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A Queue with Instantaneous Tri-Route Decision Process

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Abstract—This paper focuses on the study of several random processes associated with $M/G/1$ queue with instantaneous tri-route decision process. The stationary distribution of the output process is derived. Some particular queues with feedback and without feedback are also analysed. Some operating characteristics are studied for this queue. Optimum service rate is obtained. A numerical study is carried out to test the feasibility of the queueing model.

Keywords—Markov renewal process, Tri-route decision process, Queue lengths, Stationary distributions.

1. INTRODUCTION

A queueing system which includes the possibility for a customer to return to the counter for additional service is called a queue with feedback. The formulation of queue with feedback mechanism was given by Takács [1]. For instance, this type of queueing system occurs in computer networks, communication systems, etc. Disney, McNickle and Simon [2] have studied several random processes that occur in $M/G/1$ queue with instantaneous feedback in which the feedback decision process is a Bernoulli process. D'Avignon and Disney [3] have also considered the same queue with state dependent feedback mechanism. Further, they [4] have studied the queue $M_2/G_2/1$ with instantaneous Bernoulli feedback. The $M/G/1$ queues with renewal departure process have been characterized by Disney, Farrell and Demorais [5]. Thangaraj and Santhakumaran [6] have initiated a study on a queueing model with a pair of instantaneous independent Bernoulli feedback processes associated with the queue. In this paper, we are concerned with $M/G/1$ queue with instantaneous tri-route decision process.

2. DESCRIPTION OF THE SYSTEM

Figure 1 illustrates the flow of customers through a queueing system with a tri-route decision process. In this system, the arrival process is taken to be a Poisson process. Customers get services from the server on first-come, first-served basis. The service times at the counter form a sequence of independent identically distributed random variables. After receiving service, a decision is made whether or not to feedback. If a customer does feedback, he joins the feedback stream and re-enters at the end of the queue. Assume that there are two types of feedback. The

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feedbacks are assumed to occur instantaneously. If a customer does not feedback, he joins the departure process and leaves the system forever. Assume that the probability for simultaneous feedback is zero. The queue is infinite in capacity.

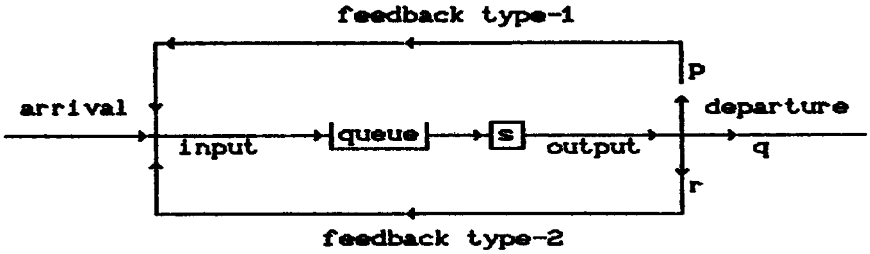


Figure 1. Flow of customers.

3. NOTATIONS

This paper refers to the system $M/G/1$ queue with an instantaneous tri-route decision process. That is, the arrival process is a Poisson process with rate $\lambda(>0)$ and service times follow general distribution with parameter $\mu(>0)$. The outside arrival epochs are $\{W_n,\ n=1,2,\dots\}$. Service completion epochs occurring at $T_0 < T_1 < \cdots$ are called output epochs. The arrival and service processes are independent processes. Service times are independent identically distributed non-negative random variables S_n with distribution function $P\{S_n \le t\} = H(t)$, where S_n denotes the service time of the n^{th} customer. We define the Laplace-Stieltje's transform of $H(t)$ by

$$H^*(s) = \int_0^\infty e^{-st}\, dH(t), \qquad \operatorname{Re} s \ge 0 \quad \text{and} \quad E(S_n) = \frac{1}{\mu}.$$

Let q, p and r be the probability that n^{th} output is a departure, feedback of type-1 and feedback of type-2, respectively, and $p+q+r=1$.

Let the time points $\{t_n\} \subset \{T_n\}$ be a sequence of departure epochs after service. The elements of the subsequences $\{\tau_n'\} \subset \{T_n\}$ and $\{\tau_n''\} \subset \{T_n\}$ are epochs of type-1 and type-2 feedback, respectively. Further note that $\{t_n\} \cup \{\tau_n'\} \cup \{\tau_n''\} = \{T_n\}$. The epoch T_n' is the time at which the n^{th} customer enters the queue. The sequence $\{T_n'\}$ is called the input process where $\{T_n'\} = \{W_n\} \cup \{\tau_n'\} \cup \{\tau_n''\}$.

There are five queue length processes associated with this queue. Let $Q(t)$ be the queue length (number in the system) at time t. Then $Q_1^-(n) = Q(W_n - 0)$, $Q_2^-(n) = Q(T_n' - 0)$, $Q_3^+(n) = Q(T_n + 0)$, $Q_4^+(n) = Q(t_n + 0)$ are, respectively, the embedded queue lengths at arrival, input, output and departure epochs of the n^{th} customer. Other relevant notations are defined wherever they occur. We strictly follow the notations as in [2] and give proofs whenever necessary.

4. QUEUE LENGTH AT DEPARTURE EPOCHS

In this section, we obtain the stationary probability distribution of $Q_4^+(n)$. Let K denote the number of services performed between the $(n-1)^{\text{st}}$ and n^{th} departures.

