we get

$$e^x dx+\frac{2xydx-x^2dy}{y^2}=0$$

$$\text{ie,}\quad e^x dx+d\left(\frac{x^2}{y}\right)=0$$

Integrating, $e^x+\frac{x^2}{y}=c.$

(2) $(x^2+y^2+x)dx-(2x^2+2y^2-y)dy=0$

Solution : The given equation is written as

$(x^2+y^2)(dx-2dy)+xdx+ydy=0.$

Divide throughout by $\frac{2}{x^2+y^2}$

$$\therefore\quad 2(dx-2dy)+\frac{2xdx+2ydy}{x^2+y^2}=0$$

ie, $2dx-4dy+d[\log(x^2+y^2)]=0$

Integrating, $2x-4y+\log(x^2+y^2)=c.$

(3) $(1+xy)y\,dx+(1-xy)xdy=0$

Solution : $y\,dx+xdy+xy^2dx-x^2ydy=0$

Divide by x^2y^2

$$\therefore\quad \frac{ydx+xdy}{x^2y^2}+\frac{1}{x}dx-\frac{1}{y}dy=0$$

$$\text{ie,}\quad \frac{ydx+xdy}{(xy)^2}+\frac{1}{x}dx-\frac{1}{y}dy=0$$

$$\text{ie,}\quad d\left(\frac{-1}{xy}\right)+\frac{1}{x}dx-\frac{1}{y}dy=0$$

$$\frac{-1}{xy} + \log x - \log y = c.$$

(4) $y\,dx - x\,dy + (1+x^2)dx + x^2 \sin y\,dy = 0.$

Solution : Divide throughout by x^2

$$\frac{-(xdy-ydx)}{x^2} + \frac{1}{x^2}dx + 1\,dx + \sin y\,dy = 0$$

ie, $$-d\left(\frac{y}{x}\right) + \frac{1}{x^2}dx + dx + \sin y\,dy = 0$$

Integrating,

$$\frac{-y}{x} + \left(\frac{-1}{x}\right) + x - \cos y = c.$$

ie, $$\frac{-y}{x} - \frac{1}{x} + x - \cos y = c$$

(5) $y\,dx - x\,dy + xy^2dy = 0$

Solution : $ydx - xdy + xy^2\,dy = 0$

Divide throughout by xy

$$\therefore \frac{ydx-xdy}{xy} + y\,dy = 0.$$

ie, $$d\left(\log\frac{x}{y}\right) + y\,dy = 0$$

Integrating,

$$\log\frac{x}{y} + \frac{y^2}{2} = c$$

(6) $ye^x\,dx - e^x\,dy + y^4\,dy = 0$

Solution : Divide throughout by y^2

$$\therefore \frac{e^x}{y}dx - \frac{e^x}{y^2}dy + y^2\,dy = 0$$

ie, $$\frac{e^x(ydx-dy)}{y^2} + y^2dy = 0$$

ie, $$d\left(\frac{e^x}{y}\right) + y^2dy = 0$$

Integrating, $$\frac{e^x}{y} + \frac{y^3}{3} = c.$$

(7) $y(x^2+y^2-1)dx + x(x^2+y^2+1)dy = 0$

Solution : The equation is written as

$$xdy - ydx + (x^2+y^2)(ydx + xdy) = 0$$

Divide throughout by (x^2+y^2) we get

$$\frac{xdy-ydx}{x^2+y^2} + (ydx + xdy) = 0.$$

ie, $$d\left(\tan^{-1}\frac{y}{x}\right) + d(xy) = 0$$

Integrating, $\tan^{-1}\frac{y}{x} + xy = c.$

(8) $(y - 2x^3)dx - x(1-xy)dy = 0$

Solution : The equation can be written as

$$ydx - xdy - 2x^3dx + x^2y\,dy = 0$$

Divide throughout by x^2

$$\therefore \frac{-(xdy-ydx)}{x^2} - 2xdx + y\,dy = 0$$

ie, $$-d\left(\frac{y}{x}\right) - 2x\,dx + y\,dy = 0$$

Integrating, $$-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$$

(b) Integrating factors using certain rules:

It is not always easy to determine the integrating factors by inspection. In such cases we find the integrating factors for the equation $Mdx + Ndy = 0$ by the following rules:

Rule (1) If M and N are homogeneous functions of the same degree and $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor for the equation $Mdx + Ndy = 0$

Rule (2) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say $f(x)$ or a constant k then $e^{\int f(x)dx}$ or $e^{\int kdx}$ is an integrating factor for $Mdx + Ndy = 0$ respectively

Rule (3) If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of y alone say $g(y)$ or a constant k then $e^{-\int g(y)\,dy}$ or $e^{-\int k\,dy}$ is an integrating factor of $Mdx + Ndy = 0$ respectively.

Rule (4) If the equation $Mdx + Ndy = 0$ is in the form $f_1(xy)ydx + f_2(xy)\,xdy = 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.

Rule (5) If the equation is of the form $x^ay^b(mydx+nxdy) + x^ry^s(pydx+qxdy) = 0$ where a, b, m, n, r, s, p, q are c k are constants such that after multiplying by $x^h y^k$, the equation becomes exact.

Worked Examples

Solve the following equations:

(1) $x^2y\,dx - (x^3+y^3)dy=0$

Solution : The equation is homogeneous.

$M = x^2y \qquad N = -(x^3+y^3)$

$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$

This is not exact.

$$Mx + Ny = (x^2y)\,x - (x^3+y^3)y$$
$$= x^3y - x^3y - y^4 = -y^4 \neq 0$$

$\therefore \frac{1}{Mx+Ny}$ is an integrating factor.

ie, $\frac{-1}{y^4}$ is an IF.

Multiply the given equation by $\frac{-1}{y^4}$

$$\therefore \frac{-x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0.$$

Now $\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}, \ \frac{\partial N}{\partial x} = \frac{3x^2}{y^4}.$

$\therefore$ The equation is exact

$\therefore$ The solution is

$$\int \frac{-x^2}{y^3}dx + \int \frac{1}{y}dy = \text{constant.}$$

y is constant

ie, $\frac{-x^3}{3y^3} + \log y = c$

(2) $(xy^2+2x^2y^3)\,dx + (x^2y - x^3y^2)dy=0$

Solution : The equation can be written as

$(xy+2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy = 0.$

This is in the form

$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

$Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y + 3x^3y^3 \neq 0$

$\therefore \frac{1}{\ \ } = \frac{1}{\ \ }$ is an I.F.

$$\therefore \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2}, \ \frac{\partial N}{\partial x} = \frac{-1}{3x^2y^2}$$

$\therefore$ The equation is exact

$\therefore$ The solution is

$$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx - \int \frac{1}{3y}dy = \text{constant.}$$

ie, $\frac{1}{3y}\left(\frac{-1}{x}\right) + \frac{2}{3}\log x - \frac{1}{3}\log y = c$

(3) $(7x^4y + 2xy^2 - x^3)dx + (x^4+xy)x\,dy = 0.$

Solution : Consider $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

$$= \frac{1}{(x^4+xy)x}\,[7x^4+4xy-5x^4-2xy]$$
$$= \frac{1}{(x^4+xy)x}\,[2x^4+2xy]$$
$$= \frac{2}{x} = \text{a function of x alone say } f(x)$$

$\therefore$ I.F. $= e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = x^2.$

Multiplying by x^2, we get

$$(7x^6y + 2x^3y^2 - x^5)dx + (x^6+x^3y)x\,dy = 0$$

$\frac{\partial M}{\partial y} = 7x^6+4x^3y, \quad \frac{\partial N}{\partial x} = 7x^6+4x^3y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence equation is exact.

$\therefore$ The solution is $\int (7x^6y + 2x^3y^2 - x^5)dx = c$

y is constant.

ie, $\frac{7x^7y}{7} + \frac{2x^4y^2}{4} - \frac{x^6}{6} = c$

ie, $6x^7y + 3x^4y^2 - x^6 = c$

(4) $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0$

$$= \frac{1}{(xy^2-x^2)}[2xy-(6xy^2+2xy-6x^2)]$$

$$= \frac{-6(xy^2-x^2)}{(xy^2-x^2)} = -6$$

$\therefore$ I.F. $= e^{\int 6dy} = e^{6y}$

Multiplying the given equation by e^{6y} we get

$$(xy^2-x^2)e^{6y}\,dx + (3x^2y^2+x^2y-2x^3+y^2)e^{6y}dy=0$$

This equation is exact.

$\therefore$ the solution is given by

$$\int (xy^2-x^2)e^{6y}\,dx + \int y^2e^{6y}\,dy = c$$

y is constant.

$$\frac{x^2y^2e^{6y}}{2} - \frac{x^3e^{6y}}{3} + \frac{y^2e^{6y}}{6} - \int \frac{e^{6y}}{6}\cdot 2y\,dy = c$$

ie, $\frac{x^2y^2e^{6y}}{2} - \frac{x^3e^{6y}}{3} + \frac{y^2e^{6y}}{6} - \frac{1}{3}\left[\frac{ye^{6y}}{6} - \int \frac{e^{6y}}{6}\,dy\right] = c$

ie, $\frac{x^3y^2e^{6y}}{2} - \frac{x^3e^{6y}}{3} + \frac{y^2e^{6y}}{6} - \frac{ye^{6y}}{18} + \frac{e^{6y}}{108} = c$

ie, $e^{6y}\left[\frac{x^2y^2}{2} - \frac{x^3}{3} + \frac{y^2}{6} - \frac{y}{18} + \frac{1}{108}\right] = c.$

(5) $(3xy+8y^5)dx + (2x^2+24xy^4)dy=0$

Solution : The equation can be written as

$$x(3ydx+2xdy) + 8y^4(ydx+3xdy)=0$$

This is of the form

$$x^ay^b(mydx+nxdy) + x^ry^s(pydx+qxdy)=0$$

Let $x^h y^k$ be the I.F.

Multiply the equation by $x^h y^k$

$\therefore (3x^{h+1}y^{k+1} + 8x^hy^{k+5})dx + (2x^{h+2}y^k + 24x^{h+1}y^{k+4})dy=0$

If this is exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore 3x^{h+1}(k+1)y^k + 8x^h(k+5)y^{k+4} = 2(h+2)x^{h+1}y^k + 24(h+1)x^hy^{k+4}$

$\therefore 3(k+1) = 2(h+2) \Rightarrow 2h-3k = -1$

$8(k+5) = 24(h+1) \Rightarrow 3h-k = 2$

Substituting these values, we get

$$(3x^2y^2+8xy^6)dx + (2x^3y+24x^2y^5)dy=0$$

This is exact.

$\therefore$ The solution is $\int (3x^2y^2+8xy^6)dx = c.$

y is constant.

ie, $x^3y^2 + 4x^2y^6 = c.$

Exercise 5.8

Solve the following equations:

(1) $xdy-ydx=0$ (2) $xdy-(y-x)dx=0$

(3) $ydx-xdy+(1+x^2)dx+x^2\sin y\,dy=0$

(4) $(1+xy)y\,dx+(1-xy)xdy=0$

(5) $(x^3+xy^2-y)dx+(y^3+x^2y+x)dy=0$

(6) $(y^2e^x+2xy)dx-x^2dy=0$

(7) $(x^4e^x-2mxy^2)dx+2mx^2ydy=0$

(8) $y(2x^2y+e^x)dx-(e^x+y^3)dy=0.$

(9) $(y+3x^2y^2e^{x^3})dx-xdy=0$

(10) $y(x^2+y)dx+x(x^2-2y)dy=0$

(11) $y(x^2+y^2-1)dx+x(x^2+y^2+1)dy=0$

(12) $a(xdy+2ydx)=xydy$

Answers 5.8

1) $y=cx$ 2) $\frac{y}{x}+\log x=c$ 3) $-\frac{y}{x}-\frac{1}{x}+x-\cos y=c$

4) $\log\frac{x}{y}-\frac{1}{xy}=c$ 5) $x^2+y^2+2\tan^{-1}\frac{y}{x}=2c$ 6) $e^x+\frac{x^2}{y}=c$

7) $x^2e^x+my^2=cx^2$ 8) $\frac{2x^3}{3}+\frac{e^x}{y}-\frac{y^2}{2}=c$ 9) $\frac{x}{y}+e^{x^3}=c$

10) $x^2y-y^2=cx$ 11) $xy+\tan^{-1}\frac{y}{x}=c$ 12) $a\log(x^2y)=c+y$

Exercise 5.9

Solve the following equations:

(1) $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

(2) $(x^2+y^2+2x)dx+2ydy=0.$

(3) $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$

(4) $(x^2y^2+xy+1)ydx + (x^2y^2-xy+1)\,xdy=0$

(5) $(xy\sin xy+\cos xy)ydx + (xy\sin xy-\cos xy)xdy=0$

(6) $(3xy^2-y^3)\,dx - (2x^2y-xy^2)dy=0$

(7) $xydx - (x^2+2y^2)dy=0$

(8) $(x^4y^4+x^2y^2+xy)\,y\,dx + (x^4y^4-x^2y^2+xy)xdy=0$

(9) $(1+xy)\,y\,dx + (1-xy)\,x\,dy=0$ (10) $(xy^2-e^{1/x^3})dx - x^2ydy=0$

(11) $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}(x+xy^2)dy=0$

(12) $(y^4+2y)dx + (xy^3+2y^4-4x)dy=0$

(13) $y(8x-9y)dx + 2x(x-3y)dy=0$

(14) $(y^3-2yx^2)\,dx + (2xy^2-x^3)dy=0$

(15) $(2x^2y^2+y)dx - (x^3y-3x)dy=0$

(16) $(y^2+2x^2y)dx + (2x^3-xy)dy=0$

(17) $(2xy-3y^4)\,dx + (3x^2+2xy^3)dy=0$

(18) $(x^3+xy^4)\,dx + 2y^3\,dy=0$

(19) $(2x^2y-3y^4)\,dx + (3x^3+2xy^3)dy=0$

(20) $(2ydx+3xdy) + 2xy(3ydx+4xdy)=0$

Answers 5.9

1) $\frac{x}{y}-2\log x+3\log y=c$ 2) $(x^2+y^2)e^x=c$ 3) $x^3y^3+x^2=cy$

4) $xy-\frac{1}{xy}+\log\left(\frac{x}{y}\right)=c$ 5) $x\sec xy=cy$ 6) $3\log x+\frac{y}{x}-2\log y=c$

7) $\frac{-x^2}{4y^2}+\log y=c$ 8) $\frac{x^2y^2}{2}+\log x-\frac{1}{xy}-\log y=c$ 9) $\log\frac{x}{y}-\frac{1}{xy}=c$

10) $\frac{y^2}{2x^2}+\frac{1}{3}e^{1/x^3}=c$ 11) $(3y+y^3)\,x^4+x^6=c$ 12) $xy+\frac{2x}{y^2}+y^2=c$

13) $x^3y(2x-3y)=c$ 14) $y(2+3\tan^{-1}\frac{x}{y})-2x=c$

15) $\frac{x^2}{5}-\frac{1}{4y}=cx^{4/7}\,y^{5/7}$ 16) $\sqrt{xy}\,(y+6x^2)=3cx^2$

17) $\frac{1}{9}y^{18/13}\,x^{-27/13}-\frac{1}{7}y^{-21/3}\,x^{-14/13}=c$

18) $\frac{1}{2}e^{x^2}(x^2-1+y^4)=c$

19) $-\frac{1}{5}x^{\frac{-10}{13}}\,y^{\frac{-15}{13}}+13\,x^{\frac{-36}{13}}\,y^{\frac{24}{13}}=c$

20) $x^2y^3+2x^3y^4=c$

5.3 Differential Equations of first order and higher degree:

A differential equation of the form

$$a_0p^n+a_1p^{n-1}+a_2p^{n-2}+\ldots+a_{n-1}p+a_n=0 \qquad \ldots(1)$$

where $p=\frac{dy}{dx}$ and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are functions of x and y is called the first order and higher degree differential equation.

We have the following methods of solving such equations.

(1) Equations solvable for p (2) Equations solvable for x

(3) Equations solvable for y (4) Clairaut's equation.

(1) Equations solvable for p:

If the L.H.S. of equation (1) can be factorised into linear factors of p of the type

$$(p-\alpha_1)\,(p-\alpha_2)\,(p-\alpha_3)\ldots\ldots(p-\alpha_n)$$

then by equating each factor to zero, we get

$$p=\alpha_1,\ p=\alpha_2,\ p=\alpha_3\ \ldots\ldots p=\alpha_n$$

Replacing p by $\frac{dy}{dx}$ we get

$$\frac{dy}{dx}=\alpha_1,\ \frac{dy}{dx}=\alpha_2,\ \frac{dy}{dx}=\alpha_3,\ldots\ \frac{dy}{dx}=\alpha_n.$$

These are first order, first degree differential equations which can be solved by any of the methods studied earlier. The solutions will be of the type $f_1(x,y,c_1)=0,\ f_2(x,y,c_2)=0, \ldots\ f_n(x,y,c_n)=0$

Then the solution is given by

$$f_1(x,y,c_1).\ f_2(x,y,c_2)\ldots f_n(x,y,c_n)=0.$$

Worked Examples

Solve the following equations:

(1) $p^2-3p+2=0$

Solution: $p^2-3p+2=0$ ie, $(p-1)(p-2)=0$

$\therefore$ p=1, p=2.

$\frac{dy}{dx}=1$, $\frac{dy}{dx}=2$

Integrating,

$y=x+c_1$, and $y=2x+c_2$.

$\therefore$ The solution is $(y-x-c_1)(y-2x-c_2)=0$.

(2) $p^2+(y-2x)p-2xy=0$.

Solution: Factorising, we get $(p+y)(p-2x)=0$ $\therefore$ $p=-y$, $p=2x$.

ie, $\frac{dy}{dx}=-y$, $\frac{dy}{dx}=2x$

ie, $\frac{dy}{y}=dx$, $dy=2xdx$

Integrating,

$\log y=-x+c_1$ and $y=x^2+c_2$.

$\therefore$ The solution is $(\log y+x-c_1)(y-x^2-c_2)=0$.

(3) $6p^3-7p^2-p+2=0$.

Solution: $6p^3-7p^2-p+2=0$.

By inspection $p=1$ is a root. Factorising, we get

$(p-1)(2p+1)(3p-2)=0$

$\therefore$ $p=1$, $p=\frac{-1}{2}$, $p=\frac{2}{3}$.

ie, $\frac{dy}{dx}=1$, $\frac{dy}{dx}=\frac{-1}{2}$, $\frac{dy}{dx}=\frac{2}{3}$.

Integrating we get $y=x+c_1$, $y=\frac{-x}{2}+c_2$ $y=\frac{2x}{3}+c_3$.

$\therefore$ The solution is $(y-x-c_1)(y+\frac{x}{2}-c_2)(y-\frac{2x}{3}-c_3)=0$.

(4) $px^2-xyp-y^2=0$.

Solution: Solving the equation for p, we get

$$p=\frac{xy\pm\sqrt{x^2y^2+4x^2y^2}}{2x^2}=\frac{xy\pm\sqrt{5}xy}{2x^2}=\frac{xy(1\pm\sqrt{5})}{2x^2}$$

$$\therefore p=\frac{y(1\pm\sqrt{5})}{2x}$$

ie, $\frac{dy}{dx}=\frac{y(1\pm\sqrt{5})}{2x}$

Separating the variables,

$$\frac{dy}{y}=\left(\frac{1\pm\sqrt{5}}{2}\right)\frac{dx}{x}$$

Integrating, we get

$$\log y=\left(\frac{1\pm\sqrt{5}}{2}\right)\log x+\text{constant.}$$

ie, $2\log y=(1\pm\sqrt{5})\log x=\log c$

ie, $\log y^2=(1\pm\sqrt{5})\log x+\log c$ is the solution.

(5) $x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}+2y^2-x^2=0$

Solution: The given equation can be written as

$x^2p^2-2xyp+(2y^2-x^2)=0$.

Solving for p, we get

$$p=\frac{2xy\pm\sqrt{4x^2y^2-4x^2(2y^2-x^2)}}{2x^2}=\frac{2xy\pm\sqrt{4x^4-4x^2y^2}}{2x^2}$$

ie, $p=\frac{2xy\pm2x\sqrt{x^2-y^2}}{2x^2}$

ie, $\frac{dy}{dx}=\frac{y\pm\sqrt{x^2-y^2}}{x}$ This is homogeneous.

Put $y=vx$ $\therefore$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ $v+x\frac{dv}{dx}=\frac{vx\pm\sqrt{x^2-v^2x^2}}{x}=v\pm\sqrt{1-v^2}$

$\therefore$ $x\frac{dv}{dx}=\pm\sqrt{1-v^2}$

$\therefore$ $\frac{dv}{\sqrt{1-v^2}}=\pm\frac{dx}{x}$

Integrating, we get

$\sin^{-1}v=\pm\log x\pm\log c$

$\therefore$ $\sin^{-1}\frac{y}{x}=\pm\log cx$. is the solution.

