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Modeling for Optical Ray Tracing and Error Analysis

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Abstract—One of the most popular mathematical tools in fields of robotics, mechanisms, and computer graphics is the 4×4 homogeneous transformation matrix. We extend the use of that matrix to the optical domain of:

(1) skew ray tracing to determine the paths of skew rays; and
(2) error analysis to investigate the various deviations of imagines due to imperfect placement of optical elements.

In order to trace a skew ray, the reflection and refraction laws of optics are formulated in the language of homogeneous transformation matrices. Then an error matrix to describe the position errors and orientation errors of optical elements is introduced in order to analyze their effects on rays' path. This ray tracing procedure can result in very powerful and fast optical design programs. The error analysis can provide the sensitivity of each error component of elements to a system's accuracy and is crucial to upgrade the precision of optical systems in design stage.

Keywords—Optics, Matrix, Ray tracing, Refraction, Reflection.

1. INTRODUCTION

The 4×4 homogeneous transformation matrix is one of the most efficient and useful tools in the robotics [1,2], mechanisms [3,4], and computer graphics [5]. This paper extends its utility to the field of geometrical optics.

In designing optical systems it is necessary to determine the paths of light rays with great accuracy. This may be done by successive application of the law of reflection or refraction. This method is known as ray tracing and is extensively employed in practice [6,7]. The generalized ray tracing, known as skew ray tracing, is very laborious and thus only carried out in the design of high precision optical systems. One of the aims of this paper is to introduce a new simple skew ray tracing method by using homogeneous transformation matrices. In order to achieve this, the reflection and refraction laws are first formulated in forms of homogeneous transformation matrices. The algorithms to analyze optical elements and systems are then discussed.

Optical systems are high precision instruments. Telescopes, microscopes, cameras are examples. Since no one can place an element at its desired position and orientation, there are always position errors and orientation errors (denoted as system errors henceforth) in an optical system. In order to upgrade the precision of an instrument, it is necessary to investigate the effects of system errors on a ray's path in design stage. This may be done by introducing an error matrix and taking the total derivative of the reflected (refracted) beam from ray tracing. In the past, the existing

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skew ray tracing needs a lot of tedious algebra and thus it is almost impossible to investigate the effects of system errors on the ray's path.

The objective of this paper is to introduce a new skew ray tracing method and then investigate the effects of system errors on a ray's path. In Section 2, the necessary mathematics is reviewed. The newly developed skew ray tracing method is discussed in Section 3. The algorithms to investigate the effects of system errors are derived in Section 4. The paths of rays passing through an optical element and a system are examined as illustrative examples in Section 5. Finally, some important conclusions are drawn in Section 6.

2. HOMOGENEOUS COORDINATE NOTATION

In homogeneous coordinate notation [1], we will make use of point vectors and planes to describe the relationships between objects. A point vector $\underline{\mathbf{a}}_r = a_{rx}\underline{\mathbf{i}} + a_{ry}\underline{\mathbf{j}} + a_{rz}\underline{\mathbf{k}}$ referred with respect to coordinate frame $(\mathbf{xyz})_j$ can be written as a column matrix ${}^j\underline{\mathbf{a}}_r = [a_{rx} \quad a_{ry} \quad a_{rz} \quad 1]^\top$. A plane $\underline{S}_q$ referred with respect to frame $(\mathbf{xyz})_j$ is represented as a row matrix ${}^j\underline{S}_q = [\,\underline{\mathbf{n}}_q - d_q\,] = [n_{qx} \quad n_{qy} \quad n_{qz} \quad -d_q]$. $\underline{\mathbf{n}}_q$ is the unit normal of this plane situated a distance d_q from the origin of $(\mathbf{xyz})_j$ in the direction of this normal. For a point ${}^j\underline{\mathbf{a}}_r$ in plane ${}^j\underline{S}_q$ the matrix product ${}^j\underline{S}_q \cdot {}^j\underline{\mathbf{a}}_r = 0$ should be satisfied. Vectors of the form $[n_{qx} \quad n_{qy} \quad n_{qz} \quad 0]^\top$ are used to represent direction. The superscript j will be omitted if $j = 0$ for reason of simplicity.

The position and orientation of a frame $(\mathbf{xyz})_j$ with respect to another frame $(\mathbf{xyz})_i$ can be described by a 4×4 homogeneous transformation matrix ${}^i\underline{\underline{\mathbf{A}}}_j$:

$$
{}^i\underline{\underline{\mathbf{A}}}_j = \begin{bmatrix} \underline{\mathbf{I}} & \underline{\mathbf{J}} & \underline{\mathbf{K}} & \underline{\mathbf{P}} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_x & J_x & k_x & P_x \\ I_y & J_y & k_y & P_y \\ I_z & J_z & k_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1}
$$

The vectors $\underline{\mathbf{I}}$, $\underline{\mathbf{J}}$ and $\underline{\mathbf{K}}$ describe the orientation of the three unit vectors of frame $(\mathbf{xyz})_j$ with respect to frame $(\mathbf{xyz})_i$. The vector $\underline{\mathbf{P}}$ is the position vector of the origin of frame $(\mathbf{xyz})_j$ with respect to frame $(\mathbf{xyz})_i$. With this notation, the inverse of ${}^i\underline{\underline{\mathbf{A}}}_j$ takes the simple form of:

$$
{}^i\underline{\underline{\mathbf{A}}}_j^{-1} = {}^j\underline{\underline{\mathbf{A}}}_i = \begin{bmatrix} \underline{\mathbf{I}}^\top & -\underline{\mathbf{I}}^\top & \underline{\mathbf{P}} \\ \underline{\mathbf{J}}^\top & -\underline{\mathbf{J}}^\top & \underline{\mathbf{P}} \\ \underline{\mathbf{K}}^\top & -\underline{\mathbf{K}}^\top & \underline{\mathbf{P}} \\ \underline{\mathbf{0}}^\top & 1 & \end{bmatrix}. \tag{2}
$$

Given a point ${}^j\underline{\mathbf{a}}_r$, its transformation ${}^i\underline{\mathbf{a}}_r$ is represented by the matrix product

$$
{}^i\underline{\mathbf{a}}_r = {}^i\underline{\underline{\mathbf{A}}}_j\,{}^j\underline{\mathbf{a}}_r. \tag{3}
$$

The corresponding plane transformation ${}^j\underline{S}_q$ to ${}^i\underline{S}_q$ is

$$
{}^i\underline{S}_q = {}^j\underline{S}_q\,{}^j\underline{\underline{\mathbf{A}}}_i = {}^j\underline{S}_q\,{}^i\underline{\underline{\mathbf{A}}}_j^{-1}. \tag{4}
$$

