Chapter 1

Fuzzy Topology

Definition 1 (Fuzzy Topology)**.** Let X be a non-empty set. A collection δ of fuzzy sets on X is called the fuzzy topology on X if it satisfies the following conditions:

(i) $\underline{0}$, $\underline{1} \in \delta$.

(ii) If $A, B \in \delta$, then $A \wedge B \in \delta$.

(iii) If $A_i \in \delta$, then $\vee_{i \in I} A_i \in \delta$.

If δ is a topology on X then, $\langle \mathcal{F}(X), \delta \rangle$ is called a fuzzy topological space.

Example. Let $X = \{a, b\}$ and A be a fuzzy set defined by $A(a) = 0.5$ and $A(b) = 0.4$. Then $\delta = \{\underline{0}, \underline{1}, A\}$ be a fuzzy topology and $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space.

Example. Let A, B be a fuzzy sets of $I = [0, 1]$ defined as

$$A(x) = \begin{cases} 0; & \text{if } 0 \leq x \leq \dfrac{1}{2} \\ 2x - 1; & \text{if } \dfrac{1}{2} \leq x \leq 1 \end{cases} \qquad \text{and} \qquad B(x) = \begin{cases} 1; & \text{if } 0 \leq x \leq \dfrac{1}{4} \\ -4x + 2; & \text{if } \dfrac{1}{4} \leq x \leq \dfrac{1}{2} \\ 0; & \text{if } \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

Then $\delta = \{\underline{0}, \underline{1}, A, B, A \vee B\}$ is a fuzzy topology on I.

Definition 2 (Open and CLosed Fuzzy Sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, the member of δ i.e., each $A \in \delta$ is called the fuzzy open set. A fuzzy set B is called a fuzzy closed set if $B^c \in \delta$.

Example. Let $X = \{a, b\}$, $B : X \to [0, 1]$ such that $B(a) = 0.5$, $B(b) = 0.6$. Then, $B^c(a) = 0.5$, $B^c(b) = 0.4$, $\delta = \{\underline{0}, \underline{1}, A\}$, $A(a) = 0.5$, $A(b) = 0.4$.
$\therefore$ B is closed under $\delta/\delta-$closed. i.e., B^c is open.

Difference between classical and fuzzy sets: Classical set contains elements that satisfy precise properties of membership while fuzzy set contains elements that satisfy imprecise properties of membership.

Definition 3 (Interior and Closure of fuzzy sets)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological spaceand A be a non-empty subset of X.

The interior of A is denoted by A° and defined as the union of all open sets contained in A. i.e., $A^\circ = \cup\{G \in \delta | G \leq A\}$. (Largest open set contained in A).

The closure of A is denoted by $\bar{A}$ and defined as the intersection of all closed sets containing A. i.e., $\bar{A} = \cap\{F | F^c \in \delta \text{ and } A \leq F\}$. (Smallest closed set containing A).

Example. Consider, $X = \{a, b, c\}$ and

$$\begin{aligned} A &: \ a \mapsto 0.2,\ b \mapsto 0.4,\ c \mapsto 0.8 \\ B &: \ a \mapsto 0.4,\ b \mapsto 0.6,\ c \mapsto 0.8 \\ C &: \ a \mapsto 0.6,\ b \mapsto 0.8,\ c \mapsto 1.0 \end{aligned}$$

Then, $\delta = \{\underline{0}, \underline{1}, A, B, C\}$ be a fuzzy topology on X. Here $U : X \to [0, 1]$ and $U : a \mapsto 0.8$, $b \mapsto 0.7$, $c \mapsto 0.8$. Find U° and $\bar{U}$.

Solution. 1. We know that, $U^\circ = \cup\{G \in \delta : g \leq U\} = \cup\{\underline{0}, A, B\} = B$. Since, $\underline{0} \leq A \leq B$.

2. At first, $A^c : a \mapsto 0.8,\ b \mapsto 0.6,\ c \mapsto 0.2$
$B^c : a \mapsto 0.6,\ b \mapsto 0.4,\ c \mapsto 0.2$
$C^c : a \mapsto 0.4,\ b \mapsto 0.2,\ c \mapsto 0.0$
$\underline{0}^c = \underline{1}$ and $\underline{1}^c = \underline{0}$
We know that $\bar{U} = \cap\{F | F^c \in \delta \text{ and } U \leq F\} = \underline{1}$.

