Theory of Relativity and Cosmology

MAT518

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Preface

This is a compilation of lecture notes with some books and my own thoughts. If there are any mistake/typing error or, for any query mail me at mehedi12@student.sust.edu.

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Special Relativity

1.1 Introduction

In macroscopic world, $u << c$, where $u =$ speed of moving objects or mechanical waves with respect to any observer, $c =$ speed of light ($3.00 \times 10^{10} m/sec$).
For example, $u \to$ speed of an artificial satellite circling the earth, then $\frac{u}{c} = .00027 << 1$; or $u \to$ speed of sound waves, then $\frac{u}{c} = .0000010 << 1$.
In this macroscopic environment, Newton developed his system of mechanics.
In the microscopic world, it is possible to find particles whose speeds are quite close to that of light.

Example. For an electron accelerated through a 10 million volt potential difference, the speed u is equals to $0.9988c$ that is,

$$\frac{u}{c} = 0.9988 \approx 1$$

Energy of this electron is

$$E = 10MeV$$

According to Newton mechanics,

$$E = \frac{1}{2}mu^2$$

If u is replaced by $2u$ then,

$$\frac{1}{2}m(2u)^2 = 4 \cdot \frac{1}{2}mu^2 = 4E$$

Let us now increase the energy of 10 MeV electron to $4 \times 10MeV = 40MeV$. Then the speed of the electron will be double to

$$\begin{aligned} 2u &= 2 \times 0.9988c \\ &= 1.9976c \end{aligned}$$

as we might expect from the Newtonian mechanics, but remains below c; it increases only from $0.9988c$ to $0.9999c$.
Hence, Newtonian mechanics work at low speeds, it fails badly as $\frac{u}{c} \to 1$.

In 1905 Albert Einstein published his special theory of relativity. In his theory, he extended and generalized Newtonian mechanics as well. He correctly predicted the results of mechanical experiments over the complete range of speed from $\frac{u}{c} = 0$ to $\frac{u}{c} \to 1$.

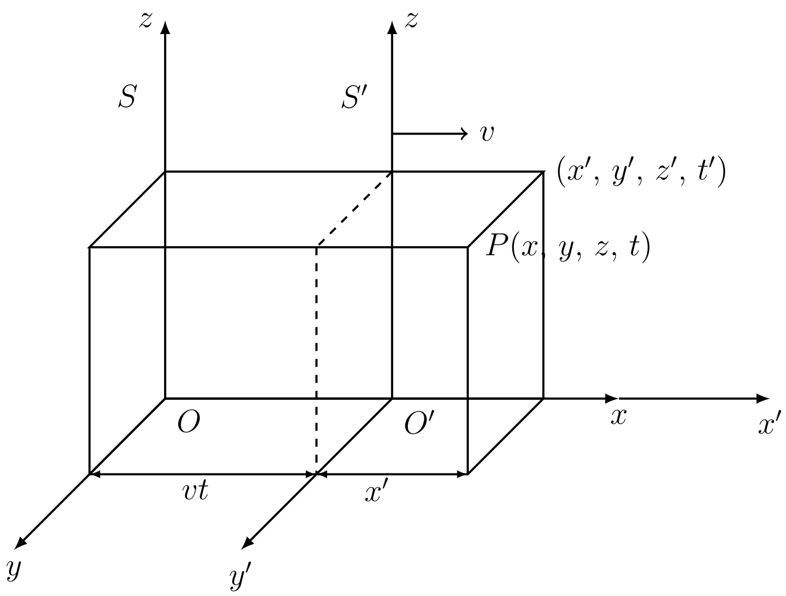
1.1.1 Some Basic Concepts

Frame of Reference: A system which is used to describe any event.
Event: Which happens independently of any reference frame and is described by four (space-time) measurements (x, y, z, t).
Inertial Frame: In which law of inertia - Newton's first law holds.
Newton's first law: A free particle is either at rest or moves along a straight line with constant velocity. Every unaccelerated frame is an inertial frame.
Inertial observer: Observer attached to inertial frame.

1.2 Galilean Transformations

Consider two frames of reference S and S', where the frame S is at rest/fixed and S' moves at a constant velocity v with respect to S (along $x - x'$ axis).

The origins of the two frames of reference are so chosen that they coincide at $t = t' = 0$. For convenience, we choose the three sets of axes to be parallel and allow their relative motion to be along the common x, x' axis.



$$
\begin{aligned}
S, S' &\rightarrow \text{ two inertial frames} \\
S &\rightarrow \text{ fixed} \\
S' &\rightarrow \text{ moves at constant velocity with respect to } S \text{ along } x - x' \text{ axis}
\end{aligned}
$$

S, S' coincide at $t = 0$, $t' = 0$

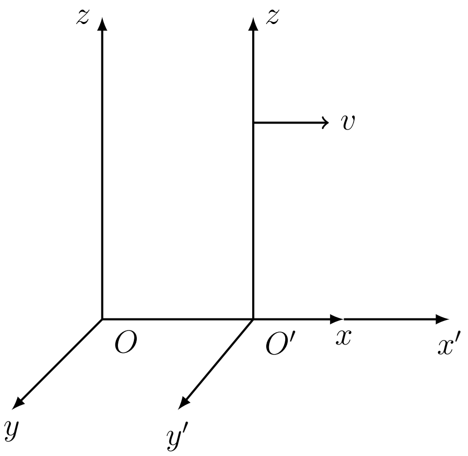
Let an event occur at P. An observer attached to S specifies the location and time of occurrence of this event as (x, y, z, t). Another observer attached to S' specifies the same event as (x', y', z', t'). According to the classical physics, the length intervals and time intervals are absolute. This gives,

$$
\left.
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
\right\} \tag{1.1}
$$

This is known as Galilean coordinate transformations.

Note. We can regard S to moving velocity $-v$ with respect to S' as we can regard S' to move with velocity v with respect to S.
length $\rightarrow$ meter sticks
clock $\rightarrow$ synchronized.

1.2.1 Newton's Laws are covariant under Galilean Transformation



Consider two events at points P and Q. In $S'-$ frame, time interval between occurrence of P and Q is

$$t'_P - t'_Q$$

In $S-$frame, time interval is

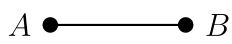
$$t_P - t_Q$$

But $t' = t$

$$\therefore\ t_P - t_Q = t_P - t_Q$$

i.e., time interval is same for both inertial frames.

Consider now a rod AB at rest in $S-$ frame.



In $S-$frame, end points measurements are x_A and x_B.
In $S'-$frame, they are x'_A and x'_B.
Transformation law implies that

$$x'_B = x_B - vt_B$$
$$x'_A = x_A - vt_A$$

Length of the rod,

$$x'_B - x'_A = x_B - x_A - v(t_B - t_A)$$

But t_B and t_A are simultaneous. i.e., $t_B = t_A$,

$$\therefore\ x'_B - x'_A = x_B - x_A$$

i.e., space interval between two points, say A and B, measured at a given instant, is the same for each observer. Classical mechanics assumes that mass of a body is constant, independent of its motion with respect to an observer.

Thus classical mechanics and Galilean transformations imply that length, mass and time - the three basic quantities in mechanics are all independent of the relative motion of the observer.

1.2.2 Velocity Transformation

We have

$$x' = x - vt$$

$$\therefore \frac{\mathrm{d}\,x'}{\mathrm{d}\,t} = \frac{\mathrm{d}\,x}{\mathrm{d}\,t} - v \qquad \text{by differentiating with respect to } t$$

Since $t' = t$, $\frac{\mathrm{d}}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,t'}$, according to Galilean transformation

$$\therefore \frac{\mathrm{d}\,x'}{\mathrm{d}\,t'} = \frac{\mathrm{d}\,x'}{\mathrm{d}\,t} = \frac{\mathrm{d}\,x}{\mathrm{d}\,t} - v \qquad \left[\text{For uniform velocity, } v \to \text{constant, } \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = 0\right]$$

Similarly,

$$\frac{\mathrm{d}\,y'}{\mathrm{d}\,t'} = \frac{\mathrm{d}\,y}{\mathrm{d}\,t}$$

and

$$\frac{\mathrm{d}\,z'}{\mathrm{d}\,t'} = \frac{\mathrm{d}\,z}{\mathrm{d}\,t}$$

Hence,[1]

$$u'_x = u_x - v$$
$$u'_y = u_y$$
$$u'_z = u_z$$

These are the transformation of the velocity component which is the simple classical velocity addition.

In more general, $\underline{u}' = \underline{u} - \underline{v}$, $\underline{v} \to$ relative velocity of the frames.

1.2.3 Acceleration Transformation

We know that from velocity transformation $u'_x = u_x - v$. Now differentiating bothsides with respect to t, we get

$$\frac{\mathrm{d}\,u'_x}{\mathrm{d}\,t'} = \frac{\mathrm{d}}{\mathrm{d}\,t'}(u_x - v)$$

$$\Rightarrow \frac{\mathrm{d}\,u'_x}{\mathrm{d}\,t} = \frac{\mathrm{d}\,u_x}{\mathrm{d}\,t}, \quad v \to \text{constant velocity, and } t' = t \ \therefore \frac{\mathrm{d}}{\mathrm{d}\,t'} = \frac{\mathrm{d}}{\mathrm{d}\,t}$$

[1] Now,

$$\frac{\mathrm{d}\,x'}{\mathrm{d}\,t'} = u'_x, \quad \text{is the } x\text{−component of the velocity measured in } S'$$
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = u_x, \quad \text{is the } x\text{−component of the velocity measured in } S \text{ and so on.}$$

Similarly,

$$\frac{\mathrm{d}\,u'_y}{\mathrm{d}\,t'} = \frac{\mathrm{d}\,u_y}{\mathrm{d}\,t'}$$

and

$$\frac{\mathrm{d}\,u'_z}{\mathrm{d}\,t'} = \frac{\mathrm{d}\,u_z}{\mathrm{d}\,t'}$$

i.e.,

$$a'_x = a_x$$
$$a'_y = a_y$$
$$a'_z = a_z$$

Hence, $\underline{\mathrm{a}}' = \underline{\mathrm{a}}$ under Galilean transformations.[2]

In classical physics, the mass is also unaffected by the motion of reference frame.
Hence $m\underline{a}$ will be the same for all inertial observers.
According to Newton's law of motion the force of a particle is given by
in $S-$frame, force $\underline{\mathrm{F}} = m\underline{\mathrm{a}}$
and in $S;-$frame, force $\underline{\mathrm{F}}' = m\underline{\mathrm{a}}'$ But $\underline{\mathrm{a}} = \underline{\mathrm{a}}'$, $\therefore$ $\underline{\mathrm{F}} = \underline{\mathrm{F}}'$, i.e., force is the same for all inertial frames.
Hence, Newton's laws of motion are exactly the same in all inertial frames.

In mechanics, the conservation principles - such as those for energy, linear momentum and angular momentum - are all consequences of Newton's laws. Hence, the laws of mechanics are the same in all inertial frames.

- Mechanical experiments carried out in one inertial frame can not tell the observer what motion of that frame is with respect to any other inertial frame.
- By comparing measurements in two inertial frames, we can tell the relative velocity between them.
- There is no way at all of determining the absolute velocity of an inertial reference frame from mechanical experiments.

