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Homework 2

Question 1

Task 1

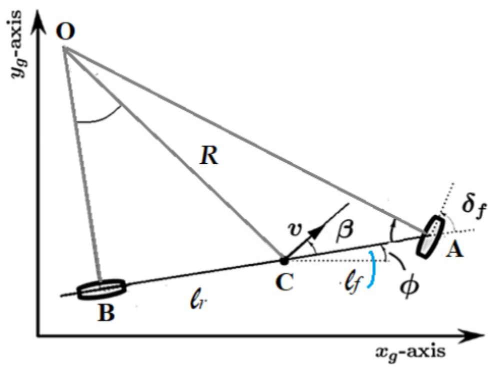

Figure 1: Bicycle Drive Kinematics

Finding ∠ BOC

Observing the $\triangle BOC$, we can state that:

$$\sin \angle BOC = \frac{Perp}{Hyp} = \frac{l_r}{R}$$

$$\angle BOC = \arcsin \frac{l_r}{R}$$

Finding ∠ COA

Now, looking at the $\triangle COA$, the angle at point A can be write as:

$$\angle A = 90 - \delta_f$$

Now at point C:

$$\angle C = 90 + \beta$$

We know that the sum of all angles of a triangle is equal to 180.

$$\angle C + \angle COA + \angle A = 180$$

Substituting angles C and A in above equation:

$$90 + \beta + \angle COA + 90 - \delta_f = 180$$
$$\angle COA = \delta_f - \beta$$

Task 2

Proof 1

Observing figure 1 △ BOC, we can write the ∠ C in terms of β as:

$$\angle C = 90 - \beta$$

Now, since we know that sum of all angles in a triangle is 180:

$$\angle B + \angle BOC + \angle C = 180$$
$$90 + \angle BOC + 90 - \beta = 180$$
$$\angle BOC = \beta$$

As we have already found the value of $\angle BOC$ in task 1, we can substitute in above eq:

$$\beta = \arcsin \frac{l_r}{R}$$
$$\sin \beta = \frac{l_r}{R}$$

Proof 2

We know $\angle BOC$ and $\angle COA$, thus we also know $\angle BOA = \delta_f$. In $\triangle BOA$:

$$\tan \delta_f = \frac{d}{OB}$$

Rearranging the above equation in terms of OB:

$$OB = \frac{d}{\tan \delta_f}$$

Moreover, in $\triangle BOC$:

$$\tan \beta = \frac{l_r}{OB}$$

Again, rearranging the above equation in terms of OB:

$$OB = \frac{l_r}{\tan\beta}$$

By comparing both equations:

$$\frac{d}{\tan\delta_f} = \frac{l_r}{\tan\beta}$$

$$\tan\beta = \frac{l_r}{d}\tan\delta_f$$

Task 3

We know that

$$v = R * \omega = R * \dot{\phi}$$

$$\dot{\phi} = \omega = \frac{v}{R}$$

From $\triangle BOC$, we know that:

$$\sin\beta = \frac{l_r}{R}$$

$$R = \frac{l_r}{\sin\beta}$$

Moreover, we know that from task 2 proof 2:

$$\tan\beta = \frac{l_r}{d}\tan\delta_f$$

$$\frac{\sin\beta}{\cos\beta} = \frac{l_r}{d}\tan\delta_f$$

$$\sin\beta = \frac{l_r}{d}\cos\beta\tan\delta_f$$

Substituting $\sin\beta$:

$$R = \frac{l_r}{\frac{l_r}{d}\cos\beta\tan\delta_f}$$

$$R = \frac{1}{\frac{1}{d}\cos\beta\tan\delta_f}$$

Finally, substituting expression of R in $\dot{\phi} = \frac{v}{R}$:

$$\dot{\phi} = \omega = \frac{v}{\frac{1}{\frac{1}{d}\cos\beta\tan\delta_f}}$$

$$\dot{\phi} = \frac{1}{d} v \cos\beta \tan\delta_f$$

Proved.

