EC813B - Recitation 6

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Question 1

Consider the neoclassical growth model in continuous time. The representative consumer has preferences:

$$\int_0^{\infty} e^{-\rho t}\left[u\left(c_t\right)-v\left(h_t\right)\right] dt$$

Assume that u is increasing and strictly concave and v is increasing and strictly convex. The budget constraint for the agent looks like: $c_t+x_t=r_t k_t+w_t h_t$. The law of motion for capital is $\dot{k}_t=x_t-\delta k_t$ The production function is $y_t=F\left(k_t, h_t\right)$ where F is constant returns to scale in the two inputs.

(a) Write down the first order conditions for the consumer's problem.

(b) Solve the firm's static maximization problem.

(c) Combine your solutions to find the Euler equation for this economy.

(d) Find conditions that characterize the steady state levels of capital, consumption and labor.

Solution

(a) We first start writing down the utility maximization problem of consumer:

$$\max_{\{c_t, h_t\}} \int_0^{\infty} e^{-\rho t}\left[u\left(c_t\right)-v\left(h_t\right)\right] dt \quad \text{s.t.} \quad c_t+x_t=r_t k_t+w_t h_t \quad \text{and} \quad \dot{k}_t=x_t-\delta k_t$$

Next, we set up the Hamiltonian (continous-time equivalent of discrete-time Lagrangian) equation[1]. The Hamiltonian equation can be written in two different forms:

- Present-value Hamiltonian

$$\mathcal{H}=e^{-\rho t}\Big[u\left(c_t\right)-v\left(h_t\right)\Big]+\lambda\left[r_t k_t+w_t h_t-c_t-\delta k_t\right]$$

[1] Please check Acemoglu, D.(2009), *Introduction to Modern Economic Growth*, for a detailed discussion of Hamiltonian equation (Chapter 7.2).

- Current-value Hamiltonian:

$$\mathcal{H}^* = e^{-\rho t}\big\{u\left(c_t\right) - v\left(h_t\right) + \mu\left[r_t k_t + w_t h_t - c_t - \delta k_t\right]\big\} \quad \text{where} \quad \mu = \lambda e^{\rho t}$$

Now, we can write the FOCs[2]:

(i) $\frac{\partial \mathcal{H}^*}{\partial c_t} = 0 = u'(c_t) - \mu \quad \Longrightarrow \quad u'(c_t) = \mu$

(ii) $\frac{\partial \mathcal{H}^*}{\partial h_t} = 0 = -v'(h_t) + \mu w_t \quad \Longrightarrow \quad v'(h_t) = \mu w_t$

(iii) $\frac{\partial \mathcal{H}^*}{\partial k_t} = -\dot{\lambda} = \rho\mu - \dot{\mu} = \mu(r_t - \delta) \quad \Longrightarrow \quad \dot{\mu} = \mu(\rho + \delta - r_t)$

(iv) $\frac{\partial \mathcal{H}^*}{\partial \mu} = r_t k_t + w_t h_t - c_t - \delta k_t = \dot{k}_t$

Combining (i) and (ii), we get $v'(h_t) = u'(c_t) w_t$. In addition, differentiating (i) w.r.t. time, t, we obtain $\dot{\mu} = u''(c_t)\dot{c}_t$.

(b) The firm's static maximization problem can be written as

$$\max_{\{h_t, k_t\}} F\left(k_t, h_t\right) - r_t k_t - w_t h_t$$

Then, we need to derive FOCs and find the following

$$r_t = F_1\left(k_t, h_t\right) \quad \text{and} \quad w_t = F_2\left(k_t, h_t\right)$$

(c) We want to find $\dot{c}_t$ as we know $\dot{k}_t$. Combining what we find in part (a) and (b), we get

$$\begin{aligned} &\dot{\mu} = u''(c_t)\dot{c}_t \quad \text{and} \quad \dot{\mu} = \mu(\rho + \delta - r_t) = u'(c_t)\big[\rho + \delta - F_1\left(k_t, h_t\right)\big] \\ \Longrightarrow \quad &\dot{c}_t = \frac{u'(c_t)}{u''(c_t)}\big[\rho + \delta - F_1\left(k_t, h_t\right)\big] \end{aligned}$$

Hence, we can define the Euler equations as follows

$$\dot{k}_t = F_1\left(k_t, h_t\right) k_t + F_2\left(k_t, h_t\right) h_t - c_t - \delta k_t \tag{1}$$

$$\dot{c}_t = \frac{u'(c_t)}{u''(c_t)}\big[\rho + \delta - F_1\left(k_t, h_t\right)\big] \tag{2}$$

with the Transversality condition[3] $\lim_{t\to\infty} e^{-\rho t} u'(c_t) k_{t+1} = 0$.