3. SKEW RAY TRACING THROUGH OPTICAL SYSTEMS

The aim of this section is to trace a skew ray through a boundary surface. Shown in Figure 1 is a boundary surface $\underline{S}_i$ separating two homogeneous media of different refraction indices c_{i-1} and c_i. A light ray originating at light source $\underline{\mathbf{a}}_{i-1} = [a_{i-1x} \quad a_{i-1y} \quad a_{i-1z} \quad 1]^\top$ is incident along unit directional vector $\underline{\ell}_{i-1} = [\ell_{i-1x} \quad \ell_{i-1y} \quad \ell_{i-1z} \quad 0]^\top$ and will be reflected (or refracted) on $\underline{S}_i$. A line constructed at the incidence point $\underline{\mathbf{a}}_i$, perpendicular to $\underline{S}_i$ and pointing from medium $i-1$ into medium i, is the surface normal $\underline{\mathbf{n}}_i$. The angle subtended by the surface normal and

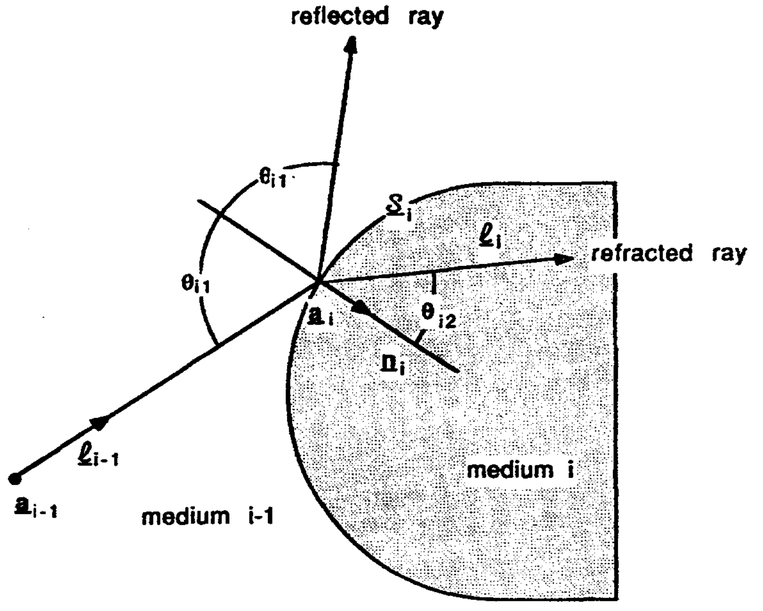

Figure 1. Ray tracing at a boundary surface.

the incident ray $\underline{\ell}_{i-1}$ is the incidence angle θ_{i1}. The angle subtended by $-\underline{\mathbf{n}}_i$ and the reflected ray is the reflection angle. The angle subtended by the surface normal and the refracted ray is refraction angle θ_{i2}.

Any point along the incident ray $\underline{\mathbf{a}}_{i-1}\underline{\mathbf{t}}$, expressed in its parametric form, is:

$$\underline{\mathbf{a}}_{i-1}\underline{\mathbf{t}} = \begin{bmatrix} \underline{\mathbf{a}}_{i-1} + \underline{\ell}_{i-1}t \\ 1 \end{bmatrix} = \begin{bmatrix} a_{i-1x} + \ell_{i-1x}t \\ a_{i-1y} + \ell_{i-1y}t \\ a_{i-1z} + \ell_{i-1z}t \\ 1 \end{bmatrix}, \tag{5}$$

where t is a positive parameter.

If the tangential plane of boundary surface is given by $\underline{S}_i = [\,\underline{\mathbf{n}}_i \quad -d_i\,] = [\,n_{ix} \quad n_{iy} \quad n_{iz} \quad -d_i\,]$, the parameter $t = t_i$ at the incidence point $\underline{\mathbf{a}}_i$ can be determined by $\underline{S}_i \cdot \underline{\mathbf{a}}_{i-1}\underline{\mathbf{t}} = 0$, yielding:

$$t_i = \frac{-\underline{\mathbf{n}}_i^{\top}\underline{\mathbf{a}}_{i-1} + d_i}{\underline{\mathbf{n}}_i^{\top}\underline{\ell}_{i-1}} = \frac{-a_{i-1x}n_{ix} - a_{i-1y}n_{iy} - a_{i-1z}n_{iz} + d_i}{\ell_{i-1x}n_{ix} + \ell_{i-1y}n_{iy} + \ell_{i-1z}n_{iz}} \tag{6}$$

and the incidence point:

$$\underline{\mathbf{a}}_i = \begin{bmatrix} \underline{\mathbf{a}}_{i-1} + \underline{\ell}_{i-1}t_i \\ 1 \end{bmatrix} = \begin{bmatrix} a_{i-1x} + \ell_{i-1x}t_i \\ a_{i-1y} + \ell_{i-1y}t_i \\ a_{i-1z} + \ell_{i-1z}t_i \\ 1 \end{bmatrix}. \tag{7}$$

The refraction angle θ_{i2} between these two optical media must satisfy Snell's law:

$$\sin\theta_{i2} = \frac{c_{i-1}}{c_i}\sin\theta_{i1}, \tag{8}$$

where the incidence angle θ_{i1} is defined by $\cos\theta_{i1} = \underline{\ell}_{i-1}^{\top}\underline{\mathbf{n}}_i = \ell_{i-1x}n_{ix} + \ell_{i-1y}n_{iy} + \ell_{i-1z}n_{iz}$. In the case of $\sin\theta_{i2} > 1$, the light ray will be totally reflected back into medium $i-1$.

To determine the reflected or refracted beam, we need the unit common normal $\underline{m}_i$ of $\underline{\mathbf{n}}_i$ and $\underline{\ell}_{i-1}$:

$$\underline{m}_i = \begin{bmatrix} \underline{m}_i \\ 0 \end{bmatrix} = \begin{bmatrix} m_{ix} \\ m_{iy} \\ m_{iz} \\ 0 \end{bmatrix} = \frac{\underline{\mathbf{n}}_i \times \underline{\ell}_{i-1}\cdot}{\sin\theta_{i1}} = \frac{1}{\sin\theta_{i1}} \begin{bmatrix} \ell_{i-1z}n_{iy} - \ell_{i-1y}n_{iz} \\ \ell_{i-1x}n_{iz} - \ell_{i-1z}n_{ix} \\ \ell_{i-1y}n_{ix} - \ell_{i-1x}n_{iy} \\ 0 \end{bmatrix}. \tag{9}$$

