EC813B - Recitation 7

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Mehmet Karaca

Question 1

Consider two countries (1 and 2) producing the same good with the same production function

$$Y = AK^{\alpha}(hN)^{1-\alpha}h_a^{\gamma}$$

where K and N are homogeneous capital and labor inputs, h is the individual human capital and h_a is the average human capital in a country. The two countries differ in terms of per capita physical and human capital. Derive the ratio of marginal products of capital between the countries as a function of y_1/y_2 and h_1/h_2, where $y \equiv Y_i/N_i$.

Solution

First, we can derive the general form for marginal product of capital.

$$\text{MPK } = \frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1}(hN)^{1-\alpha}h_a^{\gamma} = \alpha Ak^{\alpha-1}h^{1-\alpha}h_a^{\gamma} \quad \text{where} \quad k^{\alpha-1} = \left(\frac{K}{N}\right)^{\alpha-1}$$

Next, we can find per capita capital using the production function as following

$$y = \frac{Y}{N} = Ak^{\alpha}h^{1-\alpha}h_a^{\gamma} \quad \Longrightarrow \quad k = \left(\frac{y}{Ah^{1-\alpha}h_a^{\gamma}}\right)^{\frac{1}{\alpha}}$$

Now, we need to find $\frac{\text{MPK}_1}{\text{MPK}_2}$. We obtain

$$\frac{\text{MPK}_1}{\text{MPK}_2} = \frac{\alpha Ak_1^{\alpha-1}h_1^{1-\alpha}h_{a_1}^{\gamma}}{\alpha Ak_2^{\alpha-1}h_2^{1-\alpha}h_{a_2}^{\gamma}} = \left(\frac{k_1}{k_2}\right)^{\alpha-1} \cdot \left(\frac{h_1}{h_2}\right)^{1-\alpha} \cdot \left(\frac{h_{a_1}}{h_{a_2}}\right)^{\gamma}$$

Plugging in for k_1 and k_2, we get

$$\begin{aligned}\frac{\mathrm{MPK}_1}{\mathrm{MPK}_2} &= \left(\left(\frac{y_1}{Ah_1^{1-\alpha}h_{a_1}^{\gamma}}\right)\cdot\left(\frac{Ah_2^{1-\alpha}h_{a_2}^{\gamma}}{y_2}\right)\right)^{\frac{\alpha-1}{\alpha}}\cdot\left(\frac{h_1}{h_2}\right)^{1-\alpha}\cdot\left(\frac{h_{a_1}}{h_{a_2}}\right)^{\gamma} \\ &= \left(\frac{y_1}{y_2}\right)^{\frac{\alpha-1}{\alpha}}\cdot\left(\frac{h_1}{h_2}\right)^{\frac{1-\alpha+\gamma}{\alpha}}\end{aligned}$$

where $h_{a_i} = h_i$ in equilibrium.

Question 2

(Romer's Model) Consider the baseline Romer's endogenous growth model with one difference, possibility of monopoly power erosion. A firm that invents a new blueprint (idea) receives a patent, which expires at the Poisson rate λ. Once the patent expires, that machine is produced competitively and is supplied to final good producers at marginal cost. Otherwise, the model is exactly the same as the one studied in class. R&D sector produces blueprints of new varieties/types of capital goods at a rate $\frac{\dot{A}}{A} = \delta H_A$, where H_A characterizes employment in the research sector. The intermediate goods sector characterized by monopolistic competition uses the blue prints and produces intermediate capital goods for the final goods production. Final good producers employ H_Y share of total labor H and variety (a set) of capital goods x_i, $i \in [0; A]$ in the production

$$Y = H_Y^{1-\alpha}\int_0^A x_i^{\alpha}di$$

The firms are fully competitive in input and output markets. The final good is the numeraire good which may be either consumed or invested. From the consumption side, the representative household chooses its consumption and next period assets to maximize its lifetime utility

$$U = \int_0^{\infty} e^{-\rho t}\frac{C^{1-\sigma}-1}{1-\sigma}dt$$

subject to the standard budget constraint, $\dot{W} = rW + wH - C$. All capital varieties depreciate fully within one period and there is no growth in population, i.e., $H =$ constant.

(a) Set up the final goods firm maximization problem and solve it for demand for labor and demand for capital good.

(b) Set up the intermediate goods firm maximization problem and solve for the price, p_x, and quantity choices x (given the symmetry across the different varieties in the final good production x does not depend on i). Derive the profit formula π.

