STA250 Probability and Statistics

Chapter 5 Notes

Expected Value, Variance and Covariance of Random Variables

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STA250 Probability and Statistics

Reference Book

This lecture notes are prepared according to the contents of

“**PROBABILITY & STATISTICS FOR ENGINEERS & SCIENTISTS** by Walpole, Myers, Myers and Ye”

«APPLIED STATISTICS AND PROBABILITY FOR ENGINEERS»
by Montgomery and Runger

The Expected Value of a Random Variable

- **The mean or expected value μ of a random variable X with probability distribution f (x), is**

$$\mu = E(X) = \Sigma_x \, x \, f(x) \qquad \text{if discrete, or}$$

$$\mu = E(X) = \int_x x \, f(x) \, dx \qquad \text{if continuous}$$

The Expected Value of a Function of a Random Variable

- **If X is a random variable with distribution f(x). The mean $\mu_{g(X)}$ of the random variable g(X) is**

$$\mu_{g(X)} = E[g(X)] = \Sigma_x\, g(x)\, f(x) \qquad \text{if discrete, or}$$

$$\mu_{g(X)} = E[g(X)] = \int_x g(x)\, f(x)\, dx \qquad \text{if continuous}$$

Variance

- The <u>variance</u> σ^2 of a random variable X with distribution f(x) is

$$\sigma^2 = E[(X-\mu)^2] = \Sigma_x (x-\mu)^2 f(x) \qquad \text{if discrete, or}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_x (x-\mu)^2 f(x)\,dx \qquad \text{if continuous}$$

- An equivalent and easier computational formula, also easy to remember, is

$$\sigma^2 = E[X^2] - [E(X)]^2 = E[X^2] - \mu^2$$

  - $E(X^2) = \sum_x x^2 f(x)$ if discrete or,
  - $E(X^2) = \int_x x^2 f(x) dx$ if continuous

Example 1.

EXAMPLE **3.2** The probability distribution for a random variable Y is given in Table 3.3. Find the mean, variance, and standard deviation of Y.

Table 3.3 Probability distribution for Y

y	$p(y)$
0	1/8
1	1/4
2	3/8
3	1/4

$$\mu = E(Y) = \sum_{y=0}^{3} yp(y) = (0)(1/8) + (1)(1/4) + (2)(3/8) + (3)(1/4) = 1.75,$$

$$\begin{aligned}\sigma^2 &= E[(Y-\mu)^2] = \sum_{y=0}^{3} (y-\mu)^2 p(y) \\ &= (0-1.75)^2(1/8) + (1-1.75)^2(1/4) + (2-1.75)^2(3/8) + (3-1.75)^2(1/4) \\ &= .9375, \\ \sigma &= +\sqrt{\sigma^2} = \sqrt{.9375} = .97.\end{aligned}$$

Source: ***Mathematical Statistics with Applications*, Seventh Edition**
Dennis D. Wackerly, William Mendenhall III, Richard L. Scheaffer

Example 2.

EXAMPLE **3.3** Use Theorem 3.6 to find the variance of the random variable Y in Example 3.2.

Solution The mean $\mu = 1.75$ was found in Example 3.2. Because

$$E(Y^2) = \sum_y y^2 p(y) = (0)^2(1/8) + (1)^2(1/4) + (2)^2(3/8) + (3)^2(1/4) = 4,$$

Theorem 3.6 yields that

$$\sigma^2 = E(Y^2) - \mu^2 = 4 - (1.75)^2 = .9375. \qquad \blacksquare$$

Source: ***Mathematical Statistics with Applications,*** **Seventh Edition**
Dennis D. Wackerly, William Mendenhall III, Richard L. Scheaffer

Example 3.

EXAMPLE 4.6 In Example 4.4 we determined that $f(y) = (3/8)y^2$ for $0 \le y \le 2$, $f(y) = 0$ elsewhere, is a valid density function. If the random variable Y has this density function, find $\mu = E(Y)$ and $\sigma^2 = V(Y)$.

