# EC813B - Recitation 10

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## Question 1[1]

(Prelim #1, Fall 2005) Consider the following two-period OLG model. The preferences of an agent born at time $t$ are represented by

$$U(c_{t1}, c_{t2}) = \ln c_{t1} + \beta \ln c_{t2}$$

where $c_{t1}$ denotes consumption when young and $c_{t2}$ denotes consumption when old. There is no physical capital and output is produced according to the following Cobb-Douglas production function

$$Y_t = S_t^{\gamma} L_t^{1-\gamma}$$

Here $L_t$ represents the labor services of young, unskilled workers and $S_t$ represents the labor services of old, skilled workers. The number of births per period is fixed at $N$. When young, agents append a fraction $e_t$ of their non-leisure time working where they earn a wage $w_t$, and a fraction $1 - e_t$ investing in human capital. It follows that $L_t = e_t N$. The human capital they acquire is given by

$$h_{t+1} = h_t + (1 - e_t)\,\theta h_t,$$

where $\theta$ is a parameter. The idea here is that the young receive the knowledge acquired by the old as a "spillover" and can build upon it. When old, agents just work and earn a wage $v_t$ per unit of human capital. Their labor services at time $t$ depend on the human capital, $h_t$, they accumulated when young: $S_t = h_t N$.

(a) Derive the optimal fraction of time allocated to working when young, $e^*$?

(b) What is the equilibrium growth rate of human capital? What is the equilibrium growth rate of output?

(c) Suppose the government imposes tax rates $\tau_1$ and $\tau_2$ on the wages of the young and old respectively, and transfers the revenue back to them as lump-sums, $\tau_1$ and $\tau_2$ correspond-

[1] I highly suggest checking R. Wright's lecture notes on *Overlapping Generations Model* which are very helpful to understand how basic model works and further changes and inclusions (money etc.) affect the model.

ingly, such that in equilibrium $T_1 = \tau_1 e_t w_t$ and $T_2 = \tau_2 v_{t+1} h_{t+1}$. If tax is "progressive", so that $\tau_2 > \tau_1$, how will this affect the optimal fraction of time allocated to working when young and the equilibrium growth rate of output? Explain the intuition behind this.

## *Solution*

(a) Since we have a CRTS production function, we can write output in terms of per capita terms

$$y_t = \frac{Y_t}{N_t} = \frac{S_t^\gamma L_t^{1-\gamma}}{N_t} = \frac{(h_t N_t)^\gamma (e_t N_t)^{1-\gamma}}{N_t} = h_t^\gamma e_t^{1-\gamma}$$

We can write the RA's problem as

$$\max_{\{c_{t1}, c_{t2}\}} \quad \ln c_{t1} + \beta \ln c_{t2} \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= w_t e_t \\ c_{t2} &= v_{t+1} h_{t+1} = v_{t+1}\big[h_t + (1-e_t)\,\theta h_t\big] \end{aligned}$$

Then, the maximization problem becomes

$$\max_{\{e_t\}} \quad \ln\Big(w_t e_t\Big) + \beta \ln\Big(v_{t+1}\big[h_t + (1-e_t)\,\theta h_t\big]\Big)$$

The FOC w.r.t $e_t$ is

$$w_t \frac{1}{w_t e_t} - \beta v_{t+1} \theta h_t \frac{1}{v_{t+1}\big[h_t + (1-e_t)\theta h_t\big]} = 0$$

Solving for[2] $e_t$, the optimal fraction of time allocated to working when young is

$$e^* = \frac{1+\theta}{\theta(1+\beta)}$$

(b) The equilibrium growth rate of human capital is

$$\frac{h_{t+1}}{h_t} = \frac{h_t + (1-e_t)\,\theta h_t}{h_t} = 1 + \theta - e_t\theta = 1 + \theta - \frac{\theta(1+\theta)}{\theta(1+\beta)} = \frac{\beta(1+\theta)}{(1+\beta)}$$

[2]Since we have log utility, $w_t$ and $v_{t+1}$ cancel in the equation but with a different utility function you have to derive $w_t$ and $v_{t+1}$ from Firm's Profit Maximization Problem which is $\max_{\{e_t, h_t\}} Y_t - w_t L_t - v_t S_t$.

The equilibrium growth rate of output is

$$\frac{y_{t+1}}{y_t} = \frac{h_{t+1}^{\gamma} e_{t+1}^{1-\gamma}}{h_t^{\gamma} e_t^{1-\gamma}} = \frac{h_{t+1}^{\gamma}}{h_t^{\gamma}} = (1+\theta - e_t\theta)^{\gamma} = \left[\frac{\beta(1+\theta)}{(1+\beta)}\right]^{\gamma}$$

the second equality comes from the fact that $e_t$ is constant.