LEMMA 1. *Let $S_n' = S_1 + S_2 + \cdots + S_K$ be the total service time of customers consumed between the $(n-1)^{\text{st}}$ and n^{th} departures and I_n be the idle time following t_{n-1} when $Q_4^+(n-1) = 0$. Then $K \ge 1$ has probability mass function*

$$\sum_{m=0}^{k-1} \binom{k-1}{m} q p^m r^{k-m-1}, \qquad k = 1,2,3,\dots. \tag{4.1}$$

PROOF OF LEMMA 1. It is easy to see that

$$t_n = \begin{cases} t_{n-1} + I_n + S_n', & \text{when } Q_4^+(n-1) = 0, \\ t_{n-1} + S_n', & \text{when } Q_4^+(n-1) > 0. \end{cases} \tag{4.2}$$

For $M/G/1$ queue, the I_n's are independent identically distributed random variables that are exponentially distributed with parameter λ. Since customers are indistinguishable, S'_n is the total service times of N_1 type-1 feedback and N_2 type-2 feedback so that $N_1 + N_2 = K$. Hence,

$$P\{K = k\} = \sum_{m=0}^{k-1} \binom{k-1}{m} qp^m r^{k-m-1}, \qquad k = 1, 2, 3, \dots.$$

LEMMA 2. *The Laplace-Stieltje's transform, $G^*(s)$, of the distribution of S'_n is given by*

$$G^*(s) = \frac{qH^*(s)}{1 - (p+r)\,H^*(s)}, \qquad \operatorname{Re} s \geq 0. \tag{4.3}$$

PROOF OF LEMMA 2. The Laplace Stieltje's transform of S'_n is given by

$$\begin{aligned}
G^*(s) &= \sum_{k=1}^{\infty} q \sum_{m=0}^{k-1} \binom{k-1}{m} p^m r^{k-m-1} \left[H^*(s)\right]^k \\
&= qH^*(s) \sum_{m=0}^{\infty} p^m \left[\sum_{k=m+1}^{\infty} \binom{k-1}{m} r^{k-m-1} \left[H^*(s)\right]^{k-1}\right] \\
&\qquad \text{(changing the order of summation)} \\
&= qH^*(s) \sum_{m=0}^{\infty} \frac{p^m}{m!} \left[m!\left[H^*(s)\right]^m + \frac{(m+1)!}{1!} r \left[H^*(s)\right]^{m+1} + \cdots\right] \\
&= qH^*(s) \left[\sum_{m=0}^{\infty} p^m \left[H^*(s)\right]^m \left\{1 + \binom{m+1}{1} rH^*(s) + \cdots\right\}\right] \\
&= qH^*(s) \sum_{m=0}^{\infty} \left[\frac{pH^*(s)}{1 - rH^*(s)}\right]^m \frac{1}{1 - rH^*(s)} \\
&= \frac{qH^*(s)}{\left(1 - (pH^*(s))\,/\,(rH*(s))\right)} \frac{1}{1 - rH^*(s)} \\
&= \frac{qH^*(s)}{1 - (p+r)\,H^*(s)}.
\end{aligned}$$

LEMMA 3. *The process $\{Q_4^+(n),\, t_n - t_{n-1}\}$ is a Markov renewal process with kernal*

$$A(i,j,x) = P\left\{Q_4^+(n) = j,\, t_n - t_{n-1} \leq x \mid Q_4^+(n-1) = i\right\}. \tag{4.4}$$

If we define $P_j(y) = \left(e^{-\lambda y}(\lambda y)^j\right)/(j!)$, $j = 0, 1, 2, \dots,$ then

$$A(i,j,x) = \begin{cases} 0, & \text{if } j < i-1, \\ \int_0^x P_{j-i+1}(y)\,dH(y), & \text{if } i \neq 0, \quad j \geq i-1, \\ \int_0^x \left[1 - e^{-\lambda(x-y)}\right] P_j(y)\,dH(y), & \text{if } i = 0, \quad j > 0, \\ \int_0^x \left[1 - e^{-\lambda(x-y)}\right] P_0(y)\,dH(y), & \text{if } i = 0, \quad j = 0. \end{cases} \tag{4.5}$$

PROOF OF LEMMA 3. We have

$$t_n - t_{n-1} = \begin{cases} S'_n, & \text{if } Q_4^+(n-1) > 0, \\ I'_n + S'_n, & \text{if } Q_4^+(n-1) = 0, \end{cases} \tag{4.6}$$

where I'_n is the exponentially distributed idle time preceding S'_n if $Q_3^+(n-1) = 0$. The result then follows directly using arguments as in [5].

THEOREM 1. *If $\lambda E(S_n) < (1-p-r)$, then the stationary distribution of $Q_4^+(n)$ viz. $\pi^d(0), \pi^d(1), \pi^d(2), \ldots$ exists and its probability generating function is given by*

$$\psi(z) = \frac{\pi^d(0)\,(z-1)\,G^*(\lambda - \lambda z)}{z - G^*(\lambda - \lambda z)}, \qquad |z| \le 1, \tag{4.7}$$

where

$$\psi(z) = \sum_{j=0}^{\infty} \pi^d(j)\, z^j,$$

and

$$\pi^d(0) = 1 - \frac{\lambda E(S_n)}{1-p-r}. \tag{4.8}$$

PROOF OF THEOREM 1. In Lemma 3, as $x \to \infty$, we get $A(i,j,x) \to A(i,j)$ which is the $(i,j)^{\text{th}}$ element of the one step transition probability matrix for the $\{Q_4^+(n)\}$ process. That is,

$$A(i,j) = \begin{cases} K_{j-i}, & \text{if } j \ge i, \\ 0, & \text{if } j < i, \end{cases} \tag{4.9}$$

where

$$K_j = \int_0^{\infty} \frac{e^{-\lambda y} (\lambda y)^j}{j!}\, dH(y), \qquad j = 0, 1, 2, \ldots. \tag{4.10}$$

Then using standard embedded Markov chain results in [7], we get the probability generating function

$$\psi(z) = \frac{\pi^d(0)\,(z-1)\,K(z)}{z - K(z)},$$

where

$$\begin{aligned} K(z) &= \sum_{j=0}^{\infty} K_j z^j, \qquad |z| \le 1 \\ &= G^*(\lambda - \lambda z). \end{aligned}$$

