GEBZE TECHNICAL UNIVERSITY

ENGINEERING FACULTY

ELECTRONICS ENGINEERING

ELM 462

STATISTICAL SIGNAL PROCESSING

PROJECT

Deadline: 16.12.2024

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Contents

INTRODUCTION

The goal of this project is to estimate the signal from a noisy observed signal, where the relationship between the signals is represented as:

Here:

- : Observation vector of size.
- : Convolution matrix of size, representing the LTI system.
- : Ground truth signal to be estimated, of size.
- : White Gaussian noise vector with zero mean and variance.

In addition to noise properties, the prior distribution for is given as:
where is a regularization matrix, and is a regularization parameter that controls the smoothness of the estimate.

This report explores two estimators:

1. Maximum Likelihood (ML) Estimation.
2. Maximum A-Posteriori (MAP) Estimation.

The estimators are derived and implemented, and their performance is analyzed.

1. ML Estimator Derivation

The ML estimator maximizes the likelihood of the observed data

The system model is:

$$y = \mathbf{H}x + w$$

where:

- **y** is the observation vector (size 300 × 1).
- **H** is the system matrix (size 300 × 300)
- **x** is the signal to be estimated (size 300 × 1).
- $\mathbf{w \sim N(0, \sigma_w^2 I)}$ is white Gaussian noise.

ML Estimation Framework

The Maximum Likelihood (ML) estimator finds the value of **x** that maximizes the likelihood of the observed data **y**

The likelihood of **y** given **x** is:

$$\mathbf{p(y|x)} = \frac{\mathbf{1}}{(\mathbf{2\pi\sigma_w}^2)^{\mathbf{150}}} \ \mathbf{exp}\left(-\frac{\mathbf{1}}{\mathbf{2\sigma_w}^2} \ \| \mathbf{y} - \mathbf{Hx} \|^2 \right)$$

Taking the negative log-likelihood to simplify optimization:

$$\mathcal{L}(x) = \frac{1}{2\sigma_w^2} \parallel y - Hx \parallel^2 + constant$$

To minimize the negative log-likelihood:

$$\hat{x}_{ML} = \arg\min \parallel y - Hx \parallel^2$$

Solve for $\hat{x}_{ML}$:

The optimization problem is a quadratic least-squares problem. Differentiating with respect to **x** and setting the gradient to zero:

$$-H^T (y - Hx) = 0$$

$$H^T Hx = H^T y$$

$$\hat{x}_{ML} = (H^T H)^{-1} H^T y$$

Thus, the ML estimator is:

$$\hat{x}_{ML} = (H^T H)^{-1} H^T y$$

2. ML Estimate Results

Problem b - Matlab

```
% Mehmet ALTINTAŞ 1901022065
% ELM 462 PROJE 1- PROBLEM 2
% Load the provided MATLAB data
load('projectecg.mat');

% (a) ML Estimator Calculation
H_t_H_inv = inv(H' * H);
x_ml = H_t_H_inv * H' * y;

% (b) Plot Observation and ML Estimate
figure;
plot(y, 'r--', 'DisplayName', 'Observation (y)', 'LineWidth', 1.5); % Blue
dashed line
hold on;
plot(x_ml, 'y-', 'DisplayName', 'ML Estimate (x̂_{ML})', 'LineWidth', 1.5);
% Green solid line
plot(x, 'c-', 'DisplayName', 'Ground Truth (x_{true})', 'LineWidth', 1.5);
% Red solid line
legend show;
xlabel('Sample Index');
ylabel('Signal Value');
title('Observation, ML Estimate, and Ground Truth');
grid on;
```

