Fuzzy Relations

Murat Osmanoglu

Crisp Relations

- Cartesian product AxB defined as

  $$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

- $A_1 \times A_2 \times \ldots \times A_n$ consists of all n-tuples $(a_1,a_2,\ldots,a_n)$

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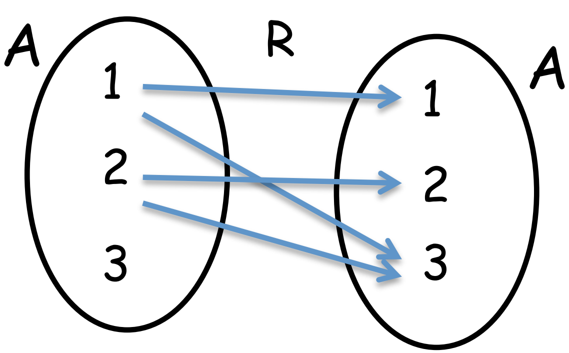

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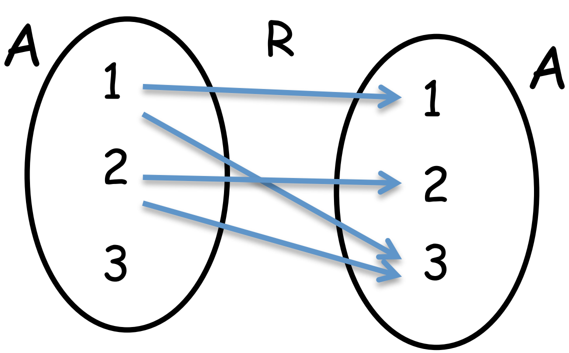

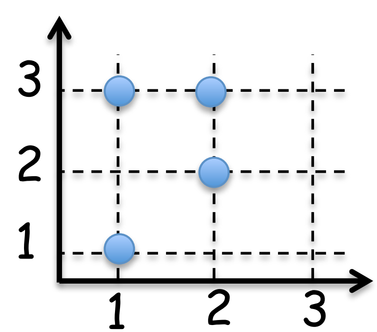

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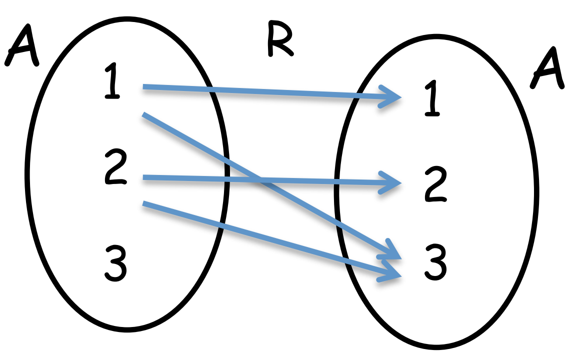

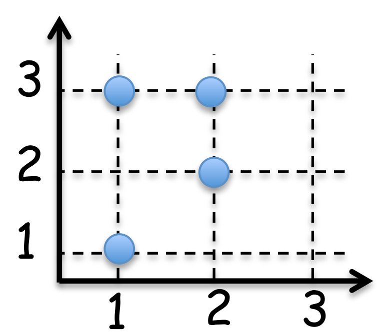

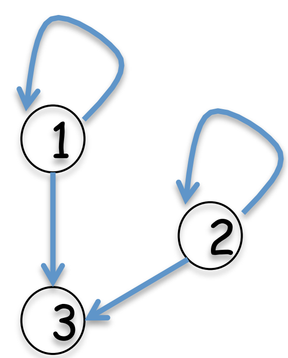

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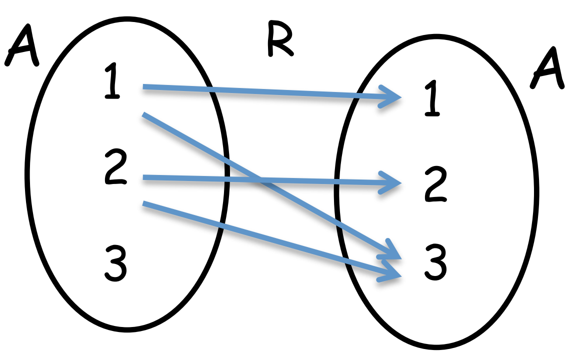

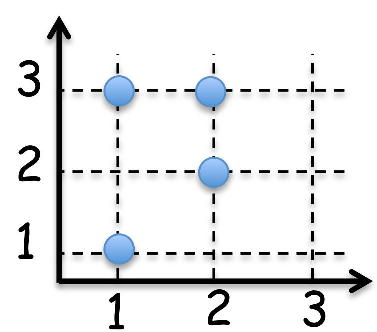

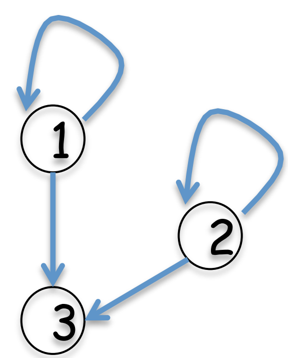

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- for all $x \in A$, there exists some $y \in B$ s.t. $(x,y) \in R$
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Crisp Relations

Operations on Relations

- Complement, $R \subseteq A \times B$, $\neg R = \{(x,y) \mid (x,y) \notin R\}$
- Intersection, $R,S \subseteq A \times B$, $R \cap S = \{(x,y) \mid (x,y) \in R \text{ and } (x,y) \in S\}$
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- Composition, $R \subseteq AxB$, $S \subseteq BxC$, $\subseteq AxC$,

  $SoR = \{(x,z) \mid (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in B\}$

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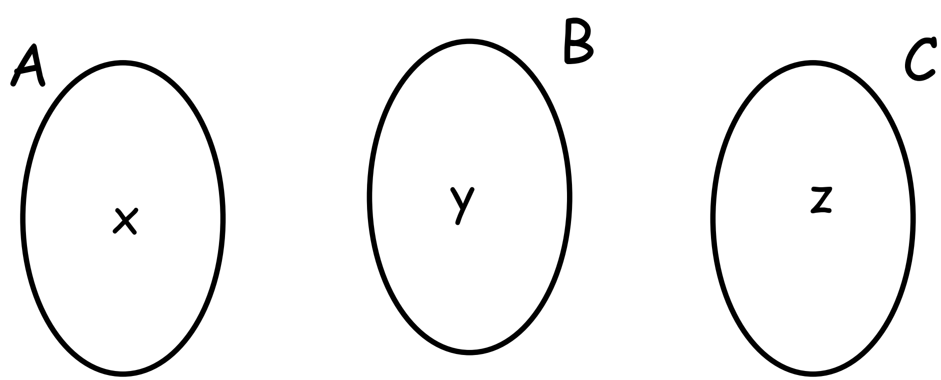

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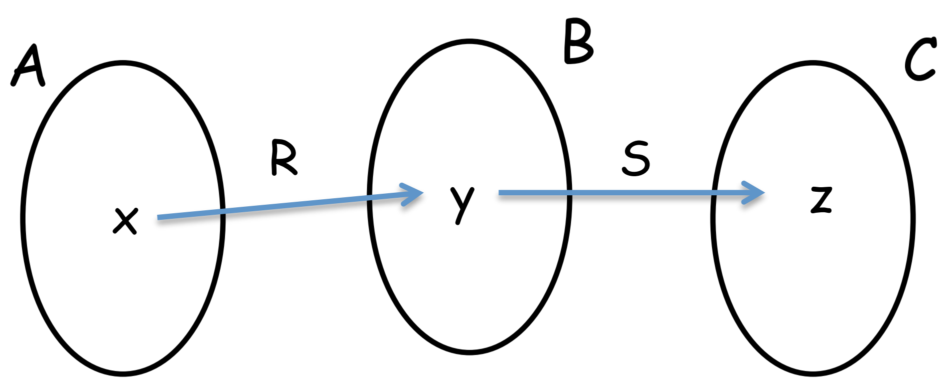
- Composition, $R \subseteq A\times B$, $S \subseteq B\times C$, $\subseteq A\times C$,

  $S\circ R = \{(x,z) \mid (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in B\}$



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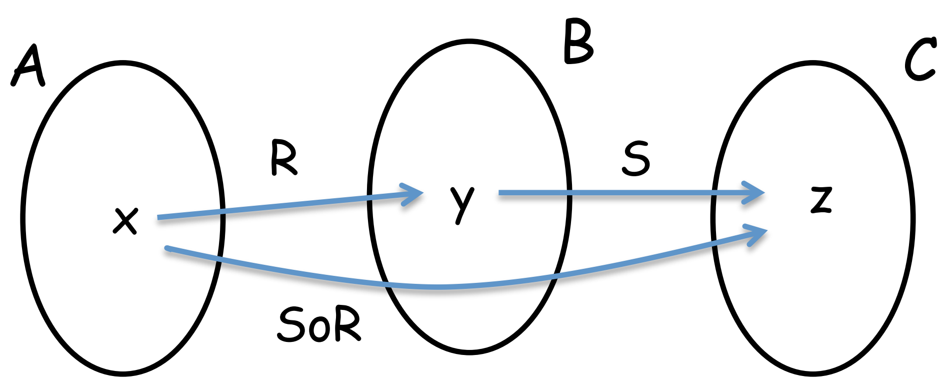