(c) With the implementation of tax, the RA's problem becomes

$$\max_{\{e_t\}} \quad \ln c_{t1} + \beta \ln c_{t2} \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= (1-\tau_1)w_t e_t + T_1 \\ c_{t2} &= (1-\tau_2)v_{t+1}h_{t+1} + T_2 = (1-\tau_2)v_{t+1}\big[h_t + (1-e_t)\,\theta h_t\big] + T_2 \end{aligned}$$

Then, the maximization problem becomes

$$\max_{\{e_t\}} \quad \ln\Big((1-\tau_1)w_t e_t + T_1\Big) + \beta \ln\Big((1-\tau_2)v_{t+1}\big[h_t + (1-e_t)\,\theta h_t\big] + T_2\Big)$$

The FOC w.r.t $e_t$ is

$$\frac{(1-\tau_1)w_t}{(1-\tau_1)w_t e_t + T_1} - \beta \frac{(1-\tau_2)v_{t+1}\theta h_t}{(1-\tau_2)v_{t+1}\big[h_t + (1-e_t)\theta h_t\big] + T_2} = 0$$

Solving for $e_t$ (plug in for $T_1$ and $T_2$), the optimal fraction of time allocated to working with tax when young is

$$e^{**} = \frac{1+\theta}{\theta} \cdot \frac{1}{1+\beta\left(\frac{1-\tau_2}{1-\tau_1}\right)}$$

Since we assume that tax is "progressive", so that $\tau_2 > \tau_1$, we get $e^{**} > e^*$.

Now, we compare the equilibrium growth rate of output. Remember we find

$$\frac{y_{t+1}}{y_t} = (1+\theta - e^{optimal}\theta)^{\gamma}$$

and $e^{**} > e^*$. Thus, the equilibrium growth rate of output decreases when government impose taxes.

# Question 2

(Prelim #2, Fall 2015) Consider an overlapping generations economy in which each individual lives for two periods. An individual born at time $t$ consumes $c_{t1}$ in period $t$ and $c_{t2}$ in period

$t+1$, and derives utility

$$U(c_{t1}, c_{t2}) = u(c_{t1}) + \beta u(c_{t2})$$

where $\beta$ is the discount factor. Individuals work only in the first period of life, supplying inelastically one unit of labor and earning a real wage of $w_t$. They consume part of their first-period income and save the rest to finance their second-period consumption. The saving of the young in period $t$ generates the capital stock that is used to produce output in period $t+1$ in combination with the labor supplied by the young generation in period $t+1$. Population grows at rate $n$. Firms act competitively and use the constant return to scale technology $y = f(k)$ where $y$ is output per worker and $k$ is capital-labor ratio.

(a) Define the goods market equilibrium for this economy in per capita terms.

(b) Calculate the steady state interest rate using the following specifications

$$U(c_{t1}, c_{t2}) = \ln c_{t1} + \beta \ln c_{t2}$$
$$f(k) = Ak^{\alpha} - \delta k$$

where $\delta$ is the depreciation rate so that $f(k)$ is net production.

(c) Provide conditions under which the decentralized equilibrium is dynamically inefficient.

## *Solution*

(a) We can define the *goods market equilibrium* as a sequence $\{R_t, s_t, c_{t1}, c_{t2}\}$ such that: $(i)$ given $\{w_t, r_t\}$, RA solves the utility maximization problem; $(ii)$ Firm maximizes profits, and $(iii)$ the market clearing condition $s_t = (1+n)k_{t+1}$ holds for all t.

(b) Now, we start with RA's problem. It can be written as

$$\max_{\{c_{t1}, c_{t2}\}} \quad \ln c_{t1} + \beta \ln c_{t2} \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= w_t - s_t \\ c_{t2} &= (1 + r_{t+1})s_t \end{aligned}$$

Then, the maximization problem becomes

$$\max_{\{s_t\}} \quad \ln(w_t - s_t) + \beta \ln((1 + r_{t+1})s_t)$$

The FOC w.r.t $s_t$ is

$$\frac{1}{w_t - s_t} = \beta \frac{1}{s_t} \quad \Longrightarrow \quad s_t = \frac{\beta}{1+\beta} w_t$$