Theorem 1.0.1. Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then, the following conditions hold:

(i) $\underline{0}^\circ = \underline{0}$ and $\underline{1}^\circ = \underline{1}$

(ii) $\forall A \in \mathcal{F}(X),\ A^\circ \leq A$

(iii) $\forall A \in \mathcal{F}(X),\ A^{\circ\circ} = A^\circ$

(iv) for $A, B \in \mathcal{F}(X)$ with $A \leq B$ implies $A^\circ \leq B^\circ$

(v) for $A, B \in \mathcal{F}(X)$, $(A \wedge B)^\circ = A^\circ \wedge B^\circ$

Proof. (i) By definition, $\underline{0}^\circ = \cup\{G \in \delta | G \leq \underline{0}\} = \underline{0}$ and $\underline{1}^\circ = \cup\{G \in \delta | G \leq \underline{1}\} = \underline{1}$

(ii) By definition, $A^\circ = \cup\{G \in \delta | G \leq A\}$. Since, the arbitrary union of open sets is open, A° is the open set of $\mathcal{F}(X)$ and also, A° is the largest open set contained in A. $\therefore\ A^\circ \leq A$.

(iii) From (ii), $A^\circ \leq A \ \Rightarrow A^{\circ\circ} \leq A^\circ$. But A° is the largest open set contained in A. So, $A^\circ \leq A^{\circ\circ}$. Hence, $A^{\circ\circ} = A^\circ$.

(iv) Let $A, B \in \mathcal{F}(X)$ such that $A \leq B$. Now, since $A^\circ \leq A$, hence $A^\circ \leq B$. But B° is the set of all open sets contained in B. So, $B^\circ \leq B$. Therefore, $A^\circ \leq B^\circ$.

(v) Let $A, B \in \mathcal{F}(X)$. Then,

$$\begin{aligned} & A^\circ \leq A,\ B^\circ \leq B \\ \Rightarrow\ & A^\circ \wedge B^\circ \leq\ A \wedge B \\ \Rightarrow\ & (A^\circ \wedge B^\circ)^\circ \leq\ (A \wedge B)^\circ \end{aligned} \tag{1.1}$$

Here, A° is the largest open set contained in A and B° is the largest open set contained in B. Hence, $A^\circ \wedge B^\circ$ is also an open set of X. So, $(A^\circ \wedge B^\circ)^\circ \leq\ (A \wedge B)^\circ$.
From (1.1),

$$A^\circ \wedge B^\circ \leq\ (A \cap B)^\circ \tag{1.2}$$

Again, Since,

$$\begin{aligned} & A \wedge B \leq A,\ B \\ \Rightarrow\ & (A \wedge B)^\circ \leq A^\circ,\ B^\circ \\ \Rightarrow\ & (A \wedge B)^\circ \leq A^\circ \wedge B^\circ \end{aligned} \tag{1.3}$$

From, (1.2) and (1.3), $A^\circ \wedge B^\circ = (A \wedge B)^\circ$.

□

Note. If A be a fuzzy open set of the topological space $\langle X, \delta \rangle$, then $A^\circ = A$, $\bar{A} = A$ iff A is closed.

Definition 4 (Fuzzy Point)**.** A fuzzy set x_a on X is called a fuzzy point on X if $\forall y \in X$,

$$x_a(y) = \begin{cases} a; & \text{if } x = y \\ 0; & \text{if } x \neq y \end{cases} \quad ; \qquad \text{where, } 0 < a \leq 1$$

The set of all fuzzy points on X is denoted by $P(X)$. The fuzzy points x_{1-a} is called the dual point of the fuzzy points x_a.

Example. $X = [0,1]$m where $X = \{x, y, z\}$. We need to find $x_a(y)$ where $y \in X$.

$$\begin{array}{llll} x_a : x \to a & y_a : x \to 0 & z_a : x \to 0 & \text{dual of } x_a,\ x_{1-a} : x \to (1-a) \\ \quad y \to 0 & \quad y \to a & \quad y \to 0 & \qquad\qquad\quad y \to 0 \\ \quad z \to 0 & \quad z \to 0 & \quad z \to a & \qquad\qquad\quad z \to 0 \end{array}$$

Definition 5 (Neighborhood of a fuzzy point)**.** Let $\langle X, \delta \rangle$ be a fuzzy topological space and $x_a \in P(X)$. Then $U \in \delta$ is called a fuzzy neighborhood of x_a if $x_a \in U$.
The set of all fuzzy neighborhood of x_a is denoted by $\mathcal{N}_\delta(x_a)$.

