Discrete Math

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April 12, 2020

Preface

This is a compilation of lecture notes with some books and my own thoughts. This document is not a holy text. So, if there is a mistake, solve it by your own judgement.

Contents

Part I

Note

Chapter 1

Logic and Proof

1.1 Proposition

Definition 1 (Proposition)**.** A proposition is a declarative sentence (that is a sentence that declares a fact) that is either true or false but not both.

Example.

1. Toronto is the capital of Canada.

2. $1 + 1 = 2$

3. $2 + 2 = 3$

Proposition 1 and 3 are false where as 2 is true.
Consider the following sentences:

1. What time it is?

2. $x + y = z$

are not propositions.

Definition 2 (Negation of p)**.** Let p be a proposition. The negation of p denoted by $\neg\, p$ (also denoted by $\bar{p}$) is the statement "It is not the case that p". The proposition $\neg\, p$ is read "not p".

"Today is friday".
The negation is, "It is not the case that today is friday." Simply, "Today is not friday."

p	$\neg\, p$
T	F
F	T

Table 1.1: Truth table for negation of a proposition

Definition 3 (Conjunction of p and q)**.** Let p and q be propositions. The conjunction of p and q, denoted by $(p \wedge q)$ is the proposition "p and q". The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

"Today is friday and it is raining."

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.2: Truth table for $p \wedge q$

Definition 4 (Disjunction of p and q)**.** Let p and q be propositions. The disjunction of p and q denoted by $(p \vee q)$ is the proposition "p or q". The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

"Students who have taken calculus or computer science can take this class."

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.3: Truth table for $p \vee q$

Definition 5 (Exclusive or of p and q)**.** Let p and q be propositions. The exclusive or of p and q denoted by $(p \oplus q)$ is the proposition that is true when exactly one of p and q is true and is false otherwise.

"Students who have taken calculus or computer science, but not both, can take this class."

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 1.4: Truth table for $p \oplus q$

Let p and q be propositions. The conditional statement $p \rightarrow q$ is the proposition "If p then q". The conditional statement $p \rightarrow q$ is false when p is true and q is false and true otherwise. In the conditional statement $p \rightarrow q$, p is called hypothesis and q is called conclusion. "If I am elected, then I will lower taxes."

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.5: Truth table for $p \rightarrow q$

Definition 6 (Converse, Contrapositive and Inverse)**.** The proposition $q \rightarrow p$ is called converse of $p \rightarrow q$. The contrapositive of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. The proposition $\neg p \rightarrow \neg q$ is called inverse of $p \rightarrow q$.

Example. "If it is raining, then the home team wins."
Contrapositive: If the home team does not win then it is not raining.
Converse: If the home team wins, then it is raining.
Inverse: If it is not raining, then the home team does not win.

Definition 7 (Biconditional)**.** Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise. "You can take the flight if and only if you buy a ticket."

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.6: Truth table for $p \leftrightarrow q$

Problem 1.1.1. Construct the truth table of the compound proposition.

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

Solution. Let us construct a table

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Problem 1.1.2. How can this English sentence be translated into a logical expression?
"You can access the internet from campus only if you are a computer science major or you are not a freshman."

Solution.

$a \rightarrow$ You can access the internet from campus
$b \rightarrow$ You are a computer science major
$c \rightarrow$ You are a freshman
$a \rightarrow (b \vee \neg c)$

- Computer represent information using bits.

- A bit is a symbol with two possible values, namely, 0 and 1, 1 represents true, 0 represents false.

- A variable is called a Boolean variable if its value is either true or false.

- A bit string is a sequence of zero or more bits. The length of this string is the number of bits in this string.

$$01\,1011\,0110$$
$$11\,0001\,1101$$
$$\overline{}$$
$$11\,10111\,111 \longleftarrow \text{bitwise OR}$$
$$01\,0001\,0100 \longleftarrow \text{bitwise AND}$$
$$10\,1010\,1011 \longleftarrow \text{bitwise XOR}$$

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Definition 8 (Tautology and Contradiction)**.** A compound proposition that is always true, no matter what the truth values of the propositions that occur in it is called tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

p	$\neg\, p$	$p \wedge \neg\, p$	$p \vee \neg\, p$
T	F	F	T
F	T	F	T

Table 1.7: Truth table for tautology and contradiction

Definition 9 (Logically Equivalent)**.** The compound proposition p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. It is denoted as $p \equiv q$

p	q	$p \vee q$	$(\neg\, p \vee q)$	$\neg\, p$	$\neg\, q$	$\neg\, p \wedge \neg\, q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Table 1.8: Truth table for logically equivalent

$$(\neg\, p \vee q) \equiv \neg\, p \wedge \neg\, q$$

Problem 1.1.3. Prove that $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

Problem 1.1.4. Show that $(\neg\, p \vee (\neg\, p \wedge q)) \equiv \neg\, p \wedge \neg\, q$

Solution.

$$
\begin{aligned}
\neg\,(p \vee (\neg\, p \wedge q)) &\equiv \neg\, p \wedge \neg\,(\neg\, p \wedge q) \\
&\equiv \neg\, p \wedge (\neg\,\neg\, p \vee \neg\, q) \\
&\equiv \neg\, p \wedge (p \vee \neg\, q) \\
&\equiv (\neg\, p \wedge p) \vee (\neg\, p \vee \neg\, q) \\
&\equiv \neg\, p \vee \neg\, q
\end{aligned}
$$

Definition 10 (Predicates)**.** Suppose a statement involving variables, such as $x > 3$
The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part - the predicate, "is greater than 3" refers to a property that the subject of the statement can have.

We can denote the statement by $P(x)$, where p denotes the predicates and x is the variable. Propositional function p at x. Once a value has been assigned to the variable x the statement $P(x)$ becomes a proposition and has a truth value.

Example. Let $P(x)$ denotes the statement "$x > 3$". What are the values of $P(4)$ and $P(2)$?
$P(4)$, which is the statement "$4 > 3$" is true and $P(2)$, which is the statement "$2 > 3$" is false.

A statement of the form $P(x_1, x_2, ; x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, ; x_n)$ and P is called an n-place predicate or a n-ary predicate.

Problem 1.1.5. Consider the following program:

```
temp:=x
x:=y
y:=temp
```

Find predicates that we can use as the precondition and the post-condition to verify the correctness of this program. Then explain how to use them to verify that for all valid input the program does what is intended?

TODO:: Solve this.

Definition 11 (Quantifiers). The universal quantification of $P(x)$ is the statement "$P(x)$ for all the values of x in the domain". THe notation $\forall x P(x)$ denotes the universal quantification of $P(x)$ and $\forall$ is called the universal quantifier.

An element for which $P(x)$ is false is called a counterexample of $\forall x P(x)$.

