ELEC431-636 project1

Consider the linear system $A\mathbf{x}=\mathbf{b}$ where the coefficient matrices and the solution vector are as follows.

$$A=\begin{bmatrix}-2 & 1 & -3\\ -1 & 2 & 1\\ 3 & -2 & 1\end{bmatrix} \qquad b=\begin{bmatrix}-9\\ 6\\ 2\end{bmatrix} \qquad x=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix} \qquad A^{-1}=\begin{bmatrix}0.500 & 0.625 & 0.875\\ 0.500 & 0.875 & 0.625\\ -0.500 & -0.125 & -0.375\end{bmatrix}$$

Let us find the norms of these matrices and vectors.

$$||A||_1=6. \qquad ||A||_2=5.12 \qquad ||A||_\infty=6. \qquad ||A||_F=5.83 \qquad ||b||_1=17. \qquad ||b||_2=11. \qquad ||b||_\infty=9.$$

$$|A^{-1}|=1.88 \qquad |A^{-1}|=1.75 \qquad |A^{-1}|=2.00 \qquad |A^{-1}|=1.79 \qquad ||x||_1=6. \qquad ||x||_2=3.74 \qquad ||x||_\infty=3.$$

Let the right side $\mathbf{b}$ change to $\mathbf{b}+\mathbf{\Delta b}$, and thus the new solution becomes $(\mathbf{x}+\mathbf{\Delta x})$ such that $A(\mathbf{x}+\mathbf{\Delta x})=\mathbf{b}+\mathbf{\Delta b}$

$$\Delta b=\begin{bmatrix}-3\\ 3\\ 2\end{bmatrix} \qquad \Delta b+b=\begin{bmatrix}-12\\ 9\\ 4\end{bmatrix} \qquad x+\Delta x\equiv(A^{-1})\,(\Delta b+b)=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix} \qquad \Delta x=\begin{bmatrix}2.12\\ 2.38\\ 0.375\end{bmatrix}$$

Then, the relative errors and the condition numbers $c_1=`||A||_1||A^{-1}||_1`$, $c_2=`||A||_2||A^{-1}||_2`$, $c_\infty=`||A||_\infty||A^{-1}||_\infty`$.

$$\frac{||\Delta b||_1}{||b||_1}=0.471 \qquad \frac{||\Delta b||_2}{||b||_2}=0.426 \qquad \frac{||\Delta b||_\infty}{||b||_\infty}=0.333 \qquad \frac{||\Delta x||_1}{||x||_1}=0.813 \qquad \frac{||\Delta x||_2}{||x||_2}=0.858 \qquad \frac{||\Delta x||_\infty}{||x||_\infty}=0.792$$

$$c_1=11.3 \qquad c_2=8.94 \qquad c_\infty=12.0$$

Note that the inequality $\|\mathbf{\Delta x}\|/\|\mathbf{x}\| \le \|A\|\,\|A^{-1}\|\cdot\|\mathbf{\Delta b}\|/\|\mathbf{b}\|$ is satisfied for all the norms.

$$\frac{||\Delta x||_1}{||x||_1}\le\left(\frac{c_1\,||\Delta b||_1}{||b||_1}=5.32\right),\ \frac{||\Delta x||_2}{||x||_2}\le\left(\frac{c_2\,||\Delta b||_2}{||b||_2}=3.81\right),\ \frac{||\Delta x||_\infty}{||x||_\infty}\le\left(\frac{c_\infty\,||\Delta b||_\infty}{||b||_\infty}=4.00\right)$$

Also, let the matrix $\mathbf{A}$ change to $A+\mathbf{\Delta}A$, and thus the new solution becomes $(\mathbf{x}+\mathbf{\Delta x})$ such that $(A+\mathbf{\Delta}A)(\mathbf{x}+\mathbf{\Delta x})=\mathbf{b}$. Then, change in the solution $\mathbf{\Delta x}$ is found as follows.

$$\Delta A=\begin{bmatrix}-1 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 1\end{bmatrix} \qquad A+\Delta A=\begin{bmatrix}-3 & 2 & -4\\ -2 & 2 & 2\\ 4 & -3 & 2\end{bmatrix} \qquad x+\Delta x\equiv((AA+\Delta A)^{-1})\,b=\begin{bmatrix}-9.\\ -10.\\ 4.\end{bmatrix} \qquad \Delta x=\begin{bmatrix}-10.\\ -12.\\ 1.\end{bmatrix}$$

Then, the relative errors are computed as follows.

$$\frac{||\Delta A||_1}{||A||_1}=0.500 \qquad \frac{||\Delta A||_2}{||A||_2}=0.478 \qquad \frac{||\Delta A||_\infty}{||A||_\infty}=0.500 \qquad \frac{||\Delta x||_1}{||x+\Delta x||_1}=1.00 \qquad \frac{||\Delta x||_2}{||x+\Delta x||_2}=1.12 \qquad \frac{||\Delta x||_\infty}{||x+\Delta x||_\infty}=1.20$$

Note that the inequality $\|\mathbf{\Delta x}\|/\|\mathbf{x}+\mathbf{\Delta x}\| \le \|A\|\,\|A^{-1}\|\cdot\|\Delta A\|/\|A\|$ is satisfied for all the norms.

$$\frac{||\Delta x||_1}{||x+\Delta x||_1}\le\left(\frac{c_1\,||\Delta A||_1}{||A||_1}=5.65\right),\ \frac{||\Delta x||_2}{||x+\Delta x||_2}\le\left(\frac{c_2\,||\Delta A||_2}{||A||_2}=4.28\right),\ \frac{||\Delta x||_\infty}{||x+\Delta x||_\infty}\le\left(\frac{c_\infty\,||\Delta A||_\infty}{||A||_\infty}=6.00\right)$$