EC813B - Recitation 11

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Question 1

Consider the Diamond overlapping generations model with logarithmic utility and Cobb-Douglas production:

$$
\begin{aligned}
U(c_{t1}, c_{t2}) &= \ln c_{t1} + \beta \ln c_{t2} \\
Y_t &= F(K_t, N_t) \\
&= A K_t^{\alpha} N_t^{1-\alpha}
\end{aligned}
$$

The population grows at rate n. Individuals supply inelastically one unit of labor in the first period of their lives and they retire in the second period.

(a) Define decentralized goods market equilibrium in terms of per capita variables.

(b) Derive the capital accumulation equation. Solve for the steady state capital stock.

(c) Describe how each of the following affects k_{t+1} as a function of k_t:

  (i) A rise in n.

  (ii) A rise in A.

  (iii) A rise in α.

(d) Now use this model to compare two types of pension systems. First consider a fully funded system. In period t, the government collects lumpsum taxes τ from the young agents. The tax revenues are invested in physical capital and returned to the old generation the next period. How will the steady state capital stock be affected by this pension system?

(e) Next, consider a pay-as-you-go pension system, in which the government collects amount τ from each young person but now the tax revenues are immediately transferred to the current old person. How will the steady state capital stock be affected by this pension system?

Solution

(a) Decentralized goods market equilibrium is a list of $\{s_t\}, \{k_{t+1}\}$, and $\{r_t, w_t\}$ such that

  (i) firms maximize profits given $\{r_t, w_t\}$,

  (ii) consumers maximize utilities given $\{r_{t+1}, w_t\}$, and

  (iii) market clears.

(b) We start with Household's maximization problem. It is set up as follows

$$\max_{\{c_{t1}, c_{t2}\}} \quad \ln c_{t1} + \beta \ln c_{t2} \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= w_t - s_t \\ c_{t2} &= (1 + r_{t+1}) s_t \end{aligned}$$

Plugging in for c_{t1} and c_{t2} we get

$$\max_{\{s_t\}} \quad \ln\big(w_t - s_t\big) + \beta \ln\big((1 + r_{t+1}) s_t\big)$$

The FOC is

$$\frac{1}{w_t - s_t} = \beta \frac{1 + r_{t+1}}{(1 + r_{t+1})\, s_t} \quad \Longrightarrow \quad s_t = \frac{\beta}{1 + \beta} w_t$$

Next, the Firm's problem is

$$\max_{\{K_t, N_t\}} \quad A K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - r_t K_t$$

The FOC is

$$\begin{aligned} r_t &= A\alpha k_t^{\alpha - 1} \\ w_t &= A(1 - \alpha) k_t^{\alpha} \end{aligned}$$

The *market clearing condition* is

$$\begin{aligned} s_t N_t &= K_{t+1} \\ s_t &= \frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = k_{t+1}(1 + n) \end{aligned}$$

Combining the FOCs from Household's and Firm's problem and the market clearing condition, we obtain the capital accumulation equation as

$$\begin{aligned}
& (1+n)k_{t+1} = s_t = \frac{\beta}{1+\beta}w_t = \frac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha} \\
\implies \quad & k_{t+1} = \frac{\beta A(1-\alpha)}{(1+\beta)(1+n)}k_t^{\alpha}
\end{aligned}$$

Now, we solve for the steady state capital stock, k^*, assuming the steady-state condition $k_t = k_{t+1} = \cdots = k^*$. We get

$$k^* = \left[\frac{\beta A(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

(c) Using the capital accumulation equation, we need to show how each of the following affects k_{t+1} as a function of k_t.

(i) A rise in n.

$$\frac{\partial k_{t+1}}{\partial n} = -\frac{\beta A(1-\alpha)}{(1+\beta)(1+n)^2}k_t^{\alpha} < 0$$

(ii) A rise in A.

$$\frac{\partial k_{t+1}}{\partial A} = \frac{\beta(1-\alpha)}{(1+\beta)(1+n)}k_t^{\alpha} > 0$$

(iii) A rise in α.

$$\begin{aligned}
\frac{\partial k_{t+1}}{\partial \alpha} &= \frac{\beta A}{(1+\beta)(1+n)}k_t^{\alpha}\ln k_t - \frac{\beta A}{(1+\beta)(1+n)}k_t^{\alpha} - \frac{\beta A\alpha}{(1+\beta)(1+n)}k_t^{\alpha}\ln k_t \\
&= \frac{\beta A k_t^{\alpha}}{(1+\beta)(1+n)}\left[(1-\alpha)\ln k_t - 1\right] > 0
\end{aligned}$$

