ENG 346
Data Structures and Algorithms for Artificial Intelligence
Matrix Operations and Numpy

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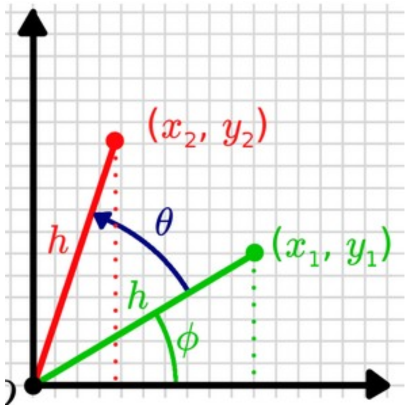
https://github.com/mehmetpekmezci/GTU-ENG-346

Matrices

- Matrices can be used to store data
  - Image data (nxm) matrix
  - Sound data (nx1) matrix
  - Any Log data (nxm) matrix,
    - e.g. : log of humidity, temperature, … values in a room
  - Geometric data
- Matrices can be used for Linear Transformations (Linear Function)
  - Rotation
  - Translation
  - Stretching

Matrices – Geometric Meaning

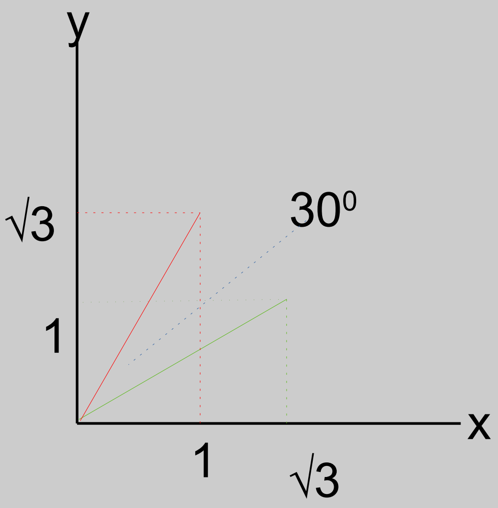
- **Rotation**



$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

NOTE: 3D rotation matrix is a 3x3 matrix, not a 2x2x2 tensor

Example :



$$\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

```
import numpy as np
theta = np.radians(30)
c, s = np.cos(theta), np.sin(theta)
R = np.array(((c, -s), (s, c)))
green_vector=np.array([np.sqrt(3),1])
print(green_vector)
red_vector=np.dot(R,green_vector)
print(red_vector)
```

https://articulatedrobotics.xyz/tutorials/coordinate-transforms/rotation-matrices-2d/

Multiplication of Matrices

Example : 4x4 matrix A=

- We can multiply A and B If A(N,**M**) . B(**M**,K)
- A(**N**,M). B(M,**K**) = C(**N,K**)

A.B =

b_{11}	b_{12}	b_{13}	b_{14}
b_{21}	b_{22}	b_{23}	b_{24}
b_{31}	b_{32}	b_{33}	b_{34}
b_{41}	b_{42}	b_{43}	b_{44}

+
+
+

a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	a_{22}	a_{23}	a_{24}
a_{31}	a_{32}	a_{33}	a_{34}
a_{41}	a_{42}	a_{43}	a_{44}

a11.b12+ a12.b22+a13.b32+a14.b42

a11.b11+ a12.b21+a13.b31+a14.b41

Matrix Example

- Example :

$$x + 2y = 4$$
$$3x - 4y = 5$$

$$\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$A \;.\, X = B$$

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \quad \begin{bmatrix} x+2y \\ 3x-4y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow \begin{matrix} x + 2y = 4 \\ 3x - 4y = 5 \end{matrix}$$

Determinants and Inverse of a Matrix

Example : 4x4 matrix A=

$$\begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline e & f & g & h \\ \hline i & j & k & l \\ \hline m & n & o & p \\ \hline \end{array}$$

Minor of element b is det(M_{12}), **M_{12} =** $\begin{array}{|c|c|c|} \hline e & g & h \\ \hline i & k & l \\ \hline m & o & p \\ \hline \end{array}$

Cofactor(A) is a matrix composed of C_{ij} .

