Pergamon

Mathl. Comput. Modelling Vol. 19, No. 12, pp. 119–124, 1994

0895-7177/94 $7.00 + 0.00

0895-7177(94)00086-7

A Path Integral Approach for Solving Euler's Equation Describing the Oscillations of an Infinite Bar

H. Gzyl
Escuela de Física y Matemática, Facultad de Ciencias
Universidad Central de Venezuela, A.P. 52120, Caracas 1050-A, Venezuela

(Received December 1992; accepted May 1993)

Abstract—We provide a path integral representation of the solution to Euler's equation, describing the transversal oscillations of a rail under action of damping, restoring, and external forces, with respect to the trajectories of a quasi-Markov process naturally associated with Euler's equation for the free, infinite rail.

1. INTRODUCTION

We have been exploring some connections between diffusion and scalar waves. See the series beginning in [1] and ending in [2]. In that line of work, the role of path integrals was to provide explicit representation for the Green functions involved.

Proceeding as in the references cited above, here we shall exploit the possibility of describing solutions to equations of the type

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0 \tag{1.1}$$

(plus appropriate initial conditions) by means of path integrals, to obtain path integral representations of the solutions to the equation

$$\mathrm{EI}\,\frac{\partial^4 v}{\partial x^4} + \mu\,\frac{\partial^2 v}{\partial t^2} + 2\mu\,\omega\,\frac{\partial v}{\partial t} + k\,v = F(t,x) \tag{1.2}$$

(plus appropriate initial conditions), which describes the vibrations of an infinite one-dimensional beam, under the presence of damping, resistive forces, and external forcing.

The way we will deal with (1.2) is to reduce it to a problem related to (1.1) by means of a Laplace transform in the time variable. We will do this in Section 2, where we do some elementary computations, and we apply the general techniques developed in [3] for the heat equation, by means of which we transform hard path integrals onto path integrals "easy to solve by simulation."

In Section 3, we consider a mixture of probabilistic approaches, useful when $\omega(x)$ is constant, and in Section 4, we examine a simple perturbative expansion, assuming that $\omega(x)$ and $k(x)$ are stationary Gaussian processes indexed by x. Let us now review the basics of the integration associated to (1.1).

The solution to (1.1), supplemented with the initial condition

$$U(0,x) = U_0(x), \tag{1.1a}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX can be written down in terms of the fundamental solution to (1.1) as

$$U(t,x) = \int p(t,x-y)\,U_0(y)\,dy, \tag{1.3}$$

where

$$p(t,x) = \int e^{-t\,\xi^4}\,e^{-ix\,\xi}\,\frac{d\xi}{2\pi}, \tag{1.4}$$

and $p(t,x)$ has the following properties: it is infinitely differentiable in $(0,\infty)\times(-\infty,\infty)$, scales as $p(t,x) = p(1,t^{-1/4}\,x)\,t^{-1/4}$, satisfies $p(t,x-y) \to \delta(x,y)$ as $t \to 0$, and for every $t > 0$, integrates to 1 over the whole space and, more importantly,

$$p(t+s,\,x-y) = \int_{-\infty}^{\infty} p(t,x-z)\,p(s,y-z)\,dz,$$

for every $t > 0$, $s \geq 0$, x, y in $\mathbb{R}$. This last property allows one to construct a consistent family of finitely additive measures P^x on $\Omega = \{\omega : [0,\infty) \to \mathbb{R}\}$, by defining them over cylindrical functions by

$$E^*[F(X(t_1),\ldots,X(t_n))] = \int p(t_1,\,x-y_1)\cdots p(t_n-t_{n-1},\,y_{n-1}-y_n)\,F(y_1,\ldots,y_n)\,dy_1\cdots dy_n.$$

Even though the P^x are not σ-additive on $(\Omega,\mathcal{L})$, where $\mathcal{L}$ is the σ-algebra generated by the cylindrical sets, much of the standard artillery can be developed for the coordinate process

$$X_t(t) : \Omega \to \mathbb{R}, \quad X_t(\omega) = \omega(t).$$

For example, much of the stochastic calculus, analogous to Îto calculus, can be extended for the process $X(t)$, which by the way, could be used to extend the results of Section 3 to include the case in which the functions ω, k appearing in (1.2) are position dependent as in [4].