According to the reflection (refraction) law of optics, the reflected (refracted) unit directional vector $\underline{\ell}_i$ can be obtained by rotating $\underline{\mathbf{n}}_i$ about $\underline{m}_i$ at an angle $\theta_i = \pi - \theta_{i1}(\theta_i = \theta_{i2})$. This leads to:

$$\begin{aligned} \underline{\ell}_i &= \begin{bmatrix} \underline{\ell}_i \\ 0 \end{bmatrix} = \begin{bmatrix} \ell_{ix} & \ell_{iy} & \ell_{iz} & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} m_{ix}^2(1-C\theta_i)+C\theta_i & m_{iy}m_{ix}(1-C\theta_i)-m_{iz}S\theta_i & m_{iz}m_{ix}(1-C\theta_i)+m_{iy}S\theta_i & 0 \\ m_{ix}m_{iy}(1-C\theta_i)+m_{iz}S\theta_i & m_{iy}^2(1-C\theta_i)+C\theta_i & m_{iz}m_{iy}(1-C\theta_i)-m_{ix}S\theta_i & 0 \\ m_{ix}m_{iz}(1-C\theta_i)-m_{iy}S\theta_i & m_{iy}m_{iz}(1-C\theta_i)+m_{ix}S\theta_i & m_{iz}^2(1-C\theta_i)+C\theta_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} n_{ix} \\ n_{iy} \\ n_{iz} \\ 0 \end{bmatrix}, \end{aligned} \tag{10}$$

where $C\theta_i$ and $S\theta_i$ denote $\cos\theta_i$ and $\sin\theta_i$ respectively. If the unit directional vector $\underline{\ell}_{i-1}$ coincides with surface normal $\underline{\mathbf{n}}_i$, then the unit directional vector of the reflected (refracted) beam is given by $\underline{\ell}_i = -\underline{\ell}_{i-1}(\underline{\ell}_i = \underline{\ell}_{i-1})$.

After this reflection (refraction) process the light ray will start its new traveling path with $\underline{\mathbf{a}}_i$ as its new light source and $\underline{\ell}_i$ as its new unit directional vector. The reflected (refracted) ray's path is then given by $\underline{\mathbf{a}}_i\underline{\mathbf{t}} = [\underline{\mathbf{a}}_i + \underline{\ell}_i t \quad 1]^T$. The preceding section illustrated how to find the path of reflected (or refracted) ray through single boundary surface. One can apply this approach successively to trace rays through optical systems. An optical system may consist of a series of media separating by j boundary surfaces. To trace light rays, one should label these boundary surfaces sequentially from 1 to j through the optical system (Figure 2). All variables and constants in space after being reflected or refracted by surface $\mathcal{S}_i$ are subscripted with number i. One can trace any ray through a system using the preceding calculation algorithms with $i = 1, i = 2, \ldots,$ until $i = j$. The output ray path is:

$$\begin{aligned} \underline{\mathbf{a}}_j\underline{\mathbf{t}} &= \begin{bmatrix} \underline{\mathbf{a}}_j + \underline{\ell}_j t \\ 1 \end{bmatrix} = \begin{bmatrix} (\underline{\mathbf{a}}_{j-1} + \underline{\ell}_{j-1}t_j) + \underline{\ell}_j t \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{a}}_o + \underline{\ell}_o t_1 + \cdots + \underline{\ell}_{j-1}t_j + \underline{\ell}_j t \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \underline{\mathbf{a}}_o + \sum\limits_{i=1}^{j} \underline{\ell}_{i-1}t_i + \underline{\ell}_j t \\ 1 \end{bmatrix} \end{aligned} \tag{11}$$

These tracing procedures are succinct and can be easily written in computer language. They can result in very powerful and fast optical design programs for personal computers. Two illustrative examples are given in the following.

3.1. Ray Tracing through an Optical Element

A Dove prism (Figure 3) is usually used to invert images without deviation or displacement if the input rays are parallel to its base. We set up this prism using the labeling method described in the last section. The input sample skew ray originates from $\underline{\mathbf{a}}_o = [0 \quad -a/\sqrt{2} \quad a \quad 1]^T$ and

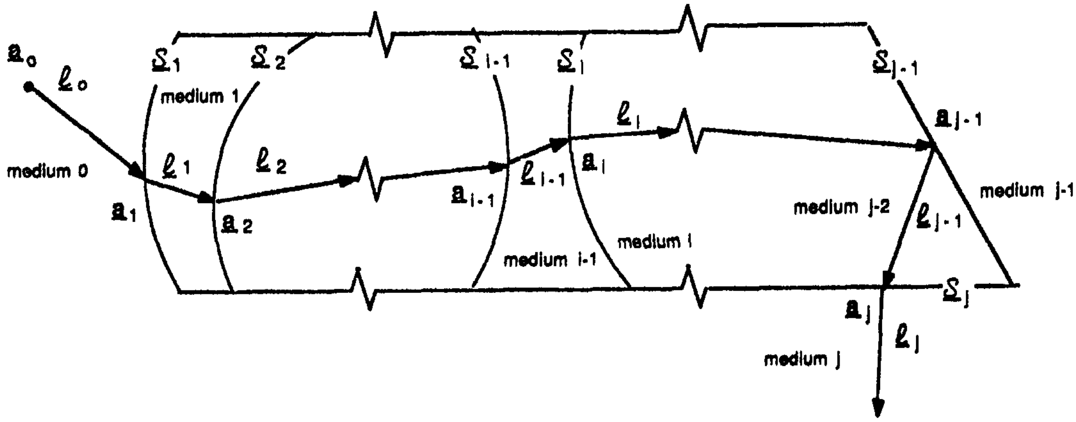

Figure 2. General notation for a ray tracing through j boundary surfaces.

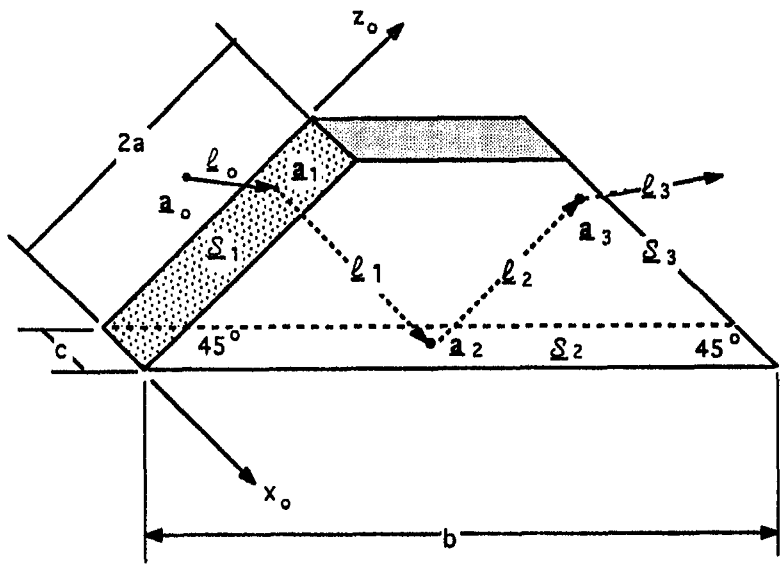

Figure 3. Ray path of a skew ray passing through a Dove prism.

is incident along $\underline{\ell}_o = [1/\sqrt{2} \quad 1/2 \quad 1/2 \quad 0]^T$. If we follow the algorithms of Section 3, we can obtain the intermediate variables and parameters (Appendix A). Note that the output unit directional vector $\underline{\ell}_3$ has the same direction with $\underline{\ell}_o$. If a few sample rays are traced then the various departures from the ideal performance of this prism can be evaluated.