Example. Let $P(x)$ be the statement "$x + 1 > x$". What is the truth value of the quantification $\forall x P(x)$. Where domain consists of all real numbers?

Because $P(x)$ is true for all real numbers x, the quantification, $\forall x P(x)$ is true.

Definition 12 (Existential Quantifier). The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier. Let $P(x)$ denote the statement "$x > 3$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

\# $\quad \forall x < 0 \, (x^2 > 0)$, $\forall y \neq 0 (y^3 \neq 0)$
$\forall x(x < 0 \to x^2 > 0)$ $\forall y(y \neq 0 \to y^3 \neq 0)$
$\exists z > 0(z^2 = 2)$
$\exists z(z > 0 \wedge z^2 = 2)$

\# Statements involving predicate and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statement and which domain of discourse is used for the variables in these propositional function.

Problem 1.1.6. Show that $\forall x \, (P(x) \wedge Q(x))$ and $\forall x \, P(x) \wedge \forall x \, Q(x)$ are logically equivalent.

Solution. Let $\forall x \, (P(x) \wedge Q(x))$ is true. This means if a is in the domain then $(P(a) \wedge Q(a))$ is true. Hence $P(a)$ and $Q(a)$ is true for every element in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true. This means that $\forall x P(x) \wedge \forall x Q(x)$ is true.

Next suppose that $\forall x P(x) \wedge \forall x Q(x)$ is true. It follows that $\forall x P(x)$ is true and $\forall x Q(x)$ is true. Hence if a is in the domain, $P(a)$ is true and $Q(a)$ is true. It follows that for all a, $P(a) \wedge Q(a)$ is true. Hence $\forall x(P(x) \wedge Q(x))$ is true.

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

1. $\neg \exists x P(x) \equiv \forall x \neg P(x)$ [For every x, $P(x)$ is false]

2. $\neg \forall x P(x) \equiv \exists x \neg P(x)$ [There is an x for which $P(x)$ is false]

Problem 1.1.7. Show that, $\neg \forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$

TODO: solve Problem

Problem 1.1.8. Express the statements "some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

Solution. Let,
$S(x) \rightarrow x$ is a student in this class.
$M(x) \rightarrow x$ has visited Mexico.
$C(x) \rightarrow x$ has visited Canada.
The solution for first statement is $\exists x(S(x) \wedge M(x))$
The solution for second statement is $\forall x(S(x) \rightarrow (C(x) \vee M(x)))$

Definition 13 (Theorem). A theorem is a statement that can be shown to be true. Less important theorems sometimes are called *proposition*.

Definition 14 (Proof). A proof is a valid argument that establish the truth of a theorem.

Definition 15 (Lemma). A less important theorem that is helpful in proof of other results is called a lemma.

Definition 16 (Corollary). A corollary is a theorem that can be established directly from a theorem that has been proved.

Definition 17 (Conjucture). A conjecture is a statement that is being proposed to be true statement, usually on the basis of some partial evidence.

Definition 18 (Direct Proof). A direct proof of a conditional statement $p \rightarrow q$ is constructed when the first step is the assumption that P is true; subsequent steps are constructed using rules of interference, with the final step showing that q must also be true.

Definition 19 (Indirect Proof). The proof that do not start with the hypothesis and end with the conclusion are called indirect proof.

Definition 20 (Proof by Contraposition). Proof by contraposition means that the conditional statement $p \rightarrow q$ can be proved by showing that the contrapositive $\neg q \rightarrow \neg p$ is true.

Problem 1.1.9. Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Proof. Direct proof:
    Let, $3n + 2$ is odd and $3n + 2 = 2k + 1$, for some integer k
This implies, $3n + 1 = 2k$
There does not seem to be any direct way to conclude that n is odd.
    Our attempt at direct proof failed.
    Proof by contraposition:
    Assume that n is even.
So, $n = 2k$ for some integer k.
So, we have,
$$3n + 2 = 3(2k) + 2 = 2(3k + 1)$$
This tells us that $3n + 2$ is even.
    Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. □

Definition 21 (Vacuous Proof). $p \rightarrow q$ is true when P is false $p \rightarrow q \equiv (\neg p \vee q)$.
If we can show that p is false, then we have a proof called a vacuous proof of the conditional statement $p \rightarrow q$.

Definition 22 (Trivial Proof). A proof of $p \to q$ that uses the fact that q is true is called a trivial proof.

Definition 23 (Proofs by Contradiction). Because the statement $r \land \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \land \neg r)$ is true for some proposition r. Proofs of this type are called proofs by contradiction.

Problem 1.1.10. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Proof. If $\sqrt{2}$ is rational, there exists integer a and b with, $\sqrt{2} = \frac{a}{b}$ where a and b have no common factors.(Every rational number can be written in lowest terms.) So, $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2$
So, a^2 is even and a is also even.
So, $a = 2c$. Then, $2b^2 = 4c^2 \Rightarrow b^2 = 2c^2$
Similarly b is also even.
Now we have a and b have no common factors, a ans b are even, that is 2 divides both a and b.

Because our assumption of $\neg p$ leads to the contradiction that 2 divides both a and b and 2 does not divide both a and b, $\neg p$ must be false.
So $\sqrt{2}$ must be irrational. $\qquad\qquad\square$

A statement of the form $\forall x P(x)$ is false we need only find a counterexample that is an example for which $P(x)$ is false.

Mistakes is proofs:

$$a = b$$
$$\Rightarrow a^2 = ab$$
$$\Rightarrow a^2 - b^2 = ab - b^2$$
$$\Rightarrow (a+b)(a-b) = b(a-b)$$
$$\Rightarrow a + b = b$$
$$\Rightarrow 2b = b$$
$$\Rightarrow 2 = 1$$

Definition 24 (Exhaustive Proof). Some theorems can be proved by examining a relatively small number of examples. Such proofs are called exhaustive proof.

Example. Prove that, $(n+1)^3 \geq 3^n$ if n is a positive integer and $n \leq 4$.
If $n = 1,\quad (1+1)^3 \geq 3^1 \Rightarrow 8 \geq 3$
$n = 2,\quad (2+1)^3 \geq 3^2 \Rightarrow 27 \geq 9$
$n = 3,\quad (3+1)^3 \geq 3^3 \Rightarrow 64 \geq 27$
$n = 4,\quad (4+1)^3 \geq 4^3 \Rightarrow 125 \geq 81$

A proof by cases must cover all possible cases that arise in a theorem.
When the phrase "without loss of generality" is used in a proof, we assert that by proving one case of a theorem, no additional argument is required to prove other specified cases.

Definition 25 (Uniqueness Proofs). The two parts of a uniqueness proof are:

- **Existence:** We show that an element x with desired property exists.

- **Uniqueness:** We show that if $y \neq x$, then y does not have the desired property. Equivalently, we can show that if x and y both has the desired property, then $x = y$.