Therefore,

$$\psi(z) = \frac{\pi^d(0)\,(z-1)\,G^*(\lambda - \lambda z)}{z - G^*(\lambda - \lambda z)}, \qquad |z| \le 1.$$

To find $\pi^d(0)$, allow $z \to 1$ in $\psi(z)$.

$$\begin{aligned} \lim_{z \to 1} \psi(z) &= \frac{\lim_{z \to 1} \left[\pi^d(0)\,(z-1)\,G^{*\prime}(\lambda - \lambda z)\,(-\lambda z) + \pi^d(0)\,G^*(\lambda - \lambda z)\right]}{\lim_{z \to 1} \left[1 - G^{*\prime}(\lambda - \lambda z)\,(-\lambda)\right]}, \\ 1 &= \frac{\pi^d(0)\,G^*(0)}{1 + \lambda G^{*\prime}(0)}, \qquad \text{using L'Hospital Rule.} \end{aligned}$$

Thus,

$$\pi^d(0) = 1 + \lambda G^{*\prime}(0),$$

where $G^*(s)$ is given by (4.3)

$$G^{*\prime}(0) = \frac{-E(S_n)}{1-p-r}.$$

Thus, $\pi^d(0) = 1 - (\lambda)/(1-p-r)\,E(S_n)$ and $\pi^d(j)$ is obtained from the following relation.

$$\pi^d(n-1) = \sum_{r=0}^{n-1} \pi^d(n-r)\,K_r + \pi^d(0)\,K_{n-1}, \qquad n = 1, 2, 3, \ldots. \tag{4.11}$$

REMARK. If $r = 0$ in Theorem 1, we obtain the results of the system identical to the one considered by Disney *et al.* [2].

5. QUEUE LENGTH AT OUTPUT EPOCHS

As we are interested in both $\{Q_3^+(n)\}$ and $\{T_n - T_{n-1}\}$, first we study the joint process $\{Q_3^+(n), T_n - T_{n-1}\}$ and then the results for $\{Q_3^+(n)\}$ are derived.

LEMMA 4. *The process $\{Q_3^+(n), T_n - T_{n-1}\}$ is a Markov renewal process with kernal*

$$A(i,j,x) = P\left\{Q_3^+(n) = j,\, T_n - T_{n-1} \le x \mid Q_3^+(n-1) = i\right\}. \tag{5.1}$$

If we define

$$P_j(y) = \frac{e^{-\lambda y}(\lambda y)^j}{j!}, \qquad j = 0,1,2,\dots,$$

then

$$A(i,j,x) = \begin{cases} 0 & \text{if } j < i-1, \\ \int_0^x \left[P_{j-1}(y)(p+r) + qP_{j-i+1}\right] dH(y) & \text{if } i \ne 0, \quad j \ge i-1, \\ \int_0^x \left[1 - e^{-\lambda(x-y)}\right]\left[P_{j-1}(y)\,(p+r) + qP_j(y)\right] dH(y) & \text{if } i = 0, \quad j > 0, \\ \int_0^x \left[1 - e^{-\lambda(x-y)}\right] P_0(y)\,q\,dH(y) & \text{if } i = 0, \quad j = 0. \end{cases} \tag{5.2}$$

PROOF OF LEMMA 4. We have

$$T_n - T_{n-1} = \begin{cases} S_n, & \text{if } Q_3^+(n) > 0, \\ I_n + S_n, & \text{if } Q_3^+(n) = 0, \end{cases} \tag{5.3}$$

where I_n is the exponentially distributed idle time preceding S_n if $Q_3^+(n-1) = 0$. The result then follows directly using arguments as in [5].

THEOREM 2. *If $\lambda E(S_n) < (1-p-r)$, the stationary distribution of $Q_3^+(n)$ viz. $\pi^0(0), \pi^0(1), \pi^0(2), \dots$ exists and its probability generating function is given by*

$$\phi(z) = \frac{\pi^0(0)\,(z-1)\,K(z)\,[q + (p+r)\,z]}{z - K(z)\,[q + (p+r)\,z]}, \qquad |z| \le 1, \tag{5.4}$$

where

$$\pi^0(0) = (1-p-r) - \lambda E(S_n).$$

PROOF OF THEOREM 2. In Lemma 4, as $x \to \infty$, we get $A(i,j,x) \to A(i,j)$ which is the $(i,j)^{\text{th}}$ element of the one step transition probability matrix for the $\{Q_3^+(n)\}$ process. That is,

$$[A(i,j)] = \begin{bmatrix} qK_0 & qK_1 + (p+r)\,K_0 & qK_2 + (p+r)\,K_1 & \dots \\ qK_0 & qK_1 + (p+r)\,K_0 & qK_2 + (p+r)\,K_1 & \dots \\ 0 & qK_0 & qK_1 + (p+r)\,K_0 & \dots \\ 0 & 0 & qK_0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{5.5}$$

Then using the standard embedded Markov chain results in [7], we obtain the probability generating function $\phi(z) = \sum_{j=0}^{\infty} \pi^0(j)\,z^j$ for the stationary probabilities $\pi^0(j)$ of $Q_3^+(n)$ as

$$\phi(z) = \frac{\pi^0(0)\,(z-1)\,K(z)\,[q + (p+r)\,z]}{z - K(z)\,[q + (p+r)\,z]}$$

and

$$\pi^0(0) = (1-p-r) - \lambda E(S_n).$$

Then $\pi^0(j)$ is obtained from the following relation:

$$\begin{aligned} &\pi^0(n-1) - q\left[K_0\pi^0(n) + K_1\pi^0(n-1) + K_2\pi^0(n-2) + \dots + K_n\pi^0(0)\right] \\ &\qquad - (p+r)\left[K_0\pi^0(n-1) + K_1\pi^0(n-2) + \dots + K_{n-1}\pi^0(0)\right] \\ &\quad = \pi^0(0)\,K_{n-1}\,q + \pi^0(0)\,K_{n-2}(p+r) - \pi^0(0)(p+r)\,K_n \\ &\qquad - \pi^0(0)\,(p+r)\,K_{n-1}, \qquad n = 1,2,3,\dots, \end{aligned} \tag{5.6}$$

where K_j is given by (4.10).