1.2.4 No universal reference frame exists at absolute rest

No inertial frame is preferred over any other, because the laws of mechanics are the same in all. Hence, there is no physically definable absolute rest frame. We say that all inertial frames are equivalent as far as mechanics is concerned.

The person riding the train cannot tell absolutely whether he alone is moving, or the earth alone is moving past him, or if some combination of motions is involved. Indeed, would one say that one on earth is at rest, that one is moving 30 km/sec (the speed of earth in its orbit about the sun) or that

[2]Which shows that the acceleration of particle is the same in all reference frames which move relative to one another with constant velocity. That means acceleration is invariant.

one's speed is much greater still (for instance the sun's speed in its orbit about the galactic center)? Actually, no mechanical experiment can be performed which will detect an absolute velocity through empty space.

We can only speak of the relative velocity of one frame with respect to another, and not of an absolute velocity of a frame. This is sometimes called Newtonian relativity.

1.3 Electromagnetism and Newton Relativity

The laws of physics include the

(i) laws of mechanics,

(ii) laws of electromagnetism.

The laws of mechanics are invariant under Galilean transformation. We inquire now whether the laws of electromagnetism are invariant under Galilean transformation.

Consider a pulse of light (i.e., an electromagnetic pulse) traveling to the right with respect to the medium through which it is propagated at a speed c. The "medium" of light propagation was given the name "ether",

Historically when the mechanical view if physics dominated physicists thinking (late 10th century and early 20th century) it was not really accepted that an electromagnetic disturbance could be propagated in empty space.

Let S denote the ether frame and regard it as an inertial one, for simplicity.
In $S-$frame an observer measures the speed of light to be exactly $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.997925 \times 10^8 m/sec.$
$S'-$frame is moving at a constant speed v with respect to $S-$frame.
In $S'-$frame an observer would measure the speed of light pulse as

$$c + v \quad \text{or} \quad c - v$$

according to the direction of relative motion by the Galilean velocity transformation.

Hence, the speed of light is not invariant under a Galilean transformation.

.... is the velocity of propagation of light in vacuum. Hence, Maxwell's equations are not invariant under the Galilean transformations.

In the Galilean transformations really do apply to optical or electromagnetic phenomena, then there is one inertial system and only one, in which the measured speed of light is exactly c; that is, there is a unique inertial system in which the so-called ether is of rest. We would then have an absolute (or rest) frame and get a way of determining the relative velocity of some other frame with respect to the absolute frame by optical experiments performed in that frame. This contradicts with the Newtonian relativity which states that there is no physically definable absolute last frame.

<u>Findings:</u>

1. Laws of mechanics are invariant under Galilean transformations.

2. Laws of electrodynamics are not invariant under Galilean transformations.

Three possibilities

1. A relativity principle exists for mechanics, but not for electrodynamics. In electrodynamics there is a preferred inertial frame, i.e., the ether frame.

2. A relativity principle exists both for mechanics and for electrodynamics, but the laws of electrodynamics as given by Maxwell are not correct.

3. A relativity principle exists both for mechanics and for electrodynamics, but the laws of mechanics as given by Newton are not correct.

Alternatives 1 and 2 are rejected on the basis of experiments. Alternative 3 is then correct one, and we already know that Newtonian mechanics breaks down at a high speed. For the new relativity principle the laws of mechanics are reformulated, and the transformation laws are corrected. The Galilean transformations are rejected, because Maxwell's equations are not invariant under Galilean transformations.

- Michelson-Morley experiment gives us the result for the existence of an absolute frame(i.e., the ether frame of reference).

- Electrodynamics needs no modification. Experiments show that the velocity of electromagnetic radiation is independent of the velocity of the source.

1.4 Ether concept

It seemed inconceivable to 19th century physicists that the light and other electromagnetic waves could be propagated without a medium. It seemed to be logical step to postulate a medium called the ether for the propagation of electromagnetic waves. To account for its undetectability, it was necessary to assume unusual properties for it, such as zero density and perfect transparency. The ether was assumed to fill all space and to be the medium with respect to which is speed of light is $c = \frac{1}{\sqrt{\epsilon_o \mu_0}} = 2.997925 \times 10^8 \, m/s$.

An observer moving through the ether with velocity $\underline{v}$ would measure a velocity $\underline{c}\,'$ for a light beam where $\underline{c}' = \underline{c} + \underline{v}$.

If an ether exists, the spinning and rotating earth should be moving through it. An observer on earth would sense an "ether wind" whose velocity is $\underline{v}$ relative to the earth.

1.5 Michelson-Morley Experiment

(An attempt to locate the ether frame)

The Michelson interferometer is fixed on the earth.

We imagine that the "ether" is fixed with respect to the sun. Then the earth (and interferometer) moves through ether at a speed of 30 km/sec, in different directions in different season. For the moment, we neglect the earth's spinning motion. The beam of light from the laboratory source S (fixed with respect to the interferometer) is split by the partially silvered mirror M into two coherent

beams:
beam 1 is transmitted through M and
beam 2 is reflected off M.

Beam 1 is reflected back to M by mirror M_1 and beam 2 by mirror M_2. Then the returning beam 1 is partially reflected and the returning beam 2 is partially transmitted by M back to a telescope at T where they interfere.
The interference is constructive or destructive depending on the phase difference of the beams.

The partially silvered mirror surface M is inclined at 45° to the beam direction. The time for beam 1 to travel from M to M_1 and back is[3]

$$\begin{aligned} t_1 &= \frac{l_1}{c-v} + \frac{l_1}{c+v} \\ &= l_1\left(\frac{2c}{c^2-v^2}\right) \\ &= \frac{2l_1}{c}\left(\frac{1}{1-\frac{v^2}{c^2}}\right) \end{aligned}$$

where c is the speed of light in the ether and l_1 the length traveled by the beam 1 of light.

Here, $c-v$ is an "up stream" speed of the beam and
$c+v$ is a "down stream" speed with respect to the apparatus.

The path of beam 2, traveling from M to M_2 and back is a cross-stream path through the ether as shown in the following figure, enabling the beam to return to the (advancing) mirror M.

The transit time (t_2) of beam 2 is given by

$$\begin{aligned} ct_2 &= MM_2 + M_2M \\ \Rightarrow ct_2 &= 2MM_2 \\ \Rightarrow ct_2 &= 2MM_2\left[l_2^2 + \left(\frac{vt_2}{2}\right)^2\right]^{\frac{1}{2}} \\ \Rightarrow ct_2 &= 2MM_2\left[l_2^2 + \left(\frac{vt_2}{2}\right)^2\right]^{\frac{1}{2}} \\ \Rightarrow \frac{c^2t_2^2}{4} &= l_2^2 + \frac{v^2t_2^2}{4} \\ \Rightarrow \frac{1}{4}(c^2-v^2)t_2^2 &= l_2^2 \\ \Rightarrow t_2 &= \frac{2l_2}{\sqrt{c^2-v^2}} = \frac{2l_2}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}$$

This difference in transit time is

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left[\frac{l_2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{l_1}{1-\frac{v^2}{c^2}}\right]$$

Suppose that the instrument is rotated through 90°, thereby making l_1 the cross-stream length and l_2 the down stream length. The corresponding transit time difference is

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\left[\frac{l_2}{1-\frac{v^2}{c^2}} - \frac{l_1}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

[3]earth moves with speed v towards the right. (v)
$\Rightarrow$ether moves with v towards the left. $(-v)$

Hence the rotation changes the differences by

$$
\begin{aligned}
\Delta t' - \Delta t &= \frac{2}{c}\left[\frac{l_2 + l_1}{1 - \frac{v^2}{c^2}} - \frac{l_2 + l_1}{\sqrt{1 - \frac{v^2}{c^2}}}\right] \\
&= \frac{2}{c}(l_1 + l_2)\left[\left(1 - \frac{v^2}{c^2}\right)^{-1} - \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right]
\end{aligned}
$$

neglecting terms higher than the second order,

$$
\begin{aligned}
&\approx \frac{2}{c}(l_1 + l_2)\left[1 - \frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2}\right] \\
&= \frac{2}{c}(l_1 + l_2)\frac{v^2}{2c^2} \\
&= \left(\frac{l_1 + l_2}{c}\right)\frac{v^2}{c^2}
\end{aligned}
$$

Therefore, the rotation should cause a shift in the fringe pattern, because it changes the phase relationship between beams 1 and 2.

Let ΔN be the number of fringe-shift due to rotation of the apparatus. If λ be the wave length of light used, so that the period of one vibration is $T = \frac{1}{\nu} = \frac{\lambda}{c}$, then

$$
\begin{aligned}
\Delta N &= \frac{\Delta t' - \Delta t}{T} \\
&\approx \frac{l_1 + l_2}{cT} \cdot \frac{v^2}{c^2} \\
&= \frac{l_1 + l_2}{\lambda} \cdot \frac{v^2}{c^2}
\end{aligned}
$$

Michelson and Morley, in their experiment, used $l_2 \approx l_1$ and $l_1 + l_2 = 22m$, so that,

$$
\Delta N = \frac{22m}{5.5 \times 10^{-7}m} \cdot 10^{-8} = 0.4
$$

where $\lambda = 5.5 \times 10^{-7}$ m, $\frac{v}{c} = 10^{-4}$, $v = 30$ km/sec, $c = 3 \times\ 10^8$ m/sec.

i.e., a shift of four-tenths of a fringe.

But in experiment, the expected fringe shift was not observed. The experimental conclusion was that there was no fringe shift at all. Hence, the experiment to locate an absolute frame (i.e., the ether frame) of reference gives null result ($\Delta N = 0$).

1.6 Attempts to Preserve the concept of a preferred Ether Frame - The Lorentz-Fitzgerald contraction hypothesis

The hypothesis was that all bodies are contracted in the direction of motion relative to the stationary ether by a factor $\sqrt{1 - \frac{v^2}{c^2}}$.

If l° be the length of a body at rest with respect to the ether and l its length when in motion with respect to the ether, then

$$
l = l^\circ\sqrt{1 - \frac{v^2}{c^2}}
$$

In Michelson-Morley experiment,

$$l_1 = l_1^\circ \sqrt{1 - \frac{v^2}{c^2}}$$

and

$$l_2 = l_2^\circ$$

$$\therefore \ \Delta t = \frac{2}{c}\left[\frac{l_2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{l_1}{1-\frac{v^2}{c^2}}\right] = \frac{2}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(l_2^\circ - l_1^\circ\right)$$

on 90° rotation,

$$l_2 = l_2^\circ \sqrt{1 - \frac{v^2}{c^2}}$$
$$l_1 = l_1^\circ$$

$$\therefore \ \Delta t' = \frac{2}{c}\left[\frac{l_2}{1-\frac{v^2}{c^2}} - \frac{l_1}{\sqrt{1-\frac{v^2}{c^2}}}\right] = \frac{2}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(l_2^\circ - l_1^\circ\right)$$
$$\therefore \ \Delta t' - \Delta t = 0 \qquad \text{i.e.,} \ \ \Delta N = 0$$

Hence, no fringe shift should be expected on the rotation of the interferometer.