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[2]The equality $-\dot{\lambda} = \rho\mu - \dot{\mu}$ in (iii) comes from the fact that $\lambda = e^{-\rho t}\mu$.

[3]Transversality condition is, typically, a necessary condition for an optimum, and it expresses the following

(d) To find conditions that characterize the steady state levels of capital, consumption and labor, we use the fact $\dot{c}_t = \dot{k}_t = 0$. So, using equations (1) and (2), we get

$$\begin{aligned} \dot{k}_t = 0 \quad &\Longrightarrow \quad \big[F_1(k^*, h^*) - \delta\big]k^* + F_2(k^*, h^*)h^* = c^* \\ \dot{c}_t = 0 \quad &\Longrightarrow \quad F_1(k^*, h^*) = \rho + \delta \end{aligned}$$

Question 2

Consider a version of the continuous time Ramsey model with CARA utility function in which households are also the producers of the final good. There is a government that taxes output at the rate τ and purchase goods and services in the per capita amount g. The production function is Cobb-Douglas and there is no technological progress. Hence, the household maximizes

$$U = \int e^{-\rho t} u\left(c_t\right) dt$$

where $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ subject to the budget constraint $\dot{k}_t = (1-\tau)Ak_t^{\alpha} - c_t - \delta k_t$ given k_0. Suppose that the government's purchases are a pure waste, so they do not provide utility and do not enhance productivity.

(a) Set up the household's optimization problem using the Hamiltonian formulation and derive the first order conditions assuming that the household takes the tax rate τ as given.

(b) Use the households' FOCs to solve for the growth rate of per capita consumption (as a function of k). What is the steady state level of k?

(c) Use a phase-diagram in the (k, c) space to show how the paths of k and c change when the government surprises people by permanently raising the values of τ and g. What happens to the steady state values of k and c?

simple idea: it cannot be optimal for the consumer to choose a capital sequence such that, in present-value utility terms, the shadow value of k_t remains positive as t goes to infinity. This could not be optimal because it would represent saving too much: a reduction in saving would still be feasible and would increase utility.

Solution

(a) We set up the current-value Hamiltonian:

$$\mathcal{H}^* = u\left(c_t\right) + \mu\big[(1-\tau)Ak_t^{\alpha} - c_t - \delta k_t\big]$$

Now, we can write the FOCs:

(i) $\frac{\partial \mathcal{H}^*}{\partial c_t} = 0 = u'(c_t) - \mu$

(ii) $\frac{\partial \mathcal{H}^*}{\partial k_t} = \rho\mu - \dot{\mu} = \mu(1-\tau)A\alpha k_t^{\alpha-1} - \mu\delta$

(iii) $\frac{\partial \mathcal{H}^*}{\partial \mu} = (1-\tau)Ak_t^{\alpha} - c_t - \delta k_t = \dot{k}_t$

(b) First, we need to derive the Euler equations. We want to find $\dot{c}_t$ as we know $\dot{k}_t$. Combining what we find, we get

- From (i), differentiating (i) w.r.t. time, t, we obtain $\dot{\mu} = u''(c_t)\dot{c}_t$.
- From (ii), we get $\dot{\mu} = \mu\big[\rho + \delta - (1-\tau)A\alpha k_t^{\alpha-1}\big]$
- Combining these, we find $\dot{c}_t = \frac{u'(c_t)}{u''(c_t)}\big[\rho + \delta - (1-\tau)A\alpha k_t^{\alpha-1}\big]$

Hence, we can define the Euler equations (plugging in for $c(t)$)[4] as follows

$$\dot{k}_t = (1-\tau)Ak_t^{\alpha} - c_t - \delta k_t \tag{3}$$

$$\dot{c}_t = -\frac{c_t}{\sigma}\big[\rho + \delta - (1-\tau)A\alpha k_t^{\alpha-1}\big] \tag{4}$$

with the Transversality condition $\lim_{t\to\infty} e^{-\rho t}u'(c_t)k_{t+1} = 0$.

