EC813B - Recitation 8

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Question 1

(Prelim #2, Fall 2016) (RBC) Consider a neoclassical growth model with human capital accumulation. There is a representative agent who has preferences given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

where the function u is differentiable, strictly increasing and strictly concave. There is a $[0,1]$ continuum of identical firms each producing output according to the following technology

$$\begin{aligned} Y_t &= K_t^{\theta} H_t^{1-\theta} \\ Y_t &= c_t + I_{K_t} + I_{H_t} \\ K_{t+1} &= \left(1-\delta_K\right) K_t + I_{K_t} \\ H_{t+1} &= \left(1-\delta_H\right) H_t + I_{H_t} \end{aligned}$$

where K_t and H_t are the physical capital and the human capital correspondingly, and the initial capital stocks are given, K_0 and H_0.

(a) Solve the social planner's problem using the dynamic programming approach. Assume that the depreciation rates are the same, $\delta_K = \delta_H = \delta$, and solve for the ratio K_{t+1}/H_{t+1} as a function of θ (use the Euler equations for K_{t+1} and H_{t+1}).

(b) Let the utility function be CRRA: $u\left(c_t\right) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$. Assume that the economy is on a balanced growth path: $c_{t+1}/c_t = \gamma$ and calculate the growth rate of the economy, γ, as a function of β, θ, σ and δ.

Solution

(a) The social planner's problem is

$$\begin{aligned} \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \\ \text{s.t.} \quad Y_t &= K_t^{\theta} H_t^{1-\theta} \\ Y_t &= c_t + I_{K_t} + I_{H_t} \\ K_{t+1} &= \left(1-\delta_K\right) K_t + I_{K_t} \\ H_{t+1} &= \left(1-\delta_H\right) H_t + I_{H_t} \end{aligned}$$

The state variables are K_t and H_t, and the control variables are K_{t+1} and H_{t+1}. Then, the Bellman's equation can be written as

$$V\left(K_t, H_t\right) = \max_{\{K_{t+1}, H_{t+1}\}} \Big\{ u(c_t) + \beta V\left(K_{t+1}, H_{t+1}\right) \Big\}$$

Using the constraints, we can plug in for c_t and get

$$V\left(K_t, H_t\right) = \max_{\{K_{t+1}, H_{t+1}\}} \left\{ u\Big(K_t^{\theta} H_t^{1-\theta} - K_{t+1} + \left(1-\delta_K\right) K_t - H_{t+1} + \left(1-\delta_H\right) H_t \Big) + \beta V\left(K_{t+1}, H_{t+1}\right) \right\}$$

Assuming that the depreciation rates are the same, $\delta_K = \delta_H = \delta$, we can rewrite the equation as follows

$$V\left(K_t, H_t\right) = \max_{\{K_{t+1}, H_{t+1}\}} \left\{ u\Big(K_t^{\theta} H_t^{1-\theta} + \left(1-\delta\right)\left(K_t + H_t\right) - K_{t+1} - H_{t+1} \Big) + \beta V\left(K_{t+1}, H_{t+1}\right) \right\}$$

The FOCs are

- For K_{t+1} : $\quad u'(c_t) = \beta V_1\left(K_{t+1}, H_{t+1}\right)$
- For H_{t+1} : $\quad u'(c_t) = \beta V_2\left(K_{t+1}, H_{t+1}\right)$

Then we need to use the ***envelope theorem*** for K_t and H_t to derive the Euler equations. First, we use the envelope theorem for K_t:

$$V_1\left(K_t, H_t\right) = u'(c_t)\Big[\theta K_t^{\theta-1} H_t^{1-\theta} + (1-\delta)\Big]$$

Update the value function for 1 period:

$$V_1\left(K_{t+1}, H_{t+1}\right) = u'(c_{t+1})\Big[\theta K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + (1-\delta)\Big]$$