Important for us is the extension of the Feynmann-Kac formula, asserting that the solution to

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + U(x)\,u = 0, \quad u(0,x) = u_0(x)$$

is given by

$$u(t,x) = E^x\left[u_0(X_t)\,\exp\left(-\int_0^t U(X_s)\,ds\right)\right], \tag{1.5}$$

from which it follows that the solution to

$$\left(\alpha + \frac{\partial^4}{\partial x^4} + U(x)\right) W(x) = u_0(x)$$

is given by

$$W(x) = \int_0^{\infty} e^{-\alpha t}\,E^x\left[u_0(X_t)\,\exp\left(-\int_0^t U(X_s)\,ds\right)\right] dt. \tag{1.6}$$

Take a look at [5,6] for further work along these lines.

2. SOLUTIONS VIA LAPLACE TRANSFORMS

Consider equation (1.2) subject to the initial conditions

$$v(0,x) = v_1(x), \quad \frac{\partial v}{\partial t}(0,x) = v_2(x);$$

we will allow the coefficients μ, ω, and k to be dependent on the coordinates, but independent of time. Also, below, we will throw in some randomness as in the study of rail vibrations carried out in [4]. If we denote by $\tilde{v}(\alpha,x)$ the Laplace transform in time of (1.2), we obtain, by substitution, that

$$\left(\mu\,\alpha^2 + 2\mu\,\omega\,\alpha + \mathrm{EI}\,\frac{\partial^4}{\partial x^4} + k\right)\tilde{v}(\alpha,x) = g(\alpha,x), \tag{2.1}$$

where $g(\alpha,x) = \tilde{F}(\alpha,x) + \mu\,(\alpha+2\omega)\,v_1(x) + \mu\,v_2(x)$.

According to (1.6), the solution to (2.1) can be written down as an integral over the paths of $X(t)$ as

$$\tilde{v}(\alpha, x) = \frac{1}{\mathrm{EI}} \int_0^\infty e^{-\beta\alpha^2 t} E^x \left[g(\alpha, X(t)) \exp\left(- \int_0^t q(X(s))\, ds \right) \right] dt, \tag{2.2}$$

where $\beta = \mu/\mathrm{EI}$ and $q(x) = (k(x) + 2\mu\,\alpha\,\omega(x))/\mathrm{EI}$. To begin with, we shall compute (2.2) when $\mu,\ k,\ \omega$ are constant. Thus

$$\begin{aligned} \tilde{v}(\alpha, x) &= \frac{1}{\mathrm{EI}} \int_0^\infty e^{-(\beta\alpha^2+q)\,t} E^x[g(\alpha, X(t))]\, dt \\ &= \frac{1}{\mathrm{EI}} \int_0^\infty e^{-(\beta\alpha^2+q)\,t} \int_{-\infty}^\infty g(\alpha, y)\, p(t, x-y)\, dy\, dt, \end{aligned} \tag{2.3}$$

which, when $g(\alpha, y)$ can be recovered from its Fourier transform $\hat{g}(\alpha, \xi)$, it can be rewritten as

$$\begin{aligned} \tilde{v}(\alpha, x) &= \frac{1}{\mathrm{EI}} \int_{-\infty}^\infty \hat{g}(\alpha, \xi) \left(\int_0^\infty e^{-(\beta\alpha^2+q)\,t} e^{-t\xi^4}\, dt \right) e^{-ix\xi} \frac{d\xi}{2\pi} \\ &= \frac{1}{\mathrm{EI}} \int_{-\infty}^\infty \frac{\hat{g}(\alpha, \xi)\, e^{-ix\xi}}{\beta\alpha^2 + q + \xi^4} \frac{d\xi}{2\pi} = \frac{1}{\mu} \int_{-\infty}^\infty \frac{\hat{g}(\alpha, \xi)\, e^{-ix\xi}}{(\alpha + \lambda_1)(\alpha + \lambda_2)} \frac{d\xi}{2\pi} \end{aligned}$$

where $-\lambda_i,\ i = 1, 2$ are the roots of $\alpha^2 + 2\alpha\,\omega + k/\mu + \xi^4/\beta = 0$.

Since $1/[(\alpha + \lambda_1)(\alpha + \lambda_2)]$ is the Laplace transform of $[e^{-\lambda_1 t} - e^{-\lambda_2 t}]/(\lambda_2 - \lambda_1)$, we can finally obtain $v(t, x)$ as the appropriate convolution. This is the result we would have obtained by a combination of Laplace and Fourier transforms applied to (1.1) directly. So far we are consistent. When $F(t, x) = Q\,\delta(x - ct)$ as in [4], and when $v_1(x) = v_2(x) = 0$, equation (2.3) becomes