$C_{ij} = (-1)^{i+j} \det(M_{ij})$

det(A)=|A| = $\mathbf{a\ .\ det}\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & f & g & h \\ \hline & j & k & l \\ \hline & n & o & p \\ \hline \end{array}\right)$ $\mathbf{-\ b\ .\ det}\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline e & & g & h \\ \hline i & & k & l \\ \hline m & & o & p \\ \hline \end{array}\right)$

$\mathbf{+\ c\ .\ det}\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline e & f & & h \\ \hline i & j & & l \\ \hline m & n & & p \\ \hline \end{array}\right)$ $\mathbf{-\ d\ .\ det}\left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline e & f & g & \\ \hline i & j & k & \\ \hline m & n & o & \\ \hline \end{array}\right)$

$$\begin{array}{|c|c|c|c|} \hline C_{11} & C_{12} & C_{13} & C_{14} \\ \hline C_{21} & C_{22} & C_{23} & C_{24} \\ \hline C_{31} & C_{32} & C_{33} & C_{34} \\ \hline C_{41} & C_{42} & C_{43} & C_{44} \\ \hline \end{array}$$

Inverse of A

$A^{-1} = 1/\det(A). (\text{Cofactor}(A))^T$

NO INVERSE
if det(A)=0 or
matrix is not square

Transpose and Inverse Properties

$A \,.\, A^{-1} = A^{-1} \,.\, A = I$

$(A^{-1})^{-1} = A$

$(A^{T})^{T} = A$

$(A^{T})^{-1} = (A^{-1})^{T}$

$(AB)^{T} = B^{T} \,.\, A^{T}$

$(A+B)^{T} = A^{T} + B^{T}$

$(AB)^{-1} = B^{-1} \,.\, A^{-1}$

$(k.A)^{-1} = 1/k \,.\, A^{-1}$

$(k.A)^{T} = k \,.\, A^{T}$

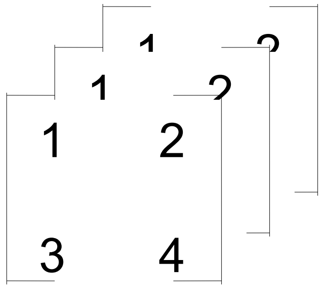
$\det(A^{T}) = \det(A)$

$\det(A^{-1}) = 1/\det(A)$

$\det(A.B)=\det(A).\det(B)$

Tensors (Multilinear Map)

- Vectors = 1^{st} order Tensor $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ (1D arrays)
- Matrices = 2^{nd} order Tensor $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (2D arrays)
- 3^{rd} order Tensor , is a stack of matrices (3D arrays)



→ 2 x 2 x 3 Tensor (I ,j, k indices)

- Tensor may have n order.

Matrix Operations

- Matrix/Tensor Construction
- Dimensions (Shape) and reshaping, array slicing, masking.
- Matrix Summation
- Transpose of the Matrix
- Matrix Multiplication
- Inverse of the Matrix
- Stacking (Vertical, Horizontal, Depth)
- Splitting (Vertical, Horizontal)

- **NOT COVERED : Eigenvalues, Eigenvectors, SVD , Tensor Dot**

Matrix/Tensor Construction

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longleftrightarrow$$

```
import numpy as np
a = np.array( [   [1, 2] , [3, 4] ] )
print(a)
```

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 01.arrays_matrices.py
[[1 2]
 [3 4]]
```

$$\begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}$$

```
np.full((2, 2), 10,dtype=int)
# 2 rows, 2 columns
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
number_of_rows=2
np.identity(number_of_rows,dtype=int)
```

$$\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

```
np.zeros(2)
# default dtype=float
```

$$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

```
np.full((2, 2), 10, dtype=float)
# 2 rows, 2 columns
```

Matrix/Tensor Construction

```python
import numpy as np
np.random.seed(0)  # seed for reproducibility
random_tensor = np.random.randint(100, size=(3, 4, 5))
print(random_tensor)
```

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 02.random.py
[[[44 47 64 67 67]
  [ 9 83 21 36 87]
  [70 88 88 12 58]
  [65 39 87 46 88]]

 [[81 37 25 77 72]
  [ 9 20 80 69 79]
  [47 64 82 99 88]
  [49 29 19 19 14]]

 [[39 32 65  9 57]
  [32 31 74 23 35]
  [75 55 28 34  0]
  [ 0 36 53  5 38]]]
```

