Example.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} - 2y = 8x(9-x), \qquad y(0)=0,\ \ y(9)=0$$

Solution. Use Euler's method with step size $h = 3$. Assume, $y'(0) = 4$.
Now,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\, x^2} = 2y + 8x(9-x) = f(x, y, y'), \qquad y(0)=0,\ \ y'(0)=4$$

Let,

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, x} = z \text{ (say) } = f_1(x, y, z); \quad y(0) = 0 \tag{1}$$

$$\frac{\mathrm{d}\, z}{\mathrm{d}\, x} = 2y + 8x(9-x) = f_2(x, y, z); \quad y'(0) = z(0) = 4 \tag{2}$$

$$\tag{3}$$

From (1), $y_{i+1} = y_i + hf_1(x_i, y_i, z_i)$
From (2), $z_{i+1} = z_i + hf_2(x_i, y_i, z_i)$
For $i = 0$:

$$\begin{aligned}
y_1 &= y_0 + hf_1(x_0, y_0, z_0) \\
&= y_0 + hf_1(0, 0, 4) \\
&= 3 \times 4 = 12 \quad [y_1 = y(3) = 12] \\
z_1 &= z_0 + hf_2(x_0, y_0, z_0) \\
&= 4 + 3f_1(0, 0, 4) \\
&= 4 + 3[2 \times 0 + 8 \times 0 \times (9-0)] = 4 \quad [z_1 = z(3) = 4]
\end{aligned}$$

For $i = 1$:

$$\begin{aligned}
y_2 &= y_1 + hf_1(x_1, y_1, z_1) \\
&= 12 + 3f_1(3, 12, 4) \\
&= 12 + 3 \times 4 = 28 \quad [y_2 = y(6) = 28] \\
z_2 &= z_1 + hf_2(x_1, y_1, z_1) \\
&= 4 + 3f_1(3, 12, 4) \\
&= 4 + 3[2 \times 12 + 8 \times 3 \times (9-3)] = 508 \quad [z_2 = z(6) = 508]
\end{aligned}$$

For $i = 3$, $y_3 = 1548 \approx y(9)$
Now choose $y'(0) = -24$ Go through the steps of Euler's method

i	x_i	y_i	z_i
0	0	0	-24
1	3	-72	-24
2	6	-144	-24
3	9	-216	

$y_3 = -216 \approx y(9)$
Now, we need interpolation technique to

$y'(0)$	$y_3 \approx y(9)$
$p_0 = 4$	$1548 = q_0$
$p_1 = -24$	$-216 = q_1$
p	q

$$p = p_0 + \frac{p_1 - p_0}{q_1 - q_0}(q - q_0); \quad q_0 \le q \le q_1$$

$$p = 4 + \frac{-24 - 4}{-216 - 1548}(q - 1548)$$

We want to find the value of p where $q = 0$

$$\therefore\ p = 4 + \frac{-24-4}{-216-1548}(0-1548) = -20.57$$

So, $y'(0) = -20.57$
Now go through the steps of Euler's method

i	x_i	y_i	z_i
0	0	0	-20.57
1	3	-61.7	-20.57
2	6	-123.42	41.17
3	9	0.09	

Hence, $y_3 = 0.09 \approx y(9)$

H.W.

(i) Does the value of y_3 get close to zero on the 1st try because the example is a linear ODE.

(ii) Try to use this method with a nonlinear ODE.

(iii) Repeat the example using RK2 and RK4.

Note. Consider the equation

$$y'' = f(x, y, y'), \quad y(a) = A,\ y(b) = B$$

Let $y' = z$, $z' = f(x, y, z)$

In order to solve this set as an initial value problem, we need to test condition at $x = a$, therefore required another condition for z at $x = a$.

Let us assume that $z(a) = \mu_1$, where μ_1 is a guess, μ_1 represents the slope $y'(x)$ at $x = a$.
Thus,

$$\left.\begin{aligned} y' &= z, \quad y(a) = A \\ z' &= f(x, y, z), \quad z(a) = \mu_1 \end{aligned}\right\} \tag{4}$$

Equation (4) can be solved for y and z, using Heun's method or RK4 method.

If $B = B_1$, the obtained required solution.
If $B \neq B_1$, the require another guess $z(a) = \mu_2$. Let the new estimate of $y(x)$ at $x = b$ be B_2. If B_2 is not equal to B, then the process may be continued until we obtain the correct estimate of $y(b)$. However, the procedure can be accepted by using an improved guess for $z(a)$ after estimate of B_1 and B_2.

Let us assume that $z(a) = \mu_3$ leads to the value of $y(b) = B$. Then

$$\mu_3 = \mu_2 - \frac{B_2 - B}{B_2 - B_1}(\mu_2 - \mu_1)$$

Now, $z(a) = \mu_3$, we get the solution of $y(x)$.

Example.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} = 6x, \qquad y(1) = 2,\ \ y(2) = 9 \text{ in } (1, 2)$$