EC813B - Recitation 13

SPRING 2022

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Question 1

(Final 2013) Consider the following model of a market with search frictions. Time is continuous, and there is a continuum of two types of agents in the economy, buyers and sellers, who are randomly matched pairwise. The utility from consuming one unit of the good is 1. A buyer who consumes one unit of goods exits the market and is replaced by another buyer, so measure of buyers at any point in time is fixed. On the other hand, the measure of sellers is endogenously determined in the model by the free entry condition. To enter the market as a seller an agent must produce one unit of consumption good at cost e (entry cost). After the transaction the seller leaves the market and must pay e to re-enter. To find a trade partner any agent must exert effort: the search cost at a point in time is rc where r is the discount rate. We focus on the steady state of the economy. Let's denote s the fraction of sellers in the steady state (the fraction of buyers is $1-s$), p the utility transferred from the buyer to the seller (price) in a match, and P the prevailing price in the market. We assume the Poisson arrival rate for any agent in the steady state is normalized to 1. Agents use symmetric Nash bargaining with their continuation values as the threat points to determine p.

(a) Taking the prevailing price P as given, write down the steady state value functions for a buyer, V_b, and a seller, V_s.

(b) Taking the value functions $V_b(P)$ and $V_s(P)$ as given, find the bargaining solution $p(P)$ and solve for the equilibrium price $p^*(P^*) = P^*$.

(c) Solve for the equilibrium of the model by using the free entry condition for sellers. To answer this question, you do not need to find the closed form solutions for all the endogenous variables. It is enough to show that the number of unknowns is the same as the number of equations.

(d) Now let us remove the search friction by limiting $r \to 0$ (the agents are infinitely patient). How does the friction of sellers s respond to a change in the search cost c ? Explain your result.

Solution

(a) The value functions are

$$
\begin{aligned}
V_b &= \frac{1}{1+r}\left[s(1-p)+(1-s)V_b - rc\right] \\
V_s &= \frac{1}{1+r}\left[(1-s)p + sV_s - rc\right]
\end{aligned}
$$

The flow values of buyer and seller are, respectively

$$
\begin{aligned}
rV_s &= s\left(1-p-V_b\right) - rc \\
rV_b &= (1-s)\left(p-V_s\right) - rc
\end{aligned}
$$

Now, we can solve for the value functions. We find

$$
\begin{aligned}
V_b &= \frac{1}{r+s}[s(1-p)-rc] \\
V_s &= \frac{1}{r+(1-s)}[(1-s)p-rc]
\end{aligned}
$$

(b) The maximization problem is

$$
\max_p \left(1-p-V_b\right)\left(p-V_s\right) \quad \text{s.t.} \quad \begin{aligned} 1-p &\geq V_b \\ p &\geq V_s \end{aligned}
$$

The FOC is

$$
-p + V_s + 1 - p - V_b = 0
$$

Thus, from the first-order condition,

$$
p(P) = \frac{1}{2}\left(1+V_s-V_b\right)
$$

By substituting $V_s - V_b$ (could be obtained from part (a)) into the FOC and imposing the equilibrium condition $p(P) = P$, we obtain:

$$
p^*(P^*) = P^* = \frac{r+(1-s)+c(1-2s)}{2r+1}
$$

(c) The equilibrium is fully characterized by the following two:

$$P^* = \frac{r + (1-s) + c(1-2s)}{2r+1}$$
$$e = P^*$$

where the second equation is from the free entry condition $V_s = e$. The unknowns in the above system of equations are P^* and s.

(d) When $r = 0$, the two equations become

$$P^* = (1-s) + c(1-2s)$$
$$e = P^*$$

Therefore, in the steady state the fraction of the sellers is

$$s = \frac{1+c-e}{1+2c}$$

and the derivative of it is

$$\frac{\partial s}{\partial c} = \frac{2e-1}{(1+2c)^2}$$

which is positive if and only if $e > 1/2$. The expression derived above shows that an increase in c has a positive impact on the equilibrium price P^* when s is small enough. This is because an increase in c undermines the buyers bargaining position if there are more buyers than sellers. When the entry cost is high enough, the fraction of seller is relatively small, so an increase in the search cost enhances the sellers' bargaining position, which in turn invites more sellers to the market.