Crisp Relations

Operations on Relations

Let $A = \{1,2,3\}$, $B = \{a,b\}$, $C = \{x,y,z\}$ and $R \subseteq A\times B$, $S \subseteq B\times C$

R	a	b
1	1	0
2	0	1
3	0	0

S	x	y	z
a	1	0	1
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$$(u_1 \wedge v_1) \vee (u_2 \wedge v_2) \vee \ldots \vee (u_n \wedge v_n)$$

SoR	x	y	z
1			
2			
3			

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Let A = {1,2,3}, B = {a,b}, C = {x,y,z} and R $\subseteq$ AxB, S $\subseteq$ BxC

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$(1 \wedge 1) \vee (0 \wedge 0)$

SoR	x	y	z
1	1		
2			
3			

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R^{-1}	1	2	3
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reflexive

not symmetric

not anti-symmetric

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- Anti-symmetry, $R \subseteq A \times A$, R is anti-symmetric if $(y,x) \notin R$ for all $(x,y) \in R$ and $x \neq y$
- Transitivity, $R \subseteq A \times A$, R is transitive if $(x,z) \in R$ for all $(x,y) \in R$ and $(y,z) \in R$,

R	1	2	3
1	1	0	1
2	0	1	1
3	0	1	1

reflexive

not symmetric

not anti-symmetric

$(1,3) \in R$ and $(3,2) \in R$, but $(1,2) \notin R$

Crisp Relations

Properties of Relations

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- if a relation is reflexive, symmetric, and transitive, it's called equivalence relation
- if a relation is reflexive and symmetric, it's called compatibility relation
- if a relation is reflexive and transitive, it's called pre-order relation
- if a relation is reflexive, anti-symmetric, and transitive, it's called order relation

Fuzzy Relation

- $R = \{((x,y), \mu_R) \mid \mu_R(x,y) \geq 0\}$ where $R \subseteq A \times B$
- $\mu_R(x,y) \in [0,1]$, degree of relationship between x and y
  (strongly related or weakly related)

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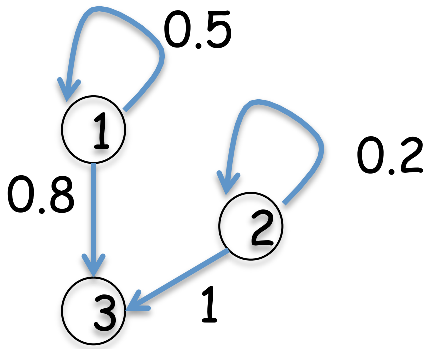

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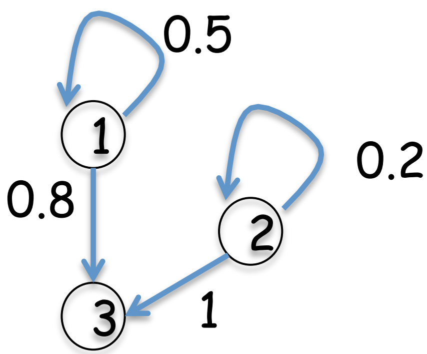

R	1	2	3
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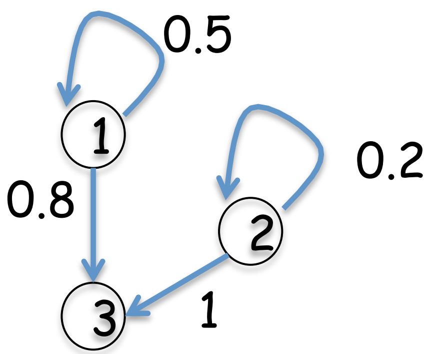

R	1	2	3
1	0.5	0	0.8
2	0	0.2	1
3	0	0	0

R	İst	Ank	Kon
İst	0	0.4	0.7
Ank	0.4	0	0.3
Kon	0.7	0.3	0

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- Cartesian product AxB defined as

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- $\mu_{A \times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$

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- $\mu_{codom(R)}(y) = \max_x \{\mu_R(x,y)\}$

Operations on Fuzzy Relations

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- Inverse, $R \subseteq AxB$, $R^{-1} \subseteq BxA$, $T = R^{-1}$, $\mu_T(x,y) = \mu_R(x,y)$

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- Composition, $R \subseteq AxB$, $S \subseteq BxC$ $T = SoR$

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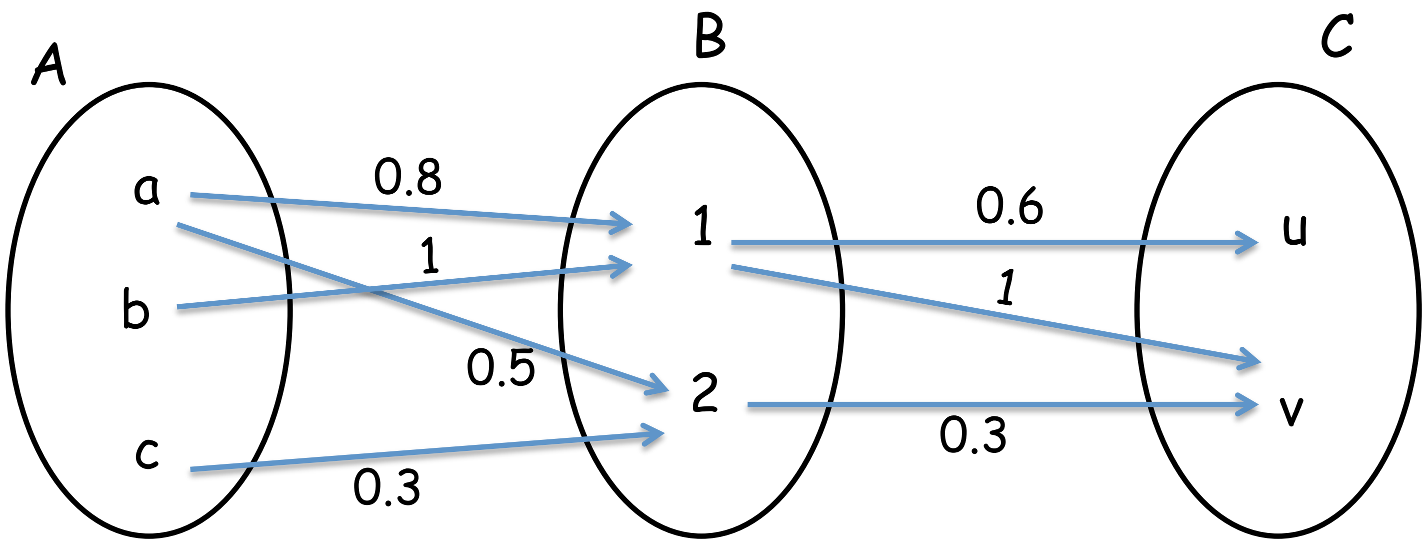

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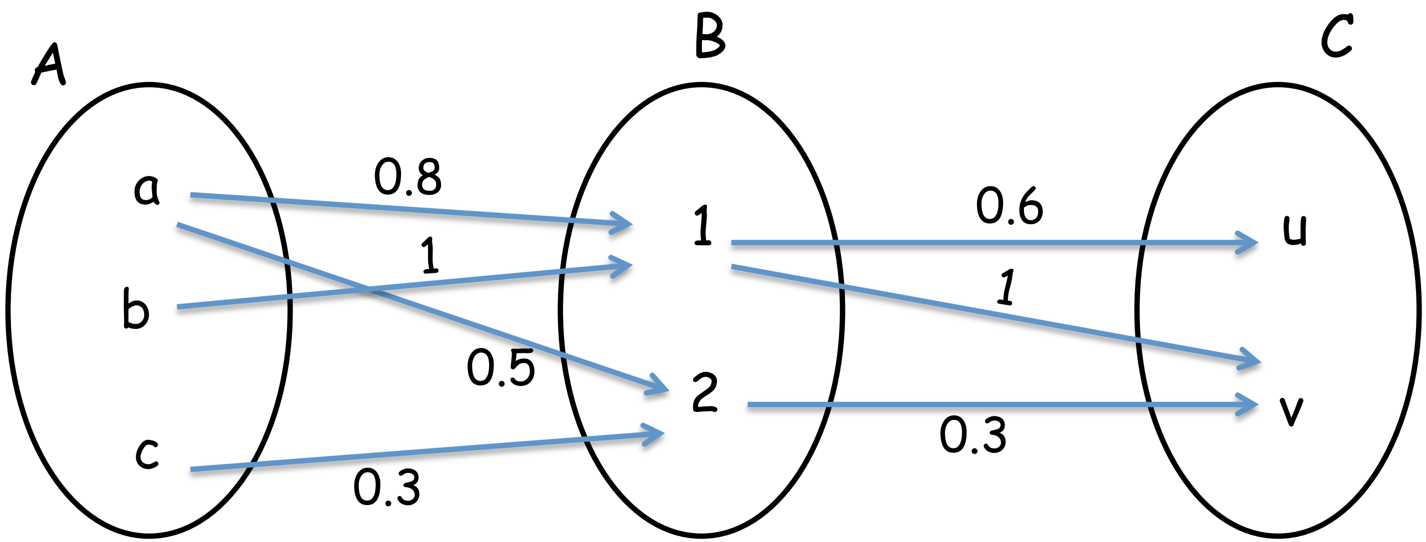