Definition 26 (Argument)**.** By an argument, we mean a sequence of statements that end with a conclusion.

By *valid*, we mean that the conclusion or final statement of the argument must follow form the truth of the preceding statements or premises of the argument.

That is an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.

Definition 27 (Premises & Conclusion)**.** An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

An argument is valid if the truth of all its premises implies that the conclusion is true.

\#   We can first establish the validity of some relatively simple argument forms called rules of interference.

$p \rightarrow$ It is below freezing now

$q \rightarrow$ It is raining now

"It is below freezing now. Therefore it is either below freezing or raining now."

$$p \qquad p \rightarrow (p \vee q) \quad \text{addition}$$

$$\overline{\qquad\qquad}$$

$$\therefore p \vee q$$

Rule of interference	Tautology	Name
p $p \rightarrow q$ ∴ q	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus pones
$\neg q$ $p \rightarrow q$ ∴ $\neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg q$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ ∴ $p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow q)] \rightarrow (p \rightarrow r)$	Hypothetical Syllogism
$p \wedge q$ $\neg \& p$ ∴ q	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive Syllogism
p ∴ $p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \wedge q$ ∴ p	$(p \wedge q) \rightarrow p$	Simplification
p q ∴ $p \wedge q$	$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg$ $p \vee r$ ∴ $q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r$	Resolution

Table 1.9: Rules of Interference

Problem 1.1.11. Show that the hypothesis "It is not sunny this afternoon and it is colder than yesterday", "We will go swimming only if it is sunny", "If we do not go swimming, then we will take a canoe trip", and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution. Let,
$p \rightarrow$ It is sunny this afternoon
$q \rightarrow$ It is colder than yesterday
$r \rightarrow$ We will go swimming
$s \rightarrow$ We will take a canoe trip
$t \rightarrow$ We will be home by sunset

We construct an argument to show that our hypothesis lead to the desired conclusion as follows:

Steps	Reason
1. $\neg p \land q$	Hypothesis
2. $\neg p$	Simplification using 1
3. $r \to p$	Hypothesis
4. $\neg r$	Modus tollens using 2 and 3
5. $\neg r \to s$	Hypothesis
6. s	Modus pones using 4 and 5
7. $s \to t$	Hypothesis
8. t	Modus pones using 6 and 7

Problem 1.1.12. Prove that, if n is an integer then $n^2 \geq n$.

Proof. We split the proof into three cases.
Case i: When $n = 0$, because $0^2 = 0$. We see that, $0^2 = 0$. It follows that $n^2 \geq n$ is true in this case.
Case ii: When $n \geq 1$. We obtain $n^2 \geq n$ for positive integer n.
Case iii: When $n \leq -1$. We obtain $n^2 \geq 0 \geq n$
    Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer then $n^2 \geq n$. $\square$

Division Algorithm   Let a be an integer and d be a positive integer. Then there are unique integers q and r with $0 \geq r < d$. Such that,

$$a = dq + r$$

$$q = \text{div } d, \qquad r = a \mod d$$

Problem 1.1.13. What are the quotients when -11 is divided by 3?

Solution. $11 = 3(-4) + 1$ remainder cannot be negative.

    If a and b are integers and m is a positive integer. Then a is congruent to b modulo m if m divides $(a - b)$

$$a \equiv b \mod m$$

$$17 \equiv 5 \mod 6$$

Fundamental Theorem of Arithmetic   Every positive integer greater that 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of non-decreasing size.

$$100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$$

$$999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37$$

Representations of Integers   Let b be a positive integer greater that 1. Then if n is a positive integer, it can be uniquely expressed in the form,

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

Where k is a non-negative integer, $a_0, a_1, \ldots, a_k$ are non-negative integers less than b and $a_k \neq 0$

Problem 1.1.14. What is the decimal expansion of the integer that has $(1\,0101\,1111)_2$ as its binary expansion?

Solution. We have,

$$(1\,0101\,1111)_2 = 1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 351$$

Hexadecimal Expansions   The base 16 expansion of an integer is called its hexadecimal expansion.

$$(2AE0B)_{16} = 2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16^1 + 11 \cdot 16^0 = (175627)_{10}$$

$$(1110\,0101)_2 = (E5)_{16}$$

Base Conversion   integer n, base b

$$\begin{aligned}
n &= bq_0 + a_0, & 0 \leq a_0 < b \\
q_0 &= bq_1 + a_1, & 0 \leq a_1 < b \\
&\cdots
\end{aligned}$$

This process terminates when we obtain a quotient equal to zero.

Problem 1.1.15. Find the base 8 or octal expansion of $(12345)_{10}$.

Solution. First divide 12345 by 8, we obtain

$$\begin{aligned}
12345 &= 8 \cdot 1543 + 1 \\
1543 &= 8 \cdot 192 + 7 \\
192 &= 8 \cdot 24 + 0 \\
24 &= 8 \cdot 3 + 0 \\
3 &= 8 \cdot 0 + 3
\end{aligned}$$

So $(12345)_{10} = (30071)_8$

Problem 1.1.16. Find the hexadecimal expansion of $(177130)_{10}$.

Solution. First divide 177130 by 16 to obtain,

$$\begin{aligned}
177130 &= 16 \cdot 11070 + 10 \\
11070 &= 16 \cdot 691 + 14 \\
691 &= 16 \cdot 43 + 3 \\
43 &= 16 \cdot 2 + 11 \\
2 &= 16 \cdot 0 + 2
\end{aligned}$$

So $(177130)_{10} = (2B3EA)_{16}$

Problem 1.1.17. Find the binary expansion of $(241)_{10}$.

Solution. First divide 241 by 2 to obtain,

$$
\begin{aligned}
241 &= 2 \cdot 120 + 1 \\
120 &= 2 \cdot 60 + 0 \\
60 &= 2 \cdot 30 + 0 \\
30 &= 2 \cdot 15 + 0 \\
15 &= 2 \cdot 7 + 1 \\
7 &= 2 \cdot 3 + 1 \\
3 &= 2 \cdot 1 + 1 \\
1 &= 2 \cdot 0 + 1
\end{aligned}
$$

So $(241)_{10} = (1111\,0001)_2$

Binary Addition   Add $(1110)_2$ and $(1011)_2$
Following the procedure specified in the algorithm
$a_0 + b_0 = 0 + 1 = 0 \cdot 2 + 1$
$c_0 = 0$ and $s_0 = 1$
Then $a_1 + b_1 + c_0 = 1 + 1 + 0 = 1 \cdot 2 + 0$
$c_1 = 1$ and $s_1 = 0$
Continuing,
$a_2 + b_2 + c_1 = 1 + 0 + 1 = 1 \cdot 2 + 0$
$c_2 = 1$ and $s_2 = 0$
$a_3 + b_3 + c_2 = 1 + 1 + 1 = 1 \cdot 2 + 1$
$c_3 = 1$ and $s_3 = 1$
Then $s_4 = c_3 = 1$
so, $(1110)_2 + (1011)_2 = (1\,1001)_2$

