EC813B - Recitation 3

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Question 1

Consider a discrete time, infinite horizon job search model. Each period an unemployed worker receives on *i.i.d.* offer ω from cumulative distribution function $F(\omega)$. The worker maximizes

$$E\sum_{t=0}^{\infty}\beta^t c_t$$

where $0 < \beta = 1/(1+r) < 1$, and

$$c_t = \begin{cases} \omega_t & \text{if she is working} \\ b & \text{if she is unemployed} \end{cases}$$

Once an offer is accepted the job is held forever.

(a) Formulate the Bellman's equation for the worker's problem. Show that mapping T defined as

$$T(R) = (1-\beta)b + \beta E \max\{\omega, R\}$$

is a contraction (use Blackwell's theorem). Derive the reservation wage equation.

(b) Now assume that each period an unemployed worker can draw two *i.i.d.* wage offers from the same cumulative distribution function $F(\omega)$. Derive the reservation wage the offer at which the individual is indifferent between accepting the offer and remaining unemployed.

(c) Compare the worker's reservation wages for both cases.

Solution

Theorem below is directly taken from *Stokey and Lucas (1989), Recursive Methods in Economic Dynamics, page 54.*

Theorem *(Blackwell's Sufficient Condition for a Contraction)* Let $X \subseteq R^l$ and let $B(X)$

be a space of bounded functions $f : X \to R$ with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying (i) (monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$ implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$, and (ii) (discounting) there exists some $\beta \in (0,1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \beta a$$

for all $f \in B(X), a \geq 0, x \in X$. Then T is a contraction mapping with modulus β.

Proof. For all $f, g \in B(X)$, $f \leq g + \|f - g\|$. Thus, $Tf \leq Tg + \beta\|f - g\|$ or $Tf - Tg \leq \beta\|f - g\|$. Switching the roles of f and g we obtain $\|Tf - Tg\| \leq \beta\|f - g\|$ and hence, T is a contraction with modulus β. ■

(a) $\Longrightarrow$ ***Bellman's equation:***

Let $V(\omega)$ denote the expected payoff to accepting an offer ω at some point in time, referred to as the value of ω. It does not depend on when the offer is accepted, given our assumptions of a stationary environment and an infinite horizon. In fact, since jobs are retained forever,

$$V(\omega) = \frac{\omega}{1 - \beta}$$

Also, let U denote the value of rejecting an offer and remaining unemployed, which also does not depend on time, and does not depend on which wage was rejected since offers are $i.i.d.$

$$U = b + \beta E \max\{V(\omega), U\}$$

since the value of rejecting an offer equals instantaneous unemployment income plus the discounted expected value of having the option to accept or reject a new offer next period.

Let $J(\omega) = \max\{V(\omega), U\}$ be the value of having offer ω in hand. Then $J(\omega)$ satisfies the following version of Bellman's equation of dynamic programming:

$$J(\omega) = \max\left\{\frac{\omega}{1 - \beta}, c + \beta EJ\right\} \tag{1}$$

The nature of the solution can be described as follows. Since $V(\omega)$ is increasing in ω and U is independent of ω, there exists a unique R satisfying $V(R) = U$ where R is the reservation wage.

$\Longrightarrow$ ***Derivation of the reservation wage equation***

Since $V(R) = R/(1-\beta)$ and $U = c + \beta EJ$, the definition of the reservation wage, $V(R) = U$, is equivalent to

$$R = (1-\beta)c + (1-\beta)\beta EJ \tag{2}$$

This expresses R in terms of the unknown value function, J. However, rewriting (1) as

$$J(w) = \begin{cases} \frac{\omega}{1-\beta} & \text{for } \omega \geq R \\ \frac{R}{1-\beta} & \text{for } \omega < R \end{cases}$$

we see that $(1-\beta)EJ = E\max(\omega, R)$. Combining this with (2) we can express the reservation wage as the solution to the following equation:

$$R = (1-\beta)c + \beta \int_0^\infty \max(\omega, R) dF(\omega).$$

It can be rewritten as

$$R = b + \frac{\beta}{1-\beta} \int_R^\infty (\omega - R) dF(\omega)$$

$\Longrightarrow$ T ***is a contraction:*** By Blackwell's theorem, it is enough to show that $T(R)$ is increasing, and $T(a+R) \leq T(R) + \beta a$. To show it is increasing, we rewrite $T(R)$ as

$$T(R) = (1-\beta)c + \beta \left[F(R)R + \int_R^\infty w dF(\omega) \right]$$

and differentiate it with respect to R. Then, we have

$$T'(R) = \beta[f(R)R + F(R) - Rf(R)] = \beta F(R) \geq 0.$$

The second condition is also easily shown by a simple algebra.

$$\begin{aligned} T(a+R) &= (1-\beta)c + \beta E\max\{\omega, R+a\} \\ &= (1-\beta)c + \beta E\max\{\omega - a, R\} + \beta a \\ &\leq (1-\beta)c + \beta E\max\{\omega, R\} + \beta a \\ &= T(R) + \beta a \end{aligned}$$

(b) Let $\omega' = \max\{\omega_1, \omega_2\}$. Then, the distribution function of ω' is

$$\begin{aligned} G(\omega') &= \Pr\left(\max\{\omega_1, \omega_2\} \le \omega'\right) \\ &= \Pr\left(\omega_1 \le \omega', \omega_2 \le \omega'\right) \\ &= \Pr\left(\omega_1 \le \omega'\right) \cdot \Pr\left(\omega_2 \le \omega'\right) \quad (\because \text{ i.i.d. draw }) \\ &= F^2(\omega'). \end{aligned}$$

Therefore, the reservation wage equation is

$$R' = b + \frac{\beta}{1-\beta} \int_{R'}^{\infty} (\omega - R) dG(\omega)$$

(c) Now, the worker has more chances to get higher wage offers. So, she becomes more picky, i.e. R' is larger than R.

$\Longrightarrow$ Suppose $R' \le R$. Then,

$$\begin{aligned} R' - R &= \frac{\beta}{1-\beta}\left[\int_{R'}^{\infty}\left[1 - F^2(\omega)\right] d\omega - \int_{R}^{\infty}[1 - F(\omega)] d\omega\right] \\ &\ge \frac{\beta}{1-\beta}\left[\int_{R}^{\infty}\left[1 - F^2(\omega)\right] d\omega - \int_{R}^{\infty}[1 - F(\omega)] d\omega\right] \quad (\because R' \le R) \\ &= \frac{\beta}{1-\beta}\int_{R}^{\infty}\left[F(\omega) - F^2(\omega)\right] d\omega \\ &= \frac{\beta}{1-\beta}\int_{R}^{\infty} F(\omega)[1 - F(\omega)] d\omega > 0 \end{aligned}$$

which contradicts with the assumption. ■

Question 2

Consider a discrete time, infinite horizon job search model. The worker maximizes

$$E \sum_{t=0}^{\infty} \beta^t c_t$$

where $0 < \beta = 1/(1+r) < 1$. An unemployed worker receives a wage offer ω each period that is a draw from a probability distribution $F(\omega)$, where $0 \le \omega \le \hat{\omega}$. When employed, the worker receives ω units of consumption goods at the beginning of the period, and then experiences separation probability δ, where $0 < \delta < 1$. When the worker becomes unemployed, she receives an unemployment benefit b for one period only. If an unemployed worker was also unemployed in the previous period, then she receives an unemployment benefit of zero. Assume that $b > 0$.

Hint: Note that you need to determine $V_e(\omega)$, the value of being employed at the wage w; V_{u^1}, the value of being unemployed in the first period of unemployment; and V_{u^0}, the value of being unemployed if the worker was unemployed in the previous period.

(a) Formulate the Bellman's equations for the worker's problem evaluating each state at the end of the period. Derive the reservation wage equation and show that the reservation wage is independent of how long the worker has been unemployed. Explain why.