We get the usual results from the Firm's problem. First, we find $w_t$ as follows

$$w_t = F_{N_t}(K_t, N_t) = \frac{\partial}{\partial N_t}\left[N_t \cdot F\left(\frac{K_t}{N_t}, 1\right)\right]$$

Taking derivatives, we get

$$\begin{aligned}
w_t &= 1 \cdot f(k_t) - N_t f'(k_t)\left(\frac{K_t}{N_t^2}\right) \\
&= f(k_t) - f'(k_t)k_t \\
&= Ak_t^\alpha - \delta k_t - \left(A\alpha k_t^{\alpha-1} - \delta\right) k_t \\
&= A(1-\alpha)k_t^\alpha
\end{aligned}$$

Second, we find $r_t$ as follows

$$\begin{aligned}
r_t &= F_{K_t}(K_t, N_t) = \frac{\partial}{\partial K_t} F(K_t, N_t) \\
&= \alpha A k_t^{\alpha-1} - \delta
\end{aligned}$$

Next, using the RA's FOC and the market clearing condition, we get

$$s_t = k_{t+1} = \frac{K_{t+1}}{N_{t+}} \cdot \frac{N_{t+1}}{N_{t+1}} = k_{t+1}(1+n) \quad \Longrightarrow \quad k_{t+1}(1+n) = \frac{\beta}{1+\beta} w_t$$

Plugging in for $w_t$, we obtain

$$k_{t+1}(1+n) = \frac{\beta}{1+\beta} A(1-\alpha)k_t^\alpha$$

Now, we solve for $k^*$ assuming the steady-state condition $k_t = k_{t+1} = \cdots = k^*$. We get

$$k^*(1+n) = \frac{\beta}{1+\beta} A(1-\alpha)(k^*)^\alpha \quad \Longrightarrow \quad k^* = \left[\frac{(1-\alpha)\beta A}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}$$

We know that $r_t = \alpha A k_t^{\alpha-1} - \delta$. Plugging in for $k^*$, we obtain

$$r^* = \alpha A \left[ \left[ \frac{(1-\alpha)\beta A}{(1+n)(1+\beta)} \right]^{\frac{1}{1-\alpha}} \right]^{\alpha-1} - \delta = \frac{\alpha(1+n)(1+\beta)}{(1-\alpha)\beta} - \delta$$

(c) The decentralized equilibrium is dynamically inefficient if $f'(k^*) = r < n$ where $n = f'(k^{GR})$. So the condition required is

$$\frac{\alpha(1+n)(1+\beta)}{(1-\alpha)\beta} - \delta < n$$

# Question 3

(OLG with Money) Consider the following infinite horizon economy. Time is discrete. There is measure 1 of newborns in every period. Everyone lives for 2 periods except for the first generation of old people (no population growth). Preferences for the generations born in and after period 0 are

$$U(c_{t1}, c_{t2}) = u(c_{t1}) + \beta u(c_{t2})$$

where $c_{ti}$ is consumption in period $t$ and stage $i$ of life, $u(\cdot)$ is increasing strictly concave and twice differentiable. The initial old generation utility is $u(c_{02})$. Each generation has $(e_t, e_t)$, the endowment of the single perishable consumption good where $e_t = \gamma^t e, \gamma > 0, t \geq 0$. That is, everyone gets the same endowment in youth and old age but each subsequent generation gets a different endowment than the last generation. Endowments grow/shrink at the gross rate $\gamma$. Initial old generation is endowed with money, the money supply is $M$. The value of money is $q_t$, and the price level is $p_t = 1/q_t$.

(a) Define and characterize a stationary competitive monetary equilibrium.

(b) Restricting attention to $u(c) = \ln c$, what restriction on $\beta$ is required for existence of an equilibrium in which $c_{t1}$ and $c_{t2}$ both grow at the (gross) rate $\gamma$? Explain why such a requirement is necessary. Given $\beta$ satisfies this requirement, what is the value of $p_{t+1}$?