Question 2

Consider a simple Mortensen-Pissarides labor search model with notation used in class (meetings are determined through a matching function and wages through bargaining between risk neutral workers and firms). Show that the generalized Nash bargaining solution with threat points U and V

$$w = \arg\max_{w'} \left[W\left(w'\right) - U\right]^{\alpha} \left[J\left(y - w'\right) - V\right]^{1-\alpha}$$

is equivalent to:

$$w = \arg\max_{w'} \left[w' - w_R\right]^{\alpha} \left[y - w' - \pi_R\right]^{1-\alpha}$$

where w_R and π_R are reservation wage and profit levels for the worker and firm ($\pi = y - w$). You do not have to calculate w_R and π_R, it is enough to express them through the value functions U and V. *(HINT: Write down the values of being matched for the worker and firm, $W(w)$ and $J(y-w)$.)*

Solution

The flow values of being matched for worker and firm are

$$rW(w) = w + \delta[U - W(w)]$$
$$rJ(\pi) = \pi + \delta[V - J(\pi)]$$

The value functions are

$$W(w) = \frac{1}{r+\delta}[w + \delta U]$$
$$J(\pi) = \frac{1}{r+\delta}[\pi + \delta V]$$

The reservation wage and profit can be found using value functions as follows

$$W(w_R) = U \quad \Longrightarrow \quad \frac{1}{r+\delta}\left[w_R + \delta U\right] = U \quad \Longrightarrow \quad w_R = rU$$
$$J(\pi_R) = V \quad \Longrightarrow \quad \frac{1}{r+\delta}[\pi_R + \delta V] = V \quad \Longrightarrow \quad \pi_R = rV$$

Now, we can find $W(w') - U$ and $J(y - w') - V$ using the value functions and plug in for w_R and π_R. Assume ($\pi' = y - w'$) and we get

$$W(w') - U = \frac{1}{r+\delta}\left[w' + \delta U\right] - U = \frac{1}{r+\delta}\left[w' - rU\right] = \frac{1}{r+\delta}\left[w' - w_R\right]$$
$$J(\pi') - V = \frac{1}{r+\delta}\left[\pi' + \delta V\right] - V = \frac{1}{r+\delta}\left[\pi' - rV\right] = \frac{1}{r+\delta}\left[\pi' - \pi_R\right]$$

Thus, we find that the generalized Nash bargaining solution with threat points U and V is equivalent to

$$w = \arg\max_{w'} \left[W(w') - U\right]^{\alpha} \left[J(y - w') - V\right]^{1-\alpha} = \arg\max_{w'} \frac{1}{r+\delta}\left[w' - w_R\right]^{\alpha} \left[\pi' - \pi_R\right]^{1-\alpha}$$

Question 3

(Final 2012) Consider The Trejos-Wright 1995 monetary model with divisible goods. There is a continuum of agents with population normalized to 1. Agents never consume their own output, and so must trade. Traders meet according to Poisson arrival rates that are proportional to the number of agents on the other side of the market. The meeting arrival rate is α. In a random meeting between two agents, the latter can produce the former's consumption goods with probability x, and they can both produce each others' consumption goods with probability 0 (no double coincidence of wants). She derives utility $u(q) = q$ from consuming $q \in \mathbb{R}_+$units of the service provided it falls in her desired interval. The discount rate is r. There is a disutility $c(q) = q^2$ to producing q units of a service. At the beginning of time, a fraction of agents $M \in (0, 1)$ are randomly given one unit of currency. Currency is indivisible and can be stored only one unit at a time. In the case of no barter, 0 threat points, and the buyer having bargaining power θ, define and find the unique monetary equilibrium. Normalize $\alpha x = 1$.