(c) Derive the formula for the price of a blueprint on the balanced growth path. ***Hint:*** *Start with the following asset-pricing equation:* $rp_A = \pi - \lambda p_A + \dot{p_A}$ *(notice that this equation is different from the one we used in class and has an extra term on the RHS reflecting the fact that the firm can "loose" the patent with the arrival rate* λ *).*

(d) Solve the representative household's problem for the growth rate of consumption.

(e) Solve for the growth rate of the economy on the BGP, g_A. How does it depend on λ? What is the value of λ that maximizes the equilibrium rate of economic growth? Briefly explain the intuition behind this result. Show that a policy of $\lambda = 0$ does not necessarily maximize social welfare (just compare your result with $g^{SP} = \frac{\delta H - \rho}{\sigma}$).

Solution

(a) The profit maximization problem of firms in the final goods sector is

$$\max_{\{H_Y, x_i\}} H_Y^{1-\alpha} \int_0^A x_i^\alpha d_i - \int_0^A p_{x_i} x_i d_i - w_Y H_Y$$

The FOCs are

- for H_Y,

$$(1-\alpha) H_Y^{-\alpha} \int_0^A x_i^\alpha d_i = w_Y \quad \Longrightarrow (1-\alpha)\frac{Y}{H_Y} = w_Y$$

- for x_i,

$$\alpha H_Y^{1-\alpha} x_i^{\alpha-1} = p_{x_i}$$

(b) The profit maximization problem of firms in the intermediate goods sector is

$$\max_{\{x_i\}} p_{x_i} x_i - r x_i \equiv \pi \quad \text{s.t.} \quad p_{x_i} = \alpha H_Y^{1-\alpha} x_i^{\alpha-1}$$

The FOC is

- for x_i,

$$r = \alpha^2 H_Y^{1-\alpha} x_i^{\alpha-1} = \alpha p_{x_i} \quad \Longrightarrow \quad p_{x_i} = \frac{r}{\alpha}$$

Now, we can derive the profit formula

$$\pi = (p_{x_i} - r)x_i = \Big(\frac{r}{\alpha} - r\Big)x_i = \Big(\frac{1-\alpha}{\alpha}\Big)rx_i$$

In a symmetric equilibrium, aggregate capital goods is

$$K = \int_0^A x d_i = Ax \quad \Longrightarrow \quad x = \frac{K}{A}$$

So, the profit maximization problem of firms in the final goods sector can be rewritten as

$$\max_{\{K\}} (AH)^{1-\alpha}K^{\alpha} - p_x K - w_Y H_Y$$

and the FOC w.r.t. K is

$$p_x = \alpha(AH)^{1-\alpha}K^{\alpha-1} = \frac{r}{\alpha}$$

where the second equality comes from the FOC of intermediate goods sector firm. Then, r can be derived as

$$\frac{r}{\alpha} = \alpha(AH)^{1-\alpha}K^{\alpha-1} \quad \Longrightarrow \quad r = \alpha^2\frac{Y}{K}$$

Lastly, plugging in for r and x, the profit function can be rewritten as following

$$\pi = \Big(\frac{1-\alpha}{\alpha}\Big)\cdot\alpha^2\frac{Y}{K}\cdot\frac{K}{A} = \alpha(1-\alpha)\frac{Y}{A}$$

(c) To derive the formula for the price of a blueprint, we first need to start with the maximization problem of a "blueprint" producer. It can written as following

$$\max_{\{H_A\}} p_A A\delta H_A - w_A H_A$$

The FOC is

- for H_A,

$$p_A A\delta = w_A$$

Since $\dot{p_A} = g_{p_A} = 0$, we can assume that p_A is constant. Thus, using the asset-pricing equation, we obtain

$$rp_A = \pi - \lambda p_A + \dot{p_A} \quad \Longrightarrow \quad r + \lambda = \frac{\pi}{p_A} - \frac{\dot{p_A}}{p_A} \quad \Longrightarrow \quad p_A = \frac{\pi}{r+\lambda}$$

(d) The utility maximization problem of the representative household is

$$\max_{\{C\}} \int_0^\infty e^{-\rho t} \frac{C^{1-\sigma}-1}{1-\sigma} dt \quad \text{s.t.} \quad \dot{W} = rW + wH - C$$