(b) Determine how a change in b affects the reservation wage, and explain your results. *Hint: use the **Leibnizs rule**: let f be be a continuous function with a continuous partial derivative with respect to a parameter a, and let p and q be differentiable functions. Consider:*

$$F(a) = \int_{p(a)}^{q(a)} f(x,a)dx$$

Then,

$$\frac{\partial F}{\partial a} = \int_{p(a)}^{q(a)} \frac{\partial f(x,a)}{\partial a}dx + f(q(a),a)\frac{\partial q(a)}{\partial a} - f(p(a),a)\frac{\partial p(a)}{\partial a}$$

Solution

(a) You may want to use the end of period discounting. That is, if there is no threat of separation and a worker receives w while employed, a value function for the employment state can be written as

$$V(w) = \frac{1}{1+r}(w + V(w))$$

Let's denote the value of each state as follows;

$$\begin{cases} V_e(w): & \text{the value of being employed at the wage at } w \\ V_{u^1}: & \text{the value of being unemployed in the first period of unemployment} \\ V_{u^0}: & \text{the value of being unemployed for more than one period} \end{cases}$$

Then, the Bellman's equation for each state can be formulated as;

$$\begin{cases} V_e(w) & = \frac{1}{1+r}\left(w + \delta V_{u^1} + (1-\delta)V_e(w)\right) \\ V_{u^1} & = \frac{1}{1+r}\left(b + \mathrm{E}\max\left\{V_e(w), V_{u^0}\right\}\right) \\ V_{u^0} & = \frac{1}{1+r}\left(0 + \mathrm{E}\max\left\{V_e(w), V_{u^0}\right\}\right) \end{cases}$$

From equations for V_{u^1} and V_{u^0}, we obtain $V_{u^1} = b/(1+r) + V_{u^0}$. Substituting it into the equation for $V_e(w)$, we have

$$\begin{aligned} V_e(w) &= \frac{1}{1+r}\left\{w + \delta\left(\frac{b}{1+r} + V_{u^0}\right) + (1-\delta)V_e(w)\right\} \\ \Longrightarrow V_e(w) &= \frac{1}{r+\delta}\left\{w + \delta\left(\frac{b}{1+r} + V_{u^0}\right)\right\} \end{aligned}$$

By the definition of the reservation wage, $V_e(\bar{w}) = V_{u^0}$, or

$$\begin{aligned} V_e(\bar{w}) = V_{u^0} &= \frac{1}{r+\delta}\left\{\bar{w} + \delta\left(\frac{b}{1+r} + V_{u^0}\right)\right\} \\ \Longrightarrow (r+\delta)V_{u^0} &= \bar{w} + \delta\frac{b}{1+r} + \delta V_{u^0} \\ \Longrightarrow rV_{u^0} &= \bar{w} + \delta\frac{b}{1+r} \end{aligned} \tag{3}$$

Also, we obtain

$$V_e(w) - V_e(\bar{w}) = V_e(w) - V_{u^0} = \frac{1}{r+\delta}(w - \bar{w}) \tag{4}$$

From the equation for V_{u^0} in the Bellman's equation,

$$\begin{aligned} V_{u^0} &= \frac{1}{1+r}\left\{0 + \mathrm{E}\max\left\{V_e(w) - V_{u^0}, 0\right\}\right\} + \frac{1}{1+r}V_{u^0} \\ \Longrightarrow rV_{u^0} &= \mathrm{E}\max\left\{V_e(w) - V_{u^0}, 0\right\} \end{aligned}$$

Using (3) and (4), we have

$$\begin{aligned} rV_{u^0} = \bar{w} + \delta\frac{b}{1+r} &= \int_{\bar{w}}^{\hat{w}} \frac{w - \bar{w}}{r+\delta}\mathrm{d}F(w) \\ \bar{w} &= -\frac{\delta b}{1+r} + \int_{\bar{w}}^{\hat{w}} \frac{w - \bar{w}}{r+\delta}\mathrm{d}F(w) \end{aligned} \tag{5}$$

