EC813B - Recitation 4

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Review questions from previous recitations for Midterm 1.

Question

Consider the model of a firm searching for a single worker to produce one unit of output per period (forever). The firm maximizes the discounted sum of profits. The output sells at price p per unit, and the firm discounts future with a discount factor β. The firm gets to sample one worker per period who demands wage ω, where ω is an i.i.d. draw from a cumulative distribution function $G(\omega)$.

(a) Show that the profit maximizing strategy is to hire a worker if $\omega < R$, where R is a reservation wage.

(b) Derive the reservation value equation for R.

(c) How does the reservation wage R depend on the discount factor β? Explain.

Solution

(a) Since jobs are retained forever, the value function of a firm in case firm finds a worker can be defined as the following

$$V(\omega) = p - \omega + \beta V(\omega) \tag{1}$$

Also, let U denote the value of not finding a worker as follows

$$U = \beta E \max\{V(\omega), U\} \tag{2}$$

Using (1), we can derive the functional form of $V(\omega)$. We get

$$V(\omega) = \frac{p - \omega}{1 - \beta}$$

Hence, $V(\cdot)$ is a decreasing function of ω. Let $J(\omega) = \max\{V(\omega), U\}$ be the value of having offer ω in hand. Then $J(\omega)$ satisfies the following version of Bellman's equation of dynamic programming:

$$J(\omega) = \max\left\{\frac{p-\omega}{1-\beta}, \beta EJ\right\} \tag{3}$$

Now, we can identify the profit maximizing strategy. Following a similar way as we did from the workers' perspective, considering $V(R) = \frac{(p-R)}{(1-\beta)}$ and $U = \beta EJ$, the definition of the reservation wage, $V(R) = U$, is equivalent to

$$R = p - (1-\beta)\beta EJ \tag{4}$$

This expresses R in terms of the unknown value function, J. However, rewriting (3) as

$$J(w) = \begin{cases} \frac{p-\omega}{1-\beta} & \text{for } \omega \geq R \\ \frac{p-R}{1-\beta} & \text{for } \omega < R \end{cases} \tag{5}$$

Thus, hiring a worker if $\omega < R$ is the profit maximizing strategy.

(b) We know that $(1-\beta)EJ = E\max(\omega, R)$. Combining this with (4) we can express the reservation wage as the solution to the following equation:

$$R = p - \beta \int_0^\infty \max(\omega, R)dG(\omega). \tag{6}$$

It can be rewritten as

$$R = p - \frac{\beta}{1-\beta}\int_0^R (R-\omega)dG(\omega) \tag{7}$$

Note: How do we get from (6) to (7)?

$\Longrightarrow$ Using $V(R) = U$ and (5), we obtain

$$\frac{(p-R)}{(1-\beta)} = \beta EJ \quad \Longrightarrow \quad \frac{(p-R)}{(1-\beta)} = \beta E \max\left\{\frac{(p-\omega)}{(1-\beta)}, \frac{(p-R)}{(1-\beta)}\right\}$$
$$\Longrightarrow (p-R) = \beta E \max\Big\{(p-\omega), (p-R)\Big\}$$
$$\Longrightarrow (p-R) = \beta E \max\Big\{(p-\omega), (p-R)\Big\} + \beta(p-R) - \beta(p-R)$$
$$\Longrightarrow (1-\beta)(p-R) = \beta E \max\Big\{(p-\omega), (p-R)\Big\} - \beta(p-R)$$
$$\Longrightarrow (p-R) = \frac{\beta}{1-\beta} E \max\Big\{(R-\omega), 0\Big\}$$
$$\Longrightarrow R = p - \frac{\beta}{1-\beta} E \max\Big\{(R-\omega), 0\Big\}$$
$$\Longrightarrow R = p - \frac{\beta}{1-\beta} \int_0^R (R-\omega) dG(\omega)$$

or

$$R = p - \frac{\beta}{1-\beta} \int_0^R G(\omega) d\omega \tag{8}$$

Note: How do we obtain (8)?
$\Longrightarrow$ The equation follows from integration by parts. The below is taken from R. Wright's lecture notes:
For any $\bar{w}$, we have

$$\int_R^{\bar{w}} (w-R) dF(w) = (\bar{w}-R)F(\bar{w}) - \int_R^{\bar{w}} F(w) dw = \int_R^{\bar{w}} [F(\bar{w}) - F(w)] dw$$

Letting $\bar{w} \to \infty$, we have the required result. For future reference, we catalogue the following properties of the surplus function: $\varphi(0) = Ew; \varphi(\infty) = 0; \varphi'(R) = -[1 - F(R)] \leq 0$, where the inequality is strict as long as $F(R) < 1$; and $\varphi''(R) = F'(R) \geq 0$, assuming the density F' exists, where the inequality is strict as long as $F'(R) > 0$.

(c) Define $H(R, \beta) = R - p + \frac{\beta}{1-\beta} \int_0^R G(\omega) d\omega$. Then, using the Implicit Function Theorem we need to find $\frac{\partial R}{\partial \beta}$. The Implicit Function Theorem states that

$$\frac{\partial R}{\partial \beta} = -\frac{\partial H/\partial \beta}{\partial H/\partial R} = -\frac{\frac{1}{(1-\beta)^2}\int_0^R G(\omega)d\omega}{1+\frac{\beta}{(1-\beta)}\cdot g(R)} < 0$$