We write down the current-value Hamiltonian as follows

$$\mathcal{H}^* = \frac{C^{1-\sigma}-1}{1-\sigma} + \mu\big[rW + wH - C\big]$$

The FOCS are

- for $C \quad \Longrightarrow C^{-\sigma} - \mu = 0$
- for $W \quad \Longrightarrow \mu r = \rho\mu - \dot{\mu}$
- for $\mu \quad \Longrightarrow rW + wH - C = \dot{W}$

Differentiating $C^{-\sigma} = \mu$, we get $-\sigma C^{-\sigma-1}\dot{C} = \dot{\mu}$. Combining with FOCs, we obtain

$$\begin{aligned} &-\sigma C^{-\sigma-1}\dot{C} = \dot{\mu} = (\rho - r)\mu \\ &\frac{\dot{C}}{C} = g_C = \frac{r-\rho}{\sigma} \end{aligned}$$

(e) The growth rate of the economy on the BGP is defined as $g_A = \frac{\dot{A}}{A} = \delta H_A$. We must have $w = w_A = w_Y$, because otherwise the household wouldn't be indifferent between working in the final good sector and research sector. Then, we find

$$w = (1-\alpha)\frac{Y}{H_Y} = p_A\delta A \quad \Longrightarrow \quad (1-\alpha)\frac{Y}{H_Y} = \frac{\pi}{r+\lambda}\delta A$$

We plug in for π and solve for H_Y. We get

$$(1-\alpha)\frac{Y}{H_Y} = \alpha(1-\alpha)\frac{Y}{A}\frac{\delta}{r+\lambda}A \quad \Longrightarrow \quad H_Y = \frac{r+\lambda}{\delta\alpha}$$

Hence, H_Y is constant. Then, a rough sketch of how we obtain g_A is as follows

- From $Y = A^{1-\alpha}H^{1-\alpha}K^{\alpha} \quad \Longrightarrow \quad g_Y = (1-\alpha)g_A + \alpha g_K$,
- From $MPK = \alpha A^{1-\alpha}H^{1-\alpha}K^{\alpha-1} \quad \Longrightarrow \quad (\alpha-1)g_K + (1-\alpha)g_A = 0 \quad \Longrightarrow \quad g_A = g_K$,
- Then, $g_Y = (1-\alpha)g_A + \alpha g_K = (1-\alpha)g_A + \alpha g_A \quad \Longrightarrow \quad g_Y = g_A$.

- We also know $\pi = \alpha(1-\alpha)\frac{Y}{A} \implies g_\pi = g_Y - g_A$.
- So, we get $g_\pi = g_Y - g_A = g_A - g_A = 0 \implies g_\pi = 0$.
- Using the asset-pricing equation:

$$rp_A = \pi - \lambda p_A + \dot{p_A} \implies r + \lambda = \frac{\pi}{p_A} - \frac{\dot{p_A}}{p_A} \implies g_\pi = g_{p_A} = 0$$

- Finally, we show that $g_A = g_C$. Remember we know $r + \lambda = \delta\alpha H_Y$ and $g_C = \frac{r-\rho}{\sigma}$. We can get

$$\begin{aligned}
&\alpha\delta H_Y - \lambda = \rho + \sigma g_C \\
&\alpha\delta(H - H_A) - \lambda = \rho + \sigma g_C \\
&\alpha\delta H - \alpha\delta H_A - \lambda = \rho + \sigma g_C \\
&\alpha\delta H - \lambda = \rho + \sigma g_C + \alpha g_A
\end{aligned}$$

  using earlier finding we know that $g_A = g_K$ and so it must be the case that $g_A = g_C$.

Therefore, we find

$$g_A = \frac{\alpha\delta H - \rho - \lambda}{\alpha + \sigma}$$

The value of λ that maximizes the equilibrium rate of economic growth would be $\lambda = 0$

$$g_A = \frac{\alpha\delta H - \rho - \lambda}{\alpha + \sigma} < \frac{\alpha\delta H - \rho}{\alpha + \sigma}$$

RHS of the inequality shows an economy without expiration of ideas (*blueprints*). Hence, it indicates that firms can preserve their monopoly power from erosion, as a result it would lead to higher profits and economic growth.