3.2. Ray Tracing through an Optical System

A ray passing through an optical system composed of a right prism (refraction index $= \sqrt{3}$) and a first-surface-mirror is examined in this section (Figure 4). A right prism is an optical element ideally suited to beam deflection and deviation. A ray originating from $\underline{a}_o(\underline{0}) = [0 \quad 0 \quad 0 \quad 1]^T$ and incident along $\underline{\ell}_o(\underline{0}) = [0 \quad 0 \quad 1 \quad 0]^T$ is refracted and reflected through this system. The nominal values of vectors $\underline{\ell}_i$, $\underline{a}_i$ and parameters t_i are given in the Appendix B. The results show that, ideally, the output ray $\underline{\ell}_4$ will be parallel to input ray $\underline{\ell}_o$ with lateral displacement vector $\underline{a}_4 - \underline{a}_1 = [0 \quad -b-f \quad b \quad 1]^T$.

4. ERROR ANALYSIS

The aim of this section is to study the effects of system errors on a ray's path. Notice that $\underline{a}_j\underline{t}$ of equation (11) is a continuous function of light source $\underline{a}_o$ and unit directional vectors $\underline{\ell}_{i-1}$.

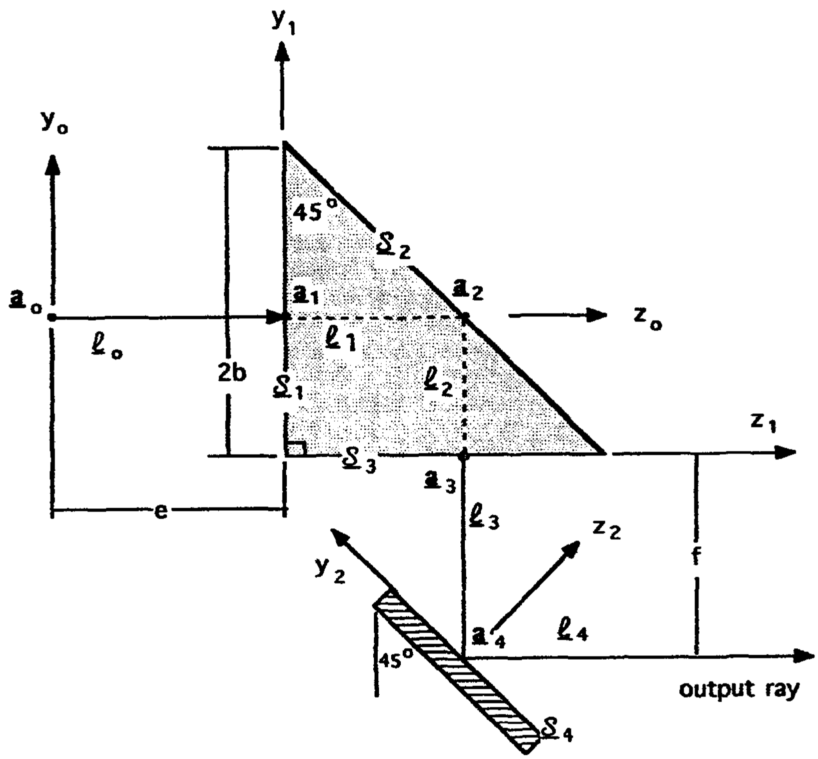

Figure 4. Progress of a ray through a optical system.

To investigate the effects of system errors on this ray, one can approximate $\underline{\mathbf{a}}_j\underline{\mathbf{t}}$ by Taylor series expansion:

$$\underline{\mathbf{a}}_j\underline{\mathbf{t}} = \underline{\mathbf{a}}_j\underline{\mathbf{t}}(\underline{\mathbf{0}}) + \Delta\underline{\mathbf{a}}_j\underline{\mathbf{t}}(\underline{\mathbf{0}}), \tag{12}$$

where $(\underline{\mathbf{0}})$ means that all parameters are evaluated under ideal conditions (i.e., no system error). $\underline{\mathbf{a}}_j\underline{\mathbf{t}}(\underline{\mathbf{0}})$ and $\Delta\underline{\mathbf{a}}_j\underline{\mathbf{t}}(\underline{\mathbf{0}})$ represent the ideal path and the deviation of $\underline{\mathbf{a}}_j\underline{\mathbf{t}}$, respectively. Explicitly they are given by:

$$\underline{\mathbf{a}}_j\underline{\mathbf{t}}(\underline{\mathbf{0}}) = \begin{bmatrix} \underline{\mathbf{a}}_j(\underline{\mathbf{0}}) + \underline{\ell}_j(\underline{\mathbf{0}})t(\underline{\mathbf{0}}) \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{a}}_o(\underline{\mathbf{0}}) + \sum_{i=1}^{j} \underline{\ell}_{i-1}(\underline{\mathbf{0}})t_i(\underline{\mathbf{0}}) + \underline{\ell}_j(\underline{\mathbf{0}})t(\underline{\mathbf{0}}) \\ 1 \end{bmatrix}, \tag{13}$$

$$\begin{aligned} \Delta\underline{\mathbf{a}}_j\underline{\mathbf{t}}(\underline{\mathbf{0}}) &= \begin{bmatrix} \Delta\underline{\mathbf{a}}_j(\underline{\mathbf{0}}) + \Delta\underline{\ell}_j(\underline{\mathbf{0}})t(\underline{\mathbf{0}}) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \Delta\underline{\mathbf{a}}_o(\underline{\mathbf{0}}) + \sum_{i=1}^{j} [\Delta\underline{\ell}_{i-1}(\underline{\mathbf{0}})t_i(\underline{\mathbf{0}}) + \underline{\ell}_{i-1}(\underline{\mathbf{0}})\Delta t_i(\underline{\mathbf{0}})] + \Delta\underline{\ell}_j(\underline{\mathbf{0}})t(\underline{\mathbf{0}}) \\ 0 \end{bmatrix}, \end{aligned} \tag{14}$$

where $\Delta\underline{\mathbf{a}}_o(\underline{\mathbf{0}}) = [\Delta x_o \quad \Delta y_o \quad \Delta z_o]^{\mathrm{T}}$ is the position error of light source. $\Delta\underline{\ell}_i(\underline{\mathbf{0}})(i = 0, 1, \ldots, j)$ and $\Delta t_i(\underline{\mathbf{0}})(i = 1, 2, \ldots, j)$ are respectively the variations of $\underline{\ell}_i$ and t_i due to system errors. Notice that $\underline{\ell}_i$ and t_i are implicit functions of $\underline{S}_i$. To determine $\Delta\underline{\ell}_i(\underline{\mathbf{0}})$ and $\Delta t_i(\underline{\mathbf{0}})$, a mathematical scheme is needed to model the position errors and orientation errors of boundary surfaces.