But if the velocity of the interferometer changes with respect to ether from v to v', we should expect a fringe shift. The predicted shift in fringes is We have,

$$t_1 = \frac{2l_1}{c}\left(\frac{1}{1-\frac{v^2}{c^2}}\right)$$

$$\therefore \ t_1 = \frac{2l_1^\circ\sqrt{1-\frac{v^2}{c^2}}}{c(1-v^2/c^2)} \qquad \text{using length contraction}$$

If v changes to v', then

$$t_1' = \frac{2l_1^\circ}{c}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)$$
$$\therefore \Delta t_1 = \frac{2l_1^\circ}{c}\left(\frac{1}{\sqrt{1-v^2/c^2}} - \frac{1}{\sqrt{1-v'^2/c^2}}\right)$$

Also

$$\Delta t_2 = \frac{2l_2^\circ}{c}\left(\frac{1}{\sqrt{1-v^2/c^2}} - \frac{1}{\sqrt{1-v'^2/c^2}}\right)$$
$$\therefore \Delta t = \Delta t_2 - \Delta t_1$$
$$\approx \frac{l_2^\circ - l_1^\circ}{c}\left(\frac{v^2}{c^2} - \frac{v'^2}{c^2}\right)$$

$$\therefore \Delta N = \frac{\Delta t}{T} = \frac{c}{\lambda}\frac{l_2^\circ - l_1^\circ}{c}\left(\frac{v^2}{c^2} - \frac{v'^2}{c^2}\right)$$
$$= \frac{l_2^\circ - l_1^\circ}{\lambda}\left(\frac{v^2}{c^2} - \frac{v'^2}{c^2}\right)$$

$$\Delta N = \frac{l_2^\circ - l_1^\circ}{\lambda}\left(\frac{v^2}{c^2} - \frac{v'^2}{c^2}\right)$$

Although the difference $\frac{v^2-v'^2}{c^2}$ should change as a result of earth's spin (the biggest change occurring in twelve hours) and the earth's rotation (the biggest change occurring in six months), neither effect was observed (i.e., $\Delta N = 0$) in direct contradiction to the contraction hypothesis.

1.7 The Ether-Drag Hypothesis

(Attempt to preserve the concept of a preferred Ether frame)

This hypothesis assumed that the ether frame was attached to all bodies of finite mass, i.e., dragged along with such bodies.

This assumption of such a "local" ether would automatically give a null result in the Michelson-Morley experiment.

However, there were two well-established effects which contradicted the ether-drag hypothesis: stellar aberration and Fizeau convection coefficient.

1.7.1 Fizeau Experiment

The set-up of the Fizeau experiment is shown diagrammatically in the following figure.

$$\begin{aligned} S &\rightarrow \text{light source} \\ M &\rightarrow \text{partially silvered mirror} \\ M_1, M_2, M_3 &\rightarrow \text{Mirrors} \\ T &\rightarrow \text{Telescope} \\ v_w &\rightarrow \text{Water velocity} \end{aligned}$$

Let the apparatus be $S-$frame. In this laboratory frame, [the velocity of light in still water is $\frac{c}{\eta}$ and the velocity of the water is v_w, η being the refraction index of water and c the free-space velocity of light.]

$$\begin{aligned} c &\rightarrow \text{velocity of light in free space} \\ \eta &\rightarrow \text{refractive index of water} \\ \frac{c}{\eta} &\rightarrow \text{velocity of light in still water} \\ v_w &\rightarrow \text{velocity of water} \end{aligned}$$

Then, the velocity of light in the moving water, as was given by Fresnel, is

$$v = \frac{c}{\eta} \pm v_w\left(1 - \frac{1}{\eta^2}\right) \tag{1.2}$$

where the factor $\left(1 - \frac{1}{\eta^2}\right)$ is called the Fresnel drag coefficient.

Since $\left(1 - \frac{1}{\eta^2}\right) < 1, \qquad \therefore\ v_2\left(1 - \frac{1}{\eta^2}\right) < v_w$

i.e., the increase or decrease of light speed is less than the speed of medium. This predicts that light is dragged partially along by a moving medium.

In Fizeau experiment, the approximate values of the parameters were as follows: $l = 1.5$m, $\eta = 1.33$, $\Lambda = 5.3 \times 10^{-7}$m and $v_w = 7$m/sec. A shift of 0.33 fringe was observed from the case

$v_w = 0$ calculated the drag coefficient and compare it with the predicted value.

We have from (1.2),

$$v = \frac{c}{\eta} \pm v_w d, \qquad d = 1 - \frac{1}{\eta^2}$$

The time for beam 1 to pass through the water is

$$t_1 = \frac{2l}{\frac{2}{n} - v_w d}$$

and for beam 2

$$t_2 = \frac{2l}{\frac{c}{\eta} + v_w d}$$

Where $2l \rightarrow$ length of whole path on which light travels in water.
Hence,

$$\Delta t = t_2 - t_1 = \frac{4lv_w d}{\left(\frac{c}{\eta}\right)^2 - v_w^2 d^2} \approx \frac{4l\eta^2 v_w d}{c^2}$$

The period of vibration of the light is $T = \frac{\lambda}{c}$

$$\therefore \Delta N = \frac{\Delta t}{T} \approx \frac{4l\eta^2 v_w d}{\lambda c}$$

Hence, the drag coefficient is

$$d = \frac{\lambda c \Delta N}{4l\eta^2 v_w} = 0.49$$

The Fresnel prediction is

$$d = 1 - \frac{1}{\eta^2} = 0.44$$

Fizeau's experiment confirms the Fresnel prediction.
If the ether was dragged with the water, then the velocity of light in the laboratory frame, using the Galilean ideas, would have been $\frac{c}{\eta} - v_w$ in one tube

and $\frac{c}{\eta} + v_w$ in the other tube

The Fizeau experiment is interpreted in terms of no ether at all, ether by the apparatus or the water moving through it. The observed partial drag is due to the motion of the refractive medium.

Hence, the ether-drag hypothesis is contradicted.

1.8 Attempts to Modify Electrodynamics

A possible interpretation of the Michelson-Morley result (null result) is that the velocity of the light c has the same value in all inertial frames.

If this is so, then the velocity of light surely cannot depend on the velocity of the light source relative to the observer. Now, if we avoid the principle of the invariance of the velocity modification of electromagnetic is to assume that the velocity of a light wave is connected with the motion of the source rather than with an ether. If the source velocity affected the velocity of the light, we should observe complicated fringe pattern changes in the Michelson-Morley experiment. No such effects were observed in the experiment. Thus, we are forced by experiment to conclude that the laws of electrodynamics are correct and do not need modification.

Clearly since the speed of light (i.e., the speed of electromagnetic radiation) is independent of the relative motion of source and observer, we cannot use the Galilean transformation. We conclude that the Galilean transformation must be replaced, and therefore, the laws of mechanics which were consistent with those transformations need to be modified.

1.9 The postulates of Special Relativity Theory

In 1905, Albert Einstein (1879-1955) provided a solution to the dilemma facing physics. He gave two postulates are as follows:

1. The laws of physics are the same in all inertial systems. No preferred inertial system exists. (the principle of Relativity)

2. The speed of light in free space has the same value c in all inertial systems. (The principle of the constancy of the speed of light).

The special relativity is based on these two principles.

The principle of Relativity leads to the results that "time" can not be absolute. In different reference systems time runs differently. Therefore, if two events are simultaneous in one inertial system, they will not be simultaneous in some other system.

1.10 Simultaneity

Suppose that two events A and B occur at the same place in one particular frame of reference. Let us have a clock at that place which registers the time of occurrence of each event. If the reading is the same for each event, then we regard the events as simultaneous.

1.11 Clock Synchronization

Two clocks (of same nature) are said to be synchronized if they always read the same time when we set them at the same place in a particular frame of reference.

Let two events A and B occur at different locations in one particular reference frame. We set two clocks of same nature at each event.

If observer A sees that two clocks to read always the same time, observer B will find still that the clocks are not synchronized. This is due to the fact that it takes time for light to travel from B to A and vice versa.

Let us put a light source, that can be turned on and off (e.g., a flash bulb), at the exact mid-point of the straight line connecting A and B.

Let us inform each observer to put his clock at $t = 0$ when the turned-on light signal reaches him.

The light will take an equal amount of time to reach A and B from the mid-point, so that this procedure does indeed synchronize the clocks that are at A and B.

The time of an event is measured by the clock whose location coincides with that of the event. Events occurring at A and B must be called simultaneous when the clocks at the respective places record the same time for them.

1.11.1 Simultaneity is genuinely a relative concept, not an absolute one

Suppose that two inertial frames S and S' are in coincident form. Two points A and B are equidistant from O in $S-$ frame. Points A, O and B coincides with respectively with A', O' and B' in $S'-$frame.

Suppose, two lightning bolts occur at A and B simultaneously with respect to observer O in $S-$frame. The two events (i.e., lightning bolts) will also be simultaneous to observer O' in $S'-$frame, because the light signals from A and B reach O' at the same time, since $AO' = O'B$.

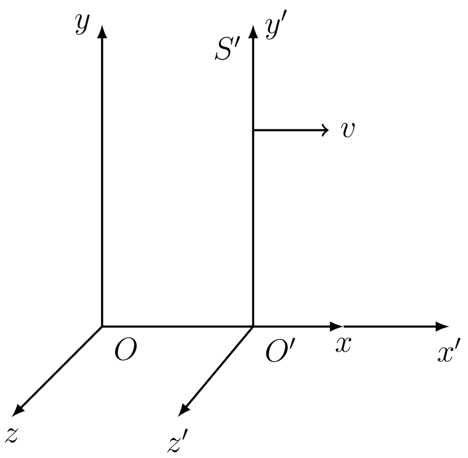
Now, let $S'-$frame moves to the right with velocity $\underline{v}$ and two lightning bolts occur at A and B simultaneously as before. Then due to the relative velocity of $S'-$frame, the light signal from B will reach O' before reaching the light signal from A.

As a result, observer O' will see that the lightning bolts are not simultaneous, though they are simultaneous to observer O.

Hence, the simultaneously is not independent of reference frame.

The concept of simultaneity enters into the measurement of length. To measure the length of a rod, one must ..... simultaneously the positions of its end points. Therefore, the length of a rod is not an absolute one.

1.12 Lorentz Transformation



we observe an event in one inertial frame S. The location and time of the event are described by the coordinates $(x,\, y,\, z,\, t)$.

In a second inertial frame S', the same event is recorded as the time-space coordinates $(x',\, y',\, z',\, t')$.

Let

$$x' = x'(x,\, y,\, z,\, t)$$
$$y' = y'(x,\, y,\, z,\, t)$$
$$z' = z'(x,\, y,\, z,\, t)$$
$$t' = t'(x,\, y,\, z,\, t)$$

We use the assumptions:

(i) Space is isotropic, i.e., all spatial direction are equivalent.

(ii) Space and time are homogenous, i.e., all points in space and time are equivalent.

(iii) S and S' coincide at $t = 0$, $t' = 0$.

Let $S'-$frame moves with relative velocity v along the common $x - x'$ axis.
The homogeneity of space and time implies that the transformation equations must be linear:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$
$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$
$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$
$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

where the subscripted coefficients are constant.
Note. If $x' = a_{11}x^2$, then $x_2' - x_1' = a_{11}(x_2^2 - x_1^2)$;
For a rod of unit length in S with end points at

(i) $x_1 = 1$ and $x_2 = 2$, we get $x_2' - x_1' = 3a_{11}$;

(ii) $x_1 = 4$ and $x_2 = 5$, we get $x_2' - x_1' = 9a_{11}$;

i.e., the measured length of the rod depends on there it is in space. Similar is the situation for t.