Solution According to Definition 4.5,

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y f(y)\, dy \\ &= \int_0^2 y \left(\frac{3}{8}\right) y^2\, dy \\ &= \left(\frac{3}{8}\right)\left(\frac{1}{4}\right) y^4 \Big]_0^2 = 1.5. \end{aligned}$$

The variance of Y can be found once we determine $E(Y^2)$. In this case,

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{\infty} y^2 f(y)\, dy \\ &= \int_0^2 y^2 \left(\frac{3}{8}\right) y^2\, dy \\ &= \left(\frac{3}{8}\right)\left(\frac{1}{5}\right) y^5 \Big]_0^2 = 2.4. \end{aligned}$$

Thus, $\sigma^2 = V(Y) = E(Y^2) - [E(Y)]^2 = 2.4 - (1.5)^2 = 0.15$. ■

Source: ***Mathematical Statistics with Applications*, Seventh Edition**
Dennis D. Wackerly, William Mendenhall III, Richard L. Scheaffer

Proporties of Expected Value and Variance

- **Let X be a random variable and a be a constant.**
  - $E(a) = a$
  - $E(X + a) = E(X) + a$
  - $E(aX) = aE(X)$
- **Let X be a random variable and a be a constant.**
  - $Var(a) = 0$
  - $Var(X + a) = Var(X)$
  - $Var(aX) = a^2Var(X)$

Example 4.

EXAMPLE 3.4 The manager of an industrial plant is planning to buy a new machine of either type A or type B. If t denotes the number of hours of daily operation, the number of daily repairs Y_1 required to maintain a machine of type A is a random variable with mean and variance both equal to $.10t$. The number of daily repairs Y_2 for a machine of type B is a random variable with mean and variance both equal to $.12t$. The daily cost of operating A is $C_A(t) = 10t + 30Y_1^2$; for B it is $C_B(t) = 8t + 30Y_2^2$. Assume that the repairs take negligible time and that each night the machines are tuned so that they operate essentially like new machines at the start of the next day. Which machine minimizes the expected daily cost if a workday consists of (a) 10 hours and (b) 20 hours?

Source: ***Mathematical Statistics with Applications,*** **Seventh Edition**
Dennis D. Wackerly, William Mendenhall III, Richard L. Scheaffer

Solution 4.

Solution The expected daily cost for A is

$$E[C_A(t)] = E\left[10t + 30Y_1^2\right] = 10t + 30E\left(Y_1^2\right)$$
$$= 10t + 30\{V(Y_1) + [E(Y_1)]^2\} = 10t + 30[.10t + (.10t)^2]$$
$$= 13t + .3t^2.$$

In this calculation, we used the known values for $V(Y_1)$ and $E(Y_1)$ and the fact that $V(Y_1) = E(Y_1^2) - [E(Y_1)]^2$ to obtain that $E(Y_1^2) = V(Y_1) + [E(Y_1)]^2 = .10t + (.10t)^2$. Similarly,

$$E[C_B(t)] = E\left[8t + 30Y_2^2\right] = 8t + 30E\left(Y_2^2\right)$$
$$= 8t + 30\{V(Y_2) + [E(Y_2)]^2\} = 8t + 30[.12t + (.12t)^2]$$
$$= 11.6t + .432t^2.$$

Thus, for scenario (a) where $t = 10$,

$$E[C_A(10)] = 160 \quad \text{and} \quad E[C_B(10)] = 159.2,$$

which results in the choice of machine B.

For scenario (b), $t = 20$ and

$$E[C_A(20)] = 380 \quad \text{and} \quad E[C_B(20)] = 404.8,$$

resulting in the choice of machine A.

In conclusion, machines of type B are more economical for short time periods because of their smaller hourly operating cost. For long time periods, however, machines of type A are more economical because they tend to be repaired less frequently. ■

Source: ***Mathematical Statistics with Applications*, Seventh Edition**
Dennis D. Wackerly, William Mendenhall III, Richard L. Scheaffer

Example 5.

The probability distribution for a random variable X is given follow,

$$f(x) = \begin{cases} cx & , \quad x = 1,2,3,4 \\ 0 & , \quad dy \end{cases}$$

a) $c = ?$
b) $E(X) = ?$
c) $Var(X) = ?$
d) $P(X = 1) = ?$
e) $P(2 < X < 4) = ?$
f) $P(X \leq 3) = ?$

Solution 5

- **For any discrete probability distribution, the following must be true**
  - $f(x) \geq 0 \qquad x \in D_x$
  - $\sum_{x \in D_x} f(x) = 1$

$$\sum_{x=1}^{4} cx = 1$$

$$c.1 + c.2 + c.3 + c.4 = 1$$

$$10.c = 1$$

$$c = \frac{1}{10}$$

$$f(x) = \begin{cases} \frac{1}{10}x & , \quad x = 1,2,3,4 \\ 0 & , \quad dy \end{cases}$$