Example. $X = \{a, b, c\}$, $\delta = \{\underline{0}, \underline{1}, A, B\}$, $A : a \to 0.0$, $B : a \to 0.2$. Find the neighborhood of $a_{0.4}$, $b_{0.7}$, $c_{0.8}$.

$$\begin{array}{ll} b \to 0.2 & b \to 0.4 \\ c \to 0.7 & c \to 0.8 \end{array}$$

Solution.

1. $a_{0.4} : a \to 0.4$; Fuzzy neighborhood of $a_{0.4} : \{\underline{1}\}$.
   $b \to 0.0$
   $c \to 0.0$

2. $b_{0.7} : a \to 0.0$; Fuzzy neighborhood of $b_{0.7} : \{\underline{1}\}$.
   $b \to 0.7$
   $c \to 0.0$

3. $c_{0.8} : a \to 0.0$; Fuzzy neighborhood of $c_{0.8} : \{B, \underline{1}\}$.
   $b \to 0.0$
   $c \to 0.8$

Theorem 1.0.2. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $A \subseteq X$. Then a fuzzy point $x_a \in A^\circ \Leftrightarrow x_a$ has a neighborhood U such that $U \subseteq A$.

Proof. Suppose, $x_a \in A^\circ$. By the definition of A°, $A^\circ = \cup\{G \in \delta | G \subseteq A\}$. $\therefore$ $x_a \in \cup\{G \in \delta | G \subseteq A\}$. Thus we have $x_a \in U$ for some $U \in \delta \ni U \subseteq A$. $\therefore$ There exists a neighborhood U of x_a such that $U \subseteq A$.

Conversely, suppose, U be a neighborhood of a fuzzy point $x_a \ni U \subseteq A$. This implies, $x_a \in U \subseteq A$. Now, since A° is the largest open set contained in A, we have $U \subseteq A^\circ$. Thus, $x_a \in A^\circ$. ☐

Definition 6 (Quasi-Coincident of a fuzzy point)**.** Let $\langle X, \delta \rangle$ be a fuzzy topological space. A fuzzy point x_a is called quasi-coincident of a fuzzy set A denoted by $x_a \propto A$ iff $x_a \not\leq A^c$ i.e., $a > A^c(x) \Rightarrow a + A(x) > 1$,

Definition 7 (Quasi-Coincident of a fuzzy set)**.** A fuzzy set A is said to be quasi-coincident with a fuzzy set B iff there exists an $x \in X$ such that $A(x) > B^c(x)$ i.e., $A(x) + B(x) > 1$ for some $x \in X$.

Definition 8 (Quasi-neighborhood)**.** An open set $U \in \delta$ is called a quasi-neighborhood of a fuzzy point x_a if x_a is a quasi-coincident of U. The set of all quasi-coincident of x_a is denoted by $\mathcal{Q}_\delta(x_a)$.

Example. Consider, $X = \{a, b, c\}$, $\delta = \{\underline{0}, \underline{1}, A, B\}$,

$$\begin{aligned} A :&\ a \mapsto 0.0,\ b \mapsto 0.2,\ c \mapsto 0.7 \\ B :&\ a \mapsto 0.6,\ b \mapsto 0.4,\ c \mapsto 0.8 \\ \text{Given, } P :&\ a \mapsto 0.0,\ b \mapsto 0.4,\ c \mapsto 0.9 \end{aligned}$$

Find the quasi-neighborhood of x_a at $a = 0.4$.

Solution. Here, $B^c :\ a \mapsto 0.4,\ b \mapsto 0.6,\ c \mapsto 0.2$. Since, $a = 0.4 \geq B^c(a) = 0.4$ so, $x_{0.4}$ is a quasi-coincident of B and $\mathcal{Q}_\delta(x_a) = \{\underline{1}, B\}$.

Theorem 1.0.3. A quasi-neighborhood of x_a is exactly a neighborhood of x_{1-a}.