If $v = 0$, then $a_{11} = a_{22} = a_{33} = a_{44} = 1$, all other coefficients being zero. The $x-$ axis coincides continuously with $x'-$axis. This gives $y' = 0$, $z' = 0$ for $y = 0$, $z = 0$. Then we have,

$$y' = a_{22}y + a_{23}z$$
$$z' = a_{32}y + a_{33}z$$
$$\text{i.e., } a_{21} = a_{24} = a_{31} = a_{34} = 0$$

Again, the plane $z = 0$ should transform to $z' = 0$ and the plane $y = 0$ to $y' = 0$. Hence,

$$y' = a_{22}y$$
$$z' = a_{33}z$$
$$\text{i.e., } a_{23} = 0 = a_{32}$$

Consider a rod at rest of unit length lying along the $y-$axis in S. According to the S' observer, the rod's length will be[4]

$$y' = a_{22} \times 1 = a_{22}$$

Consider the same rod at rest along the y' axis in S'. To the S observer, the rod's length will be[5]

$$y = \frac{1}{a_{22}} y' = \frac{1}{a_{22}} \times 1 = \frac{1}{a_{22}}$$

The first postulate of special relativity implies that these measurements are identical. Therefore,

$$\frac{1}{a_{22}} = a_{22} \qquad \Rightarrow \; a_{22} = 1$$

With the similar argument, $a_{33} = 1$.
Thus,

$$\begin{aligned} y' &= y \\ z' &= z \end{aligned}$$

Other two transformation equations are

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{aligned}$$

Since space is isotropic, we get that t' does not depend on y and z.[6]
Hence, $a_{42} = 0 = a_{43}$.
Also, a point with $x' = 0$ appears to move in the positive $x-$axis with speed v.
So, $x' = 0$ corresponds to $x = vt$, and we expect

$$\begin{aligned} x' &= a_{11}(x - vt) \\ &= a_{11}x - a_{11}vt \\ &= a_{11}x + a_{14}t \\ \text{i.e.,} \quad a_{14} &= -va_{11} \end{aligned}$$

Therefore, the transformation equations reduce to

$$\left.\begin{aligned} x' &= a_{11}(x - vt) \\ y' &= y \\ z' &= z \\ t' &= a_{41}x + a_{44}t \end{aligned}\right\} \tag{1.3}$$

[4] $y' = a_{22}y$, $y'_2 - y'_1 = a_{22}(y_2 - y_1)$, $y' = a_{22} \times 1$
[5] $y'_2 - y'_1 = a_{22}(y_2 - y_1) = a_{22}y$, $y = \frac{1}{a_{22}}$
[6]

Note. Otherwise, if we place clocks at $+y$, $-y$, then $t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \neq a_{11}x - a_{12}y + a_{13}z + a_{14}t$ Similar is the case at $+z$, $-z$. That is, clocks placed symmetrically in the $y - z$ plane about the $x-$axis would appear to disagree as observed from S', which contradicts the isotropy of space.

We now recall the second postulate of special relativity i.e., the speed of light in free space has the same value c in all inertial frames.

Consider a spherical electromagnetic wave leaving the origin at $t = 0$. The wave propagation is described by

$$x^2 + y^2 + z^2 = c^2t^2 \quad \text{for } S \tag{1.4}$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \text{for } S' \tag{1.5}$$

Substituting (1.3) into (1.5), we get

$$\begin{aligned} &a_{11}^2(x - vt)^2 + y^2 + z^2 = c^2\left(a_{41}x + a_{44}t\right)^2 \\ \Rightarrow &\left(a_{11}^2 - c^2a_{41}^2\right)x^2 + y^2 + z^2 - (2a_{11}^2v + 2c^2a_{41}a_{44})xt = (c^2a_{44}^2 - a_{11}^2v^2)t^2 \end{aligned}$$

This must be the same as (1.4)

1.13 Lorentz Transformation

Consider, two frames of reference S and S', S' is moving with uniform velocity v along the $x'-$axis relative to S. Let the observers at the origin O and O' be observing the same event at any point P whose coordinates are (x, y, z, t) and (x', y', z', t') in S and S' respectively.

Let us consider that the $x-$axis of two systems coincides permanently. The event P is a light signal and is produced when both t and t' are zero.

Let us assume that at time $t = 0$, a spherical wave of light signal leaves O which coincides with O' at the moment. Since the velocity of light in both systems is the same and so the speed of propagation is the same in all directions and equal to c in terms of either set of coordinates. Its progress is therefore described by either of the two equations,

$$x^2 + y^2 + z^2 = c^2t^2 \qquad \Rightarrow x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{1.6}$$

and

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \qquad \Rightarrow x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{1.7}$$

As velocity of S' is only along $x-$axis. Thus, from symmetry,

$$y = y' \qquad \text{and} \qquad z = z' \tag{1.8}$$

Then from equations (1.6) and (1.7),

$$x^2 - c^2t^2 = \lambda(x'^2 - c^2t'^2) \tag{1.9}$$

where λ is any undetermined constant.

Now, for the transformation equation relating to x and x'. Let us put,

$$x' = \gamma(x - vt) \tag{1.10}$$

γ being independent of x and t.

The reasons for trying the above relation are

(i) The transformation must reduce to Galilean transformation for low speed. i.e., for speed $\frac{v}{c} \to 0$.

(ii) The transformation must be linear and simple.

Since the motion is relative, we may assume that, S is moving relative to S' with velocity $(-v)$ along the direction of $x-$axis. Therefore,

$$x = \gamma(x' + vt') \tag{1.11}$$

Now, putting the value of x' from (1.10) in (1.11), we get,

$$\begin{aligned} x &= \gamma\left[\gamma(x - vt) + vt'\right] \\ \Rightarrow \gamma v t' &= x - \gamma^2 x + \gamma^2 vt \\ \Rightarrow t' &= \gamma\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right] \end{aligned} \tag{1.12}$$

Now, substituting the value of x' from (1.10) and t' from (1.12) in (1.9), we get,

$$x^2 - c^2t^2 = \lambda\left[\gamma^2(x - vt)^2 - c^2\gamma^2\left\{t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right\}^2\right]$$

$$\Rightarrow x^2 - c^2t^2 - \lambda\gamma^2\left(x^2 - 2vxt + v^2t^2\right) + \lambda c^2\gamma^2\left\{t^2 - 2\frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)t + \frac{x^2}{v^2}\left(1 - \frac{1}{\gamma^2}\right)^2\right\} = 0 \tag{1.13}$$

Since, this equation is identity, the coefficients of various powers of x and t must vanish separately. Equating coefficients of xt to zero, we have,

$$\begin{aligned} 2\lambda\gamma^2 v - \frac{2\lambda c^2\gamma^2}{v}\left(1 - \frac{1}{\gamma^2}\right) &= 0 \\ \Rightarrow 2\gamma^4 v^2 - 2c^2\gamma^4 + 2c^2\gamma^2 &= 0 \\ \Rightarrow \gamma^2(v^2 - c^2) + c^2 &= 0 \end{aligned} \tag{1.14}$$

Equating coefficients of t^2 to zero, we have,

$$\begin{aligned} -c^2 - \lambda\gamma^2 v^2 + \lambda c^2\gamma^2 &= 0 \\ \Rightarrow \lambda(v^2 - c^2)\gamma^2 + c^2 &= 0 \end{aligned} \tag{1.15}$$

From equation (1.14) and (1.15), we have $\lambda = 1$.
Equation (1.14) gives,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.16}$$

From (1.12) we have,

$$\begin{aligned} t' &= \gamma\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right] \\ t' &= \gamma\left[t - \frac{x}{v}\left(1 - \frac{c^2 - v^2}{c^2}\right)\right] \\ t' &= \gamma\left[t - \frac{xv^2}{c^2 v}\right] \\ \therefore\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{1.17}$$

Thus, using the values of γ from (1.16) we get, from (1.8), (1.10), (1.17)

$$
\begin{aligned}
x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned}
$$

These are called Lorentz transformation equations.

1.14 Mass Transformation Relation

Let S, S' be two frames of reference, where S' moves with velocity v along $x-$axis with respect to S frame.

In $S-$frame, an object of mass m_1 is moving with velocity u along $x-$axis.

In $S'-$frame, the mass and velocity is m_1' and u_1' respectively, then

$$
\begin{aligned}
u_1 &= \frac{u_1' + v}{1 + \frac{v}{c^2}u_1'} \\
\Rightarrow u_1\left(1 + \frac{v}{c^2}u_1'\right) &= u_1' + v \\
\Rightarrow u_1'\left(1 - \frac{v}{c^2}u_1\right) &= u_1 - v \\
\therefore u_1' &= \frac{u_1 - v}{1 - \frac{v}{c^2}u_1}
\end{aligned}
$$

Now,

$$
\begin{aligned}
1 - \frac{{u_1'}^2}{c^2} &= 1 - \frac{(u_1 - v)^2}{c^2\left(1 - \frac{v}{c^2}u_1\right)^2} \\
\Rightarrow 1 - \frac{{u_1'}^2}{c^2} &= \frac{c^2\left(1 - \frac{v}{c^2}u_1\right)^2 - (u_1 - v)^2}{c^2\left(1 - \frac{v}{c^2}u_1\right)^2} \\
\Rightarrow c^2\left(1 - \frac{{u_1'}^2}{c^2}\right)\left(1 - \frac{v}{c^2}\right)^2 &= c^2\left(1 + \frac{v^2}{c^4}{u_1}^2 - 2\frac{v}{c^2}u_1\right) - (u_1^2 + v^2 - 2u_1v) \\
\Rightarrow c^2\left(1 - \frac{{u_1'}^2}{c^2}\right)\left(1 - \frac{v}{c^2}\right)^2 &= c^2\left(1 - \frac{u_1^2 + v^2}{c^2} + \frac{u_1^2v^2}{c^4}\right) \\
\therefore \left(1 - \frac{{u_1'}^2}{c^2}\right)\left(1 - \frac{v}{c^2}\right)^2 &= \left(1 - \frac{u_1^2 + v^2}{c^2} + \frac{u_1^2v^2}{c^4}\right)
\end{aligned}
$$

Again,

$$\begin{aligned}
\gamma_1' u_1' &= \frac{u_1 - v}{\sqrt{1 - \frac{{u_1'}^2}{c^2}}\left(1 - \frac{v}{c^2} u_1\right)} \\
&= \frac{u_1 - v}{\sqrt{1 - \frac{u_1^2 + v^2}{c^2} + \frac{u_1^2 v^2}{c^4}}} \\
&= \frac{u_1 - v}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u_1^2}{c^2}\right)}} \\
&= \gamma \cdot \gamma_1 (u_1 - v) \\
\therefore\ \frac{\gamma_1' u_1'}{\gamma_1} &= \gamma(u_1 - v)
\end{aligned} \tag{1.18}$$

where

$$\begin{aligned}
\gamma &= \frac{1}{1 - \frac{v^2}{c^2}} \\
\gamma_1 &= \frac{1}{1 - \frac{u_1^2}{c^2}} \\
\gamma_1' &= \frac{1}{1 - \frac{{u_1'}^2}{c^2}}
\end{aligned}$$

Consider a particle is moving along the $x-$axis with constant velocity in the $S-$frame such that their mass and momentum remain unchanged.

$$\therefore \sum m_1 = \text{constant} \qquad \text{and} \qquad \sum m_1 u_1 = \ \text{constant}$$

Since, the values of γ and u is same for all objects, so we get,

$$\begin{aligned}
&\sum m_1 \gamma\, v = \ \text{constant}, \qquad \text{and} \qquad \sum m_1 u_1 \gamma = \ \text{constant} \\
\Rightarrow &\sum m_1 \gamma (u_1 - v) = \text{constant} \qquad [\text{subtracting}] \\
\Rightarrow &\sum \left(m_1 \frac{\gamma_1' u_1'}{\gamma_1} \right) = \text{constant} \qquad [\text{from } (1.18)]
\end{aligned}$$

Applying the law of conservation of momentum in S' frame to get,

$$\sum m_1' u_1' = \text{constant} \tag{1.19}$$

Comparing (1.18) and (1.19),

$$\begin{aligned}
\frac{m_1 \gamma_1'}{\gamma_1} &= m_1' \\
\Rightarrow \frac{m_1}{\gamma_1} &= \frac{m_1'}{\gamma_1'} = m_0 \ (\text{say}) \\
\therefore\ m_1 = \frac{m_0}{\sqrt{1 - \frac{u_1^2}{c^2}}}, &\quad m_1' = \frac{m_0}{\sqrt{1 - \frac{{u_1'}^2}{c^2}}}
\end{aligned}$$

This proves that if an object of mass m is moving in a frame of reference with velocity u,

$$m = \frac{m_0}{\sqrt{q - \frac{u^2}{c^2}}}$$

If $u = c$, the rest mass of a particle is, $m_0 = 0$.