$$\mu_T(x,z) = \{((a,u),), ((a,v),), ((b,u),), ((b,v),), ((c,u),), ((c,v),)\}$$

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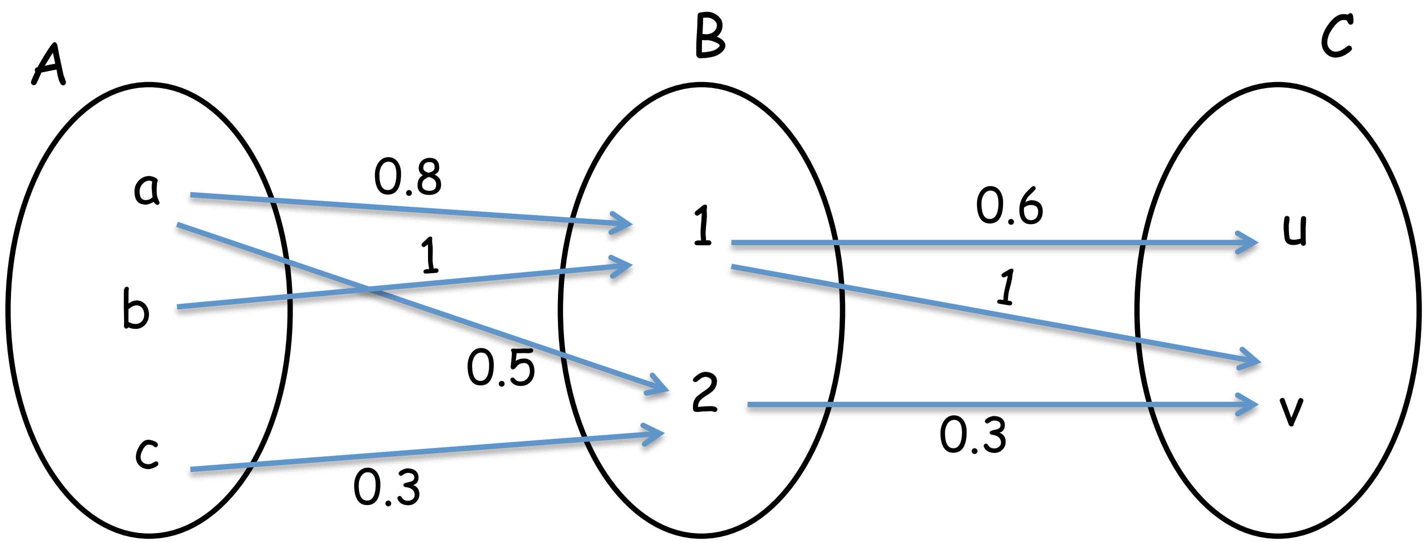

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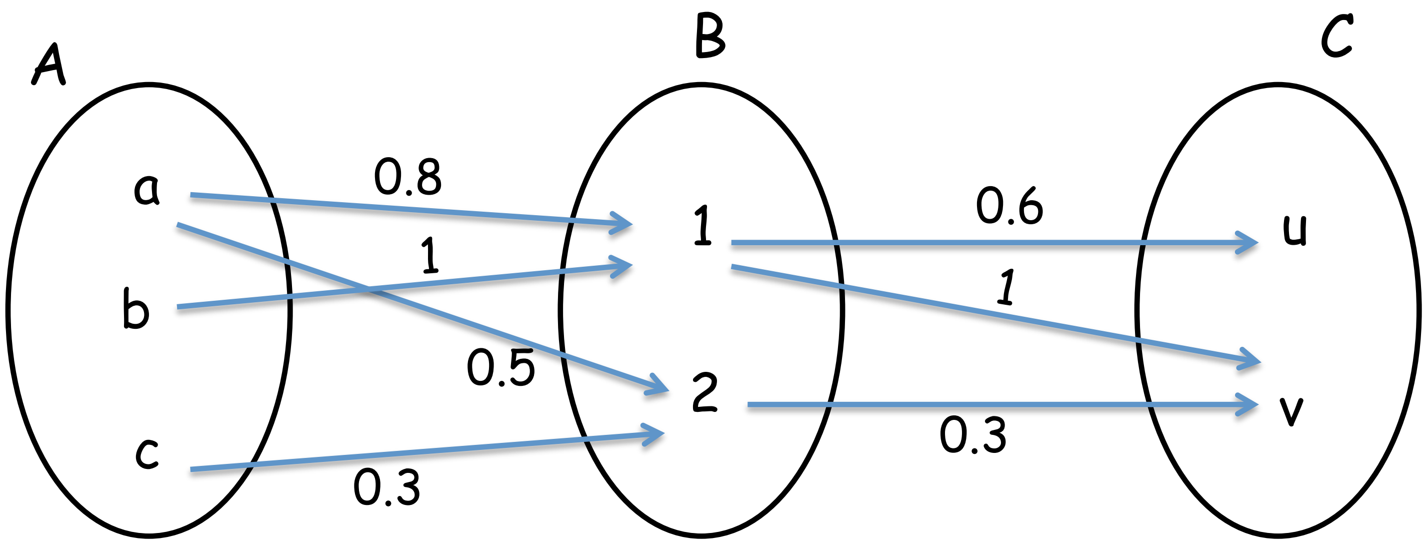

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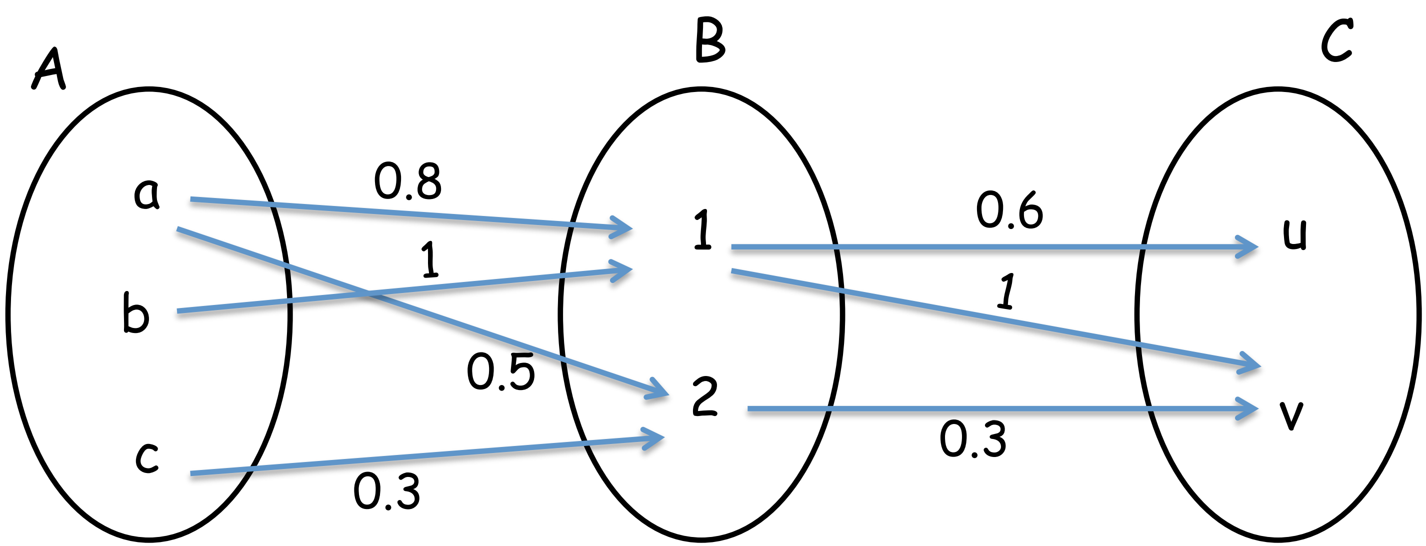