An optical system may consist of several (for example, n) elements. To describe the position and orientation of an element R, one needs a coordinate frame $(\mathbf{xyz})_R$ $(R = 1, 2, \ldots, n)$ imbedded in element R. Notice that no one can place any element at its ideal position and orientation in assembling an optical system. Consequently, the actual position and orientation of frame $(\mathbf{xyz})_R$ will differ from its ideal position and orientation (denoted as $(\mathbf{xyz})_{\hat{R}}$). Mathematically, the position and orientation deviations between these two frames can be described by a translation error vector $[\Delta x_R \quad \Delta y_R \quad \Delta z_R]^{\mathsf{T}}$ and a rotational angular error vector $[\Delta\Gamma_R \quad \Delta\Psi_R \quad \Delta\Phi_R]^{\mathsf{T}}$ with respect to the three axes of frame $(\mathbf{xyz})_{\hat{R}}$ (Figure 5). The position and orientation deviations between these two frames can be expressed by the following matrix form:

$$
\begin{aligned}
{}^{\hat{R}}\underline{\underline{\mathbf{A}}}_R &= \underline{\underline{\mathbf{T}}}\mathbf{rans}(\Delta x_R, \Delta y_R, \Delta z_R)\underline{\underline{\mathbf{R}}}\mathbf{ot}(z_{\hat{R}}, \Delta\Phi_R)\underline{\underline{\mathbf{R}}}\mathbf{ot}(y_{\hat{R}}, \Delta\Psi_R)\underline{\underline{\mathbf{R}}}\mathbf{ot}(x_{\hat{R}}, \Delta\Gamma_R) \\
&= \begin{bmatrix} C\Delta\Phi_R C\Delta\Psi_R & C\Delta\Phi_R S\Delta\Psi_R S\Delta\Gamma_R - S\Delta\Phi_R C\Delta\Gamma_R & C\Delta\Phi_R S\Delta\Psi_R C\Delta\Gamma_R + S\Delta\Phi_R S\Delta\Gamma_R & \Delta x_R \\ S\Delta\Phi_R C\Delta\Psi_R & S\Delta\Phi_R S\Delta\Psi_R S\Delta\Gamma_R + C\Delta\Phi_R C\Delta\Gamma_R & S\Delta\Phi_R S\Delta\Psi_R C\Delta\Gamma_R - C\Delta\Phi_R S\Delta\Gamma_R & \Delta y_R \\ -S\Delta\Psi_R & C\Delta\Psi_R S\Delta\Gamma_R & C\Delta\Psi_R C\Delta\Gamma_R & \Delta z_R \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{15}
$$

where $\underline{\underline{\mathbf{T}}}\mathbf{rans}$ and $\underline{\underline{\mathbf{R}}}\mathbf{ot}$ are translation and rotation operators [1]. Due to the fact that these error vectors are very small, equation (15) can be approximated by the first order Taylor series expansion:

$$
{}^{\hat{R}}\underline{\underline{\mathbf{A}}}_R = \begin{bmatrix} {}^{\hat{R}}\underline{\mathbf{I}}_R & {}^{\hat{R}}\underline{\mathbf{J}}_R & {}^{\hat{R}}\underline{\mathbf{K}}_R & {}^{\hat{R}}\underline{\mathbf{P}}_R \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\Delta\Phi_R & \Delta\Psi_R & \Delta x_R \\ \Delta\Phi_R & 1 & -\Delta\Gamma_R & \Delta y_R \\ -\Delta\Psi_R & \Delta\Gamma_R & 1 & \Delta z_R \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{16}
$$

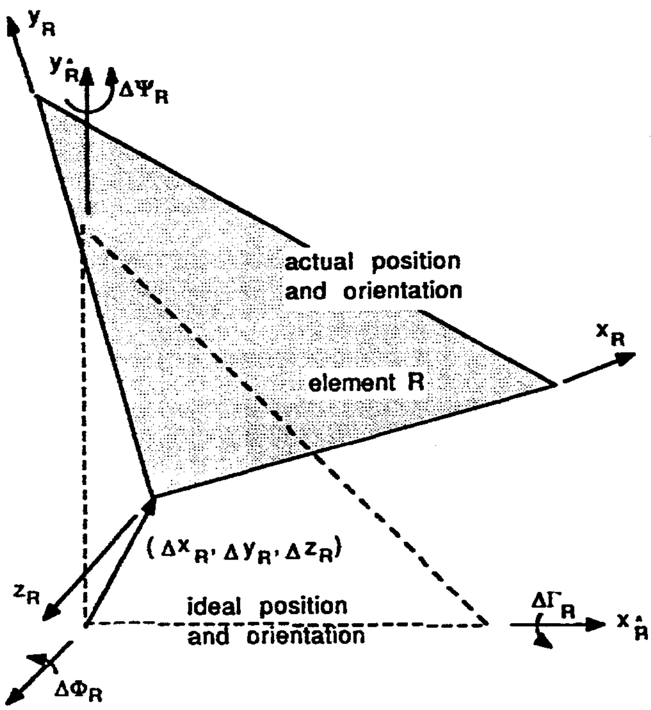

Figure 5. Position error and orientation error of an optical element.