Proof. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $U \in \delta$ be a quasi-neighborhood of x_a. By the definition of quasi-neighborhood of x_a,

$$\begin{aligned} & a > U^c(x), \quad \text{for some } x \in X, \\ \Leftrightarrow\, & a > 1 - U(x), \quad \text{for some } x \in X, \\ \Leftrightarrow\, & a + U(x) > 1, \quad \text{for some } x \in X, \\ \Leftrightarrow\, & 1 - a < U(x), \quad \text{for some } x \in X, \\ \Leftrightarrow\, & x_{1-a} \in U, \\ \Leftrightarrow\, & U \text{ is a neighborhood of } x_{1-a} \end{aligned}$$

□

Proposition 1. Let, $\langle X, \delta \rangle$ be a fuzzy topological space and $A, B \subseteq X$. Then $A \leq B$ iff A and B^c are not quasi-coincident.

Proof. Suppose, $A \leq B$, then, $A(x) \leq B(x)$, for all $x \in X$.
Now, $A(x) + B^c(x) = A(x) + 1 - B(x) \leq 1$, for all $x \in X$ [Since, $A(x) \leq B(x)$]
Hence, A and B^c are not quasi-coincident.

Conversely, suppose $A(x)$ and $B^c(x)$ are not quasi-coincident. Then,

$$\begin{aligned} & A(x) + B^c(x) \leq 1 \\ \Rightarrow\, & A(x) + 1 - B(x) \leq 1 \\ \Rightarrow\, & A(x) - B(x) \leq 0 \\ \Rightarrow\, & A(x) \leq B(x) \end{aligned}$$

□

Theorem 1.0.4. Let $\langle X, \delta \rangle$ be a fuzzy topological space and $A \in \mathcal{F}(X)$. Then, the following conditions hold:

1. $x_a \in A^\circ$ iff $x_{1-a} \notin \bar{A}^c$.
2. $x_a \in \bar{A}$ iff each neighborhood of its dual point x_{1-a} is quasi-coincident with A.

Proof.

1. Let $x_a \in A^\circ$. Then by definition of A°, there exists $B \in \delta$ such that $x_a \in B \subseteq A$ i.e., B is a neighborhood of x_a and hence B is a quasi-neighborhood of x_{1-a}. Hence $x_{1-a} \nleq B^c$ i.e., $x_{1-a} \notin B^c$. Since, $B \subseteq A$ and $\bar{A}$ is the smallest closed set containing A, we have, $B \subseteq A \subseteq \bar{A}$ implies $\bar{A}^c \subseteq B^c$. Hence we can show that $x_{1-a} \notin \bar{A}^c$.

   Conversely, suppose $x_{1-a} \notin \bar{A}^c$. Then there is a neighborhood B of x_a which is not quasi-coincident with A^c. Thus,

   $$\begin{aligned} & B(x) + A^c(x) \leq 1 \quad \forall x \in X \\ \Rightarrow\, & B(x) \leq A(x) \quad \forall x \in X \end{aligned}$$

   $\therefore$ $B^c \subseteq A$ and so $x_a \in B \subseteq A$ i.e., $x_a \in A^\circ$.

2. Let N be the neighborhood of x_{1-a}. Now, N is a quasi-coincident with A implies

   $$\begin{aligned} & N(x) + A(x) > 1, \quad \forall x \in X \\ \Rightarrow\, & N \text{ and } A \text{ intersect at } x \\ \Rightarrow\, & x_a \in \bar{A} \end{aligned}$$

   Conversely, suppose $x_a \in \bar{A}$. The, N and A intersect at x. This implies,

   $$\begin{aligned} & N(x) + A(x) > 1, \quad \forall x \in X \\ \Rightarrow\, & N \text{ is a quasi-coincident with } A \text{ at } x \\ \Rightarrow\, & \text{each neighborhood } N \text{ of } x_{1-a} \text{ is quasi-coincident with } A \end{aligned}$$

□

Definition 9 (Subspace)**.** Let $\langle X, \delta \rangle$ be a fuzzy topological space and $Y \subseteq X$, $Y \neq \varnothing$. Define $\delta_{\restriction_Y} = \{U_{\restriction_Y} | U \in \delta\}$. Then $\delta_{\restriction_Y}$ is a fuzzy topology on Y. The fuzzy topological space $\langle Y, \delta_{\restriction_Y} \rangle$ is called a subspace of $\langle X, \delta \rangle$.

