EC813B - Recitation 5

SPRING 2022

Mehmet Karaca

Question 1

(Prelim #2, Fall 2004) Consider the following discrete-time version of the Solow growth model with embodied technical change. The economy is closed and populated by consumers with a fixed savings rate s :

$$S_t = sY_t$$

where S_t is aggregate savings and Y_t is output. Each consumer inelastically supplies one unit of labor, such that supply N_t coincides with population size. The economy is also populated by a representative firm with a Cobb-Douglas production technology:

$$Y_t = z_t K_t^{\alpha} N_t^{1-\alpha}$$

where K_t is the capital stock, and technical change z_t and population are assumed to grow at constant rates γ and n :

$$z_{t+1} = \gamma z_t \text{ and } N_{t+1} = nN_t$$

Finally, the capital stock evolves according to a technology $K_{t+1} = (1-\delta)K_t + I_t$.

(a) What is the necessary transformation to obtain the intensive form of the production function $y_t = k_t^{\alpha}$? (HINT: Remember, that in the Cobb-Douglas case we can interpret technological progress as labor-augmenting, since the z_t term can be factored out.)

(b) Give the definition of a balanced growth path. What is the growth rate of output and per capita output along the balanced growth path.

(c) Find the steady state equilibrium for y_t and k_t. Display this equilibrium graphically.

Solution

See **Acemoglu, D.**[a] **(2009),** ***Introduction to Modern Economic Growth***, for a detailed and fairly mathematical discussion of the Solow growth model (Chapter 2).

[a] You can find his course material from previous years here.

(a) We start with reorganizing the production function

$$Y_t = z_t K_t^\alpha N_t^{1-\alpha} = K_t^\alpha \left(z_t^{\frac{1}{1-\alpha}} N_t\right)^{1-\alpha}$$

Then we get

$$\frac{Y_t}{\left(z_t^{\frac{1}{1-\alpha}} N_t\right)} = \left(\frac{K_t}{z_t^{\frac{1}{1-\alpha}} N_t}\right)^\alpha \quad \Longrightarrow \quad y_t = k_t^\alpha$$

which is the necessary transformation.

Note: *See the discussion in R. Wright's lecture notes (Neoclassical Growth Model, page 5) about the claim in the HINT.*

We verify that balanced growth requires labor-augmenting technical progress for Cobb-Douglas production function. In general, we have

$$Y = f\left(e^{\gamma_K t} K, e^{\gamma_L t} L\right) = e^{\gamma_K t} K \phi \left[e^{(\gamma_L - \gamma_K)t} \frac{L}{K}\right]$$

where γ_K and γ_L are the rates of capital- and labor-augmenting technological progress and $\phi(\omega) = f(1, \omega)$. Let the growth rate of L be n and of K be $\gamma = \sigma \frac{Y}{K} - \delta$. Then $\frac{L}{K} = Ae^{(n-\gamma)t}$, and

$$\frac{Y}{K} = e^{\gamma_K t} \phi \left[e^{(\gamma_L - \gamma_K + n - \gamma)t}\right]$$

Now γ constant implies $\frac{Y}{K}$ constant, and so either $\gamma_k = 0$ and $\gamma = \gamma_L + n$; or $\gamma_K \neq 0$ but the change in $e^{\gamma_K t}$ exactly offsets the change in $\phi\left[e^{(\gamma_L - \gamma_K + n - \gamma)t}\right]$ over time. In the latter case, if we differentiate and rearrange $\frac{d}{dt}\frac{Y}{K} = 0$, we have

$$\frac{\omega \phi'(\omega)}{\phi(\omega)} = \frac{-\gamma_K}{\gamma_L - \gamma_K + n - \gamma} = \text{ constant}$$

This can be integrated to yield $\phi(\omega) = A\omega^\alpha$, where A and α are constants. This means that

$$Y = K e^{\gamma_K t} \phi \left[e^{(\gamma_L - \gamma_K)t} \frac{L}{K}\right] = A \left(e^{\gamma_K t} K\right)^{1-\alpha} \left(e^{\gamma_L t} L\right)^\alpha$$

or, in other words, the production function is Cobb-Douglas.

(b) **Definition:** A *balanced growth path* can be defined as an equilibrium path along which capital and output grow at the same rate (constant) when the labor input is fixed for any production function other than Cobb-Douglas (in the Cobb-Douglas case we can interpret technical change as labor- or capital-augmenting or neutral, since the z_t term can be factored out).