It should be noted that if element R is placed at ideal position and orientation, ${}^{\hat{R}}\underline{\underline{\mathbf{A}}}_R$ will be reduced to the unit matrix. The actual position and orientation of frame $(\mathbf{xyz})_R$, with respect to the base frame $(\mathbf{xyz})_o$, is given by:

$$
{}^{o}\underline{\underline{\mathbf{A}}}_R = \underline{\underline{\mathbf{A}}}_R = \begin{bmatrix} \underline{\mathbf{I}}_R & \underline{\mathbf{J}}_R & \underline{\mathbf{K}}_R & \underline{\mathbf{P}}_R \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{o}\underline{\underline{\mathbf{A}}}_{\hat{R}} = {}^{\hat{R}}\underline{\underline{\mathbf{A}}}_R. \tag{17}
$$

The actual position of boundary surface $\underline{S}_i$, expressed with respect to frame $(\mathbf{xyz})_o$, can now be obtained from the following transformation:

$$\underline{S}_i = [\,\underline{\mathbf{n}}_i \quad -d_i\,] = {}^R\underline{S}\,{}^R\underline{\underline{\mathbf{A}}}_o = {}^R\underline{S}_i\,{}^o\underline{\underline{\mathbf{A}}}_R^{-1}. \tag{18}$$

Equation (18) gives implicitly the actual position of boundary surface $\underline{S}_i = [\,\underline{\mathbf{n}}_i \quad -d_i\,]$. With $\underline{\mathbf{n}}_i$ and d_i in terms of system errors, one can evaluate $\Delta\underline{\ell}_i(\underline{0})$, $\Delta t_i(\underline{0})$, and $\Delta\underline{\mathbf{a}}_i(\underline{0})$ of equation (14) by taking total derivatives of $\underline{\ell}_i$, t_i, and $\underline{\mathbf{a}}_i$ sequentially.

4.1. Error Analysis of Optical System

The effects of error components of system shown in Figure 4 are investigated as an illustrative example. The deviations of t_i, $\underline{\mathbf{a}}_i$, and $\underline{\ell}_i$ due to system errors are listed in the Appendix C. The result of $\Delta\underline{\ell}_3(\underline{0})$ reveals that the right prism cannot bend the input beam as it would in an ideal situation if its unit directional vector $\underline{\ell}_o$ is not perpendicular to boundary surface $\underline{S}_1$. To increase the precision of this system, these two error components $\Delta\Gamma_2$ (rotational error along $\mathbf{z}_{\hat{2}}$) and Δz_2 (translation error along $\mathbf{z}_{\hat{2}}$) of the reflecting mirror should be kept as small as possible. Notice that $\Delta\underline{\ell}_i(\underline{0})$, $\Delta t_i(\underline{0})$, and $\Delta\underline{\mathbf{a}}_i(\underline{0})$ are linear functions of system error components. The leading coefficient is the sensitivity representing the weight with which each error component will affect the system's accuracy. In tolerance allocation, one can define the tolerance of the error component to be proportional to the inverse of its sensitivity, then the more sensitive the error component is, the less effect it will have on system's accuracy. If the dominant error component can be controlled, then the precision of this optical system can certainly be upgraded.

5. CONCLUSIONS

The 4×4 homogeneous transformation matrix is one of the most popular mathematical tools in the robotics, mechanisms, and computer graphics. This paper extends its use to the domain of skew ray tracing and error analysis of optical elements and systems. The reflection and refraction laws of optics are formulated in the language of homogeneous transformation matrices. By using this ray tracing technique the paths of rays through optical elements and systems can be easily traced. Furthermore, the ray's path can be expressed implicitly in terms of the position errors and orientation errors of elements. The deviation values of incidence points and unit direction vectors of a ray can be determined and the sensitivity of each error component to the system's accuracy can be evaluated. This error analysis will provide economical tolerance bands of optical elements in tolerance allocation.

APPENDIX A

The intermediate variables and parameters are given by:

$$c_o = c_2 = c_3 = 1, \tag{A1}$$

$$c_1 = \sqrt{2}, \tag{A2}$$

$$\underline{\mathbf{n}}_1 = [0 \quad 1 \quad 0], \tag{A3}$$

$$\underline{\mathbf{n}}_2 = \left[0 \quad \frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}}\right], \tag{A4}$$

$$\underline{\mathbf{n}}_3 = [0 \quad 0 \quad 1], \tag{A5}$$

$$\underline{S}_1 = [0 \quad 1 \quad 0 \quad 0], \tag{A6}$$

$$\underline{S}_2 = \left[0 \quad \frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right], \tag{A7}$$

$$\underline{S}_3 = \left[0 \quad 0 \quad 1 \quad \frac{-b}{\sqrt{2}}\right], \tag{A8}$$

$$t_1 = \sqrt{2a}, \tag{A9}$$

$$\underline{\mathbf{a}}_1 = \left[a \quad 0 \left(1+\frac{1}{\sqrt{2}}\right) a \quad 1\right]^{\mathsf{T}}, \tag{A10}$$

$$\theta_{11} = \frac{\pi}{3}, \tag{A11}$$

$$\theta_{12} = \sin^{-1}\left(\sqrt{6}/4\right), \tag{A12}$$

$$\underline{\mathbf{m}}_1 = \left[\frac{1}{\sqrt{3}} \quad 0 \quad -\sqrt{\tfrac{2}{3}} \quad 0\right]^{\mathsf{T}}, \tag{A13}$$

$$\underline{\ell}_1 = \left[\frac{1}{2} \quad \frac{\sqrt{10}}{4} \quad \frac{\sqrt{2}}{4} \quad 0\right]^{\mathsf{T}}, \tag{A14}$$

$$t_2 = \frac{\left(\sqrt{10}+\sqrt{2}+\sqrt{5}+1\right)a}{2}, \tag{A15}$$

$$\underline{\mathbf{a}}_2 = \begin{bmatrix} \dfrac{\left[4+\left(\sqrt{2}+1\right)\left(\sqrt{5}+1\right)\right]a}{4} \\ \dfrac{\left(\sqrt{2}+2\right)\left(\sqrt{5}+5\right)a}{4} \\ \dfrac{\left(10+5\sqrt{2}+2\sqrt{5}+\sqrt{10}\right)a}{8} \\ 1 \end{bmatrix}, \tag{A16}$$

$$\theta_{21} = \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right), \tag{A17}$$

$$\underline{\mathbf{m}}_2 = \left[\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \quad \frac{-1}{\sqrt{5+\sqrt{5}}} \quad \frac{-1}{\sqrt{5+\sqrt{5}}} \quad 0\right]^{\mathsf{T}}, \tag{A18}$$

$$\underline{\ell}_2 = \left[\frac{1}{2} \quad \frac{\sqrt{2}}{4} \quad \frac{\sqrt{10}}{4} \quad 0\right]^{\mathsf{T}}, \tag{A19}$$

$$t_3 = \frac{\left(\sqrt{10}+\sqrt{2}+\sqrt{5}+1\right)a}{2} + \frac{2b}{\sqrt{5}}, \tag{A20}$$