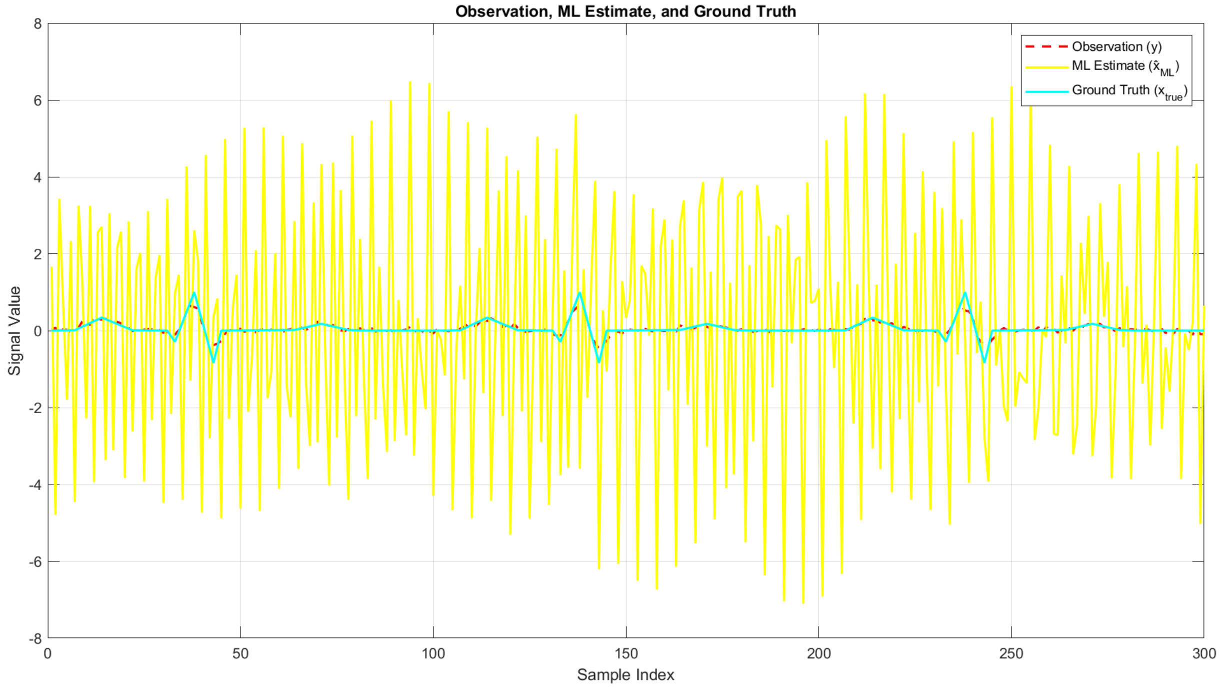

Figure 1. Observation, ML Estimate, and Ground Truth

The ML estimate was computed using the provided data. The following observations were made:

1. Observation vs. ML Estimate vs. Ground Truth: The ML estimate was plotted alongside the observed signal and the ground truth. The ML estimate attempts to recover the true signal but is sensitive to noise.
2. Performance: The ML estimate showed notable deviations due to the influence of noise.

3. MAP Estimator Derivation

The MAP estimator incorporates prior information about x, unlike the ML estimator. Here, we know:

- **$x \sim N(0, (\lambda L^T L)^{-1})$:** a Gaussian prior on x with mean zero and covariance **$(\lambda L^T L)^{-1}$** where $\lambda > 0$ is regularization parameter.
- **$w \sim N(0, \sigma_w^2 I)$**

MAP Estimation Framework

The MAP estimator maximizes the posterior probability **$p(x|y)$:**

$$\hat{x}_{MAP} = \arg\max p(x \mid y)$$

Using Bayes' theorem:

$$p(x \mid y) \propto p(y \mid x) p(x)$$

Take the logarithm of the posterior (for simplification):

$$\log p(x \mid y) \propto \log p(y \mid x) + \log p(x)$$

Likelihood term (p(y|x):

$$\log p(y \mid x) = -\frac{1}{2\sigma_w^2} \| y - Hx \|^2 + constant$$

Prior term (p(x):

$$\log p(x) = -\frac{1}{2} x^T (\lambda L^T L) x + constant$$

Combining the two terms:

$$\log p(x \mid y) \propto -\frac{1}{2\sigma_w^2} \| y - Hx \|^2 - \frac{\lambda}{2} x^T (L^T L) x$$

Solve for $\hat{x}_{MAP}$

To maximize the posterior, minimize the negative log-posterior:

$$\hat{x}_{MAP} = arg\ min \left[\frac{1}{2\sigma_w^2} \| y - Hx \|^2 + \frac{\lambda}{2} x^T (L^T L) x \right]$$

Expanding and rearranging:

$$\hat{x}_{MAP} = \arg\min \left[x^T H^T H x - 2x^T H^T y + \lambda x^T L^T L x \right]$$