Next, we can define the growth rate of per capita consumption as $g_c = \frac{\dot{c}_t}{c_t}$. Using equation (4), we get

$$g_c = \frac{1}{\sigma}\big[(1-\tau)A\alpha k_t^{\alpha-1} - \rho - \delta\big]$$

To characterize the steady state levels of capital, we use the fact $\dot{c}_t = \dot{k}_t = 0$. So, using equations (3) and (4), we get

$$\dot{k}_t = 0 \quad\Longrightarrow\quad 0 = (1-\tau)A(k^*)^{\alpha} - c^* - \delta k^*$$

$$\dot{c}_t = 0 \quad\Longrightarrow\quad 0 = (1-\tau)A\alpha(k^*)^{\alpha-1} - \rho - \delta$$

[4] Given that $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, $u'(c) = c^{-\sigma}$ and $u''(c) = -\sigma c^{-\sigma-1}$.

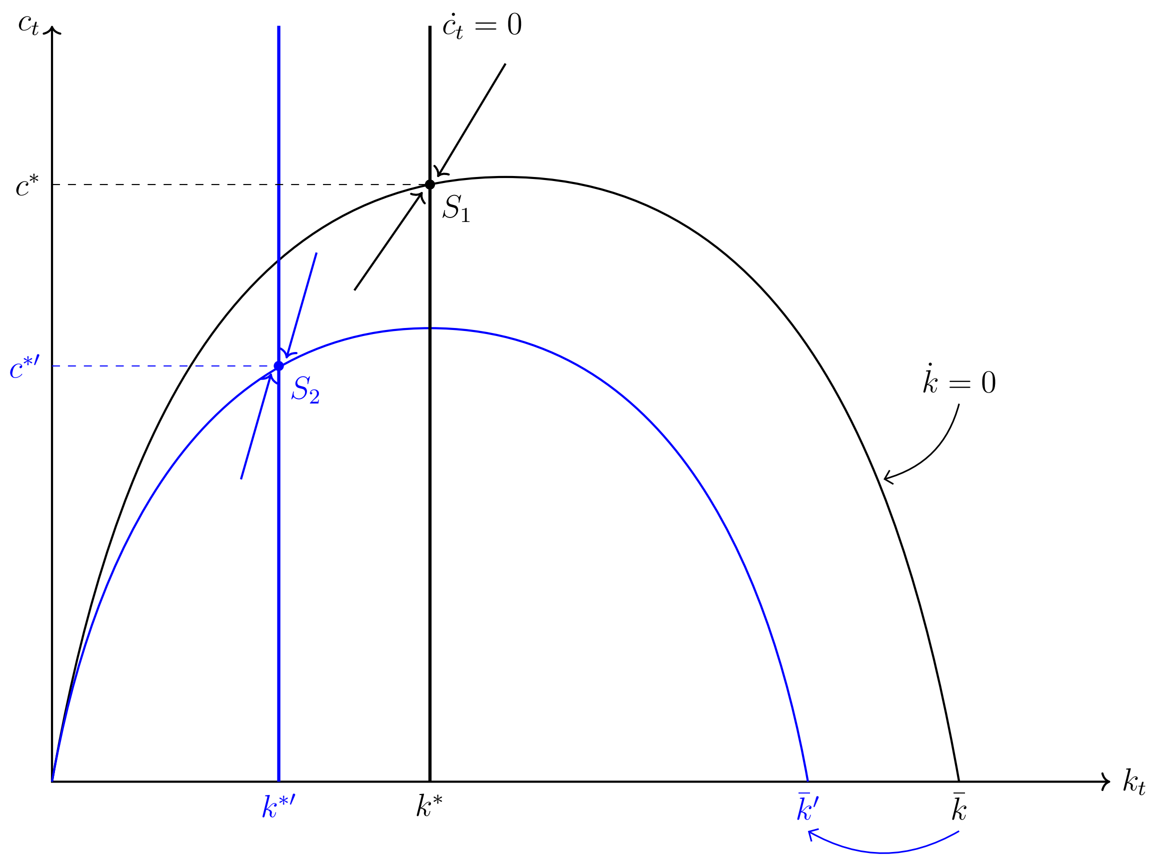
then we can use the second equation to find k^*. Solving for k^*, we obtain

$$k^* = \left(\frac{(1-\tau)A\alpha}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} \tag{5}$$