From part (a), we know that

$$y_t = \frac{Y_t}{\left(z_t^{\frac{1}{1-\alpha}} N_t\right)} \quad \text{and} \quad y_{t+1} = \frac{Y_{t+1}}{\left(z_{t+1}^{\frac{1}{1-\alpha}} N_{t+1}\right)}$$

Denote the growth rate of per capita output $g = \frac{y_{t+1}}{y_t}$. We get

$$g = \frac{y_{t+1}}{y_t} = \frac{Y_{t+1}}{Y_t} \cdot \frac{z_t^{\frac{1}{1-\alpha}} N_t}{z_{t+1}^{\frac{1}{1-\alpha}} N_{t+1}}$$

Reorganizing the equation we find

$$g = \frac{\frac{Y_{t+1}}{Y_t}}{\frac{z_{t+1}^{\frac{1}{1-\alpha}}}{z_t^{\frac{1}{1-\alpha}}} \cdot \frac{N_{t+1}}{N_t}} \quad \Longrightarrow \quad \frac{Y_{t+1}}{Y_t} = g \cdot \frac{z_{t+1}^{\frac{1}{1-\alpha}}}{z_t^{\frac{1}{1-\alpha}}} \cdot \frac{N_{t+1}}{N_t}$$

We are given $z_{t+1} = \gamma z_t$ and $N_{t+1} = nN_t$. Thus, the growth rate of output is

$$\frac{Y_{t+1}}{Y_t} = g(\gamma)^{\frac{1}{1-\alpha}} n$$

(c) We need to do this in two steps. First, we need to find k_{t+1} as a function of k_t. Secondly, we use the steady-state condition $k_t = k_{t+1} = \cdots = k^*$.

We start with rewriting the law of motion of capital as following

$$K_{t+1} = (1-\delta)K_t + I_t \quad \Longrightarrow \quad K_{t+1} = (1-\delta)K_t + sY_t$$

since in a closed economy $I_t = S_t$. Then, we divide each side by $z_t^{\frac{1}{1-\alpha}} N_t$. We get

$$\frac{K_{t+1}}{z_t^{\frac{1}{1-\alpha}} N_t} \cdot \frac{z_{t+1}^{\frac{1}{1-\alpha}} N_{t+1}}{z_{t+1}^{\frac{1}{1-\alpha}} N_{t+1}} = (1-\delta)\frac{K_t}{z_t^{\frac{1}{1-\alpha}} N_t} + s\frac{Y_t}{z_t^{\frac{1}{1-\alpha}} N_t}$$

Reorganize the equation

$$k_{t+1}(\gamma)^{\frac{1}{1-\alpha}} n = (1-\delta)k_t + sy_t$$

and plug in $y_t = k_t^\alpha$, we obtain

$$k_{t+1} = \frac{1}{(\gamma)^{\frac{1}{1-\alpha}} n} \Big[(1-\delta)k_t + s k_t^\alpha \Big]$$

Now, we use the steady-state condition and solve for k^*

$$k^* = \frac{1}{(\gamma)^{\frac{1}{1-\alpha}} n} \Big[(1-\delta)k^* + s(k^*)^\alpha \Big]$$

