Mehdi Raza Khorasani Digital Signal Processing
Homework 2
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Question 1

Part (a)

From the definition of Z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

So we get:

$$X(z) = (3)z^{-3} + (0)z^{-2} + (0)z^{-1} + (6)z^{0} + (1)z^{1} + (-4)z^{2}$$

$$\boxed{X(z) = 3z^{-3} + 6 + z - 4z^{2}}$$

Part (b)

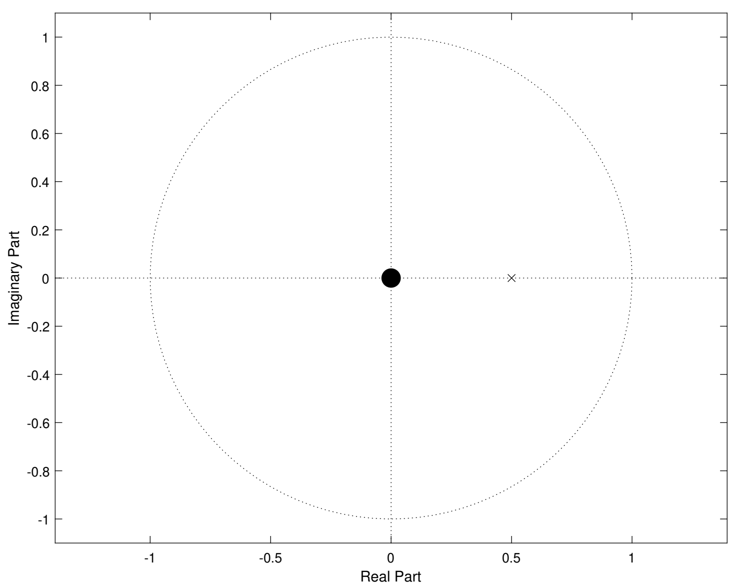

Figure 1: PZ plot for Q1

Proof 1

We know that if system is FIR, then it is non-recursively and defined as follows:

$$y(n) = \sum_{k=-\infty}^{\infty} b_k x(n-k)$$

Taking Z-transform, we get:

$$Y(z) = \sum_{k=-\infty}^{\infty} b_k X(z) z^{-k}$$

From above equation, we can observe the following constraint for FIR system:

1. It can contain any number of zeros.
2. It does not contain any poles.
3. Its ROC will be the whole z plane (due to absense of poles).

Since the given system violates the condition 2 and 3, it is not FIR. So it can be safely termed **IIR**.

Proof 2

We have:

$$H(z) = G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

where $G = \dfrac{b_o}{a_o}$.
In our case:

$$N = M = 1$$

Then:

$$H(z) = G z^0 \frac{z}{z - 0.5}$$

$$H(z) = G \frac{z}{z - 0.5}$$

We know that:

$$H(z) = \frac{Y(z)}{X(z)}$$

So the system function can be written as:

$$\frac{Y(z)}{X(z)} = \frac{b_o}{a_o}\left(\frac{z}{z - 0.5}\right)$$

Multiply and divide by z^{-1}:

$$\frac{Y(z)}{X(z)} = \frac{b_o}{a_o}\left(\frac{1}{1 - 0.5z^{-1}}\right)$$

Re-arranging:

$$a_o Y(z) - 0.5 a_o Y(z) z^{-1} = b_o X(z)$$

Converting to LCCDE form:

$$a_o y(n) - 0.5 a_o y(n-1) = b_o x(n-1)$$

$$\boxed{a_o y(n) = 0.5 a_o y(n-1) + b_o x(n-1)}$$

The given LCCDE is recursively defined hence the system is indeed IIR. Verified!

Question 2

Given:

$$y(n) = 0.8y(n-1) - 0.6y(n-2) + x(n) + 2x(n-1)$$

Part (a)

Taking Z-Transform on both sides:

$$Y(z) = 0.8z^{-1}Y(z) - 0.6Y(z)z^{-2} + X(z) + 2X(z)z^{-1}$$

$$Y(z)(1 + 0.6z^{-2} - 0.8z^{-1}) = X(z)(1 + 2z^{-1})$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 0.8z^{-1} + 0.6z^{-2}}$$

$$= \frac{z^2 + 2z}{z^2 - 0.8z + 0.6}$$

$$H(z) = \frac{z(z+2)}{z^2 - 0.8z + 0.6}$$

Part (b)

