

Question 1

Task 1

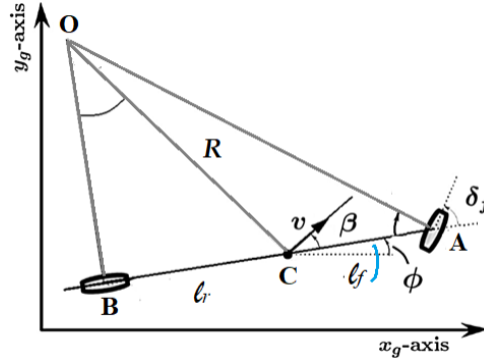


Figure 1: Bicycle Drive Kinematics

Finding $\angle BOC$

Observing the $\triangle BOC$, we can state that:

$$\sin \angle BOC = \frac{Perp}{Hyp} = \frac{l_r}{R}$$

$$\angle BOC = \arcsin \frac{l_r}{R}$$

Finding $\angle COA$

Now, looking at the $\triangle COA$, the angle at point A can be write as:

$$\angle A = 90 - \delta_f$$

Now at point C:

$$\angle C = 90 + \beta$$

We know that the sum of all angles of a triangle is equal to 180.

$$\angle C + \angle COA + \angle A = 180$$

Substituting angles C and A in above equation:

$$90 + \beta + \angle COA + 90 - \delta_f = 180$$

$$\angle COA = \delta_f - \beta$$

Task 2

Proof 1

Observing figure 1 $\triangle BOC$, we can write the $\angle C$ in terms of β as:

$$\angle C = 90 - \beta$$

Now, since we know that sum of all angles in a triangle is 180:

$$\angle B + \angle BOC + \angle C = 180$$

$$90 + \angle BOC + 90 - \beta = 180$$

$$\angle BOC = \beta$$

As we have already found the value of $\angle BOC$ in task 1, we can substitute in above eq:

$$\beta = \arcsin \frac{l_r}{R}$$

$$\sin \beta = \frac{l_r}{R}$$

Proof 2

We know $\angle BOC$ and $\angle COA$, thus we also know $\angle BOA = \delta_f$. In $\triangle BOA$:

$$\tan \delta_f = \frac{d}{OB}$$

Rearranging the above equation in terms of OB:

$$OB = \frac{d}{\tan \delta_f}$$

Moreover, in $\triangle BOC$:

$$\tan \beta = \frac{l_r}{OB}$$

Again, rearranging the above equation in terms of OB:

$$OB = \frac{l_r}{\tan \beta}$$

By comparing both equations:

$$\begin{aligned}\frac{d}{\tan \delta_f} &= \frac{l_r}{\tan \beta} \\ \tan \beta &= \frac{l_r}{d} \tan \delta_f\end{aligned}$$

Task 3

We know that

$$v = R * \omega = R * \dot{\phi}$$

$$\dot{\phi} = \omega = \frac{v}{R}$$

From $\triangle BOC$, we know that:

$$\begin{aligned}\sin \beta &= \frac{l_r}{R} \\ R &= \frac{l_r}{\sin \beta}\end{aligned}$$

Moreover, we know that from task 2 proof 2:

$$\begin{aligned}\tan \beta &= \frac{l_r}{d} \tan \delta_f \\ \frac{\sin \beta}{\cos \beta} &= \frac{l_r}{d} \tan \delta_f \\ \sin \beta &= \frac{l_r}{d} \cos \beta \tan \delta_f\end{aligned}$$

Substituting $\sin \beta$:

$$\begin{aligned}R &= \frac{l_r}{\frac{l_r}{d} \cos \beta \tan \delta_f} \\ R &= \frac{1}{\frac{1}{d} \cos \beta \tan \delta_f}\end{aligned}$$

Finally, substituting expression of R in $\dot{\phi} = \frac{v}{R}$:

$$\dot{\phi} = \omega = \frac{v}{\frac{1}{d} \cos \beta \tan \delta_f}$$

$$\dot{\phi} = \frac{1}{d} v \cos \beta \tan \delta_f$$

Proved.