6. PARTICULAR CASES

COROLLARY 1. *If we assume that the service time distribution in Theorem 2 is*

$$H(t) = 1 - e^{-\mu t}, \qquad t \geq 0, \quad \mu > 0.$$

In this case, $E(S_n) = 1/\mu$.

$$\pi^0(0) = (1-p-r) - \frac{\lambda}{\mu},$$

and

$$\pi^0(j) = \left[1 - \frac{\lambda}{(1-p-r)\,\mu}\right]\left[\frac{\lambda}{(1-p-r)\,\mu}\right]^{j-1}\left[(p+r) + \frac{\lambda}{\mu}\right], \qquad j = 1,2,3,\ldots. \tag{6.1}$$

COROLLARY 2. *If we assume that* $r = 0$ *in Corollary 1, we obtain the results of* $M/M/1$ *(FIFO) queue with instantaneous Bernoulli feedback and infinite capacity, that is*

$$\pi^0(0) = q - \frac{\lambda}{\mu}, \quad \pi^0(j) = \left[1 - \frac{\lambda}{q\mu}\right]\left[\frac{\lambda}{q\mu}\right]^{j-1}\left[p + \frac{\lambda}{\mu}\right], \qquad j = 1,2,3,\ldots, \tag{6.2}$$

which coincides with the results in [2, p. 639].

COROLLARY 3. *If* $p = 0$ *and* $r = 0$ *in Corollary 1, we obtain the results of* $M/M/1$ *(FIFO) queue with infinite capacity. That is,*

$$\pi^0(j) = \left[1 - \frac{\lambda}{\mu}\right]\left[\frac{\lambda}{\mu}\right]^{j}, \qquad j = 0,1,2,3,\ldots. \tag{6.3}$$

7. OPERATING CHARACTERISTICS

By straightforward calculations, the following results are obtained for this queueing system.

(i) If N denotes the number of customers in the system at output process, then the average number of customers is given by

$$E(N) = \frac{[\mu(p+r)+\lambda]\,q}{q\mu - \lambda}. \tag{7.1}$$

(ii) The variance of N is given by

$$V(N) = \frac{q\left[(p+r)\,\mu + \lambda\right]\left[\lambda(p+r) + \mu q^2\right]}{[q\mu - \lambda]^2}. \tag{7.2}$$

(iii) If L denotes the queue length at output process excluding the customer in service, then

$$E(L) = \frac{[\mu(p+r)+\lambda]\,\lambda}{[\mu q - \lambda]\,\mu}. \tag{7.3}$$

(iv) Average length of nonempty queue is calculated to be

$$E(L \mid L > 0) = \frac{q\mu}{\mu q - \lambda}. \tag{7.4}$$

(v) Probability of the system size being greater than or equal to n is calculated to be

$$P(N \geq n) = \left[(p+r) + \frac{\lambda}{\mu}\right]\frac{\lambda^n}{[\mu q]^n}. \tag{7.5}$$

8. OPTIMUM SERVICE RATE

The results in this section are of independent interest. For the queueing system $M/G/1$ queue with arrival rate λ, we get the optimum service rate μ, from the following cost function.

$$T = C_o\mu + C_1E(N), \tag{8.1}$$

where

T = total cost,

C_0 = the cost per unit service rate,

C_1 = holding cost of a customer in the system per unit time.

THEOREM 3. *If $\lambda < \mu(1-p-r)$, then the cost function T attains optimum value at*

$$\mu^0 = \frac{\lambda}{(1-p-r)} + \sqrt{\frac{\lambda C_1}{C_0(1-p-r)}}. \tag{8.2}$$

PROOF OF THEOREM 3. Substituting the value of $E(N)$ from (7.1) and on simplification, we get

$$T = C_0\mu + C_1q\left[\frac{\mu}{\mu q - \lambda} - 1\right].$$

Differentiating T with respective to μ, we get

$$\frac{dT}{d\mu} = C_0 - \frac{\lambda C_1 q}{[\mu q - \lambda]^2}.$$

Equating the above equation to zero and solving for μ, we get the stationary point μ^0. Thus, μ^0 is given by (8.2). The cost function T attains minimum when $\frac{d^2T}{d\mu^2} > 0$. This is achieved when $\lambda < \mu(1-p-r)$.

REMARK. The optimum service rate μ^0 depends on departure probabilities and arrival rate.

COROLLARY 4. *If there is only one feedback, i.e., $r = 0$ in Theorem 3, then*

$$\mu^0 = \frac{\lambda}{q} + \sqrt{\frac{\lambda C_1}{C_0 q}}, \qquad q\mu > \lambda, \quad q = 1-p, \tag{8.3}$$

which coincides with Theorem 3 of Thangaraj and Santhakumaran [6].

COROLLARY 5. *If there is no feedback, i.e., $p = 0$ and $r = 0$ in Theorem 3, then*

$$\mu^0 = \lambda + \sqrt{\frac{\lambda C_1}{C_0}}, \qquad \mu > \lambda, \tag{8.4}$$

which coincides with the $M/M/1$ case without feedback.

9. DEPARTURE PROCESS

Disney *et al.* [5] have studied the departure process. For our model, we have also studied the departure process $\{t_n\}$.