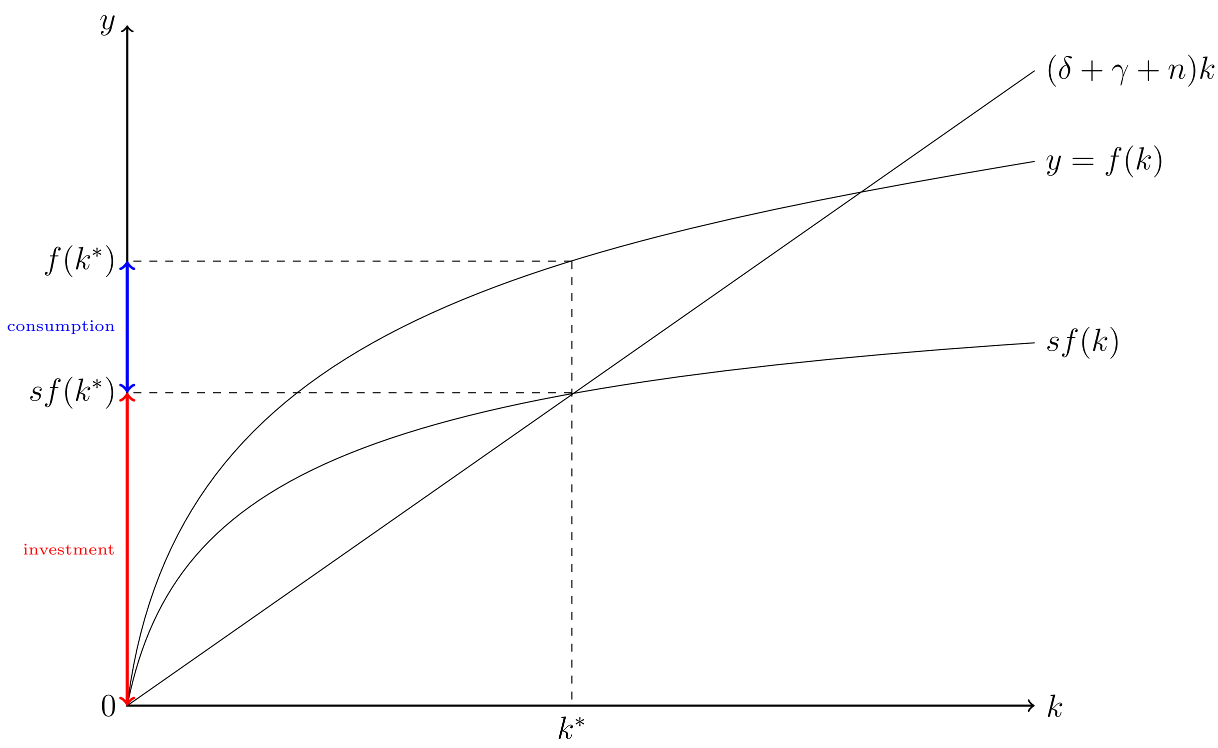
We find

$$k^* = \left(\frac{s}{(\gamma)^{\frac{1}{1-\alpha}} n - (1-\delta)} \right)^{\frac{1}{1-\alpha}}$$

Using $y^* = (k^*)^\alpha$ for steady state, we get

$$y^* = \left(\frac{s}{(\gamma)^{\frac{1}{1-\alpha}} n - (1-\delta)} \right)^{\frac{\alpha}{1-\alpha}}$$

Here is a graph of the equilibrium:



Question 2

(Prelim #2, Spring 2005) Consider the following Solow Growth model. Firms produce final goods with a constant returns to scale technology

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

where $0 < \alpha < 1$. Capital accumulation evolves according to

$$\dot{K}_t = I_t - \delta K_t$$

where $0 < \delta < 1$. The population N_t grows at rate $n > 0$. The level of technology A_t grows at rate $g > 0$. Consumers have a fixed savings rate of $0 < s < 1$.

(a) At what rate does this economy grow? What transformation ensures the intensive form $y = k^{\alpha}$?

(b) Derive the differential equation that governs the behavior of this economy.

(c) Find the steady state.

Solution

(a) This is a continous time version of the Solow growth model.[1] However, the analysis is pretty much the same with discrete version since we have a Cobb-Douglas production function.

We assume $S_t = sY_t$ as it is the usual assumption in the Solow growth model. We start with reorganizing the production function

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} = K_t^{\alpha} \left(A_t^{\frac{1}{1-\alpha}} N_t\right)^{1-\alpha}$$

Then we get

$$\frac{Y_t}{\left(A_t^{\frac{1}{1-\alpha}} N_t\right)} = \left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t}\right)^{\alpha} \quad \Longrightarrow \quad y_t = k_t^{\alpha}$$

which is the necessary transformation.

[1] A detailed discussion on how to transition from discrete to continous time can be found in Acemoglu, D. (2009) Introduction to Modern Economic Growth, Chapter 2, page 47.

(b) Define $\dot{k_t} = \frac{\partial K_t}{\partial t}$. Then we can write

$$\dot{k_t} = \dot{\left(\frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t}\right)} = \frac{\dot{K_t} \cdot \left(A_t^{\frac{1}{1-\alpha}} N_t\right) - K_t \cdot \dot{\left(A_t^{\frac{1}{1-\alpha}} N_t\right)}}{\left(A_t^{\frac{1}{1-\alpha}} N_t\right)^2}$$

where

$$\dot{\left(A_t^{\frac{1}{1-\alpha}} N_t\right)} = \frac{1}{1-\alpha} \cdot A_t^{\left(\frac{1}{1-\alpha}-1\right)} \cdot \dot{A_t} \cdot N_t + A_t^{\frac{1}{1-\alpha}} \dot{N_t}$$

