Exercise 2.16

Problem statement

Explain, in general, why equivalent algebraic expressions may lead to different answers. Can you devise an interval-arithmetic package that does not have this shortcoming, or is this task impossible? (Warning: This problem is very difficult.)

Preliminaries

The width of an interval $[a, b]$ is defined as

$$w = b - a \tag{1}$$

A general operation $\circ$ on two intervals $[a_1, b_1]$ and $[a_2, b_2]$ is defined as

$$[a_1, b_1] \circ [a_2, b_2] = [\min(a_1 \circ a_2, a_1 \circ b_2, b_1 \circ a_2, b_1 \circ b_2), \max(a_1 \circ a_2, a_1 \circ b_2, b_1 \circ a_2, b_1 \circ b_2)] \tag{2}$$

In consequence, we can define some elementary operations. Addition:

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2] \tag{3}$$

Multiplication

$$[a_1, b_1] \times [a_2, b_2] = [\min(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2), \max(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2)] \tag{4}$$

Division

$$\frac{[a_1, b_1]}{[a_2, b_2]} = [a_1, b_1] \times \left[\frac{1}{a_2}, \frac{1}{b_2}\right] \tag{5}$$

Division comes with a few caveats. First, as noted in Exercise 2.10, dividing by a 0 width interval makes no sense and we updated the code to raise an error in that case. Second, if $a_2 \leq 0 \leq b_2$ the resulting interval will be undefined.

We also defined the unit interval

$$\mathbb{1} = [1, 1] \tag{6}$$

One hiccup we noticed was that for an interval A, $A/A \neq \mathbb{1}$. Thus when given two resistors [1] R_1 and R_2, two equivalent expressions

$$R_{\|} = \frac{R_1 R_2}{R_1 + R_2} \tag{7}$$

[1] each R is an interval, so for an interval $R_x = 5 \pm 0.05\ \Omega \iff R_x = [4.95, 5.05]$

and

$$R_{\parallel} = \frac{\mathbb{1}}{\frac{\mathbb{1}}{R_1} + \frac{\mathbb{1}}{R_2}} \tag{8}$$

produce different results. In ex 2.15 we arrived at the conclusion that the second form (8) produces better results than the first (7). We haven't explained however why algebraically equivalent expressions produce different results.

Comments

Obviously some extra reading is required before attempting this ourselves

- (pdf) A short very readable paper
- https://en.wikipedia.org/wiki/Interval_arithmetic
- Scheme Wiki
- Bill the Lizard's blog
- (pdf) Some paper

Intervals have now multiplicative inverse, meaning there are no non-degenerate intervals $[v_1, u_1]$ and $[v_2, u_2]$ so that $[v_1, u_1] \times [v_2, u_2] = \mathbb{1}$. The same holds for additive inverses, there are no non-degenerate intervals $[v_1, u_1]$ and $[v_2, u_2]$ so that $[v_1, u_1] + [v_2, u_2] = [0, 0]$. The distributive law fails as well, for 3 intervals A, B and C

$$A \times (B + C) \neq A \times B + A \times C$$

So far it's quite unlikely it is possible to design an interval arithmetics package that would please Lem and Eva.

Wikipedia mentions the following *The so-called dependency problem is a major obstacle to the application of interval arithmetic. Although interval methods can determine the range of elementary arithmetic operations and functions very accurately, this is not always true with more complicated functions. If an interval occurs several times in a calculation using parameters, and each occurrence is taken independently then this can lead to an unwanted expansion of the resulting intervals.* It then proceeds to show the example of the function $f(x) = x^2 + x$, which in practice applied on the interval $[-1, 1]$ should return $\left[-\frac{1}{4}, 2\right]$ but using the algebraic rules defined above returns the more broad interval $[-1, 2]$.

While the returned interval is always correct — in the sense the resulting interval contains the real one, thus the width w of the resulting operation is larger than needed.

In conclusion no package for interval arithmetic can be created to avoid this problem.