Binary Substraction

$$
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
1.\quad -0 & -0 & -1 & -1 \\
\hline
0 & 1 & 0 & 11
\end{array}
$$

$$
\begin{array}{c|c}
1\ 0101 & 21 \\
-1\ 0100 & 20 \\
\hline
0\ 0001 & 1
\end{array}
\qquad
\begin{array}{c|c}
1101 & 13 \\
-0011 & 3 \\
\hline
1010 & 10
\end{array}
$$

2. (i) 1's complement method:

To subtract a smaller number form a larger number, the 1's complement method is as follows:

Example. Subtract $(1010)_2$ from $(1111)_2$

$$
\begin{array}{r}
1111 \\
+\ 0101 \quad \leftarrow \text{1's complement of} (1010)_2 \\
\hline
\text{carry} \leftarrow \quad 1\ 0100 \\
\text{Add carry} \rightarrow \quad\quad 1 \\
\hline
0101
\end{array}
$$

(ii) 2's complement subtraction:

The 2's complement of a binary number can be obtained by adding 1 to it's 1's complement.

Example. Subtract $(10\,1101)_2$ from $(11\,0110)_2$

$$
\begin{array}{r}
11\ 0110 \\
+\ 01\ 0011 \quad \leftarrow \text{2's complement of} (10\,1101)_2 \\
\hline
\text{carry} \leftarrow \quad \underline{1}00\ 1001
\end{array}
$$

The carry is discarded. Thus answer is $(1001)_2$

Note: The 2's complement method is also for subtractions of a large number from a smaller one.

Chapter 2

Graph Theory

2.1 Graph and Multigraph

A graph G consists of two things:

(i) A set $V = V(G)$ whose elements are called vertices, point or nodes of G.

(ii) A set $E = E(G)$ of unordered pairs of distinct vertices called edges of G.

We denote such a graph by $G(V, E)$.
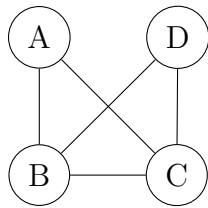
Figure 2.1: G_1
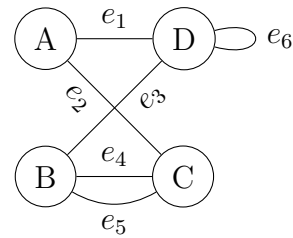
Figure 2.2: G_2

2.2 Multigraphs

Look at the graph 2.2, the edges e_4 and e_5 are called multiple edges since they connect the same endpoints and the edge e_6 is called a loop since its endpoints are the same vertex. Such a diagram is called multigraph.

2.3 Degree

The degree of a vertex v in a graph G written $deg(v)$ is equal to the number of edges in G which contains v, that is which are incident on v. A vertex with degree zero is called isolated. In graph G_2, $deg(D) = 4$, $deg(C) = 3$.

A multigraph is said to be finite if it has a finite number of vertices and a finite number of edges. The finite graph with one vertex and no edges is called the trivial graph.

2.4   Subgraph

Consider a graph $G = G(V, E)$. A graph $H = H(V', E')$ is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G.

2.5   Isomorphic Graphs and Homeomorphic Graph

Graphs $G(V, E)$ and $G^*(V^*, E^*)$ are said to be homeomorphic if the can be obtained from the same graph or isomorphic graphs by dividing an edge with additional vertices.
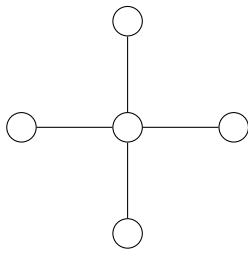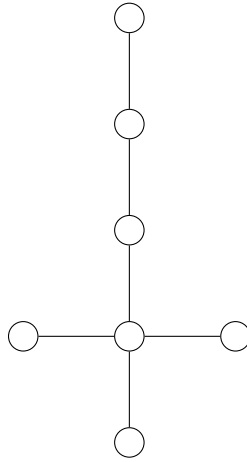
Figure 2.3: Graph A   Figure 2.4: Graph B   Figure 2.5: Graph C
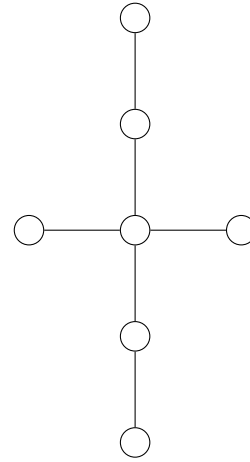
Graphs 2.4 and 2.5 are homeomorphic since they can be obtained from 2.3 by adding appropriate vertices.

Graphs G and G^* are said to be isomorphic if there exists a one-to-one correspondence $f : V \to V^*$ such that $\{u, v\}$ is an edge of G if and only if $\{f(u), f(v)\}$ is an edge of G^*.
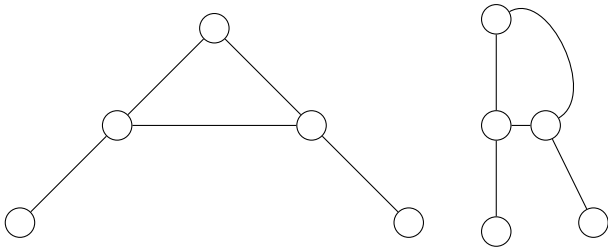
Figure 2.6: Isomorphic Graphs   Figure 2.7: Isomorphic Graphs
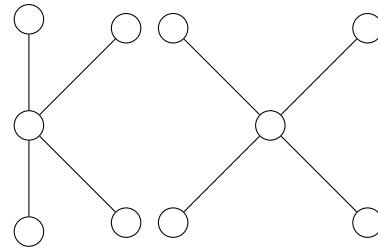
2.6   Paths, Connectivity

A path in a multigraph G consists of vertices and edges of the form,

$\qquad v_0, e_1, v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_{n-1}, e_n, v_n.$

Where each edge e_i contains the vertices v_{i-1} and v_i. The number n of edges is called the length of the path.

- The path is closed if $v_0 = v_n$.

- A simple path is a path in which all vertices are different.

- A path in which all edges are different will be called a trail.

- A cycle is a closed path in which all vertices are distinct except $v_0 = v_n$.
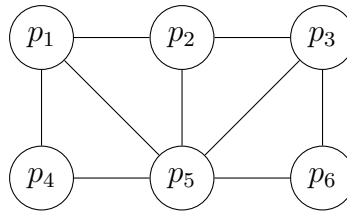
$\alpha = \{p_4, p_1, p_2, p_5, p_1, p_2, p_3, p_6\}$ is not a trail
$\beta = \{p_4, p_1, p_5, p_2, p_6\}$ is not a path
$\gamma = \{p_4, p_1, p_5, p_2, p_3, p_5, p_6\}$ is not a trail but not simple
$\delta = \{p_4, p_1, p_5, p_3, p_6\}$ is simple but not shortest

- A graph G is connected if there is a path between any two of its vertices.