(6) $(1-y^2+\frac{y^4}{x^2})p^2-2\frac{y}{x}p+\frac{y^2}{x^2}=0$.

Solution: the given equation can be written as

$p^2-p^2y^2+\frac{p^2y^4}{x^2}-\frac{2yp}{x}+\frac{y^2}{x^2}=0.$

ie, $p^2-\frac{2yp}{x}+\frac{y^2}{x^2}=p^2y^2-\frac{p^2y^4}{x^2}.$

ie, $(p-\frac{y}{x})^2=p^2y^2(1-\frac{y^2}{x^2})$

ie, $\frac{(px-y)^2}{x^2}=\frac{p^2y^2(x^2-y^2)}{x^2}$

ie, $(px-y)^2=p^2y^2(x^2-y^2)$

$\therefore px-y=\pm py\sqrt{x^2-y^2}$

$\therefore p(x\pm y\sqrt{x^2-y^2})=y.$

$\therefore p=\frac{y}{x\pm y\sqrt{x^2-y^2}}$

$\therefore \frac{1}{p}=\frac{x\pm y\sqrt{x^2-y^2}}{y}$

ie, $\frac{dx}{dy}=\frac{x\pm y\sqrt{x^2-y^2}}{y}$

ie, $\frac{dx}{dy}=\frac{x}{y}\pm\sqrt{x^2-y^2}$

Put $x=vy$ $\therefore \frac{dx}{dy}=v+y\frac{dv}{dy}$

$\therefore v+y\frac{dv}{dy}=v\pm\sqrt{v^2y^2-y^2}$

ie, $y\frac{dv}{dy}=\pm y\sqrt{v^2-1}$

$\therefore \frac{dv}{dy}=\pm\sqrt{v^2-1}$

$\therefore \frac{dv}{\sqrt{v^2-1}}=\pm dy$

Integrating,

$\log(v+\sqrt{v^2-1})=\pm y+c.$

ie, $\log\left(\frac{x}{y}+\frac{\sqrt{x^2-y^2}}{y}\right)=\pm y+c.$

ie, $\log\left(\frac{x+\sqrt{x^2-y^2}}{y}\right)=\pm y+c.$

(7) $p^2+2py\cot x=y^2.$

Solution: Add $y^2\cot^2x$ on both sides.

$\therefore p^2+2py\cot x+y^2\cot^2x=y^2+y^2\cot^2x.$

ie, $(p+y\cot x)^2=y^2(1+\cot^2x)=y^2\operatorname{cosec}^2x.$

$\therefore p+y\cot x=\pm y\operatorname{cosec} x.$

$\therefore p=-y\cot x\pm y\operatorname{cosec} x.$

$\therefore \frac{dy}{dx}=-y\cot x+y\operatorname{cosec} x$ and $\frac{dy}{dx}=-y\cot x-y\operatorname{cosec} x.$

ie, $\frac{dy}{y}=-\cot x+\operatorname{cosec} x.$ and $\frac{dy}{y}=-\cot x-\operatorname{cosec} x.$

Integrating, we get

$\log y=-\log\sin x+\log(\operatorname{cosec} x-\cot x)+\log c$

$\therefore y=\frac{c(\operatorname{cosec} x-\cot x)}{\sin x}$

Similarly, $y=\frac{c}{\sin x(\operatorname{cosec} x-\cot x)}$

Exercise 5.10

Solve the following equations:

(1) $p^2-5p+6=0$ (2) $p^2-7p+12=0.$

(3) $p^3+6p^2+11p+6=0$ (4) $p^2+2py\cot x-y^2=0$

(5) $x^2(\frac{dy}{dx})^2+3xy(\frac{dy}{dx})+2y^2=0$

(6) $4y^2p^2+2pxy(3x+1)+3x^3=0.$

(7) $(p+y+x)(xp+y+x)(p+2x)=0$

(8) $xp^2+(y-x)p-y=0$ (9) $p^2-2p\cosh x+1=0$

(10) $p^2+x^3y-x^3p-yp=0.$

(11) $x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0$

(12) $p^2+2px-3x^2=0$ (13) $p^2y+(x-y)p=x$

(14) $p^2+p(x+y)+xy=0$

(15) $p^3-(x^2+xy+y^2)p^2+(x^3y+x^2y^2+xy^3)p-x^3y^3=0$

(16) [illegible]

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(17) $\left(p^2-\frac{1}{a^2-x^2}\right)(p-\sqrt{y/x})=0$

(18) $2p^3-(2x+4\sin x-\cos x)p^2-(x\cos x-4x\sin x+\sin 2x)p$ $+x\sin 2x=0.$

(19) $p^3-(y+2x-e^{x-y})p^2+(2xy-2xe^{x-y}-ye^{x-y})p+2xye^{x-y}=0.$

(20) $(a^2-x^2)p^3+bx(a^2-x^2)p^2-p-bx=0$

(21) $(p-xy)(p-x^2)(p-y^2)=0$ (22) $xy(p^2+1)=(x^2+y^2)p.$

Answers 5.10

(1) $(y-2x-c)(y-3x-c)=0.$ (2) $(y-3x-c)(y-4x-c)=0$

(3) $(y+x-c)(y+2x-c)(y+3x-c)=0$

(4) $y(1\pm\cos x)=k$ (5) $(yx^2-c)(yx-c)=0$

(6) $(2y^2+x^2-2c)(y^2+x^3-c)=0$

(7) $(y-x-1+ce^{-x})(2xy+x^2+c)(y+x^2+c)=0$

(8) $(yx-c)(y-x-c)=0$ (9) $(y-e^x-c)(y+e^{-x}-c)=0$

(10) $(y-ce^x)(4y-x^3+c)=0$ (11) $(yx^3-k)(y-kx^2)=0$

(12) $(y-\frac{x^2}{2}-c)(y+\frac{3x^2}{2}-c)=$ (13) $(x^2+y^2-c)(y-x-c)=0$

(14) $(y+\frac{x^2}{2}-c)(x+\log y-c)=0$

(15) $(3y-x^2-c)(xy+1+cy)(y-c\,e^{-x^2/2})=0$

(16) $(y-\frac{x^2}{2}-c)(x-\log y+c)(x+y-1-ce^{-x})=0$

(17) $((y-\sin^{-1}\frac{x}{a}-c)(y-\cos^{-1}\frac{x}{a}-c)(\sqrt{y}-\sqrt{x}-c)=0$

(18) $(2y-x^2-c)(2y+\sin x-c)(y+2\cos x+c)=0$

(19) $(y-ce^x)(y-x^2-c)(e^x+e^y-c)=0$

(20) $(y+\frac{bx^2}{2}-c)[\frac{x^2}{a^2}-\sin^2(y-c)]=0$

(21) $(\log y-\frac{x^2}{2}-c)(y-\frac{x^3}{3}-c)[y(x+c)+1]=0$

(22) $(x^2-y^2-c)(y-cx)=0.$

(2) Equations solvable for x :

Solve the differential equation for x in terms of y and p say

$$x=f(y,p) \qquad \ldots(1)$$

Differentiating (1) w.r.t.y we get

$$\frac{dx}{dy}=\phi(y,p,\frac{dp}{dy})$$

Replace $\frac{dx}{dy}$ by $\frac{1}{p}$

$$\therefore\quad \frac{1}{p}=\phi(y,p,\frac{dp}{dy})$$

This is a differential equation in y and p.

Find the solution of this equation in the form say $F(p,y,c)=0$. ..(2)

Eliminate p between (1) and (2) to obtain the solution. If it is not possible to eliminate p between (1) and (2), solve (1) and (2) for x and y in terms of p.

Worked Examples

Solve the following equations:

(1) $x=y+p^2$.

Solution : Differentiating w.r.t. y we get

$$\frac{dx}{dy}=1+2p\frac{dp}{dy}$$

$$\text{ie, }\ \frac{1}{p}=1+2p\frac{dp}{dy}$$

$$\therefore\quad 2p\frac{dp}{dy}=\frac{1}{p}-1$$

$$\frac{dp}{dy}=\frac{1-p}{2p^2}$$

$$\therefore\quad \frac{2p^2}{1-p}dp=dy$$

Integrating,

$$\int\frac{2p^2}{1-p}dp=\int dy+\text{constant.}$$

$$\text{ie, }\ \int\frac{2(p^2-1+1)}{1-p}dp=y+c$$

$$2\int\frac{p^2-1}{1-p}dp+2\int\frac{1}{1-p}dp=y+c$$

$$\text{ie, }2\int\frac{(p+1)(p-1)}{(1-p)}dp+2\int\frac{1}{1-p}dp=y+c$$

ie, $-2\int(p+1)\,dp + 2\int \frac{1}{1-p}\,dp = y+c$

ie, $\frac{-2p^2}{2} - 2p - 2\log(1-p) = y+c$

$-p^2 - 2p - 2\log(1-p) = y+c$

$\therefore$ The solution is given by eliminating p between this equation and the given equation. From the given equation $p = \sqrt{x-y}$,

Substituting we get $-(x-y) - 2\sqrt{x-y} - 2\log(1-\sqrt{x-y}) = c$.

(2) $x = y + a\log p$

Solution: $x = y + a\log p$.

Differentiating w.r.t. y we get

$$\frac{dx}{dy} = 1 + a\left(\frac{1}{p}\right)\frac{dp}{dy}$$

ie, $\frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy}$

$\therefore\ 1 = p + a\frac{dp}{dy}$

$\therefore\ \frac{dp}{dy} = \frac{(1-p)}{a}$

ie, $\frac{a\,dp}{1-p} = dy$

Integrating, $-a\log(1-p) = y + c$

$\therefore\ y = -a\log(1-p) - c$

$\therefore\ x = -a\log(1-p) - c + a\log p$.

$\therefore$ The solution is given by $x = a\log\left(\frac{p}{1-p}\right) - c$ and

$y = -a\log(1-p) - c$

(3) $y = 2px + y^2p^3$

Solution: $2x = \frac{y}{p} - y^2p^2$

$\therefore 2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2y^2p\frac{dp}{dy} - 2p^2y$

ie $\frac{2}{p} - \frac{1}{p} + 2p^2y + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + 2p^2y\right) = 0$

$\therefore\ \left(\frac{1}{p} + 2p^2y\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$

$\therefore\ 1 + \frac{y}{p}\frac{dp}{dy} = 0$

ie, $\frac{dp}{p} + \frac{dy}{y} = 0$

Integrating, $\log p + \log y = \log c$ ie, $p = \frac{c}{y}$

Substituting this value of p in the given equation we get

$$y = 2\left(\frac{c}{y}\right)x + y^2\left(\frac{c}{y}\right)^3$$

ie, $y = \frac{2cx}{y} + \frac{y^2c^3}{y^3}$

ie, $y^2 = 2cx + c^3$ is the solution

(4) $p^3 - p(y+3) + x = 0$

Solution: $x = p(y+3) - p^3$

$\therefore\ \frac{dx}{dy} = p + (y+3)\frac{dp}{dy} - 3p^2\frac{dp}{dy}$

ie, $\frac{1}{p} = p + (y+3)\frac{dp}{dy} - 3p^2\frac{dp}{dy}$

$\therefore\ \frac{1}{p} - p = \frac{dp}{dy}(y+3-3p^2)$

ie, $\frac{1-p^2}{p} = \frac{dp}{dy}(y+3-3p^2)$

$\frac{dp}{dy} = \frac{1-p^2}{p[y+3(1-p^2)]}$

$\therefore\ \frac{dy}{dp} = \frac{p[y+3(1-p^2)]}{(1-p^2)}$

ie, $\frac{dy}{dp} = \frac{p}{1-p^2}.y + 3p$.

ie, $\frac{dy}{dp} - \frac{p}{1-p^2}.y = 3p$.

This is linear

$IF = e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\log(p^2-1)} = \sqrt{p^2-1}$

ie, $y\sqrt{p^2-1} = \frac{3}{2}\int \sqrt{p^2-1}\ 2p\,dp + \text{constant}$

$$= \frac{3}{2} \frac{(p^2-1)^{3/2}}{3/2} + \text{constant.}$$

ie, $y\sqrt{p^2-1} = (p^2-1)\sqrt{p^2-1} + \text{constant.}$

or $\quad y = (p^2-1) + \frac{c}{\sqrt{p^2-1}}$

From the given equation, substituting for y, we get

$$x = p\,[\,p^2-1+\frac{c}{\sqrt{p^2-1}}+3\,]-p^3$$

ie, $x = p^3+2p+\frac{pc}{\sqrt{p^2-1}}-p^3$

ie, $x = 2p+\frac{pc}{\sqrt{p^2-1}}$

∴ The solution is given by

$x = 2p+\frac{pc}{\sqrt{p^2-1}}$ and $y = p^2-1+\frac{c}{\sqrt{p^2-1}}$.

Exercise 5.11

Solve the following equations :

(1) $y-2px+yp^2=0$

(2) $y = 2px + p^2y$

(3) $y = 3px + 6p^2y^2$

(4) $x = 4p + 4p^3$

(5) $y^2 \log y = xyp + p^2$

(6) $x^2 + p^2x = yp$

(7) $p^2 - 4xyp + 8y^2 = 0$

(8) $ayp^2 + (2x-b)p - y = 0$

(9) $ap^2 + py - x = 0$

(10) $4p^3 + 3xp = y$

(11) $p^3 - 2xyp + 4y^2 = 0$

(12) $p = \tan\left(x - \frac{p}{1+p^2}\right) = 0$

(13) $\left(\frac{dy}{dx}\right)^3 y^2 - 2x\frac{dy}{dx} + y = 0$

Answers 5.11

(1) $y^2-2cx+c^2=0$

(2) $2cx-y^2+c^2=0$

(3) $y^3 = 3cx + 6c^2$

(4) $x = 4p+4p^3,\ y = 2p^2+3p^4-c$

(5) $\log y = kx + k^2$

(6) $x = \frac{-p^2}{3} + c\sqrt{p},\ y = \frac{\left(c\sqrt{p}-\frac{1}{3}p^2\right)^2}{p} + p(c\sqrt{p}-\frac{1}{3}p^2)$

(7) $y = \frac{c^2}{4}\left(x - \frac{c^2}{4}\right)^2$

(8) $ak^2 + (2x-b)k - y^2 = 0$

(9) $x = \frac{p(c-a\cosh^{-1}p)}{\sqrt{p^2-1}},\quad y = \frac{c-a\cosh^{-1}p}{\sqrt{p^2-1}} - ap$

(10) $y = \frac{-8}{7}p^3 + \frac{c}{\sqrt{p}},\quad x = \frac{-12}{7}p^2 + \frac{c}{3p\sqrt{p}}$

(11) $16y = c(c-2x)^2$

(12) $y = c - \frac{1}{1+p^2},\ x = \tan^{-1}p + \frac{p}{1+p^2}$

(13) $2cx = c^3 + y^2$

(3) Equations solvable for y :

Solve the given equation for y in terms of x and p.

Let $y = f(x,p)$ ..(1)

Differentiating this equation w.r.t. x we get

$$\frac{dy}{dx} = \phi(x, p, \frac{dp}{dx})$$

Replacing $\frac{dy}{dx}$ by p we get

$$p = \phi\,(x, p, \frac{dp}{dx})$$

This is a differential equation in x and p which can be solved by one of the earlier methods. Let the solution be given by

$$F(p, x, c) = 0 \qquad ...(2)$$

Eliminating p between (1) and (2), we get the solution of the given equation.

If it is not possible to eliminate p, then solve equations (1) and (2) in terms of p, which determine the solution.

Worked Examples

Solve the following equations :

(1) $y = 3x + a \log p$

Solution : Differentiate the given equation w.r.t. x

$$\therefore \frac{dy}{dx} = 3 + a\left(\frac{1}{p}\frac{dp}{dx}\right)$$

Replacing $\frac{dy}{dx}$ by p we get

$$p = 3 + \frac{a}{p}\frac{dp}{dx}$$

ie, $p^2 - 3p = a\frac{dp}{dx}$.

Separating the variables, we get

$$\frac{a\,dp}{p^2 - 3p} = dx$$

ie, $\frac{a\,dp}{p(p-3)} = dx$

Consider $\frac{1}{p(p-3)} = \frac{A}{p} + \frac{B}{p-3}$

$\therefore \quad 1 = A(p-3) + Bp$

Put p=0 $\therefore \quad 1 = A(-3) \quad \therefore A = \frac{-1}{3}$

Put p=3 $\therefore \quad 1 = 3B \quad \therefore B = \frac{1}{3}$

$$\therefore \frac{a\,dp}{p(p-3)} = a\left[\frac{-1}{3p} + \frac{1}{3(p-3)}\right] dp = dx$$

Intergrating, $a\left[\frac{-1}{3}\log p + \frac{1}{3}\log (p-3)\right] = x + c$

ie, $a \log\left(\frac{p-3}{p}\right) = 3x + 3c$

$$\therefore \quad \frac{p-3}{p} = e^{\frac{3}{a}(x+c)}$$

$$\text{ie,} \quad 1 - \frac{3}{p} = e^{\frac{3}{a}(x+c)}$$

$$\therefore \quad 1 - e^{\frac{3}{a}(x+c)} = \frac{3}{p}$$

$$\therefore \quad p = \frac{3}{1 - e^{\frac{3}{a}(x+c)}}$$

Substituting this value of p in the given equation, we get

$y = 3x + a \log\left[\frac{3}{1 - e^{\frac{3}{a}(x+c)}}\right]$ which is the required solution.

(2) $xp^2 - 2py + ax = 0$

Solution : $y = \frac{ax + xp^2}{2p}$

Differentiating w.r.t. x we get

$$\frac{dy}{dx} = \frac{2p(a + x\,.\,2p\frac{dp}{dx} + p^2) - (ax + xp^2)\,2\frac{dp}{dx}}{4p^2}$$

$$\text{ie,} \quad p = \frac{2ap + 4xp^2\frac{dp}{dx} + 2p^3 - 2ax\frac{dp}{dx} - 2xp^2\frac{dp}{dx}}{4p^2}$$

ie, $4p^3 - 2ap - 2p^3 = \frac{dp}{dx}(4p^2x - 2ax - 2xp^2)$

ie, $2p^3 - 2ap = \frac{dp}{dx}(2p^2x - 2ax)$

ie, $p^3 - ap = \frac{dp}{dx}(p^2x - ax)$

ie, $p(p^2 - a) = \frac{dp}{dx}\,.\,x(p^2 - a)$

Cancelling $p^2 - a$ we get

$$p = \frac{dp}{dx}\,.\,x$$

$$\therefore \quad \frac{dx}{x} = \frac{dp}{p}$$

Integrating, $\log x = \log p + \log c$

ie, $x = pc$

or $p = \frac{x}{c}$

Substituting $p = \frac{x}{c}$ in the given equation, we get

$$x\left(\frac{x^2}{c^2}\right) - 2\left(\frac{x}{c}\right)y + ax = 0$$

ie, $x^3 - 2cxy + ac^2x = 0$

(3) $y - 2xp = f(xp^2)$

Solution : $y = 2px + f(xp^2)$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + f'(xp^2)\,(p^2 + 2xp\,\frac{dp}{dx})$$

ie, $p = 2p + 2x\frac{dp}{dx} + f'(xp^2)(p^2 + 2xp\frac{dp}{dx})$

ie, $p + 2x\frac{dp}{dx} + f'(xp^2)p^2 + f'(xp^2)\,2xp\frac{dp}{dx} = 0$

ie, $2x\frac{dp}{dx}[1 + p f'(xp^2)] + p[1 + pf'(xp^2)] = 0$

$\therefore \left[2x\frac{dp}{dx} + p\right][1 + pf'(xp^2)] = 0$

$\therefore \left[2x\frac{dp}{dx} + p\right] = 0$ or $[1 + pf'(xp^2)] = 0$

Consider only $2x\frac{dp}{dx} + p = 0$

ie, $\frac{dx}{x} + 2\frac{dp}{p} = 0$

Integrating $\log x + 2\log p = \log c$

ie, $xp^2 = c$

or $p = \sqrt{\frac{c}{x}}$

Substituting in the given equation we get

$y = 2\sqrt{cx} + f(c)$ which is the required solution.

(4) $y + px = x^4p^2$

Solution : $y = -px + x^4p^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + x^4 2p\frac{dp}{dx} + p^2 4x^3$$

ie, $p = -p - x\frac{dp}{dx} + 2x^4p\frac{dp}{dx} + 4p^2x^3$

ie, $\left(2p + x\frac{dp}{dx}\right) - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$

ie, $(1 - 2px^3)(2p + x\frac{dp}{dx}) = 0$

$\therefore 2p + x\frac{dp}{dx} = 0$

$\therefore \frac{dp}{p} = -2\frac{dx}{x}$

Integrating $\log p = -2\log x + \log c$

$$p = \frac{c}{x^2}$$

Substituting in the given equation,

$y + \frac{c}{x} = c^2$ is the required solution.