Lastly, we compare equilibrium results with a policy of $\lambda = 0$ with the outcome of social planner's problem. We get

$$g_A = \frac{\alpha\delta H - \rho}{\alpha + \sigma} < g^{SP} = \frac{\delta H - \rho}{\sigma}$$

Therefore, we show that a policy of $\lambda = 0$ does not necessarily maximize social welfare.

Question 3

(RCE) Consider a standard neoclassical growth economy. There is a representative agent who has preferences given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

where the function u is differentiable, strictly increasing and strictly concave. There is a $[0,1]$ continuum of identical firms each producing output according to the following technology

$$\begin{aligned}
y_t &= k_t^{\alpha} K_t^{\gamma} \\
c_t + i_t &\leq y_t \\
k_{t+1} &\leq (1-\delta)k_t + i_t \\
(c_t, k_{t+1}, i_t) &\geq (0,0,0)
\end{aligned}$$

where k_t is the capital input of a specific firm and K_t is the aggregate capital stock in the economy, and the initial capital stock is given.

(a) Solve the social planner's problem.

(b) Define a recursive competitive equilibrium.

(c) Compare the equilibrium allocation and the solution to the social planner's problem. Explain.

Solution

(a) The social planner's problem is

$$\begin{aligned}
&\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \\
&\text{s.t.} \quad c_t = k_t^{\alpha+\gamma} + (1-\delta)k_t - k_{t+1}
\end{aligned}$$

Then, the Bellman's equation can be written as

$$V\left(k_t\right) = \max_{k_{t+1}} \left\{ u\Big(k_t^{\alpha+\gamma} + (1-\delta)k_t - k_{t+1}\Big) + \beta V\left(k_{t+1}\right)\right\}$$

which gives the first order condition as

$$-u'(c_t) + \beta V'(k_{t+1}) = 0$$

or

$$u'(c_t) = \beta u'(c_{t+1}) \left\{(\alpha + \gamma) k_{t+1}^{\alpha+\gamma-1} + 1 - \delta\right\}$$

The social planner's problem can be solved by the equation above with a given k_o and TVC.

(b) A RCE for this economy is a list of

(i) individual decision rule, $k_{t+1} = k(k_t, K_t)$

(ii) aggregate decision rule, $K_{t+1} = K(K_t)$

(iii) pricing function, $r_t = r(K_t)$

(iv) value function, $V(k_t, K_t)$

such that,

(1) individual maximizes discounted utility if behaving in accordance to $k(\cdot)$, given the factor price.

(2) firms maximizes profits and the factor market clears, $K_t = k_t$.

(3) consistency between aggregate and individual decision rules (rational expectations), $K(K_t) = k(k_t, K_t)$.

(c) Each firm's profit maximization problem can be written as

$$\max_{k_t} k_t^{\alpha} K_t^{\gamma} - r_t k_t$$

which gives the optimal rule as

$$\alpha k_t^{\alpha-1} K_t^{\gamma} = r_t$$

By the market clearing condition, $k_t = K_t$, the pricing function can be written as

$$r(K_t) = \alpha K_t^{\alpha+\gamma-1}$$

A representative agent maximizes

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t.} \quad c_t = r_t k_t + (1-\delta)k_t - k_{t+1}$$

and she has to know how a factor price and K_t evolve over time. Then, the Bellman's equation is

$$V(k_t, K_t) = \max_{k_{t+1}} \Big\{ u\Big(r(K_t) k_t + (1-\delta)k_t - k_{t+1} \Big) + \beta V\Big(k_{t+1}, K(K_t) \Big) \Big\}$$

FOC is derived as

$$-u'(c_t) + \beta V'(k_{t+1}) = 0$$

or

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1}) \{ r(K_{t+1}) + 1 - \delta \} \\ &= \beta u'(c_{t+1}) \left\{ \alpha K_{t+1}^{\alpha+\gamma-1} + 1 - \delta \right\} \end{aligned}$$

The steady state capital stock in the social planner's problem and RCE can be derived as

$$\begin{aligned} K^{SP} &= \left\{ \frac{1}{\alpha+\gamma} \left(\frac{1}{\beta} - 1 + \delta \right) \right\}^{\frac{1}{\alpha+\gamma-1}} \\ K^{RCE} &= \left\{ \frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta \right) \right\}^{\frac{1}{\alpha+\gamma-1}} \end{aligned}$$

implying that $K^{SP} > K^{RCE}$ if $\frac{1}{\alpha+\gamma-1} < 0$ or $\alpha + \gamma < 1$.