EC813B - Recitation 1

SPRING 2022

Mehmet Karaca

Question 1

For $x, y \in \mathbb{R}$

$$\boldsymbol{\rho_1}(x,y) = (x-y)^2$$
$$\boldsymbol{\rho_2}(x,y) = \sqrt{|x-y|}$$
$$\boldsymbol{\rho_3}(x,y) = \left|x^2-y^2\right|$$
$$\boldsymbol{\rho_4}(x,y) = |x-2y|$$
$$\boldsymbol{\rho_5}(x,y) = \frac{|x-y|}{1+|x-y|}$$

Determine which of these is (or are) not a metric (or metric).

Solution

There are three conditions to be a metric. For $x, y \in \mathbb{R}$, define $\rho(x,y)$. If

(i) $\rho(x,y) \geq 0$ ($x = y \Longleftrightarrow \rho(x,y) = 0$),

(ii) $\rho(x,y) = \rho(y,x)$, and

(iii) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ (Triangle inequality)

then $\rho(x,y)$ is a metric.

Considering these conditions, we can find

(1) $\boldsymbol{\rho_1(x,y) = (x-y)^2}$ is **not** a metric. It satisfies the first two conditions but violates triangle inequality. We can show it with a simple counter-example. Suppose $x = 1, y = 0$, and $z = 1/2$. Using triangle inequality, we obtain $(x-y)^2 \geq (x-z)^2 + (z-y)^2$ instead of $\rho_1(x,y) \leq \rho_1(x,z) + \rho_1(z,y)$. ■

(2) $\boldsymbol{\rho_2(x,y) = \sqrt{|x-y|}}$ is a metric. It satisfies all three conditions. Here is a proof of

triangle inequality:

$$\begin{aligned}
|x-y| \le |x-z| + |z-y| &\le |x-z| + 2\sqrt{|x-z|.|z-y|} + |z-y| \\
&= \left(\sqrt{|x-z|} + \sqrt{|z-y|}\right)^2 \\
\implies \sqrt{|x-y|} &\le \sqrt{\left(\sqrt{|x-z|} + \sqrt{|z-y|}\right)^2} \\
\implies \sqrt{|x-y|} &\le \sqrt{|x-z|} + \sqrt{|z-y|}
\end{aligned}$$

$\blacksquare$

(3) $\boldsymbol{\rho_3(x,y) = |x^2 - y^2|}$ is **not** a metric. Showing that it violates the first condition should be enough. We can show it with a counter-example. Suppose $x = 1$ and $y = -1$. We get $\rho_3(x,y) = 0$ but $x \neq y$. $\blacksquare$

(4) $\boldsymbol{\rho_4(x,y) = |x - 2y|}$ is **not** a metric. Showing that it violates the first condition should be enough. We can show it with a counter-example. Suppose $x = 1$ and $y = 1$. We get $\rho_4(x,y) \neq 0$ but $x = y$. $\blacksquare$

(5) $\boldsymbol{\rho_5(x,y) = \frac{|x-y|}{1+|x-y|}}$ is a metric. It satisfies all three conditions. Here is a proof of triangle inequality:

$$\begin{aligned}
&\implies \rho_5(x,y) \le \rho_5(x,z) + \rho_5(z,y) \\
&\frac{|x-y|}{1+|x-y|} \le \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} \\
&0 \le \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} - \frac{|x-y|}{1+|x-y|} \\
&0 \le \frac{\overbrace{\Big(|x-z| + |z-y| - |x-y|\Big)}^{\ge 0} + \overbrace{\Big(2|x-z|\cdot|z-y|\Big)}^{\ge 0} + \overbrace{\Big(|x-z|\cdot|z-y|\cdot|x-y|\Big)}^{\ge 0}}{\Big(1+|x-z|\Big)\cdot\Big(1+|z-y|\Big)\cdot\Big(1+|x-y|\Big)}
\end{aligned}$$

$\blacksquare$

Question 2

Prove that in any metric space, closed subsets of compact sets are compact.

Solution

Here are two different ways to prove:

(i) Suppose that $F \subset X$ where F is closed and X is compact. If (x_n) is a sequence in F, then there is a subsequence (x_{n_k}) that converges to $x \in X$ since X is compact. Then $x \in F$ since F is closed, so F is compact. ■

(ii) Let K be a compact metric space and F a closed subset. Then its complement F^c is open. Thus if $\{V_\alpha\}$ is an open cover of F we obtain an open cover Ω of K by adjoining F^c. Since K is compact, Ω has a finite subcover; removing F^c if necessary, we obtain a finite subcollection of $\{V_\alpha\}$ which covers F. This is the desired open cover. ■

Question 3

Use the Contraction Mapping Theorem (CMT) to show that the following sequence converges:

$$\left(\frac{1}{3}, \frac{1}{3+\frac{1}{3}}, \frac{1}{3+\frac{1}{3+\frac{1}{3}}}, \ldots\right)$$

Work in the metric space $< [0, \infty), \boldsymbol{\rho}(x, y) = |x - y| >$

(a) Formulate the CMT. Define a mapping that corresponds to this sequence and show that it is a contraction mapping.

(b) Find the limit of the sequence.

Solution

Definition and theorem below are directly taken from *Stokey and Lucas (1989), Recursive Methods in Economic Dynamics, page 49.*

Definition Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into inself. T is a *contraction mapping* (with modulus β) if for some $\beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$, for all $x, y \in S$.