Taking the gradient and setting it to zero:

$$H^T H x + \lambda L^T L x = H^T y$$

Solve for $\hat{x}_{MAP}$

$$\hat{x}_{MAP} = (H^T H + \lambda L^T L)^{-1} H^T y$$

Thus, the MAP estimator is:

$$\hat{x}_{MAP} = (H^T H + \lambda L^T L)^{-1} H^T y$$

4. MAP Estimate Results for Different

Problem d- Matlab

```
% Mehmet ALTINTAŞ 1901022065
% ELM 462 PROJE 1- PROBLEM 4
% Define lambda values
lambdas = [1, 10, 100, 1000];

% (c) Compute MAP Estimates
map_estimates = cell(length(lambdas), 1);

for i = 1:length(lambdas)
    lambda = lambdas(i);
    regularization_term = lambda * (L' * L);
    map_matrix = inv(H' * H + regularization_term);
    x_map = map_matrix * H' * y;
    map_estimates{i} = x_map;
end

% (d) Plot MAP Estimates for Different Lambda Values
figure;
hold on;
for i = 1:length(lambdas)
    plot(map_estimates{i}, '--', 'DisplayName', sprintf('MAP Estimate
(\\lambda = %d)', lambdas(i)), 'LineWidth', 1.5);
end
plot(x, '-', 'DisplayName', 'Ground Truth (x_{true})', 'LineWidth', 2);
legend show;
xlabel('Sample Index');
ylabel('Signal Value');
title('MAP Estimates for Different \\lambda Values');
grid on;
```

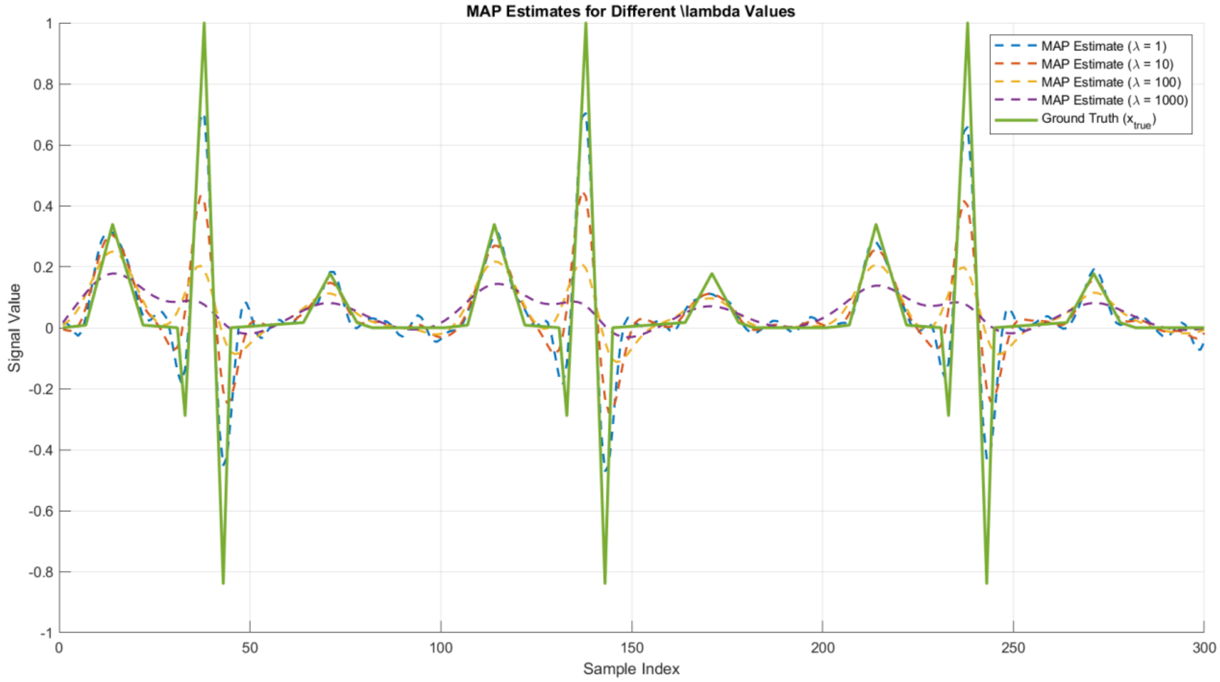

Figure 2. MAP Estimates for Different λ Values

The MAP estimates were computed for . The following results were observed:

1. Effect of:
   - Smaller: The MAP estimate closely follows the ground truth, but it is still sensitive to noise.
   - Larger: The MAP estimate is smoother, reducing noise sensitivity but deviating from the ground truth.
2. Visualization: MAP estimates were plotted for different values alongside the ground truth.