- The distance between vertices u and v in G written $d(u, v)$ is the length of the shortest path between u and v and the diameter of G, written $diam(g)$ is the maximum distance between any two points in G.
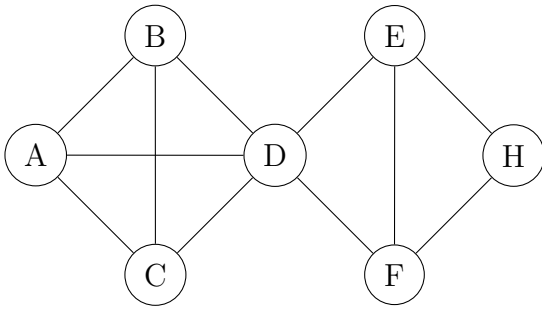
Figure 2.8: G_1
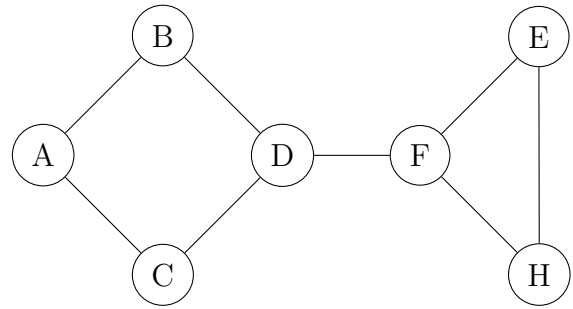
Figure 2.9: G_2

In G_1(figure 2.8), $d(A, F) = 2$ and $diam(G_1) = 3$
In G_2(figure 2.9), $d(A, F) = 3$ and $diam(G_2) = 4$

- A vertex v in G is called a cutpoint if $G - v$ is disconnected.

- An edge e is called a bridge if $G - e$ is disconnected.
  (D in graph G_1 is cutpoint and $e = \{D, F\}$ is a bridge in G_2.)

- A multigraph is said to be traversable if it can be drawn without any breaks in the curve and without repeating any edges.

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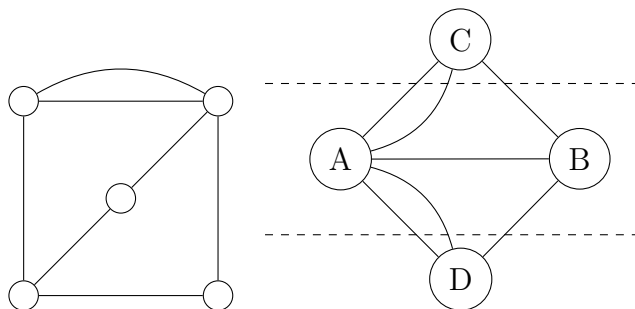
A multigraph with more than two odd vertices cannot be traversable. The famous königsberg bridge problem has four odd vertices.

- A graph G is called an Eulerian graph if there exists a closed traversable trail.

- A finite connected graph is Eulerian if and only if each vertex has even degree.

- A Hamiltonian circuit in a graph G named after the nineteenth-century Irish mathematician William Hamilton, is a closed path that visits every vertex in G exactly once.

  A graph G that admits a Hamiltonian circuit is called a Hamiltonian graph.
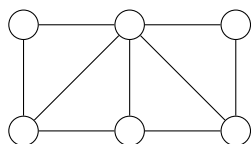
Figure 2.10: Hamiltonian and non-Eulerian
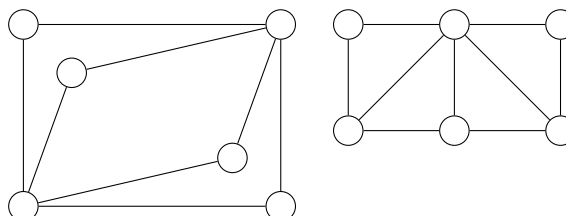
Figure 2.11: Eulerian and non-Hamilton

- A graph G is called a labeled graph if its edges are assigned data of one kind.

  A graph G is called a weighted graph if each e of G is assigned a non-negative number $w(e)$ called the weight or length of v.

- A graph G is said to be complete if every vertex in G is connected to every other vertex in G. The complete graph with n vertices is denoted by k_n.
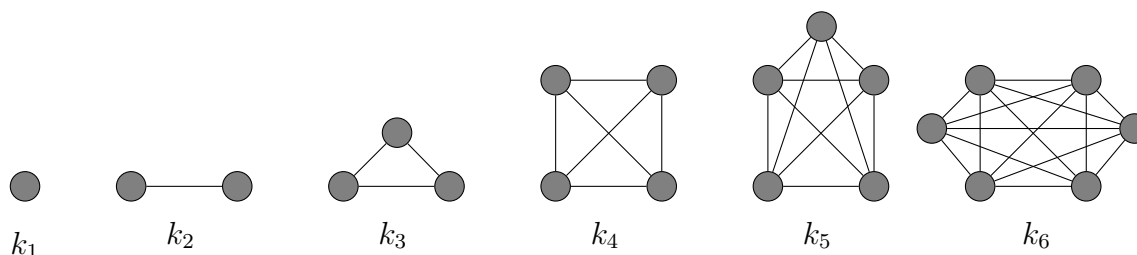
Figure 2.12: Some complete graphs

- A graph G is k-regular if every vertex has degree k.
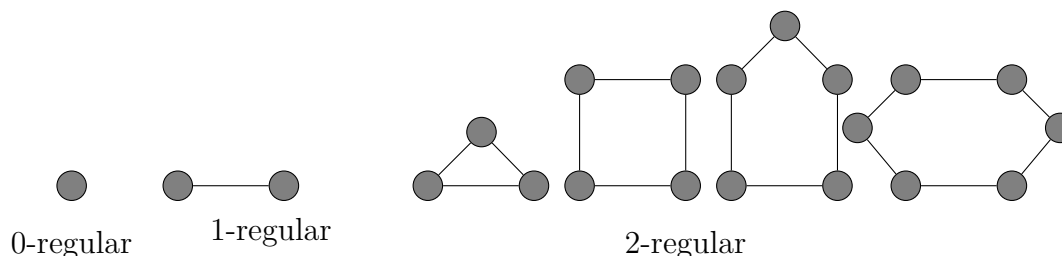
Figure 2.13: Some regular graphs

- A graph G is said to be bipartite if its vertices V can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N.