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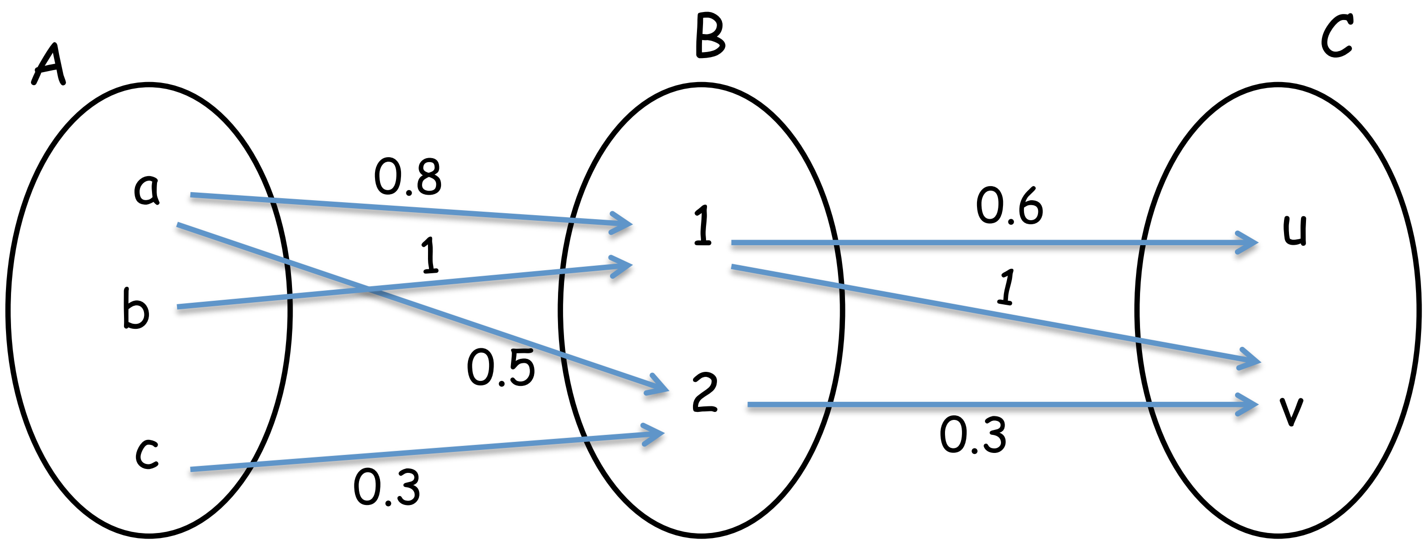

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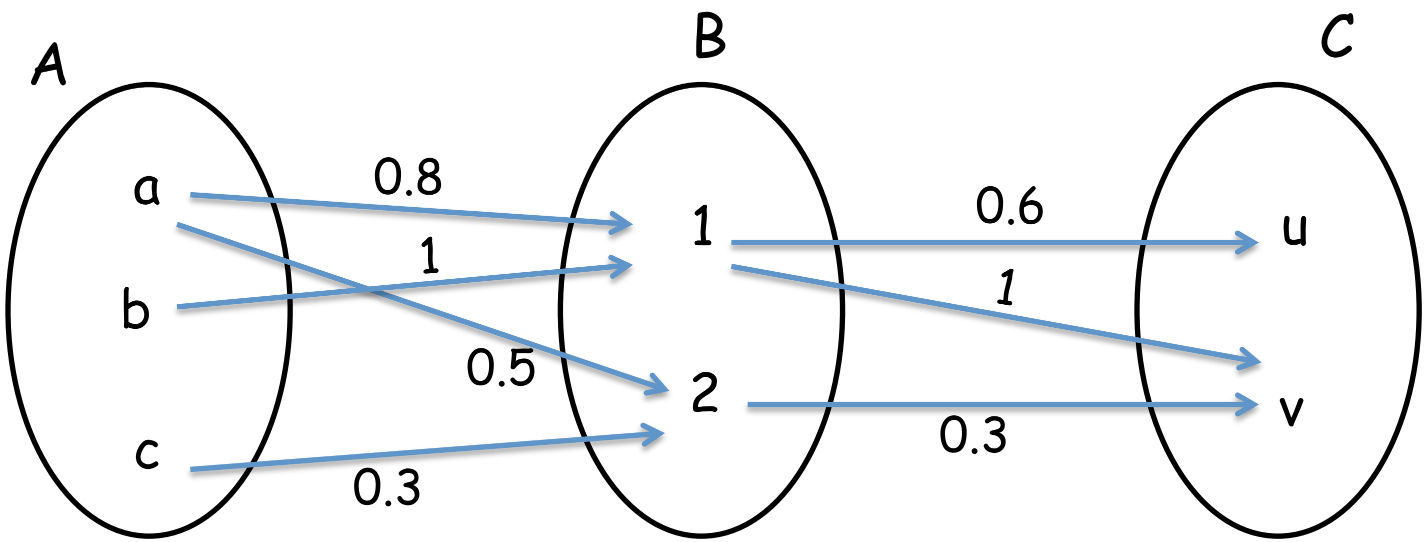

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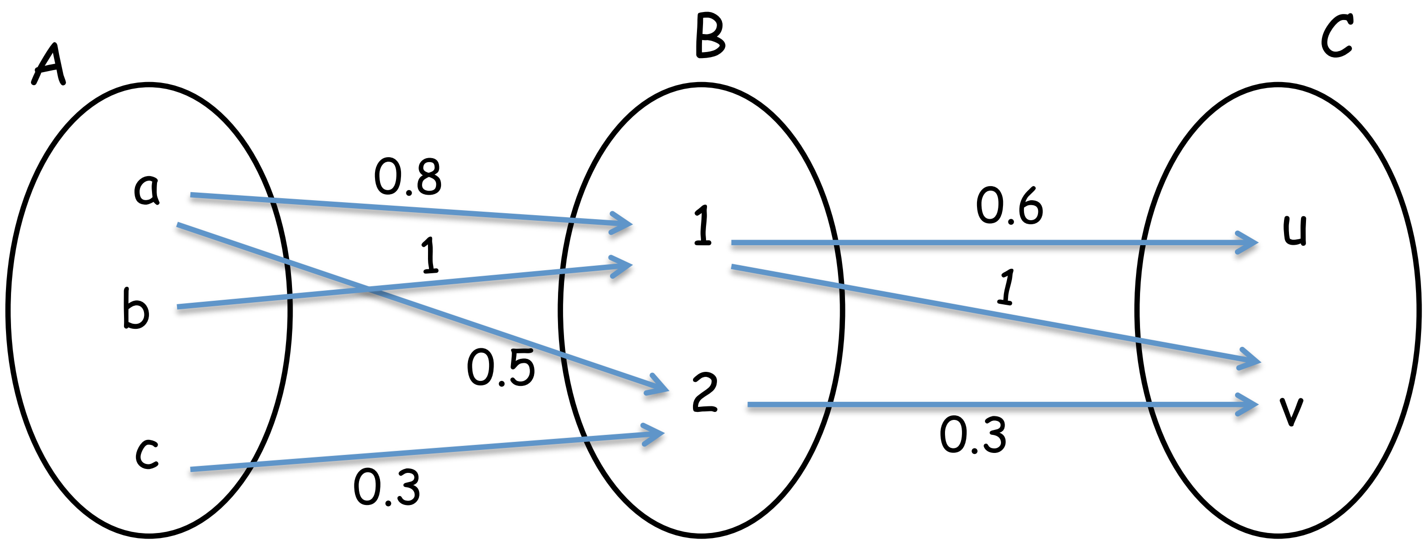

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$$= 0.3 \times \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cup 0.5 \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cup 0.9 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

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R_A	
a	1
b	1
c	0.7

R_B	1	2	3
	1	0.8	1

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- Let A be a fuzzy set, B be a fuzzy or crisp set, and $R \subseteq A \times B$ is a crisp relation

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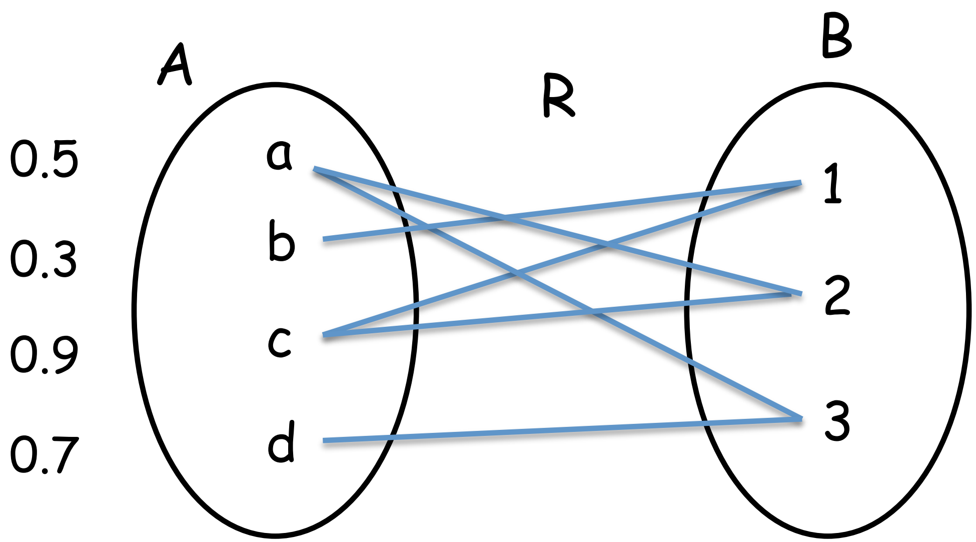

Extension of Fuzzy Sets

- Let A be a fuzzy set, B be a fuzzy or crisp set, and R $\subseteq$ AxB is a crisp relation