LEMMA 5. *The departure process from the $M/G/1$ queue with tri-route decision process is a Poisson process if and only if the inter-departure time is exponentially distributed.*

PROOF OF LEMMA 5. From Lemma 2, we have $G^*(s)$, the Laplace-Stieltje's transform of the distribution function of $\lambda S_n'$ is given by (4.3). Suppose S_n' (the inter-departure time) is exponentially distributed with parameter 'a,' then $G^*(s) = a/(a+s)$. If that is the case, $H^*(s)$ must satisfy the equation

$$\frac{qH^*(s)}{1-(p+r)\,H^*(s)} = \frac{a}{a+s}, \tag{9.1}$$

$$H^*(s) = \frac{a/(1-p-r)}{a/(1-p-r)+s}, \tag{9.2}$$

which implies $H(t)$ is exponential.

REMARKS.

(1) If $r = 0$ in Lemma 5, we get that the departure process is Poisson as in [2, p. 640].

(2) In view of the different service mechanism in [6], the departure process is only a renewal process but not a Poisson process. But in this paper the departure process is Poisson as observed in [2, p. 640]. Thus, the departure process is very much affected by the service mechanism, but not the arrival or the feedback mechanism.

10. OUTPUT AND INPUT PROCESSES

From Section 5, we have seen that the output process is a Markov renewal process whose distributions are given by (5.2). The following results are deduced from them.

THEOREM 4. *The output process* $\{T_n - T_{n-1}\}$ *is a renewal process if and only if* $q = 1$ *and* $H(t) = 1 - e^{-\mu t}$, $t \geq 0$.

PROOF OF THEOREM 4. Assume that $q = 1$ and $H(t) = 1 - e^{-\mu t}$. Then the output process and the departure processes are identical. On similar lines with [2], we can prove that this departure process is a Poisson process and thus 'if' part follows.

Assuming $q \neq 1$ and following [2], it is easy to prove analogously the 'only if' part for our case that $\{T_n - T_{n-1}\}$ is not a renewal process.

This completes the proof of Theorem 4.

COROLLARY 6. *The output process* $\{T_n - T_{n-1}\}$ *for the* $M/M/1$ *queue is a Poisson process if and only if* $q = 1$.

PROOF OF COROLLARY 6. The proofs are on similar lines with [2]. Define

$$F(x) = P\{T_n - T_{n-1} \leq x\}. \tag{10.1}$$

$F(x) = \pi AU$ where U is a column vector all of whose elements are 1, π is the vector of stationary probabilities given in Corollary 1 for $\{Q_3^+(n)\}$ and A is the matrix of $A(i,j,x)$. Then from Theorem 2 one obtains

$$F(x) = \left\{q - \frac{\lambda}{\mu}\right\} \int_0^x \left[1 - e^{-\lambda(x-y)}\right] dH(y) + \left\{(p+r) + \frac{\lambda}{\mu}\right\} H(x), \tag{10.2}$$

for any $M/G/1$ queue with instantaneous tri-route decision process.

For $H(y) = 1 - e^{-\mu y}$, it follows that

$$F(x) = 1 - \frac{(q\mu - \lambda)}{(\mu - \lambda)} e^{-\lambda x} - \frac{(p+r)\,\mu}{\mu - \lambda} e^{-\mu x}. \tag{10.3}$$

THEOREM 5. *If* $H(x) = 1 - e^{-\mu x}$, *the input process* $\{Q_2^-(n),\, T_n' - T_{n-1}'\}$ *is a Markov renewal process with kernal,*

$$Y(i,j,x) = P\left\{Q_2^-(n) = j,\, T_n' - T_{n-1}' \leq x \mid Q_2^-(n-1) = i\right\}, \tag{10.4}$$

given by

$$Y(i,j,x) = \begin{cases} 0, & j > i+1, \\ \int_0^x (e^{-\lambda s} - qe^{-\lambda s})\,(p+r)^i\, dH^{(i+1)}(s), & j = 0, \quad i \geq 0, \\ \int_0^x e^{-\lambda s} \left[\frac{(p+r)\,\lambda}{\lambda+\mu}\right] \left[1 - e^{-(\lambda+\mu(x-s))} + (p+r)\right] & \\ \qquad \times [q]^{i-j}\, dH^{(i-j+1)}(s), & 1 \leq j \leq i, \\ e^{-\mu x}\left[1 - e^{-\lambda x}\right], & j = i+1, \end{cases} \tag{10.5}$$

where $dH^{(i+1)}(s) = \left\{\mu[\mu s]^n \frac{e^{-\mu s}}{n!}\right\} ds.$

PROOF OF THEOREM 5. The proof is on similar lines with [2, p. 641].

Now if $Y(x)$ is the matrix whose elements are $Y(i,j,x)$, π is the vector of probabilities found in Theorem 2 and U is a vector all of whose elements are 1, then it is easy to see that $F(x) = P\{T'_n - T'_{n-1} \leq x\} = \pi Y(x)\, U$ and

$$G(x,y) = P\left\{T'_n - T'_{n-1} \leq x,\, T'_n - T'_{n-1} \leq y\right\} = \pi Y(x)\, Y(y)\, U, \tag{10.6}$$

where $F(x)$ is given by (10.3). Of course, if $\{T'_n - T'_{n-1}\}$ is to be a renewal process, then it is necessary (but not sufficient) that $G(x,y) = F(x)\, F(y)$.

From this we can conclude the following result.