Now, using the first equation, we plug in for c^*, we get

$$c^* = \rho \left(\frac{(1-\tau)A\alpha}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} \tag{6}$$

(c) From equation (5) and (6), it can be seen that the effect of permanent increase in τ decreases both k^* and c^*. Here is phase-diagram to display the steady-state and the effect of an increase in τ.



We assume an increase in taxes goes hand in hand with government spending so as to maintain a balanced budget. Thereby the after-tax wealth of the household is reduced. As the households are now less wealthy private consumption drops.

Question 3

(Prelim #1, Spring 2005) Consider the following consumer's problem. The consumer has lifetime utility that is given by

$$\int_0^\infty e^{-\rho t} u(c,a)dt$$

where c is consumption, a is the stock of financial assets, and $0 < \rho < 1$ is the subjective discount rate. The period utility is $u(c,a) = \ln c + \gamma \ln a$, for $\gamma > 0$. The consumer faces a budget constraint:

$$\dot{a} = ra + y - c$$

where $\dot{a} = da/dt, y > 0$ is a constant labor income, and $0 < r < 1$ is the market interest rate. The consumer is initially endowed with $a(0) = a_0 > 0$ of financial assets. Finally, assume that $\rho > (1+\gamma)r$.

(a) What are the state and control variables for this problem?

(b) Write the current value Hamiltonian and its first-order necessary conditions.

(c) Using these first-order conditions, solve for consumption growth $\dot{c}$.

(d) Find the steady state level of consumption and the stock of assets.

(e) Illustrate the solution to the consumer's problem using a phase diagram in the c and a plane. (Hint: use the solution for consumption growth and the budget constraint, and draw the phase diagram). Intuitively, is the steady state a saddle point?

Solution

(a) The control variable is c_t and the state variable is a_t.

(b) The current-value Hamiltonian is as follows

$$\mathcal{H}^* = \ln c + \gamma \ln a + \mu\big[ra + y - c\big]$$

and the FOCs are

(i) $\frac{\partial \mathcal{H}^*}{\partial c} = 0 = \frac{1}{c} - \mu$

(ii) $\frac{\partial \mathcal{H}^*}{\partial a} = \rho\mu - \dot{\mu} = \gamma\frac{1}{a} + \mu r$

(iii) $\frac{\partial \mathcal{H}^*}{\partial \mu} = ra + y - c = \dot{a}$

(c) Differentiating (i) w.r.t. time, t, we obtain $\dot{\mu} = -\frac{1}{c^2}\dot{c}$, and using (ii), we get $\dot{\mu} = \mu(\rho - r) - \frac{\gamma}{a}$. Combining these, we find