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By $k_{m,n}$ we mean that each vertex of M is connected to each vertex of N, a complete bipartite graph.
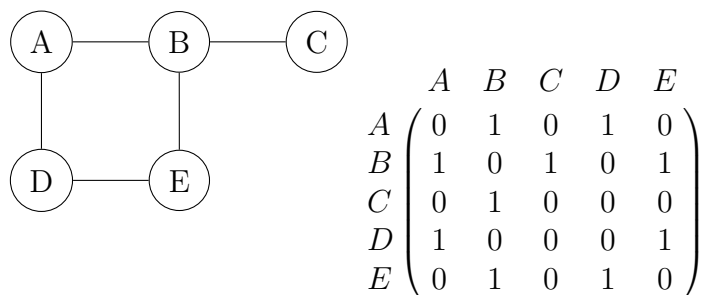

$$k_{2,3}$$
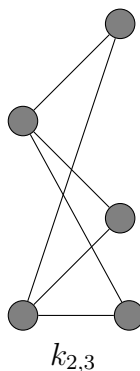
- A graph or multigraph which can be drawn in the plane so that its edges do not cross is said to be planar.
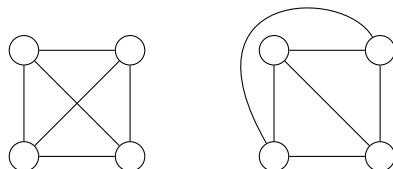
- A particular planar representation of a finite planar multigraph is called a map.

- Let G be a connected planar graph with p vertices and q edges, where $p \geq 3$. Then $q \leq 3p - 6$.

- Suppose G is a graph with m vertices and suppose the vertices have been ordered say, $v_1, v_2, \ldots, v_m$. Then the adjacency matrix $A = [a_{ij}]$ of the graph G is the $m \times m$ matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{Otherwise} \end{cases}$$

Adjacency matrix is symmetric.

$$
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

- A vertex coloring or simply a coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors.

  The minimum number of colors needed to paint G is called the chromatic number of G and is denoted by $\lambda(G)$.
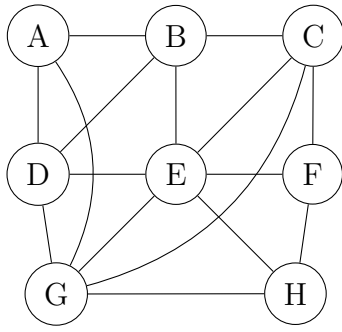
1. Ordering the vertices of G according to decreasing degrees. Here they are E, G, B, C, D, A, F, H

2. Assign first color c_1 to first vertex and assign c_1 to each vertex which is not adjacent to first vertex.

3. Repeat step 2 with second color c_2.

4. Repeat step 3 and 3 until no vertex left.

5. Exit.

First color c_1 to E, A
Second color c_2 to G, B, F
Third color c_3 to C, D, H
Since every vertex is adjacent to every other vertex, k_n requires n colors in any coloring. Thus $\lambda(k_n) = n$.

2.7   Four Color Theorem

Any planar graph is four colorable.

2.7.1   Dual maps and the four color theorem

If the regions of any map M are colored so that adjacent regions have different colors, then no more than four colors are required. Note to self:: TO BE ADDED

2.8   Spanning Tree

A subgraph T of a connected graph G is called a spanning tree of G if T is a tree and T includes all the vertices of G.

Suppose G is a connected weighted graph. Then any spanning tree T of G is assigned a total weight obtained by adding the weights of the edges in T.
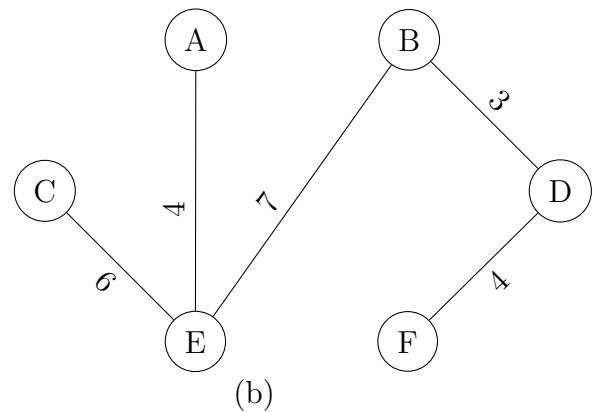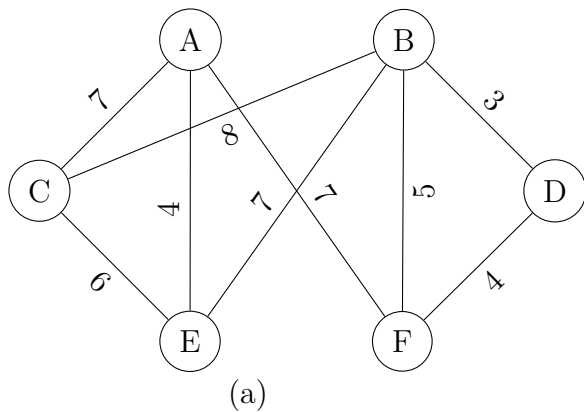
A minimal spanning tree of G is a spanning tree whose total weight is as small as possible.

2.8.1   Kruskal's algorithm for minimal spanning tree

step 1: Arrange the edges of G in order of increasing weights.

step 2: Starting only with the vertices of G and proceeding sequentially, add each edge which does not result in a cycle until $(n-1)$ edges are added.

step 3: Exit.

(a)

(b)

(c)
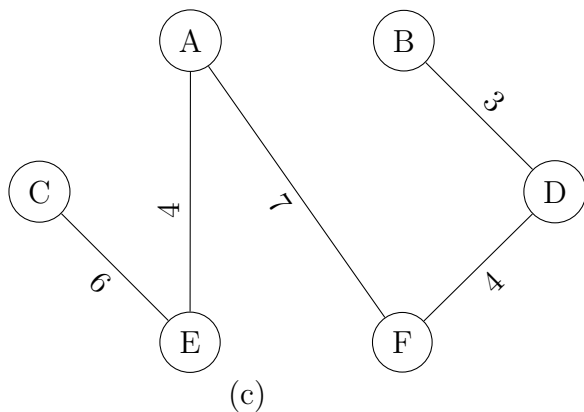
First we order the edges by decreasing weights and then we successively delete edges without disconnecting Q until five edges remain. This yields the following data:

Edges	BC	AF	AC	BE	CE	BF	AE	DF	BD
Weight	8	7	7	7	6	5	4	4	3
Delete	Yes	Yes	Yes	No	No	Yes	No	No	No

Thus the minimal spanning tree of Q which is obtained contains the edges

$$BE, CE, AE, DF, BD$$

2.9 Traversing Binary Tree

There are three standard ways of traversing a binary tree T with root R. these three algorithm called pre order, in order and post order.