$$\begin{aligned} \tilde{v}(\alpha, x) &= \frac{1}{\mathrm{EI}} \int_0^\infty e^{-(\beta\alpha^2+q)\,t} \int_{-\infty}^\infty e^{-\alpha y/c}\, p(t,\, x-y)\, dy\, dt \\ &= \frac{e^{-\alpha x/c}}{\mathrm{EI}} \int_0^\infty e^{-(\beta\alpha^2+q)\,t} \int_{-\infty}^\infty e^{-\alpha y/c}\, p(t, y)\, dy\, dt \\ &= \frac{e^{-\alpha x/c}}{\mathrm{EI}} \int_0^\infty e^{-(\beta\alpha^2+q)\,t} e^{-(\alpha/c)^4 t}\, dt = \frac{e^{-\alpha x/c}}{\mathrm{EI}\ [\beta\alpha^2 + q + (\frac{\alpha}{c})^4]}, \end{aligned}$$

which can be inverted after a partial fraction decomposition.

When $k(x)$ and $\omega(x)$ are not constants, there is not much one can be do with (2.2), except transform it into something else. One such possibility was explored by Bertrand and Gaveau [3] for the heat equation. We follow up their lead. Assume, to simplify, that $k(x) = 0$ and $v_1(x) = v_2(x) = 0$. Furthermore, assume there exist a finite positive measure $\mu(dx)$ and a function $\phi(y)$, such that

$$\omega(x) = \int e^{i\,(y\,x + \phi(y))}\, \mu(dy),$$

and let $m = \int \mu\,(dy)$. Let us denote by $\nu(ds, dy)$ a Poisson random measure on $[0, \infty)$ with intensity $t\,\mu\,(dy)$ which is independent of $X(t)$. That is, on some other probability space we have defined an expectation $\mathcal{E}$, such that for any function $F(s, y)$,

$$\mathcal{E} \exp\left(i \int_0^t \int F(s, y)\, \nu\,(ds, dy) \right) = \exp\left(\int_0^t ds \int (e^{iF(s,y)} - 1)\, \mu\,(dy) \right). \tag{2.4}$$

With this random Poisson measure we associate the Poisson process

$$Y(t) = y_0 + \int_0^t ds \int y\,\nu\,(ds, dy),$$

and write the expectation in (2.2) as

$$E^0\left[g(\alpha,\, x+X(t))\,\exp\left(-\alpha\int_0^t \omega(x+X(s))\,ds\right)\right]. \tag{2.5}$$

Now, by making use of the independence of the increments of $X(t)$, we will proceed as in [3]; that is, we set $a=\pi - i\ln\alpha$ and write

$$\begin{aligned}
\exp\left(-\alpha\int_0^t \omega(x+X(s))\,ds\right) &= e^{mt}\exp\left[-\int_0^t ds\int\left(e^{i\,(x+X(s))\,y}\,e^{i\,(\phi(y)+a)}-1\right)\,\mu\,(dy)\right]\\
&= e^{mt}\,\mathcal{E}\exp\left[-i\int_0^t\int((x+X(s))\,y+\phi(y)+a)\;\nu\,(ds,dy)\right]\\
&= e^{mt}\,\mathcal{E}\left\{\exp\left(-i\int_0^t\int(\phi(y)+a)\,\nu\,(ds,dy)\right)\right.\\
&\qquad\left.\times\exp\left(-i\int_0^t(x+X(s))\,dY_s\right)\right\},
\end{aligned} \tag{2.6}$$

where we made use of (2.4) at the second step. Now, write $g(\alpha,x)$ as

$$g(\alpha,x)=\int\hat{g}(\alpha,y')\,e^{-ixy'}\,dy',$$

and insert this and (2.6) into (2.5). Then, exchanging integrals, we obtain

$$\begin{aligned}
&E^x\left[g(\alpha,X(t))\,\exp\left(-\alpha\int_0^t\omega(X(s))\,ds\right)\right]\\
&\quad= e^{mt}\int\hat{g}(\alpha,y')\,dy'\mathcal{E}\left[\exp\left(i\int_0^t\int(\phi(y)+a)\,\nu\,(ds,dy)\right)\right.\\
&\qquad\left. E^0\left[\exp\left(-i\int_0^t(x+X(s))\,dy_s-i\,y'(x+X(t))\right)\right]\right].
\end{aligned} \tag{2.7}$$

To compute the innermost expectation we proceed as in [3], i.e., write

$$\int_0^t X(s)\,dY_s+y'\,X(t)=-\int_0^t[Y_t-Y_s-y']\,dX(s)$$

and, therefore

$$\begin{aligned}
&E^0\left[\exp\left(-i\int_0^t(x+X(s))\,dY_s-i\,y'\,(x+X(t))\right)\right]\\
&\quad=\exp[-i\,x\,(y'+Y(t))]\,E^0\left[\exp\left(-i\int_0^t(Y_t-Y_s-y')\,dX(s)\right)\right]\\
&\quad=\exp[-i\,x\,(y'+Y(t))]\,\exp\left(-i\int_0^t(Y_t-Y_s-y')^4\,ds\right)
\end{aligned}$$

where obvious use is made of the independence of the increments of $X(t)$ and of (1.4) in its infinitesimal version.