Task 4

MoR in direction of cycle

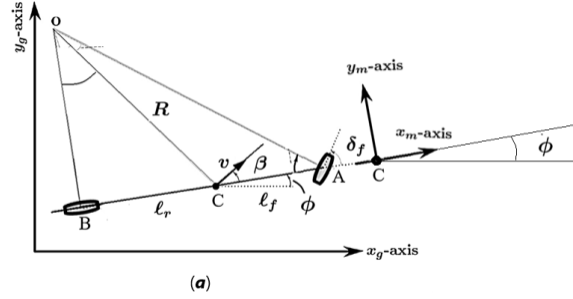


Figure 2: MoR in the direction of cycle

In this mobile frame of reference, we can write the velocities expressions as:

$$\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} V \cos \beta \\ V \sin \beta \end{bmatrix}$$

Now in order to relate to the global FoR we will simply multiply with the rotation matrix:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} V \cos \beta \\ V \sin \beta \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V(\cos \phi \cos \beta - \sin \phi \sin \beta) \\ V(\sin \phi \cos \beta + \cos \phi \sin \beta) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V \cos(\phi + \beta) \\ V \sin(\phi + \beta) \end{bmatrix}$$

MoR in direction of β

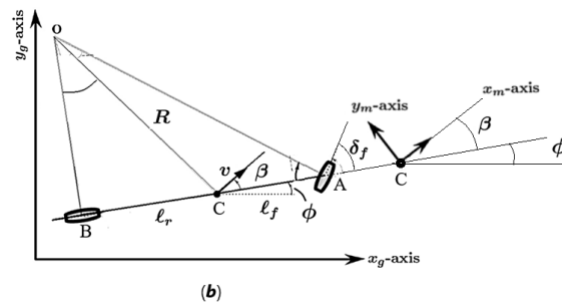


Figure 3: MoR in the direction of β

In this mobile frame of reference, we can write the velocities expressions as:

$$\begin{bmatrix} \dot{x}_m \\ \dot{y}_m \end{bmatrix} = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{bmatrix} V \cos \beta \\ V \sin \beta \end{bmatrix}$$

Now relating to the global FoR we will simply multiply with the rotation matrix:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \dot{x}_m \\ \dot{y}_m \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} V \cos \beta \\ V \sin \beta \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V(\cos \phi \cos \beta - \sin \phi \sin \beta) \\ V(\sin \phi \cos \beta + \cos \phi \sin \beta) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V \cos(\phi + \beta) \\ V \sin(\phi + \beta) \end{bmatrix}$$

Task 5

Rectangular Integration

We have the expressions for the kinematic model:

$$\dot{x} = V \cos(\phi + \beta)$$

$$\dot{y} = V \sin(\phi + \beta)$$

For rectangular integration, we know that:

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} f(t)dt$$

Substituting the function value:

$$\begin{aligned} x_{n+1} &= x_n + \int_{t_n}^{t_{n+1}} V \cos(\phi + \beta) dt \\ x_{n+1} &= x_n + V \cos(\phi + \beta) \int_{nT}^{(n+1)T} dt = x_n + V \cos(\phi + \beta) * ((n+1)T - nT) \\ x_{n+1} &= x_n + VT \cos(\phi + \beta) \end{aligned}$$

Similarly for y:

$$y_{n+1} = y_n + VT \sin(\phi + \beta)$$

Now for ϕ_n :

$$\phi_{n+1} = \phi_n + \Delta\phi = \phi_n + T\omega_n$$

In summary,

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_{n+1} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_n + \begin{bmatrix} VT \cos(\phi + \beta) \\ VT \sin(\phi + \beta) \\ T\omega_n \end{bmatrix}$$

Trapezoidal Integration

We have the expressions for the kinematic model:

$$\begin{aligned} \dot{x} &= V \cos(\phi + \beta) \\ \dot{y} &= V \sin(\phi + \beta) \end{aligned}$$

For Trapezoidal integration, we know that:

$$x_{n+1} \approx x_n + \frac{1}{2}(\Delta x_1 + \Delta x_2)$$

where;

$$\begin{aligned} \Delta x_1 &= Tf(x(t_n), u(t_n)) \\ \Delta x_2 &= Tf(x(t_{n+1}), u(t_{n+1})) \end{aligned}$$

$$\Delta x_2 \approx T f(x(t_n) + \Delta x_1, u(t_n))$$

For x:

$$x_{n+1} \approx x_n + \frac{1}{2}(TV \cos(\phi_n + \beta_n) + TV \cos(\phi_n + \beta_n + \Delta \phi_n))$$
$$x_{n+1} \approx x_n + \frac{TV}{2}(\cos(\phi_n + \beta_n) + \cos(\phi_n + \beta_n + T\omega_n))$$

Similarly, for y:

$$y_{n+1} \approx y_n + \frac{TV}{2}(\sin(\phi_n + \beta_n) + \sin(\phi_n + \beta_n + T\omega_n))$$

Now for ϕ_n :

$$\phi_{n+1} \approx \frac{T}{2}(\omega_n + \omega_{n+1}) = \phi_n + T\omega_n$$

In summary,

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_{n+1} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_n + T \begin{bmatrix} \frac{V}{2}(\cos(\phi_n + \beta_n) + \cos(\phi_n + \beta_n + T\omega_n)) \\ \frac{V}{2}(\sin(\phi_n + \beta_n) + \sin(\phi_n + \beta_n + T\omega_n)) \\ \omega_n \end{bmatrix}$$