(d) In a fully funded pension system, Household's maximization problem is

$$\max_{\{c_{t1},c_{t2}\}} \quad \ln c_{t1} + \beta \ln c_{t2} \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= w_t - (s_t+\tau) \\ c_{t2} &= (1+r_{t+1})(s_t+\tau) \end{aligned}$$

In a fully funded social security system, the government at date t raises some amount τ from the young, for example, by compulsory contributions to their social security accounts. These funds are invested in the only productive asset of the economy, the capital stock, and the workers receive the returns, given by $(1+r_{t+1})\tau$, when they are old. Then, we can define $\phi_t = s_t+\tau$ as the total amount invested in capital accumulation.

Plugging in for c_{t1} and c_{t2} we get

$$\max_{\{\phi_t\}} \quad \ln\big(w_t - \phi_t\big) + \beta \ln\big((1+r_{t+1})\phi_t\big)$$

The FOC is

$$\frac{1}{w_t - \phi_t} = \beta \frac{1+r_{t+1}}{(1+r_{t+1})\,\phi_t} \quad \Longrightarrow \quad \phi_t = \frac{\beta}{1+\beta} w_t$$

Firm's problem is the same as in the case without the pension system and the market clearing condition becomes

$$\begin{aligned} \phi_t N_t &= K_{t+1} \\ \phi_t &= \frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}}\frac{N_{t+1}}{N_t} = k_{t+1}(1+n) \end{aligned}$$

Combining the FOCs from Household's and Firm's problem and the market clearing condition as in part (b), we obtain the capital accumulation equation as

$$k_{t+1} = \frac{\beta A(1-\alpha)}{(1+\beta)(1+n)} k_t^{\alpha}$$

which is the same as the case without the pension system. Hence, the steady state capital stock, k^*, is the same as well.

(e) In a pay-as-you-go pension system, Household's maximization problem is

$$\max_{\{c_{t1}, c_{t2}\}} \quad \ln c_{t1} + \beta \ln c_{t2} \quad \text{ s.t. } \quad \begin{aligned} c_{t1} &= w_t - (s_t + \tau) \\ c_{t2} &= (1+r_{t+1})s_t + (1+n)\tau \end{aligned}$$

Notice that in this environment the rate of return on social security payments is n rather than $(1+r_{t+1})$, because unfunded social security is a pure transfer system. Only s_t—rather than $(s_t+\tau)$ as in the fully funded case—goes into capital accumulation.

Plugging in for c_{t1} and c_{t2} we get

$$\max_{\{s_t\}} \quad \ln\big(w_t - (s_t+\tau)\big) + \beta \ln\big((1+r_{t+1})s_t + (1+n)\tau\big)$$

The FOC is

$$\frac{1}{w_t - (s_t+\tau)} = \beta \frac{1+r_{t+1}}{(1+r_{t+1})s_t + (1+n)\tau}$$

and, we get

$$s_t = \frac{\beta}{1+\beta} w_t - \left[\frac{\beta\left(1+r_{t+1}\right)+(1+n)}{(1+\beta)\left(1+r_{t+1}\right)}\right]\tau = \frac{\beta}{1+\beta} w_t - \frac{\beta}{1+\beta}\tau - \frac{(1+n)}{(1+\beta)\left(1+r_{t+1}\right)}\tau$$

Firm's problem is the same as in the case without the pension system. Combining the FOCs from Household's and Firm's problem and the market clearing condition as in part (b), we obtain the capital accumulation equation as

$$k_{t+1} = \frac{\beta A(1-\alpha)}{(1+\beta)(1+n)} k_t^\alpha - \frac{\beta}{(1+\beta)(1+n)}\tau - \frac{1}{(1+\beta)\left(1+A\alpha k_{t+1}^{\alpha-1}\right)}\tau$$

$$\underbrace{k_{t+1} + \frac{\tau}{(1+\beta)\left(1+A\alpha k_{t+1}^{\alpha-1}\right)}}_{g(k_{t+1})} = \underbrace{\frac{\beta A(1-\alpha)}{(1+\beta)(1+n)} k_t^\alpha}_{f(k_t)} - \frac{\beta\tau}{(1+\beta)(1+n)}$$

Therefore, the steady state capital stock is reduced after the social security is introduced.