Plugging this result in (2.7), we obtain

$$\begin{aligned}
E^x\left[g(\alpha,X(t))\,\exp\left(-\alpha\int_0^t\omega(X(s))\,ds\right)\right] &= e^{mt}\int\hat{g}(\alpha,y')\,dy'\\
\times\,\mathcal{E}\left[\exp\left(-i\int_0^t\int(\phi(y)+a+x\,y)\,\nu\,(ds,dy)\right)\exp\left(-\int_0^t(Y_t-Y_s-y')^4\,ds\right)\right]&e^{-ixy'}\,dy'.
\end{aligned}$$

The whole point of this lengthy computation is to transform the integral over paths of the complicated (and hard to simulate) process $X(t)$ into an integral over paths of a random Poisson process.

3. A PARTICULAR CASE

When the damping coefficient ω is constant but the elastic coefficient k depends on x, a trick due to Kaplan [7] yields a solution to (1.2).

Note, to begin with, that a solution to (1.2) with $\omega = 0$, i.e., to

$$\mathrm{EI}\,\frac{\partial^4 v}{\partial x^4} + \mu\,\frac{\partial^2 v}{\partial t^2} + k\,v = F(t,x), \tag{3.1}$$

can be obtained using (1.5) as follows. Taking the Laplace transform of (3.1), we obtain (2.1), with $\omega = 0$, i.e.,

$$\left(\mu\,\alpha^2 + \mathrm{EI}\,\frac{\partial^4}{\partial x^4} + k(x)\right)\tilde{v}(\alpha,x) = g(\alpha,x),$$

where $g(\alpha,x) = \tilde{F}(\alpha,x) + \alpha\,\mu v_1(x) + \mu\,v_2(x)$, the solution to which is given by (2.2) with $\omega = 0$, i.e.,

$$\tilde{v}(\alpha,x) = \frac{1}{\mathrm{EI}}\int_0^\infty e^{-\beta\,\alpha^2\,t} E^x\left\{g(\alpha,x(t))\,\exp\left(-\int_0^t q(X(s))\,ds\right)\right\}dt, \tag{3.2}$$

where $\beta = \mu/\mathrm{EI}$ and $q(x) = k(x)/\mathrm{EI}$. Let $v(t,x)$ denote the Laplace antitransform of (3.2). Let $N(t)$ denote a Poisson process of intensity ω on a probability space $(\Omega_1, \mathcal{F}, \mathcal{P})$ independent of $X(t)$. Define now

$$u(t,x) = \mathcal{E}[v(T(t),\,x)],$$

where $T(t) = \varepsilon_0\,\mu\int_0^t (-1)^{N(s)}\,ds$ and $\varepsilon_0 = \pm 1$, with probability $\frac{1}{2}$ independently of all other random variables involved.

The result of Kaplan asserts that $u(t,x)$ satisfies (1.2), i.e.,

$$\mu\,\frac{\partial^2 u}{\partial t^2} + \mu\,\omega\,\frac{\partial u}{\partial t} + \mathrm{EI}\,\frac{\partial^4 u}{\partial x^4} + k(x)\,u = \tilde{F}(t,x).$$

4. AN APPROXIMATE SOLUTION

In this section, we provide a variation on a classical theme. We shall consider the model dealt with in [4], i.e., we take $\omega(x) = \omega_0 + \omega_1(x)$, $k(x) = k_0 + k_1(x)$, where $|\omega_1(x)| \ll \omega_0$, and $|k_1(x)| \ll k_0$ are stationary random processes. We begin by rewriting (2.2) as

$$\tilde{v}(\alpha,x) = \frac{1}{\mathrm{EI}}\int_0^\infty e^{-At}\,E^x\left[g(\alpha,X(t))\,e^{-\int_0^t q_1(X(s))\,ds}\right]dt, \tag{4.1}$$

where $A = (\mu\,\alpha^2 + k_0 + 2\alpha\,\omega_0)/\mathrm{EI}$ and $q_1 = (k_1(x) + 2\mu\,\alpha\,\omega_1(x))$.