(b) Define

$$\phi(w,b) = \bar{w} + \frac{\delta b}{1+r} - \int_{\bar{w}}^{\hat{w}} \frac{w-\bar{w}}{r+\delta} \mathrm{d}F(w) \tag{6}$$

Then, you may want to derive

$$\frac{\partial \bar{w}}{\partial b} = -\frac{\phi_b}{\phi_{\bar{w}}}$$

Differentiating (6) w.r.t b and $\bar{w}$,

$$\phi_b = \frac{\delta}{1+r}$$

$$\begin{aligned}\phi_{\bar{w}} = 1 - &\left\{ \int_{\bar{w}}^{\hat{w}} \partial \left(\frac{w-\bar{w}}{r+\delta} \right) / \partial \bar{w} \ \mathrm{d}F(w) + \frac{\hat{w}-\bar{w}}{r+\delta} \cdot 0 - \left(\frac{\bar{w}-\bar{w}}{r+\delta} \right) \left(-\frac{1}{r+\delta} \right) \right\} \\ &= 1 + \int_{\bar{w}}^{\hat{w}} \frac{1}{r+\delta} \mathrm{d}F(w) > 0\end{aligned}$$

Therefore,

$$\frac{\partial \bar{w}}{\partial b} < 0$$

The reservation wage decreases as b increases. In order to receive the unemployment benefit, the worker must be employed first. In other words, the unemployment benefit can be interpreted as one type of benefit from being employed so the job position becomes attractive as b goes up.

Question 3

Consider a discrete time, infinite horizon job search model. The worker maximizes

$$E \sum_{t=0}^{\infty} \beta^t c_t$$

where $0 < \beta = 1/(1+r) < 1$. When a worker is umemployed she receives an unemployment benefit b at the beginning of each period while unemployed. Then with probability a she receives a wage offer w that is a draw from a probability distribution $F(w)$, where $0 \leq w \leq \hat{w}$.

If the worker accepts the offer she becomes employed, that is the worker receives w units of consumption goods at the beginning of the period, and then experiences separation with probability δ, where $0 < \delta < 1$. If the match survives, the worker gets promoted in the same period with probability γ. Promotion means that the wage is increased by the factor $\phi > 1$: that is the new wage next period is ϕw. The worker with promotion gets ϕw at the beginning of each period and then she gets separated with the same probability δ and becomes unemployed. (promotion does not affect the probability of getting laid-off)

(a) Write down the relevant value functions for the three states, V_u (unemployed), $V_e(w)$ (employed without promotion), $V_p(w)$ (employed with promotion), use them to get the asset value equations and show that the worker will always like to be promoted, i.e. evaluate $V_p(w) - V_e(w)$ and show it is positive. Use end of period discounting. HINT: Simplify the asset value equation for $V_e(w)$ ignoring the terms proportional to δ^2 and $\delta\gamma$ and then derive $V_p(w) - V_e(w)$.

(b) Suppose now that we could observe a continuum (of mass 1) of such individuals. Draw a diagram showing the flow rates of individuals across states where $n_u; n_e; n_p$ are corresponding measures of agents in each state. Derive a system of equations that can be solved for the steady state proportions of the population that are in each state. You do not have to solve it.

(c) Solve for the reservation wage equation and explain how this wage changes with ϕ : you do not have to differentiate the equation: providing intuition is enough.

HINT: You have to substitute $V_p(w)$ out (use $V_p(w) - V_e(w)$ derived in a) and get equations in $V_e(w)$ and V_u. Then follow the class derivations. Remember that $V_e(w^*) = V_u$, where w^* is the reservation wage.