$$\theta_{31} = \cos^{-1}\left(\frac{\sqrt{10}}{4}\right), \tag{A21}$$

$$\theta_{32} = \frac{\pi}{3}, \tag{A22}$$

$$\underline{\mathbf{a}}_3 = \begin{bmatrix} \dfrac{\left(\sqrt{10}+\sqrt{5}+\sqrt{2}+3\right)a}{2} + \dfrac{b}{\sqrt{5}} \\ \left(\dfrac{3\sqrt{5}}{4}+\dfrac{3\sqrt{10}}{8}+\dfrac{11\sqrt{2}}{8}+\dfrac{11}{4}\right)a + \dfrac{b}{\sqrt{10}} \\ \left(\dfrac{\sqrt{5}}{2}+\dfrac{\sqrt{10}}{4}+\dfrac{5\sqrt{2}}{4}+\dfrac{5}{2}\right)a + \dfrac{b}{\sqrt{2}} \\ 1 \end{bmatrix}, \tag{A23}$$

$$\underline{\mathbf{m}}_3 = \left[\frac{-1}{\sqrt{3}} \quad \frac{2}{\sqrt{6}} \quad 0 \quad 0\right]^{\mathsf{T}}, \tag{A24}$$

$$\underline{\ell}_3 = \left[\frac{1}{\sqrt{2}} \quad \frac{1}{2} \quad \frac{1}{2} \quad 0\right]^{\mathsf{T}}. \tag{A25}$$

APPENDIX B

The nominal values of various vectors and parameters are given by:

$$ {}^{o}\underline{\underline{\mathbf{A}}}_{\hat{1}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{B1} $$

$$ {}^{o}\underline{\underline{\mathbf{A}}}_{\hat{2}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -(f+b) \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & (e+b) \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{B2} $$

$$ {}^{\hat{1}}\underline{S}_1 = [0 \quad 0 \quad 1 \quad 0], \tag{B3} $$

$$ {}^{\hat{1}}\underline{S}_2 = \left[0 \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}b}\right], \tag{B4} $$

$$ {}^{\hat{1}}\underline{S}_3 = [0 \quad -1 \quad 0 \quad 0], \tag{B5} $$

$$ {}^{\hat{1}}\underline{S}_4 = [0 \quad 0 \quad -1 \quad 0], \tag{B6} $$

$$ {}^{o}\underline{S}_1 = \underline{S}_1 = {}^{\hat{1}}\underline{S}_1 {}^{\hat{1}}\underline{\underline{\mathbf{A}}}_o = [\underline{\mathbf{n}}_1 \quad -d_1] = [0 \quad 0 \quad 1 \quad -e], \tag{B7} $$

$$ t_1(\underline{\mathbf{0}}) = e, \tag{B8} $$

$$ \underline{\mathbf{a}}_1(\underline{\mathbf{0}}) = [0 \quad 0 \quad e]^{\mathsf{T}}, \tag{B9} $$

$$ \theta_{11}(\underline{\mathbf{0}}) = \theta_{12}(\underline{\mathbf{0}}) = 0^o, \tag{B10} $$

$$ \underline{\ell}_1(\underline{\mathbf{0}}) = [0 \quad 0 \quad 1]^{\mathsf{T}}, \tag{B11} $$

$$ {}^{o}\underline{S}_2 = \underline{S}_2 = {}^{\hat{1}}\underline{S}_2 {}^{\hat{1}}\underline{\underline{\mathbf{A}}}_o = [\underline{\mathbf{n}}_2 \quad -d_2] = \left[0 \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{-(b+e)}{\sqrt{2}}\right], \tag{B12} $$

$$ t_2(\underline{\mathbf{0}}) = b, \tag{B13} $$

$$ \underline{\mathbf{a}}_2(\underline{\mathbf{0}}) = [0 \quad 0 \quad e+b]^{\mathsf{T}}, \tag{B14} $$

$$ \theta_{21}(\underline{\mathbf{0}}) = \theta_{22}(\underline{\mathbf{0}}) = 45^o, \tag{B15} $$

$$ \underline{\mathbf{m}}_2(\underline{\mathbf{0}}) = [1 \quad 0 \quad 0]^{\mathsf{T}}, \tag{B16} $$

$$ \underline{\ell}_2(\underline{\mathbf{0}}) = [0 \quad -1 \quad 0]^{\mathsf{T}}, \tag{B17} $$

$$ {}^{o}\underline{S}_3 = \underline{S}_3 = {}^{\hat{1}}\underline{S}_3 {}^{\hat{1}}\underline{\underline{\mathbf{A}}}_o = [\underline{\mathbf{n}}_3 \quad -d_3] = [0 \quad -1 \quad 0 \quad -b], \tag{B18} $$

$$ t_3(\underline{\mathbf{0}}) = b, \tag{B19} $$

$$ \underline{\mathbf{a}}_3(\underline{\mathbf{0}}) = [0 \quad -b \quad e+b]^{\mathsf{T}}, \tag{B20} $$

$$ \theta_{31}(\underline{\mathbf{0}}) = \theta_{32}(\underline{\mathbf{0}}) = 0^o, \tag{B21} $$

$$ \underline{\ell}_3(\underline{\mathbf{0}}) = [0 \quad -1 \quad 0]^{\mathsf{T}}, \tag{B22} $$

$$ {}^{o}\underline{S}_4 = \underline{S}_4 = {}^{\hat{2}}\underline{S}_4 {}^{\hat{2}}\underline{\underline{\mathbf{A}}}_o = [\underline{\mathbf{n}}_4 \quad -d_4] = \left[0 \quad \frac{-1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad \frac{e-f}{\sqrt{2}}\right], \tag{B23} $$

$$ t_4(\underline{\mathbf{0}}) = f, \tag{B24} $$

$$ \underline{\mathbf{a}}_4(\underline{\mathbf{0}}) = [0 \quad -b-f \quad e+b]^{\mathsf{T}}, \tag{B25} $$

$$ \theta_{41}(\underline{\mathbf{0}}) = 45^o, \tag{B26} $$

$$ \underline{\mathbf{m}}_4(\underline{\mathbf{0}}) = [-1 \quad 0 \quad 0]^{\mathsf{T}}, \tag{B27} $$

$$ \underline{\ell}_4(\underline{\mathbf{0}}) = [0 \quad 0 \quad 1]^{\mathsf{T}}. \tag{B28} $$