Exercise 5.12

Solve the following equations :

(1) $y = 2px + p^4x^2$

(2) $x^3p^2 + x^2py + a^3 = 0$

(3) $y = 2px - p^2$

(4) $x - yp = ap^2$

(5) $y = x + 2\tan^{-1}p$

(6) $y = 2px + y^2p^3$

(7) $y = 2px + p^2$

(8) $y = a\sqrt{1 + p^2}$

(9) $x^2 + p^2x = yp$

(10) $p^3 + mp^2 = a(y + mx)$

(11) $y - p\sin p = \cos p$

(12) $y = 3px + 6p^2y^2$

(13) $y = xp^2 + p$

(14) $4p^3 + 3px = y$

(15) $y = \frac{1}{\sqrt{1 + p^2}} + b$

(16) $y = p\tan p + \log\cos p$

Answers 5.12

(1) $(y - c^2)^2 = 4cx$

(2) $c(c + xy) + a^3x = 0$

(3) $x = \frac{2}{3}p + cp^{-2},\ y = cp^{-1} + \frac{1}{3}p^2$

(4) $x = \frac{p}{\sqrt{1 - p^2}}(a\sin^{-1}p) + c,\ y = \frac{1}{\sqrt{1 - p^2}}(c + a\sin^{-1}p) - ap$

(5) $x = \log\left[\frac{1 - p}{\sqrt{1 + p^2}}\right] - \tan^{-1}p + c,\ y = x + 2\tan^{-1}p$

(6) $y^2 = 2cx + c^3$

(7) $x = \frac{-2p}{3} + cp^{-2},\ y = \frac{-p^2}{3} + 2cp^{-1}$

(8) $x = a\log(y + \sqrt{y^2 - a^2}) + c$

(9) $x = \frac{-p^2}{3} + c\sqrt{p},\ y = \frac{1}{p}\left(c\sqrt{p} - \frac{p^2}{3}\right)^2 + p\left(c\sqrt{p} - \frac{p^2}{3}\right)$

(10) $ax = \frac{3}{2}p^2 - mp + m^2\log(p+m) + c$

$ay = p^3 + mp^2 - ma\left[\frac{3}{2}p^2 - mp + m^2\log(p+m)\right] + c$

(11) $x = \sin p + c,\ y = p\sin p + \cos p$

(12) $y^3 = 3cx + 6c^2$

(13) $x = (\log p - p + c)(p-1)^2,\ y = xp^2 + p$

(14) $x = \frac{-12}{7} p^2 + \frac{c}{3} p^{\frac{-3}{2}},\ y = \frac{8}{7} p^3 + cp^{\frac{-1}{2}}$

(15) $(x+c)^2 + (y-b)^2 = 1$

(16) $x = \tan p + c,\ y = p \tan p + \log \cos p$

(4) Clairaut's Equation :

An equation of the form $y = px + f(p)$ where $p = \frac{dy}{dx}$ is called the **Clairaut's equation**. Apart from the general solution and the particular solution, a differential equation, may have a singular solution. The singular solution does not contain the arbitary constants and is not obtained from the general solution by giving particular values for the constants. In solving Clairaut's equation, we shall study how to find both general and singular solutions.

General and Singular solutions of Clairaut's equation :

$$y = px + f(p) \qquad ..(1)$$

Differentiating (1) w.r.t. 'x' we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

ie, $p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$\therefore x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0$

ie, $\frac{dp}{dx}[x + f'(p)] = 0$

$\therefore \frac{dp}{dx} = 0$ or $x + f'(p) = 0 \qquad ..(2)$

$\frac{dp}{dx} = 0 \Rightarrow p = c$

Substituting $p = c$ in (1) we get the general solution $y = cx + f(c)$

The singular solution is obtained by eliminating p between (1) and (2) ie, from (2) find the value of p and substitute in the given equation (1).

Working Rule :

(1) Reduce the given equation to the Clairaut's form $y = px + f(p)$.

(2) General solution is obtained by replacing p by c in $y = px + f(p)$ i.e., $y = cx + f(c)$.

(3) If $\phi(x, y, c) = 0$ in the general solution, then the singular solution can be obtained by eliminating c from $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$. This is known as c-eliminent method.

(4) If $F(x, y, p) = 0$ in the given differential equation then the singular solution can be obtained by eliminating p from $F(x, y, p) = 0$ and $\frac{\partial F}{\partial p} = 0$. This is known as p-eliminent method.

Worked Examples

Find the general and singular solutions of the following equaions :-

(1) $y = px + \frac{a}{p}$

Solution : The given equation is in the Clairaut's form $y = px + f(p)$ where $f(p) = \frac{a}{p}$.

$\therefore$ G.S. is $y = cx + \frac{a}{c}$

To find the singular solution, consider

$$x + f'(p) = 0$$

$$\Rightarrow x - \frac{a}{p^2} = 0$$

$$\Rightarrow \frac{a}{p^2} = x \Rightarrow p2 = \frac{a}{x} \Rightarrow p = \sqrt{\frac{a}{x}}$$

Substitute $p = \sqrt{\frac{a}{x}}$ in the given equation

$$\therefore y = \sqrt{\frac{a}{x}}.x + a\sqrt{\frac{x}{a}}$$

ie, $y = \sqrt{ax} + \sqrt{ax}$

ie, $y = 2\sqrt{ax}$

or $y^2 = 4ax$ is the singular solution.

(2) $y = px + p - p^2$

Solution : $y = px + (p - p^2)$ is the Clairaut's form where $f(p) = p - p^2$

$\therefore$ The general solution is

$y = cx + (c - c^2)$

$f'(p) = 1 - 2p$

$\therefore \; x + f'(p) = 0 \Rightarrow x + 1 - 2p = 0$

$\Rightarrow p = \frac{x+1}{2}$

Substituting $p = \frac{x+1}{2}$ in the given equation,

$$y = \left(\frac{x+1}{2}\right)x + \left[\frac{x+1}{2} - \frac{(x+1)^2}{4}\right]$$

ie, $y = \frac{(x+1)}{2}\left[x + 1 - \frac{x+1}{2}\right]$

$= \frac{(x+1)}{2}\left[\frac{2x+2-x-1}{2}\right]$

$= \frac{(x+1)}{2}\,\frac{(x+1)}{2} = \frac{(x+1)^2}{4}$

or $4y = (x+1)^2$ is the singular solution.

(3) $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$

Solution : The given equation can be written as $y = xp + p^2$ which is the Clairaut's form where $f(p) = p^2$.

$\therefore$ G.S. is $y = xc + c^2$

$f'(p) = 2p$

$\therefore \; x + f'(p) = 0 \Rightarrow x + 2p = 0$

$\Rightarrow p = \frac{-x}{2}$

Substituting $p = \frac{-x}{2}$ in the given equation,

$$y = x\left(\frac{-x}{2}\right) + \left(\frac{-x}{2}\right)^2$$

ie, $y = \frac{-x^2}{2} + \frac{x^2}{4}$

ie, $y = \frac{-x^2}{4}$ or $x^2 + 4y = 0$ is the singular solution.

(4) $(y - px)(p - 1) = p$

Solution : The given equation can be written as $y - px = \frac{p}{p-1}$

ie, $y = px + \frac{p}{p-1}$ which is the Clairaut's form where $f(p) = \frac{p}{p-1}$

$\therefore$ G.S. is $y = cx + \frac{c}{c-1}$

$x + f'(p) = 0 \Rightarrow x + \frac{(p-1)-p}{(p-1)^2} = 0$

$\Rightarrow x - \frac{1}{(p-1)^2} = 0$

$\Rightarrow (p-1)^2 = \frac{1}{x}$

$\Rightarrow p - 1 = \frac{1}{\sqrt{x}}$

$\Rightarrow p = 1 + \frac{1}{\sqrt{x}}$

$\Rightarrow p = \frac{\sqrt{x}+1}{\sqrt{x}}$

$\therefore$ Singular solution is

$$y = \left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)x + \frac{\frac{\sqrt{x}+1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}$$

ie, $y = \sqrt{x}(\sqrt{x}+1) + \sqrt{x} + 1$

ie, $y = x + 2\sqrt{x} + 1$

ie, $y = (\sqrt{x}+1)^2$

(5) $p = \log(px - y)$

Solution : The given equation can be written as

$px - y = e^p$

or $\quad y = px - e^p$ which is the Clairaut's form,

where $f(p) = -e^p$

$\therefore$ G.S. is $y = cx - e^c$

$x + f'(p) = 0 \Rightarrow x - e^p = 0 \Rightarrow x = e^p$

$\Rightarrow p = \log x$

Substituting in $y = px - e^p$, we get

$y = x \log x - x$

or $\quad y = x(\log x - 1)$ is the singular solution.

(6) $\sin px \cos y = \cos px \sin y + p$

Solution : The given equation can be written as

$\sin px \cos y - \cos px \sin y = p$

ie, $\quad \sin (px - y) = p$

$\therefore \quad px - y = \sin^{-1} p$

$\therefore \quad y = px - \sin^{-1} p$ which is the Clairaut's form

where $\quad f(p) = -\sin^{-1} p$

$\therefore$ G.S. is $\quad y = cx - \sin^{-1} c$

$$x + f'(p) = 0 \Rightarrow x - \frac{1}{\sqrt{1-p^2}} = 0$$

$$\Rightarrow \sqrt{1-p^2} = \frac{1}{x}$$

$$\Rightarrow p^2 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

$$\Rightarrow p = \frac{\sqrt{x^2-1}}{x}$$

$\therefore$ Singular solution is

$$y = \frac{\sqrt{x^2-1}}{x}\, x - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$$

ie, $y = \sqrt{x^2-1} - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

(7) $x^2 (y - px) = p^2 y$

Solution : This is not in the Clairaut's form. This can be transformed by using the substitutions

$x^2 = u \quad$ and $\quad y^2 = v$

$\therefore \quad 2x dx = du$ and $2y dy = dv$

$$\therefore \quad \frac{2y dy}{2x dx} = \frac{dv}{du}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x}{y} \cdot \frac{dv}{du}$$

ie, $\quad p = \frac{x}{y} P$

Substituting in the given equation, we get

$$x^2\left(y - \frac{x^2}{y} P\right) = \frac{x^2}{y^2} P^2 y$$

ie, $\quad \frac{y^2 - x^2 P}{y} = \frac{P^2}{y}$

ie, $\quad v - uP = P^2$

or $\quad v = uP + P^2$ which is in the Clairaut's form

where $\quad f(P) = P^2$

$\therefore$ G.S. is $v = uc + c^2$

ie, $\quad y^2 = cx^2 + c^2$

$u + f'(P) = 0 \Rightarrow u + 2P = 0$

$\Rightarrow P = \frac{-u}{2}$

$\therefore$ Singular solution is

$$v = u\left(\frac{-u}{2}\right) + \left(\frac{-u}{2}\right)^2$$

ie, $\quad v = \frac{-u^2}{2} + \frac{u^2}{4} = \frac{-u^2}{4}$

or $\quad 4v + u^2 = 0$

ie, $\quad 4y^2 + x^2 = 0$

(8) $(px - y)(py + x) = p$

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Solution : Put $x^2 = u$ and $y^2 = v$

$\therefore$ $2x\,dx = du$ and $2y\,dy = dv$

$\therefore \frac{dv}{du} = \frac{y}{x} \cdot \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{x}{y} \frac{dv}{du}$

ie, $p = \frac{x}{y} P.$

Substituting in the given equation,

$$\left(\frac{x^2}{y} P - y\right)(xP + x) = \left(\frac{xP}{y}\right)$$

ie, $(x^2P - y^2)(P + 1)x = Px$

ie, $(x^2P - y^2)(P + 1) = P$

ie, $(uP - v)(P + 1) = P$

$\therefore \quad uP - v = \frac{P}{P+1}$

ie, $v = uP - \frac{P}{P+1}$ which is in the Clairaut's form

where $f(P) = \frac{-P}{P+1}$

$\therefore$ G.S. is $v = uc - \frac{c}{c+1}$

$f'(P) = \frac{(P+1)(-1) + P}{(P+1)^2} = \frac{-1}{(P+1)^2}$

$u + f'(P) = 0 \Rightarrow u - \frac{1}{(P+1)^2} = 0.$

$\Rightarrow (P+1)^2 = \frac{1}{u}$

$\Rightarrow P + 1 = \frac{1}{\sqrt{u}}$

$\Rightarrow P = \frac{1}{\sqrt{u}} - 1.$

$\therefore$ Singular solution is

$$v = u\left(\frac{1}{\sqrt{u}} - 1\right) - \frac{\frac{1-\sqrt{u}}{\sqrt{u}}}{\frac{1}{\sqrt{u}}}$$

ie, $v = \sqrt{u} - u - 1 + \sqrt{u}$

$v = -(u - 2\sqrt{u} + 1)$

ie, $v = -(\sqrt{u} - 1)^2$

ie, $y^2 = -(x-1)^2$

or $(x-1)^2 + y^2 = 0.$

(9) $y = 2px - p^3y^2$

Solution: Put $y^2 = v$ $\therefore 2y \frac{dy}{dx} = \frac{dv}{dx}$

ie, $2y\,p = P.$

$\therefore \quad p = \frac{P}{2y}$

$\therefore$ The given equation becomes

$$y = 2\left(\frac{P}{2y}\right)x - y^2\left(\frac{P}{2y}\right)^3$$

ie, $y^2 = Px - \frac{P^3}{8}$

ie, $v = Px - \frac{P^3}{8}$ which is in the Clairaut's form where $f(P) = \frac{-P^3}{8}$

$\therefore$ G.S is $v = cx - \frac{c^3}{8}$

ie, $y^2 = cx - \frac{c^3}{8}$

$x + f'(P) = 0 \Rightarrow x - \frac{3P^2}{8} = 0$

$\Rightarrow 3P^2 = 8x$

$\Rightarrow P = \sqrt{\frac{8x}{3}}.$

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$\therefore$ Singular solution is

$$v=\left(\sqrt{\frac{8x}{3}}\right)x-\frac{1}{8}\left(\sqrt{\frac{8x}{3}}\right)^3$$

ie, $v = x\sqrt{\frac{8x}{3}}-\frac{8x}{8.3}\sqrt{\frac{8x}{3}}.$

ie, $y^2 = x\sqrt{\frac{8x}{3}}-\frac{1}{3}x\sqrt{\frac{8x}{3}}$

$=\frac{2x}{3}\sqrt{\frac{8x}{3}}.$

or $y^4=\frac{4x^2}{9}\cdot\frac{8x}{3}$

ie, $27y^4 = 32x^3$.

(10) $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$

Solution: Put $x+y = u$

$\therefore 1+\frac{dy}{dx}=\frac{du}{dx}$ ie, $1+p=\frac{du}{dx}$

Put $x^2+y^2=v$

$\therefore 2x+2y\frac{dy}{dx}=\frac{dv}{dx}$

$\therefore 2(x+yp)=\frac{dv}{dx}$

$\therefore \frac{dv}{du}=\frac{2(x+yp)}{1+p}$

ie, $P=\frac{2(x+yp)}{1+p}$

Substituting in the given equation, we get

$$v\left[\frac{4(x+yp)^2}{P^2}\right]-2\left[\frac{2u(x+yp)^2}{P}\right]+(x+yp)^2=0.$$

Dividing throughout by $(x+yp)^2$ we get

$$\frac{4v}{P^2}-\frac{4u}{P}+1=0.$$

$$v-uP+\frac{P^2}{4}=0.$$

ie, $v = uP-\frac{P^2}{4}$ which is in Clairaut's form where $f(P)=\frac{-P^2}{4}$.

$\therefore$ G.S. is $v=uc-\frac{c^2}{4}$ or $(x^2+y^2)=c(x+y)-\frac{c^2}{4}$.

$u+f'(P)=0 \Rightarrow u-\frac{2P}{4}=0.$

$\Rightarrow P=2u$

$\therefore$ Singular solution is

$v=2u^2-\frac{4u^2}{4}$

ie, $v=2u^2-u^2$

ie, $v=u^2$

or $x^2+y^2=(x+y)^2$

ie, $x^2+y^2=x^2+y^2+2xy$.

or $2xy=0$.

Exercise 5.13

Find the general and singular solutions of the following equations:

1) $y=px+\sqrt{a^2p^2+b^2}$
2) $y=px+p^2$
3) $(y-px)^2=a^2(1+p^2)$
4) $y=px+2p^2$
5) $y=px-p^2$
6) $y=px+\sqrt{4+p^2}$
7) $y-1=xp-p^2$
8) $p^2(x^2-a^2)-2xyp+y^2+a^4=0$.
9) $y^2-2pxy+p^2(x^2-1)=k^2$
10) $y^2-2xyp+x^2p^2=\frac{4}{p^2}$
11) $(px-y)(py+x)=h^2p$ (use $u=x^2$, $v=y^2$)
12) $axyp^2+(x^2-ay^2-b)p-xy=0$ (use $u=x^2, v=y^2$)
13) $e^{3x}(p-1)+p^3e^{2y}=0$ (use $e^x=u, e^y=v$)
14) $(px-y)(x-py)=2p$ (use $x^2=u$, $y^2=v$)
15) $x^2\left(\frac{dy}{dx}\right)^2+y(2x+y)\frac{dy}{dx}+y^2=0$ (use $y=u$ and $xy=v$)
16) $y=px+\frac{p}{x}$ (use $x^2=u$ and $y^2=v$)

16-0 [illegible]

Linear differential equations with constant co-efficients:

(3) A differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X$$

$$a_0 \neq 0$$

or

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X \quad \cdots (1)$$

where $P_1, P_2, \cdots P_n$ and X are functions of x or constants, is called a linear differential equation of nth order.

And if $P_1, P_2, \cdots, P_n$ are all constants only and X is some function of x, then the equation (1) is said to be a linear differential equation of nth order with constant co-efficients.

If $P_1, P_2, \cdots, P_n$ are all constants only and $X = 0$, then the equation (1) is called a homogeneous linear differential equation of n-th order with constant co-efficients.

Note: Generally, we write the equation (1) as

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n) y = X \quad \cdots (A)$$

or

$$f(D)\, y = X \quad \cdots (B)$$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n$

and where $D = \frac{d}{\ }$, $D^2 = \frac{d^2}{\ }$, $\cdots$ $D^n = \frac{d^n}{\ }$

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Theorem:

If $y = y_1$, $y = y_2$, ..., $y = y_n$ are linearly independent solutions of

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n) y = 0 \quad \cdots (1)$$

or of

$$f(D) y = 0 \quad \cdots \quad (2)$$

Then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is the general or complete solution of the differential equation (1) where $c_1, c_2, \ldots, c_n$ are n arbitrary constants.

Proof:

Since $y = y_1$, $y = y_2$, .., $y = y_n$ are (given) solutions of (1) or of (2), we have

$$\left.\begin{aligned} f(D) y_1 &= 0 \\ f(D) y_2 &= 0 \\ &\vdots \\ f(D) y_n &= 0 \end{aligned}\right\} \quad \cdots\cdots (*)$$

Now putting $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ in (2) we get,

$$f(D)(c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n) = 0$$

$$\Rightarrow c_1 \{f(D) y_1\} + c_2 \{f(D) y_2\} + c_3 \{f(D) y_3\} + \cdots + c_n \{f(D) y_n\} = 0$$

$$\Rightarrow c_1 (0) + c_2 (0) + c_3 (0) + \cdots + c_n (0) = 0$$

by (*).

Hence, $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ satisfies the differential equation (1) or (2).

So, $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is a solution of (1) ~~and~~ or (2).

Also since it contains n arbitrary constants, it is the general or complete solution of the given differential equation.