Hence, we can write

$$\begin{aligned}
\dot{k_t} &= \frac{\dot{K_t}}{A_t^{\frac{1}{1-\alpha}} N_t} - \frac{K_t}{A_t^{\frac{1}{1-\alpha}} N_t} \cdot \left[\frac{\frac{1}{1-\alpha} \cdot A_t^{\left(\frac{1}{1-\alpha}-1\right)} \cdot \dot{A_t} \cdot N_t}{A_t^{\frac{1}{1-\alpha}} N_t} + \frac{A_t^{\frac{1}{1-\alpha}} \dot{N_t}}{A_t^{\frac{1}{1-\alpha}} N_t}\right] \\
\implies \quad \dot{k_t} &= \frac{sY_t}{A_t^{\frac{1}{1-\alpha}} N_t} - \frac{\delta K_t}{A_t^{\frac{1}{1-\alpha}} N_t} - k_t \cdot \left[\frac{1}{1-\alpha} \cdot \frac{\dot{A_t}}{A_t} + \frac{\dot{N_t}}{N_t}\right] \\
\dot{k_t} &= sy_t - \delta k_t - k_t\Big[\frac{1}{1-\alpha} \cdot g + n\Big]
\end{aligned}$$

Hence, the differential equation that governs the behavior of this economy is the following

$$\dot{k_t} = sk_t^{\alpha} - k_t\Big[\frac{1}{1-\alpha} \cdot g + n + \delta\Big]$$

(c) To find the steady-state equlibrium, solving for $\dot{k_t} = \frac{\partial K_t}{\partial t} = 0$ ensures that $k_t = k_{t+1} = \cdots = k^*$ at the steady state. Thus, we write

$$s(k^*)^{\alpha} - k^*\Big[\frac{1}{1-\alpha} \cdot g + n + \delta\Big] = 0$$

solving for k^*, we get

$$k^* = \left(\frac{s}{\frac{1}{1-\alpha} \cdot g + n + \delta}\right)^{\frac{1}{1-\alpha}}$$

The value for y^* is

$$y^* = \left(\frac{s}{\frac{1}{1-\alpha} \cdot g + n + \delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Question 3

(Prelim #1, Spring 2012) Consider a representative agent model where the representative consumer has preferences given

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

where $0 < \beta < 1$ and c_t is consumption in period t. The production technology is given by $y_t = \alpha_t k_t$ where $\alpha_t = \alpha_1$ for $t = 0, 2, 4 \cdots$, and $\alpha_t = \alpha_2$ for $t = 1, 3, 5 \cdots$. Assume that $\alpha_1 \beta > 1$ and $\alpha_2 \beta < 1$. Also assume 100% depreciation.

(a) Solve for the planners value function and allocation by using dynamic programming and guess-and-verify methods. Show that the capital stock, output, and consumption increase in even periods and decrease in odd periods. (HINT: note that in general the value function will be different in even and odd periods, $V_i(k_t) = A_i + B_i \ln k_t$, where $i =$ even, odd, solving for B_i should be enough to show the results.)

(b) Show that trend consumption increases (that is, consumption increases from period t to period $t+2$ for all t) if $\alpha_1 \alpha_2 \beta^2 > 1$ and decrease if $\alpha_1 \alpha_2 \beta^2 < 1$. Explain your results.

Solution

(a) The planner's problem is

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{s.t.} \quad c_t = \alpha_t k_t - k_{t+1} \quad \text{since} \quad \delta = 1$$

Then using the hint, we can write the value function in general form as follows

$$V(k_t) = \max_{\{k_{+1}\}} \Big\{ \ln(\alpha_t k_t - k_{t+1}) + \beta V(k_{t+1}) \Big\}$$

We begin writing value functions explicitly assuming $V_i(k_t) = A_i + B_i \ln k_t$ since value function will be different in even and odd periods.