5. Final MSE Comparison

$$\mathbf{MSE} = \frac{1}{300}\,\|\mathbf{x}-\hat{\mathbf{x}}\|^2$$

5.1 Mean Square Error for ML Estimator

The ML estimate is computed as:

$$\hat{\mathbf{x}}_{ML} = (\mathbf{H}^T\mathbf{H})^{-1}\,\mathbf{H}^T\mathbf{y}$$

The error vector is:

$$\Delta\hat{\mathbf{x}}_{ML} = \mathbf{x} - \hat{\mathbf{x}}_{ML}$$

The MSE for the ML estimator becomes:

$$\mathbf{MSE}_{ML} = \frac{1}{300}\,\|\mathbf{x}-\hat{\mathbf{x}}_{ML}\|^2 = \frac{1}{300}\sum_{i=1}^{300}(\,x_i - \hat{\mathbf{x}}_{ML,i})^2$$

By substituting the computed $\hat{\mathbf{x}}_{ML}$ and ground truth x the numerical value obtained is:

$$\mathbf{MSE}_{ML} = \frac{1}{300}\sum_{i=1}^{300}(\,x_i - \hat{\mathbf{x}}_{ML,i})^2 = \mathbf{10.8078}$$

5.2 Mean Square Error for MAP Estimator

The MAP estimate is computed as:

$$\hat{\mathbf{x}}_{MAP} = (\mathbf{H}^T\mathbf{H} + \lambda\mathbf{L}^T\mathbf{L})^{-1}\,\mathbf{H}^T\mathbf{y}$$

The error vector becomes

$$\Delta\hat{\mathbf{x}}_{MAP} = \mathbf{x} - \hat{\mathbf{x}}_{MAP}$$

The MSE for the MAP estimator is:

$$\mathbf{MSE}_{MAP} = \frac{1}{300}\,\|\mathbf{x}-\hat{\mathbf{x}}_{MAP}\|^2 = \frac{1}{300}\sum_{i=1}^{300}(\,x_i - \hat{\mathbf{x}}_{MAP,i})^2$$

Substituting Results for Different λ:

- **For λ=1:**

$$\mathbf{MSE_{MAP}} = \frac{1}{300}\,\|\mathbf{x}-\hat{\mathbf{x}}_{\mathbf{MAP}}\|^2 = \frac{1}{300}\sum_{i=1}^{300}(\,x_i - \hat{x}_{MAP,\lambda=1,i})^2 = \mathbf{0.0052}$$

- **For λ=10:**

$$\mathbf{MSE_{MAP}} = \frac{1}{300}\,\|\mathbf{x}-\hat{\mathbf{x}}_{\mathbf{MAP}}\|^2 = \frac{1}{300}\sum_{i=1}^{300}(\,x_i - \hat{x}_{MAP,\lambda=10,i})^2 = \mathbf{0.0148}$$

- **For λ=100:**

$$\mathbf{MSE_{MAP}} = \frac{1}{300}\,\|\mathbf{x}-\hat{\mathbf{x}}_{\mathbf{MAP}}\|^2 = \frac{1}{300}\sum_{i=1}^{300}(\,x_i - \hat{x}_{MAP,\lambda=100,i})^2 = \mathbf{0.0269}$$

- **For λ=1000:**

$$\mathbf{MSE_{MAP}} = \frac{1}{300}\,\|\mathbf{x}-\hat{\mathbf{x}}_{\mathbf{MAP}}\|^2 = \frac{1}{300}\sum_{i=1}^{300}(\,x_i - \hat{x}_{MAP,\lambda=1000,i})^2 = \mathbf{0.0348}$$

	ML	MAP (λ = 1)	MAP (λ = 10)	MAP (λ = 100)	MAP (λ = 1000)
MSE	**10.8078**	**0.0052**	**0.0148**	**0.0269**	**0.0348**

Table 1. Mean Square Error

6. Conclusion

The project successfully demonstrated the application of ML and MAP estimators for ECG signal estimation. The following conclusions were drawn:

- ML Estimator Performance: The ML estimator provided a reasonable estimate but was highly sensitive to noise due to the lack of prior information.
- MAP Estimator Performance: The MAP estimator significantly improved the signal estimation by incorporating prior knowledge through regularization. The performance depended on the choice of the regularization parameter .
- Impact of Regularization: A smaller value led to a better fit to the ground truth but was more sensitive to noise. A larger value produced a smoother estimate but deviated from the actual signal.
- Best Practices: Balancing the regularization parameter is crucial in obtaining an optimal estimate that minimizes both noise sensitivity and deviation from the ground truth.

These results highlight the importance of incorporating prior knowledge into estimation processes, especially in noisy environments.

7. References

- S. M. Kay, Fundamentals of Statistical Signal Processing: Estimation Theory, Prentice Hall, 1993.