STA250 Probability and Statistics

Chapter 12 Notes

Hypothesis Testing 3

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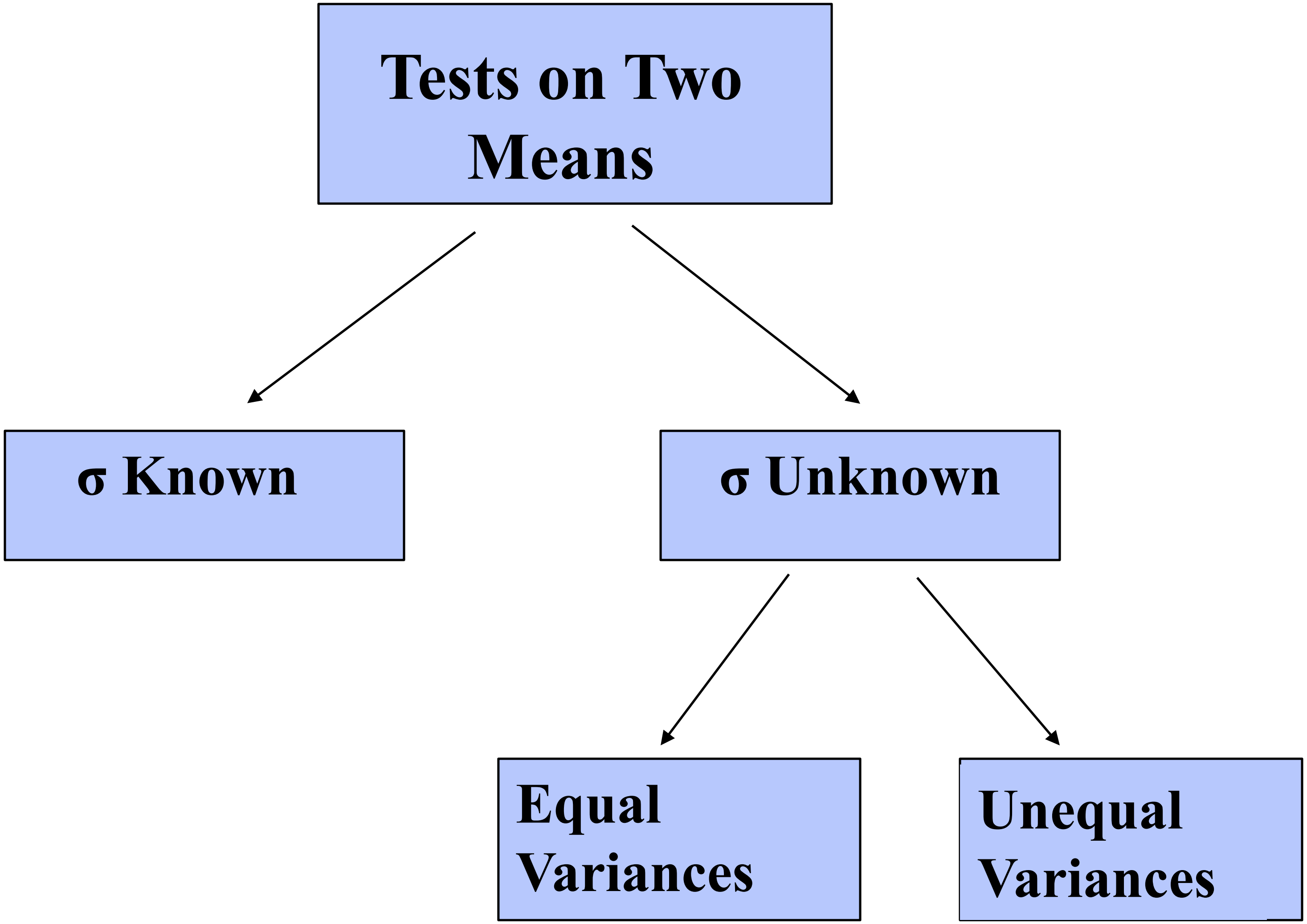
STA250 Probability and Statistics

Reference Book

This lecture notes are prepared according to the contents of

“**PROBABILITY & STATISTICS FOR ENGINEERS & SCIENTISTS** by Walpole, Myers, Myers and Ye”

Hypothesis Tests for the Two Means



Hypothesis Tests for the Two Means (σ Known)

- **Suppose we have an unknown population distribution with mean μ and known variance σ^2, and the two-sided hypothesis on two means can be written generally as,**

  - $H_0: \mu_1 - \mu_2 = d_0$
  - $H_1: \mu_1 - \mu_2 \neq d_0$

  The test statistics is given by,

$$\boxed{z = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2 / n_1 + \sigma_2^2 / n_2}} > z_{\alpha/2} \quad or \quad z < -z_{\alpha/2}}$$

then reject H_0.