Combine the FOC and the envelope condition to get the Euler equation for K_{t+1}:

$$u'(c_t) = \beta u'(c_{t+1})\Big[\theta K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + (1-\delta)\Big] \tag{1}$$

Second, we use the envelope theorem for H_t:

$$V_2\left(K_t, H_t\right) = u'(c_t)\Big[(1-\theta) K_t^{\theta} H_t^{-\theta} + (1-\delta)\Big]$$

Update the value function for 1 period:

$$V_2\left(K_{t+1}, H_{t+1}\right) = u'(c_{t+1})\Big[(1-\theta) K_{t+1}^{\theta} H_{t+1}^{-\theta} + (1-\delta)\Big]$$

Combine the FOC and the envelope condition to get the Euler equation for H_{t+1}:

$$u'(c_t) = \beta u'(c_{t+1})\Big[(1-\theta) K_{t+1}^{\theta} H_{t+1}^{-\theta} + (1-\delta)\Big] \tag{2}$$

Next, using Euler equations (1) and (2), we solve for the ratio K_{t+1}/H_{t+1}:

$$\frac{u'(c_t)}{u'(c_t)} = 1 = \frac{\beta u'(c_{t+1})\Big[\theta K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + (1-\delta)\Big]}{\beta u'(c_{t+1})\Big[(1-\theta) K_{t+1}^{\theta} H_{t+1}^{-\theta} + (1-\delta)\Big]}$$

and we find

$$\frac{K_{t+1}}{H_{t+1}} = \frac{\theta}{(1-\theta)}$$

(b) We can use one of the Euler equations. We can rewrite the Euler equation for K_{t+1} as follows

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\Big[\theta K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + (1-\delta)\Big]$$

Given that the utility function is CRRA, $u\left(c_t\right) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$, we obtain

$$\frac{c_t^{-\sigma}}{c_{t+1}^{-\sigma}} = \beta\Big[\theta K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + (1-\delta)\Big]$$

and we plugging in for K_{t+1}/H_{t+1}, we solve for the growth rate of the economy, $\gamma = c_{t+1}/c_t$:

$$\gamma = \left\{\beta\left[\theta\left(\frac{\theta}{(1-\theta)}\right)^{\theta-1} + (1-\delta)\right]\right\}^{\frac{1}{\sigma}}$$

Question 2

(RBC) Consider the following neoclassical growth model. The representative consumer maximize her utility given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to the budget constraint

$$c_t + i_t \leq \left(1-\tau_t\right) r_t k_t + w_t$$

where the investment i_t equals $k_{t+1} - (1-\delta)k_t, \tau_t$ is the tax rate, r_t is the rental rate of capital, and w_t is wage income. Note that we assume that each worker has one unit of labor endowment per period. Per capita production is given as $y_t = F\left(k_t, 1\right) = f\left(k_t\right)$ where F is CRTS and f is increasing and strictly concave.

(a) Write Bellman's equation and derive Euler equation.

(b) Assuming there is no direct tax imposed on firms' activity, derive the equilibrium rental rate of capital and the wage rate.

(c) Now, let us assume $f(k) = k^\alpha$, and that the tax rate does not change in the steady state. Derive the steady state capital stock as a function of τ.

(d) The dictator of the society wants to maximize the tax revenue. Suppose the economy is in the steady state. What is the optimal tax rate which does not change over time? Interpret your result.

Solution

(a) The Bellman's equation is

$$V\left(k_t\right) = \max_{\{k_{t+1}\}} \left\{u(c_t) + \beta V\left(k_{t+1}\right)\right\}$$

Using the budget constraint, we can plug in for c_t and get

$$V(k_t) = \max_{\{k_{t+1}\}} \left\{ u\Big((1-\tau_t)r_t k_t + w_t - k_{t+1} + (1-\delta)k_t\Big) + \beta V(k_{t+1}) \right\}$$