Task 4

MoR in direction of cycle

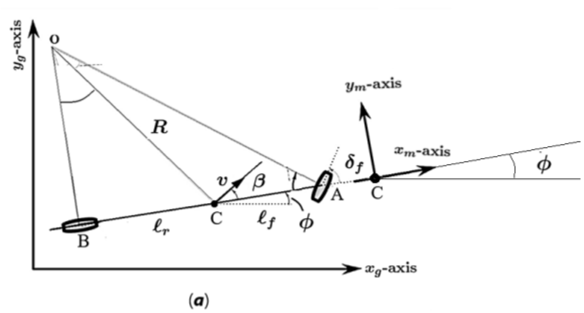

Figure 2: MoR in the direction of cycle

In this mobile frame of reference, we can write the velocities expressions as:

$$\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} V\cos\beta \\ V\sin\beta \end{bmatrix}$$

Now in order to relate to the global FoR we will simply multiply with the rotation matrix:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} V\cos\beta \\ V\sin\beta \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V(\cos\phi\cos\beta - \sin\phi\sin\beta) \\ V(\sin\phi\cos\beta + \cos\phi\sin\beta) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V\cos(\phi+\beta) \\ V\sin(\phi+\beta) \end{bmatrix}$$

MoR in direction of β

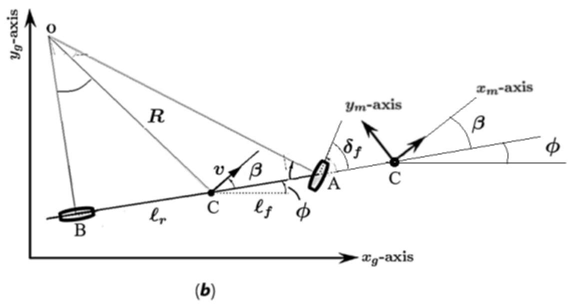

Figure 3: MoR in the direction of β

In this mobile frame of reference, we can write the velocities expressions as:

$$\begin{bmatrix} \dot{x_m} \\ \dot{y_m} \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{bmatrix} V\cos\beta \\ V\sin\beta \end{bmatrix}$$

Now relating to the global FoR we will simply multiply with the rotation matrix:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{x_m} \\ \dot{y_m} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} V\cos\beta \\ V\sin\beta \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V(\cos\phi\cos\beta - \sin\phi\sin\beta) \\ V(\sin\phi\cos\beta + \cos\phi\sin\beta) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V\cos(\phi+\beta) \\ V\sin(\phi+\beta) \end{bmatrix}$$

Task 5

Rectangular Integration

We have the expressions for the kinematic model:

$$\dot{x} = V\cos(\phi+\beta)$$
$$\dot{y} = V\sin(\phi+\beta)$$

For rectangular integration, we know that:

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} f(t)dt$$

Substituting the function value:

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} V\cos(\phi+\beta)dt$$

$$x_{n+1} = x_n + V\cos(\phi+\beta)\int_{nT}^{(n+1)T} dt = x_n + V\cos(\phi+\beta) * ((n+1)T - nT)$$

$$x_{n+1} = x_n + VT\cos(\phi+\beta)$$

Similarly for y:

$$y_{n+1} = y_n + VT\sin(\phi+\beta)$$

Now for ϕ_n:

$$\phi_{n+1} = \phi_n + \Delta\phi = \phi_n + T\omega_n$$

In summary,

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_{n+1} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_{n} + \begin{bmatrix} VT\cos(\phi+\beta) \\ VT\sin(\phi+\beta) \\ T\omega_n \end{bmatrix}$$

Trapezoidal Integration

We have the expressions for the kinematic model:

$$\dot{x} = V\cos(\phi+\beta)$$

$$\dot{y} = V\sin(\phi+\beta)$$

For Trapezoidal integration, we know that:

$$x_{n+1} \approx x_n + \frac{1}{2}(\Delta x_1 + \Delta x_2)$$

where;

$$\Delta x_1 = Tf(x(t_n), u(t_n))$$

$$\Delta x_2 = Tf(x(t_{n+1}), u(t_{n+1}))$$

$$\Delta x_2 \approx Tf(x(t_n) + \Delta x_1, u(t_n))$$

For x:

$$x_{n+1} \approx x_n + \frac{1}{2}(TVcos(\phi_n + \beta_n) + TVcos(\phi_n + \beta_n + \Delta\phi_n))$$

$$x_{n+1} \approx x_n + \frac{TV}{2}(cos(\phi_n + \beta_n) + cos(\phi_n + \beta_n + T\omega_n))$$

Similarly, for y:

$$y_{n+1} \approx y_n + \frac{TV}{2}(sin(\phi_n + \beta_n) + sin(\phi_n + \beta_n + T\omega_n))$$