Theorem *(Contraction Mapping Theorem)* If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β then (i) T has exactly one fixed point v in S, and (ii) for any $v_0 \in S, \rho\left(T^n v_0, v\right) \leq \beta^n \rho\left(v_0, v\right)$, $n = 0, 1, 2 \ldots$

Proof. Define $v_{n+1} = Tv_n = T^n v_0$. Then $\rho(v_{n+1}, v_n) \leq \beta^n \rho(v_1, v_0)$. Hence, for $m > n$

$$\begin{aligned}
\rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}) + \ldots + \rho(v_{n+1}, v_n) \\
&\leq \left[\beta^{m-1} + \ldots + \beta^n\right] \rho(v_1, v_0) \\
&\leq \beta^n \left[\beta^{m-n-1} + \ldots + 1\right] \rho(v_1, v_0) \\
&\leq \frac{\beta^n}{1-\beta} \rho(v_1, v_0) \to 0 \text{ as } n \to \infty
\end{aligned}$$

Hence, $\{v_n\}$ is a Cauchy sequence and converges to $v \in S$ since S is complete. Furthermore, since

$$\begin{aligned}
\rho(Tv, v) &\leq \rho(Tv, T^n v_0) + \rho(T^n v_0, v) \\
&\leq \beta \rho\left(v, T^{n-1} v_0\right) + \rho(T^n v_0, v)
\end{aligned}$$

where last terms $\to 0$ as showed above. Furthermore v is unique since suppose $\widehat{v} \neq v$ is the another fixed point. Then

$$0 < \rho(\widehat{v}, v) = \rho(T\widehat{v}, Tv) \leq \beta \rho(\widehat{v}, v).$$

This only holds when $\rho(\widehat{v}, v) = 0$ or $\widehat{v} = v$. ■

(a) We formulate CMT as above. Now, we define a mapping that corresponds to this sequence and show that it is a contraction mapping.

Define $T(x) \equiv \frac{1}{3+x}$ and remember we work in the metric space $< [0, \infty), \boldsymbol{\rho}(x, y) = |x - y| >$. Using *Definition*, $T(x)$ needs to satisfy $\rho(Tx, Ty) \leq \beta \rho(x, y)$ for some $\beta \in (0, 1)$.

$$\begin{aligned}
\implies \rho(Tx, Ty) = |T(x) - T(y)| &= \left| \frac{1}{3+x} - \frac{1}{3+y} \right| \\
&= \left| \frac{(y-x)}{(3+x) \cdot (3+y)} \right| = \frac{|y-x|}{|9 + 3(x+y) + xy|}
\end{aligned}$$

$$\implies \frac{|y-x|}{|9 + 3(x+y) + xy|} \leq \frac{1}{9}|y-x|$$

$$\implies \rho(Tx, Ty) \leq \frac{1}{9} \rho(x, y)$$

Thus, it is a contraction mapping. ■

(b) To find the limit of the sequence, we use CMT which states that a contraction mapping has a unique fixed point and any sequence converges to this fixed point. We start with

$$T(x_0) = x_0 \qquad \Longrightarrow \qquad \frac{1}{3+x_0} = x_0 \qquad \Longrightarrow \qquad x_0^2 + 3x_0 - 1 = 0$$

We can find the solution for the equation using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Hence, the sequence converges to $x_0 = \frac{-3+\sqrt{13}}{2}$.

Question 4

Consider a one-period economy where the representative consumer has preference given by the utility function $u(c, l)$, where c is consumption and l is leisure. The consumer has an endowment of 1 unit of time which can be allocated between work and leisure. The representative firm produces consumption goods according to $y = n$, where y is output and n is labor input. The government purchases an exogenous quantity of the consumption good, g, and finances this expenditure by imposing a proportional tax t on the consumers labor income. That is, the consumers after-tax wage income is $(1-t)\omega(1-l)$, where t is tax rate and w is real wage rate.

(a) Write down the government's budget constraint.

(b) Is the competitive equilibrium Pareto optimal? If it is, show why, and if it is not, show why not.

Solution

(a) The government's budget constraint is $g = t \cdot w(1-l)$

(b) A competitive equilibrium is a set of quantities $\{c, l, n\}$, prices $\{\omega\}$ and government policy $\{t\}$ such that:

(1) the representative agent chooses c and l optimally given ω and t:

$$\max_{c,l} \; u(c,l)$$
$$s.t. \quad c = (1-t)\omega(1-l) \;\; where \;\; 0 \leq l \leq 1$$

(2) the representative firm chooses n optimally given ω :

$$\max_{n} f(n) - \omega n$$

(3) the government balances its budget:

$$g = t \cdot w(1 - l)$$

(4) markets clear:

$$n = 1 - l \text{ (labor market)}$$
$$c = y \text{ (goods market)}$$

First, the competitive equilibrium allocation can be found finding the market clearing condition. We start with F.O.C. for the agent's problem:

$$u_1(c, l) \cdot \omega \cdot (1 - t) = u_2(c, l)$$

then, we find F.O.C. for the firm's problem:

$$\omega = f_1(n)$$

Thus, the market clearing conditions:

$$u_1 \left[f\left(1 - l\right), l \right] \cdot f_1\left(1 - l\right) \cdot (1 - t) = u_2 \left[f\left(1 - l\right), l \right] \tag{1}$$

Second, we find the pareto optimal allocation solving the social planner's problem:

$$\max_{c,l} \; u(c, l)$$
$$s.t. \quad c = f\left(1 - l\right) \;\; where \;\; 0 \leq l \leq 1$$

which implies

$$u_1 \left[f\left(1 - l\right), l \right] \cdot f_1\left(1 - l\right) = u_2 \left[f\left(1 - l\right), l \right] \tag{2}$$

As you may see from equation (1) and (2), $l^{CE} \neq l^{PO}$ if $t > 0$. The tax distorts the agent's choice by changing the relative prices in the economy.