Solution

The flow values of a buyer and a seller are, respectively

$$\begin{aligned} rV_b &= (1 - M)(Q + V_s - V_b) \\ rV_s &= M\left(-Q^2 + V_b - V_s\right) \end{aligned}$$

By rearranging terms, we get

$$\begin{aligned} r(V_b - V_s) &= (1 - M)Q + MQ^2 - (V_b - V_s) \\ V_b - V_s &= \frac{1}{1 + r}[(1 - M)Q + MQ^2] \end{aligned}$$

Plugging in for $(V_b - V_s)$, we find the value functions as follows

$$\begin{aligned} V_b &= \frac{1 - M}{r}\left[Q - \frac{(1 - M)Q + MQ^2}{1 + r}\right] \\ V_s &= \frac{M}{r}\left[-Q^2 + \frac{(1 - M)Q + MQ^2}{1 + r}\right] \end{aligned}$$

Q is the solution of the following generalized Nash bargaining.

$$\max_{q} \quad \theta \ln\left(q+V_s\right)+(1-\theta)\ln\left(-q^2+V_b\right) \quad \text{s.t.} \quad \begin{aligned} q+V_s &\geq V_b \\ -q^2+V_b &\geq V_s \end{aligned}$$

From the F.O.C.:

$$\theta\left(V_b-q^2\right)=2q(1-\theta)\left(q+V_s\right)$$

By substituting V_b and V_s into the F.O.C given as above and imposing the equilibrium condition $q^*=Q^*$, we obtain:

$$\theta\left\{\frac{1-M}{r}\left[Q-\frac{(1-M)Q+MQ^2}{1+r}\right]-Q^2\right\}=2Q(1-\theta)\left\{Q+\frac{M}{r}\left[-Q^2+\frac{(1-M)Q+MQ^2}{1+r}\right]\right\}$$

$$\theta\left\{\frac{1-M}{r}\left[1-\frac{(1-M)+MQ}{1+r}\right]-Q\right\}=2(1-\theta)\left\{Q+\frac{M}{r}\left[-Q^2+\frac{(1-M)Q+MQ^2}{1+r}\right]\right\}$$

Solving for Q^*, we get

$$\Big[2(1-Q)M(1+r-M)\Big]Q^2-(2-\theta)\Big[r(1+r)+M(1-M)\Big]Q+\theta(1-M)(r+M)=0$$

This is a quadratic equation where Q is unknown variable. By applying the formula, we get the following solution:

$$Q^*=\frac{(2-\theta)\phi-\sqrt{\psi}}{4M(r+1-M)(1-\theta)}$$

where

$$\begin{aligned} \phi &= r(1+r)+M(1-M) \\ \psi &= (2-\theta)^2\phi^2-8(r+1-M)(r+M)M(1-M)\theta(1-\theta) \end{aligned}$$

Question 4

(Prelim #3, Spring 2011) Consider the following simple version of Trejos and Wright (1995). Time is discrete ($\Delta=1$) and there is a continuum of agents with population normalized to 1. Any particular agent specializes in the production of one service (a non-storable good) but likes other services in an interval of size $x \in (0,1)$. She derives utility $u(q)$ from consuming $q \in \mathbb{R}_+$units of the service she likes (as in class let us assume $bx=1$, where b is the meeting probability), $u(q)$ is strictly concave, and satisfies the Inada conditions. An agent discounts the future at rate r (discount factor $1=(1+r)$). There is a disutility $c(q)=q$ to producing q units of a service. At the beginning of time, a fraction of agents $M \in (0,1)$ are randomly given one unit of currency. Currency is indivisible and can be stored only one unit at a time. Agents are exogenously matched in the following way. Agents with money (we will term them buyers) are randomly matched in pairs with agents without money. Thus, the probability that a buyer is matched with a seller whose good she desires is $(1 - M)$. Also, the probability that a seller is matched with a buyer who wants her good is M. Barter is ruled out. Buyers have all the bargaining power so they make take-it-or-leave-it offers. There exists a government whose role is to redistribute the trade surplus using tax τ and transfer T. Every period in each successful trade meeting the buyer should pay tax τ and the government transfers T to each of the unmatched or non-trading matched individuals. τ and T are paid in terms of consumption goods. Government budget is always balanced.

