EC813B - Recitation 2

SPRING 2022

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Question 1

Consider a metric space $\langle(0,1),\rho\rangle$ with $\rho(x,y)=|x-y|$. Show that function $f(x)=\frac{x^2}{3}$ is a contraction on $(0,1)$. Does it have a fixed point in this metric space? Does this violate the Contraction Mapping Theorem? Why?

Solution

Using the definition of contraction mapping, $f(x)$ needs to satisfy $\rho(fx,fy)\leq\beta\rho(x,y)$ for some $\beta\in(0,1)$.

$$\Longrightarrow |f(x)-f(y)|=\left|\frac{x^2}{3}-\frac{y^2}{3}\right|=\frac{(x+y)|x-y|}{3}<\frac{2}{3}|x-y|$$

Hence, it is a contraction mapping. And, there is **no** fixed point in $(0,1)$. The solutions for the equation $f(x)=\frac{x_0^2}{3}=x_0$ is $x=0$ and $x=3$ which are not in $(0,1)$. But, this is not a violation of CMT because the metric space is **not** complete (*Definition: A metric space (S,ρ) is* ***complete*** *if every Cauchy sequence in S converges to an element in S.*) because there is no fixed point within $(0,1)$.

Question 2

Prove the following theorem by using the **definition** of closed sets. Let $\langle X,\rho\rangle$ be a metric space. Then

(a) X and $\emptyset$ are closed.

(b) If $\{Y_\alpha\}_{\alpha\in A}$ is an arbitrary collection of closed sets of X then $\bigcap_{\alpha\in A}Y_\alpha$ is a closed set.

(c) The union of a finite number of closed sets is closed.

Solution

Note (or Warning): *This question was from my Midterm (2018 - Midterm I) and I sketched these proofs as much as I remembered using definitions of a closure point, closure of a set, and closed set. There might be some mistakes or nuances that I missed. Please let me know if you think there is something wrong. So, I can fix it.*

Definition 1: A point in $Y \subset X$ is a **closure point** if $\exists y \in Y$, $\forall \varepsilon > 0$ such that $y \in S(x, \varepsilon) \cap Y$, $\forall x \in X$.
Definition 2: Collection of all closure points is the **closure** of the set.
Definition 3: Set Y is a **closed set** if $Y = \overline{Y}$ where $\overline{Y}$ is the closure of set Y.

(a) $\implies$ Consider $x \in X$. Then, $\exists y \in X$, $\forall \varepsilon > 0$ such that $y \in S(x, \varepsilon) \cap X$. Since X is the space itself, $y \in X$. Thus, X contains this closure point y.Regarding y is arbitrarily chosen, X contains all of its closure points. Hence, X is closed. ■

$\implies$ Consider $x \in X$. Then, $\forall \varepsilon > 0$, $\exists S(x, \varepsilon)$ such that $\emptyset \in S(x, \varepsilon) \cap X$. Since empty set is a subset of every set, the definition of a closure point holds for empty set. Thus, $\emptyset$ is closed. ■

(b) $\implies$ Using the definition of closed sets, we know that $Y = \overline{Y}$ where $\overline{Y}$ is the closure of set Y. Then, each Y_α is closed because $\{Y_\alpha\}_{\alpha \in A}$ is a collection of closed sets. This implies that

$$Y_\alpha = \overline{Y}_\alpha \implies \bigcap_{\alpha \in A} Y_\alpha = \bigcap_{\alpha \in A} \overline{Y}_\alpha.$$

Now, this means that we obtain the intersection of closures which contains all the closure points. Hence, $\bigcap_{\alpha \in A} Y_\alpha$ is closed. ■

(c) By definition, closed sets contain all of their closure points. Implying that the union of closed sets $\bigcup_{\alpha \in A} Y_\alpha$ is a set of closure points. Then, the union contains all the closure points. Thus, it is equal to its closure. Therefore, it is closed. ■

Question 3

Let F be a closed set in a metric space $\langle X, \rho \rangle$. Suppose that a sequence of points $\{x_n\}$ of F converges to a point $x \in X$. Prove that $x \in F$.

Solution

$\Longrightarrow x$ is a closure point because $\forall \varepsilon > 0$, $S(x, \varepsilon) \cap F$ is not empty. So, $x \in \overline{F} = F$ where the last equality is from closedness of F. ■

Question 4

Use the Contraction Mapping Theorem (CMT) to prove that $\sqrt{2}$, which satisfies the equation $x^2 = 2$, exists and is unique:

(a) Formulate the CMT.

(b) Notice that $x^2 = 2$ is equivalent to

$$\frac{1}{2}\left(x + \frac{2}{x}\right) = x.$$

Assume that $x > 1$ and define a function (mapping):

$$\Phi : [1, \infty) \to [1, \infty) \text{ by } \Phi(x) = \frac{1}{2}\left(x + \frac{2}{x}\right).$$

Given that the half-line $X = [1, \infty)$ is a complete metric space (the distance is defined by $\rho(x, y) = |x - y|$), prove everything you have to prove for the CMT to hold.