COROLLARY 7. *The input process to the $M/M/1$ queue with instantaneous tri-route decision process is not a renewal process unless $q = 1$.*

PROOF OF COROLLARY 7. If $q = 1$, then the input process is just the arrival process, which is Poisson. If the input process is a renewal process for $q \neq 1$, then it must be true that $\pi Y(x)\, U = F(x)$ for every x, $\pi Y(x)\, Y(y)\, U = F(x)\, F(y)$ for every x and y where $F(x)$ is given by (10.3) and U is a column of 1's. Thus, $[\pi - (\pi Y(x))/(F(x))]\, Y(y)\, U = 0$ for every x and y. After some algebraic manipulations, we get

$$\begin{aligned} \left[\pi - \frac{\pi Y(x)}{F(x)}\right] Y(y)\, U = {} & \left[\frac{F(x) - (1 - e^{-\mu x})}{F(x)}\right] \\ & \times \left[\frac{\left((p+r)\,\lambda e^{-\lambda y} - (p+r)\,\mu e^{-\mu y}\right)}{\mu - \lambda} + (p+r)\, e^{-(\mu+\lambda)\, y}\right]. \end{aligned} \tag{10.7}$$

If $q \neq 1$,

$$\begin{aligned} F(x) - (1 - e^{-\mu x}) &= e^{-\mu x} - \frac{(\mu q e^{-\lambda x})}{\mu - \lambda} - \frac{u(p+r)\, e^{-\mu x}}{\mu - \lambda} \\ &< e^{-\mu x} - \frac{(\mu q - \lambda)}{\mu - \lambda}\, e^{-\lambda x} - \frac{\mu(p+r)\, e^{-\mu x}}{\mu - \lambda} = 0. \end{aligned}$$

Thus, to show that the input process is not renewal, it suffices to show that for some y

$$\frac{\left((p+r)\,\lambda e^{-\lambda y}\right)}{\mu - \lambda} - \frac{\mu(p+r)\, e^{-\lambda x}}{\mu - \lambda} + (p+r)\, e^{-(\mu+\lambda)\, y} \neq 0. \tag{10.8}$$

The first two terms of the Taylor expansion of this expression are zero. But the coefficient of the third term is

$$\frac{(p+r)\,\lambda}{\mu - \lambda}\, \frac{\lambda^2}{2} - \frac{(p+r)\,\mu}{\mu - \lambda}\, \frac{\mu^2}{2} + (p+r)\, \frac{(\mu+\lambda)^2}{2} = \frac{(p+r)\,\lambda\mu}{2} \neq 0. \tag{10.9}$$

So it can not be identically zero unless $q = 1$.

COROLLARY 8. *Either the feedback process is not a Poisson process or the arrival process and feedback processes are not independent processes for the $M/M/1$ queue with instantaneous tri-route decision process.*

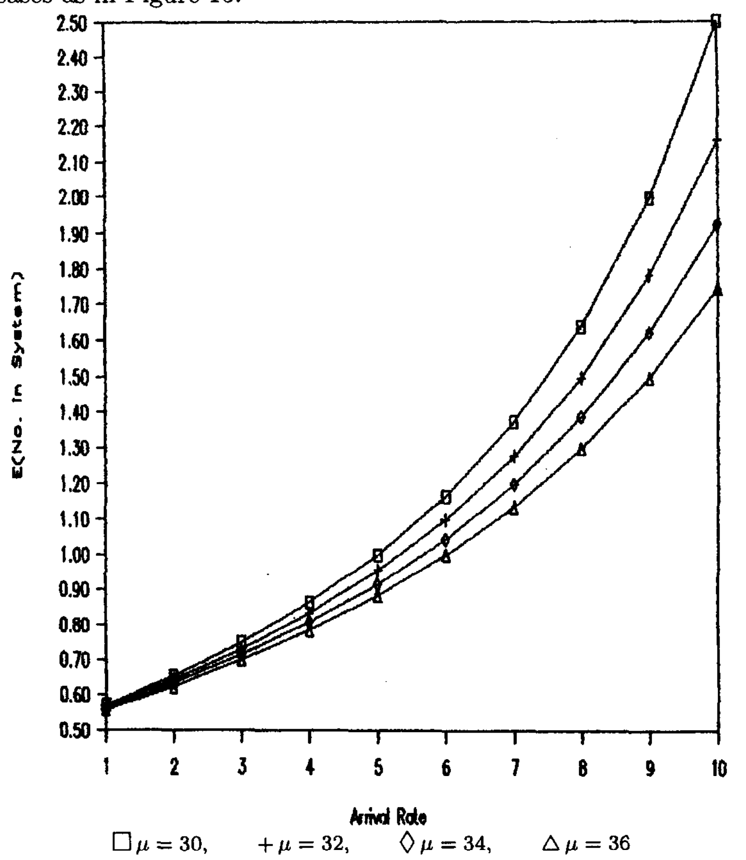
PROOF OF COROLLARY 8. This result follows from [2].

11. NUMERICAL STUDY

Figure 2 shows the E(No. in system) versus arrival rate. Our computation experience shows that as arrival rate increases, the E(No. in system) is monotonically increasing function. Figures 3–6 describe the E(No. in system), Sd(No. in system), E(Time in queue) and E(Time in system) versus departure probability, respectively. As departure probability increases from 0 to 0.1, the expected number in system, expected time in queue and expected time in system are uniformly increasing and decreasing from 0.1 to 1. This shows that E(No. in system), Sd(No. in system), E(Time in queue) and E(Time in system) are unimodal concave function in the interval 0 to 1.