Solution

(a) The value functions for each state can be written as;

$$\begin{cases} V_u & = \frac{1}{1+r}\left(b + a\mathrm{E}\max\left\{V_e(w), V_u\right\} + (1-a)V_u\right) \\ V_e(w) & = \frac{1}{1+r}\left(w + \delta V_u + (1-\delta)\left\{\gamma V_p(w) + (1-\gamma)V_e(w)\right\}\right) \\ V_p(w) & = \frac{1}{1+r}\left(\phi w + \delta V_u + (1-\delta)V_p(w)\right) \end{cases}$$

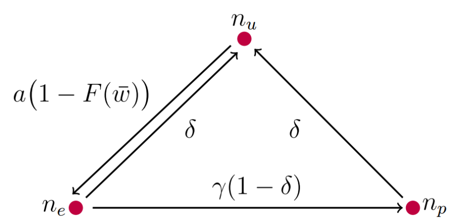
which can be rearranged as;

$$
\begin{cases}
rV_u & = b + a\,\mathrm{E}\max\left\{V_e(w) - V_u, 0\right\} \\
rV_e(w) & = w - \delta\left(V_e(w) - V_u\right) + \gamma(1-\delta)\left(V_p(w) - V_e(w)\right) \\
rV_p(w) & = \phi w - \delta\left(V_p(w) - V_u\right)
\end{cases}
\tag{7}
$$

From the equations for $rV_e(w)$ and $rV_p(w)$, we obtain

$$
V_p(w) - V_e(w) = \frac{(\phi - 1)w}{r + \delta + \gamma(1-\delta)} > 0
$$

(b) We know $n_u + n_e + n_p = 1$. The flow rates of individuals across states can be drawn as;



In steady-state,

$$
\begin{cases}
a(1 - F(\bar{w}))n_u = (\delta + \gamma(1-\delta))n_e \\
\gamma(1-\delta)n_e = \delta n_p \\
a(1 - F(\bar{w}))n_u = \delta\left(n_e + n_p\right)
\end{cases}
$$

(c) Substituting the expression for $V_p(w) - V_e(w)$ into the equation for $rV_e(w)$ in (7), we have

$$
\begin{aligned}
rV_e(w) &= w - \delta\left(V_e(w) - V_u\right) + \gamma(1-\delta)\frac{(\delta - 1)w}{r + \delta + \gamma(1-\delta)} \\
\implies V_e(w) &= \frac{1}{r+\delta} \cdot \frac{1}{r + \delta + \gamma(1-\delta)}\left((r + \delta + \phi\gamma(1-\delta))w + \delta(r + \delta + \gamma(1-\delta))V_u\right)
\end{aligned}
$$

Using the definition of the reservation wage $V_e(\bar{w}) = V_u$, we obtain

$$
\begin{aligned}
rV_u &= \bar{w} + \gamma(1-\delta)\frac{(\phi - 1)\bar{w}}{r + \delta + \gamma(1-\delta)} \\
&= \frac{r + \delta + \phi\gamma(1-\delta)}{r + \delta + \gamma(1-\delta)}\bar{w}
\end{aligned}
$$

and

$$V_e(w) - V_u = \frac{r + \delta + \phi\gamma(1 - \delta)}{(r + \delta)(r + \delta + \gamma(1 - \delta))}(w - \bar{w})$$

Substituting these expressions into the equation for rV_u in (7), we have

$$\begin{aligned} \tfrac{r+\delta+\phi\gamma(1-\delta)}{r+\delta+\gamma(1-\delta)}\bar{w} &= b + a\tfrac{r+\delta+\phi\gamma(1-\delta)}{(r+\delta)(r+\delta+\gamma(1-\delta))}\,\mathrm{E}\max\{w - \bar{w}, 0\} \\ \Longrightarrow \bar{w} &= \tfrac{r+\delta+\gamma(1-\delta)}{r+\delta+\phi\gamma(1-\delta)}b + \tfrac{a}{r+\delta}\int_{\bar{w}}^{\hat{w}}(w - \bar{w})\mathrm{d}F(w) \end{aligned}$$

Question 4

DIAMOND COCONUT ECONOMY with MATCH-SPECIFIC PREFERENCES

Consider a discrete time, infinite horizon Diamond economy. There is a mass of 1 of ex-ante identical individuals with infinite lives. The common discount rate is r, consumption of own produce yields 0 utilities, and consumption of anyone else's output yields uF on $[0, \bar{u}]$ utilities. The economy consists of two islands: a trading island and a production island. On the production island, individuals come across a tree with a coconut with probability α each period. The cost of obtaining the coconut is 0 in every case. On the trading island, people with coconuts meet another with probability γ, a constant. Travel between islands is instantaneous. Everyone has a boat and starts off with one of their own coconuts.