Auxiliary Equation:

Consider the differential equation

$$f(D)y = 0 \quad \cdots\cdots \quad (1)$$

where

$$f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n \quad \text{——} (2)$$

Let $y = e^{mx}$ be a trial solution of (1),

Then, we get

$$\left.\begin{aligned} Dy &= \frac{dy}{dx} = m e^{mx} \\ D^2 y &= \frac{d^2y}{dx^2} = m^2 e^{mx} \\ D^3 y &= \frac{d^3y}{dx^3} = m^3 e^{mx} \\ &\vdots \\ D^n y &= \frac{d^ny}{dx^n} = m^n e^{mx} \end{aligned}\right\} \quad \cdots\cdots (3)$$

Hence from (1), $f(D)y = (D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n)y = 0$

$$\Rightarrow D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \cdots + P_n y = 0$$

$$\Rightarrow m^n e^{mx} + P_1 m^{n-1} e^{mx} + P_2 m^{n-2} e^{mx} + \cdots + P_n e^{mx} = 0$$

$$\Rightarrow (m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n) e^{mx} = 0$$

$\therefore \quad m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0 \quad \cdots (4)$

since $e^{mx} \neq 0$.

or $\boxed{f(m) = 0}$

The equation (4) or $f(m) = 0$ is known as The auxiliary equation of $f(D)y = 0$

Case-I:

If $m = m_1, m = m_2, m = m_3, \ldots, m = m_n$ are The real roots of Auxiliary Equation (4), Then for $m_1 \neq m_2 \neq m_3 \neq \cdots \neq m_n$ that is $m_1, m_2, \ldots, m_n$ are distinct The general solution of the differential equation $f(D)y = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

Case-II:

If $m = m_1, m = m_2, m = m_3, \ldots, m = m_n$ are all real roots of the auxiliary equation (4), Then for (1) $m_1 = m_2$, $m_3 \neq m_4 \neq m_5 \neq \cdots \neq m_n$ The general solution of The differential equation $f(D)y = 0$ is

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

and for (2) $m_1 = m_2 = m_3$, $m_4 \neq m_5 \neq \cdots \neq m_n$

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

and for (iii)

$$m_1 = m_2, \quad m_3 = m_4; \quad m_5 \neq m_6 \neq \cdots \neq m_n$$

$$y = (c_1 + c_2 x) e^{m_1 x} + (c_3 + c_4 x) e^{m_3 x} + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$$

etc.

Case- III:

If $\alpha \pm i\beta$ be the imiginary roots and other roots are real and distinct. That is

(1) for $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, $m_3 \neq m_4 \neq \cdots \neq m_n$

Then
$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}.$$

(2) for $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ and $m_3 = m_4 = m_5 \neq m_6 \cdots \neq m_n$

Then
$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + (c_3 + c_4 x + c_5 x^2) e^{m_3 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$$

(3) If auxiliary equation has two equal pairs of imiginary roots and other roots are real and distinct that is

$m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and $m_5 \neq m_6 \neq \cdots \neq m_n$ Then

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$$

Ex-1:

Solve $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$ ... (1)

Let $y = e^{mx}$ be a trial solution of (1), Then the auxiliary equation is

$$m^4 - m^3 - 9m^2 - 11m - 4 = 0$$

$$\Rightarrow m^3(m-4) + 3m^2(m-4) + 3m(m-4) + 1(m-4) = 0$$

$$\Rightarrow (m-4)(m^3 + 3m^2 + 3m + 1) = 0$$

$$\Rightarrow (m-4)(m+1)^3 = 0$$

$$\therefore m = -1, -1, -1, 4$$

Hence, The general solution of (1) is

$$y = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4 e^{4x}$$ Ans:

Ex-2: $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0$ ... (1)

A.E is $m^3 - 13m - 12 = 0$

$\Rightarrow m = -1, -3, 4$

$\therefore y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{4x}$ Ans.

Ex-3: Solve $(D^3 - 2D^2 - 4D + 8)y = 0$

A.E is $m^3 - 2m^2 - 4m + 8 = 0$

$\Rightarrow m = -2, 2, 2$

$\therefore y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}$ Ans.

Ex:-3 Solve $(D^4 + 5D^2 + 6)y = 0$

A.E $m^4 + 5m^2 + 6 = 0$

$\Rightarrow m = \pm\sqrt{3}i, \pm\sqrt{2}i$

$\therefore y = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x + c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x$ Ans.

General solution of $f(D)y = X$, where $X \neq 0$.

To show that if $y = Y$ is a complete solution of $f(D)y = 0$ and $y = u$ is a particular solution of $f(D)y = X$ then $y = Y + u$ is a general solution of $f(D)y = X$.

<u>Proof:</u> The given differential equation is

$$f(D)y = X \quad \text{- - - - (1)}$$

Since $y = Y$ is a solution of $f(D)y = 0$, we get

$$f(D)Y = 0 \quad \text{- - - - (2)}$$

Also since $y = u$ is a solution of (1), we have

$$f(D)u = X \quad \text{- - - (3)}$$

Now adding (2) and (3), we get

$$f(D)Y + f(D)u = 0 + X$$

$$\Rightarrow f(D)(Y+u) = X$$

Thus, $y = Y + u$ is a solution of $f(D)y = X$.
Now Y being the general solution of $f(D)y = 0$ contains n arbitrary constants and as such $Y+u$ also contains n arbitrary constants.
Therefore $y = Y + u$ is a general solution of $f(D)y = X$.

* In the general solution $y = Y + u$ of the equation $f(D)y = X$, Y is called the complementary function (C.F) and u is called the particular integral (P.I) and hence the general solution $= (C.F) + (P.I)$

** Note that (C.F) is the general solution of $f(D)y=0$ since $P.I=0$ for $x=0$.

particular integral of $f(D)y = X$

The given differential equation is
$$f(D)y = X \quad ----(1)$$
If $y=u$ be the particular integral of (1), then

$$f(D)u = X \quad --(2)$$

putting $u = \frac{1}{f(D)}X$ in (2)

$$f(D)\frac{1}{f(D)}X = X$$

$$\Rightarrow X = X \quad \text{which is true.}$$

That means that $\frac{1}{f(D)}X$ is a particular integral of $f(D)y = X$.

particular Integral $= \frac{1}{f(D)}X$ when $X = e^{ax}$

$$= \frac{1}{f(D)}e^{ax}$$

$$= \frac{1}{f(a)}e^{ax} \quad \text{if } f(a) \neq 0$$

particular integral $= \frac{1}{f(D^2)}\sin ax$

$$= \frac{1}{f(-a^2)}\sin ax \quad \text{if } f(-a^2) \neq 0.$$

particular integral $= \frac{1}{f(D^2)}\cos ax$

$$= \frac{1}{f(-a^2)}\cos ax \quad \text{if } f(-a^2) \neq 0.$$

particular integral $= \frac{1}{f(D)} (e^{ax} v)$, v is a function of x

$$= e^{ax} \frac{1}{f(D+a)} v$$

Exceptional cases:

(1) $\frac{1}{f(D)} e^{ax}$ when $f(a) = 0$

$$P.I = \frac{1}{f(D)} e^{ax}$$

$$= x^n \frac{1}{f^n(a)} e^{ax} \quad \text{if } f^{n-1}(a) = 0,$$

(2) $P.I = \frac{1}{f(D^2)} \sin ax$ when $f(-a^2) = 0$

$$= x \frac{1}{f'(D^2)} \cdot \sin ax$$

(3) $P.I = \frac{1}{f(D^2)} \cos ax = x \frac{1}{f'(D^2)} \cos ax$

Example-1: Solve $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = e^x$ ——— (1)

Auxiliary equation of (1) is

$$m^2 - 2km + k^2 = 0$$

$\Rightarrow (m-k)^2 = 0 \quad \therefore m = k, k$

Hence $C.F = (c_1 + c_2 x) e^{kx}$

Now $P.I = \frac{1}{f(D)} e^x = \frac{1}{D^2 - 2kD + k^2} e^x$

$$= \frac{1}{(D-k)^2} e^x = \frac{1}{(1-k)^2} e^x \quad k \neq 1$$

Therefore the general solution $y = C.F + P.I$

$\Rightarrow y = (c_1 + c_2 x) e^{kx} + \frac{1}{(1-k)^2} e^x.$

Important:

since $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$

Which clearly shows that

for D^2 put $-a^2$

for $D^3 = D^2 \cdot D$ put $-a^2 D$

for $D^4 = D^2 \cdot D^2$ put $-a^2(-a^2) = a^4$

Again for $\frac{1}{D+\alpha} \sin ax = \frac{D-\alpha}{D^2-\alpha^2} \sin ax$

$= (D-\alpha)\left[\frac{1}{D^2-\alpha^2} \sin ax\right]$

$= (D-\alpha)\left[\frac{1}{-a^2-\alpha^2} \sin ax\right]$

$= \frac{1}{-a^2-\alpha^2} (D-\alpha) \sin ax$

$= \frac{1}{-a^2-\alpha^2} (a\cos ax - \alpha \sin ax)$

...mple-2: Solve $(D^3+D^2+D+1)y = \sin 2x$

Solⁿ: Auxiliary equation is

$$m^3+m^2+m+1=0$$

$$\Rightarrow m^2(m+1)+1(m+1)=0$$

$$\Rightarrow (m+1)(m^2+1)=0$$

$$\Rightarrow m=-1,\quad m=\pm i$$

Hence, $\underline{C.F = c_1 e^{-x} + c_2\cos x + c_3 \sin x}$ $\quad \cdots (*)$

Now, $P.I = \dfrac{1}{D^3+D^2+D+1}\sin 2x$

$$= \frac{1}{(D+1)(D^2+1)}\sin 2x$$

$$= \frac{1}{D+1}\left[\frac{1}{D^2+1}\sin 2x\right]$$

$$= \frac{1}{D+1}\left[\frac{1}{-2^2+1}\sin 2x\right]$$

$$= -\frac{1}{3}\left[\frac{1}{D+1}\sin 2x\right]$$

$$= -\frac{1}{3}\left[\frac{D-1}{(D+1)(D-1)}\sin 2x\right]$$

$$= -\frac{1}{3}(D-1)\left[\frac{1}{D^2-1}\sin 2x\right]$$

$$= -\frac{1}{3}(D-1)\cdot\left[\frac{1}{-2^2-1}\sin 2x\right]$$

$$= \frac{1}{15}(D-1)\sin 2x = \frac{1}{15}(2\cos 2x - \sin 2x)$$

Hence, the general solution $y = C.F + P.I$

$$\Rightarrow y = c_1 e^{-x} + c_2\cos x + c_3\sin x + \frac{1}{15}(2\cos 2x - \sin 2x)$$

Ans.

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Important note:

(9)

$$De^{ax} = \frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$D^2 e^{ax} = \frac{d}{dx}(ae^{ax}) = a^2 e^{ax}$$

$$D^3 e^{ax} = \frac{d}{dx}(a^2 e^{ax}) = a^3 e^{ax}$$

$$\vdots$$

$$D^r e^{ax} = a^r e^{ax}$$

If $f(D) = P_0 + P_1 D + P_2 D^2 + \cdots + P_n D^n$ ....(1)

$$= \sum_{r=0}^{n} P_r D^r$$

Then $f(D) e^{ax} = \sum_{r=0}^{n} P_r D^r a^{ex}$

$$= \sum_{r=0}^{n} P_r (a^r e^{ax})$$

$$= \left(\sum_{r=0}^{n} P_r a^r\right) e^{ax}$$

$= f(a) e^{ax}$ ——— by (1).

$\therefore f(D) e^{ax} = f(a) e^{ax}$ ——— (2)

prove that $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$.

proof: We know that

$f(D) e^{ax} = f(a) e^{ax}$ ... (3)

Taking inverse operator in both side, we get

$$\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

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(A2)

$$e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

$$\Rightarrow \frac{e^{ax}}{f(a)} = \frac{1}{f(D)}e^{ax}$$

$$\Rightarrow \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)} \quad \underline{\text{proved}}$$

prove that $(D-a)^r(x^r e^{ax}) = \lfloor r \, e^{ax}$

<u>proof:</u>

$$(D-a)(x^r e^{ax}) = D(x^r e^{ax}) - ax^r e^{ax}$$

$$= e^{ax}(Dx^r) + x^r(De^{ax}) - ax^r e^{ax}$$

$$= e^{ax}(rx^{r-1}) + x^r(ae^{ax}) - ax^r e^{ax}$$

$$= rx^{r-1}e^{ax} + \underline{ax^r e^{ax}} - \underline{ax^r e^{ax}}$$

$$= rx^{r-1}e^{ax} \quad \text{----- (1)}$$

Again

$$(D-a)^2(x^r e^{ax}) = (D-a)\, r.\, x^{r-1}e^{ax}$$

$$= r(D-a)x^{r-1}e^{ax}$$

$$= r.(r-1)\, x^{r-2}e^{ax} \quad (\text{as } (1))$$

proceeding in the similar way, we get

$$(D-a)^3(x^r e^{ax}) = r(r-1)(r-2)\, x^{r-3}e^{ax}$$

$$(D-a)^4(x^r e^{ax}) = r(r-1)(r-2)(r-3)\, x^{r-4}e^{ax}$$

$$\vdots$$

$$(D-a)^r(x^r e^{ax}) = r(r-1)(r-2)(r-3)\cdots 3.2.1\, x^{r-r}e^{ax}$$

$$\Rightarrow \quad (D-a)^n (x^n e^{ax}) = \underline{|n} \; e^{ax}$$

proved

If $f(D) = \phi(D)(D-a)^n$, $n \in \mathbb{N}$ and $\phi(a) \neq 0$ then show that

$$\frac{1}{f(D)} e^{ax} = \frac{x^n e^{ax}}{\underline{|n} \; \phi(a)}.$$

proof: We are given

$$f(D) = \phi(D)(D-a)^n$$

$$\therefore \quad f(D)(x^n e^{ax}) = \phi(D)(D-a)^n (x^n e^{ax})$$

$$= \phi(D)[(D-a)^n (x^n e^{ax})]$$

$$= \phi(D)[\underline{|n} \; e^{ax}] \quad \left[\because (D-a)^n (x^n e^{ax}) = \underline{|n} \; e^{ax}\right]$$

$$= \underline{|n} \; \phi(D) e^{ax} = \underline{|n} \; \phi(a) e^{ax}$$

Taking inverse operator, we get

$$\frac{1}{f(D)} f(D)(x^n e^{ax}) = \frac{1}{f(D)} [\underline{|n} \; \phi(a) e^{ax}]$$

$$\Rightarrow \quad x^n e^{ax} = \underline{|n} \; \phi(a) \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \quad \frac{x^n e^{ax}}{\underline{|n} \; \phi(a)} = \frac{1}{f(D)} e^{ax}$$

$$\therefore \quad \frac{1}{f(D)} e^{ax} = \frac{x^n e^{ax}}{\underline{|n} \; \phi(a)}$$

proved

(A4)

If $f(D) = (D-a)^n \phi(D)$, $n \in \mathbb{N}$ and $\phi(a) \neq 0$ Then show that

$$\frac{1}{f(D)} e^{ax} = \frac{x^n e^{ax}}{f^{(n)}(a)} = \frac{x^n e^{ax}}{\lfloor n \, \phi(a)}.$$

proof:

We know that for $f(D) = (D-a)^n \phi(D)$ —— ①

$$\frac{1}{f(D)} e^{ax} = \frac{x^n e^{ax}}{\lfloor n \, \phi(a)} \quad \text{—— (A)}$$

By using Leibnitz theorem, we obtain from ①

$$f^{(n)}(D) = \frac{d^n}{dD^n}\left[(D-a)^n \phi(D)\right]$$

$$= (D-a)^n \phi_n(D) + {}^nC_1 \, n (D-a)^{n-1} \cdot \phi_{n-1}(D) + \cdots$$

$$+ {}^nC_r \frac{\lfloor n}{\lfloor n-r} (D-a)^{n-r} \phi_{n-r}(D) + \cdots + \lfloor n \, \phi(D)$$

Which gives where $\phi_n(D) = \frac{d^n}{dD^n}\phi(D)$

$$f^{(n)}(a) = 0 + 0 + \cdots + 0 + \cdots + \lfloor n \, \phi(a)$$

$$\Rightarrow f^{(n)}(a) = \lfloor n \, \phi(a)$$

Hence, from (A), we get

$$\frac{1}{f(D)} e^{ax} = \frac{x^n e^{ax}}{f^{(n)}(a)}$$

Example-1:

Solve $2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = e^x + 1$

Soln: The given differential equation can be written as

$$2D^3y - 3D^2y + y = e^x + 1$$

$$(2D^3 - 3D^2 + 1)y = e^x + e^0 \quad \text{---} \quad (1)$$

The auxiliary equation of (1) is

$$2m^3 - 3m^2 + 1 = 0$$

$$\Rightarrow 2m^2(m-1) - m(m-1) - (m-1) = 0$$

$$\Rightarrow (m-1)(2m^2 - m - 1) = 0$$

$$\Rightarrow (m-1)[2m(m-1) + (m-1)] = 0$$

$$\Rightarrow (m-1)(m-1)(2m+1) = 0$$

$$\therefore m = 1, 1, -\tfrac{1}{2}$$

Hence, $C.F = (C_1 + C_2x)e^x + C_3 e^{-\frac{1}{2}x}$ —— (A)

Now $P.I = \frac{1}{f(D)}(e^x + e^0)$

$$= \frac{1}{(D-1)^2(2D+1)}(e^x + e^0)$$

$$= \frac{1}{(D-1)^2(2D+1)}e^x + \frac{1}{(D-1)^2(2D+1)}e^0$$

$$= \frac{1}{2D^3 - 3D^2 + 1}e^x + \frac{1}{(0-1)^2(2.0+1)}e^0$$

$$= x.\frac{1}{6D^2 - 6D}e^x + \frac{1}{1}$$

$$= x.x\frac{1}{12D - 6}e^x + 1$$

$$= x^2.\frac{1}{12D-6}e^x + 1 = x^2\frac{1}{12.1-6}e^x + 1$$

$$= \frac{1}{6}x^2e^x + 1 \quad \text{——} \quad (B)$$

$\therefore y = C.F + P.I$ Ans:

Example-1:

Solve $2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = e^x + 1$

Sol^n: The given differential equation can be written as

$$2D^3y - 3D^2y + y = e^x + 1$$

$$(2D^3 - 3D^2 + 1)y = e^x + e^0 \quad \text{---} \quad ①$$

The auxiliary equation of ① is

$$2m^3 - 3m^2 + 1 = 0$$

$$\Rightarrow 2m^2(m-1) - m(m-1) - (m-1) = 0$$

$$\Rightarrow (m-1)(2m^2 - m - 1) = 0$$

$$\Rightarrow (m-1)[2m(m-1) + (m-1)] = 0$$

$$\Rightarrow (m-1)(m-1)(2m+1) = 0$$

$$\therefore \quad m = 1, 1, -\frac{1}{2}$$

Hence, $C.F = (c_1 + c_2 x)e^x + c_3 e^{-\frac{1}{2}x}$ ——— Ⓐ

Now $P.I = \frac{1}{f(D)}(e^x + e^0)$

$$= \frac{1}{(D-1)^2(2D+1)}(e^x + e^0)$$

$$= \frac{1}{(D-1)^2(2D+1)}e^x + \frac{1}{(D-1)^2(2D+1)}e^0$$

$$= \frac{1}{2D^3 - 3D^2 + 1}e^x + \frac{1}{(0-1)^2(2\cdot 0+1)}e^0$$

$$= x \cdot \frac{1}{6D^2 - 6D}e^x + \frac{1}{1}$$

$$= x \cdot x\frac{1}{12D - 6}e^x + 1$$

$$= x^2 \cdot \frac{1}{12D - 6}e^x + 1 = x^2 \frac{1}{12\cdot 1 - 6}e^x + 1$$

$$= \frac{1}{6}x^2 e^x + 1 \quad \text{———} \quad (B)$$

$$\therefore \quad y = C.F + P.I$$

Ans.

(4)

Solve $D^2y + 4Dy + 4y = e^{2x} + e^{-2x}$ ——— (1)

The auxiliary equation is

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow (m+2)^2 = 0$$

$$m = -2, -2$$

$$C.F = (C_1 + C_2 x)e^{-2x} \text{ ——— } (A)$$

$$P.I = \frac{1}{f(D)}(e^{2x} + e^{-2x})$$

$$= \frac{1}{f(D)} e^{2x} + \frac{1}{f(D)} e^{-2x}$$

$$= \frac{1}{(D+2)^2} e^{2x} + \underset{\sim\sim\sim}{\frac{1}{(D+2)^2}} e^{-2x}$$

$$= \frac{1}{16} e^{2x} + x \cdot \frac{1}{2(D+2)} \cdot e^{-2x}$$

$$= \frac{1}{16} e^{2x} + x \cdot x \cdot \frac{1}{2} e^{-2x}$$

$$= \frac{1}{16} e^{2x} + \frac{1}{2} x^2 e^{-2x} \text{ ——— } (3)$$

Hence, The general solution

$$y = C.F + P.I$$

$$= (C_1 + C_2 x)e^{-2x} + \frac{1}{16} e^{2x} + \frac{1}{2} x^2 e^{-2x}$$

Ans:

Solve: $(D^2 + 4D + 4)y = 2\sinh x$
$$= e^{2x} - e^{-2x}$$

Solve: $(D^6 + 1)y = \sin \frac{3}{2}x \sin \frac{1}{2}x$
$$= \frac{1}{2}(\cos x - \cos 2x)$$

Solve: $(D^3 - 5D^2 + 7D - 2)y = e^{2x} \cosh x$
$$= e^{2x} \cdot \frac{1}{2}(e^{x} + e^{-x})$$

$$= \frac{1}{2}(e^{3x} + e^{x})$$

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To show that $\frac{1}{f(D)}(e^{ax}v) = e^{ax}\frac{1}{f(D+a)}v$.