Zeros

Zeros are as follows:

$$z = 0$$

$$z = -2$$

Poles

Calculating roots of the equation below:

$$z^2 - 0.8z + 0.6 = 0$$

Poles are as follows:

$$p = 0.4 + 0.6633j$$

$$p = 0.4 - 0.6633j$$

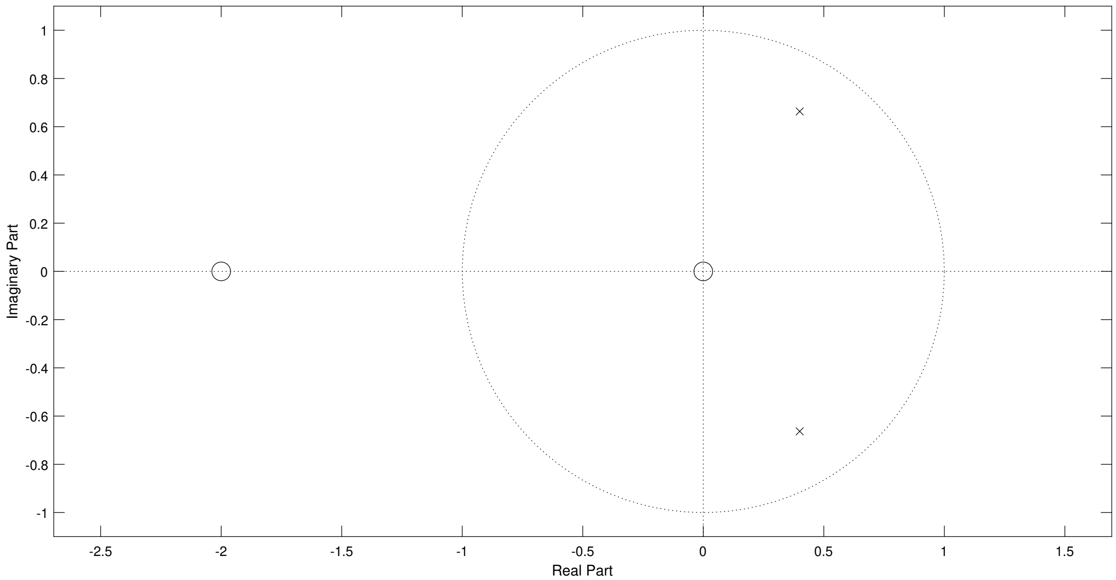

Figure 2: PZ plot

Part (c)

The outer most pole is located at:

$$r = \sqrt{0.4^2 + 0.6633^2} = 0.7746$$

Since system is causal, the ROC consists of $r > 0.7746$. This implies that the ROC will include the unit circle (we may also observe this from the PZ plot). So system is stable.

Question 3

Given:

$$H(z) = \frac{z^2 + z}{z^2 + z - 0.75}$$

Part (a)

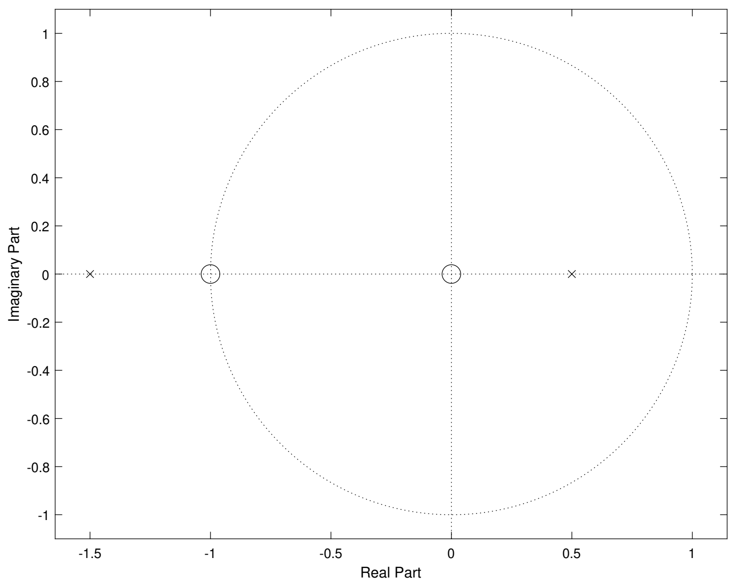

Figure 3: PZ plot for Q3

Zeros

Zeros are calculated as follows:

$$z(z+1) = 0$$

Then,

$$\boxed{z = 0}$$

$$\boxed{z = -1}$$

Poles

Poles are calculated as follows:

$$z^2 + z - 0.75 = 0$$

$$\boxed{p = -1.5}$$

$$\boxed{p = 0.5}$$

Part (b)

The outer most pole lies at:

$$r = 1.5$$

With the system obeying causality, its ROC will be $r > 1.5$. So ROC doesn't includes the unit circle. So system is **unstable**

Part (c)

The system function can be written as:

$$\frac{Y(z)}{X(z)} = \frac{z^2 + z}{z^2 + z - 0.75}$$

Multiply and divide by z^{-2}

$$\frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 + z^{-1} - 0.75z^{-2}}$$

$$Y(z) + Y(z)z^{-1} - 0.75Y(z)z^{-2} = X(z) + X(z)z^{-1}$$

Above equation can now be converted into LCCDE form:

$$y(n) + y(n-1) - 0.75y(n-2) = x(n) + x(n-1)$$

$$\boxed{y(n) = -y(n-1) + 0.75y(n-2) + x(n) + x(n-1)}$$

Question 4

Given:

$$X(z) = \frac{z}{z^2 + z - 0.75}$$
$$\frac{X(z)}{z} = \frac{1}{z^2 + z - 0.75}$$