2.9.1 Pre order

1. Process the root R.

2. Traverse the left subtree of R in pre order.

3. Traverse the right subtree of R in pre order.

2.9.2 In order

1. Traverse the left subtree of R in in order.

2. Process the root R.

3. Traverse the right subtree of R in in order.

2.9.3 Post order

1. Traverse the left subtree of R in post order.

2. Traverse the right subtree of R in post order.

3. Process the root R.

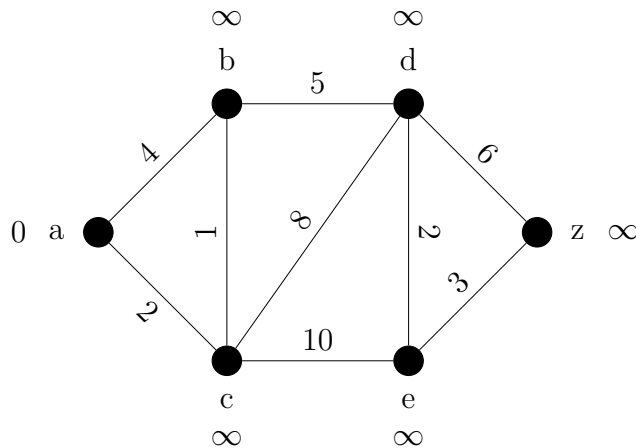**Example.** Traverse the tree of the following figure.
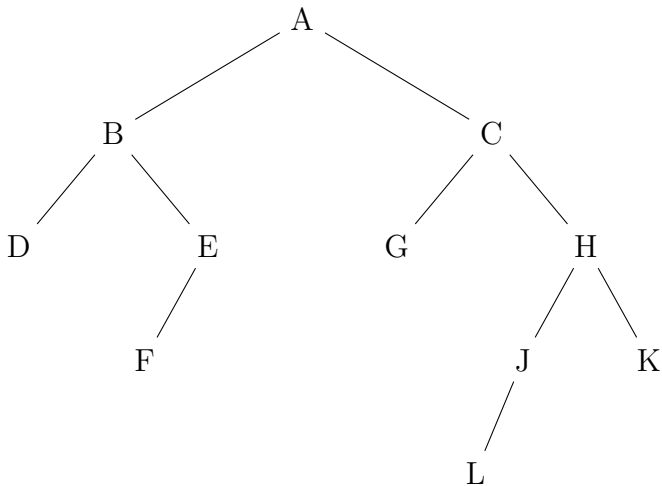
Preorder traversal = A, B, D, E, F, C, G, H, J, L, K
Inorder traversal = D, B, F, E, A, G, C, L, J, H, K
Postorder traversal = D, F, E, B, G, L, J, K, H, C, A

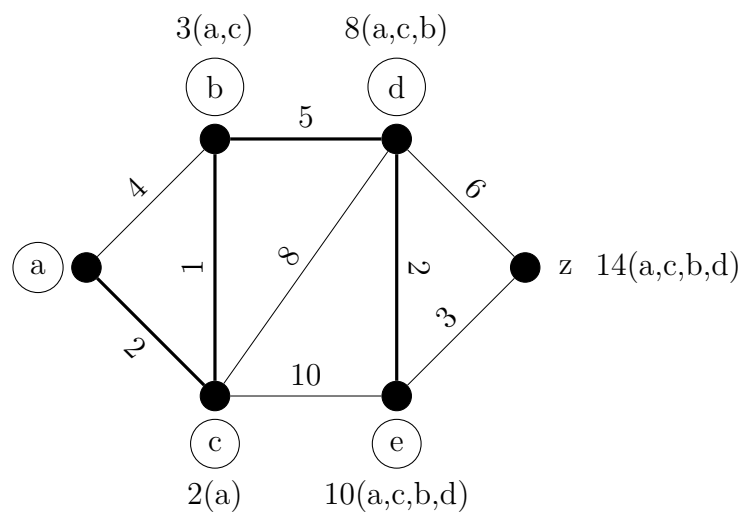**Example.** Using Dijkstra's algorithm to find shortest path from a to z.

step 1:

step 2:

4(a)

∞

b

d

5

4

1

8

2

6

selected → a

2

10

z ∞

3

2

c

e

2(a)

∞

step 3:

3(a,c)

10(a,c)

b

d

5

4

1

8

2

6

a

2

10

z ∞

3

c

e

2(a)

12(a,c)

step 4:

3(a,c)

8(a,c,b)

b

d

5

4

1

8

2

6

a

2

10

z ∞

3

c

e

2(a)

12(a,c)

step 5:

23

3(a,c) 8(a,c,b)

b d

5

4

a 1 8 2 z 14(a,c,b,d)

6

3

2

10

c e

2(a) 10(a,c,b,d)

step 6:

3(a,c) 8(a,c,b)

b d

5

4

a 1 8 2 z 14(a,c,b,d)

6

3

2

10

c e

2(a) 10(a,c,b,d)

step 7:

3(a,c) 8(a,c,b)

b d

5

4

a 1 8 2 z 13(a,c,b,d,e)