Matrix Dimensions

```
import numpy as np

print("INITIAL ARRAY")
a = np.array( [   [1, 2] , [3, 4] , [5,6] ] )
print(f"MATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")

print("\nRESHAPED TO (2,3) ARRAY")
b=a.reshape((2,3))
print(f"MATRIX DIM : {b.shape}")
print(f"MATRIX : {b}")


print("\nRESHAPED TO (6,1) ARRAY")
c=a.reshape((6,))
print(f"MATRIX DIM : {c.shape}")
print(f"MATRIX : {c}")
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 03.matrix.dim.py
INITIAL ARRAY
MATRIX DIM : (3, 2)
MATRIX : [[1 2]
 [3 4]
 [5 6]]

RESHAPED TO (2,3) ARRAY
MATRIX DIM : (2, 3)
MATRIX : [[1 2 3]
 [4 5 6]]

RESHAPED TO (6,1) ARRAY
MATRIX DIM : (6,)
MATRIX : [1 2 3 4 5 6]
```

Array Slicing

```
import numpy as np

print("INITIAL ARRAY")
a = np.array( [   [1, 2,11] , [3, 4,12] , [5,6,13] ] )
print(f"MATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")


b=np.copy(a[1,:2])
print(f"\nMATRIX DIM : {b.shape}")
print(f"MATRIX : {b}")

a[1:,1]=8
print(f"\nMATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")

a[2:,1:]=b
print(f"\nMATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")
```

```
mpekmezci@cobalt:~$ python3 04.array.slicing.py
INITIAL ARRAY
MATRIX DIM : (3, 3)
MATRIX : [[ 1  2 11]
 [ 3  4 12]
 [ 5  6 13]]

MATRIX DIM : (2,)
MATRIX : [3 4]

MATRIX DIM : (3, 3)
MATRIX : [[ 1  2 11]
 [ 3  8 12]
 [ 5  8 13]]

MATRIX DIM : (3, 3)
MATRIX : [[ 1  2 11]
 [ 3  8 12]
 [ 5  3  4]]
```

Array Masking

```
import numpy as np

print("INITIAL ARRAY")
a = np.arange(15)
print(f"MATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")

wanted_indices=[3,8,12]

b=np.copy(a[wanted_indices])
print(f"\nMATRIX DIM : {b.shape}")
print(f"MATRIX : {b}")
```

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 05.array.masking.py
INITIAL ARRAY
MATRIX DIM : (15,)
MATRIX : [ 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14]

MATRIX DIM : (3,)
MATRIX : [ 3  8 12]
```

Matrix Summation

```
import numpy as np

print("INITIAL ARRAY")
a = np.array( [   [1, 2] , [3, 4] , [5,6] ] )
print(f"MATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")

b=a+a
print(f"\n MATRIX:{b} \n")
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 06.summation.py
INITIAL ARRAY
MATRIX DIM : (3, 2)
MATRIX : [[1 2]
 [3 4]
 [5 6]]

 MATRIX:[[ 2  4]
 [ 6  8]
 [10 12]]
```

Matrix Transpose

```
import numpy as np

print("INITIAL ARRAY")
a = np.array( [   [1, 2] , [3, 4] , [5,6] ] )
print(f"MATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")

b=a.T
print(f"\n MATRIX:{b} \n")
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ vi 07.transpose.py
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 07.transpose.py
INITIAL ARRAY
MATRIX DIM : (3, 2)
MATRIX : [[1 2]
 [3 4]
 [5 6]]

 MATRIX:[[1 3 5]
 [2 4 6]]
```

Matrix Multiplication

```
import numpy as np

print("INITIAL ARRAY")
a = np.array( [   [1, 2] , [3, 4] , [5,6] ] )
print(f"MATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")

b=np.matmul(a.T , a)
print(f"\n MATRIX:{b} \n")
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ vi 08.multiplication.py
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 08.multiplication.py
INITIAL ARRAY
MATRIX DIM : (3, 2)
MATRIX : [[1 2]
 [3 4]
 [5 6]]

 MATRIX:[[35 44]
 [44 56]]
```

Matrix Inverse

```
import numpy as np

print("INITIAL ARRAY")
a = np.array( [   [1,2,3] , [5,7,11] , [13,17,23] ] )
print(f"MATRIX DIM : {a.shape}")
print(f"MATRIX : {a}")

b=np.linalg.inv(a)
print(f"\n INVERSE MATRIX:{b} \n")


c=np.matmul(b,a)
print(f"\n A INVERSE * A :{c} \n")
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 11 \\ 13 & 17 & 23 \end{bmatrix}$$