$$\dot{c} = \gamma \frac{c^2}{a} - c(\rho - r)$$

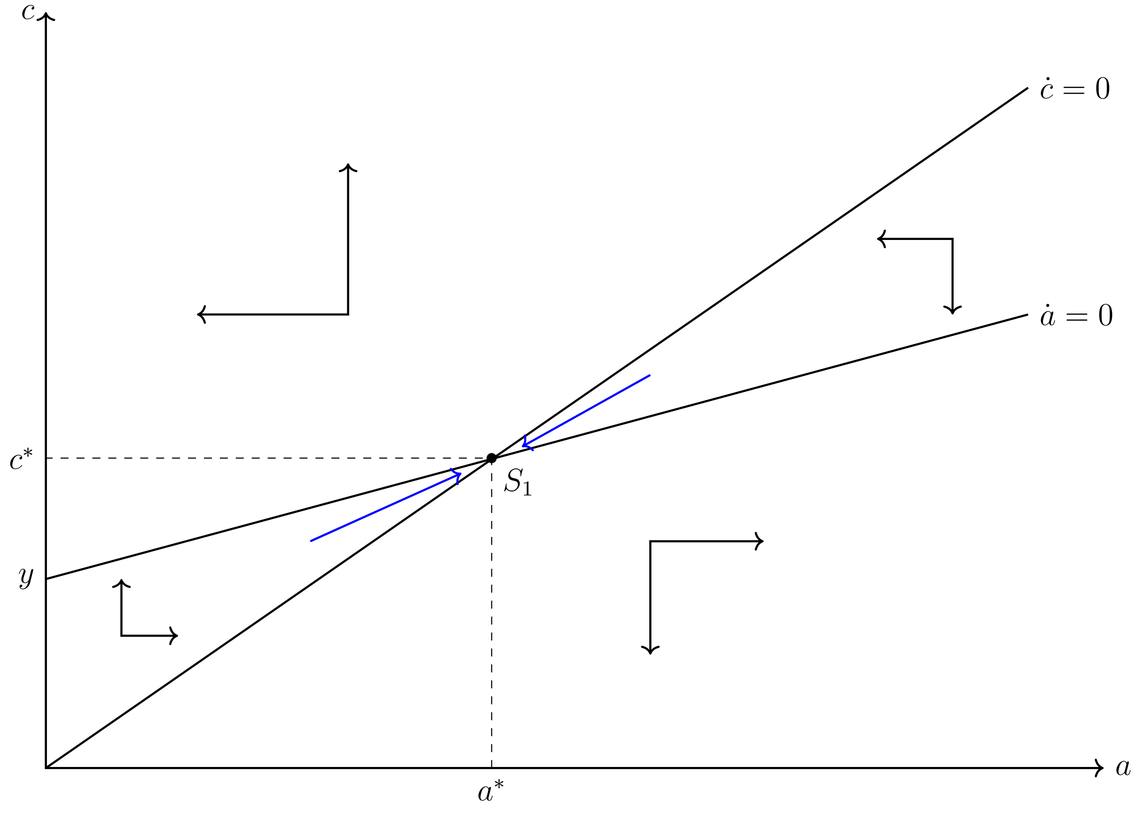
(d) To characterize the steady state, we use the fact $\dot{c} = \dot{a} = 0$. So, we get

$$\dot{a} = 0 \quad \Longrightarrow \quad 0 = ra + y - c$$
$$\dot{c} = 0 \quad \Longrightarrow \quad 0 = \gamma \frac{c^2}{a} - c(\rho - r)$$

Now, we can solve for c^* and a^*. We obtain

$$a^* = \frac{\gamma y}{\rho - r(1+\gamma)} \quad \text{and} \quad c^* = \frac{y(\rho - r)}{\rho - r(1+\gamma)}$$

(e) Here is phase-diagram to display the steady-state



More on Ramsey Model Phase-Diagram

Here is a nice explanation of the phase-diagram of Ramsey model[5].