## *Solution*

(a) A *competitive monetary equilibrium* is a sequence $\{R_t, q_t, c_{t1}, c_{t2}\}$ such that: $c_{02} = e_2 + q_0 M$; given $\{q_t\}, (c_{t1}, c_{t2})$ solves the maximization problem for all $t \geq 1$; and the market clearing condition $c_{t1} + c_{t-1,2} = e_1 + e_2$ holds for all t or $m_t = M$ as long as $q_t > 0$.[3]

To characterize the equilibrium, we start with RA's problem. It can be written as

$$\max_{\{c_{t1},c_{t2}\}} u\left(c_{t1}\right) + \beta u\left(c_{t2}\right) \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= e_t - q_t m_t \\ c_{t2} &= e_t + q_{t+1} m_t \end{aligned}$$

Then, the maximization problem becomes

$$\max_{\{m_t\}} u\left(e_t - q_t m_t\right) + \beta u\left(e_t + q_{t+1} m_t\right)$$

The FOC w.r.t $m_t$ is

$$q_t u'(c_{t1}) = \beta q_{t+1} u'(c_{t2}) \quad \Longrightarrow \quad \frac{u'(c_{t1})}{u'(c_{t2})} = \beta \frac{q_{t+1}}{q_t} = \mu(c_{t1}, c_{t2})$$

Then, the equilibrium $\{q_t\}$ is such that

(i) $\mu(e_t - q_t m_t, e_t + q_{t+1} m_t) = \beta \frac{q_{t+1}}{q_t}$, and

(ii) $\{q_t\}$ is bounded.

(b) We have $c_{t1} = \gamma c_{t-1,1}$ and $e_t = \gamma^t e$. We start with plugging in for $c_{t1}$

$$\begin{aligned} e_t - q_t m_t &= \gamma(e_{t-1} - q_{t-1} m_t) \\ \gamma^t e - q_t M &= \gamma(\gamma^{t-1} e - q_{t-1} M) \quad \Longrightarrow \quad (\text{use } m_t = M) \\ q_t &= \gamma q_{t-1} \\ \frac{q_t}{q_{t-1}} &= \gamma \end{aligned}$$

From the RA's FOC, we get

$$\frac{q_t}{q_{t-1}} = \gamma = \frac{\mu(c_{t1}, c_{t2})}{\beta}$$

Using $u(c) = \ln c$, we obtain

$$\gamma = \frac{c_{t2}}{\beta c_{t1}}$$

Now, plugging in for $c_{t1}$ and $c_{t2}$ we can write

$$\gamma\beta(e_t - q_t M) = e_t + q_{t+1} M$$

[3] We know (by Walras' law) that the goods market clears if and only if the money market clears.

Solving for $M$, we get

$$M = \frac{(\gamma\beta - 1)e_t}{(\gamma\beta q_t + q_{t+1})}$$

We need $M > 0$ for the existence of equilibrium. Thus, the condition required for $\beta$ is

$$\beta > \frac{1}{\gamma}$$

Now, we can find $p_{t+1}$. We know that $p_{t+1} = \frac{1}{q_{t+1}}$. Hence we can start with

$$\begin{aligned}
\frac{q_{t+1}}{q_t} &= \frac{e_t + q_{t+1}M}{\beta(e_t - q_t M)} \\
\gamma\beta(e_t - q_t M) &= e_t + q_{t+1}M \\
\gamma\beta(\gamma^t e - q_t M) &= \gamma^t e + \gamma q_t M \\
(\gamma\beta - 1)\gamma^t e &= (1+\beta)\gamma q_t M \\
q_t &= \frac{(\gamma\beta - 1)\gamma^t e}{(1+\beta)\gamma M}
\end{aligned}$$

Finally, we can find $p_{t+1} = \frac{1}{q_{t+1}}$ as

$$p_{t+1} = \frac{(1+\beta)\gamma M}{(\gamma\beta - 1)\gamma^{t+1} e}$$

# Question 4

(Prelim #1, Fall 2004) Consider an overlapping generations economy where $t = 0, 1, 2, \ldots$. All agents in the generation born at $t$ have the same utility function,

$$U(c_{t1}, c_{t2}) = c_{t1} + \frac{\beta}{1-\alpha} c_{t2}^{1-\alpha}$$

where $\alpha < 1$, and endowment $e_t = (e_1, e_2)$. The stock of fiat money grows at rate $\gamma$, so that $M_{t+1} = (1+\gamma)M_t$ for all $t$, and new money is distributed via lump-sum transfers to old agents.

(a) Solve for the "money demand" function.

(b) Give necessary and sufficient conditions such that a monetary equilibrium exists.

(c) Prove that in the case $e_2 = 0$, any monetary equilibrium satisfies the difference equation $z_{t+1} = a_0 + a_1 z_l$ where $z_t$ is the log of real balances at t, while $a_0$ and $a_1$ are constants for you to determine.