(a) Write down the value functions and government budget constraint.

(b) Show that the take-it-or-leave-it offer assumption yields V_s as a function of r, M and T. Define welfare.

(c) Assume $M = 1/2$, and show that redistribution is welfare improving when τ is not too large.

Solution

(a) We can write the value functions as following

$$V_b = \frac{1}{1+r}\left\{(1-M)\left[u(q-\tau)+V_s\right]+M\left[V_b+u(T)\right]\right\}$$
$$V_s = \frac{1}{1+r}\left\{M\left[-c(q)+V_b\right]+(1-M)\left[V_s+u(T)\right]\right\}$$

Rearranging we can write the flow values as

$$rV_b = (1-M)\left[u(q-\tau)+V_s-V_b\right]+Mu(T)$$
$$rV_s = M\left[-q+V_b-V_s\right]+(1-M)u(T)$$

The government budget constraint is

$$M(1-M)\tau = \left[M^2+(1-M)^2\right]T$$

(b) We assume buyers have all the bargaining power so they make take-it-or-leave-it offers.

Then, the maximization problem is

$$\max_q \left(u(q-\tau)+V_s\right)^1 \left(-q+V_b\right)^0 \quad \text{s.t.} \quad \begin{aligned} u(q-\tau)+V_s &\geq u(\tau)+V_b \\ -q+V_b &\geq u(\tau)+V_s \end{aligned}$$

Second constraint is binding because seller does not have any bargaining power. Thus, we get

$$\begin{aligned} q &= V_b - V_s - u(T) \\ V_b - V_s &= q + u(T) \end{aligned}$$

Then, imposing consistency condition $q = Q$ on the flow value equations, we get

$$\begin{aligned} rV_s &= M[-Q+Q+u(T)] + (1-M)u(T) = u(T) \\ V_s &= \frac{1}{r}u(T) \end{aligned}$$

Next

$$\begin{aligned} V_b &= \frac{1}{r}\left[(1-M)\left(u(Q-\tau)+V_s-V_b\right)+Mu(T)\right] \\ &= \frac{1}{r}[(1-M)(u(Q-\tau)-Q-u(\tau))+Mu(T)] \\ &= \frac{1}{r}[(1-M)(u(Q-\tau)-Q)-(1-2M)u(T)] \end{aligned}$$

Finally, we can define welfare as

$$W = MV_b + (1-M)V_s$$

(c) Assume $M = 1/2$. From the government budget constraint, we get

$$M(1-M)\tau = \left[M^2+(1-M)^2\right]T \quad \Longrightarrow \quad T = \frac{1}{2}\tau$$

Then we can write welfare as

$$W = \frac{1}{2}V_b + \frac{1}{2}V_s = \frac{1}{2}\left\{\frac{1}{r}[(1-M)(u(Q-\tau)-Q)-(1-2M)u(T)] + \frac{1}{r}u(T)\right\}$$
$$W = \frac{1}{2r}\Big\{(1-M)(u(Q-\tau)-Q)-(1-2M)u(T)+u(T)\Big\}$$
$$W = \frac{1}{2r}\Big\{(1-M)(u(Q-\tau)-Q)+2Mu(T)\Big\}$$
$$W = \frac{1}{2r}\left\{\frac{1}{2}[u(Q-\tau)-Q]+u(T)\right\}$$
$$W = \frac{1}{2r}\left\{\frac{1}{2}[u(Q-\tau)-Q]+u\left(\frac{1}{2}\tau\right)\right\}$$

From the maximization problem we find the FOC and take the inverse of $u(\cdot)$. We get

$$u'(q-\tau)=1 \quad\Longrightarrow\quad u'^{-1}(1)=q-\tau \quad\Longrightarrow\quad q=u'^{-1}(1)+\tau$$