If $u = 0$, $m = m_0$; where m_0 is the mass of the object at rest.

Part II

General Relativity

Chapter 2

Questions

2.1 General Relativity

Problem 2.1 (7)**.** What are Bianchi identities, Einstein tensor and Ricci tensor? Show that the covariant derivatives of Einstein tensor is zero.

Solution. Bianchi identities:

$$R^{\lambda}_{\mu\nu\sigma,\rho} + R^{\lambda}_{\mu\sigma\rho,\nu} + R^{\lambda}_{\mu\rho\nu,\sigma} = 0$$

This tensor equation is called Bianchi identities.

Ricci tensor: Ricci tensor is,

$$R_{\alpha\beta} = R^{\lambda}_{\mu\nu\sigma} = R_{\beta\alpha}$$

which is the contraction of $R^{\mu}_{\alpha\nu\beta}$ on the first and third indices.
Other contraction would in principle also be possible. On the first and second, the first and forth etc. But because $R_{\alpha\beta\mu\nu}$ is antisymmetric on α and β and on μ and ν, all these contractions either vanish or reduce to $+ = R_{\alpha\beta}$. Therefore, the Ricci tensor is the only contraction of the Riemann tensor.

Einstein tensor: The Einstein tensor G is a tensor of order 2 define over Pseudo-Riemannian manifolds.
In index-free notation it is defined as,

$$G = R - \frac{1}{2}gR$$

where, R is the Ricci tensor, g is the metric tensor and R is the scalar curvature.
In component form, the previous equation reads as,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

The Einstein tensor is symmetric.

The covariant derivatives of Einstein tensor is zero: From the contracted Bianchi identities, we known that,

$$\nabla_l R^l_m - \frac{1}{2}\delta^l_m \nabla_l R = 0$$

where, δ is the Kronecker delta. Since, the mixed Kronecker delta is equivalent to the mixed tensor,

$$\delta^l_m = g^l_m$$

And since the covariant derivative of the metric tensor is zero (so it can be moved in or out of the scope of any such derivative), then,

$$\nabla_l R^l_m - \frac{1}{2}\nabla_l g^l_m R = 0$$

Factor out the covariant derivative,

$$\nabla_l \left(R^l_m - \frac{1}{2} g^l_m R \right) = 0$$

then, raise the index m throughout

$$\nabla_l \left(R^{lm} - \frac{1}{2} g^{lm} R \right) = 0$$

The expression in parentheses is the Einstein tensor $\nabla_l G^{lm} = 0$. Which implies the covariant derivative of Einstein tensor is zero.

Problem 2.2 (7)**.** Define a flat space-time. Show that the vanishing of curvature tensor is a necessary and sufficient condition for a space-time to be flat.

Solution. Flat space time: A region of world said to be flat if it is possible to construct it in a Galilean frame of reference. The space time is said to be flat if such coordinate can be found in it for which $g_{\mu\nu}$ are constant.

Necessary and sufficient condition for the flat space-time:
The necessary condition for flat space time is the vanishing of Riemann Christoffel tensor. This condition will also be sufficient if the converse is also true, i.e., if the Riemann Christoffel tensor vanishes the space time must be flat.

The construction of a uniform vector field by parallel displacement of a vector all over the region is possible if,

$$R^\lambda_{\mu\nu\sigma} = 0 \tag{2.1}$$

Let, $A^\mu_{(\alpha)}$ be four uniform vector fields given by $\alpha = 1, 2, 3, 4$ (here, α is not a tensor suffix). Then equation (2.1) becomes

$$\begin{aligned} \left(A^\mu_{(\alpha)}\right)_{,\sigma} &= \frac{\partial\, A^\mu_{(\alpha)}}{\partial\, x^\sigma} + \Gamma^\mu_{\lambda\sigma} A^\lambda_{(\alpha)} = 0 \\ \Rightarrow \frac{\partial\, A^\mu_{(\alpha)}}{\partial\, x^\sigma} &= -\Gamma^\mu_{\lambda\sigma} A^\lambda_{(\alpha)} \end{aligned} \tag{2.2}$$

Now consider the transformation law of coordinates,

$$\mathrm{d}\, x^\mu = A^\mu_{(\alpha)}\, \mathrm{d}\, \bar{x}^\alpha (\alpha = 1, 2, 3, 4)$$

Since, $\mathrm{d}\, s^2$ is invariant, we have,

$$\mathrm{d}\, s^2 = \overline{g}_{\alpha\beta} \cdot \overline{\mathrm{d}\, x^\alpha} \cdot \overline{\mathrm{d}\, x^\beta} = g_{\mu\nu} \cdot \mathrm{d}\, x^\mu \cdot \mathrm{d}\, x^\nu$$

$$\begin{aligned}
\mathrm{d}\, s^2 &= \overline{g}_{\alpha\beta} \cdot \overline{\mathrm{d}\, x^\alpha} \cdot \overline{\mathrm{d}\, x^\beta} \\
&= g_{\mu\nu} \cdot \mathrm{d}\, x^\mu \cdot \mathrm{d}\, x^\nu \\
&= g_{\mu\nu} A^\mu_{(\alpha)} \overline{\mathrm{d}\, x^\alpha} A^\nu_{(\beta)} \overline{\mathrm{d}\, x^\beta}
\end{aligned}$$

i.e.,[1]

$$\overline{g}_{\alpha\beta} = g_{\mu\nu} A^\mu_{(\alpha)} A^\nu_{(\beta)} \tag{2.3}$$

Differentiating the above equation with respect to x^σ, we get,

$$\begin{aligned}
\frac{\partial\, \overline{g}_{\alpha\beta}}{\partial\, x^\sigma} &= g_{\mu\nu} A^\mu_{(\alpha)} \frac{\partial\, A^\nu_{(\beta)}}{\partial\, x^\sigma} + g_{\mu\nu} A^\nu_{(\beta)} \frac{\partial\, A^\mu_{(\alpha)}}{\partial\, x^\sigma} + A^\mu_{(\alpha)} A^\nu_{(\beta)} \frac{\partial\, g_{\mu\nu}}{\partial\, x^\sigma} \\
&= -g_{\mu\nu} A^\mu_{(\alpha)} A^\lambda_{(\beta)} \Gamma^\nu_{\lambda\sigma} - g_{\mu\nu} A^\nu_{(\beta)} A^\lambda_{(\alpha)} \Gamma^\mu_{\lambda\sigma} + A^\mu_{(\alpha)} A^\nu_{(\beta)} \frac{\partial\, g_{\mu\nu}}{\partial\, x^\sigma} \quad \text{[using (2.2)]}
\end{aligned}$$

Now changing the dummy suffixes, we get,

$$\begin{aligned}
&= A^\mu_{(\alpha)} A^\nu_{(\beta)} \left[-g_{\mu\nu} \Gamma^\lambda_{\nu\sigma} - g_{\mu\nu} \Gamma^\lambda_{\mu\sigma} + \frac{\partial\, g_{\mu\nu}}{\partial\, x^\sigma} \right] \\
&= A^\mu_{(\alpha)} A^\nu_{(\beta)} \left[-\Gamma_{\mu,\nu\sigma} - \Gamma_{\nu,\mu\sigma} + \frac{\partial\, g_{\mu\nu}}{\partial\, x^\sigma} \right] \\
&= A^\mu_{(\alpha)} A^\nu_{(\beta)} \left[-\frac{\partial\, g_{\mu\nu}}{\partial\, x^\sigma} + \frac{\partial\, g_{\mu\nu}}{\partial\, x^\sigma} \right] \\
&= 0
\end{aligned}$$

Integrating we get, $\overline{g}_{\alpha\beta} = C\,(Constant)$, which is constant throughout the region. From definition, it follows that the space-time is flat. Hence, the vanishing of Riemann-Christoffel tensor is necessary and sufficient conditions for flat space-time.

Problem 2.3 (6)**.** Give an account of Einstein's principle of equivalence. What are observable consequences of General theory of Relativity?/ Discuss that it acts as a bridge to pass from special to general theory of Relativity.

Solution. Einstein's principle of equivalence: In the theory of general relativity, the principle of equivalence which necessarily leads to the introduction of a curved space time.
Simply, the equivalence principle is the equivalence of gravitational and inertial mass, that means inertial and gravitational mass are same.

Consequences of General Theory of Relativity: Some of the consequence of general relativity are:
Gravitational time dilation: Clocks run slower in deeper gravitational walls.
Precession: Precession of orbits are in unexpected way in Newton's theory of gravity. (This has been observed in the orbit of Mercury and binary pulsars.)
Light Deflection: Rays of light bend in the presence of a gravitational field.
Frame-Dragging: Rotating masses "drag along" the spacetime around them.
Metric expansion of space: The universe is expanding, and the far parts of it are moving away from us faster than the speed of light.

[1] Comparing this with $\mathrm{d}\, s^2$ invariant

Gravitational Lensing: The curvature of spacetime means that the path of light is deflected around massive objects. This effect is known as gravitational lensing, and it can affect the shape of an event's light cone allowing light to travel into previously forbidden region.
Black holes: Black holes occurs when an object becomes so massive that even light can't escape its gravitational wall. We know that super massive black holes reside at the center of most galaxies, including our own. These are millions of times the mass of sun.

Problem 2.4 (8)**.** Write a short note on the energy momentum tensor $T^{\mu\nu}$ and discuss the reasons which led Einstein to choose the field equations in the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu}$$

or,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}$$

show further that these equations reduce in linear approximation to Newtonian equations $\nabla^2\psi = -8\pi$

Solution. Energy momentum tensor, $T^{\mu\nu}$: The energy momentum tensor is a tensor quantity in physics that describes the density and flux of energy and momentum in spacetime.
The stress energy tensor is defined as tensor, $T^{\mu\nu}$, of order two that gives the flux of the $\mu-$th component of the momentum vector across a surface with constant x^ν coordinate.
In general relativity, the energy momentum tensor is symmetric.