Now for ϕ_n:

$$\phi_{n+1} \approx \frac{T}{2}(\omega_n + \omega_{n+1}) = \phi_n + T\omega_n$$

In summary,

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_{n+1} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_n + T \begin{bmatrix} \frac{V}{2}(cos(\phi_n + \beta_n) + cos(\phi_n + \beta_n + T\omega_n)) \\ \frac{V}{2}(sin(\phi_n + \beta_n) + sin(\phi_n + \beta_n + T\omega_n)) \\ \omega_n \end{bmatrix}$$

Task 6

```
Ts = 0.01;
ITER = 1e3;
omega = 2*pi/10;
v = 0.5; % bicycle velocity
d = 2; % length of chasis
lr = d*0.6;
t=linspace(-pi,pi,ITER);
%resulting forward kinematic velocities
x_f = zeros(1, ITER); y_f = zeros(1, ITER);

% steering angle Δ f
Δ_f = 2*cos(omega*t);
%plot(t,Δ_f); figure();

%velocity-chassis angle
beta = atan2(lr*tan(Δ_f),d);

% phi
phi = (1/d).*cos(beta).*tan(Δ_f);
```

```

for n=1:ITER-1
    x_f(n+1) = x_f(n) + Ts*v*cos(phi(n)+ beta(n));
    y_f(n+1) = y_f(n) + Ts*v*sin(phi(n)+ beta(n));

    %x_f(n+1) = x_f(n) + 0.5*(Ts*v*cos(phi(n)+beta(n)) + ...
        Ts*v*cos(phi(n)+beta(n)+Ts*omega)); %*cos(Δ_f)
    %y_f(n+1) = y_f(n) + 0.5*(Ts*v*sin(phi(n)+beta(n)) + ...
        Ts*v*sin(phi(n)+beta(n)+Ts*omega));

end
hold on
% plot(x, y, 'linewidth', 6)
plot(x_f, y_f, 'g-','linewidth', 2)
legend("Bicycle's Trajectory")
xlabel('X'); ylabel('Y');
hold off
```

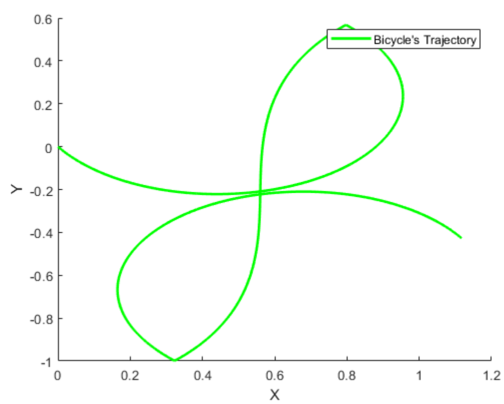

Figure 4: Global Kinematic Behavior

Another approach for simulating the global kinematic behavior can be found here. In this approach, we traverse the robot on a given trajectory using both the models.

Question 2

Task 7

Velocity along Global x

The velocity along global X FoR will be given as:

$$\dot{x} = v_s cos(\delta_f) cos(\phi)$$

Velocity along Global y

The velocity along global Y FoR will be given as:

$$\dot{y} = v_s cos(\delta_f) sin(\phi)$$

The above two expressions can be re-written in matrix notation as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} cos(\delta_f)cos(\phi) & 0 \\ cos(\delta_f)sin(\phi) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} vs_s \\ \omega \end{bmatrix}$$

Task 8

The angular velocity can be calculated using the instantaneous radius of curvature and linear velocity of the chassis.
We have:

$$v_s = R\omega$$
$$\omega = \frac{v_s}{R} \tag{1}$$

For R
Using trigonometry, we have:

$$sin(\delta_f) = \frac{d}{R}$$
$$R = \frac{d}{sin(\delta_f)}$$

Substitute in Eq. (1)

$$\dot{\phi} = \omega = \frac{v_s}{d} sin(\delta_f)$$

Proved!

Question 3

Task 9

For velocity along global X

The velocity is given as:

$$\dot{x} = v cos(\phi)$$

For velocity along global Y

The velocity is given as:

$$\dot{y} = vsin(\phi)$$

Equations can be re-written in matrix notation as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} cos(\phi) & 0 \\ sin(\phi) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Task 10

We have:

$$V = R\omega$$

$$\dot{\phi} = \omega = \frac{v}{R} \tag{2}$$

From trigonometry on figure 8:

$$tan(\delta_f) = \frac{d}{R}$$

$$R = \frac{d}{\delta_f}$$

Substitute in Eq. 2

$$\dot{\phi} = \frac{v}{d} tan(\delta_f)$$