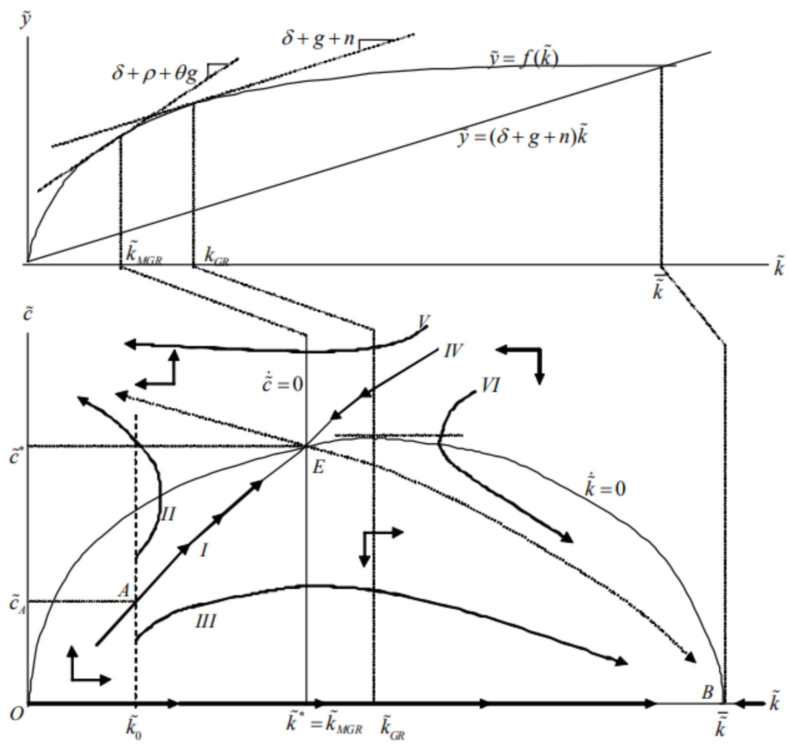


Figure 1: Phase-diagram of the Ramsey model and its relation with Solow growth model

Technical feasibility of the path also requires that its initial value for $\tilde{k}$ equals the historically given (pre-determined) value $\tilde{k}_0$. In contrast, for $\tilde{c}$ we have no exogenously given initial value. This is because $\tilde{c}_0$ is a so-called *jump variable* or *forward-looking variable*, by which is meant an endogenous variable which can immediately shift to another value when expectations about the future change. The transversality condition of the households makes up for this lack of an initial condition.

Figure shows some possible paths that could be solutions as t increases. We are especially interested in the paths which start out at the historically given $\tilde{k}_0$, that is, start out at some point on the stippled vertical line in the figure. If the economy starts out with a high value of $\tilde{c}$, it will follow a curve like II in the figure. The low level of saving implies that the capital stock goes to zero in finite time. If the economy starts out with a low level of $\tilde{c}$, it will follow a curve like III in the figure. The high level of saving implies that the capital intensity

[5] This is taken from Christian Groth's publicly available lecture notes and can be found here.

converges to $\overline{\tilde{k}}$. Such a $\overline{\tilde{k}}$ exists in view of (A1) and is higher than the golden rule value $\tilde{k}_{GR}$. This suggests that there exists an initial level of consumption somewhere in between, which gives a path like I. Indeed, since the curve II emerged with a high $\tilde{c}_0$, then by lowering this $\tilde{c}_0$ slightly, a path will emerge in which the maximal value of $\tilde{k}$ on the $\dot{\tilde{k}} = 0$ locus is greater than curve II's maximal $\tilde{k}$ value. We continue lowering $\tilde{c}_0$ until the path's maximal $\tilde{k}$ value is exactly equal to $\tilde{k}^*$. The path which emerges from this, namely the path I, starting at the point A, is special in that it converges towards the steady-state point E. No other path starting at the stippled line, $\tilde{k} = \tilde{k}_0$, has this property. Those starting above A did not, as we just saw. Consider a path starting below A, like path III. Either this path never reaches the consumption level $\tilde{c}_A$ and then it can not converge to E, of course. Or, after a while its consumption level reaches $\tilde{c}_A$, but at the same time it has $\tilde{k} > \tilde{k}_0$. From then on, as long as $\tilde{k} \leq \tilde{k}^*$, for every value of $\tilde{c}$ path III has in common with path I, path III has a higher $\tilde{k}$ and a lower $\tilde{c}$ than path I. Hence, path III diverges from point E.

Equivalently, had we considered values of $\tilde{k}_0 > \tilde{k}^*$, there would also be a unique value of $\tilde{c}_0$ such that the path starting from $\left(\tilde{k}_0, \tilde{c}_0\right)$ would converge to E (see path IV). All other values of $\tilde{c}_0$ would give paths that diverge from E.

The point E is a ***saddle point***. By this is meant a steady-state point with the following property: there exists exactly two paths, one from each side of $\tilde{k}^*$, that converge towards the steady-state point; all other paths (in a neighborhood of the steady state) move away from the steady state and asymptotically approach one of the two diverging paths through E. The two converging paths together make up the so-called *stable arm*; on their own they are referred to as *saddle paths*. The two diverging paths (along the dotted North-West and South-East curve) together make up the *unstable arm*.