Second part: Field equations in classical mechanics are given by,

$$\nabla^2\varphi = -4\pi\nu\rho \tag{2.4}$$

where φ, ν, ρ stand for gravitational potential, Newtonian constant of gravitation and density of material distribution respectively.
According to principle of equivalence, φ can be interpreted either as potential function or, metric tensor $g_{\mu\nu}$.
In order to get an analogue of the equation (2.4) in general theory of relativity, φ must be replaced by the metric tensor $g_{\mu\nu}$. That is to say, we consider $g_{\mu\nu}$ to be gravitational potential. It follows from (2.4) that, the field equation in general theory of relativity are expressible in terms of second order derivatives of $g_{\mu\nu}$. The most appropriate tensor which contains second derivatives is the Ricci tensor $R_{\mu\nu}$.

Hence, left side of (2.4) will be either $R_{\mu\nu}$ or, its linear combination. While describing the relativistic field equations we must keep in mind that the field equations must be invariant under the tensor law of transformation.
It means that both sides of (2.4) must be expressed in terms of tensor. Hence, ρ in (2.4) must be replaced by second rank tensors. This tensor is commonly known as energy momentum tensor.
All the above facts are met in the equation,

$$R_{\mu\nu} + aRg_{\mu\nu} = -kT_{\mu\nu} \tag{2.5}$$

where, k stands for Einstein constant of gravitation and $k = 8\pi$. From (2.5)

$$R^\mu_\nu + aRg^\mu_\nu = -kT^\mu_\nu$$
$$\Rightarrow (R^\mu_\nu + aRg^\mu_\nu)_{,\mu} = (-kT^\mu_\nu)_{,\mu} = 0$$

[for $(-kT^\mu_\nu)_{,\mu} = -kT^\mu_{\nu,\mu} = -k \cdot 0 = 0$ Since, divergence of energy tensor is zero]

$$\Rightarrow a = -\frac{1}{2}$$

$\therefore$ (2.5) becomes,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu}$$

Taking cosmological constant Λ into account, we obtain,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -kT_{\mu\nu} \tag{2.6}$$

This is the required field equation in general theory of relativity.
For $\Lambda = 0$, (2.6) gives,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu} \tag{2.7}$$

Multiplying (2.7) by $g^{\mu\nu}$, we get,

$$R - \frac{1}{2}R \cdot 4 = -kT$$
$$\Rightarrow R = kT$$

Putting the value of R in (2.7), we get,

$$R_{\mu\nu} - \frac{1}{2}KTg_{\mu\nu} = -kT_{\mu\nu} \tag{2.8}$$

This is an alternative form of the field equation given by (2.7).
For empty space, so that, $T = 0$, then (2.8) gives,

$$R_{\mu\nu} = 0 \tag{2.9}$$

Thus, the field equation in empty space are given by (2.9).

Problem 2.5. State and comment on the basic hypothesis and postulates of the general theory of relativity and discuss how the principle of equivalence and covariance follow from the guiding principle in the development if general relativity.

Solution. The special theory of relativity has its origin in the development of electromagnetic while general relativity is the relativistic theory of gravitation. The special theory of relativity only accounts for inertial frame of reference in the free space where gravitational effects can be neglected. In these systems, the law of inertial holds good and the physical laws retain the same form.
The special theory of relativity deals with the problems of uniform rectilinear motion. We wish to extend the principle of relativity in such a way that it may hold for non-inertial systems and consequently the theory may explain the non-inertial phenomenon like the phenomenon of gravitation. The extended theory is known as the general theory of relativity.

The theory of gravitation also deals with non-inertial systems unlike special theory which deals only inertial systems; for this reason, the theory of gravitation is called "General theory of relativity".

Principle of Covariance: The generalized principle of relativity states, "the laws of nature retain their same form in all coordinate systems", this statement is called the principle of covariance. This principle is the foundation of theory of general relativity. According to this principle, we must express all the physical laws of nature by means of equations in the covariant form, which are independent of the coordinate system. This can be done by expressing the laws of nature in the tensor equations because the tensor equations has exactly the same form in the coordinate systems.

Principle of equivalence: The principle of equivalence states that the gravitational forces and inertial forces are equivalent and are indistinguishable from each other. It also follows from the principle of equivalence that inertial mass and gravitational mass are equal.

Problem 2.6. Show that geodesics equations of motion are reducible to Newtonian equations of motion in case of a weak static field.

Solution. Consider, the motion of a test particle in case of a weak static field. The motion of a test particle is governed by geodesic equation as given below:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\cdot\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \tag{2.10}$$

Since the field is static i.e., it does not change with time.
Hence, velocity component can be taken as,

$$\frac{\mathrm{d}x^1}{\mathrm{d}s}, \frac{\mathrm{d}x^2}{\mathrm{d}s}, \frac{\mathrm{d}x^3}{\mathrm{d}s} = 0 \quad \text{and} \quad \frac{\mathrm{d}x^0}{\mathrm{d}s} = 1 \tag{2.11}$$

Our coordinates are Galilean coordinate, $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$. A weak static field is characterized by taking

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{such that } g_{\mu\nu} = 0 \text{ for } \mu \neq \nu.$$

Here, $\eta_{\mu\nu}$ is a metric tensor for Galilean values and $h_{\mu\nu}$ is a function of x, y and z.
The derivation of the metric from unity is represented through $h_{\mu\nu}$. The quantities $h_{\mu\nu}$ are taken to be small so that the powers of $h_{\mu\nu}$ higher than the first are neglected, we have,

$$\eta_{11} = \eta_{22} = \eta_{33} = -\eta_{00} = -1, \quad \eta_{\mu\nu} = 0 = g_{\mu\nu}(\mu \neq \nu) \tag{2.12}$$

$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ gives,

$$1 = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}$$

By virtue of (2.11) gives,

$$1 = g_{00}\,c\frac{\mathrm{d}t}{\mathrm{d}s}c\frac{\mathrm{d}t}{\mathrm{d}s}$$

$$\Rightarrow 1 = (1+h_{00})c^2\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s}$$

$$\Rightarrow \mathrm{d}s^2 = (1+h_{00})c^2\,\mathrm{d}t^2$$

Taking first approximation,

$$\mathrm{d}\, s^2 = c^2 \,\mathrm{d}\, t^2 \quad \Rightarrow \quad \mathrm{d}\, s = c \,\mathrm{d}\, t \tag{2.13}$$

By virtue of (2.11), (2.10) becomes,

$$\begin{aligned}
&\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, s^2} + \Gamma^\alpha_{00} \frac{\mathrm{d}\, x^0}{\mathrm{d}\, s} \cdot \frac{\mathrm{d}\, x^0}{\mathrm{d}\, s} = 0 \\
\Rightarrow\ &\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, s^2} = -\Gamma^\alpha_{00} \left(\frac{\mathrm{d}\, x^0}{\mathrm{d}\, s}\right)^2 \\
\Rightarrow\ &\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, s^2} = -\Gamma^\alpha_{00} 1^2 \\
\Rightarrow\ &\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, s^2} = -\Gamma^\alpha_{00} \\
\Rightarrow\ &-\Gamma^\alpha_{00} = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, s^2} \\
\Rightarrow\ &-\Gamma^\alpha_{00} = \frac{\mathrm{d}}{\mathrm{d}\, s}\left(\frac{\mathrm{d}\, x^\alpha}{\mathrm{d}\, s}\right) \\
\Rightarrow\ &-\Gamma^\alpha_{00} = \frac{\mathrm{d}}{c\,\mathrm{d}\, t}\left(\frac{\mathrm{d}\, x^\alpha}{c\,\mathrm{d}\, t}\right) \\
\Rightarrow\ &-\Gamma^\alpha_{00} = \frac{1}{c^2}\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, t^2} \ \text{[by (2.13)]} \\
\Rightarrow\ &\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, t^2} = -c^2 \Gamma^\alpha_{00}
\end{aligned}$$

It is easy to show that $\Gamma^0_{00} = 0$

Hence,

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, t^2} = -c^2 \Gamma^\alpha_{00},\ (\alpha = 1, 2, 3) \tag{2.14}$$

Now,

$$\begin{aligned}
\Gamma^\alpha_{00} &= g^{\alpha\beta}\Gamma^0_{00,\beta} \\
&= g^{\alpha\alpha}\Gamma^0_{00,\alpha} \\
&= g^{\alpha\alpha}\frac{1}{2}\left(2\frac{\partial g_{0\alpha}}{\partial x^\alpha} - \frac{\partial g_{00}}{\partial x^\alpha}\right) \\
&= g^{\alpha\alpha}\frac{1}{2}\left(-\frac{\partial g_{00}}{\partial x^\alpha}\right) \quad \text{[using (2.12)]} \\
&= \frac{1}{2g_{\alpha\alpha}}\left[-\frac{\partial}{\partial x^\alpha}(1 + h_{00})\right] \\
&= \frac{1}{2}(-1 + h_{\alpha\alpha})^{-1}\left(-\frac{\partial h_{00}}{\partial x^\alpha}\right) \\
&= \frac{1}{2}(1 - h_{\alpha\alpha})^{-1}\left(\frac{\partial h_{00}}{\partial x^\alpha}\right) \\
&= \frac{1}{2}\frac{\partial h_{00}}{\partial x^\alpha}
\end{aligned}$$

Now (2.14) becomes,

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, t^2} = -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^\alpha},\ (\alpha = 1, 2, 3) \tag{2.15}$$

Newtonian equations of motions are

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\, t^2} = -\frac{\partial \varphi}{\partial x^\alpha},\ (\alpha = 1, 2, 3) \tag{2.16}$$

where φ is potential function.
The equation (2.15) and (2.16) become identical if

$$
\begin{aligned}
-\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^\alpha} &= -\frac{\partial \varphi}{\partial x^\alpha} \\
\therefore \frac{\partial h_{00}}{\partial x^\alpha} &= \frac{2}{c^2}\frac{\partial \varphi}{\partial x^\alpha} \\
\Rightarrow \int \frac{\partial h_{00}}{\partial x^\alpha}\,\mathrm{d}\,x^\alpha &= \frac{2}{c^2}\int \frac{\partial \varphi}{\partial x^\alpha}\,\mathrm{d}\,x^\alpha \\
\Rightarrow \int \mathrm{d}\,h_{00} &= \frac{2}{c^2}\int \mathrm{d}\,\varphi \\
\Rightarrow h_{00} &= \frac{2\varphi}{c^2} + k_1 \\
\Rightarrow 1 + h_{00} &= \frac{2\varphi}{c^2} + k_2 \\
\Rightarrow g_{00} &= \frac{2\varphi}{c^2} + k
\end{aligned}
$$

Choosing φ such that when $g_{00} = 1$, $\varphi = 0$, so that $k = 1$, then

$$
g_{00} = 1 + \frac{2\varphi}{c^2}
$$

Hence geodesic equations are reducible to Newtonian equations of motion in case of weak static field if

$$
\begin{aligned}
g_{00} &= 1 + \frac{2\varphi}{c^2} \\
g_{00} &= 1 + 2\varphi, \quad \text{if } c = 1.
\end{aligned}
$$

Problem 2.7. Derive Schwarzschild interior solution for a spherically symmetric distribution of matter with constant density.