Very close to **0**

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 09.inverse.
INITIAL ARRAY
MATRIX DIM : (3, 3)
MATRIX : [[ 1  2  3]
 [ 5  7 11]
 [13 17 23]]

 INVERSE MATRIX:[[-2.16666667  0.41666667  0.08333333]
 [ 2.33333333 -1.33333333  0.33333333]
 [-0.5         0.75       -0.25      ]]


 A INVERSE * A :[[ 1.00000000e+00  1.08246745e-15  1.58206781e-15]
 [-4.99600361e-16  1.00000000e+00 -6.10622664e-16]
 [ 2.77555756e-16  9.43689571e-16  1.00000000e+00]]
```

Horizontal Stacking

```
import numpy as np

a = np.array( [   [1, 2,3] , [4, 5, 6 ] , [7,8,9] ] )
print(f"MATRIX A : {a}")

b = np.array( [   [11, 12,13] , [14, 15, 16 ] , [17,18,19] ] )
print(f"MATRIX B : {b}")

c=np.hstack((a,b))
print(f"\n HSTACK:{c} \n")
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 10.hstack.py
MATRIX A : [[1 2 3]
 [4 5 6]
 [7 8 9]]
MATRIX B : [[11 12 13]
 [14 15 16]
 [17 18 19]]

 HSTACK:[[ 1  2  3 11 12 13]
 [ 4  5  6 14 15 16]
 [ 7  8  9 17 18 19]]
```

Vertical Stacking

```
import numpy as np

a = np.array( [   [1, 2,3] , [4, 5, 6 ] , [7,8,9] ] )
print(f"MATRIX A : {a}")

b = np.array( [   [11, 12,13] , [14, 15, 16 ] , [17,18,19] ] )
print(f"MATRIX B : {b}")

c=np.vstack((a,b))
print(f"\n VSTACK:{c} \n")
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 11.vstack.py
MATRIX A : [[1 2 3]
 [4 5 6]
 [7 8 9]]
MATRIX B : [[11 12 13]
 [14 15 16]
 [17 18 19]]

 VSTACK:[[ 1  2  3]
 [ 4  5  6]
 [ 7  8  9]
 [11 12 13]
 [14 15 16]
 [17 18 19]]
```

Depth Stacking

```
import numpy as np

a = np.array( [   [1, 2,3] , [4, 5, 6 ] , [7,8,9] ] )
print(f"MATRIX A : {a}")

b = np.array( [   [11, 12,13] , [14, 15, 16 ] , [17,18,19] ] )
print(f"MATRIX B : {b}")

c=np.dstack((a,b))
print(f"\n DSTACK:{c} \n")
```

1	2	3
4	5	6
7	8	9

11	12	13
14	15	16
17	18	19

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 12.dstack.py
MATRIX A : [[1 2 3]
 [4 5 6]
 [7 8 9]]
MATRIX B : [[11 12 13]
 [14 15 16]
 [17 18 19]]

 DSTACK:[[[ 1 11]
  [ 2 12]
  [ 3 13]]

 [[ 4 14]
  [ 5 15]
  [ 6 16]]

 [[ 7 17]
  [ 8 18]
  [ 9 19]]]
```

Horizontal Splitting

```
import numpy as np

a = np.array( [   [1, 2,3,4] , [5, 6, 7,8 ] ] )
print(f"MATRIX A : {a}")

b,c=np.hsplit(a,2)
print(f"\n First Split:{b} \n")
print(f"\n Second Split:{c} \n")
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ vi 13.hsplit.py
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 13.hsplit.py
MATRIX A : [[1 2 3 4]
 [5 6 7 8]]

 First Split:[[1 2]
 [5 6]]


 Second Split:[[3 4]
 [7 8]]
```

Vertical Splitting

```
import numpy as np

a = np.array( [   [1, 2] ,[3,4] , [5, 6], [7,8 ] ] )
print(f"MATRIX A : {a}")

b,c=np.vsplit(a,2)
print(f"\n First Split:{b} \n")
print(f"\n Second Split:{c} \n")
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

```
mpekmezci@cobalt:~/workspace/GTU-ENG-346-PRIVATE/matrix-numpy-codes$ python3 14.vsplit.py
MATRIX A : [[1 2]
 [3 4]
 [5 6]
 [7 8]]

 First Split:[[1 2]
 [3 4]]


 Second Split:[[5 6]
 [7 8]]
```