Where v is function of x.

proof:

On differention of the product function $e^{ax}v$,

$$D(e^{ax}v) = e^{ax}Dv + a e^{ax}v = e^{ax}(D+a)v$$

$$D^2(e^{ax}v) = e^{ax}D^2v + ae^{ax}Dv + ae^{ax}Dv + a^2e^{ax}v$$

$$= e^{ax}(D^2v + 2aDv + a^2v)$$

$$= e^{ax}(D^2 + 2aD + a^2)v$$

$$D^2(e^{ax}v) = e^{ax}(D+a)^2v$$

similarly,

$$D^3(e^{ax}v) = e^{ax}(D+a)^3v$$

$$\vdots$$

$$D^n(e^{ax}v) = e^{ax}(D+a)^nv$$

Therefore,

$$f(D)(e^{ax}v) = e^{ax}f(D+a)v$$

Taking the inverse operators, we have

$$\frac{1}{f(D)}(e^{ax}v) = e^{ax}\frac{1}{f(D+a)}v$$

proved

Solve:

$$D^2y - 9y = 6e^{3x} + xe^{3x}$$

Sol^n: The given equation can be written as

$$(D^2-9)y = e^{3x}(6+x) \quad \text{——— (1)}$$

(4)

The auxiliary equation of (1) is

$$m^2 - 9 = 0 \Rightarrow m^2 - 3^2 = 0 \Rightarrow (m+3)(m-3) = 0$$

$$\therefore \; m = 3, -3$$

Hence, $C.F = C_1 e^{3x} + C_2 e^{-3x}$ ——— Ⓐ

Now, $P.I = \frac{1}{f(D)} e^{3x}(6+x)$

$$= \frac{1}{D^2-9} e^{3x}(6+x)$$

$$= e^{3x} \frac{1}{(D+3)^2-9}(6+x)$$

$$= e^{3x} \frac{1}{D^2+6D+9-9}(6+x)$$

$$= e^{3x} \frac{1}{D^2+6D}(6+x)$$

$$= e^{3x} \frac{1}{6D(1+\frac{D}{6})}(6+x)$$

$$= e^{3x} \frac{1}{6D}(1+\frac{D}{6})^{-1}(6+x)$$

$$= e^{3x} \frac{1}{6D}(1-\frac{D}{6} - \cdots\cdots)(6+x)$$

$$= e^{3x} \frac{1}{6D}[6+x - \frac{1}{6}(0+1) - 0\cdots]$$

$$= e^{3x} \frac{1}{6D}(6+x-\frac{1}{6})$$

$$= \frac{1}{6} e^{3x}(6x + \frac{1}{2}x^2 - \frac{1}{6}x)$$

$$= \frac{1}{36} e^{3x}(35x + 3x^2)$$

Therefore, the general solution

$$y = C.F + P.I$$

$$= C_1 e^{3x} + C_2 e^{-3x} + \frac{1}{36} e^{3x}(35x+3x^2)$$

Ans:

Example: Solve $(D^3+2D^2+D)y = e^{2x}+x^2+x$

$f(D)y = e^{2x}+x^2+x$

A-E is $m^3+2m^2+m=0$

$\Rightarrow m(m^2+2m+1)=0$

$\Rightarrow m(m+1)^2=0$

$\Rightarrow m=0, \quad m=-1,-1$

$\therefore$ $C.F = c_1+(c_2+c_3x)e^{-x}$

$$P.I = \frac{1}{f(D)}(e^{2x}+x^2+x)$$

$$= \frac{1}{D(D+1)^2}(e^{2x}+x^2+x)$$

$$= \frac{1}{D(D+1)^2}e^{2x} + \frac{1}{D(D+1)^2}(x^2+x)$$

$$= \frac{1}{2(2+1)^2}e^{2x} + \frac{1}{D}(1+D)^{-2}(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}(1-2D+3D^2 \cdots\cdots)(x^2+x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}[x^2+x-2(2x)-2(1)+3(2)+0 \cdots]$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}(x^2+x-4x-2+6+0)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}(x^2-3x+4)$$

$$= \frac{e^{2x}}{18} + \left(\frac{1}{3}x^3-\frac{3}{2}x^2+4x\right)$$

Hence, The general solution is given be

$$y = C.F + P.I = c_1+(c_2+c_3x)e^{-x} + \frac{e^{2x}}{18}\left(\frac{1}{3}x^3-\frac{3}{2}x^2+4x\right) \quad \text{Ans}$$

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(6)

\# $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \cos x$

A-E is $m^3 - 2m + 4 = 0 \Rightarrow m = -2,\ 1 \pm i$

$\therefore$ C.F $= c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$.

$$P.I = \frac{1}{D^3 - 2D + 4} e^x \cos x = e^x \frac{1}{(D+1)^3 - 2(D+1) + 4} \cdot \cos x$$

$$= e^x \frac{1}{D^3 + 3D^2 + D + 3} \cos x \qquad \left[\frac{1}{f(D)} \cos ax = \frac{1}{f(-a^2)} \cos ax\right]$$

$f(-a^2) = 0$

$$= e^x \cdot x \frac{1}{3D^2 + 6D + 1} \cos x$$

$$= xe^x \cdot \frac{1}{3(-1) + 6D + 1} \cos x$$

$$= xe^x \frac{1}{-3 + 6D + 1} \cos x$$

$$= xe^x \frac{1}{6D - 2} \cos x$$

$$= \frac{1}{2} xe^x \cdot \frac{1}{3D - 1} \cos x$$

$$= \frac{1}{2} xe^x \frac{3D+1}{(3D)^2 - 1} \cos x$$

$$= \frac{1}{2} xe^x (3D+1) \cdot \left[\frac{1}{9D^2 - 1} \cos x\right]$$

$$= \frac{1}{2} xe^x (3D+1) \left[\frac{1}{9(-1) - 1} \cos x\right]$$

$$= -\frac{1}{20} xe^x (3D+1) \cos x$$

$$= -\frac{1}{20} xe^x (-3 \sin x + \cos x)$$

Now the general solution is

$$y = C.F + P.I = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) - \frac{1}{20} xe^x(-3\sin x + \cos x)$$

Calso

pentonic (TM)

main copy, 24-3-19 (5)

Example: Solve $(D^2+a^2)y = \sin ax$

Solⁿ: A.E is $m^2+a^2=0$

(6)

$$\Rightarrow m = \pm ai = 0 \pm ai$$

Hence, $C.F = c_1 \cos ax + c_2 \sin ax$

Now $P.I = \frac{1}{f(D)} \sin ax$

$$= \frac{1}{D^2+a^2} \sin ax \quad \text{.......... (A)}$$

$f(-a^2)=0$

$\therefore$ case of failure

$$= x \cdot \frac{1}{2D} \sin ax$$

$$= \frac{1}{2} x \frac{1}{D} \sin ax$$

$$= \frac{x}{2} \left(-\frac{\cos ax}{a}\right)$$

$$\therefore P.I = -\frac{x}{2a} \cos ax \qquad \therefore y = C.F + P.I$$

Alternative way to find P.I

From (A),

$$P.I = \frac{1}{D^2+a^2} \sin ax$$

$$= \text{Imaginary part of } \left(\frac{1}{D^2+a^2} e^{iax}\right) \quad \because f(a)=0$$

$$= \text{Imaginary part of } \left(x \cdot \frac{1}{2D} e^{iax}\right)$$

$$= \text{Imaginary part of } \left(x \cdot e^{iax} \cdot \frac{1}{2ia}\right)$$

$$= \text{Imaginary part of } \left[\frac{x}{2a} \frac{1}{i} (\cos ax + i \sin ax)\right]$$

$$= \text{" \quad " \quad "} \left[+\frac{x}{2a} \frac{-i^2}{i} (\cos ax + i \sin ax)\right]$$

$$= \text{" \quad " \quad "} \left[-\frac{x}{2a} i(\cos ax + i \sin ax)\right]$$

$$= \text{" \quad " \quad "} \left[-\frac{x}{2a} (i \cos ax - \sin ax)\right]$$

$$= -\frac{x}{2a} \cos ax.$$

1)

Example: Solve $[D^4 + (l^2+k^2)D^2 + l^2k^2]y = \cos\frac{1}{2}(l+k)x \cos\frac{1}{2}(l-k)x$.

Soln: A.E is

$$m^4 + (l^2+k^2)m^2 + l^2k^2 = 0$$

$$\Rightarrow m^4 + l^2m^2 + k^2m^2 + l^2k^2 = 0$$

$$\Rightarrow m^2(m^2+l^2) + k^2(m^2+l^2) = 0$$

$$\Rightarrow (m^2+l^2)(m^2+k^2) = 0$$

$$\Rightarrow m = \pm il, \quad m = \pm ik$$

$\therefore$ C.F $= c_1\cos lx + c_2 \sin lx + c_3 \cos kx + c_4 \sin kx$

Now P.I $= \frac{1}{f(D)}[\cos\frac{1}{2}(l+k)x \cos\frac{1}{2}(l-k)x]$

$$= \frac{1}{2}\cdot\frac{1}{f(D)}[2\cos\frac{1}{2}(l+k)x \cos(l-k)x]$$

$$= \frac{1}{2}\cdot\frac{1}{(D^2+l^2)(D^2+k^2)}[\cos lx + \cos kx]$$

$$= \frac{1}{2}\left[\frac{1}{(D^2+l^2)(D^2+k^2)}\cos lx + \frac{1}{(D^2+l^2)(D^2+k^2)}\cos kx\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+l^2}\left\{\frac{1}{D^2+k^2}\cos lx\right\} + \frac{1}{D^2+k^2}\left\{\frac{1}{D^2+l^2}\cos kx\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+l^2}\left\{\frac{1}{-l^2+k^2}\cos lx\right\} + \frac{1}{D^2+k^2}\left\{\frac{1}{-k^2+l^2}\cos kx\right\}\right]$$

$$= \frac{1}{2(-l^2+k^2)}\left[\frac{1}{D^2+l^2}\cos lx - \frac{1}{D^2+k^2}\cos kx\right]$$

Case of failure

$$= \frac{1}{2(-l^2+k^2)}\left[x\cdot\frac{1}{2D}\cos lx - x\cdot\frac{1}{2D}\cos kx\right]$$

$$= \frac{1}{2(-l^2+k^2)}\left[\frac{x}{2}\cdot\frac{\sin lx}{l} - \frac{x}{2}\cdot\frac{\sin kx}{k}\right]$$

$$= \frac{x}{4(k^2-l^2)}\left[\frac{1}{l}\sin lx - \frac{1}{k}\sin kx\right]$$

Hence, The complete solution is

$y = C.F + P.I$.

$\boxed{\text{Try it in another way}}$

Example: Solve $(D^2 - 2D + 4)y = e^x \cos x$

Sol^n: A.E is $m^2 - 2m + 4 = 0$

$\Rightarrow$ ~~...~~ $= 0$

$\Rightarrow m^2 - 2m + 1 + 3 = 0$

$\Rightarrow (m-1)^2 = -3 = (\sqrt{3}\,i)^2$

$\Rightarrow m = 1 \pm i\sqrt{3}$

$\therefore$ C.F $= e^x[c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x]$

$$P.I = \frac{1}{f(D)} e^x \cos x$$

$$= \frac{1}{D^2 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x \qquad \left[\because \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V\right]$$

$$= e^x \cdot \frac{1}{D^2 + 2D + 1 - 2D - 2 + 4} \cos x$$

$$= e^x \cdot \frac{1}{D^2 + \cancel{\;} + 3} \cos x = e^x \frac{1}{D^2 + 3} \cos x$$

$$= e^x \cdot \frac{1}{-1^2 + 3} \cos x = \frac{1}{2} e^x \cos x$$

$\therefore y = C.F + P.I$

Alternative way

$$P.I = \frac{1}{D^2 - 2D + 4} e^x \cos x = \text{Real part of } \frac{1}{D^2 - 2D + 4} e^x e^{ix}$$

$$= \text{Real part of } \frac{1}{D^2 - 2D + 4} [e^{(1+i)x}]$$

$$= \text{Real part of } \left[e^{(1+i)x} \cdot \frac{1}{(1+i)^2 - 2(1+i) + 4} \right]$$

$$= \text{Real part of } \left[e^x \cdot e^{ix} \cdot \frac{1}{1 + 2i + i^2 - 2 - 2i + 4} \right]$$

$$= \text{Real part of } \left[e^x \cdot e^{ix} \cdot \frac{1}{2} \right]$$

$$= \text{Real part of } \left[\frac{e^x}{2} (\cos x - i \sin x) \right]$$

$$= \frac{1}{2} e^x \cos x.$$

(4)

Example: $(D^2-4D+4)y = 3x^2 e^{2x}\sin 2x$

Solⁿ: A.E is $m^2-4m+4=0$

$\Rightarrow (m-2)^2=0$

$\Rightarrow m = 2, 2$

$C.F = (c_1+c_2x)e^{2x}$

Now $P.I = \frac{1}{(D-2)^2}\cdot(3x^2 e^{2x}\sin 2x)$

$= 3\cdot\frac{1}{(D-2)^2}\left[x^2\cdot e^{2x}\cdot e^{2ix}\right]$ its real part

$=$ ~~Real~~ imaginary part of $\frac{3}{(D-2)^2}\left[x^2 e^{(1+i)2x}\right]$

$=$ ~~Real~~ imaginary part of $3\left[e^{(1+i)2x}\cdot\frac{1}{[D+2+2i-2]^2}x^2\right]$

$=$ imaginary part of $3\left[e^{2x}\cdot e^{2ix}\cdot\frac{1}{(D+2i)^2}x^2\right]$

$=$ imaginary part of $3\left[-e^{2x}e^{2ix}\frac{1}{4}\left(1+\frac{D}{2i}\right)^{-2}x^2\right]$ —— (A)

$=$ imaginary part of $3\left[-e^{2x}e^{2ix}\frac{1}{4}\left(1+iD+\frac{3}{4}D^2-D^2..\right)x^2\right]$

$=$ " " $3\left[-e^{2x}e^{2ix}\frac{1}{4}\left(x^2+2ix-\frac{3}{2}\right)\right]$

$= \frac{-3}{8}e^{2x}\left[4x\cos 2x + (2x^2-3)\sin 2x\right]$

$\therefore y = C.F + P.I$

$\boxed{\text{obtain P.I in different way}}$

Homogeneous linear equations

Def^n: An equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = X \quad \text{---(1)}$$

where $P_1, P_2, \dots, P_n$ are constants and X is a function of x is called the homogeneous linear equations. ~~Type of~~ Eq.(1) is known as cauchy type homogeneous linear equations.

Important substitution:

If we put $x = e^z$

$\Rightarrow z = \ln x$

Then the equation (1) is transformed into an equation with constant co-efficients changing the independent variable from x to z.

Since $z = \ln x$

$$\therefore \frac{dz}{dx} = \frac{1}{x}$$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x} \cdot \frac{d^2y}{dz^2}\frac{dz}{dx} + \frac{dy}{dz}\cdot\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right] \Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{d}{d}$$

Again, we can write

$x\frac{dy}{dx} = Dy \qquad D = \frac{d}{dz}$

$x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \dots$

similarly

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$\vdots$$

$$x^n \frac{d^n y}{dx^n} = D(D-1)(D-2)\cdots\{D-(n-1)\}y$$

putting all these values in Eq.(1), we get

$$[D(D-1)(D-2)\cdots(D-n+1) + P_1\{D(D-1)(D-2)\cdots(D-n+2)\}$$
$$+ \cdots + P_{n-1}D + P_n]y = Z \text{ (say)}.$$

or

$$f(D)y = Z \quad \text{———} (*)$$

Which is clearly ordinary differential equation of n-th order with constant co-efficients.

Important Note:

(1) Transform the ODE with variable (independent) co-efficients (Cauchy type equation) into the ODE with constant co-efficients using the transformations $z = \ln x$, Then

$$x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y,$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y, \cdots\cdots$$

$$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\cdots(D-n+1)y$$

(2) Then solve the transformed equation by obtaing C.F and P.I to have $y = C.F + P.I =$ expressions of z

(3) Finally, substitute the value of $z = \ln x$ and obtain $y = \left[(C.F) + (P.I)\right]_{z=\ln x}$ = Expression of x.

Example- solve $x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$

Sol^n: The given differential equation is

$$x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x} \quad \text{---(1)}$$

Let us put $\boxed{z = \ln x}$ That gives $x = e^z$

and $x \frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

and

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Hence, Equation (1) reduces to

$$D(D-1)y - 2y = e^{2z} + e^{-z}$$

$$\Rightarrow [D(D-1) - 2]y = e^{2z} + e^{-z} \quad \text{———(2)}$$

which is ~~of~~ with constant co-efficients.

Now, A.E is $m(m-1) - 2 = 0$

$$\Rightarrow m^2 - m - 2 = 0$$

$$\Rightarrow m^2 - 2m + m - 2 = 0$$

$$\Rightarrow m(m-2) + 1(m-2) = 0$$

$$\Rightarrow (m+1)(m-2) = 0$$

$$\therefore m = -1, m = 2$$

Hence,

$$C.F = C_1 e^{-z} + C_2 e^{2z} \quad \text{———(3)}$$

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8)

Now $P.I = \frac{1}{f(D)}(e^{2z} + e^{-z})$ —— (*)

$= \frac{1}{(D-2)(D+1)}(e^{2z} + e^{-z})$

$= \frac{1}{D-2}\left[\frac{1}{D+1} e^{2z}\right] + \frac{1}{D+1}\left[\frac{1}{D-2} e^{-z}\right]$

$= \frac{1}{D-2}\left[e^{2z} \cdot \frac{1}{2+1}\right] + \frac{1}{D+1}\left[e^{-z} \cdot \frac{1}{-1-2}\right]$

$= \frac{1}{3} \frac{1}{D-2} e^{2z} - \frac{1}{3} \frac{1}{D+1} e^{-z}$

$= \frac{1}{3} \cdot z \cdot \frac{1}{1} e^{2z} - \frac{1}{3} \cdot z \cdot \frac{1}{1} e^{-z}$

$= \frac{z}{3}\left[e^{2z} - e^{-z}\right]$

Hence, complete integral $y = C.F + P.I$

$\Rightarrow y = c_1 e^{-z} + c_2 e^{2z} + \frac{1}{3} z \left[e^{2z} - e^{-z}\right]$

$= c_1 \left(\frac{1}{x}\right) + c_2 x^2 + \frac{1}{3} \ln x \left(x^2 - \frac{1}{x}\right)$

Ans.

Alternative way to find P.I

From (*)

$P.I = \frac{1}{D^2 - D - 2}(e^{2z} + e^{-z})$

$= \frac{1}{D^2 - D - 2} e^{2z} + \frac{1}{D^2 - D - 2} e^{-z}$

$= z \frac{1}{2D-1} e^{2z} + z \cdot \frac{1}{2D-1} e^{-z}$

$= z \cdot \frac{1}{2(2)-1} e^{2z} + z \cdot \frac{1}{2(-1)-1} e^{-z}$

$= \frac{z}{3}\left[e^{2z} - e^{-z}\right]$ Ans.

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Example: solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$ --- (1)

Sol: putting $x = e^z$ or $z = \ln x$ and that gives

$$x\frac{dy}{dx} = Dy \quad \text{where } D = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Hence, Equation (1) becomes,

$$[D(D-1) - 3D + 4]y = 2\cdot e^{2z}$$

$$\Rightarrow [D^2 - D - 3D + 4]y = 2e^{2z}$$

$$\Rightarrow (D^2 - 4D + 4)y = 2e^{2z} \quad \text{--- (2)}$$

A.E of (2) is

$$m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$$

so $C.F = (c_1 + c_2 z)e^{2z}$.

Now $P.I = \frac{1}{f(D)}(2e^{2z})$

$$= \frac{1}{D^2 - 4D + 4}(2e^{2z}) \quad \text{(case of failure)}$$

$$= 2\cdot z\cdot\frac{1}{2D-4}e^{2z} \quad \text{(Again case of failure)}$$

$$= 2z\cdot z\cdot\frac{1}{2}e^{2z}$$

$$= z^2 e^{2z}$$

Hence, the complete integral

$$y = C.F + P.I = (c_1 + c_2 z)e^{2z} + z^2 e^{2z}$$

$$= (c_1 + c_2\ln x)x^2 + (\ln x)^2\cdot x^2$$

Ans.