## *Solution*

(a) We start with RA's problem as follows

$$\max_{\{c_{t1},c_{t2}\}} \quad c_{t1} + \frac{\beta}{1-\alpha} c_{t2}^{1-\alpha} \quad \text{s.t.} \quad \begin{aligned} c_{t1} &= e_1 - q_t m_t \\ c_{t2} &= e_2 + q_{t+1}(m_t + \tau_t) \end{aligned}$$

The problem becomes

$$\max_{\{m_t\}} \quad (e_1 - q_t m_t) + \frac{\beta}{1-\alpha} \left[e_2 + q_{t+1}(m_t + \tau_t)\right]^{1-\alpha}$$

The FOC is

$$q_t = \beta q_{t+1} \left[e_2 + q_{t+1}\left(m_t + \tau_t\right)\right]^{-\alpha} \quad \Longrightarrow \quad \frac{q_{t+1}}{q_t} = \frac{1}{\beta} \left[e_2 + q_{t+1}\left(m_t + \tau_t\right)\right]^{\alpha}$$

Since new money is distributed via lump-sum transfers, define $\tau_t = M_{t+1} - M_t = \gamma M_t$. Using market clearing condition $m_t = M_t$ we get

$$\frac{q_{t+1}}{q_t} = \frac{1}{\beta} \left[e_2 + q_{t+1}\left(M_t + \gamma M_t\right)\right]^{\alpha}$$

The "money demand" function is

$$M_t = \left[\left(\beta \frac{q_{t+1}}{q_t}\right)^{\frac{1}{\alpha}} - e_2\right] \frac{1}{(1+\gamma) q_{t+1}}$$

(b) We start with writing $\mu(\cdot)$ in terms of real money balances, $S_t = q_t M_t$.

$$\begin{aligned}
\frac{q_{t+1}}{q_t} &= \frac{1}{\beta}\left[e_2 + q_{t+1}\left(m_t + \tau_t\right)\right]^{\alpha} \\
\frac{q_{t+1}}{q_t} &= \frac{1}{\beta}\left[e_2 + q_{t+1}\left(M_t + \gamma M_t\right)\right]^{\alpha} \\
\frac{q_{t+1}}{q_t} &= \frac{1}{\beta}\left[e_2 + q_{t+1}(1+\gamma) M_t\right]^{\alpha} \\
\frac{M_{t+1}}{M_{t+1}}\frac{M_t}{M_t}\frac{q_{t+1}}{q_t} &= \frac{1}{\beta}\left[e_2 + q_{t+1} M_{t+1}\right]^{\alpha} \\
\frac{M_t}{M_{t+1}}\frac{S_{t+1}}{S_t} &= \frac{1}{\beta}\left[e_2 + S_{t+1}\right]^{\alpha} \\
\frac{1}{(1+\gamma)}\frac{S_{t+1}}{S_t} &= \frac{1}{\beta}\left[e_2 + S_{t+1}\right]^{\alpha}
\end{aligned}$$

Assume that at the steady state $S_t = S_{t+1} = S^*$. Now we solve for $S^*$ and we get

$$S^* = \left(\frac{\beta}{(1+\gamma)}\right)^{\frac{1}{\alpha}} - e_2$$

For existence of monetary equilibrium, we must have $S^* > 0$. Thus, the condition required is

$$\left(\frac{\beta}{(1+\gamma)}\right) > e_2^{\alpha} \quad \Longrightarrow \quad \beta > (1+\gamma) e_2^{\alpha}$$

(c) We find earlier that

$$\frac{1}{(1+\gamma)}\frac{S_{t+1}}{S_t} = \frac{1}{\beta}\left[e_2 + S_{t+1}\right]^{\alpha}$$

Rearranging the equation, we find

$$S_{t+1} = \left(\frac{\beta}{(1+\gamma)}\right)^{\frac{1}{\alpha}}\left(\frac{S_{t+1}}{S_t}\right)^{\frac{1}{\alpha}} - e_2$$

Assuming $e_2 = 0$, we get

$$S_{t+1} = \left(\frac{\beta}{(1+\gamma)}\right)^{\frac{1}{\alpha}}\left(\frac{S_{t+1}}{S_t}\right)^{\frac{1}{\alpha}}$$

Taking log of both sides, we get

$$\log S_{t+1} = \frac{1}{\alpha}\left[\log\beta - \log(1+\gamma)\right] + \frac{1}{\alpha}\left[\log S_{t+1} - \log S_t\right]$$
$$\log S_{t+1} = \frac{1}{\alpha - 1}\left[\log\beta - \log(1+\gamma)\right] - \frac{1}{\alpha - 1}\log S_t$$

Thus, we showed that equilibrium satisfies the difference equation $z_{t+1} = a_0 + a_1 z_l$.