Creating partial fractions:

$$\frac{1}{(z+1.5)(z-0.5)} = \frac{A_1}{(z+1.5)} + \frac{A_2}{(z-0.5)}$$
$$1 = A_1(z-0.5) + A_2(z+1.5)$$

For A_1

Put $z = -1.5$

$$1 = -2A_1 + 0$$
$$\boxed{A_1 = -0.5}$$

For A_2

Put $z = 0.5$

$$0.5 = 2A_2 + 0$$
$$\boxed{A_2 = 0.25}$$

Then Partial fraction expansion becomes:

$$\boxed{\frac{X(z)}{z} = \frac{-0.5}{(z+1.5)} + \frac{0.25}{(z-0.5)}}$$

$$X(z) = \frac{-0.5z}{(z+1.5)} + \frac{0.25z}{(z-0.5)}$$

Multiply and divide by z^{-1}:

$$X(z) = \frac{-0.5}{(1+1.5z^{-1})} + \frac{0.25}{(1-0.5z^{-1})}$$

From the lookup table, the inverse transform is given as:

$$\boxed{x(n) = [-0.5(-1.5)^n + 0.25(0.5)^n]u(n)}$$

Question 5

Part (a)

Since the fiter is high pass, we may place pole at $\omega = \pi$ and zero at $\omega = 0$. One possible arrangement is shown in the PZ plot:

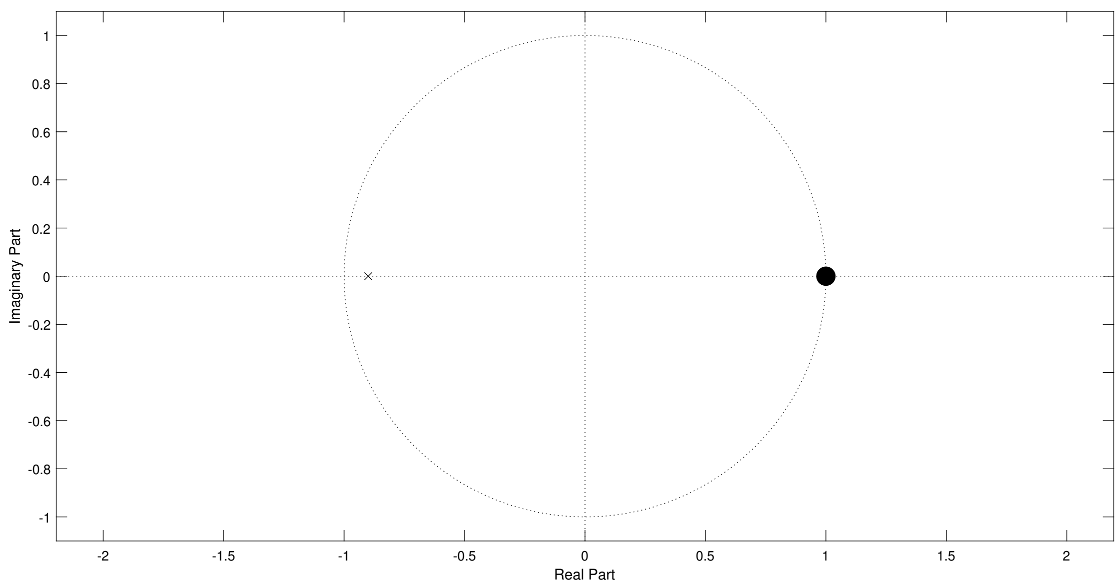

Figure 4: PZ plot for Q5

Part (b)

1. This is a band-pass filter for frequencies near $\omega = \pm\pi/2$. It is so because we see poles at frequency $\pm\pi/2$ (Poles are placed near frequencies to be emphasized). Similarly, we see zeros at $\omega = \pi$ and $\omega = 0$ (zeros are placed near frequencies to be de-emphasized). Hence this filter will block low and high frequencies ($\omega = 0$ and $\omega = \pi$) and pass frequencies around $\omega = \pm\pi/2$.

2. At $\omega = 0$, the response is:

$$|H(\omega)| = b_o \frac{\textit{product of length of vector: zero to unit circle}}{\textit{product of length of vector: pole to unit circle}}$$

   Hence the response in this case will be (Assuming poles at $r = 0.9$):

$$|H(\omega)| = b_o \frac{0 \times 2}{\textit{product of length of vector: pole to unit circle}}$$

$$|H(\omega)| = 0$$

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The complete source code can be found at: GitHub/mehhdiii/