Exercise 1.13

Problem statement

Prove that $\texttt{fib}(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, where

$$\varphi = \frac{1+\sqrt{5}}{2} \tag{1}$$

Hint Let

$$\psi = \frac{1-\sqrt{5}}{2} \tag{2}$$

Use induction and the definition of the Fibonacci numbers

$$\texttt{fib}(n) = \begin{cases} 0 & \text{if} \quad n = 0, \\ 1 & \text{if} \quad n = 1, \\ \texttt{fib}(n-1) + \texttt{fib}(n-2) & \text{otherwise} \end{cases} \tag{3}$$

to prove that

$$\texttt{fib}(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{4}$$

Few preliminaries

If $\texttt{fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$ as per equation 4 and the recurrence relation from 3 then the following needs to be proven

$$\frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \tag{5}$$

We also have the following properties for φ and ψ

$$\begin{aligned} \varphi^2 = \varphi + 1, \quad \varphi = \tfrac{1}{\varphi} + 1 \\ \psi^2 = \psi + 1, \quad \psi = \tfrac{1}{\psi} + 1 \end{aligned} \tag{6}$$

Unfortunately these aren't handed as hints, though without them the calculations can get rather hairy.

Solution

Let's do a quick numerical sanity check

n	$\varphi^n/\sqrt{5}$	`fib`(n)
0	0.447	0
1	0.724	1
1	1.171	1
2	1.894	2
3	3.065	3
5	4.960	5
8	8.025	8
13	12.985	13
21	21.010	21
34	33.994	34
55	55.004	55
89	88.998	89
144	144.001	144
233	232.999	233

Values looks alright, but just looking at a handful of values isn't going to cut it here. Let's see if the formula $\texttt{fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$ holds true for $n \in \{0, 1, 2\}$

$$\texttt{fib}(0) = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = 0$$

First step looks all good

$$\texttt{fib}(1) = \frac{\varphi - \psi}{\sqrt{5}} = \left(\frac{1+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}} = \frac{2\sqrt{5}}{2}\frac{1}{\sqrt{5}} = 1$$

The function works alright for $n = 1$ too. Let's check it for $n = 2$

$$\texttt{fib}(2) = \texttt{fib}(1) + \texttt{fib}(0) = \frac{\varphi - \psi}{\sqrt{5}} + \frac{1-1}{\sqrt{5}} = \frac{(\varphi+1) - (\psi+1)}{\sqrt{5}} = \frac{\varphi^2 - \psi^2}{5} = 1$$

The recurrence relation 5 works for $n \in \{0, 1, 2\}$, but we still need to prove that it *generally* holds true. So let's do exactly that

$$\begin{array}{rcc} \frac{\varphi^n - \psi^n}{\sqrt{5}} & = & \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\ \Leftrightarrow \quad \varphi^n - \psi^n & = & \varphi^{n-1} - \psi^{n-1} + \varphi^{n-1} - \psi^{n-1} \\ & = & (\varphi^{n-1} + \varphi^{n-2}) - (\psi^{n-1} + \psi^{n-2}) \\ & = & \varphi^{n-2}(\varphi^2) - \psi^{n-2}(\psi^2) \\ & = & \varphi^n - \psi^n \quad \checkmark \end{array}$$

Now that we have shown that the recurrence relation 5 correctly returns the n^{th} Fibonacci numbers for *all* n, now we just need to see how $(\varphi^n - \psi^n)/\sqrt{5}$ compares to $\varphi^n/\sqrt{5}$, i.e is the relation

$$\frac{\varphi^n}{\sqrt{5}} \simeq \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{7}$$

true, and is it within less than < 0.5, so the values will round up to $\texttt{fib}(n)$?

$$\begin{aligned}\frac{\varphi^n}{\sqrt{5}} \simeq & \frac{\varphi^n - \psi^n}{\sqrt{5}} \\ & \frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} \\ & \frac{\varphi^n}{\sqrt{5}} - \left(\frac{1-\sqrt{5}}{2}\right)^n \frac{1}{\sqrt{5}} \\ & \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{(-0.61803)^n}{\sqrt{5}}\end{aligned}$$

As $(-0.61803)^n / \sqrt{5}$ is a very small quantity that only gets smaller, this equation holds true for all n, therefore QED.

Comments

Having a non-recursive formula for a function like the one that returns the n-th Fibonacci number can result in massive speedups in computing the function. We went from having a naive tree-recursive implementations, to a linear iterative approach, and finally to a single expression evaluation.