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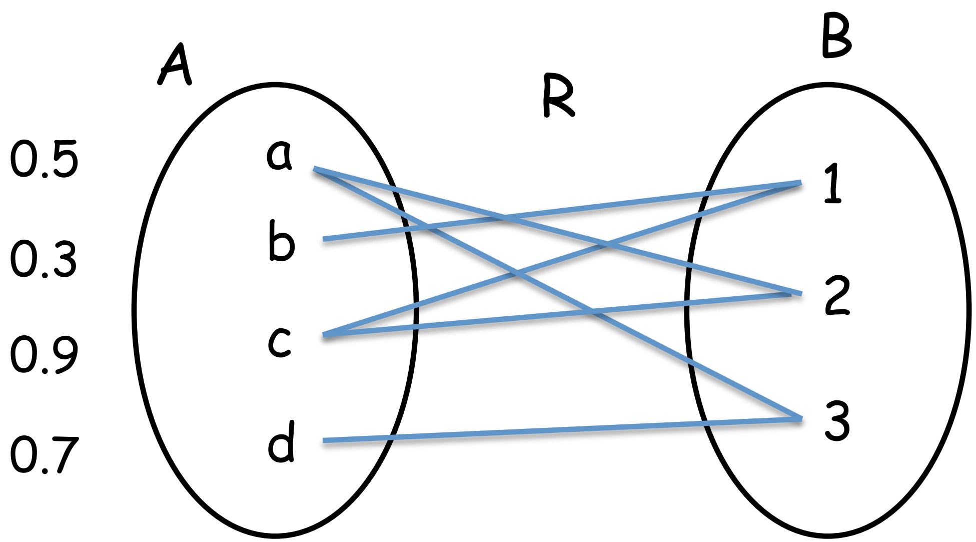

A = 'people having a particular disease'

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Extension of Fuzzy Sets

- Let *A* be a fuzzy set, B be a fuzzy or crisp set, and R $\subseteq$ AxB is a crisp relation
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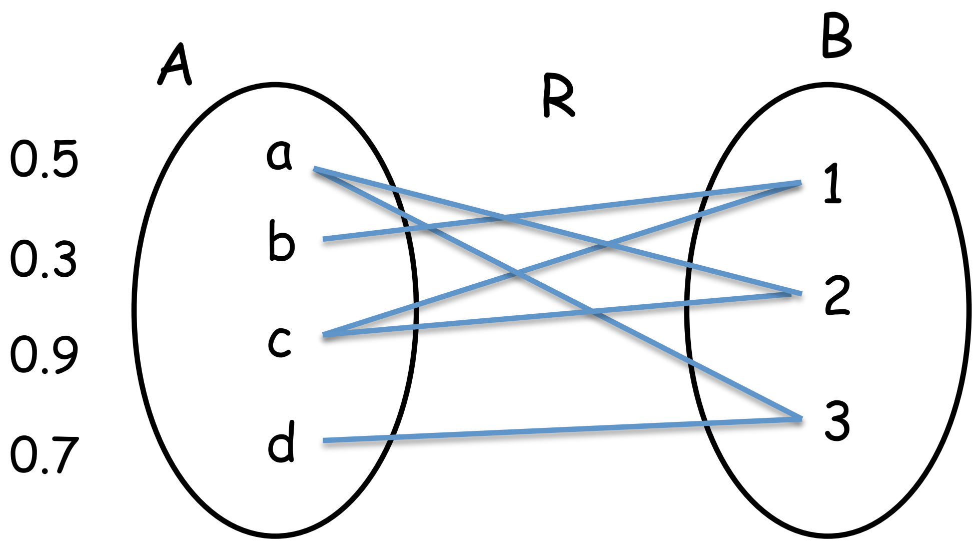

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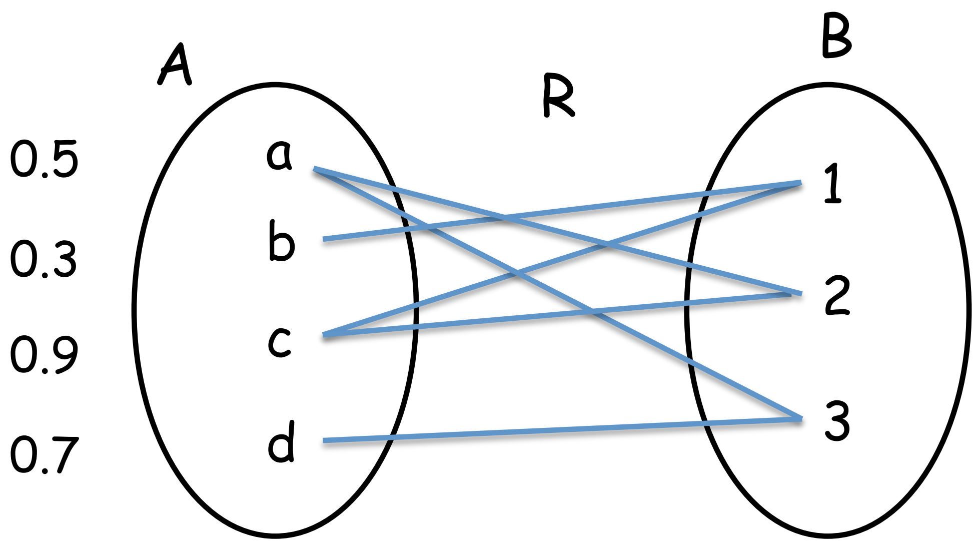

A = 'people having a particular disease'

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B′ = {(1, 0.9), (2,), (3,)}

max $\{\mu_A(b), \mu_A(c)\}$

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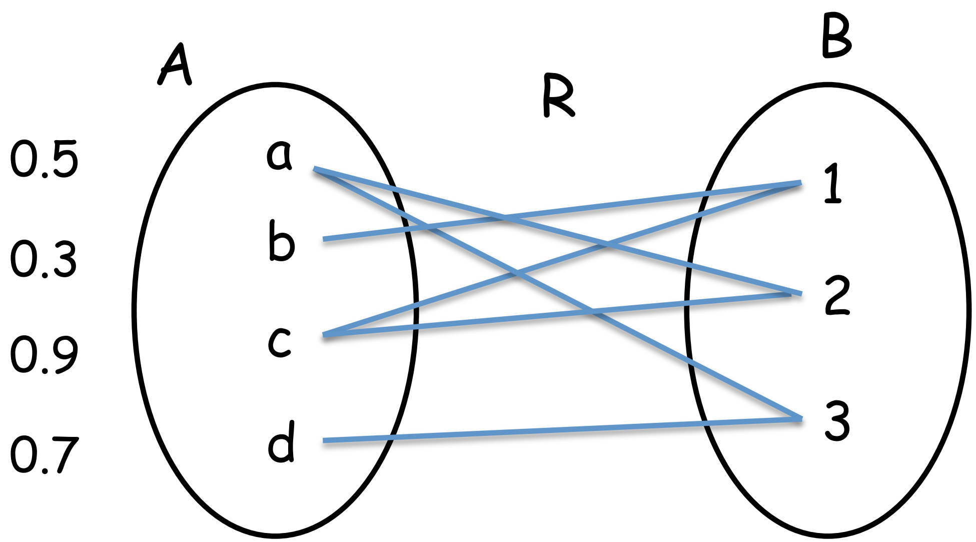

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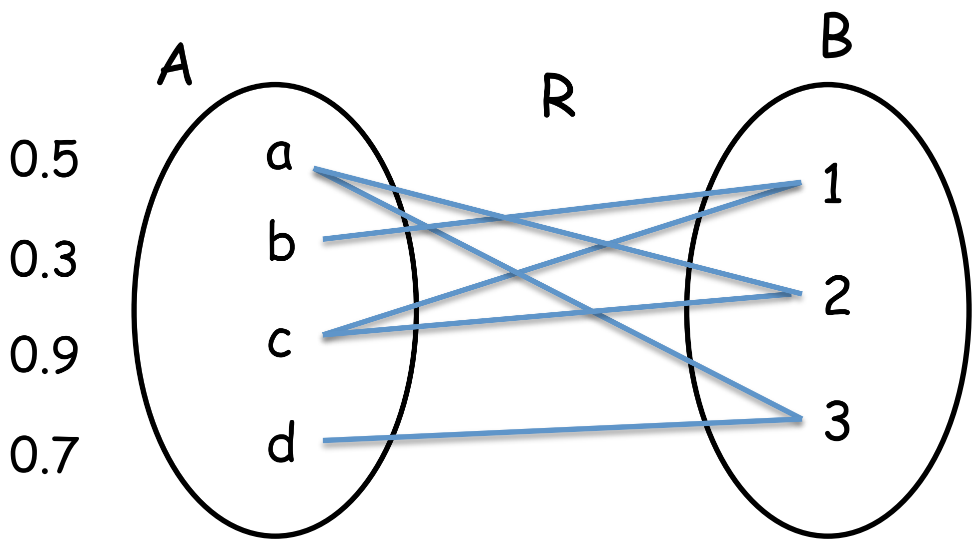