Solution 5 continue…

X	1	2	3	4
$P(X=x)$	0.1	0.2	0.3	0.4

b) $E(X) = \sum_{x=1}^{4} x f(x)$

$= 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.3 + 4 \times 0.4$

$= 0.1 + 0.4 + 0.9 + 1.6$

$= 3$

c) $Var(X) = E(X^2) - \left(E(X)\right)^2$

$$E(X^2) = \sum_{x=1}^{4} x^2 f(x)$$

$= 1 \times 0.1 + 2^2 \times 0.2 + 3^2 \times 0.3 + 4^2 \times 0.4$

$= 0.1 + 0.8 + 2.7 + 6.4$

$= 10$

$Var(X) = E(X^2) - \left(E(X)\right)^2$

$= 10 - 3^2$

$= 1$

d) $P(X = 1) = 0.1$

e) $P(2 < X \leq 4) = P(X = 3) + P(X = 4)$

$= 0.3 + 0.4 = 0.7$

f) $P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$

$= 0.1 + 0.2 + 0.2 = 0.6$

Example 6.

The probability distribution for a random variable *X* is given follow,

$$f(x) = \begin{cases} cx & , \quad 0 \leq x \leq 5 \\ 0 & , \quad dy \end{cases}$$

a) $c = ?$
b) $E(X) = ?$
c) $Var(X) = ?$
d) $P(1 \leq X \leq 3) = ?$
e) $P(2 \leq X < 4) = ?$
f) $P(X \leq 3) = ?$

Solution 6.

a) $\int_0^5 cxdx = 1$

$c\int_0^5 xdx = 1$

$c\frac{x^2}{2}\Big|_0^5 = 1$

$c = \frac{2}{25}$

b) $f(x) = \begin{cases} \frac{2}{25}x & , \quad 0 \leq x \leq 5 \\ 0 & , \quad dy \end{cases}$

$$E(X) = \int_0^5 x\frac{2}{25}xdx = \int_0^5 \frac{2}{25}x^2dx$$

$$= \frac{2}{25}\int_0^5 x^2dx = \frac{2}{25}\frac{x^3}{3}\Big|_0^5 = \frac{10}{3}$$

c) $Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \int_0^5 x^2\frac{2}{25}xdx = \int_0^5 \frac{2}{25}x^3dx = \frac{2}{25}\frac{x^4}{4}\Big|_0^5 = \frac{25}{2}$$

$$Var(X) = \frac{25}{2} - \left(\frac{10}{3}\right)^2 = \frac{25}{18}$$

d) $P(1 \leq X \leq 3) = \int_1^3 \frac{2}{25}xdx = \frac{2}{25}\frac{x^2}{2}\Big|_1^3 = \frac{2}{25}\left(\frac{9}{2} - \frac{1}{2}\right) = \frac{8}{25}$

e) $P(2 \leq X \leq 4) = \int_2^4 \frac{2}{25}xdx = \frac{2}{25}\frac{x^2}{2}\Big|_2^4 = \frac{2}{25}\left(\frac{16}{2} - \frac{4}{2}\right) = \frac{12}{25}$

f) $P(X \leq 3) = \int_0^3 \frac{2}{25}xdx = \frac{2}{25}\frac{x^2}{2}\Big|_0^3 = \frac{2}{25}\left(\frac{9}{2} - 0\right) = \frac{9}{25}$

Example 7.

- $Y = 3X - 5, \quad E(X) = 4, \quad Var(X) = 2$, find $E(Y)$ *and* $Var(Y)$.

- **Solution:**

$$E(Y) = E(3X - 5)$$

$$= 3E(X) - 5$$

$$= 3.4 - 5 = 12 - 5 = 7$$

$$Var(Y) = Var(3X - 5)$$

$$= 9Var(X) = 9 \times 2 = 18$$

Example 8.

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

***Solution*:** By Theorem 4.1, the attendant can expect to receive

$$
\begin{aligned}
E[g(X)] = E(2X-1) &= \sum_{x=4}^{9}(2x-1)f(x) \\
&= (7)\left(\frac{1}{12}\right) + (9)\left(\frac{1}{12}\right) + (11)\left(\frac{1}{4}\right) + (13)\left(\frac{1}{4}\right) \\
&\quad + (15)\left(\frac{1}{6}\right) + (17)\left(\frac{1}{6}\right) = \$12.67.
\end{aligned}
$$

Expected Value for a Joint Distribution

- **If X and Y are random variables with joint probability distribution f(x, y). The mean or expected value $\mu_{g(X,Y)}$ of the random variable g(X, Y) is**

$\mu_{g(X,Y)} = E[g(X,Y)] = \Sigma_x \Sigma_y\, g(x,y)\, f(x,y)$ if discrete, or

$\mu_{g(X,Y)} = E[g(X,Y)] = \int_x \int_y g(x,y)\, f(x,y)\, dy\, dx$ if continuous

- **Note that the mean of a distribution is a single value, so it doesn't make sense to talk of the mean the distribution f(x,y).**

Example 9.