Now, rewrite the expected value under the integral sign as

$$\begin{aligned} &E^x\left[g(\alpha,\,X(t))\,X(t))\,e^{-\int_0^t q_1(X_s)\,ds}\right] \\ &\quad= E^x[g(\alpha,\,X(t))] - \int_0^t E^x\left[q_1(X(s))\,E^{X(s)}[g(\alpha, X(t-s))]\right]ds \\ &\qquad+ \int_0^t E^x\left[q_1(X(s))\,E^{X(s)}\left[\int_0^{t-s} q_1(X(u))\,E^{X(u)}[g(\alpha,\,X(t-s-u))]\,du\right]\right]ds. \end{aligned} \tag{4.2}$$

This can be obtained by iterating the obvious identity

$$e^{-\int_0^t q_1(X_s)\,ds} = 1 - \int_0^t q_1(X(u))\,e^{-\int_0^t q_1(X_s)\,ds}\,du$$

two times, dropping the remainder and using the simple Markov property of $X(t)$ a couple of times. Actually, this is a fancy way of obtaining and iterating the identity

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A+B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A+B} \approx \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A}.$$

Since the relation between $v(t,x)$ and $\tilde{v}(\alpha,x)$ is linear, taking Laplace transforms and averaging over random elements in $\omega_1(x)$ and $k_1(x)$, commute with each other. Thus, the average wave form has Laplace transform

$$\begin{aligned}\langle\tilde{v}(\alpha,x)\rangle = \frac{1}{\mathrm{EI}} \int_0^\infty e^{-At} E^x[g(\alpha,\, X(t))]\, dt - \frac{1}{\mathrm{EI}} \int_0^\infty e^{-At}\, \bar{q}_1 \left(\int_0^t E^x\, E^{X(s)}[g(\alpha,\, X(t-s))]\, ds \right) dt \\ + \frac{1}{\mathrm{EI}} \int_0^\infty e^{-At}\, dt \int_0^t ds \int_0^{t-s} du\, E^x \left[E^{X(s)} \left[Q(X(0) - X(u))\, E^{X(u)} \left[g(\alpha,\, X(t-s-u)) \right] \right] \right].\end{aligned}$$

In the second term in the right hand side, the stationarity of $k(x)$ and $\omega(x)$ is used to obtain a constant $\bar{q}_1$, and in the third term, we have made use of $\langle q_1(x)\, q_1(y)\rangle = Q(x-y)$ as well as the fact that $P^x(X(0) = x) = 1$. Note as well that applying the simple Markov property

$$E^x\, E^{X(s)}[g(\alpha,\, X(t-s))] = E^x[g(\alpha,\, X(t))],$$

the first two terms collapse into one. The new version of $\langle\tilde{v}(\alpha,x)\rangle$ is

$$\begin{aligned}\langle\tilde{v}(\alpha,x)\rangle &= \frac{1}{\mathrm{EI}} \int_0^\infty (1 - \bar{q}_1\, t)\, e^{-At}\, E^x[g(\alpha,\, X(t))]\, dt \\ &+ \frac{1}{\mathrm{EI}} \int_0^\infty e^{-At}\, dt \int_0^t ds \int_0^{t-s} du\, E^x\, E^{X(s)} \left[Q(X(0) - X(u))\, E^{X(u)} \left[g(\alpha,\, X(t-s-u)) \right] \right]. \qquad (4.3)\end{aligned}$$

The interesting fact about (4.3) is that each term can be evaluated iterating the procedure carried out in Section 2. But we leave this computation for the interested reader and stop here.

REFERENCES

1. H. Gzyl and R. Jiménez, Probabilistic approach to wave propagation problems—I, *Mathl. Comput. Modelling* **13** (11), 57–65 (1990).
2. C. Betz and H. Gzyl, Probabilistic approach to wave propagation problems—IV, *Mathl. Comput. Modelling* **15** (11), 129–140 (1991).
3. J. Bertrand and B. Gaveau, Transformation canonique et renormalization pour certaines equations d'evolution, *J. Funct. Analysis* **50**, 81–99 (1983).
4. L. Fryba, S. Nakagiri and L. Yoshikawa, Stochastic analysis of a beam on random foundation, with uncertain damping subject to a moving load, In *Proceed. of 1991 UTAM Symp.*, Springer-Verlag (to appear).
5. K.J. Hochberg, A signed measure on path space related to Wiener measure, *The Ann. Probab.* **6** (3), 433–458 (1975).
6. K. Nishioka, Stochastic calculus for a class of evolution equations, *Jap. Jour. Math.* **11**, 9–102 (1985).
7. S. Kaplan, Differential equations in which the Poisson process plays a role, *Bull. Anal. Math. Soc.* **70**, 264–268 (1960).