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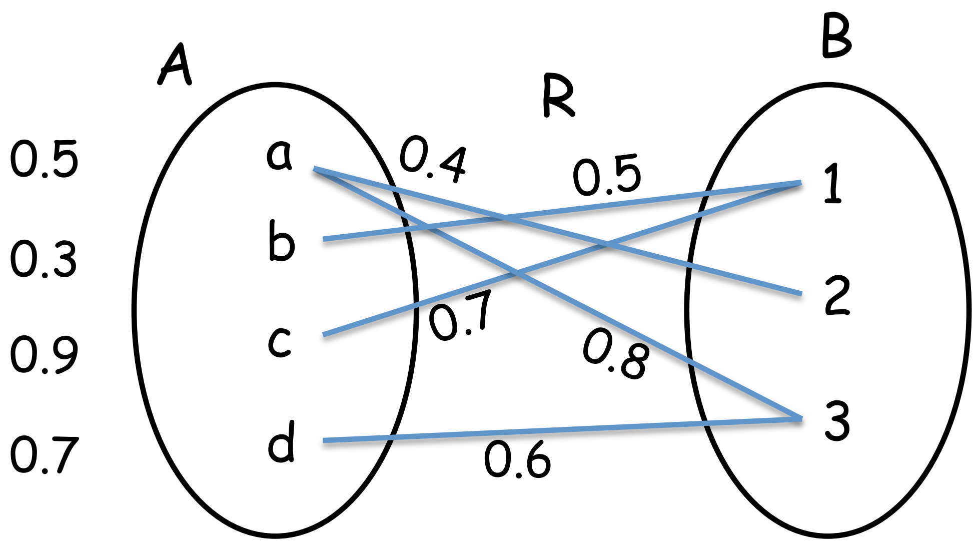

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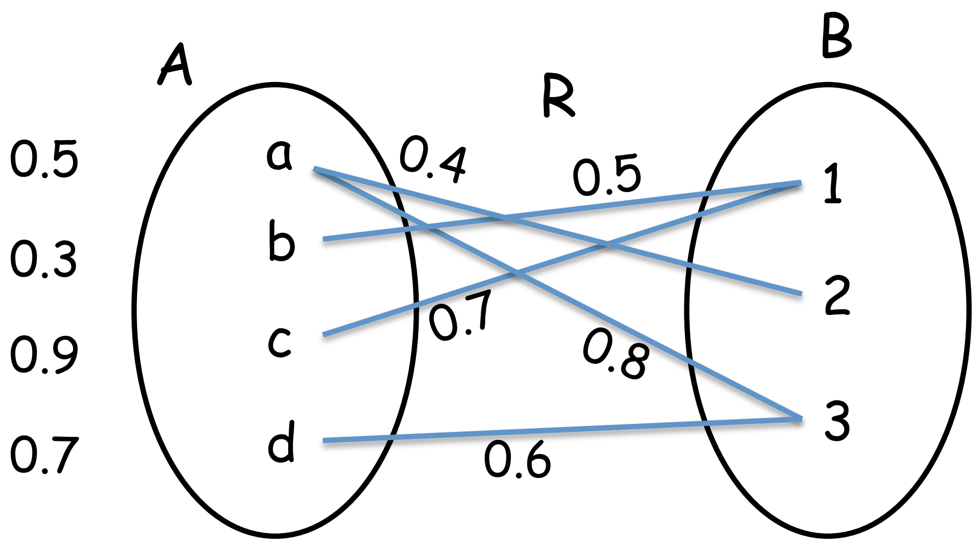

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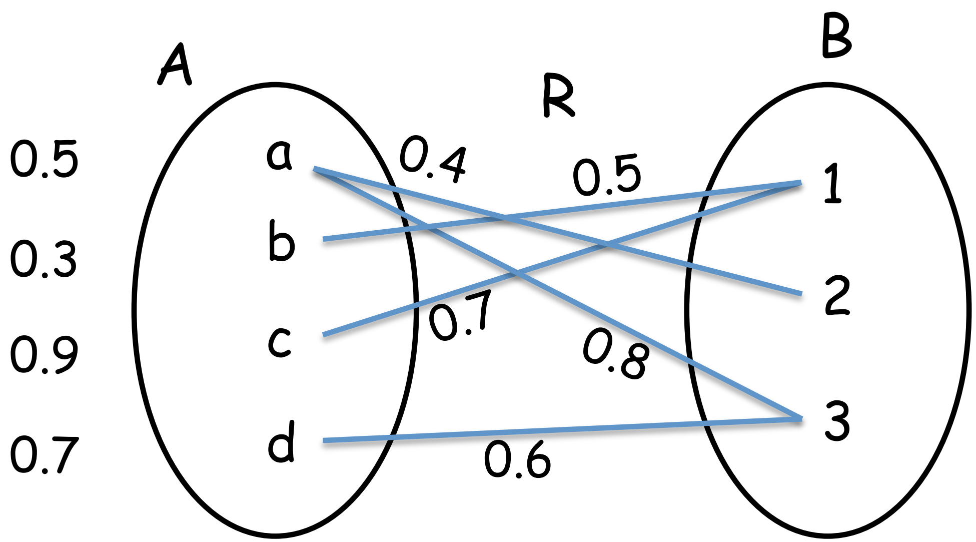

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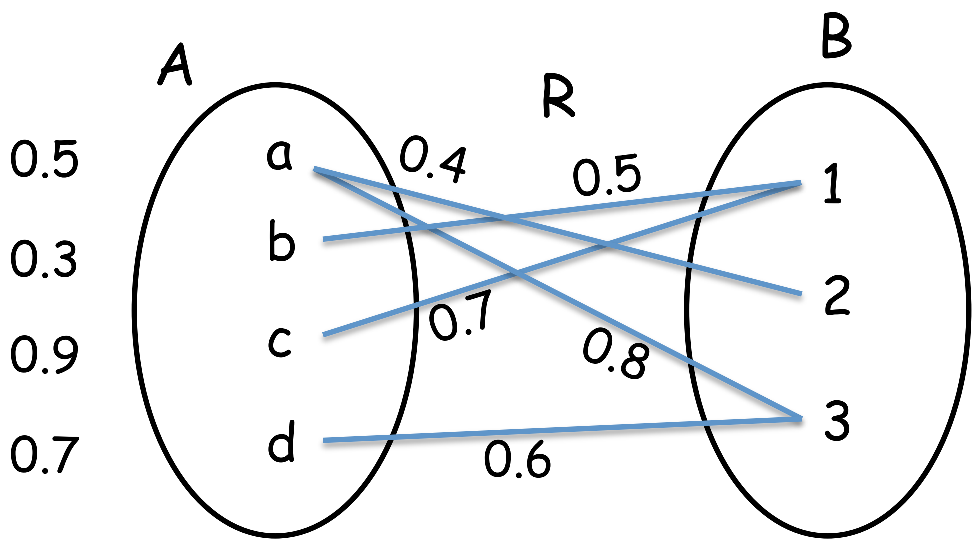

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B' = {(1, 0.7), (2,), (3,)}

$$\max \begin{cases} \min\{\mu_A(b), \mu_R(b,1)\} \\ \min\{\mu_A(c), \mu_R(c,1)\} \end{cases}$$

Extension of Fuzzy Sets

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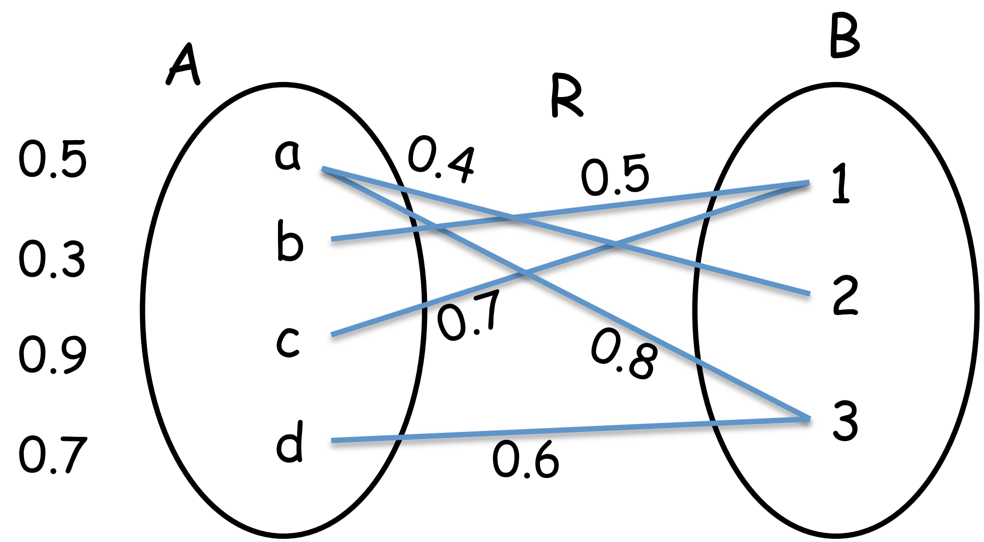

A = 'people having a particular disease'

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B′ = 'possibility of people having been in contact with the sick people getting the disease'