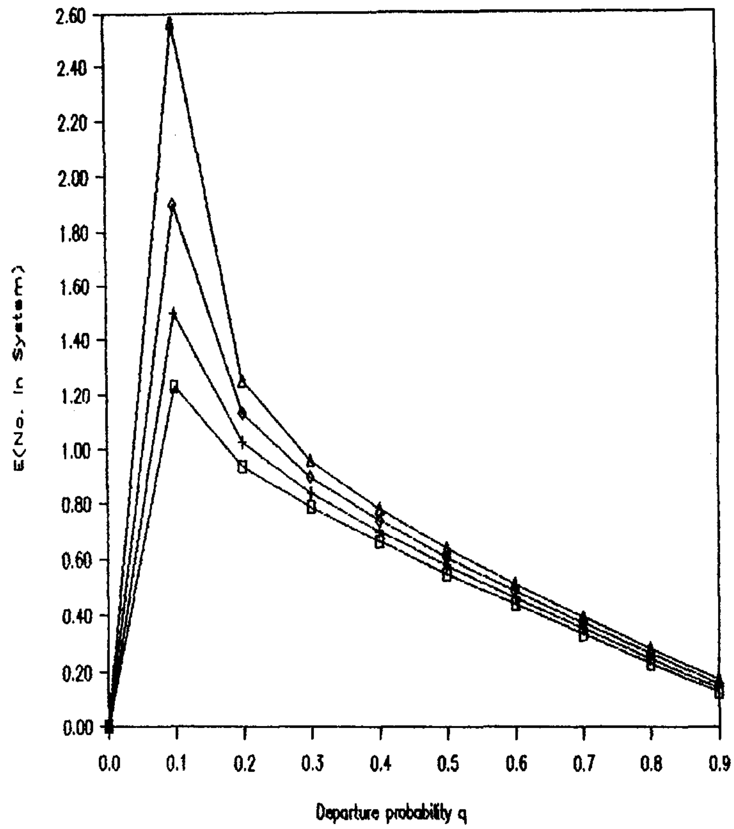
Figures 7 and 8 depict the E(No. in system) versus feedback probability. By seeing Figures 7 and 8, we conclude that the E(No. in system) increases when the feedback probability and λ increase. Figures 9 and 10 reveal the departure probability versus $P\{\text{System size } N \geq 0\}$ and $P\{\text{System size } N \geq 1\}$. From this, we say that whenever the departure probability increases, the probability of system size decreases. But probability of system size $N \geq 0$ is linearly decreasing as departure probability increases. From Figure 11, we note that the Sd(No. in system) is monotonically increasing as the arrival rate increases, irrespective of the interruption of feedbacks.

Observing Figure 12, we see that irregular variation in Sd(No. in system) versus feedback probability. Figures 13 and 14 describe the E(Time in system) and E(Time in queue) versus arrival rate, respectively. They show that the E(Time in queue) and E(Time in system) are unimodal monotone convex functions. Figure 15 shows that the E(Time in queue) versus feedback probability. As λ increases, the E(Time in queue) decreases. But the expected time in queue is an increasing function when feedback probabilities increase, because the feedback customers increase the expected time in queue of the future arriving customers. Figure 16 indicates the E(Time in system) versus the feedback probability. As λ increases, the expected time in the system also decreases as in Figure 15.



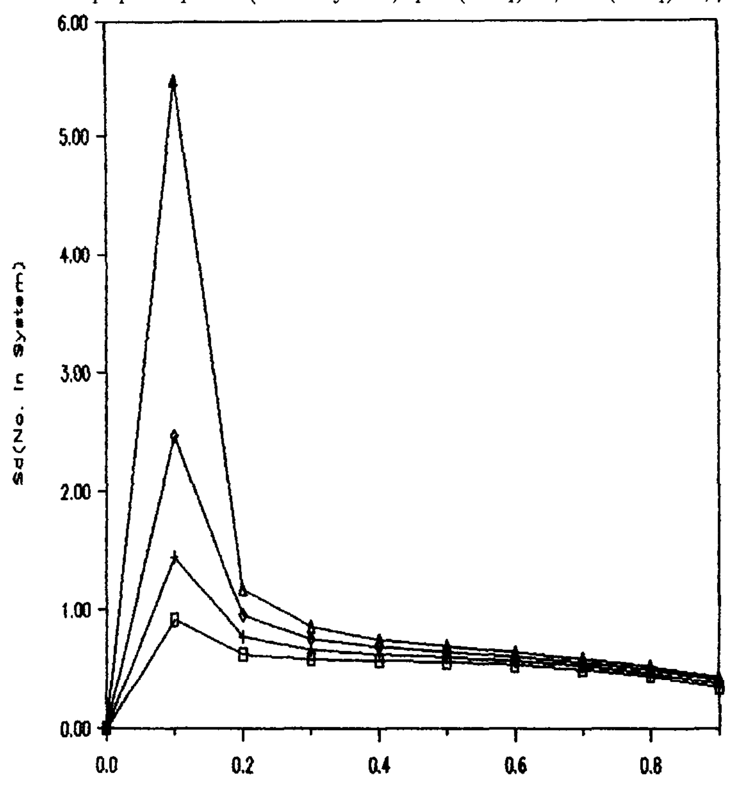
$\square\, \mu = 30$, $+\, \mu = 32$, $\Diamond\, \mu = 34$, $\triangle\, \mu = 36$

Figure 2. Arrival rate vs. E(No. in system). $q = 0.5$, $p = 0.25$, $r = 0.25$.