APPENDIX C

The deviation values of various vectors and parameters due to system errors are:

$$ {}^{o}\underline{\underline{\mathbf{A}}}_1 = \begin{bmatrix} 1 & -\Delta\Phi_1 & \Delta\Psi_1 & \Delta x_1 \\ \Delta\Phi_1 & 1 & -\Delta\Gamma_1 & \Delta y_1 - b \\ -\Delta\Psi_1 & \Delta\Gamma_1 & 1 & \Delta z_1 + e \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{C1} $$

$$ {}^{o}\underline{\underline{\mathbf{A}}}_2 = \begin{bmatrix} 1 & -\Delta\Phi_2 & \Delta\Psi_2 & \Delta x_2 \\ \dfrac{\Delta\Phi_2 - \Delta\Psi_2}{\sqrt{2}} & \dfrac{1+\Delta\Gamma_2}{\sqrt{2}} & \dfrac{1-\Delta\Gamma_2}{\sqrt{2}} & \dfrac{\Delta y_2 + \Delta z_2}{\sqrt{2}} - (f+b) \\ \dfrac{-\Delta\Phi_2 - \Delta\Psi_2}{\sqrt{2}} & \dfrac{-1+\Delta\Gamma_2}{\sqrt{2}} & \dfrac{1+\Delta\Gamma_2}{\sqrt{2}} & \dfrac{\Delta z_2 + \Delta y_2}{\sqrt{2}} + e + b \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{C2} $$

$$ \underline{\underline{S}}_1 = [\,\Delta\Psi_1 \quad -\Delta\Gamma_1 \quad 1 \quad -(b\Delta\Gamma_1 + \Delta z_1 + e)\,], \tag{C3} $$

$$ \underline{\underline{S}}_2 = \left[\,\frac{\Delta\Psi_1 - \Delta\Phi_1}{\sqrt{2}} \quad \frac{1-\Delta\Gamma_1}{\sqrt{2}} \quad \frac{1+\Delta\Gamma_1}{\sqrt{2}} \quad \frac{-(e+b)(\Delta\Gamma_1+1) - (\Delta y_1 + \Delta z_1}{\sqrt{2}}\,\right], \tag{C4} $$

$$ \underline{\underline{S}}_3 = [\,\Delta\Phi_1 \quad -1 \quad -\Delta\Gamma_1 \quad \Delta y_1 + e\Delta\Gamma_1 - b\,], \tag{C5} $$

$$ \underline{\underline{S}}_4 = \left[\,-\Delta\Psi_2 \quad \frac{\Delta\Gamma_2 - 1}{\sqrt{2}} \quad \frac{-1-\Delta\Gamma_2}{\sqrt{2}} \quad \Delta z_2 + \frac{(f+e+2b)\Delta\Gamma_2 + e - f}{\sqrt{2}}\,\right], \tag{C6} $$

$$ \Delta\underline{\ell}_o(\underline{0}) = [\,\Delta\Psi_o \quad -\Delta\Gamma_o \quad 0\,]^{\mathsf{T}}, \tag{C7} $$

$$ \Delta t_1(\underline{0}) = -\Delta z_o + b\Delta\Gamma_1 + \Delta z_1, \tag{C8} $$

$$ \Delta\underline{\mathbf{a}}_1(\underline{0}) = [\,\Delta x_o + e\Delta\Psi_o \quad \Delta y_o - e\Delta\Gamma_o \quad b\Delta\Gamma_1 + \Delta z_1\,]^{\mathsf{T}}, \tag{C9} $$

$$ \Delta\underline{\ell}_1(\underline{0}) = [\,0.577\Delta\Psi_o + 0.4226\Delta\Gamma_1 \quad -0.577\Delta\Gamma_o - 0.4226\Delta\Gamma_1 \quad 0\,]^{\mathsf{T}}, \tag{C10} $$

$$ \Delta t_2(\underline{0}) = 1.577b\Delta\Gamma_o - 0.577b\Delta\Gamma_1 - \Delta y_o + \Delta y_1, \tag{C11} $$

$$ \Delta\underline{\mathbf{a}}_2(\underline{0}) = \begin{bmatrix} \Delta x_o + (0.577b + e)\Delta\Psi_o + 0.422b\Delta\Psi_1 \\ \Delta y_o - (0.577b + e)\Delta\Gamma_o - 0.422b\Delta\Gamma_1 \\ -\Delta y_o + \Delta y_1 + \Delta z_1 + 1.577b\Delta\Gamma_o + 0.422b\Delta\Gamma_1 \end{bmatrix}, \tag{C12} $$

$$ \Delta\underline{\ell}_2(\underline{0}) = \begin{bmatrix} 0.816\Delta\Psi_o - 0.609\Delta\Psi_1 + 1.207\Delta\Phi_1 \\ 0 \\ 0.577\Delta\Gamma_o - 1.577\Delta\Gamma_1 \end{bmatrix}, \tag{C13} $$

$$ \Delta t_3(\underline{0}) = \Delta y_o - \Delta y_1 - (e + 0.577b)\Delta\Gamma_o + 0.577b\Delta\Gamma_1, \tag{C14} $$

$$ \Delta\underline{\mathbf{a}}_3(\underline{0}) = \begin{bmatrix} \Delta x_o + (1.394b + e)\Delta\Psi_o - 0.187b\Delta\Psi_1 + 1.207b\Delta\Phi_1 \\ \Delta y_1 - b\Delta\Gamma_1 \\ -\Delta y_o + \Delta y_1 + \Delta z_1 + 2.155b\Delta\Gamma_o - 1.155b\Delta\Gamma_1 \end{bmatrix}, \tag{C15} $$

$$ \Delta\underline{\ell}_3(\underline{0}) = \begin{bmatrix} 1.414\Delta\Psi_o - 1.055\Delta\Psi_1 + 1.359\Delta\Phi_1 \\ 0 \\ -\Delta\Gamma_o + \Delta\Gamma_1 \end{bmatrix}, \tag{C16} $$

$$ \Delta t_4(\underline{0}) = \Delta y_o + 2\Delta y_1 + \Delta z_1 + \sqrt{2}\Delta z_2 + (2.155b - f)\Delta\Gamma_o + (f - 2.155b)\Delta\Gamma_1, \tag{C17} $$

$$ \Delta\underline{\mathbf{a}}_4(\underline{0}) = \begin{bmatrix} \Delta x_o + (1.39b + e + 1.42f)\Delta\Psi_o + (0.31b - 1.06f)\Delta\Psi_1 + (1.21b + 1.36f)\Delta\Phi_1 \\ \Delta y_o - \Delta y_1 - \Delta z_1 + 1.414\Delta z_2 + (f - 2.155b)\Delta\Gamma_o + (1.155b - f)\Delta\Gamma_1 \\ -\Delta y_o + \Delta y_1 + \Delta z_1 + (2.155b - f)\Delta\Gamma_o + (f - 1.155b)\Delta\Gamma_1 \end{bmatrix}, \tag{C18} $$

$$\Delta\underline{\ell}_4(\underline{0}) = \begin{bmatrix} 1.414\Delta\Psi_o - 1.055\Delta\Psi_1 + 1.359\Delta\Phi_1 \\ \Delta\Gamma_o - \Delta\Gamma_1 - 2\Delta\Gamma_2 \\ 0 \end{bmatrix}. \tag{C19}$$

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