Let *X* and *Y* be the random variables with joint probability distribution indicated in the following Table. Find the expected value of *g*(*X*, *Y*) = *XY*.

	$f(x,y)$	$x=0$	$x=1$	$x=2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$
\begin{aligned}
E(XY) &= \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y) \\
&= (0)(0)f(0,0) + (0)(1)f(0,1) \\
&\quad + (1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0) \\
&= f(1,1) = \frac{3}{14}.
\end{aligned}
$$

Example 10.

Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^2 \frac{y(1+3y^2)}{4}\, dxdy = \int_0^1 \frac{y+3y^3}{2}\, dy = \frac{5}{8}.$$

Note that if $g(X,Y) = X$ in Definition 4.2, we have

$$E(X) = \begin{cases} \sum\limits_x \sum\limits_y x f(x,y) = \sum\limits_x x g(x) & \text{(discrete case)}, \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\ dy\ dx = \int_{-\infty}^{\infty} x g(x)\ dx & \text{(continuous case)}, \end{cases}$$

Functions of Two or More Random Variables

- **The expected value of the sum of two random variables is equal to the sum of the expected values.**

$$E(X \pm Y) = E(X) \pm E(Y)$$

- **The expected value of the product of two <u>independent</u> random variables is equal to the product of the expected values.**

$$E(XY) = E(X)\,E(Y)$$

Functions of Two or More Random Variables

- **The variance of the sum of two random variables and subtracts of two random variables;**
  - $Var(X+Y) = \ Var(X){+}Var(Y) + 2Cov(X,Y)$
  - $Var(X-Y) = \ Var(X){+}Var(Y) - 2Cov(X,Y)$
  - $Cov\ (X,Y) = Cov(Y,X)$
  - $Cov\ (X,X) = Var(X)$

Covariance

- **If X and Y are random variables with joint probability distribution f(x, y), the <u>covariance</u>, $Cov(X, Y)$, of X and Y is defined as**

$$Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

- **The better computational formula for covariance is**

$$Cov(X,Y) = E(XY) - \mu_X \mu_Y$$

- **Note that although the standard deviation σ can't be negative, the covariance $Cov(X, Y)$, can be negative.**
- **Covariance will be useful later when looking at the linear relationship between two random variables.**

We denote $Cov(X, Y)$ by σ_{XY}

Example 11.

This example describes a situation involving the number of blue refills X and the number of red refills Y . Two refills for a ballpoint pen are selected at random from a certain box, and the following is the joint probability distribution:

	$f(x,y)$	x = 0	1	2	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
	$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Find the covariance of X and Y.

Remember that $$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y)$$

$$= (0)(0)f(0,0) + (0)(1)f(0,1)$$

$$+ (1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0)$$

$$= f(1,1) = \frac{3}{14}.$$

Example 11. Continue…

$f(x,y)$		$x=0$	$x=1$	$x=2$	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$g(x)$		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$\mu_X = \sum_{x=0}^{2} xg(x) = (0)\left(\frac{5}{14}\right) + (1)\left(\frac{15}{28}\right) + (2)\left(\frac{3}{28}\right) = \frac{3}{4},$$

$$\mu_Y = \sum_{y=0}^{2} yh(y) = (0)\left(\frac{15}{28}\right) + (1)\left(\frac{3}{7}\right) + (2)\left(\frac{1}{28}\right) = \frac{1}{2}.$$

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$

Example 12.

The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x, y) = \begin{cases} 8xy, & 0 \le y \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.

