EC813B - Recitation 9

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Question 1[1]

Consider a version of the neoclassical growth model that admits no population growth, no technological progress, and a representative household with preferences given by

$$U = \int e^{-\rho t} u\big(c(t)\big) dt$$

where $u\big(c(t)\big) = \frac{c(t)^{1-\theta}-1}{1-\theta}$ (CRRA). The main difference from the standard model is that there are multiple capital goods. In particular, suppose that the production function of the economy is given by

$$Y(t) = F\left(K_1(t), \ldots, K_M(t), L(t)\right),$$

where K_m denotes the m^{th} type of capital, and L is labor. To focus on the effect of multiple types of capital, we can assume that each household has one unit of labor endowment per period. F is homogeneous of degree 1 in all of its variables. Capital in each sector accumulates in the standard fashion,

$$\dot{K}_m(t) = I_m(t) - \delta_m K_m(t)$$

for $m = 1, \ldots, M$. The resource constraint of the economy at time t is

$$C(t) + \sum_{m=1}^{M} I_m(t) \leq Y(t).$$

(a) Write budget constraint of the representative household with M separate assets in this economy.

(b) Characterize the BGP by specifying the profit-maximizing decision of firms in each sector and the dynamic optimization problem of consumers. Find the Euler equations for this economy.

(c) Find conditions that characterize the steady state levels of capital and consumption.

[1] This is a simplified version of Exercise 8.38 from Acemoglu, D. (2009), *Introduction to Modern Economic Growth*, Chapter 8.

Solution

(a) The budget constraint of the representative household can be written as follows

$$c(t)+\sum_{m=1}^{M} I_m(t)=w(t)+\sum_{m=1}^{M} r_m(t)K_m(t)$$

and using capital accumulation function, we get

$$\sum_{m=1}^{M}\dot{K}_m(t)=w(t)+\sum_{m=1}^{M} r_m(t)K_m(t)-c(t)-\sum_{m=1}^{M}\delta_m K_m(t)$$

where $r_m(t)$ is the rate of return for capital m and $w(t)$ is the labor income.

We can further refine the budget constraint considering that all types of capital are perfect substitutes. This assumption is required by no-arbitrage condition. If there is risk-free arbitrage opportunities, consumers can benefit from taking on a short position in the asset with the lowest return and a long position in the asset with the highest return. As a result, the capital market clearing condition would not hold. Hence, in any equilibrium we will need to have

$$r_1(t)=r_2(t)=\cdots=r_m(t)=\cdots=r(t)\quad\forall m.$$

Then the budget constraint of the representative household can be rewritten as

$$\sum_{m=1}^{M}\dot{K}_m(t)=w(t)+r(t)\sum_{m=1}^{M} K_m(t)-c(t)-\sum_{m=1}^{M}\delta_m K_m(t)$$

which is similar to the budget constraint of the standard neoclassical growth model.

(b) We start with the representative firm's profit maximization problem. The profit function can be written as

$$\max_{L(t),[K_M(t)]_{t=0}^{\infty}}\pi_m(t)=F\Big(K_1(t),\ldots,K_M(t),L(t)\Big)-w(t)L(t)-\sum_{m=1}^{M} r_m(t)K_m(t)$$

Next, the usual FOCs for wages and rental rates can be obtained

$$F_L\Big(K_1(t),\ldots,K_M(t),L(t)\Big) = w(t)$$
$$F_{K_m}\Big(K_1(t),\ldots,K_M(t),L(t)\Big) = r_m(t)$$

Now, the maximization problem of the representative household is almost the same as in the standard neoclassical growth model. We use the fact

$$r_1(t) = r_2(t) = \cdots = r_m(t) = \cdots = r(t) \quad \forall m.$$

Then the representative household's maximization problem can be written as

$$\max_{[c(t),K_1(t),\ldots,K_M(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{c(t)^{1-\theta}-1}{1-\theta}dt \quad \text{s.t.} \quad \sum_{m=1}^{M}\dot{K}_m(t) = w(t)+r(t)\sum_{m=1}^{M}K_m(t)-c(t)-\sum_{m=1}^{M}\delta_m K_m(t)$$

We set up the current-value Hamiltonian:

$$\mathcal{H}^* = u\big(c(t)\big) + \mu\Big[w(t) + r(t)\sum_{m=1}^{M}K_m(t) - c(t) - \sum_{m=1}^{M}\delta_m K_m(t)\Big]$$

Now, we can write the FOCs:

(i) $\frac{\partial \mathcal{H}^*}{\partial c(t)} = 0 = u'(c_t) - \mu$

(ii) $\frac{\partial \mathcal{H}^*}{\partial K_m(t)} = \rho\mu - \dot{\mu} = \mu\Big[r(t) - \delta_m\Big]$