Question 2

Consider a variation of the OLG model studied in class in which people may die at the end of their first period of life. There is a p probability that people will live in the second period, and they do not know whether they will live or die until it happens. There is no time discounting and expected utility is

$$EU = \ln c_1 + p \ln c_2.$$

The production function is $y = k^\alpha$ and there is no population growth. Assume that when people die, their wealth is distributed among the remaining members of their generation. Assume also that there are enough people in each generation so that there is no uncertainty about the size of bequests (the law of large numbers holds). Also assume that interest is earned by the capital that belongs to the deceased prior to it being divided up amongst the remaining population.

(a) Write down the single period and lifetime budget constraint of an individual. Call the amount received as a bequest, b. Solve for the individuals optimal savings in period 1 as a function of b, r and w.

(b) Solve for b as a function of the amount of capital in the second period.

(c) Put everything together into a difference equation for k. How does a decrease in p affect the steady state capital stock?

Solution

(a) First we need to make an assumption on the utility function because $\ln 0$ is not well defined. Define the utility function as:

$$U(c) = \begin{cases} \ln c & \text{when a consumer is alive} \\ 0 & \text{otherwise} \end{cases}$$

Thus, as given in the question, the expected utility of a representative consumer of generation t is given by (with no discounting):

$$EU = \ln c_{t1} + p \ln c_{t2} + (1-p)0$$

Also, note that consumers of generation t have the following budget constraints:

$$\begin{aligned} c_{t1} &= w_t - s_t \\ c_{t2} &= (1 + r_{t+1})\, s_t + b_t \end{aligned}$$

where b_t indicates the value or amount of bequest left to survivers of generation t. By the law of large numbers, each survivor will get

$$b_t = \frac{1}{p}(1-p)\,(1 + r_{t+1})\, s_t$$

Now, we can start with Household's maximization problem. It can be written as

$$\max_{\{c_{t1}, c_{t2}\}} \quad \ln c_{t1} + p \ln c_{t2} \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= w_t - s_t \\ c_{t2} &= (1 + r_{t+1}) s_t + b_t \end{aligned}$$

Plugging in for c_{t1} and c_{t2} we get

$$\max_{\{s_t\}} \quad \ln\big(w_t - s_t\big) + p \ln\big((1 + r_{t+1}) s_t + b_t\big)$$

The FOC is

$$\frac{1}{w_t - s_t} = \frac{p(1+r_{t+1})}{(1+r_{t+1})s_t + b_t} \quad \Longrightarrow \quad s_t = \frac{p(1+r_{t+1})w_t - b_t}{(1+p)(1+r_{t+1})}$$

(b) We start with solving Firm's problem. It can be written as

$$\max_{\{N_t, K_t\}} Y_t - w_t N_t - r_t K_t$$

From the FOCs we obtain

$$\begin{aligned} w_t &= (1-\alpha)K_t^{\alpha} = (1-\alpha)k_t^{\alpha} \\ r_t &= \alpha K_t^{\alpha-1} = \alpha k_t^{\alpha-1} \end{aligned}$$

The *market clearing condition* is

$$\begin{aligned} s_t N_t &= K_{t+1} \\ s_t &= \frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = k_{t+1} \\ \Longrightarrow \quad k_t &= K_t \quad \text{(no population growth)} \end{aligned}$$

Combining the FOCs from Household's and Firm's problem and the market clearing condition, we get

$$b_t = \frac{(1-p)}{p} \left(1 + \alpha k_{t+1}^{\alpha-1}\right) k_{t+1}$$

(c) Combine the FOCs and the market clearing condition to derive a difference equation for k. We obtain

$$\begin{aligned} k_{t+1} &= \frac{p(1+r_{t+1})w_t - b_t}{(1+p)(1+r_{t+1})} \\ k_{t+1} &= \frac{p\left(1+\alpha k_{t+1}^{\alpha-1}\right)(1-\alpha)k_t^{\alpha} - \frac{(1-p)}{p}\left(1+\alpha k_{t+1}^{\alpha-1}\right)k_{t+1}}{(1+p)\left(1+\alpha k_{t+1}^{\alpha-1}\right)} \\ k_{t+1} &= \frac{p^2}{1+p^2}(1-\alpha)k_t^{\alpha} \end{aligned}$$

Now, we solve for the steady state capital stock, k^*, assuming the steady-state condition

$k_t = k_{t+1} = \cdots = k^*$. We get

$$k^* = \left(\frac{p^2}{1+p^2}(1-\alpha) \right)^{\frac{1}{1-\alpha}}$$

and one can easily check that

$$\frac{\partial k^*}{\partial p} = \frac{1}{1-\alpha} \left[(1-\alpha) \frac{2p}{(1+p^2)^2} \right]^{\frac{\alpha}{1-\alpha}} > 0.$$