The FOC is

- For k_{t+1} : $\quad u'(c_t) = \beta V_1(k_{t+1})$

Then we need to use the ***envelope theorem*** to derive the Euler equations. We get

$$V_1(k_t) = u'(c_t)\Big[(1-\tau_t)r_t + (1-\delta)\Big]$$

Update the value function for 1 period:

$$V_1(k_{t+1}) = u'(c_{t+1})\Big[(1-\tau_{t+1})r_{t+1} + (1-\delta)\Big]$$

Combine the FOC and the envelope condition to get the Euler equation for k_{t+1}:

$$u'(c_t) = \beta u'(c_{t+1})\Big[(1-\tau_{t+1})r_{t+1} + (1-\delta)\Big] \tag{3}$$

(b) The profit maximization problem of firm is as follows

$$\max_{\{k_t, n_t\}} F(k_t, n_t) - r_t k_t - w_t n_t$$

The FOCs are

- For k_t : $\quad r_t = F_1(k_t, n_t)$
- For n_t : $\quad w_t = F_2(k_t, n_t)$

In order to find the equilibrium rental rate of capital and the wage rate, we write down the market clearing conditions:

(i) $n_t = 1$

(ii) $k_t = k_{t+1}$

and the equilibrium rental rate of capital and the wage rate are

$$r^* = F_1(k^*, 1) \quad \text{and} \quad w^* = F_2(k^*, 1)$$

(c) Assuming $f(k) = k^\alpha$, and $\tau_t = \tau_{t+1} = \cdots = \tau$, we use the Euler equation and the equilibrium rental rate of capital to derive the steady state capital stock as a function of τ. We can rewrite r_t as

$$r_t = F_1(k_t, 1) = f'(k_t) = \alpha k_t^{\alpha-1}$$

and plugging in for r_t in (3), we get

$$u'(c_t) = \beta u'(c_{t+1})\Big[(1-\tau_{t+1})\alpha k_{t+1}^{\alpha-1} + (1-\delta)\Big]$$

We know that the steady-state level of consumption is $c_t = c_{t+1} = \cdots = c^*$ and the tax rate does not change in the steady state. Then, we obtain

$$1 = \beta\Big[(1-\tau)\alpha(k^*)^{\alpha-1} + (1-\delta)\Big]$$

Solving for k^*, the steady-state capital stock is

$$k^* = \left(\frac{\alpha(1-\tau)}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

(d) We need to find the optimal tax rate to maximize the tax revenue, $\tau r^* k^*$. The maximization problem can be written as

$$\max_{\{\tau\}} \tau r^* k^*$$

We plug in for r^* and k^*. The maximization problem becomes

$$\max_{\{\tau\}} \tau\alpha(k^*)^{\alpha-1}k^* \quad \Longrightarrow \quad \max_{\{\tau\}} \tau\alpha(k^*)^{\alpha} \quad \Longrightarrow \quad \max_{\{\tau\}} \tau\alpha\left(\frac{\alpha(1-\tau)}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{\alpha}{1-\alpha}}$$

It can be rewritten as follows

$$\max_{\{\tau\}} \tau(1-\tau)^{\frac{\alpha}{1-\alpha}} \cdot \alpha\left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{\alpha}{1-\alpha}}$$

The FOC is

- For τ : $\quad \left[(1-\tau)^{\frac{\alpha}{1-\alpha}} - \tau\frac{\alpha}{1-\alpha}(1-\tau)^{\frac{\alpha}{1-\alpha}-1}\right] \cdot \alpha\left(\frac{\alpha}{\frac{1}{\beta}-(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} = 0$

Then, we solve for τ

$$(1-\tau)^{\frac{\alpha}{1-\alpha}} = \tau\frac{\alpha}{1-\alpha}(1-\tau)^{\frac{\alpha}{1-\alpha}-1} \quad \Longrightarrow \quad \frac{\tau}{1-\tau} = \frac{1-\alpha}{\alpha}$$

Hence, the optimal tax rate is $\tau^* = 1-\alpha$.