Solution **:** We first compute the marginal density functions. They are

$$g(x) = \begin{cases} 4x^3, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases} \qquad h(y) = \begin{cases} 4y(1-y^2), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

From these marginal density functions, we compute

$$\mu_X = E(X) = \int_0^1 4x^4 \, dx = \frac{4}{5} \quad \mu_Y = \int_0^1 4y^2(1-y^2) \, dy = \frac{8}{15}.$$

Example 12. Continue…

From the joint density function given before, we have

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2 \; dx \; dy = \frac{4}{9}.$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

Correlation Coefficient

- **If X and Y are random variables with covariance Cov(X,Y) and standard deviations σ_X and σ_Y respectively, the <u>correlation coefficient</u> ρ_{XY} is defined as**

$$\rho_{XY} = Cov(X,Y) \ / \ \sigma_X \sigma_Y$$

- $-1 \leq \rho_{XY} \leq 1$
- If $\rho_{XY} = 1$ or -1, then there is an exact linear relationship between X and Y.
- the <u>correlation coefficient</u> shows the level of relationship between X and Y .

Notes that:

- $Cov(X,Y) = 0$, if X and Y random variables are independent.
- ρ_{XY}=0 , if $Cov(X,Y)$ *is* 0, we can say no linear relationship between X and Y .

Example 13.

Find the correlation coefficient between *X* and *Y* in Example 11.

	$f(x,y)$	x = 0	1	2	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
	$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$

$$E(X^2) = (0^2)\left(\frac{5}{14}\right) + (1^2)\left(\frac{15}{28}\right) + (2^2)\left(\frac{3}{28}\right) = \frac{27}{28}$$

$$E(Y^2) = (0^2)\left(\frac{15}{28}\right) + (1^2)\left(\frac{3}{7}\right) + (2^2)\left(\frac{1}{28}\right) = \frac{4}{7},$$

Example 13. Continue…

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}. \quad \sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$

$$E(X^2) = (0^2)\left(\frac{5}{14}\right) + (1^2)\left(\frac{15}{28}\right) + (2^2)\left(\frac{3}{28}\right) = \frac{27}{28}$$

$$E(Y^2) = (0^2)\left(\frac{15}{28}\right) + (1^2)\left(\frac{3}{7}\right) + (2^2)\left(\frac{1}{28}\right) = \frac{4}{7},$$

$$\sigma_X^2 = \frac{27}{28} - \left(\frac{3}{4}\right)^2 = \frac{45}{112} \qquad \sigma_Y^2 = \frac{4}{7} - \left(\frac{1}{2}\right)^2 = \frac{9}{28}.$$

Therefore, the correlation coefficient between *X* and *Y* is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-9/56}{\sqrt{(45/112)(9/28)}} = -\frac{1}{\sqrt{5}}.$$

Example 14.

Find the correlation coefficient of X and Y in Example 12. Remember

$$f(x, y) = \begin{cases} 8xy, & 0 \leq y \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$g(x) = \begin{cases} 4x^3, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere,} \end{cases} \qquad h(y) = \begin{cases} 4y(1 - y^2), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\mu_X = E(X) = \int_0^1 4x^4 \, dx = \frac{4}{5} \qquad \mu_Y = \int_0^1 4y^2(1 - y^2) \, dy = \frac{8}{15}.$$

$$E(X^2) = \int_0^1 4x^5 \, dx = \frac{2}{3}$$

$$E(Y^2) = \int_0^1 4y^3(1 - y^2) \, dy = 1 - \frac{2}{3} = \frac{1}{3},$$

Example 14. Continue….

Find the correlation coefficient of *X* and *Y* in Example 12. Remember

$$\mu_X = E(X) = \int_0^1 4x^4 \, dx = \frac{4}{5} \qquad \mu_Y = \int_0^1 4y^2(1-y^2)\, dy = \frac{8}{15}.$$

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2 \, dx \, dy = \frac{4}{9}.$$

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

$$E(X^2) = \int_0^1 4x^5 \, dx = \frac{2}{3} \qquad E(Y^2) = \int_0^1 4y^3(1-y^2)\, dy = 1 - \frac{2}{3} = \frac{1}{3},$$

we conclude that

$$\sigma_X^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75} \qquad \sigma_Y^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}.$$

$$\rho_{XY} = \frac{4/225}{\sqrt{(2/75)(11/225)}} = \frac{4}{\sqrt{66}}.$$

Example 14. Continue….

the covariance and correlation of X and Y in Example 14:

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$ Nearly 0.02

$$\rho_{XY} = \frac{4/225}{\sqrt{(2/75)(11/225)}} = \frac{4}{\sqrt{66}}.$$ **Nearly 0.5**

the covariance and correlation of X and Y in Example 13:

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$ **Nearly -0.16**

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{-9/56}{\sqrt{(45/112)(9/28)}} = -\frac{1}{\sqrt{5}}.$$ Nearly -0.45

Next Lesson

- **Some Discrete Probability Distributions**

See you☺