(a) Assume that everyone else accepts any coconut with probability Ω. Write down the asset value (Bellman's) equations for the problem faced by an individual in this environment. (Note: The individual believes that anyone he meets will take his coconut with probability Ω; however, the individual chooses whether or not to accept the coconut of anyone he meets based on the utility he will get from eating it.)

(b) Let u represent the critical utility from consumption for which the individual is willing to give up his coconut. Express $\hat{u}$ in terms of the value to being on the trading island and the value to being on the production island.

(c) Derive a reservation utility equation for the individual taking Ω as given. (It should not contain any endogenous variables other than $\hat{u}$ and Ω).

(d) Define steady-state symmetric equilibrium and provide an equation that characterizes the equilibrium reservation utility, u^*.

HINT: Express Ω in terms of the reservation utility of everyone else.

(e) Draw a diagram showing the population flows between the islands. Write down the steady-state equations and solve for the population on the trading island as a function of u^*.

Solution

(a) Let V_i be the value function of being in sector $i \in \{T, P\}$ where T and P represent the trading and production island respectively.

$$V_P = \frac{1}{1+r}\left\{\alpha V_T + (1-\alpha)V_P\right\}$$
$$V_T = \frac{1}{1+r}\left\{\gamma\Omega E\max\left\{u+V_P, V_T\right\} + (1-\gamma\Omega)V_T\right\}$$

Transforming the above in the form of asset value equations,

$$rV_P = \alpha\left(V_T - V_P\right) \tag{8}$$

$$rV_T = \gamma\Omega E\max\left\{u + V_P - V_T, 0\right\} \tag{9}$$

(b) Critical Utility:

$$\hat{u} = V_T - V_P \tag{10}$$

(c) From (8) and (10), we know $rV_P = \alpha\hat{u}$ and $rV_T = r\left(\hat{u} + V_P\right)$. Therefore,

$$rV_T = (r+\alpha)\hat{u} \tag{11}$$

Substituting (11) in (9),

$$(r+\alpha)\hat{u} = \gamma\Omega\int_{\hat{u}}^{\bar{u}}(u-\hat{u})dF(u)$$
$$\implies \hat{u} = \frac{\gamma\Omega}{r+\alpha}\int_{\hat{u}}^{\bar{u}}(u-\hat{u})dF(u).$$

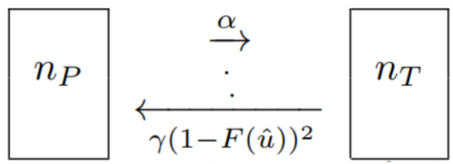
(d) A steady-state symmetric equilibrium is a reservation utility $\hat{u}$ such that when everyone else follows the implied trading strategy (i.e., accept when $u \geq \hat{u}$ and reject otherwise), $\hat{u}$ also represents each individual's optimal strategy. Mathematically,

$$\Omega = 1 - F(\hat{u})$$

and so now the reservation wage becomes

$$\hat{u} = \frac{\gamma[1 - F(\hat{u})]}{r + \alpha} \int_{\hat{u}}^{\bar{u}} (u - \hat{u}) dF(u).$$

(e) The diagram is shown as follows:



And, the steady-state n_T^* and n_P^* are the solutions of the following system of equations:

$$\alpha n_P = \gamma[1 - F(\hat{u})]^2$$
$$n_P + n_T = 1$$

Thus,

$$n_T^* = \frac{\alpha}{\alpha + \gamma[1 - F(\hat{u})]^2} \text{ and } n_P^* = 1 - n_T^*.$$