(iii) $\frac{\partial \mathcal{H}^*}{\partial \mu} = w(t) + r(t)\sum_{m=1}^{M}K_m(t) - c(t) - \sum_{m=1}^{M}\delta_m K_m(t) = \sum_{m=1}^{M}\dot{K}_m(t)$

Combining what we find, we get

- From (i), differentiating (i) w.r.t. time, t, we obtain $\dot{\mu} = u''(c(t))\dot{c(t)}$.
- From (ii), we get $\dot{\mu} = \mu\big[\rho + \delta_m - r(t)\big]$

Plugging in for CRRA utility function, we get

$$\frac{\dot{c(t)}}{c(t)} = \frac{1}{\theta}\big[r(t) - \rho - \delta_m\big]$$

Then using the FOCs from firm's problem and $r_1(t) = r_2(t) = \cdots = r_m(t) = \cdots = r(t)$, the Euler equation can be rewritten as

$$\frac{\dot{c(t)}}{c(t)} = \frac{1}{\theta}\Big[F_{K_m}\Big(K_1(t),\ldots,K_M(t),L(t)\Big) - \delta_m - \rho\Big]$$

Now, we can use the fact that each household has one unit of labor endowment per period. Then we rewrite the Euler equations as follows

$$\sum_{m=1}^{M}\dot{K}_m(t) = F_L\Big(K_1(t),\ldots,K_M(t),1\Big) + F_{K_m}\Big(K_1(t),\ldots,K_M(t),1\Big)\sum_{m=1}^{M}K_m(t) - c(t) - \sum_{m=1}^{M}\delta_m K_m(t)$$

$$\dot{c(t)} = \frac{c(t)}{\theta}\Big[F_{K_m}\Big(K_1(t),\ldots,K_M(t),1\Big) - \delta_m - \rho\Big]$$

(c) To find conditions that characterize the steady state levels of capital and consumption, we use the fact $\dot{c(t)} = \dot{K_m}(t) = 0$. So, from the Euler equations, we get

$$\dot{c(t)} = 0 \quad\Longrightarrow\quad F_{K_m}\Big(K_1^*,\ldots,K_M^*,1\Big) = \rho + \delta_m$$

$$\dot{K_m}(t) = 0 \quad\Longrightarrow\quad F_L\Big(K_1^*,\ldots,K_M^*,1\Big) + \Big[F_{K_m}\Big(K_1^*,\ldots,K_M^*,1\Big) - \delta_m\Big]\sum_{m=1}^{M}K_m^* = c^*$$

Question 2[2]

Consider the following endogenous growth model due to Uzawa and Lucas. The economy admits a representative household, and preferences are given by

$$\int_0^\infty \exp(-\rho t)\frac{C(t)^{1-\theta}-1}{1-\theta}dt,$$

where $C(t)$ is consumption of the final good, which is produced as

$$Y(t) = AK(t)^\alpha H_P^{1-\alpha}(t),$$

where $K(t)$ is capital, $H(t)$ is human capital, and $H_P(t)$ denotes human capital used in production. The accumulation equations are

$$\dot{K}(t) = I(t) - \delta K(t),$$

[2]This is a simplified version of Exercise 11.21 from Acemoglu, D. (2009), *Introduction to Modern Economic Growth*, Chapter 11.

and

$$\dot{H}(t) = BH_E(t) - \delta H(t),$$

where $H_E(t)$ is human capital devoted to education (further human capital accumulation), and for simplicity the depreciation of human capital is assumed to be at the same rate δ as physical capital. The resource constraints of the economy are $I(t) + C(t) \leq Y(t)$, and $H_E(t) + H_P(t) \leq H(t)$.

(a) Interpret the second resource constraint.

(b) Denote the fraction of human capital allocated to production by $h(t) \equiv H_P(t)/H(t)$ and calculate the growth rate of final output as a function of $h(t)$.

(c) Assume that $h(t)$ is constant, and characterize the BGP of the economy.

Solution

(a) The second resource constraint implies that each individual has the ability of allocating a fixed amount of time between employment and education (further human capital accumulation). In particular, this constraint states that part of the human capital in this economy can be used for further human capital accumulation. Hence this resource constraint shows how the economy generates human capital.