- For a single-sided test with $H_1: \mu > \mu_0$, reject H_0 if $z > z_\alpha$.
- For a single-sided test with $H_1: \mu < \mu_0$, reject H_0 if $z < -z_\alpha$.

Hypothesis Tests for the Two Means (σ UnKnown but Equal Variances)

- **Suppose that both distribution are normal and $\sigma_1 = \sigma_2 = \sigma$, *the pooled t-test (often called the two-samples t-test)* may be used.**

For the two-sided hypothesis

$$H_0\colon\ \mu_1 = \mu_2,$$
$$H_1\colon\ \mu_1 \neq \mu_2,$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}},$$

where

$$s_p^2 = \frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

- **Critical Region:**

H_1:	$\mu_1 - \mu_2 < d_0$	$t < -t_\alpha$
H_1:	$\mu_1 - \mu_2 > d_0$	$t > t_\alpha$
H_1:	$\mu_1 - \mu_2 \neq d_0$	$t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

$\Rightarrow$ **H_0 is Reject**

Example

An experiment was performed to compare the abrasive wear of two different laminated materials. Twelve pieces of material 1 were tested by exposing each piece to a machine measuring wear. Ten pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The samples of material 1 gave an average (coded) wear of 85 units with a sample standard deviation of 4, while the samples of material 2 gave an average of 81 with a sample standard deviation of 5. Can we conclude at the 0.05 level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be approximately normal with equal variances.

Solution

Let μ_1 and μ_2 represent the population means of the abrasive wear for material 1 and material 2, respectively.

1. H_0: $\mu_1 - \mu_2 = 2$.
2. H_1: $\mu_1 - \mu_2 > 2$.
3. $\alpha = 0.05$.
4. Critical region: $t > 1.725$, where $t = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$ with $v = 20$ degrees of freedom.

Solution

continue...

5. Computations:

$$\bar{x}_1 = 85, \qquad s_1 = 4, \qquad n_1 = 12,$$
$$\bar{x}_2 = 81, \qquad s_2 = 5, \qquad n_2 = 10.$$

Hence

$$s_p = \sqrt{\frac{(11)(16) + (9)(25)}{12 + 10 - 2}} = 4.478,$$
$$t = \frac{(85 - 81) - 2}{4.478\sqrt{1/12 + 1/10}} = 1.04,$$
$$P = P(T > 1.04) \approx 0.16. \qquad \text{(See Table A.4.)}$$

6. Decision: Do not reject H_0. We are unable to conclude that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units.

Hypothesis Tests for the Two Means (σ UnKnown but Unequal Variances)

- **Suppose that both distribution are normal and $\boldsymbol{\sigma_1 \neq \sigma_2}$.**

For the two-sided hypothesis

$$H_0\colon\ \mu_1 = \mu_2,$$
$$H_1\colon\ \mu_1 \neq \mu_2,$$

- **The test statistics is given by,**

$$T' = \frac{(\bar{X}_1 - \bar{X}_2) - d_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

has an approximate t-distribution with approximate degrees of freedom

$$v = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1-1) + (s_2^2/n_2)^2/(n_2-1)}.$$

- **Critical Region:**

$H_1\colon\ \mu_1 - \mu_2 < d_0$ $\quad t < -t_\alpha$

$H_1\colon\ \mu_1 - \mu_2 > d_0$ $\quad t > t_\alpha$

$H_1\colon\ \mu_1 - \mu_2 \neq d_0$ $\quad t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

⇨ $\boldsymbol{H_0}$ **is Reject**

Hypothesis Tests for the Two Means (Paired Observations)

- **Samples are not independent on the paired t-test. More than one sample is taken from the same units.**
- **The two-sided hypothesis,**
  - $H_0: \mu_D = d_0$
  - $H_0: \mu_D \neq d_0$
- **The test statistics is given by,**

$$t = \frac{\overline{d} - d_0}{s_d/\sqrt{n}}.$$

Critical regions are constructed using the t-distribution with $n-1$ degrees of freedom.

- **Critical Region:**

H_1:	$\mu_D < d_0$	$t < -t_\alpha$
H_1:	$\mu_D > d_0$	$t > t_\alpha$
H_1:	$\mu_D \neq d_0$	$t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

$\boldsymbol{H_0}$ **is Reject**

Example:

Blood Sample Data: In a study conducted in the Forestry and Wildlife Department at Virginia Tech, J. A. Wesson examined the influence of the drug succinylcholine on the circulation levels of androgens in the blood. Blood samples were taken from wild, free-ranging deer immediately after they had received an intramuscular injection of succinylcholine administered using darts and a capture gun. A second blood sample was obtained from each deer 30 minutes after the first sample, after which the deer was released. The levels of androgens at time of capture and 30 minutes later, measured in nanograms per milliliter (ng/mL), for 15 deer are given in Table 10.2.

