Chapter 1

Subnormal Series of Group

Definition 1. A subnormal (or, subinvariant) series of a group G is a finite sequence of subgroups

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_r = \{e\} \tag{1.1}$$

or,

$$\{e\} = G_r \lhd G_{r-1} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

such that each G_i is a normal subgroup of G_{i-1}, where $i = 1, 2, \ldots, r$.

Note. In the above series, r is called the length of the subnormal series; observe that the number of terms in the subnormal series is $(r + 1)$.

Remark. In the definition of a subnormal series, it is not demanded that each G_i is a proper subgroup of G_{i-1}.

Example. $S_4 \rhd V \rhd 1$ is a subnormal series for the group S_4 where

$$V = \{(1), (12)(34), (13)(24), (14)(23)\} \text{ and } 1 = \{(1)\}.$$

Example. $S_4 \rhd V \rhd C \rhd 1$ is another subnormal series for the group S_4, where

$$C = \{(1), (12)(34)\}.$$

Definition 2. A normal (or, invariant) series of a group G is a subnormal series such that each of its terms is a normal subgroup of G.

Example. The group S_3 has the normal subgroup $N = \{(1), (123), (132)\}$.
So, $S3 \supseteq N \supseteq \{(1)\}$ is a normal series.

Example. The group S_4 has the normal subgroup

$$V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}.$$

So, $S_4 \rhd V_4 \rhd \{(1)\}$ is a normal series.
Besides, we observe that $W = \{(1), (12)(34)\}$ is a normal subgroup of V_4 (because V_4 is abelian), but W is not a normal subgroup of S_4.
So, $S_4 \rhd V_4 \rhd W \rhd \{(1)\}$ is a subnormal series, but not a normal series.

Note. Some authors use the term 'normal series' for our 'subnormal series'.

Definition 3. Given two subnormal series of G, one is a refinement of the other if each term of the latter one series occurs as a term of the former series.

Example. The subnormal series $S_4 \triangleright V \triangleright C \triangleright 1$ is a refinement of the subnormal series $S_4 \triangleright V \triangleright 1$.

Definition 4. The (normal) subgroups $G_0, G_1, G_2, \ldots, G_r$ are called the terms of the series and the factor groups $G_{i-1}/G_i (i = 1, 2, \ldots, r)$ are called the factors of the series. The series (1.1) is called a proper subnormal series if every G_i is a proper normal subgroup of $G_{i-1}(i = 1, 2, \ldots, r)$.

Definition 5. The series

$$\{e\} = J_0 \triangleleft J_1 \triangleleft J_2 \triangleleft \cdots \triangleleft J_m = G$$

is said to be a proper refinement of the series

$$\{e\} = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_n = G$$

(of the same group G) if there is a $j \in \{0, 1, \ldots, m\}$ such that $H_i \neq J_j$ holds for $i \in \{0, 1, \ldots, n\}$.

Definition 6. Two subnormal series of a given group G, say

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_r = 1$$

and

$$G = H_0 \triangleright H_1 \triangleright H_2 \triangleright \cdots \triangleright H_s = 1$$

are called isomorphic (or equivalent) if there is a one-one correspondence between the set of non-trivial factor groups G_{i-1}/G_i and the set of non-trivial factor groups H_{j-i}/H_j such that the corresponding factor groups are isomorphic.

Example. The following two subnormal series (of the cyclic group C_6 of order 6)

$$C_6 \triangleright C_3 \triangleright 1 \text{ and } C_6 \triangleright C2 \triangleright 1$$

are isomorphic, for

$$C_6/C_3 \cong C_2/1 \text{ and } C_3/1 \cong C_6/C_2.$$

Example. Let $G = \langle a \rangle$ be a cyclic group of order 24 so that $o(a) = 24$. Consider the following two normal series

$$G = \langle a \rangle \triangleright \langle a^2 \rangle \triangleright \langle a^6 \rangle \triangleright \langle a^{12} \rangle \triangleright \{e\} \tag{1.2}$$

and

$$G = \langle a \rangle \triangleright \langle a^3 \rangle \triangleright \langle a^6 \rangle \triangleright \langle a^{12} \rangle \triangleright \{e\} \tag{1.3}$$

The factors of (1.2) are

$$\langle a \rangle / \langle a^3 \rangle, \langle a^3 \rangle / \langle a^6 \rangle, \langle a^6 \rangle / \langle a^{12} \rangle \text{ and } \langle a^{12} \rangle / \{e\},$$

which are of orders 2, 3, 2, 2 respectively. Since these are of prime orders, they are simple. Similarly, the factors of (1.3) are

$$\langle a\rangle/\langle a3\rangle, \langle a^3\rangle/\langle a^6\rangle, \langle a^6\rangle/\langle a^{12}\rangle \text{ and } \langle a^{12}\rangle/\{e\},$$

which are of orders 3, 2, 2, 2 respectively. These are again simple. So, (1.2) and (1.3) both are composition series of G. We see that both of these series are of same length (viz. 4)
Since two cyclic groups of same order are isomorphic, we then have

$$\langle a\rangle/\langle a^2\rangle \cong \langle a^3\rangle/\langle a^6\rangle, \langle a^2\rangle/\langle a^6\rangle \cong \langle a\rangle/\langle a^3\rangle,$$

$$\langle a^6\rangle/\langle a^{12}\rangle \cong \langle a^6\rangle/\langle a^{12}\rangle \text{ and } \langle a^{12}\rangle/\{e\} \cong \langle a^{12}\rangle/\{e\}.$$

Thus there is a one-one correspondence between the factors of (1.2) and those of (1.3) such that the corresponding factors are isomorphic.

Theorem 1.1 (Schreier's Refinement Theorem). Any two subnormal series of a given group possess isomorphic refinements.