Example: Solve $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0$ —— (1)

Sol^n: putting $z = \ln x$, $x = e^z$ that gives $x \frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Hence, (1) becomes

$$[D(D-1) + 3D + 1]y = 0$$

$$\Rightarrow (D^2 - D + 3D + 1)y = 0$$

$$\Rightarrow (D^2 + 2D + 1)y = 0 \quad \text{———— (2)}$$

A.E of (2) is

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

Here C.F $= (c_1 + c_2 z)e^{-z}$, Here P.I $= 0$

Hence, The complete integral is

$$y = C.F = (c_1 + c_2 z)e^{-z}$$

$$= (c_1 + c_2 \ln x)\left(\frac{1}{x}\right) \quad \text{Ans.}$$

** Example: Solve $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x + \sin x$ —— (1)

Sol^n: putting $z = \ln x$, $x = e^z$ that gives

$x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$ and hence (1) becomes

$$[D(D-1) + 4D + 2]y = e^z + \sin e^z$$

$$\Rightarrow (D^2 - D + 4D + 2)y = e^z + \sin e^z$$

$$\Rightarrow (D^2 + 3D + 2)y = e^z + \sin e^z \quad \text{———— (2)}$$

A.E of (2) is

$$m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$$

$$\therefore C.F = c_1 e^{-z} + c_2 e^{-2z} \quad \text{———— (3)}$$

Now. $P.I = \frac{1}{f(D)}[e^z + \sin e^z]$

$= \frac{1}{(D+1)(D+2)}[e^z + \sin e^z]$

$= \frac{1}{(D+1)(D+2)} e^z + \frac{1}{(D+1)(D+2)} \sin e^z$

$= e^z \cdot \frac{1}{(2\times 3)} + \frac{1}{(D+1)(D+2)} \sin e^z$

$= \frac{1}{6} e^z + \frac{1}{(D+1)(D+2)} \sin e^z$ ——— (4)

let $\boxed{\frac{1}{D+1} \sin e^z = u}$

$\Rightarrow (D+1)u = \sin e^z$ (by taking inverse operature)

$\Rightarrow Du + u = \sin e^z$

$\Rightarrow \frac{du}{dz} + u = \sin e^z$ which is linear (5)

$I.F = e^{\int 1 dz} = e^z$

Hence solution of (5) becomes

$u(I.F) = \int (\sin e^z)(I.F)\, dz$

$\Rightarrow u(e^z) = \int e^z \sin e^z\, dz$ | let $t = e^z$, $dt = e^z dz$

$= \int \sin t\, dt = -\cos t$

$= -\cos e^z$

$\Rightarrow \boxed{u = -e^{-z} \cos e^z}$

Now $\frac{1}{(D+1)(D+2)} \sin e^z = \frac{1}{D+2}\left[\frac{1}{D+1} \sin e^z\right]$

$= \frac{1}{D+2}(u) = \frac{1}{D+2}[-e^{-z} \cos e^z]$ ——— (6)

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Now let

$$\boxed{\frac{1}{D+2}\left[-e^{-z}\cos e^{z}\right] = v}$$

$\Rightarrow (D+2)v = -e^{-z}\cos e^{z}$

$\Rightarrow \frac{dv}{dz} + 2v = -e^{-z}\cos e^{z} \quad \cdots (7)$

Which is linear.

$I.F = e^{\int 2\,dz} = e^{2z}$

So, the solution of (7) becomes

$v(I.F) = \int(-e^{-z}\cos e^{z})(I.F)\,dz$

$\Rightarrow v(e^{2z}) = -\int e^{-z}\cos e^{z}(e^{2z})\,dz$

$= -\int e^{z}\cos e^{z}\,dz$

$= -\int \cos w\,dw$

$= -\sin w = -\sin e^{z}$

| $w = e^{z}$
| $dw = e^{z}\,dz$

$\Rightarrow \boxed{v = -e^{-2z}\sin e^{z}}$

Hence from (6), we have

$\frac{1}{(D+1)(D+2)}\sin e^{z} = -e^{-2z}\sin e^{z}$

So, from equation (4), we get

$P.I = \frac{1}{6}e^{z} - e^{-2z}\sin e^{z}$

The complete integral

$y = c_1 e^{-z} + c_2 e^{-2z} + \frac{1}{6}e^{z} - e^{-2z}\sin e^{z}$

$= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x^2}\right) + \frac{1}{6}(x) - \left(\frac{1}{x^2}\right)\sin x$

Ans.

Example: Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$ (10

Sol: Putting $z=\ln x$ or $x=e^z$ that gives

6

$x\frac{dy}{dx}=Dy,\quad x^2\frac{d^2y}{dx^2}=D(D-1)y$

Hence, (1) becomes

$$[D(D-1)+4D+2]y=e^{e^z}$$

$$\Rightarrow [D^2-D+4D+2]y=e^{e^z}$$

$$\Rightarrow [D^2+3D+2]y=e^{e^z} \quad \text{------ (A)}$$

A.E. is $m^2+3m+2=0$

$\Rightarrow m^2+2m+m+2=0$

$\Rightarrow m(m+2)+1(m+2)=0$

$\Rightarrow (m+1)(m+2)=0 \quad \therefore m=-1,-2$

$\therefore$ $C.F=c_1e^{-z}+c_2e^{-2z}$ ---- (2)

$$P.I=\frac{1}{f(D)}e^{e^z}=\frac{1}{(D+1)(D+2)}e^{e^z}$$

$$=\left(\frac{1}{D+1}-\frac{1}{D+2}\right)e^{e^z} \quad \text{------ (B)}$$

Let $\frac{1}{D+1}e^{e^z}=u$

$\Rightarrow e^{e^z}=(D-1)u$ (Taking inverse operator)

$\Rightarrow \frac{du}{dz}+u=e^{e^z}$ ---- (3)

Which is linear differential equation

$$I.F=e^{\int dz}=e^z$$

Hence, Solution of (3) $u(I.F)=\int(e^{e^z})(I.F)dz$

$u(e^{z}) = \int e^{e^{z}} e^{z}\, dz$ $\quad t = e^{z}$, $dt = e^{z} dz$

$= \int e^{t} dt = e^{t} = e^{e^{z}}$

$\therefore u = e^{-z} e^{e^{z}} \Rightarrow \boxed{\frac{1}{D+1} e^{e^{z}} = e^{-z} e^{e^{z}}}$

Again, let $v = \frac{1}{D+2} e^{e^{z}}$

Taking inverse operator, we get

$(D+2) v = e^{e^{z}}$

$\Rightarrow \frac{dv}{dz} + 2v = e^{e^{z}}$ ---- (4)

Which is a linear differential equation

$I.F = e^{\int 2dz} = e^{2z}$

Now, the solution of (4) becomes,

$v(I.F) = \int e^{e^{z}} (I.F) dz = \int e^{e^{z}} (e^{2z}) dz$

$\Rightarrow v(e^{2z}) = \int e^{z} e^{z} e^{e^{z}} dz$ $\quad t = e^{z}$, $dt = e^{z} dz$

$= \int t e^{t} dt$

$= e^{t}(t-1) = e^{e^{z}}(e^{z}-1)$

$\therefore v = e^{-2z} e^{e^{z}} (e^{z}-1)$

$\Rightarrow \boxed{\frac{1}{D+2} e^{e^{z}} = e^{-2z} e^{e^{z}} (e^{z}-1)}$

Hence, from (B),

$P.I = e^{-z} e^{e^{z}} - e^{-2z} e^{e^{z}} (e^{z}-1)$ - -

$y = C.F + P.I$

Equation reducible to homogeneous form.

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + P_2 (a+bx)^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_{n-1}(a+bx)\frac{dy}{dx} + P_n y = X \quad \dots (1)$$

where $P_1, P_2, \dots, P_n$ are constants.

putting, $a+bx = e^z$ or $z = \ln(a+bx)$

$$\frac{dz}{dx} = \frac{1}{a+bx}\cdot b = \frac{b}{a+bx}$$

Now $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$

$$\Rightarrow (a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\cdot Dy$$

Again differentiating w.r.to x, we get

$$\frac{d}{dx}\left[(a+bx)\frac{dy}{dx}\right] = b\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\Rightarrow (a+bx)\frac{d^2y}{dx^2} + b\frac{dy}{dx} = b\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$\Rightarrow (a+bx)\frac{d^2y}{dx^2} + b\frac{dy}{dx} = b\frac{b}{a+bx}\frac{d^2y}{dz^2}$$

$$\Rightarrow (a+bx)^2\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dz^2} - b(a+bx)\frac{dy}{dx}$$

$$= b^2\frac{d^2y}{dz^2} - b\cdot b\frac{dy}{dz}$$

$$= b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = b^2 D(D-1)y$$

similarly

$$(a+bx)^3\frac{d^3y}{dx^3} = b^3 D(D-1)(D-2)y$$

$\vdots$

$$(a+bx)^n\frac{d^ny}{dx^n} = b^n D(D-1)(D-2)\cdots(D-n+1)y$$

putting all these values in equation (1), it becomes ODE with constant co-efficients.

Example: Solve $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$ ——— (1)

putting $z = \ln(x+a)$ here $\boxed{b=1}$

$\therefore (x+a)\frac{dy}{dx} = Dy$

$(x+a)^2 \frac{d^2y}{dx^2} = D(D-1)y$

so, (1) becomes

$$[D(D-1) - 4D + 6]y = e^z - a$$

$$\Rightarrow (D^2 - D - 4D + 6)y = e^z - a$$

$$\Rightarrow (D^2 - 5D + 6)y = e^z - a \quad \text{——— (A)}$$

A.E $m^2 - 5m + 6 = 0 \quad \therefore m = 2, 3.$

$C.F = c_1 e^{2z} + c_2 e^{3z}.$

$$P.I = \frac{1}{D^2 - 5D + 6}(e^z - a e^{0.z})$$

$$= e^z . \frac{1}{2} - a . \frac{1}{6} = \frac{1}{6}(3e^z - a)$$

$\therefore y = C.F + P.I$

$$= c_1 e^{2z} + c_2 e^{3z} + \frac{1}{6}(3e^z - a)$$

Method of variation of parameters

This is the method to obtain the particular integral (P.I) from the complementary function (C.F) of ODE. This method was first discussed by the Johan Bernouli (1667–1748). After him the France Mathematician Joshef Lagrange (1736–1813) discussed the method in details and the method became an elegant method.

Theorem: If $y_c = Ay_1 + By_2$ is a particular complementary function (C.F) of the differential equation $y'' + py' + qy = X$ then its particular integral $y_p = u(x)y_1 + v(x)y_2$ provided

$$u'(x)y_1 + v'(x)y_2 = 0 \quad \text{and} \quad u'(x)y_1' + v'(x)y_2' = X$$

Theorem: If $y = y_1(x)$ and $y = y_2(x)$ are two solution of $y'' + py' + qy = 0$, then the general solution of $y'' + py' + qy = X$ is $y = u(x)y_1 + v(x)y_2$

where

$$u(x) = -\int \frac{y_2 X}{W}\, dx \quad \text{and} \quad v(x) = \int \frac{y_1 X}{W}\, dx$$

in which W is the Wronskian of y_1 and y_2 that is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Example: Solve $(D^2+1)y = \operatorname{cosec} x$ by the method of variation of parameters.

Solⁿ: The given differential equation

$$(D^2+1)y = \operatorname{cosec} x \quad \cdots (1)$$

$$X = \operatorname{cosec} x$$

Here A.E is $m^2+1=0$

$$\Rightarrow m = \pm i$$

So the complementary function

$$y_c = A\cos x + B\sin x = Ay_1 + By_2 \quad — (2)$$

So $y_1 = \cos x$

$y_2 = \sin x$

Now $W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1.$$

Let us suppose the general solution of (1) is

$$y = u(x)y_1 + v(x)y_2 \quad \cdots (A)$$

Then $u(x) = -\int \frac{y_2 X}{W}\,dx$ $\qquad$ $v(x) = \int \frac{y_1 X}{W}\,dx$

$$= -\int \frac{\sin x(\operatorname{cosec} x)}{1}\,dx \qquad = \int \frac{\cos x(\operatorname{cosec} x)}{1}\,dx$$

$$= \int \cot x\,dx$$

$$= -x + c_1 \qquad = \ln(\sin x) + c_2$$

Hence from (A) we get

$$y = (-x+c_1)\cos x + [\ln(\sin x) + c_2]\sin x$$

$$= c_1\cos x + c_2\sin x - x\cos x + \sin x\ln(\sin x)$$

Ans.

Example-1: Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + n^2 y = \sec nx$.

Sol^n: The given differential equation is

$$\frac{d^2y}{dx^2} + n^2 y = \sec nx \quad \text{----(1)}$$
$$= X$$

Here A.E is $m^2 + n^2 = 0 \Rightarrow m = \pm in$

Hence $C.F = c_1 \cos nx + c_2 \sin nx$

$$\Rightarrow y_c = c_1 y_1 + c_2 y_2 \quad \text{-------(2)}$$

Where $y_1 = \cos nx \qquad y_2 = \sin nx$,

Now $W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos nx & \sin nx \\ -n\sin nx & n\cos nx \end{vmatrix}$

$$= n(\cos^2 nx + \sin^2 nx) = n \neq 0$$

$\therefore \boxed{W = n}$

Let us suppose the general solution of (1) is

$$y = u y_1 + v y_2 \quad \text{----(3)}$$

Where $u = -\int \frac{y_2 X}{W} dx$

$$= -\frac{1}{n} \int \sin nx \cdot \sec nx \, dx$$

$$= -\frac{1}{n} \int \frac{\sin nx}{\cos nx} dx = \frac{1}{n^2} \ln(\cos nx) + c_1$$

$\Rightarrow \boxed{u = \frac{1}{n^2} \ln(\cos nx) + c_1}$

Again: $v = \int \frac{y_1 X}{W} dx = \frac{1}{n} \int \cos nx \sec nx \, dx$

$$= \frac{1}{n} \int dx = \frac{1}{n} x + c_2.$$

$\Rightarrow \boxed{v = \frac{x}{n} + c_2}$

Now putting the values of u and v in (3) we get

$$y = \left[\frac{1}{n^2}\ln(\cos nx) + c_1\right]\cos nx + \left[\frac{x}{n} + c_2\right]\sin nx$$

$$= c_1\cos nx + c_2\sin nx + \frac{1}{n^2}\cos nx\ln(\cos nx) + \frac{1}{n}x\sin nx$$

Which is the required solution.

H.W

$$\frac{d^2y}{dx^2} + 9y = \sec 3x$$

$$\frac{d^2y}{dx^2} + 4y = \sec 2x$$

$$\frac{d^2y}{dx^2} + 16y = \sec 4x$$

$$\frac{d^2y}{dx^2} + 25y = \sec 5x$$

$$\frac{d^2y}{dx^2} + y = \sec x.$$

Example-2: Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x.$$

Sol^n: Here, we are given the differential equation

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x = X \quad \text{--- (1)}$$

A.E is

$$m^2 + 4 = 0 \Rightarrow m = \pm i(2)$$

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Hence The C.F $= c_1 \cos 2x + c_2 \sin 2x$

$\Rightarrow \quad y_c = c_1 y_1 + c_2 y_2$ ----

where
$$y_1 = \cos 2x \;,\; y_2 = \sin 2x \quad --- (2)$$

Now The Wronskian

$$W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2(\cos^2 2x + \sin^2 2x) = 2 \neq 0$$

$\therefore \boxed{W = 2}$

Let us suppose the general solution of (1) is

$$y = u(x) y_1 + v(x) y_2 \quad --- (3)$$

where $u(x) = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin 2x \cdot (4 \tan 2x)}{2} dx$

$$= -2 \int \sin 2x \left(\frac{\sin 2x}{\cos 2x} \right) dx$$

$$= -2 \int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -2 \int \frac{1 - \cos^2 2x}{\cos 2x} dx$$

$$= 2 \int (\cos 2x - \sec 2x) dx$$

$\therefore u(x) = \sin 2x - \ln(\sec 2x + \tan 2x) + c_1$

and $v(x) = \int \frac{y_1 X}{W} dx$

$$= \int \frac{\cos 2x \cdot (4 \tan 2x)}{2} dx$$

(4)

$$v(x) = 2\int \cos 2x \tan 2x \, dx$$

$$= 2\int \sin 2x \, dx$$

$$\therefore v(x) = -\cos 2x + c_2$$

Now putting the values of $u(x)$ and $v(x)$ in eq. (3), we get

$$y = [\sin 2x - \ln(\sec 2x + \tan 2x) + c_1]\cos 2x + [-\cos 2x + c_2]\sin 2x$$

$$= c_1 \cos 2x + c_2 \sin 2x + [\sin 2x - \ln(\sec 2x + \tan 2x)]\cos 2x - \cos 2x \sin 2x$$

Which is the required solution of the given differential equation.

Example: Solve $y'' + y = \sec x \tan x$ by the method of variations of parameters.

Soln:

$$y'' + y = \sec x \tan x = X \quad \text{---} (1)$$

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$C.F = c_1 \cos x + c_2 \sin x$$

$$\Rightarrow y_c = c_1 y_1 + c_2 y_2$$

where $y_1 = \cos x$; $y_2 = \sin x$ --- (2)

The Wronskian ~~is~~

$$W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$

$$\therefore \boxed{W = 1}$$

Meth

...t us suppose the general solution of (1) is

$$y = u(x) y_1 + v(x) y_2 \quad (3)$$

Rul

Where $u(x) = -\int \frac{y_2 X}{W} dx$

$= -\int \frac{\sin x \, (\sec x \tan x)}{1} dx$

$= -\int \sin x \cdot \left(\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\right) dx$

$= -\int \frac{\sin^2 x}{\cos^2 x} dx$

$= -\int \frac{1-\cos^2 x}{\cos^2 x} dx$

$= \int (1 - \sec^2 x) dx$

$\therefore u(x) = x - \tan x + c_1$

Again $v(x) = \int \frac{y_1 X}{W} dx$

$= \int \cos x \, (\sec x \tan x) dx$

$= \int \cos x \left(\frac{1}{\cos x} \frac{\sin x}{\cos x}\right) dx$

$= \int \frac{\sin x}{\cos x} dx$

$\therefore v(x) = -\ln(\cos x) + c_2$

Hence from (3), we get

$y = (x - \tan x + c_1)\cos x + [-\ln(\cos x) + c_2]\sin x$

$= c_1 \cos x + c_2 \sin x + (x - \tan x)\cos x - \sin x \ln(\cos x)$

Ans:

(6)

Example: Solve $(D^2+1)y = \sec^4 x$ by the method of variation of parameters.

Sol^n: Here, we are given

$$(D^2+1)y = \sec^4 x \quad ----(1)$$
$$= X$$

A.E. is $\quad m^2+1=0 \Rightarrow m = \pm i$

$$C.F = c_1 \cos x + c_2 \sin x$$

$$\Rightarrow y_c = c_1 y_1 + c_2 y_2$$

Where $\quad y_1 = \cos x \qquad y_2 = \sin x \quad --(2)$

Here wronskian $\quad W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$\Rightarrow W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\Rightarrow \boxed{W=1}$$

Let us suppose the general solution of (1) is

$$y = u y_1 + v y_2 \quad ---(3)$$

Here $\quad u = -\int \frac{y_2 X}{W} dx$

$$= -\int \sin x (\sec^4 x)\, dx$$

$$= -\int \sin x \left(\frac{1}{\cos^2 x}\right)\left(\frac{1}{\cos x}\right)\left(\frac{1}{\cos x}\right) dx$$

$$= -\int \sec^2 x \sec x \tan x \, dx$$

Let $t = \sec x$
$dt = \sec x \tan x dx$

$$= -\int t^2 dt = -\frac{1}{3}t^3 + c_1$$

$$\therefore u = -\frac{1}{3}\sec^3 x + c_1$$

Again, we know

$$v = \int \frac{y_1 X}{W} dx$$

$$= \int \cos x \, (\sec^4 x) \, dx$$

$$= \int \sec^3 x \, dx$$

$$= \frac{1}{2}[\sec x \tan x + \ln(\sec x + \tan x)] + c_2$$

Hence, from (3) by putting the values of u and v we get

$$y = [-\frac{1}{3}\sec^3 x + c_1]\cos x + \{\frac{1}{2}[\sec x \tan x + \ln(\sec x + \tan x)] + c_2\}\sin x$$

$$= c_1 \cos x + c_2 \sin x - \frac{1}{3}\sec^2 x + \frac{1}{2}[\sec x \tan x + \ln(\sec x + \tan x)]\sin x$$

Ans:

Now we will solve some problems without using the Wronskian. This is may be treated as the tradition method.

Example: By use of the method of variation of parameters solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos(e^{-x})$$

Soln: The given equation can be written as

$$(D^2 - 3D + 2)y = \cos(e^{-x}) \quad \text{--- (1)}$$

A.E is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$

Hence $C.F = c_1 e^x + c_2 e^{2x}$

$\Rightarrow y_c = c_1 e^x + c_2 e^{2x}$ ---

(8) Let us suppose the general solution of the given equation (1) is of the form

$$y = ue^{x} + ve^{2x} \quad \text{----} (2)$$

We have to determine u and v which are functions of x only.