Example. Let, $X = \{a, b, c\}$ and $Y = \{b, c\}$. Let $\delta = \{\underline{0}, \underline{1}, A, B\}$ where

$$\begin{aligned} A: \; & a \mapsto 0.2, \; b \mapsto 0.4, \; c \mapsto 1.0 \\ B: \; & a \mapsto 0.1, \; b \mapsto 0.4, \; c \mapsto 0.8 \end{aligned}$$

Then $\delta_{\restriction_Y} = \{\underline{0}_{\restriction_Y}, \underline{1}_{\restriction_Y}, A_{\restriction_Y}, B_{\restriction_Y}\}$ where,

$$\begin{aligned} A_{\restriction_Y}: \; & b \mapsto 0.4, \; c \mapsto 1.0 \\ B_{\restriction_Y}: \; & b \mapsto 0.4, \; c \mapsto 0.8 \end{aligned}$$

is a fuzzy topology on Y and hence $\langle Y, \delta_{\restriction_Y} \rangle$ is a fuzzy subspace of $\langle X, \delta \rangle$.

Example. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 3, 4\}$. Find a non-trivial fuzzy topology on X and hence, find a fuzzy subspace of $\langle X, \delta \rangle$.

Solution. Let $\delta = \{\underline{0}, \underline{1}, A, B\}$ be a fuzzy topology on X where,

$$\begin{aligned} A: \; & 1 \mapsto 0.3, \; 2 \mapsto 0.1, \; 3 \mapsto 0.6, \; 4 \mapsto 0.2 \\ B: \; & 1 \mapsto 0.7, \; 2 \mapsto 0.4, \; 3 \mapsto 0.1, \; 4 \mapsto 0.2 \end{aligned}$$

Then $\delta_{\restriction_Y} = \{\underline{0}_{\restriction_Y}, \underline{1}_{\restriction_Y}, A_{\restriction_Y}, B_{\restriction_Y}\}$ where,

$$\begin{aligned} A_{\restriction_Y}: \; & 1 \mapsto 0.3, \; 3 \mapsto 0.6, \; 4 \mapsto 0.2 \\ B_{\restriction_Y}: \; & 1 \mapsto 0.7, \; 3 \mapsto 0.1, \; 4 \mapsto 0.2 \end{aligned}$$

is a fuzzy topology on Y and hence $\langle Y, \delta_{\restriction_Y} \rangle$ is a fuzzy subspace of $\langle X, \delta \rangle$.

Remark. Let $\langle X, \tau \rangle$ be a fuzzy topological space. The two fuzzy sets A and B in X are said to be intersecting $\Leftrightarrow$ there exists a point $x \in X$ such that $(A \wedge B)(x) \neq 0$.
For such a case, we say that, A and B intersect at x.

Again, if A and B are quasi-coincident at x, then, $A(x) + B(x) > 1$ i.e., both $A(x)$ and $B(x)$ are not zero and here A and B intersect at x.

- $x_a \to$ quasi-coincident of A if $a > A^c(y)$ for some $y \in X$.
- $U \in \delta \to$ quasi-neighborhood if x_a is a quasi-coincident of U.

Definition 10 (Adherent point)**.** A fuzzy point x_a is called an adherent point of a fuzzy set A iff every quasi-neighborhood of x_a is a quasi-coincident with A.

Problem 1.1. Give an example of an adherent point.

Definition 11 (Accumulation Point)**.** A fuzzy point x_a is called an accumulation point of a fuzzy set A iff x_a is an adherent point of A and every quasi-neighborhood of x_a and A are quasi-coincident at some point different from sup x_a, whenever, $x_a \in A$.

Definition 12 (Base)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $\mathcal{B}$ is a base for δ iff every open set $G \in \delta$ is the union of members of $\mathcal{B}$ i.e., $G = \cup B_i$, $\forall \, B_i \in \mathcal{B}$.

Definition 13 (Subbase)**.** Let $\langle \mathcal{F}(X), \delta \rangle$ be a fuzzy topological space. Then $S \in X$ is called a subbase iff finite intersection of member of $\mathcal{S}$ form a base for δ.