B′ = {(1, 0.7), (2, 0.4), (3,)}

$$\max \left\{ \min\{\mu_A(a), \mu_R(a,2)\} \right.$$

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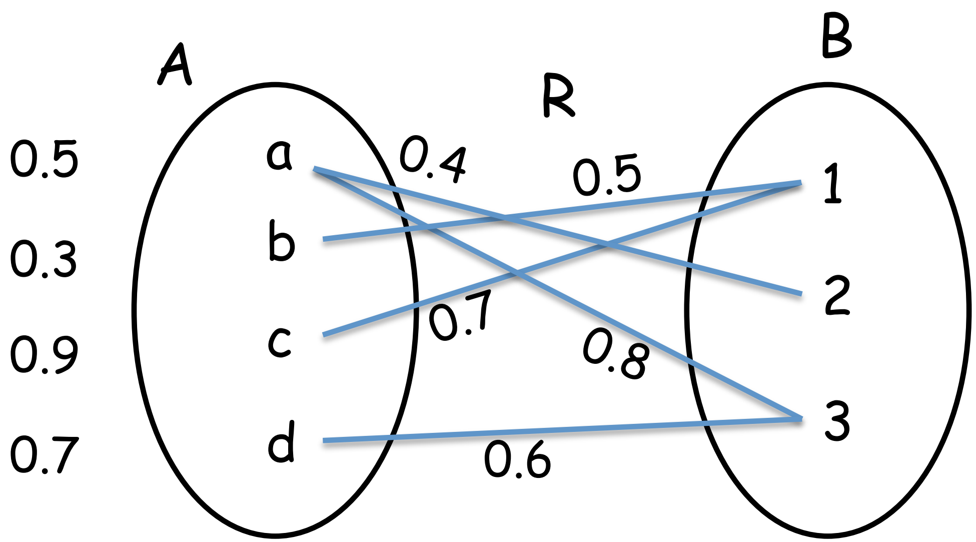

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B' = {(1, 0.7), (2, 0.4), (3, 0.6)}

$$\max \begin{cases} \min\{\mu_A(a), \mu_R(a,3)\} \\ \min\{\mu_A(d), \mu_R(d,3)\} \end{cases}$$

Extension of Fuzzy Sets

- $A = \{(x, 0.4), (y, 0.3), (z, 1.0), (t, 0.7)\}$

  $B = \{a, b, c\}$

R	a	b	c
x	0.3	0	0.8
y	0	1.0	0.6
z	0.7	0	0
t	0.4	0.1	0.5

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- B' induced by A and R

  $B = \{(a, \ ), (b, \ ), (c, \ )\}$

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- B′ induced by A and R

  $B = \{(a, 0.7), (b, 0.3), (c, 0.5)\}$

- B′ induced by A and $R_{0.5}$

  $B = \{(a,), (b,), (c,)\}$

Extension of Fuzzy Sets

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- B′ induced by A and R

  $B = \{(a, 0.7), (b, 0.3), (c, 0.5)\}$

$R_{0.5}$	a	b	c
x	0	0	1
y	0	1	1
z	1	0	0
t	0	0	1

- B′ induced by A and $R_{0.5}$

  $B = \{(a,), (b,), (c,)\}$

Extension of Fuzzy Sets

- $A = \{(x, 0.4), (y, 0.3), (z, 1.0), (t, 0.7)\}$

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R	a	b	c
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- B' induced by A and R

  $B = \{(a, 0.7), (b, 0.3), (c, 0.5)\}$

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z	1	0	0
t	0	0	1

- B′ induced by A and R

  $B = \{(a, 0.7), (b, 0.3), (c, 0.5)\}$

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Fuzzy Distance

- Let A and B be fuzzy sets, the fuzzy distance metric d(A,B) can be defined by the same formula

$$\mu_{d(A,B)}(\delta) = \max_{d(a,b)}\{\min[\mu_A(a), \mu_A(b)]\} \text{ s.t. } \delta = d(a,b)$$

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- A = {(1, 0.3), (2, 1.0), (3, 0.6)} and B = {(3, 0.7), (4, 1.0), (5, 0.4)}

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d(A,B) = {(0,), (1,), (2,), (3,), (4,)}

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$$d(A,B) = \{(0, 0.6), (1, 0.7), (2,), (3,), (4,)\}$$

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$$\max \left\{ \begin{array}{l} \min\{\mu_A(2), \mu_B(3)\} \\ \min\{\mu_A(3), \mu_B(4)\} \end{array} \right.$$

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Properties of Fuzzy Relations

Reflexivity

- Let $R \subseteq A \times A$ be a fuzzy relation. If for all $x \in A$, $\mu_R(x,x) = 1$, then R is called 'reflexive'

Properties of Fuzzy Relations

Reflexivity

- Let $R \subseteq A x A$ be a fuzzy relation. If for all $x \in A$, $\mu_R(x,x) = 1$, then R is called 'reflexive'
- $A = \{0, 1, 2, 3\}$ and $R \subseteq A x A$ $R = \{(x,y) \mid x,y \in A$ and x close to $y\}$

Properties of Fuzzy Relations

Reflexivity

- Let $R \subseteq AxA$ be a fuzzy relation. If for all $x \in A$, $\mu_R(x,x) = 1$, then R is called 'reflexive'
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R	0	1	2	3
0	1.0	0.8	0.6	0.4
1	0.8	1.0	0.8	0.6
2	0.6	0.8	1.0	0.8
3	0.4	0.6	0.8	1.0

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Properties of Fuzzy Relations

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Properties of Fuzzy Relations

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Properties of Fuzzy Relations

Symmetry

- Let $R \subseteq AxA$ be a fuzzy relation. If for all $(x,y) \in AxA$, $\mu_R(x,y) = \mu_R(y,x)$ then R is called 'symmetric'

- If for all $(x,y) \in AxA$ and $x \neq y$, $\mu_R(x,y) \neq \mu_R(y,x)$ or $\mu_R(x,y) = \mu_R(y,x) = 0$ then R is called 'anti-symmetric'

- $A = \{0, 1, 2, 3\}$ and $R,S \subseteq AxA$

Properties of Fuzzy Relations

Symmetry

- Let $R \subseteq A \times A$ be a fuzzy relation. If for all $(x,y) \in A \times A$, $\mu_R(x,y) = \mu_R(y,x)$ then R is called 'symmetric'
- If for all $(x,y) \in A \times A$ and $x \neq y$, $\mu_R(x,y) \neq \mu_R(y,x)$ or $\mu_R(x,y) = \mu_R(y,x) = 0$ then R is called 'anti-symmetric'
- $A = \{0, 1, 2, 3\}$ and $R, S \subseteq A \times A$

R	0	1	2	3
0	1.0	0.8	0.6	0.4
1	0.8	1.0	0.8	0.6
2	0.6	0.8	1.0	0.8
3	0.4	0.6	0.8	1.0

S	0	1	2	3
0	0	0.8	0.3	0
1	0	1.0	0	0
2	0.6	0	1.0	0.5
3	0	0.6	0.8	0.9

Properties of Fuzzy Relations

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- $A = \{0, 1, 2, 3\}$ and $R, S \subseteq A \times A$

R	0	1	2	3
0	1.0	0.8	0.6	0.4
1	0.8	1.0	0.8	0.6
2	0.6	0.8	1.0	0.8
3	0.4	0.6	0.8	1.0

symmetric

S	0	1	2	3
0	0	0.8	0.3	0
1	0	1.0	0	0
2	0.6	0	1.0	0.5
3	0	0.6	0.8	0.9

Properties of Fuzzy Relations

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- If for all $(x,y) \in AxA$ and $x \neq y$, $\mu_R(x,y) \neq \mu_R(y,x)$ or $\mu_R(x,y) = \mu_R(y,x) = 0$ then R is called 'anti-symmetric'
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2	0.6	0.8	1.0	0.8
3	0.4	0.6	0.8	1.0

symmetric

S	0	1	2	3
0	0	0.8	0.3	0
1	0	1.0	0	0
2	0.6	0	1.0	0.5
3	0	0.6	0.8	0.9

anti-symmetric

Properties of Fuzzy Relations

Transitivity

- Let $R \subseteq AxA$ be a fuzzy relation. If for all (x,y), (y,z),$(x,z) \in AxA$, $\mu_R(x,z) \geq max_y\{ min[\mu_R(x,y),\mu_R(y,x)] \}$ then R is called 'transitive'

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- $A = \{0, 1, 2\}$ and $R \subseteq AxA$

R	0	1	2
0	0.2	1.0	0.4
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2	0	1.0	0.3

R^2	0	1	2
0			
1			
2			

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0	0.2		
1			
2			

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1	0	0.6	0.3
2	0	1.0	0.3

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0	0.2	1.0	0.4
1	0	0.6	0.3
2	0	1.0	0.3

R^2	0	1	2
0	0.2	0.6	
1			
2			

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2	0	1.0	0.3

R^2	0	1	2
0	0.2	0.6	0.3
1			
2			

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2	0	1.0	0.3

R^2	0	1	2
0	0.2	0.6	0.3
1	0	0.6	0.3
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Properties of Fuzzy Relations

Fuzzy Equivalence Relation

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R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0

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3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0

reflexive

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2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0

reflexive

symmetric

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2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0

R^2	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0

reflexive

symmetric

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- A = {0, 1, 2, 3, 4} and R $\subseteq$ AxA