From (2) we have

$$Dy = ue^{x} + 2ve^{2x} + e^{x}Du + e^{2x}Dv \quad \text{---} (3)$$

Assume that u and v are chosen ~~in~~ for which $e^{x}Du + e^{2x}Dv = 0$ —— (4)

So, (3) $\Rightarrow$

$$Dy = ue^{x} + 2ve^{2x} \quad \text{———} (5)$$

Again differentiating (5) w.r.to x, we get

$$D^2y = ue^{x} + 4ve^{2x} + e^{x}Du + 2e^{2x}Dv \quad \text{———} (6)$$

Now using (2), (5) and (6) in (1), we get

$$ue^{x} + 4ve^{2x} + e^{x}Du + 2e^{2x}Dv - 3(ue^{x} + 2ve^{2x}) + 2(ue^{x} + ve^{2x}) = \cos(e^{-x})$$

$$\Rightarrow e^{x}Du + 2e^{2x}Dv = \cos(e^{-x}) \quad \text{———} (7)$$

Subtracting (4) from (7), we get

$$e^{2x}Dv = \cos(e^{-x}) \Rightarrow Dv = \frac{\cos(e^{-x})}{e^{2x}}$$

Similarly we can obtain from -2(4) ~~and~~ (7) we get

$$Du = -e^{-x}\cos(e^{-x})$$

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Now

$$Du = -e^{-x}\cos(e^{-x})$$

$$\therefore \quad u = -\int e^{-x}\cos(e^{-x})\,dx \qquad \Big| \; t = e^{-x},\; dt = -e^{-x}dx$$

$$= \int \cos t\,dt = \sin t + c_1$$

$$\boxed{u = \sin(e^{-x}) + c_1}$$

Again

$$Dv = \frac{\cos(e^{-x})}{e^{2x}}$$

$$\therefore \quad v = \int \frac{\cos(e^{-x})}{e^{2x}}\,dx = \int e^{-2x}\cos(e^{-x})\,dx$$

$$= -\int e^{-x}\cos(e^{-x})\,d(e^{-x})$$

$$v = -e^{-x}\sin(e^{-x}) - \cos(e^{-x}) + c_2$$

By putting the values of u and v in (2) we get,

$$y = [\sin(e^{-x}) + c_1]e^{x} + [-e^{-x}\sin(e^{-x}) - \cos(e^{-x}) + c_2]e^{2x}$$

$$= c_1e^{x} + c_2e^{2x} + e^{x}\sin(e^{-x})$$
$$+ [-e^{-x}\sin(e^{-x}) - \cos(e^{-x})]e^{2x}$$

Ans:

(10)

Example: Apply the method of variation of parameters to solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$ —— (1)

Sol^n: We find the C.F i.e solution of

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \quad \text{(homogeneous equation)} \text{ —— (2)}$$

putting $z=e^x \Rightarrow x=\ln z$, Then

$$x\frac{dy}{dx}=Dy, \quad x^2\frac{d^2y}{dx^2}=D(D-1)y$$

Then (2) becomes

$$[D(D-1)+D-1]y=0$$

$$\Rightarrow (D^2-D+D-1)y=0$$

$$\Rightarrow (D^2-1)y=0$$

A.E is $m^2-1=0 \Rightarrow m=1,-1$

$$\therefore y=c_1e^z+c_2e^{-z}$$

$$y=c_1x+c_2\left(\frac{1}{x}\right) \text{ —— (3)}$$

Let us suppose the general solution of (1) is

$$y=ux+\frac{v}{x} \text{ —— (4)}$$

Let us suppose the general solution

$$\frac{dy}{dx}=u-\frac{v}{x^2}+x\frac{du}{dx}+\frac{1}{x}\cdot\frac{dv}{dx}$$

$$\Rightarrow Dy=u-\frac{v}{x^2}+xDu+\frac{1}{x}Dv$$

$$\Rightarrow Dy=u-\frac{v}{x^2} \text{ ---- (5)}$$

When $xDu+\frac{1}{x}Dv=0$ —— (6)

Again from (5)

$$D^2y=Du+2\frac{v}{x^3}-\frac{1}{x^2}Dv \text{ ---- (7)}$$

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MAT-2128 (1)

15-4-19

Method of undetermined co-efficients

$f(D)y = X$

Rule-1: if $X(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ Then

$$y_p = c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$$

Rule-2: if $X(x) = e^{ax}$ Then $y_p = Ae^{ax}$

Rule-3: if $X(x) = \sin ax / \cos ax$ Then $y_p = c_1 \sin ax + c_2 \cos ax$

Rule-4: if $X(x) = x^n e^{ax}$ Then

$$y_p = c_1 x^n e^{ax} + c_2 x^{n-1} e^{ax} + \cdots + c_{n-1} x e^{ax} + c_n e^{ax}$$

Rule-5: if $X(x) = x^n \sin ax$ Then

$$y_p = b_1 x^n \sin ax + c_1 x^n \cos ax + b_2 x^{n-1} \sin ax + c_2 x^{n-1} \sin ax + \cdots + b_n \sin ax + c_n \cos ax$$

Rule-6: if $X(x) = x^n e^{ax} \sin bx$ Then

$$y_p = b_1 x^n e^{ax} \sin bx + c_1 x^n e^{ax} \cos bx + b_2 x^{n-1} e^{ax} \sin bx + c_1 x^{n-1} e^{ax} \sin bx + \cdots + b_n e^{ax} \sin bx + c_n e^{ax} \cos bx$$

Ex:-1: Solve by the method of undetermined co-efficients $(D^2-1)y = e^{2x}$

Solⁿ: The given differential equation is

$$(D^2-1)y = e^{2x} \quad \cdots\cdots (1)$$

Here $X = e^{2x}$

2)

The A.E is

$$m^2 - 1 = 0 \Rightarrow m = 1, -1.$$

Hence, $y_c = c_1 e^{-x} + c_2 e^{x}$ ——— (2)

Since $X(x) = e^{2x}$, we assume

$$y_p = A e^{2x} \quad --- \quad (3)$$

$\therefore$ $D y_p = 2A e^{2x}$

$\Rightarrow D^2 y_p = 4A e^{2x}$

Hence from (1), we get

$$4Ae^{2x} - Ae^{2x} = e^{2x}$$

$\Rightarrow 3A e^{2x} = e^{2x} \Rightarrow (3A-1) e^{2x} = 0$

$\Rightarrow 3A - 1 = 0 \Rightarrow A = 1/3$

From (3), we get $y_p = \frac{1}{3} e^{2x}$

Therefore, the general solution of (1) is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{x} + \frac{1}{3} e^{2x} \quad \text{Ans.}$$

Ex-2: Solve $(D^2 - 1)y = e^x$ by the method of undetermined co-efficients.

Solⁿ: Here given equation is

$$(D^2 - 1)y = e^x \quad ---- \quad (1) \qquad X = e^x.$$

The A.E is

$$m^2 - 1 = 0 \Rightarrow m = -1, 1$$

Hence, $y_c = c_1 e^{-x} + c_2 e^{x}$ --- (2)

Since $X = e^x$ is included in y_c, we assume

$y_p = A x e^{x}$ ----- (3) is the P.I.

Now $\quad Dy_p = A(xe^x + e^x)$

$$D^2y_p = A[(xe^x + e^x) + e^x]$$

$$= A(xe^x + 2e^x)$$

putting the value of y_p, D^2y_p in (1) we get

$$A(xe^x + 2e^x) - Axe^x = e^x$$

$$\Rightarrow 2Ae^x = e^x \Rightarrow 2A = 1 \Rightarrow \boxed{A = \frac{1}{2}}$$

Hence from (3), we get

$$y_p = \frac{1}{2}xe^x$$

Therefore the general solution of (1) is

$$y = y_c + y_p = c_1e^{-x} + c_2e^x + \frac{1}{2}xe^x \quad \text{Ans.}$$

Ex-3: Solve $(D^2 + 2D - 3)y = \sin x$ by the method of undetermined co-efficients.

Soln:- Here we are given

$$(D^2 + 2D - 3)y = \sin x \quad \cdots \quad (1)$$

$$X = \sin x.$$

The A.E is

$$m^2 + 2m - 3 = 0 \Rightarrow m = -3, 1$$

Hence, $\quad y_c = c_1e^{-3x} + c_2e^x \quad$ ——— (2)

since $X = \sin x$ and it is not in y_c, we assume

$$y_p = A\sin x + B\cos x \quad \cdots \quad (3)$$

Now $\quad Dy_p = A\cos x - B\sin x \quad \cdots \quad (4)$

$$D^2y_p = -A\sin x - B\cos x \quad \cdots \quad (5)$$

(4) By using (3), (4) and (5) in (1), we get

$$(-A\sin x - B\cos x) + 2(A\cos x - B\sin x) - 3(A\sin x + B\cos x) = \sin x$$

$$\Rightarrow (-A - 2B - 3A)\sin x + (-B + 2A - 3B)\cos x = \sin x$$

$$\Rightarrow (-4A - 2B)\sin x + (2A - 4B)\cos x = \sin x + 0.\cos x$$

Equating the co-efficients of like terms, we get

$$-4A - 2B = 1 \qquad \cancel{2A+B}.$$
$$2A - 4B = 0. \quad \Rightarrow$$

$$4A + 2B + 1 = 0$$
$$2A - 4B + 0 = 0$$

$$\frac{A}{+4} = \frac{B}{2-0} = \frac{1}{-16-4} = \frac{1}{-20}$$

$$\therefore A = -\frac{1}{5} \qquad B = -\frac{1}{10}$$

Hence, from (3), ~~The~~ by putting the values of A, B we get

$$y_p = -\frac{1}{5}\sin x - \frac{1}{10}\cos x$$

Thus, the general solution of (1) is

$$y = y_c + y_p = c_1 e^{-3x} + c_2 e^{x} - \frac{1}{5}\sin x - \frac{1}{10}\cos x$$

Ans.

Ex-4: Solve

(a) $(D^2 - 2D + 1)y = x\sin x$, $y_c = (c_1 + c_2 x)e^x$

$$y_p = b_1 x\sin x + c_1 x\cos x + b_2 \sin x + c_2 \cos x$$

(b) $y'' - 3y' = 8e^{3x} + 4\sin x$

Soln: Here we are given

$$(D^2 - 3D)y = 8e^{3x} + 4\sin x \quad \cdots (1)$$

The A.E is $m^2 - 3m = 0 \Rightarrow m(m-3) = 0 \Rightarrow m = 0,$ ~~m~~ $m = 3.$

$\therefore\ y_c = c_1 + c_2 e^{3x}$ ...... (2)

Since $X(x) = 8e^{3x} + 4\sin x$ and of which e^{3x} is in y_c. Hence, we assume the

$y_p = Axe^{3x} + B\sin x + C\cos x$ ... (3)

$Dy_p = A(3xe^{3x} + e^{3x}) + B\cos x - C\sin x$

$D^2 y_p = A[3(3xe^{3x} + e^{3x}) + 3e^{3x}] - B\sin x - C\cos x$

putting, all the values of y_p, Dy_p and $D^2 y_p$ in (1) we get

$$A(6e^{3x} + 9xe^{3x}) - B\sin x - C\cos x - 3A(3xe^{3x} + e^{3x}) - 3B\cos x + 3C\sin x = 8e^{3x} + 4\sin x$$

$\Rightarrow\ 3Ae^{3x} + (3C - B)\sin x - (3B + C)\cos x = 8e^{3x} + 4\sin x$

Equating the co-efficients of like terms, we get

$3A = 8 \Rightarrow \boxed{A = 8/3}$

$3C - B = 4$ and $3B + C = 0$

$\Rightarrow A = \frac{8}{3},\ B = -2/5,\ C = \frac{6}{5}$

Hence from (3), we get

$y_p = \frac{8}{3}xe^{3x} - \frac{2}{5}\sin x + \frac{6}{5}\cos x$.

Thus, the general solution of (1) is

$y = y_c + y_p = c_1 + c_2 e^{3x} + \frac{8}{3}xe^{3x} - \frac{2}{5}\sin x + \frac{6}{5}\cos x$

Ans.

(6)

H.W

Solve $(D^2-4D+4)y = (x-x^3)e^{2x}$ ——— (1)

Hints:

$$y_c = (c_1+c_2x)e^{2x} \quad \text{———} (2)$$

since $X = (x-x^3)e^{2x}$

$= \underline{xe^{2x}} - x^3e^{2x}$

$\downarrow$

Hence, $y_p = (A_1x^2e^{2x} + B_1x^3e^{2x})$

$+ (A_2x^4e^{2x} + B_2x^5e^{2x})$

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$$\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = \sin x \cos x.$$

Here $P_0 = \sin x$, $P_1 = -\cos x$, $P_2 = 2 \sin x$.
Also $P_2 - P_1' + P_0'' = 2 \sin x - \sin x - \sin x = 0$.
Therefore the equation is exact. First integral is

$$P_0 \frac{dy}{dx} + (P_1 + P_0')\, y = \int \sin x \cos x \, dx + c_1$$

i.e. $\sin x \dfrac{dy}{dx} - 2 \cos x \cdot y = \frac{1}{2} \sin^2 x + c_1$

or $\dfrac{dy}{dx} - \dfrac{2 \cos x}{\sin x}\, y = \frac{1}{2} \sin x + \dfrac{c_1}{\sin x}$.

Linear. I.F. $= e^{-2 \log \sin x} = \dfrac{1}{\sin^2 x} = \operatorname{cosec}^2 x$.

The solution is

$$y \operatorname{cosec}^2 x = c_2 + \int \left(\tfrac{1}{2} \sin x + \frac{c_1}{\sin x} \right) \operatorname{cosec}^2 x \, dx$$

$$= c_2 + \int \left(\tfrac{1}{2} \operatorname{cosec} x + c_1 \operatorname{cosec}^3 x \right) dx$$

$$= c_2 + \tfrac{1}{2} \log \tan \tfrac{1}{2}x - c_1 \left(\tfrac{1}{2} \operatorname{cosec} x \cot x + \tfrac{1}{2} \log \tan \tfrac{1}{2}x \right).$$

1·7. Non-linear Equations

Exactness. So far we have been discussing exactness of linear equations. The equations which are not linear may also be exact, in such a but thereis no simple test for their exactness. We group terms way that they become perfect differential and their integrals may be written directly. Much depends on success of trial for such arragements.

The method will be fully illustrated in the following examples.

Ex 1. *Solve* $2y \dfrac{d^3y}{dx^3} + 2\left(y + 3 \dfrac{dy}{dx}\right) \dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 2.$

Solution. The given equation may be written as

$$2y \frac{d^3y}{dx^3} + 2y \frac{d^2y}{dx^2} + 6 \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2 \qquad \ldots(1)$$

The first term may be obtained by differentiating the term $2y \dfrac{d^2y}{dx^2}$.

$$\text{But } \frac{d}{dx}\left(2y \frac{d^2y}{dx^2}\right) = 2y \frac{d^3y}{dx^3} + 2 \frac{dy}{dx}\frac{d^2y}{dx^2}. \qquad \ldots(2)$$

So leaving apart from (1) the terms on the right of (2) we are left with $2y \dfrac{d^2y}{dx^2} + 4 \dfrac{dy}{dx}\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2$

and the first term of it is obtained by differentiating $2y \dfrac{dy}{dx}$.

$$\text{But } \frac{d}{dx}\left(2y \frac{dy}{dx}\right) = 2y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 \qquad \ldots(3)$$

The remaining term $4 \dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\, 2\left(\dfrac{dy}{dx}\right)^2$. ...(4)

Thus combining terms on the right of (2), (3) and (4), we get left hand side of (1). Thus (1) can be written as

$$\frac{d}{dx}\left(2y \frac{d^2y}{dx^2}\right) + \frac{d}{dx}\left(2y \frac{dy}{dx}\right) + \frac{d}{dx}\, 2\left(\frac{dy}{dx}\right)^2 = 2.$$

$$\text{Integrating, } 2y \frac{d^2y}{dx^2} + 2y \frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 = 2x + c_1. \qquad \ldots(5)$$

In (5) the first term is obtained by the differentiation of

$2y \dfrac{dy}{dx}$ but $\dfrac{d}{dx}\left(2y \dfrac{dy}{dx}\right) = 2y \dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2$

The remaining term $2y \dfrac{dy}{dx} = \dfrac{d}{dx}(y^2)$.

Therefore (5) may be written as

$$\frac{d}{dx}\left(2y \frac{dy}{dx}\right) + \frac{d}{dx}(y^2) = 2x + c_1$$

$$\text{Integrating, } 2y \frac{dy}{dx} + y^2 = x^2 + c_1 x + c_2. \qquad \ldots(6)$$

Now putting $y^2 = u$, $2y \dfrac{dy}{dx} = \dfrac{du}{dx}$.

$\therefore$ the equation (6) becomes $\dfrac{du}{dx} + u = x^2 + c_1 x + c_2$

This is linear equation, I.F. $= e^x$.

$\therefore$ The solution is $ue^x = c_3 + \int (x^2 + c_1 x + c_2)\, e^x \, dx$

or $y^2 e^x = c_3 + e^x (x^2 - 2x + 2) + c_1 e^x (x - 1) + c_2 e^x$

or $y^2 = x^2 + k_1 + k_2 x + k_3 e^{-x}$ is required solution.

Note. The following scheme may be noted :—

$$2yy''' + 2yy'' + 6y'y'' + 2(y')^2 = 2$$

$\dfrac{d}{dx}(2yy'') = 2yy''' \quad + 2y'y''$

$\overline{\quad 2yy''' + 4y'y'' + 2(y')^2}$

$\dfrac{d}{dx}(2yy') = \quad 2yy'' \quad + 2(y')^2$

$\overline{\quad 4y'y''}$

$\dfrac{d}{dx}(2y'^2) = \quad 4y'y''$

$\overline{\quad \times}$

Therefore, the equation is

$$\frac{d}{dx}(2yy'') + \frac{d}{dx}(2yy') + \frac{d}{dx}(2y'^2) = 2. \qquad \ldots(A)$$

Integrating, $2yy'' + 2yy' + 2y'^2 = 2x + c_1$

For $\quad 2yy''+2yy'+2y'^2=2x+c_1$

$$\begin{array}{ll} \dfrac{d}{dx}(yy'')= & 2yy'' \qquad +2(y')^2 \\ \hline & 2yy' \\ \dfrac{d}{dx}(y^2) & 2yy' \\ \hline & \times \end{array}$$

$\therefore$ (A) can be written as $\dfrac{d}{dx}(2yy')+\dfrac{d}{dx}(y^2)=2x+c_1$.

Integrating, $\quad 2yy'+y^2=x^2+c_1x+c_2$

which is just (6) and may be integrated as above.

Ex. 2. *Solve* $x^2y\dfrac{d^2y}{dx^2}+\left(x\dfrac{dy}{dx}-y\right)^2-3y^2=0$.

[Agra 71, 67, 63, 58; Raj. 65, 63, 58]

Solution. The equation may be written as

$$x^2yy''+x^2(y')^2-2xyy'-2y^2=0,$$

$$\begin{array}{ll} \dfrac{d}{dx}(x^2yy')= & x^2yy''+x^2y'^2 \quad +2xyy' \\ \hline & \qquad -4xyy'-2y^2 \\ \dfrac{d}{dx}(-2xy^2)= & \qquad -4xyy'-2y^2 \\ \hline & \qquad \times \end{array}$$

Thus the equation may be written as

$$\frac{d}{dx}(x^2yy')+\frac{d}{dx}(-2xy^2)=0.$$

Integrating, $\quad x^2yy'-2xy^2=c_1$

or $\quad y\dfrac{dy}{dx}-\dfrac{2}{y}y^2=\dfrac{c_1}{x^2}$.

Put $y^2=u$, $2y\dfrac{dy}{dx}=\dfrac{du}{dx}$; the equation thus becomes

$$\frac{du}{dx}-\frac{4}{x}u=\frac{2c_1}{x^2}, \text{ linear, I.F.}=e^{-\int\left(\frac{4}{x}\right)dx}=\frac{1}{x^4}.$$

Hence the solution is $u.\dfrac{1}{x^4}=c_2+\displaystyle\int\frac{2c_1}{x^2}\cdot\frac{1}{x^4}\,dx$

or $\quad y^2\dfrac{1}{x^4}=c_2-\dfrac{2c_1}{5}\dfrac{1}{x^5}$

or $\quad xy^2=c_2x^5-\dfrac{2c_1}{5}$ or $xy^2=k_1x^5+k_2$.

Ex. 3. *Solve* $2y\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}\cdot\dfrac{dy}{dx}=-\dfrac{1}{x^2}$.