6

3

2

10

c e

2(a) 10(a,c,b,d)
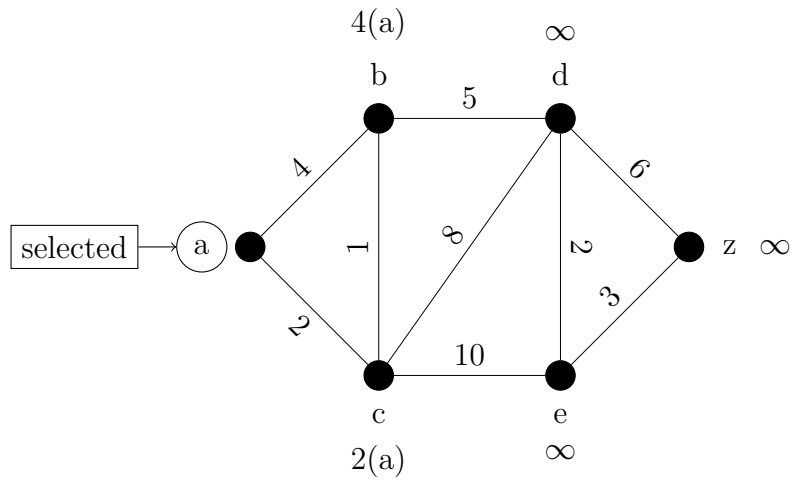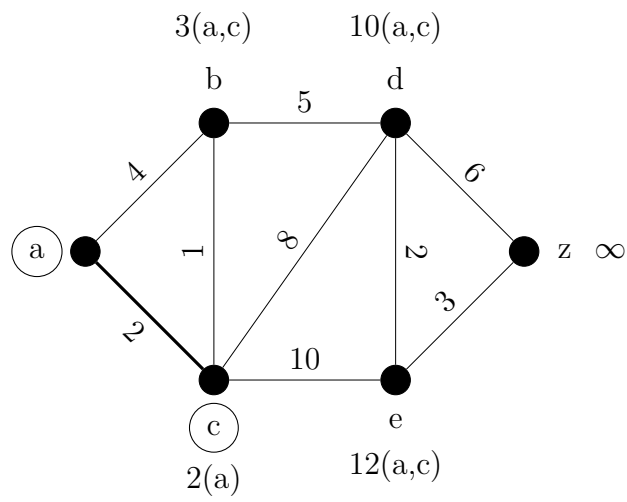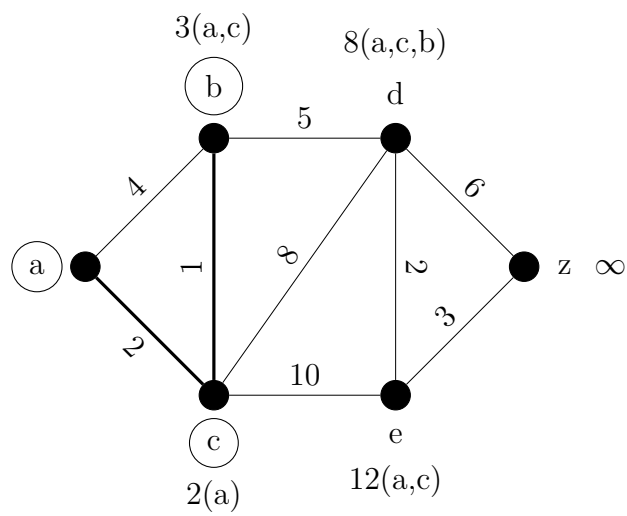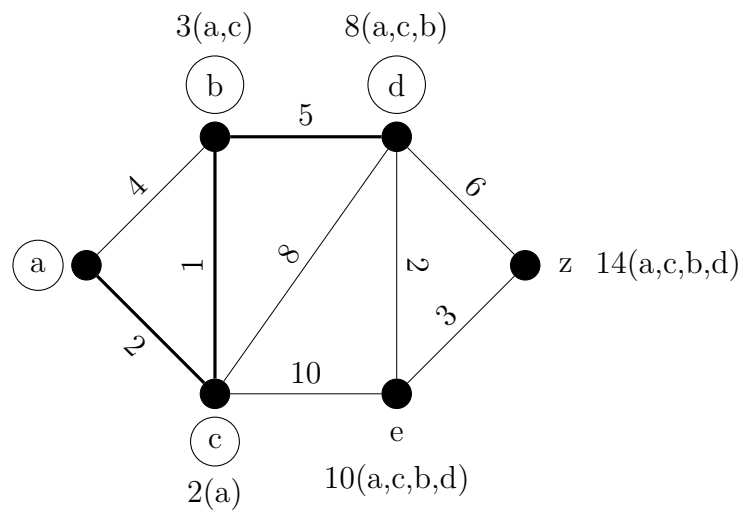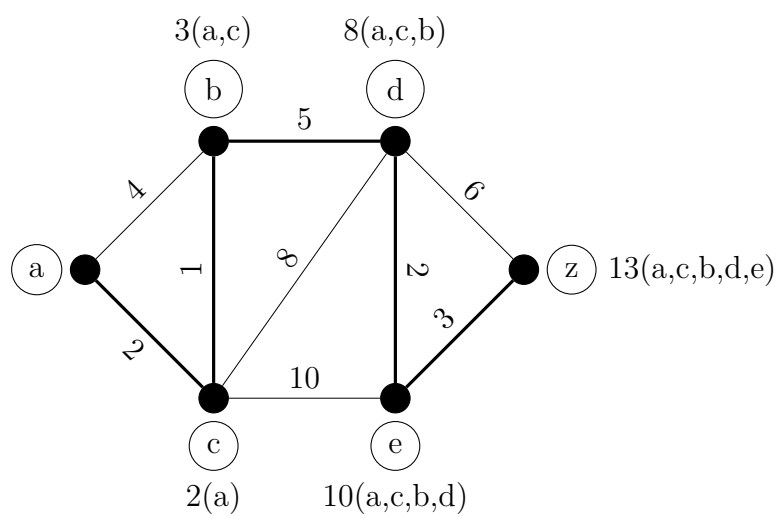
The shortest path: $a \to c \to b \to d \to e \to z$

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2.10   Graphical Representation of an Expression

In compiler construction an expression such as $'(x - y) * (w + z) * (x - y) * (w - z)'$ can be represented by the directed acyclic graph[1].
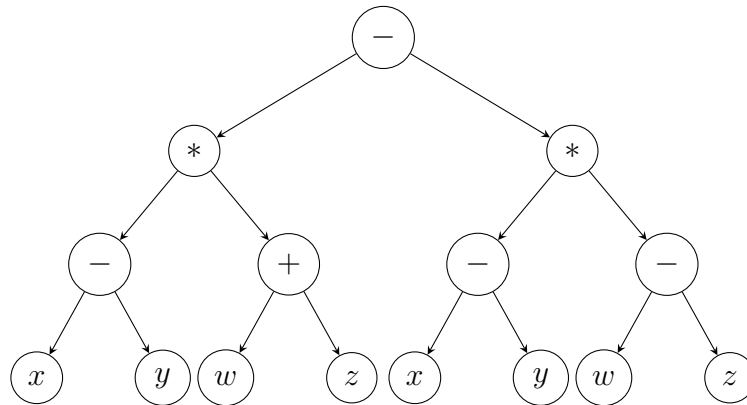
Example. Draw the tree which corresponds to the expression $E = (2x + y)(5a - b)^3$

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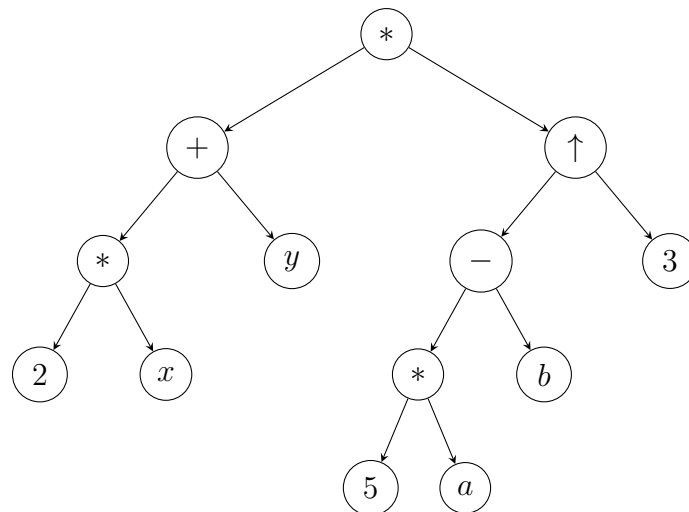
[1]See tree traversal, infix postfix expression, expression tree.