Solution

(a) Please see *Recitation 1 - Question 3* for a detailed formulation of the CMT.

(b) First, we need to show that given function, $\Phi(x)$, is a contraction mapping. Using the definition of contraction mapping, $\Phi(x)$ needs to satisfy $\rho(\Phi(x), \Phi(y)) \leq \beta\rho(x, y)$ for some $\beta \in (0, 1)$.

$$\begin{aligned}
\Longrightarrow \rho(\Phi(x), \Phi(y)) = |\Phi(x) - \Phi(y)| &= \left|\frac{1}{2}\left(x + \frac{2}{x}\right) - \frac{1}{2}\left(y + \frac{2}{y}\right)\right| \\
&= \frac{1}{2}\left|\frac{(x-y)(xy-2)}{xy}\right| = \frac{1}{2} \cdot \frac{|x-y| \cdot |xy-2|}{|xy|}
\end{aligned}$$

$$\Longrightarrow \rho(\Phi(x), \Phi(y)) \leq \beta\rho(x, y) = \frac{1}{2}|x - y|$$

Thus, it is a contraction mapping. ■

Second, we need to show that contraction mapping, $\Phi(x)$, has a fixed point.

$$\Phi(x_0) = x_0 \quad \Longrightarrow \quad \frac{1}{2}\Big(x_0 + \frac{2}{x_0}\Big) = x_0 \quad \Longrightarrow \quad x_0^2 = 2$$

The solution is $x_0 = \sqrt{2}$. Thus, we show that CMT holds. ■

A Brief Introduction to Job Search Theory

Why do we need Job Search Theory?[a]

- The central question for labor economics and macroeconomics: what determines the level of employment and unemployment in the economy?
- The usual answer given considers labor supply, labor demand, and unemployment (as *"leisure"*) together.
- However, it is neither realistic nor a useful framework for analysis.
- Alternatively, we introduce labor market frictions.
- Related questions raised by the presence of frictions:
  - is the level of employment efficient/optimal?
  - how is the composition and quality of jobs determined, is it efficient?
  - distribution of earnings across workers.
- The main challenge is to find a suitable way to model labor market frictions.
- There are some alternatives that can be considered as job market frictions to use in models:
  - incentive problems, efficiency wages
  - wage rigidities, bargaining, non-market clearing prices
  - search

- We proceed with search models, where workers look for jobs taking as given such things as the distribution of wage offers across firms, without regard to the origin of this distribution.

- We are interested for now in the optimization problem of a single agent - such as a worker looking for a job at a good wage - and we make no reference to the problems being solved by other individuals - such as the firms who presumably set the wages - or the conditions that must be satisfied for the decisions of all individuals to be consistent.

- It makes sense to understand the optimization problem of a single individual before she is embed into an equilibrium setting. For example, we usually study consumer theory before analyzing market demand before analyzing general equilibrium.

[a]Please see "Ljungqvist L. and T. Sargent. *Recursive Macroeconomic Theory*, MIT Press, Cambridge, MA, 2000." for a detailed explanation.

Question 5

Consider a discrete time, infinite horizon job search model. Each period an unemployed worker receives on *i.i.d.* offer ω from cumulative distribution function $F(\omega)$. The worker maximizes

$$E\sum_{t=0}^{\infty}\beta^t c_t$$

where $0<\beta=1/(1+r)<1$, and

$$c_t=\begin{cases}\omega_t & \text{if she is working}\\ b & \text{if she is unemployed}\end{cases}$$

Once an offer is accepted the job is held forever.

(a) Formulate the Bellman's equation for the worker's problem. Show that mapping T defined as

$$T(R)=(1-\beta)b+\beta E\max\{\omega,R\}$$

is a contraction (use Blackwell's theorem). Derive the reservation wage equation.

(b) Now assume that each period an unemployed worker can draw two *i.i.d.* wage offers from the same cumulative distribution function $F(\omega)$. Derive the reservation wage the offer at which the individual is indifferent between accepting the offer and remaining unemployed.

(c) Compare the worker's reservation wages for both cases.

Solution

Theorem below is directly taken from *Stokey and Lucas (1989), Recursive Methods in Economic Dynamics, page 54.*

Theorem *(Blackwell's Sufficient Condition for a Contraction)* Let $X \subseteq R^l$ and let $B(X)$ be a space of bounded functions $f : X \to R$ with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying (i) (monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$ implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$, and (ii) (discounting) there exists some $\beta \in (0, 1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a$$

for all $f \in B(X), a \geq 0, x \in X$. Then T is a contraction mapping with modulus β.