Solution. The equation may be written as

$$2yy'''+6y''y'=-(1/x^2). \qquad \text{...(1)}$$

$$\begin{array}{ll} \dfrac{d}{dx}(2yy'')= & 2yy'''+2y''y' \\ \hline & \qquad 4y''y' \\ \dfrac{d}{dx}(2y'^2)= & \qquad 4y''y' \\ \hline & \qquad \times \end{array}$$

Thus equation is $\dfrac{d}{dx}(2yy'')+\dfrac{d}{dx}(2y'^2)=-\dfrac{1}{x^2}$

Integrating, $\quad 2yy''+2y'^2=\dfrac{1}{x}+c_1 \qquad$ ...(2)

$$\begin{array}{ll} \text{Now } \dfrac{d}{dx}(2yy')= & 2yy''+2y'^2 \\ \hline & \qquad \times \end{array}$$

$\because$ (2) may be written as $\dfrac{d}{dx}(2yy')=\dfrac{1}{x}+c_1$.

Integrating, $\quad 2yy'=\log x+c_1x+c_2$

or $\quad 2y\dfrac{dy}{dx}=\log x+c_1x+c_2$.

Integrating it, $y^2=\displaystyle\int\log x\,dx+\tfrac{1}{2}c_1x^2+c_2x+c_3$.

Now $\displaystyle\int\log x\,dx=\int 1.\log x\,dx=\log x.x-\int x.\frac{1}{x}\,dx$

$$=x\log x-x.$$

$\therefore \quad y^2=x\log x+\tfrac{1}{2}c_1x^2+(c_2-1)x+c_3$

or $\quad y^2=x\log x+k_1x^2+k_2x+k_3$ is the solution.

Ex. 4. *Show that* $x^2\dfrac{d^3y}{dx^3}+x\dfrac{d^2y}{dx^2}+(2xy-1)\dfrac{dy}{dx}+y^2=0$ *is exact and first integral is* $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+xy^2=c$.

Solution. The given equation may be written as

$$x^2y'''+xy''+2xyy'-y'+y^2=0$$

$$\begin{array}{llll} \dfrac{d}{dx}(x^2y'')= & x^2y'''+2xy'' & & \\ & -xy''+2xyy' & -y' & +y^2 \\ \dfrac{d}{dx}(-xy')= & -xy'' & -y' & \\ & 2xyy' & & +y^2 \\ \dfrac{d}{dx}(xy^2)= & 2xyy' & & +y^2 \\ \hline & \times & & \end{array}$$

$$\frac{d}{dx}(x^2y'')+\frac{d}{dx}(-xy')+\frac{d}{dx}(xy^2)=0.$$

Integrating directly (form shows that equation is exact), the first integral is $x^2y''-xy'+xy^2+c$.

This proves the result.

Ex. 5. *Show that the equation*

$$\left(y^2+2x^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2(x+y)\left(\frac{dy}{dx}\right)^2+x\frac{dy}{dx}+y=0$$

Solution The equation may be written as

$$y^2y''+2x^2y'y''+2x(y')^2+2y(y')^2+xy'+y=0$$

$$\frac{d}{dx}(y^2y')=y^2y'' \qquad +2y(y')^2$$

$$\overline{\qquad 2x^2y'y''+2x(y')^2 \qquad +xy'+y}$$

$$\frac{d}{dx}(x^2y'^2)=2x^2y'y''+2x(y')^2$$

$$\overline{\qquad xy'+y}$$

$$\frac{d}{dx}(xy)= \qquad xy'+y$$

$$\overline{\qquad \times}$$

Therefore the equation may be written as

$$\frac{d}{dx}(y^2y')+\frac{d}{dx}(x^2y'^2)+\frac{d}{dx}(xy)=0 \text{ (exact form).}$$

Integrating, $y^2y'+x^2y'^2+xy=c$ or $y^2\frac{dy}{dx}+x^2\left(\frac{dy}{dx}\right)^2+xy=c.$

Ex. 6. *Solve* $(2y+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}\left(1+\frac{dy}{dx}\right)=0.$

Solution. The equation may be written as

$$2yy''+xy''+2y'+2y'^2=0. \qquad \ldots(1)$$

$$\frac{d}{dx}(2yy')=2yy'' \qquad +2y'^2$$

$$\overline{\qquad xy''+2y'}$$

$$\frac{d}{dx}(xy')= \qquad xy''+y'$$

$$\overline{\qquad y'}$$

$$\frac{dy}{dx}= \qquad y'$$

$$\overline{\qquad \times}$$

The equation becomes $\frac{d}{dx}(2yy')+\frac{d}{dx}(xy')+\frac{dy}{dx}=0.$

Integrating, $\qquad 2yy'+xy'+y=c_1. \qquad \ldots(2)$

$$\frac{d}{dx}(y^2)=2yy'$$

$$\overline{\qquad xy'+y}$$

$$\frac{d}{dx}(xy)= \qquad xy'+y$$

$$\overline{\qquad \times}$$

$\therefore$ (2) can be written as $\frac{d}{dx}(y^2)+\frac{d}{dx}(xy)=c_1.$

Integrating, $y^2+xy=c_1x+c_2$ is the complete solution.

Ex. 7. *Solve* $\quad xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2+y\frac{dy}{dx}=0.$

Solution. The equation may be written as

$$xyy''+xy'^2+yy'=0. \qquad \ldots(1)$$

$$\frac{d}{dx}(xyy')=xyy''+xy'^2+yy'$$

$$\overline{\qquad \times}$$

$\therefore$ the given equation is $\frac{d}{dx}(xyy')=0.$

Integrating, $xy\frac{dy}{dx}=c_1$ or $y\,dy=c_1\frac{dx}{x}.$

Integrating. $\frac{1}{2}y^2=c_1\log x+c_2$ or $y^2=k_1\log x+k_2$.

Ex. 8. *Show that the equation*

$$y+3x\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)^2+\left(x^2+2y^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=0$$

is exact and find its first integral. **[Delhi Hons. 63 ; Pb. 60]**

Solution. The equation may be written as

$$x^2y''+2y^2y'y''+2yy'^2+3xy'+y=0$$

$$\frac{d}{dx}(x^2y')=x^2y'' \qquad +2xy'$$

$$\overline{\qquad 2y^2y'y''+2yy'^2+\ xy'+y}$$

$$\frac{d}{dx}(y^2y'^2)=2y^2y'y''+2yy'^2$$

$$\overline{\qquad xy'+y}$$

$$\frac{d}{dx}(xy)= \qquad xy'+y$$

$$\overline{\qquad \times}$$

The equation may be written as

$$\frac{d}{dx}(x^2y')+\frac{d}{dx}(y^2y'^2)+\frac{d}{dx}(xy)=0 \text{ (exact form).}$$

Integrating, $x^2\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2+xy=c_1$ is the first integral.

Ex. 9. *Solve* $\cos y \dfrac{d^2y}{dx^2} - \sin y \left(\dfrac{dy}{dx}\right)^2 + \cos y \dfrac{dy}{dx} = x+1.$

Solution. The equation may be written as

$$\cos y . y'' - \sin y . y'^2 + \cos y . y' = x+1,$$

$$\frac{d}{dx}(\cos y . y'') = \cos y . y'' = \sin y . y'^2$$

$$\cos y . y'$$

$$\frac{d}{dx}(\sin y) = \qquad \cos y \; y'$$

$$\times$$

Hence the equation is $\dfrac{d}{dx}(\cos y . y') + \dfrac{d}{dx}(\sin y) = x+1.$

Integrating, $\cos y \dfrac{dy}{dx} + \sin y = \dfrac{(x+1)^2}{2} + c_1$

Putting $\sin y = u$, $\cos y \dfrac{dy}{dx} = \dfrac{du}{dx}$, the equation becomes

$\dfrac{du}{dx} + u = \dfrac{(x+1)^2}{2} + c_1$, Linear ; I.F. $= e^x$.

$$\therefore \quad ue^x = \int \tfrac{1}{2}(x+1)^2 e^x \, dx + \int c_1 e^x \, dx + c_2$$

$$= e^x \left[\frac{(x+1)^2}{2} - (x+1) + 1 + c_1 \right] + c_2$$

or $\quad \sin y = \dfrac{(x+1)^2}{2} - x + c_1 + c_2 e^{-x}$

or $\quad 2 \sin y = x^2 + k_1 + k_2 e^{-x}$ is the complete solution.

Ex. 10. *Solve* $2x^2 \cos y \dfrac{d^2y}{dx^2} - 2x^2 \sin y \left(\dfrac{dy}{dx}\right)^2$

$$+ x \cos y \frac{dy}{dx} - \sin y = \log x.$$

[Raj. 56

Solution. The equation may be written as

$$2x^2 \cos y . y'' - 2x^2 \sin y . y'^2 + x \cos y . y' - \sin y = \log x \quad \ldots(1)$$

$$\frac{d}{dx}(2x^2 \cos y . y') = 2x^2 \cos y . y'' - 2x^2 \sin y . y'^2 + 4x \cos y . y'$$

$$-3x \cos y \; y' - \sin y$$

$$\frac{d}{dx}(-3x \sin y) = \qquad -3x \cos y . y' - 3 \sin y$$

$$2 \sin y$$

Equation is not exact.
So dividing by x^2, the equation becomes

$$\cos y . y'' - 2 \sin y . y^2 + \frac{1}{x} \cos y . y' - \frac{1}{x^2} \sin y = \frac{1}{x^2} \log x. \quad \ldots(2)$$

$$\frac{d}{dx}(2 \cos y . y') = 2 \cos y . y'' - 2 \sin y . y'^2$$

$$\frac{1}{x} \cos y . y' - \frac{1}{x^2} \sin y$$

$$\frac{d}{dx}\left(\frac{\sin y}{x}\right) = \qquad \frac{1}{x} \cos y . y' - \frac{1}{x^2} \sin y$$

$$\times$$

Therefore (2) is exact and can be written as

$$\frac{d}{dx}(2 \cos y . y') + \frac{d}{dx}\left(\frac{\sin y}{x}\right) = \frac{1}{x^2} \log x.$$

Integrating, $2 \cos y \dfrac{dy}{dx} + \dfrac{\sin y}{x} = -\dfrac{1}{x}(\log x + 1) + c_1.$

Putting $\sin y = u$, $\cos y \dfrac{dy}{dx} = \dfrac{du}{dx}$,

$$\frac{du}{dx} + \frac{1}{2x} u = -\frac{1}{2x}(\log x + 1) + \tfrac{1}{2} c_1.$$

Linear, I.F. $= \sqrt{x}$. Hence the solution is

$$u\sqrt{x} = \int \left[-\frac{1}{2\sqrt{x}}(\log x + 1) + \tfrac{1}{2} c_1 \sqrt{x} \right] dx + c_2$$

$$= \int -\tfrac{1}{2}(z+1) e^{z/2} \, dz + \frac{c_1 x^{3/2}}{3} + c_2, \text{ where } x = e^z$$

$$= -e^{z/2}[(z+1) - 2] + \frac{c_1 x^{3/2}}{3} + c_2$$

or $\quad \sin y . \sqrt{x} = -\sqrt{x}[\log x - 1] + \dfrac{c_1 x^{3/2}}{3} + c_2$

or $\quad \sin y = -\log x + 1 + \dfrac{c_1 x}{3} + \dfrac{c_2}{\sqrt{x}}.$

Ex. 11. *Solve* $x^2 y \dfrac{d^2y}{dx^2} + \left(x \dfrac{dy}{dx} - y\right)^2 = 0.$ [Raj. 53]

Solution. The equation may be written as

$$x^2 yy'' + x^2 y'^2 - 2xyy' + y^2 = 0. \quad \ldots(1)$$

$$\frac{d}{dx}(x^2 yy') = x^2 yy'' + x^2 y'^2 + 2xyy'$$

$$-4xyy' + y^2$$

$$\frac{d}{dx}(-2xy^2) = \qquad -4xyy' + 2y^2$$

$$3y^2$$

Therefore the equation in its present form is not exact. Now dividing by x^2, it becomes

$$yy'' + y'^2 - \frac{2yy'}{x} + \frac{y^2}{x^2} = 0.$$

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$$\frac{d}{dx}(yy')=yy''+y'^2$$

$$-\frac{2yy'}{x}+\frac{y^2}{x^2}$$

$$\frac{d}{dx}\left(-\frac{y^2}{x}\right)= \qquad -\frac{2yy'}{x}+\frac{y^2}{x^2}$$

$$\times$$

Therefore (2) is exact. It can be written as

$$\frac{d}{dx}(yy')+\frac{d}{dx}\left(-\frac{y^2}{x}\right)=0.$$

Integrating, $y\frac{dy}{dx}-\frac{y^2}{x}=c_1$. Put $y^2=u$, $2y\frac{dy}{dx}=\frac{du}{dx}$

i.e. $\frac{du}{dx}-\frac{2}{x}u=2c_1$. Linear; I.F. $=e^{-\int\frac{2}{x}dx}=\frac{1}{x^2}$.

$$\therefore \quad u.\frac{1}{x^2}=\int 2c_1\frac{1}{x^2}\,dx+c_2$$

or $\quad y^2.\frac{1}{x^2}=-\frac{2c_1}{x}+c_2$ or $y^2=x(c_2x-2c_1)$.

Ex. 12. *Solve* $x^3\frac{d^2y}{dx^2}+(4x^2-3x)\frac{dy}{dx}+(2x-3)\,y=0$ *without using the condition of exactness.*

Solution. The condition of exactness, *i.e.* $P_2-P_1'+P_0''=0$ is satisfied but we would solve the equation without using this condition.

The equation may be written as

$$x^3y''+4x^2y'-3xy'+2xy-3y=0$$

$$\frac{d}{dx}(x^3y')=x^3y''+3x^2y'$$

$$x^2y'-3xy'+2xy-3y$$

$$\frac{d}{dx}(x^2y)= \qquad x^2y' \qquad +2xy$$

$$-3xy' \qquad -3y$$

$$\frac{d}{dx}(-3xy)= \qquad -3xy' \qquad -3y$$

$$\times$$

Hence the equation can be written as

$$\frac{d}{dx}(x^3y')+\frac{d}{dx}(x^2y)+\frac{d}{dx}(-3xy)=0.$$

Integrating, $x^3\frac{dy}{dx}+x^2y-3xy=c_1$

or $\quad \frac{dy}{dx}+\left(\frac{1}{x}-\frac{3}{x^2}\right)y=\frac{c_1}{x^3}$. Linear, I.F. $=xe^{3/x}$.

$$\therefore \quad yxe^{3/x}=\int\frac{c_1}{x^3}.xe^{3/x}\,dx+c_2=-\tfrac{1}{3}c_1e^{3/x}+c_2$$

or $\quad xy=-\frac{1}{3}c_1+c_2e^{-3/x}$ which is the required solution.

Note. It is always possible to apply the above method of trial to all linear equations which satisfy the condition of exactness of § 1·8 p. 15.

Ex. 13. *Solve*

$$2\sin x\frac{d^2y}{dx^2}+2\cos x\frac{dy}{dx}+2\sin x\frac{dy}{dx}+2y\cos x=\cos x.$$

Solution. The equation may be put as

$$2\sin x.y''+2\cos x.y'+2\sin x.y'+2y\cos x=\cos x$$

$$\frac{d}{dx}(2\sin x.y')=2\sin x.y''+2\cos x.y'$$

$$2\sin x.y'+2y\cos x$$

$$\frac{d}{dx}(2\sin x.y)= \qquad 2\sin x.y'+2y\cos x$$

$$\times$$

Thus the equation can be written as

$$\frac{d}{dx}(2\sin x.y')+\frac{d}{dx}(2\sin x.y)=\cos x.$$

Integrating, $2\sin x\frac{dy}{dx}+2\sin x.y=\sin x+c_1$

or $\quad \frac{dy}{dx}+y=\frac{1}{2}+c_1\operatorname{cosec} x$. Linear, I F. $=e^x$.

$$\therefore \quad ye^x=c_2+\int\left(\tfrac{1}{2}+c_1\operatorname{cosec} x\right)e^x\,dx.$$

Ex. 14. *Solve* $x\frac{d^3y}{dx^3}-x\frac{d^2y}{dx^2}-\frac{dy}{dx}=0.$

Solution. The equation is free from y. So putting $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$x\frac{d^2p}{dx^2}-x\frac{dp}{dx}\quad p=0.$$

This satisfies condition of exactness, *i.e.* $P_2-P_1'+P_0''=0$.

Hence the first integral is $P_0\frac{dp}{dx}+(P_1-P_0')\,p=c$

or $\quad x\frac{dp}{dx}+(-x-1)\,p=c$ or $\frac{dp}{dx}-\left(1+\frac{1}{x}\right)p=\frac{c}{x}$.

Linear equation, I.F. $= e^{-\int\left(1+\frac{1}{x}\right)dx} = \frac{1}{x}e^{-x}$.

Hence $p \cdot \frac{1}{x} e^{-x} = \int \frac{c}{x} \cdot \frac{e^{-x}}{x} dx + c'$.

or $p = \frac{dy}{dx} = c'xe^x + c . xe^x \int \frac{1}{x^2} e^{-x} dx$

which on integration further gives the solution.

Ex. 15. *Find the first intergal of*

$$\frac{dy}{dx}\frac{d^2y}{dx^2} - x^2y\frac{dy}{dx} - xy^2 = 0.$$

Solution. The equation is $\frac{1}{2}\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 - \frac{1}{2}\frac{d}{dx}(x^2y^2) = 0.$

Integrating, $\left(\frac{dy}{dx}\right)^2 = c_1 + x^2y^2$. This is first integral.

1·8. Equation of the form $\frac{d^ny}{dx^n} = f(x)$

The equation can be integrated successively to give the required solution.

1·9. Equations of the form $\frac{d^2y}{dx^2} = f(y)$

To integrate such equations, the equation is multiplied by $2\frac{dy}{dx}$. The equation thus becomes

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}$$

which on integration gives

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\frac{dy}{dx}dx + c_1$$

i.e. $\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\, dy + c_1$

which can be integrated further.

Ex. 1. *Solve* $\frac{d^ny}{dx^n} = x^m$.

Solution. Integrating the equation directly,

$$\frac{d^{n-1}y}{dx^{n-1}} = \frac{x^{m+1}}{m+1} + k_1.$$

Integrating again, $\frac{d^{n-2}y}{dx^{n-2}} = \frac{x^{m+2}}{(m+1)(m+2)} + k_1x + k_2.$

... ...

$$y = \frac{x^{m+n}}{(m+1)(m+2)\ldots(m+n)} + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n$$

$$= \frac{m!\, x^{m+n}}{(m+n)!} + (c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n)$$

where constants are suitably adjusted.

Ex. 2. *Solve* $\frac{d^4y}{dx^4} = x + e^{-x} - \cos x.$

Solution. Integrating the equation once,

$$\frac{d^3y}{dx^3} = \frac{x^2}{2} - e^{-x} - \sin x + c_1.$$

Integrating again, $\frac{d^2y}{dx^2} = \frac{x^3}{6} + e^{-x} + \cos x + c_1x + c_2.$

Again integrating, $\frac{dy}{dx} = \frac{x^4}{24} - e^{-x} + \sin x + c_1\frac{x^2}{2} + c_2x + c_3.$

Integrating once again, the solution is

$$y = \frac{x^5}{120} + e^{-x} - \cos x + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4.$$

Ex. 3. *Solve* $\frac{d^2y}{dx^2} = x^2 \sin x.$

Solution. Integrating, $\frac{dy}{dx} = x^2(-\cos x) + \int 2x \cos x\, dx + c_1$

or $\frac{dy}{dx} = -x^2 \cos x + 2x \sin x + 2\cos x + c_1.$

Integrating again, the solution is

$$y = \int(-x^2 \cos x + 2x \sin x + 2\cos x)\, dx + c_1x + c_2$$

$$= -x^2 \sin x - 4x \cos x + 6 \sin x + c_1x + c_2.$$

Ex. 4. *Solve* $\frac{d^3y}{dx^3} = \sin^2 x = \frac{1}{2}(1 - \cos 2x).$

Solution. Integrating, $\frac{d^2y}{dx^2} = \frac{1}{2}x - \frac{1}{4}\sin 2x + c_1.$

$\therefore \frac{dy}{dx} = \frac{1}{4}x^2 + \frac{1}{8}\cos 2x + c_1x + c_2$

and finally $y = \frac{1}{12}x^3 + \frac{1}{16}\sin 2x + \frac{1}{2}c_1x^2 + c_2x + c_3.$

Ex. 5. *Solve* $\frac{d^3y}{dx^3} = \log x.$

Solution. Integrating successively, $\frac{d^2y}{dx^2} = \int \log x\, dx + k_1$

or $\frac{d^2y}{dx^2} = x \log x - x + k_1$ etc.

Finally, $36y = 6x^3 \log x - 11x^3 + c_1x^2 + c_2x + c_3.$