Plugging in we obtain

$$W = \frac{1}{2r}\left\{\frac{1}{2}\left[u\left(u'^{-1}(1)\right)-u'^{-1}(1)-\tau\right]+u\left(\frac{1}{2}\tau\right)\right\}$$
$$W = \frac{1}{2r}\left\{\frac{1}{2}\left[u\left(u'^{-1}(1)\right)-u'^{-1}(1)\right]+u\left(\frac{1}{2}\tau\right)-\frac{1}{2}\tau\right\}$$

We take the derivative with respect to τ. We find that we need to have

$$\frac{1}{2}u'\left(\frac{1}{2}\tau\right)-\frac{1}{2}>0$$

From Inada condition $\lim_{\tau\to 0}u'(x)=\infty$, $\frac{\partial W}{\partial \tau}>0$ when τ is not too large.

Question 5

Consider and economy with a continuum of agents and K divisible and perishable goods. Time is discrete and indexed by t. Agents can be of K distinct types. A type i agent derives utility from the consumption of good i and is able go produce good $i+1$ (modulo K) and there is a uniform distribution of agents across types. Utility in every period is given by $u(x)-y$ where x is the amount consumed and y is the amount produced. The function u is defined on $[0,\infty)$ and we assume that $u(\cdot)$ is increasing, twice differentiable, $u(0)=0, u''<0$

and $u'(0) = \infty$. Agents maximize expected discounted utility with a discount factor of $\beta = (0,1)$. In the economy there is also a storable, indivisible and intrinsically useless object, which we denote as fiat money. We assume that an agent can hold at most one unit of money at a time. Agents can identify the sectors but inside each sector they are anonymously and pairwise matched under a uniform random matching technology. At date 0 , money is randomly distributed to a fraction m of agents. Let $V_0(m)$ be the value function of an agent without money and let $V_1(m)$ be the corresponding value function for an agent with money.

(a) Write down the value functions (use the beginning of the period discounting).

(b) Calculate the quantity produced by a seller in a trade meeting assuming take-it-or-leave-it offers by the buyer. How does it depend on m?

Solution

(a) The value functions can be written as

$$V_0(m) = m\left[-y + \beta V_1(m)\right] + (1-m)\beta V_0(m)$$
$$V_1(m) = m\beta V_1(m) + (1-m)\left[u(x) + \beta V_0(m)\right]$$

(b) We need to find y^*. We start with the maximization problem which can be written as

$$\max\left[u(x) + \beta V_0(m) - V_1(m)\right] \quad \text{s.t.} \quad -y + \beta V_1(m) \geq \beta V_0(m)$$

Use the constraint

$$\beta V_1(m) - y^* \geq \beta V_0(m) \quad \Longrightarrow \quad y^* = \beta\left[V_1(m) - V_0(m)\right]$$

Using value functions, we get

$$V_1(m) - V_0(m) = my^* + (1-m)u\left(y^*\right) = m\beta\left[V_1(m) - V_0(m)\right] + (1-m)u\left(y^*\right)$$
$$V_1(m) - V_0(m) = \frac{1-m}{1-m\beta}u\left(y^*\right)$$

Using $y^* = \beta\left[V_1(m) - V_0(m)\right]$, we find

$$y^* = \frac{\beta(1-m)}{1-m\beta}u\left(y^*\right)$$

To find how y^* depend on m, we can define a function $F(y^*, m)$ as

$$F(y^*, m) = \frac{\beta(1-m)}{1+m\beta} u(y^*) - y^*$$

Then, to find $\frac{\partial y^*}{\partial m}$, we use the implicit function theorem. Thus, we get

$$\frac{\partial y^*}{\partial m} = -\frac{F_m}{F_{y^*}} = -\frac{\phi'(m) u(y^*)}{\phi(m) u'(y^*) - 1} < 0$$

where $\phi(m) = \frac{\beta(1-m)}{1+m\beta}$ and its derivative is

$$\phi'(m) = \frac{-\beta(1+m\beta) - \beta(1-m)\beta}{(1+m\beta)^2} = \frac{-\beta(1+\beta)}{(1+m\beta)^2} < 0$$