(b) Plugging in $H_P(t) = h(t)H(t)$ in the production function, we get

$$Y(t) = AK(t)^{\alpha}\Big(h(t)H(t)\Big)^{1-\alpha}$$

then, we can find the output growth as the following

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{H}(t)}{H(t)} + (1-\alpha)\frac{\dot{h}(t)}{h(t)}$$

(c) We first start with the representative consumer's problem. The maximization problem is

$$\begin{aligned}
&\max_{[C(t),h(t),K(t),H(t)]_t} \int_0^{\infty} \exp(-\rho t)\frac{C(t)^{1-\theta}-1}{1-\theta}dt \\
\text{s.t.}\quad &\dot{K}(t) = r(t)K(t) + w(t)h(t)H(t) - C(t) - \delta K(t) \\
&\dot{H}(t) = \Big(B\big(1-h(t)\big) - \delta\Big)H(t)
\end{aligned}$$

Then we write the current-value Hamiltonian as the following

$$\mathcal{H}^* = \frac{C^{1-\theta}-1}{1-\theta} + \mu_K\Big[r(t)K + w(t)hH - C - \delta K\Big] + \mu_H\Big[\Big(B(1-h)-\delta\Big)H\Big]$$

Now, we can write the FOCs:

(i) $\frac{\partial \mathcal{H}^*}{\partial C} = 0 \Longrightarrow C^{-\theta} = \mu_K(t)$

(ii) $\frac{\partial \mathcal{H}^*}{\partial h} = 0 \Longrightarrow \mu_K(t) = \mu_H(t)\frac{B}{w(t)}$

(iii) $\frac{\partial \mathcal{H}^*}{\partial K} = \rho\mu_K(t) - \dot{\mu}_K(t) \Longrightarrow \frac{\dot{\mu}_K(t)}{\mu_K(t)} = \rho + \delta - r(t)$

(iv) $\frac{\partial \mathcal{H}^*}{\partial H} = \rho\mu_H(t) - \dot{\mu}_H(t) \Longrightarrow \frac{\mu_K(t)}{\mu_H(t)}w(t)h + B\big(1-h\big) - \delta = \rho - \frac{\dot{\mu}_H(t)}{\mu_H(t)}$

Using (i) and (iii), we get the usual Euler equation for consumption

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}\big(r(t) - \rho - \delta\big) \tag{1}$$

From (ii), we find $\mu_H(t) = \mu_K(t)\frac{w(t)}{B}$ and, then, plug in for $\mu_H(t)$ in (iv) and using (iii), we get

$$B + \frac{\dot{w}(t)}{w(t)} = r(t) \tag{2}$$

This condition ensures an interior solution to the consumer's problem. The rate of return from investing in physical capital, $r(t)$, should be equal to the sum of the rate of return from investing in human capital, B, and the depreciation of wages. If this condition is not satisfied, then consumers would invest either only in human capital or only in physical capital, and there would be a corner solution.

Assuming there is an interior solution (previous condition holds), we can characterize the equilibrium. Next, we need to solve firm's profit maximization problem which is the usual optimization problem and get the equilibrium prices $r(t)$ and $w(t)$.

From the FOCs of the optimization problem we get

$$r(t) = F_{K(t)}\Big(K(t), H_P(t)\Big) = A\alpha\left(\frac{K(t)}{h(t)H(t)}\right)^{\alpha-1} \tag{3}$$

$$w(t) = F_{H_P(t)}\Big(K(t), H_P(t)\Big) = A(1-\alpha)\left(\frac{K(t)}{h(t)H(t)}\right)^{\alpha} \tag{4}$$

Now, we can plug in all the needed values into Euler equation and accumulation equations for capital and human capital. First we start with physical capital

$$\dot{K}(t) = A\alpha \left(\frac{K(t)}{h(t)H(t)}\right)^{\alpha-1} K(t) + A(1-\alpha)\left(\frac{K(t)}{h(t)H(t)}\right)^{\alpha} h(t)H(t) - C(t) - \delta K(t)$$

and we finally get

$$\dot{K}(t) = AK(t)^{\alpha}\Big(h(t)H(t)\Big)^{1-\alpha} - C(t) - \delta K(t) \tag{5}$$

We state the accumulation equation for human capital

$$\dot{H}(t) = \Big(B\big(1-h(t)\big) - \delta\Big)H(t) \tag{6}$$

Then, we need to plug $w(t)$ into the indifference condition (2), we get

$$B + \alpha\left(\frac{\dot{K}(t)}{K(t)} - \frac{\dot{h}(t)}{\dot{h}(t)} - \frac{\dot{H}(t)}{H(t)}\right) = A\alpha\left(\frac{K(t)}{h(t)H(t)}\right)^{\alpha-1} \tag{7}$$

Finally, we plug $r(t)$ into (1), we obtain

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}\left(A\alpha\left(\frac{K(t)}{h(t)H(t)}\right)^{\alpha-1} - \rho - \delta\right) \tag{8}$$

Therefore, we can characterize an equilibrium path with equations (5), (6), (7), and (8) along with transversality conditions

$$\lim_{t\to\infty} H(t)\exp\left(-\int_0^t r(s)ds\right) = 0$$
$$\lim_{t\to\infty} K(t)\exp\left(-\int_0^t r(s)ds\right) = 0$$

and given the initial values $K(0)$ and $H(0)$.