Problem 2.8. Deduce Einstein's field equations for interior material world in the form

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}
$$

explaining the significance of the symbols used. Hence, obtain Poission's equation on approximation for a very weak static field.

Problem 2.9. Derive Einstein tensor. Prove that the divergence of $\{R^\mu_\nu - \frac{1}{2}g^\mu_\nu R\}$ is identically zero.

Solution. The Bianchi identities are given by,

$$
R^\lambda_{\mu\nu\sigma,\rho} + R^\lambda_{\mu\sigma\rho,\nu} + R^\lambda_{\mu\rho\nu,\sigma} = 0 \tag{2.17}
$$

Using anti-symmetry property in the second term we get,

$$
R^\lambda_{\mu\nu\sigma,\rho} - R^\lambda_{\mu\rho\sigma,\nu} + R^\lambda_{\mu\rho\nu,\sigma} = 0 \tag{2.18}
$$

contracting with respect to λ to σ we get,

$$
R^\lambda_{\mu\nu\lambda,\rho} - R^\lambda_{\mu\rho\lambda,\nu} + R^\lambda_{\mu\rho\nu,\lambda} = 0 \tag{2.19}
$$

By definition of Ricci tensor,

$$R^{\lambda}_{\mu\nu\lambda} = R_{\mu\nu} \qquad \text{and} \qquad R^{\lambda}_{\mu\rho\lambda} = R_{\mu\rho}$$

So equation (2.19) becomes

$$R_{\mu\nu,\rho} - R_{\mu\rho,\nu} + R^{\lambda}_{\mu\rho\nu,\lambda} = 0 \tag{2.20}$$

Since, derivatives of fundamental tensors are zero, we may write the equation (2.20) as,

$$\begin{aligned}
& g^{\mu\rho} R_{\mu\nu,\rho} - g^{\mu\rho} R_{\mu\rho,\nu} + g^{\mu\rho} R^{\lambda}_{\mu\rho\nu,\lambda} = 0 \\
\Rightarrow\ & \left(g^{\mu\rho} R_{\mu\nu}\right)_{,\rho} - \left(g^{\mu\rho} R_{\mu\rho}\right)_{,\nu} + \left(g^{\mu\rho} R^{\lambda}_{\mu\rho\nu}\right)_{,\lambda} = 0 \\
\Rightarrow\ & R^{\rho}_{\nu,\rho} - R_{,\nu} + R^{\lambda}_{\nu,\lambda} = 0
\end{aligned}$$

changing the dummy indices ρ and λ to μ, we get,

$$\begin{aligned}
& R^{\mu}_{\nu,\mu} - R_{,\nu} + R^{\mu}_{\nu,\mu} = 0 \\
\Rightarrow\ & 2R^{\mu}_{\nu,\mu} - R_{,\nu} = 0 \qquad (2.21)
\end{aligned}$$

But,

$$R_{,\nu} = \frac{\partial\, R}{\partial\, x^{\nu}} = \frac{\partial}{\partial\, x^{\mu}} \left(g^{\mu}_{\nu}\, R\right) = \left(g^{\mu}_{\nu}\, R\right)_{,\mu}$$

So (2.21) becomes,

$$\begin{aligned}
& 2R^{\mu}_{\nu,\mu} - \left(g^{\mu}_{\nu}\, R\right)_{,\mu} = 0 \\
\Rightarrow\ & \left(R^{\mu}_{\nu} - \frac{1}{2} g^{\mu}_{\nu}\, R\right)_{,\mu} = 0 \\
\Rightarrow\ & G^{\mu}_{\nu,\mu} = 0
\end{aligned}$$

where, $G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2} g^{\mu}_{\nu}\, R$ is called Einstein tensor.

2nd Part:

$$\begin{aligned}
& \text{div } \left(G^{\mu}_{\nu}\right) = G^{\mu}_{\nu,\mu} = 0 \\
\Rightarrow & \text{div } \left(R^{\mu}_{\nu} - \frac{1}{2} g^{\mu}_{\nu}\, R\right) = 0
\end{aligned}$$

Therefore, the divergence of Einstein's tensor is identically zero.

Problem 2.10. Derive the equation for a planetary orbit and obtain expression for the advance of the perihelion to an orbit.

Solution. We are to determine the differential equations of the path of a planet moving round the sun. In comparison to the sun, the planets may be regarded as small free particles, their space-time trajectories are given by geodesic equations.

$$\frac{\mathrm{d}^{\,2} x^{\alpha}}{\mathrm{d}\, s^2} + \Gamma^{\alpha}_{ij} \frac{\mathrm{d}\, x^{i}}{\mathrm{d}\, s} \cdot \frac{\mathrm{d}\, x^{j}}{\mathrm{d}\, s} = 0 \tag{2.22}$$

The field surrounding the sun may be taken as the field of an isolated particle at rest at the origin. So we consider the line element,

$$\mathrm{d}\, s^2 = -e^{\lambda}\, \mathrm{d}\, r^2 - r^2 \left(\mathrm{d}\, \theta^2 + \sin^2\theta\ \mathrm{d}\ \theta^2\right) + e^2\, \mathrm{d}\, t^2 \tag{2.23}$$

with $e^{\nu} = e^{-\lambda} = 1 - \dfrac{2GM}{r}$.
We have,

$$g_{\mu\nu} = \begin{pmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & e^2 \end{pmatrix}, g_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu$$

$$\therefore\ g_{\mu\nu} = \begin{pmatrix} -e^{\lambda} & 0 & 0 & 0 \\ 0 & -r^{-2} & 0 & 0 \\ 0 & 0 & \frac{1}{-r^2\sin^2\theta} & 0 \\ 0 & 0 & 0 & e^{-2} \end{pmatrix}, g^{\mu\nu} = 0 \quad \text{for } \mu \neq \nu$$

Our coordinates are $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = ct$.
Non-vanishing Christoffel's symbols are,

$$\Gamma^1_{11} = \frac{\lambda'}{2},\ \Gamma^1_{22} = -re^{-\lambda},\ \Gamma^1_{44} = \frac{\nu'}{2}e^{\nu-\lambda},\ \Gamma^2_{12} = \frac{1}{r},\ \Gamma^3_{23} = \cot\,\theta,\ \Gamma^3_{13} = \frac{1}{r},$$

$$\Gamma^1_{33} = -r\sin^2\,\theta\, e^{-\lambda},\ \Gamma^4_{44} = \frac{\nu'}{2},\ \Gamma^2_{33} = -\sin\,\theta\,\cos\,\theta$$

Here the velocity c of light is taken to be unit in order to use astronomical units.
For $\alpha = 1$,

$$\frac{d^2x^1}{ds^2} + \Gamma^1_{ij}\frac{dx^i}{ds}\cdot\frac{dx^j}{ds} = 0$$

$$\text{i.e., } \frac{d^2r}{ds^2} + \Gamma^1_{11}\left(\frac{dx^1}{ds}\right)^2 + \Gamma^1_{22}\left(\frac{dx^2}{ds}\right)^2 + \Gamma^1_{33}\left(\frac{dx^3}{ds}\right)^2 + \Gamma^1_{44}\left(\frac{dx^4}{ds}\right)^2 = 0$$

$$\Rightarrow \frac{d^2r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dx^1}{ds}\right)^2 - re^{-\lambda}\left(\frac{dx^2}{ds}\right)^2 - r\sin^2\,\theta\, e^{-\lambda}\left(\frac{dx^3}{ds}\right)^2 + \frac{\nu'}{2}e^{\nu-\lambda}\left(\frac{dx^4}{ds}\right)^2 = 0 \tag{2.24}$$

For $\alpha = 2$,

$$\frac{d^2x^2}{ds^2} + \Gamma^2_{ij}\frac{dx^i}{ds}\cdot\frac{dx^j}{ds} = 0$$

$$\Rightarrow \frac{d^2\theta}{ds^2} + \Gamma^2_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^2_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma^2_{33}\left(\frac{dx^3}{ds}\right)^2 = 0$$

$$\Rightarrow \frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\,\theta\,\cos\,\theta\left(\frac{d\varphi}{ds}\right)^2 = 0 \tag{2.25}$$

For $\alpha = 3$,

$$\frac{d^2x^3}{ds^2} + \Gamma^3_{ij}\frac{dx^i}{ds}\cdot\frac{dx^j}{ds} = 0$$

$$\Rightarrow \frac{d^2\varphi}{ds^2} + 2\Gamma^3_{13}\frac{dx^1}{ds}\frac{dx^3}{ds} + 2\Gamma^3_{23}\frac{dx^2}{ds}\frac{dx^3}{ds} = 0$$

$$\Rightarrow \frac{d^2\varphi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} - \cot\,\theta\,\frac{d\theta}{ds}\frac{d\varphi}{ds} = 0 \tag{2.26}$$

For $\alpha = 4$,

$$\frac{\mathrm{d}^2x^4}{\mathrm{d}s^2} + \Gamma^4_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}s}\cdot\frac{\mathrm{d}x^j}{\mathrm{d}s} = 0$$
$$\Rightarrow \frac{\mathrm{d}^2t}{\mathrm{d}s^2} + 2\Gamma^4_{14}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^4}{\mathrm{d}s} = 0$$
$$\Rightarrow \frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \nu'\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} = 0 \tag{2.27}$$

Choosing coordinates such that the planet moves initially in the plane $\theta = \frac{\pi}{2}$ so that $\cos\theta = 0$, $\sin\theta = 1$, $\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0$.
Substituting these values in (2.24), (2.25), (2.26) and (2.27),

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2} + \frac{\lambda'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - re^{-\lambda}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 + \frac{\nu'}{2}e^{\nu-\lambda}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 0 \tag{2.28}$$
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = 0 \tag{2.29}$$
$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0 \tag{2.30}$$
$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \nu'\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} = 0 \tag{2.31}$$

The equation (2.29) shows that a particle which starts moving in the plane $\theta = \frac{\pi}{2}$ contains to move in the same plane.
From (2.30),

$$r^2\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + 2r\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right) = 0$$

Integrating this we get,

$$r^2\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \text{ constant } = L \qquad [\text{ Some author use } h \text{ for } L.] \tag{2.32}$$

where $L =$ constant $=$ orbital angular momentum per unit mass.
From (2.31), we get

$$e^2\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + e^2\cdot\nu'\cdot\frac{\mathrm{d}r}{\mathrm{d}s}\cdot\frac{\mathrm{d}t}{\mathrm{d}s} = 0$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}s}\left(e^2\frac{\mathrm{d}t}{\mathrm{d}s}\right) = 0$$

Integrating,

$$e^2\frac{\mathrm{d}t}{\mathrm{d}s} = \text{ constant } = E \tag{2.33}$$

For a massive particle, E is the energy.
Along the geodesic $-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \varepsilon$ is a constant.
For a massive particle

$$\lambda = \tau, \quad \varepsilon = -1 \quad [\text{metric} - + \text{ represents -1}]$$

Part III

Cosmology

Chapter 3

Cosmology

Problem 3.1. State the principle of Cosmology. Derive the Robertson-Walker metric.

Solution. Cosmological Principle: On a large-scale ($200Mpc$), the universe appears to be homogeneous and isotropic. This is known as the Cosmological Principle. By homogeneity, we mean that the universe is the same at all points in space and by isotropy the universe is the same in all spatial direction about any point. This means that there is no preferred direction or a preferred location in the universe.