Step 1

$$V_e(k_t) = \max \{\ln(\alpha_1 k_t - k_{t+1}) + \beta V_o(k_{t+1})\}$$
$$V_e(k_t) = \max \Big\{ \ln(\alpha_1 k_t - k_{t+1}) + \beta \Big[A_o + B_o \ln(k_{t+1}) \Big] \Big\}$$

Now, we get the first order conditions w.r.t k_{t+1} to find optimal k^*_{t+1}

$$-\frac{1}{\alpha_1 k_t - k_{t+1}} + \beta\Big(B_o \frac{1}{k_{t+1}}\Big) = 0 \quad \Longrightarrow \quad k^*_{t+1} = \frac{\alpha_1 k_t}{1+\frac{1}{\beta B_o}} = \frac{\alpha_1 \beta B_o}{1+\beta B_o} \cdot k_t$$

Step 2

Next, we plug in the value we found for k^*_{t+1}, again assuming the value function has the form $V_i(k_t) = A_i + B_i \ln k_t$.

$$V_e(k_t) = A_e + B_e \ln k_t$$

$$V_e(k_t) = \ln\Big(\alpha_1 k_t - \frac{\alpha_1 \beta B_o}{1+\beta B_o} \cdot k_t\Big) + \beta\left(A_o + B_o \ln\Big(\frac{\alpha_1 \beta B_o}{1+\beta B_o} \cdot k_t\Big)\right)$$

We need to separate elements of the equation related with k_t from constants. We obtain

$$V_e(k_t) = \ln(\alpha_1) + \ln(k_t) + \ln\Big(\frac{1}{1+\beta B_o}\Big) + \beta A_o + \beta B_o \ln\Big(\frac{\alpha_1 \beta B_o}{1+\beta B_o}\Big) + \beta B_o \ln(k_t)$$

Reorganize the equation

$$V_e(k_t) = \underbrace{\ln(\alpha_1) + \ln\Big(\frac{1}{1+\beta B_o}\Big) + \beta A_o + \beta B_o \ln\Big(\frac{\alpha_1 \beta B_o}{1+\beta B_o}\Big)}_{constant} + (1+\beta B_o)\ln(k_t)$$

So we get $B_e = 1 + \beta B_o$ and it is symmettric for odd periods following $B_o = 1 + \beta B_e$. Using these, we find $B_e = B_o = \frac{1}{1-\beta}$.

Step 3

Finally, we use $k^*_{t+1} = \frac{\alpha_1 \beta B_o}{1+\beta B_o} \cdot k_t$ and plug in for B_o (or the other way around for B_e), and we get

$$k^*_{t+1} = \alpha_1 \beta k_t \quad \text{and} \quad k^*_{t+1} = \alpha_2 \beta k_t$$

Given that $\alpha_1 \beta > 1$ and $\alpha_2 \beta < 1$, we find

$$k^*_{t+1} = \alpha_1 \beta k_t > k_t \quad \text{and} \quad k^*_{t+1} = \alpha_2 \beta k_t < k_t$$

So, the capital stock increases in even periods and decreases in odd periods. The output is $y_t = \alpha_t k_t$. We can easily find that $y_{t+1} = \alpha_1 k_{t+1} = (\alpha_1)^2 \beta k_t > y_t = \alpha_1 k_t$ for even periods, and $y_{t+1} = \alpha_2 k_{t+1} = (\alpha_2)^2 \beta k_t < y_t = \alpha_2 k_t$ for odd periods. Finally, the result for consumption follows from similar reasoning. (*Hint: Find* $c_{t+1} - c_t$.)

(b) We need to show

$$\begin{aligned} c_{t+2} - c_t > 0 \quad \text{if} \quad \alpha_1\alpha_2\beta^2 > 1 \\ c_{t+2} - c_t < 0 \quad \text{if} \quad \alpha_1\alpha_2\beta^2 < 1 \end{aligned}$$

We find earlier that

$$\begin{aligned} k^*_{t+1} = \alpha_1\beta k_t \quad \text{if } t \text{ is even} \\ k^*_{t+1} = \alpha_2\beta k_t \quad \text{if } t \text{ is odd} \end{aligned}$$

Hence, we get $k^*_{t+2} = \alpha_1\alpha_2\beta^2 k_t$ for all t.

$$\begin{aligned} c_{t+2} - c_t = \left[(\alpha_1 - \alpha_2\beta)\cdot(\alpha_1\alpha_2\beta^2 - 1)\right]k_t > 0 \quad \text{if} \quad \alpha_1\alpha_2\beta^2 > 1 \\ c_{t+2} - c_t = \left[(\alpha_1 - \alpha_2\beta)\cdot(\alpha_1\alpha_2\beta^2 - 1)\right]k_t < 0 \quad \text{if} \quad \alpha_1\alpha_2\beta^2 < 1 \end{aligned}$$

Thus, the trend consumption increases if $\alpha_1\alpha_2\beta^2 > 1$ and decrease if $\alpha_1\alpha_2\beta^2 < 1$.