Task 6

```
1 Ts = 0.01;
2 ITER = 1e3;
3 omega = 2*pi/10;
4 v = 0.5; % bicycle velocity
5 d = 2; % length of chasis
6 lr = d*0.6;
7 t=linspace(-pi,pi,ITER);
8 %resulting forward kinematic velocities
9 x_f = zeros(1, ITER); y_f = zeros(1, ITER);
10
11 % steering angle Δ f
12 Δ_f = 2*cos(omega*t);
13 %plot(t,Δ_f); figure();
14
15 %velocity-chassis angle
16 beta = atan2(lr*tan(Δ_f),d);
17
18 % phi
19 phi = (1/d).*cos(beta).*tan(Δ_f);
```

```
20
21 for n=1:ITER-1
22     x_f(n+1) = x_f(n) + Ts*v*cos(phi(n)+ beta(n));
23     y_f(n+1) = y_f(n) + Ts*v*sin(phi(n)+ beta(n));
24
25     %x_f(n+1) = x_f(n) + 0.5*(Ts*v*cos(phi(n)+beta(n)) + ...
26         Ts*v*cos(phi(n)+beta(n)+Ts*omega)); %*cos(Δ_f)
27     %y_f(n+1) = y_f(n) + 0.5*(Ts*v*sin(phi(n)+beta(n)) + ...
28         Ts*v*sin(phi(n)+beta(n)+Ts*omega));
29
30 end
31 hold on
32 % plot(x, y, 'linewidth', 6)
33 plot(x_f, y_f, 'g-', 'linewidth', 2)
34 legend("Bicycle's Trajectory")
35 xlabel('X'); ylabel('Y');
36 hold off
```

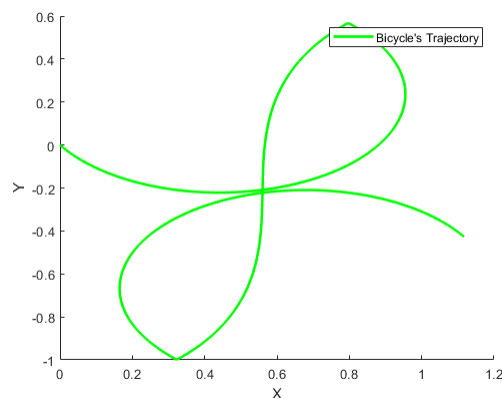


Figure 4: Global Kinematic Behavior

Another approach for simulating the global kinematic behavior can be found [here](#). In this approach, we traverse the robot on a given trajectory using both the models.

Question 2

Task 7

Velocity along Global x

The velocity along global X FoR will be given as:

$$\dot{x} = v_s \cos(\delta_f) \cos(\phi)$$

Velocity along Global y

The velocity along global Y FoR will be given as:

$$\dot{y} = v_s \cos(\delta_f) \sin(\phi)$$

The above two expressions can be re-written in matrix notation as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\delta_f) \cos(\phi) & 0 \\ \cos(\delta_f) \sin(\phi) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_s \\ \omega \end{bmatrix}$$

Task 8

The angular velocity can be calculated using the instantaneous radius of curvature and linear velocity of the chassis.

We have:

$$\begin{aligned} v_s &= R\omega \\ \omega &= \frac{v_s}{R} \end{aligned} \tag{1}$$

For R

Using trigonometry, we have:

$$\begin{aligned} \sin(\delta_f) &= \frac{d}{R} \\ R &= \frac{d}{\sin(\delta_f)} \end{aligned}$$

Substitute in Eq. (1)

$$\dot{\phi} = \omega = \frac{v_s}{d} \sin(\delta_f)$$

Proved!

Question 3

Task 9

For velocity along global X

The velocity is given as:

$$\dot{x} = v \cos(\phi)$$

For velocity along global Y

The velocity is given as:

$$\dot{y} = v \sin(\phi)$$

Equations can be re-written in matrix notation as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\phi) & 0 \\ \sin(\phi) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Task 10

We have:

$$\begin{aligned} V &= R\omega \\ \dot{\phi} &= \omega = \frac{v}{R} \end{aligned} \tag{2}$$

From trigonometry on figure 8:

$$\begin{aligned} \tan(\delta_f) &= \frac{d}{R} \\ R &= \frac{d}{\delta_f} \end{aligned}$$

Substitute in Eq. [2](#)

$$\dot{\phi} = \frac{v}{d} \tan(\delta_f)$$