R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0

R^2	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0

reflexive

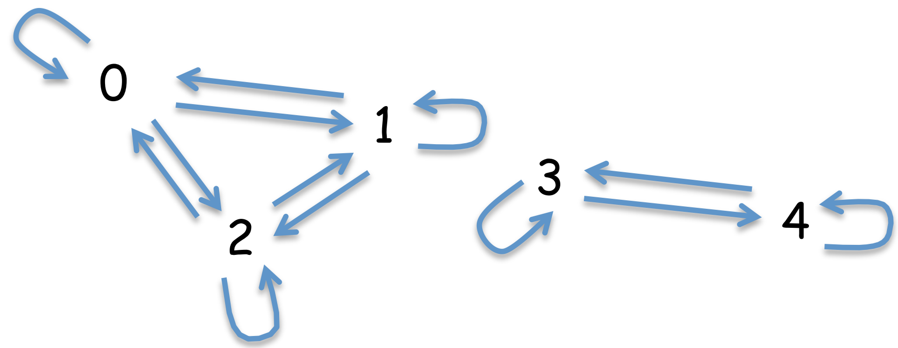
symmetric

transitive

Properties of Fuzzy Relations

Fuzzy Equivalence Relation

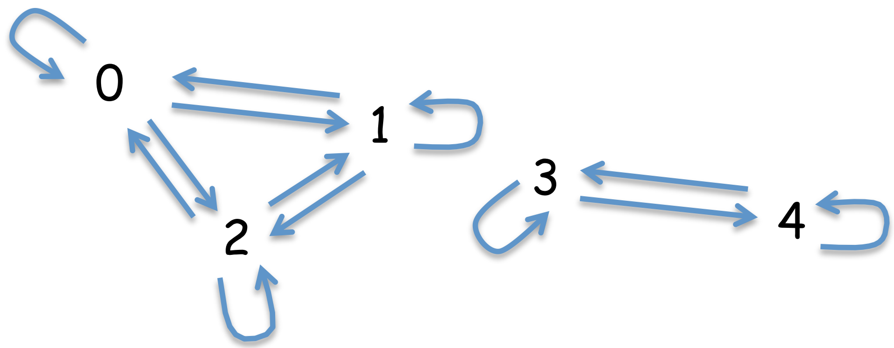
R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0



Properties of Fuzzy Relations

Fuzzy Equivalence Relation

R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0



$\pi(A/R) = \{\{0,1,2\},\{3,4\}\}$

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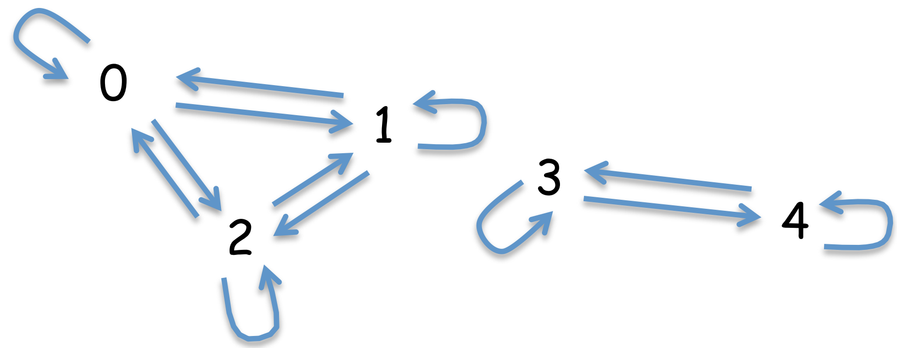
Fuzzy Equivalence Relation

R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0



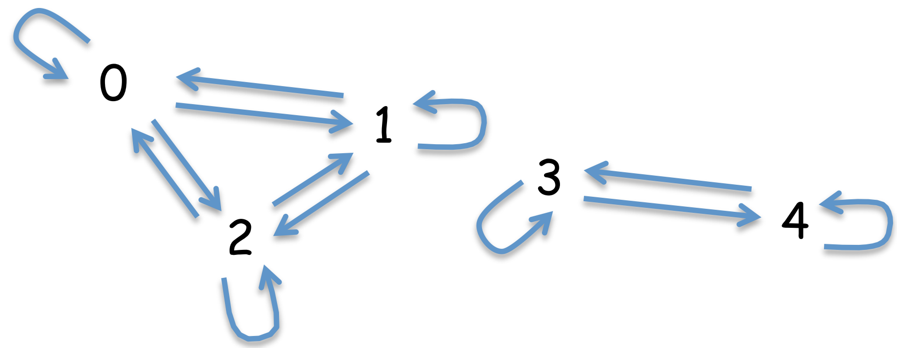
$\pi(A/R) = \{\{0,1,2\},\{3,4\}\}$

$R_{0.6}$	0	1	2	3	4
0	1	1	1	0	0
1	1	1	1	0	0
2	1	1	1	0	0
3	0	0	0	1	1
4	0	0	0	1	1

Properties of Fuzzy Relations

Fuzzy Equivalence Relation

R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0



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0	1	1	1	0	0
1	1	1	1	0	0
2	1	1	1	0	0
3	0	0	0	1	1
4	0	0	0	1	1

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Properties of Fuzzy Relations

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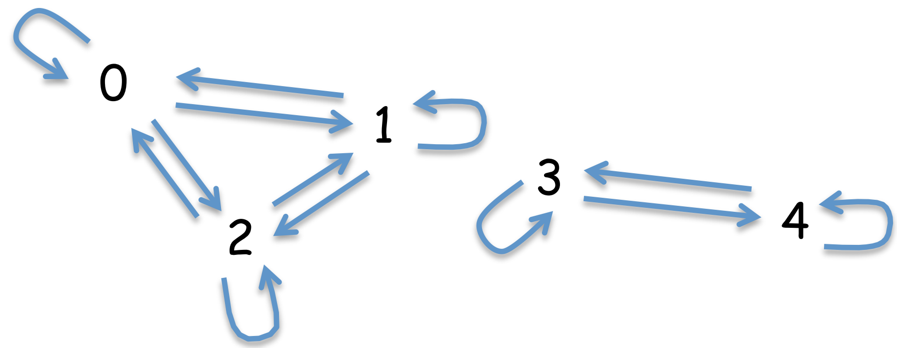
R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
4	0	0	0	0.6	1.0



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$R_{0.6}$	0	1	2	3	4
0	1	1	1	0	0
1	1	1	1	0	0
2	1	1	1	0	0
3	0	0	0	1	1
4	0	0	0	1	1

$R_{1.0}$	0	1	2	3	4
0	1	0	1	0	0
1	0	1	0	0	0
2	1	0	1	0	0
3	0	0	0	1	0
4	0	0	0	0	1

$\pi(A/R_{0.6}) = \{\{0,1,2\},\{3,4\}\}$

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R	0	1	2	3	4
0	1.0	0.6	1.0	0	0
1	0.6	1.0	0.6	0	0
2	1.0	0.6	1.0	0	0
3	0	0	0	1.0	0.6
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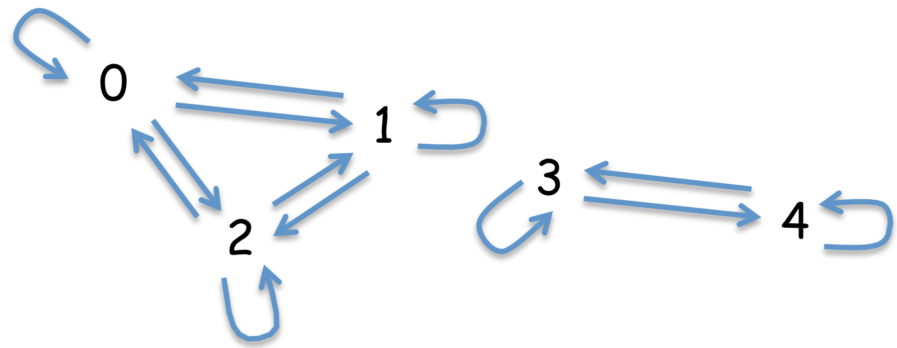

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$R_{0.6}$	0	1	2	3	4
0	1	1	1	0	0
1	1	1	1	0	0
2	1	1	1	0	0
3	0	0	0	1	1
4	0	0	0	1	1

$R_{1.0}$	0	1	2	3	4
0	1	0	1	0	0
1	0	1	0	0	0
2	1	0	1	0	0
3	0	0	0	1	0
4	0	0	0	0	1

$\pi(A/R_{0.6}) = \{\{0,1,2\},\{3,4\}\}$

$\pi(A/R_{1.0}) = \{\{0,2\},\{1\},\{3\},\{4\}\}$

Properties of Fuzzy Relations

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- $A = \{0, 1, 2, 3\}$ and $R \subseteq A \times A$

R	0	1	2	3
0	1.0	0	0.5	0
1	0.7	1.0	0.7	0
2	0	0	1.0	0
3	1.0	0.9	1.0	1.0

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reflexive

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reflexive

anti-symmetric

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1	0.7	1.0	0.7	0
2	0	0	1.0	0
3	1.0	0.9	1.0	1.0

R^2	0	1	2	3
0	1.0	0	0.5	0
1	0.7	1.0	0.7	0
2	0	0	1.0	0
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3	1.0	0.9	1.0	1.0

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0	1.0	0	0.5	0
1	0.7	1.0	0.7	0
2	0	0	1.0	0
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reflexive

transitive

anti-symmetric

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- A fuzzy upper bound of a subset A' : $\cap_{x \text{ in } A'} R_{>[x]}$
- $A = \{0, 1, 2, 3\}$ and $R \subseteq A \times A$

R	0	1	2	3
0	1.0	0	0.5	0
1	0.7	1.0	0.7	0
2	0	0	1.0	0
3	1.0	0.9	1.0	1.0

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