Robertson Walker metric

Let x, y, z and w be the Cartesian co-ordinates in E_4 with x, y, z being the usual spatial co-ordinates of E_3. Then the hyper surface has the equation when $k = 1$.

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 &= R^2(t) \\ \Rightarrow r^2 + w^2 &= R^2(t) \end{aligned} \tag{3.1}$$

where r, θ, ϕ be the spherical polar co-ordinates in E_3.
Differentiating the above equation, we get

$$\begin{aligned} r\,\mathrm{d}\,r &= -w\,\mathrm{d}\,w \\ \Rightarrow r^2\,\mathrm{d}\,r^2 &= w^2\,\mathrm{d}\,w^2 \end{aligned} \tag{3.2}$$

The special metric on the hyper surface is given by

$$\begin{aligned} \mathrm{d}\,l^2 &= \mathrm{d}\,r^2 + r^2\,\mathrm{d}\,\theta^2 + r^2\sin^2\theta\,\mathrm{d}\,\varphi^2 + \mathrm{d}\,w^2 \\ &= \mathrm{d}\,r^2 + \frac{r^2\,\mathrm{d}\,r^2}{R^2 - r^2} + r^2\left(\mathrm{d}\,\theta^2 + \sin^2\theta\,\mathrm{d}\,\varphi^2\right) \\ &= \frac{R^2\,\mathrm{d}\,r^2}{R^2 - r^2} + r^2\,\mathrm{d}\,\Omega^2 \qquad \text{Where, } \mathrm{d}\,\Omega^2 = \mathrm{d}\,\theta^2 + \sin^2\theta\,\mathrm{d}\,\varphi^2 \end{aligned}$$

A change in angle θ produce a displacement $r\,\mathrm{d}\,\theta$ while a change in r in any direction gives a displacement of $\frac{R\,\mathrm{d}\,r}{(R^2 - r^2)^{1/2}}$.
When the variable $\sigma = \frac{\pi}{r}$ is used, the metric equation reduce to

$$\mathrm{d}\,l^2 = R^2(t)\left[\frac{\mathrm{d}\,\sigma^2}{1 - \sigma^2} + \sigma^2\,\mathrm{d}\,\Omega^2\right]$$

Now we can write the complete metric equation in co-ordinating time

$$\mathrm{d}\,s^2 = c^2\,\mathrm{d}\,t^2 - R^2(t)\left[\frac{\mathrm{d}\,\sigma^2}{1-\sigma^2} + \sigma^2\,\mathrm{d}\,\Omega^2\right]$$

This is Robertson-Walker metric for $k = 1$.
More generally, the curvature k could be negative or zero.
General form,

$$\mathrm{d}\,s^2 = c^2\,\mathrm{d}\,t^2 - R^2(t)\left[\frac{\mathrm{d}\,\sigma^2}{1-k\sigma^2} + \sigma^2\,\mathrm{d}\,\Omega^2\right]$$

This is the Robertson-Walker metric for isotropic and homogeneous space time.

Problem 3.2. What is black hole? Write down and discuss the most general black hole solution. How do you reduce this to Kerr and Reissnar-Nordstrom black hole solutions?

Solution. A black hole is a region of space time where gravity is so strong that nothing - no particles or, even electromagnetic radiation such as light can escape from it. The theory of general relativity predicts that a sufficiently compact mass can deform spacetime to form a black hole.

The Kerr-Newmann black hole solution is the most general black hole solution. In 1963, Kerr had obtained a metric for the space time of mass, which is convenient form for this line element is,

$$\mathrm{d}\,s^2 = \frac{\Delta}{\rho^2}(\mathrm{d}\,T - h\sin^2\theta\,\mathrm{d}\,\varphi)^2 - \frac{\sin^2\theta}{\rho}\left[(R^2+h^2)\,\mathrm{d}\,\varphi - h\,\mathrm{d}\,T\right]^2 - \frac{\rho^2}{\Delta}\,\mathrm{d}\,R^2 - \rho^2\,\mathrm{d}\,\varphi^2 \tag{3.3}$$

where,

$$h = \frac{H}{M} = \text{ angular momentum per unit mass}$$
$$\Delta = R^2 - 2FMR + h^2$$
$$\rho^2 = R^2 + h^2\cos^2\theta \text{ and } (T,\,R,\,\theta,\,\varphi) \text{ co-ordinate; } h,\,M \text{ are parameter.}$$

The Kerr-Newmann Black hole space time is described by the metric

$$\begin{aligned}\mathrm{d}\,s^2 = {} & (r^2+a^2\cos^2\theta)\left(\frac{\mathrm{d}\,r^2}{r^2-2mr+e^2+a^2} + \mathrm{d}\,\theta\right) + \sin^2\theta\left\{r^2+a^2+\frac{a^2\sin^2\theta(2mr-e^2)}{r^2+a^2\cos^2\theta}\right\}\mathrm{d}\,\varphi^2 \\ & - \left(1-\frac{2mr-e^2}{r^2+a^2\cos^2\theta}\right)\mathrm{d}\,t^2 + \frac{2a\sin^2\theta(2mr-e^2)}{r^2+a^2\cos^2\theta}\,\mathrm{d}\,t\,\mathrm{d}\,\varphi\end{aligned}$$

where, m is the mass, a is angular momentum per unit mass, e is electric charge.

This metric includes:

(i) Kerr-Black hole space time when $e = 0$.

(ii) Reisner-Nordstrom black hole space time if $a = 0$.

(iii) Schwarzschild black hole space time for $e = a = 0$.

Problem 3.3. Deduce the Friedmann model of the universe for $P = 0$ and find out its graph for different values of k.

Solution. The dynamical equations of cosmology that describe the evaluation of the scale factor $R(t)$ follows from the Einstein's field equations are

$$\dot{R}^2 + k = \frac{8\pi\rho R^2}{3} \tag{3.4}$$

$$\dot{\rho} + 3(P+\rho)\frac{\dot{R}}{R} = 0 \tag{3.5}$$

The standard Friedmann model arise from the case $p = 0$.
In the case $p = 0$, we have from the equation (3.5),

$$\begin{aligned}
&\dot{\rho} + \frac{3\rho\dot{R}}{R} = 0 \\
\Rightarrow\ &\log\ \rho + 3\log\ R = \log\ c \qquad \text{[After integration]} \\
\Rightarrow\ &\log\ \left(\rho R^3\right) = \log\ c \\
\Rightarrow\ &\rho R^3 = c
\end{aligned}$$

If the present age of the universe is t_0, then $\rho_0 = \rho(t_0)$, $\quad , R_0 = R(t_0)$, so $\rho_0 {R_0}^3 = c$
Hence

$$\rho R^3 = \rho_0 {R_0}^3 \tag{3.6}$$

Using (3.6) in (3.4) we get,

$$\begin{aligned}
\dot{R}^2 + K &= \frac{8\pi\rho R^3}{3R} \\
&= \frac{8\pi\rho_0 {R_0}^3}{3R} \\
&= \frac{A^2}{R} \qquad \text{where} \qquad A^2 = \frac{8\pi\rho_0 {R_0}^3}{3}
\end{aligned} \tag{3.7}$$

Now Hubbles constant $H(t)$ is defined by

$$H(t) = \frac{\dot{R}}{R(t)} \qquad \text{and} \qquad H_0 = \frac{\dot{R}}{R_0} \tag{3.8}$$

Equation (3.4) gives,

$$\begin{aligned}
\frac{\dot{R}^2}{{R_0}^2} + \frac{k}{{R_0}^2} &= \frac{8\pi\rho_0}{3} \\
\Rightarrow \frac{k}{{R_0}^2} &= \frac{8\pi\rho_0}{3} - {H_0}^2 \\
\Rightarrow \frac{k}{{R_0}^2} &= \frac{8\pi}{3}\left(\rho_0 - \frac{3{H_0}^2}{8\pi}\right) \\
\Rightarrow \frac{k}{{R_0}^2} &= \frac{8\pi}{3}\left(\rho_0 - \rho_c\right)
\end{aligned} \tag{3.9}$$

where, ρ_c is the critical density given by

$$\rho_c = \frac{3{H_0}^2}{8\pi} \tag{3.10}$$

The three fried models arise are,

(i) Flat model: When $k = 0$, then $\rho_0 = \rho_c$ and equation (3.7) becomes

$$
\begin{aligned}
\dot{R}^2 &= \frac{A^2}{R} \\
\Rightarrow\ \dot{R} &= \frac{A}{\sqrt{R}} \\
\Rightarrow\ \sqrt{R}\,\mathrm{d}\,R &= A\,\mathrm{d}\,t \\
\Rightarrow\ R^{3/2} &= \frac{3}{2}At \\
\Rightarrow\ R(t) &= \left(\frac{3}{2}A\right)^{2/3} t^{2/3}
\end{aligned}
$$

This is also known as the Einstein's de-silter model.

(ii) Closed model: When $k = 1$, then $\rho_0 > \rho_c$ and equation (3.7) becomes

$$
\begin{aligned}
\dot{R}^2 + 1 &= \frac{A^2}{R} \\
\Rightarrow\ \dot{R} &= \frac{\sqrt{A^2 - R}}{\sqrt{R}} \\
\Rightarrow\ \mathrm{d}\,t &= \frac{\sqrt{R}}{\sqrt{A^2 - R}}\,\mathrm{d}\,R \\
\Rightarrow\ t &= \int \frac{A\sin\frac{\psi}{2}}{A\cos\frac{\psi}{2}} A^2 \sin\frac{\psi}{2}\cos\frac{\psi}{2}\,\mathrm{d}\,\psi \quad [\text{let, } R = A^2\sin^2\psi/2] \\
\Rightarrow\ t &= \frac{A^2}{2}\int(1 - \cos\psi)\,\mathrm{d}\,\psi \\
\Rightarrow\ t &= \frac{A^2}{2}(\psi - \sin\psi)
\end{aligned}
$$

So,

$$
R = \frac{A^2}{2}(1 - \cos\psi) \qquad \text{and} \qquad t = \frac{A^2}{2}(\psi - \sin\psi)
$$

These two equations gives cycloid and shown in figure.

(iii) Open model: When $k = -1$, then $\rho_0 < \rho_c$ and equation (3.7) becomes

$$
\begin{aligned}
\dot{R}^2 - 1 &= \frac{A^2}{R} \\
\Rightarrow\ \dot{R} &= \frac{\sqrt{A^2 + R}}{\sqrt{R}} \\
\Rightarrow\ \mathrm{d}\,t &= \frac{\sqrt{R}}{\sqrt{A^2 + R}}\,\mathrm{d}\,R \\
\Rightarrow\ t &= \int \frac{A\sinh\frac{\psi}{2}}{A\cosh\frac{\psi}{2}} A^2 \sinh\frac{\psi}{2}\cosh\frac{\psi}{2}\,\mathrm{d}\,\psi \quad [\text{let, } R = A^2\sinh^2\psi/2] \\
\Rightarrow\ t &= \frac{A^2}{2}\int(\cosh\psi - 1)\,\mathrm{d}\,\psi \\
\Rightarrow\ t &= \frac{A^2}{2}(\sinh\psi - \psi)
\end{aligned}
$$

So,

$$
R = \frac{A^2}{2}(\cosh\psi - 1) \qquad \text{and} \qquad t = \frac{A^2}{2}(\sinh\psi - \psi)
$$