Proof. For all $f, g \in B(X)$, $f \leq g+\|f-g\|$. Thus, $Tf \leq Tg+\beta\|f-g\|$ or $Tf-Tg \leq \beta\|f-g\|$. Switching the roles of f and g we obtain $\|Tf-Tg\| \leq \beta\|f-g\|$ and hence, T is a contraction with modulus β. ■

(a) $\Longrightarrow$ ***Bellman's equation:***

Let $V(\omega)$ denote the expected payoff to accepting an offer ω at some point in time, referred to as the value of ω. It does not depend on when the offer is accepted, given our assumptions of a stationary environment and an infinite horizon. In fact, since jobs are retained forever,

$$V(\omega) = \frac{\omega}{1-\beta}$$

Also, let U denote the value of rejecting an offer and remaining unemployed, which also does not depend on time, and does not depend on which wage was rejected since offers are $i.i.d.$

$$U = b + \beta E \max\{V(\omega), U\}$$

since the value of rejecting an offer equals instantaneous unemployment income plus the discounted expected value of having the option to accept or reject a new offer next period.

Let $J(\omega) = \max\{V(\omega), U\}$ be the value of having offer ω in hand. Then $J(\omega)$ satisfies the following version of Bellman's equation of dynamic programming:

$$J(\omega) = \max\left\{\frac{\omega}{1-\beta}, c + \beta EJ\right\} \tag{1}$$

The nature of the solution can be described as follows. Since $V(\omega)$ is increasing in ω and U is independent of ω, there exists a unique R satisfying $V(R) = U$ where R is the reservation wage.

$\Longrightarrow$ ***Derivation of the reservation wage equation***

Since $V(R) = R/(1-\beta)$ and $U = c + \beta EJ$, the definition of the reservation wage, $V(R) = U$, is equivalent to

$$R = (1-\beta)c + (1-\beta)\beta EJ \tag{2}$$

This expresses R in terms of the unknown value function, J. However, rewriting (1) as

$$J(w) = \begin{cases} \frac{\omega}{1-\beta} & \text{for } \omega \geq R \\ \frac{R}{1-\beta} & \text{for } \omega < R \end{cases}$$

we see that $(1-\beta)EJ = E\max(\omega, R)$. Combining this with (2) we can express the reservation wage as the solution to the following equation:

$$R = (1-\beta)c + \beta \int_0^\infty \max(\omega, R) dF(\omega).$$

It can be rewritten as

$$R = b + \frac{\beta}{1-\beta} \int_R^\infty (\omega - R) dF(\omega)$$

$\Longrightarrow$ T ***is a contraction:*** By Blackwell's theorem, it is enough to show that $T(R)$ is increasing, and $T(a+R) \leq T(R) + \beta a$. To show it is increasing, we rewrite $T(R)$ as

$$T(R) = (1-\beta)c + \beta\left[F(R)R + \int_R^\infty w dF(\omega)\right]$$

and differentiate it with respect to R. Then, we have

$$T'(R) = \beta[f(R)R + F(R) - Rf(R)] = \beta F(R) \geq 0.$$

The second condition is also easily shown by a simple algebra.

$$
\begin{aligned}
T(a+R) &= (1-\beta)c + \beta E \max\{\omega, R+a\} \\
&= (1-\beta)c + \beta E \max\{\omega - a, R\} + \beta a \\
&\leq (1-\beta)c + \beta E \max\{\omega, R\} + \beta a \\
&= T(R) + \beta a
\end{aligned}
$$

(b) Let $\omega' = \max\{\omega_1, \omega_2\}$. Then, the distribution function of ω' is

$$
\begin{aligned}
G(\omega') &= \Pr(\max\{\omega_1, \omega_2\} \leq \omega') \\
&= \Pr(\omega_1 \leq \omega', \omega_2 \leq \omega') \\
&= \Pr(\omega_1 \leq \omega') \cdot \Pr(\omega_2 \leq \omega') \quad (\because \text{ i.i.d. draw }) \\
&= F^2(\omega').
\end{aligned}
$$

Therefore, the reservation wage equation is

$$
R' = b + \frac{\beta}{1-\beta} \int_{R'}^{\infty} (\omega - R) d\omega G(\omega)
$$

(c) Now, the worker has more chances to get higher wage offers. So, she becomes more picky, i.e. R' is larger than R.

$\Longrightarrow$ Suppose $R' \leq R$. Then,

$$
\begin{aligned}
R' - R &= \frac{\beta}{1-\beta}\left[\int_{R'}^{\infty} \left[1 - F^2(\omega)\right] d\omega - \int_{R}^{\infty} [1 - F(\omega)] d\omega\right] \\
&\geq \frac{\beta}{1-\beta}\left[\int_{R}^{\infty} \left[1 - F^2(\omega)\right] d\omega - \int_{R}^{\infty} [1 - F(\omega)] d\omega\right] \quad (\because R' \leq R) \\
&= \frac{\beta}{1-\beta} \int_{R}^{\infty} \left[F(\omega) - F^2(\omega)\right] d\omega \\
&= \frac{\beta}{1-\beta} \int_{R}^{\infty} F(\omega)[1 - F(\omega)] d\omega > 0
\end{aligned}
$$

which contradicts with the assumption. ■