Assuming that the populations of androgen levels at time of injection and 30 minutes later are normally distributed, test at the 0.05 level of significance whether the androgen concentrations are altered after 30 minutes.

Example: continue...

Table 10.2: Data for Case Study 10.1

Deer	Androgen (ng/mL) At Time of Injection	Androgen (ng/mL) 30 Minutes after Injection	d_i
1	2.76	7.02	4.26
2	5.18	3.10	−2.08
3	2.68	5.44	2.76
4	3.05	3.99	0.94
5	4.10	5.21	1.11
6	7.05	10.26	3.21
7	6.60	13.91	7.31
8	4.79	18.53	13.74
9	7.39	7.91	0.52
10	7.30	4.85	−2.45
11	11.78	11.10	−0.68
12	3.90	3.74	−0.16
13	26.00	94.03	68.03
14	67.48	94.03	26.55
15	17.04	41.70	24.66

Solution:

Let μ_1 and μ_2 be the average androgen concentration at the time of injection and 30 minutes later, respectively. We proceed as follows:

1. H_0: $\mu_1 = \mu_2$ or $\mu_D = \mu_1 - \mu_2 = 0$.
2. H_1: $\mu_1 \neq \mu_2$ or $\mu_D = \mu_1 - \mu_2 \neq 0$.
3. $\alpha = 0.05$.
4. Critical region: $t < -2.145$ and $t > 2.145$, where $t = \frac{\bar{d}-d_0}{s_D/\sqrt{n}}$ with $v = 14$ degrees of freedom.
5. Computations: The sample mean and standard deviation for the d_i are

$$\bar{d} = 9.848 \quad \text{and} \quad s_d = 18.474.$$

   Therefore,

$$t = \frac{9.848 - 0}{18.474/\sqrt{15}} = 2.06.$$

6. Though the t-statistic is not significant at the 0.05 level, from Table A.4,

$$P = P(|T| > 2.06) \approx 0.06.$$

   As a result, there is some evidence that there is a difference in mean circulating levels of androgen.

Two Samples: Tests on Two Proportions

- **For n_1 and n_2 sufficiently large, that the point estimator $\hat{p}_1 - \hat{p}_2$ was approximately normally distributed with mean,**

$$\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2$$

and variance

$$\sigma^2_{\hat{p}_1-\hat{p}_2} = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$$

Upon pooling the data from both samples, **the pooled estimate of the proportion p** is,

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

Two Samples: Tests on Two Proportions

- **where x_1 *and* x_2 are the number of successes in each of the two samples. The z-value for testing $p_1 = p_2$ is determined from the formula,**

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

- $H_1: p_1 \neq p_2$ $\Longrightarrow$ the critical region is $z < -z_{\frac{\alpha}{2}}$ *or* $z > z_{\alpha/2}$
- $H_1: p_1 < p_2$ $\Longrightarrow$ the critical region is $z < -z_{\alpha}$
- $H_1: p_1 > p_2$ $\Longrightarrow$ the critical region is $z > z_{\alpha}$

Example

A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. The construction site is within the town limits, and for this reason many voters in the county believe that the proposal will pass because of the large proportion of town voters who favor the construction. To determine if there is a significant difference in the proportions of town voters and county voters favoring the proposal, a poll is taken. If 120 of 200 town voters favor the proposal and 240 of 500 county residents favor it, would you agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters? Use an $\alpha = 0.05$ level of significance.

Solution

Let p_1 and p_2 be the true proportions of voters in the town and county, respectively, favoring the proposal.

1. H_0: $p_1 = p_2$.
2. H_1: $p_1 > p_2$.
3. $\alpha = 0.05$.
4. Critical region: $z > 1.645$.
5. Computations:

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{120}{200} = 0.60, \quad \hat{p}_2 = \frac{x_2}{n_2} = \frac{240}{500} = 0.48, \quad \text{and}$$
$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{120 + 240}{200 + 500} = 0.51.$$

Therefore,

$$z = \frac{0.60 - 0.48}{\sqrt{(0.51)(0.49)(1/200 + 1/500)}} = 2.9,$$
$$P = P(Z > 2.9) = 0.0019.$$

6. Decision: Reject H_0 and agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters.

Best Wishes

- **Thank you for your attention…**

See you☺