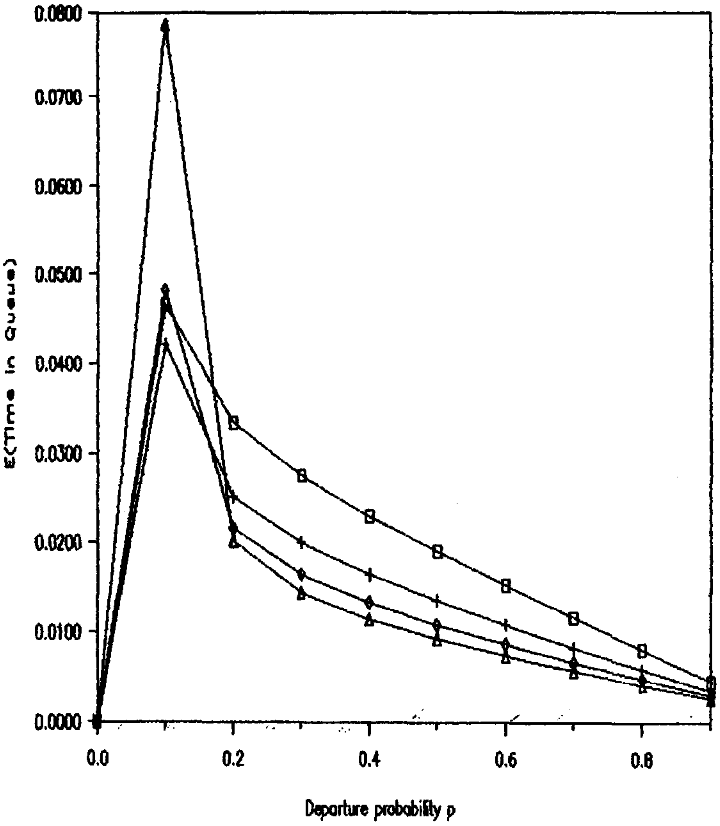
□ $\lambda = 1.0$, $+\,\lambda = 1.5$, ◇ $\lambda = 2.0$, △ $\lambda = 2.5$

Figure 3. Dep. prob. q vs. E(No. in system). $p = (1-q)0.5$, $r = (1-q)0.5$, $\mu = 40$.



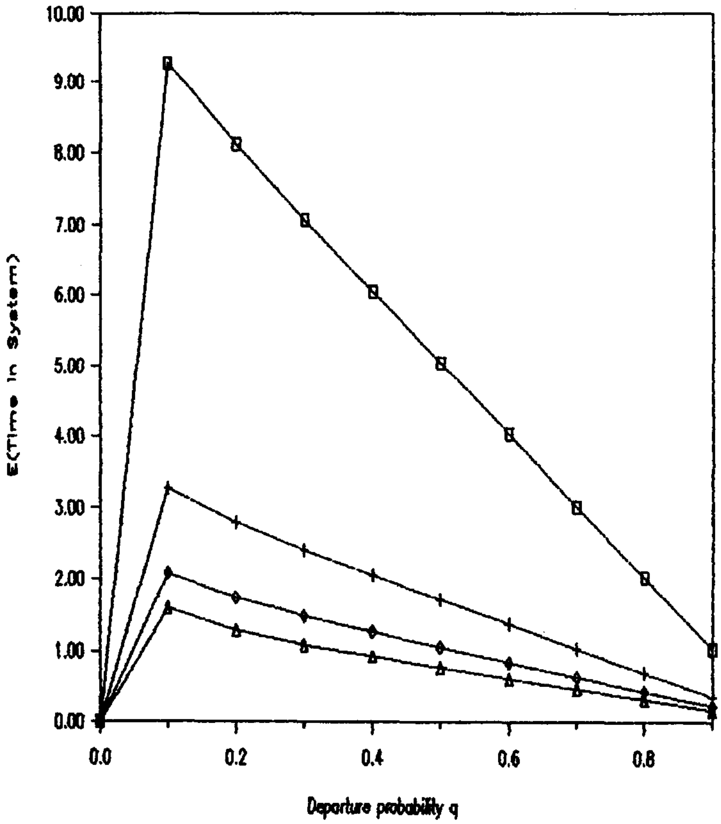
□ $\lambda = 1.0$, $+\,\lambda = 1.5$, ◇ $\lambda = 2.0$, △ $\lambda = 2.5$

Figure 4. Dep. prob. q vs. Sd(No. in system). $p = (1-q)0.5$, $r = (1-q)0.5$, $\mu = 30$.



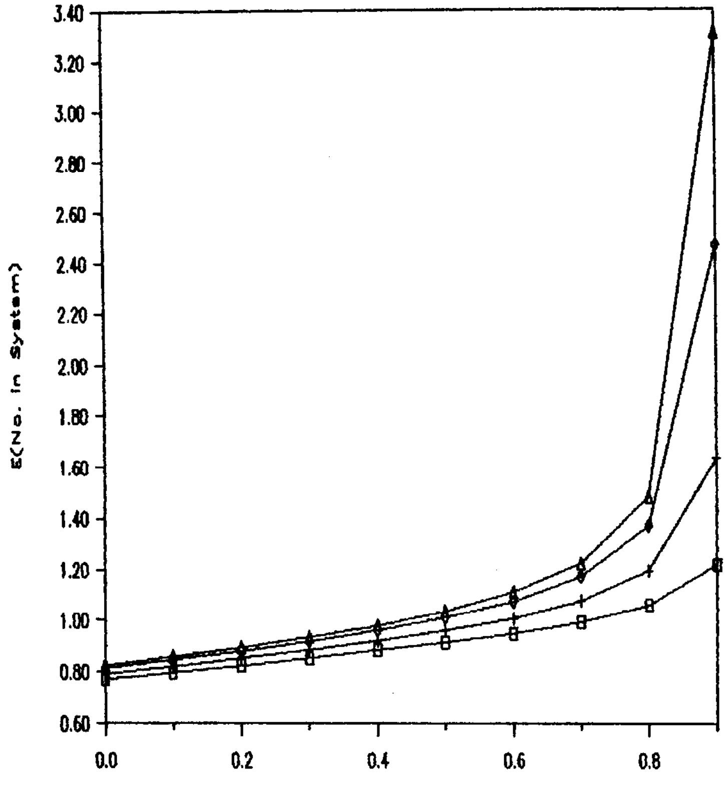
□ $\lambda = 1.0$, $+\lambda = 1.5$, ◇ $\lambda = 2.0$, △ $\lambda = 2.5$

Figure 5. Dep. prob. q vs. E(Time in queue). $p = (1-q)0.5$, $r = (1-q)0.5$, $\mu = 30$.



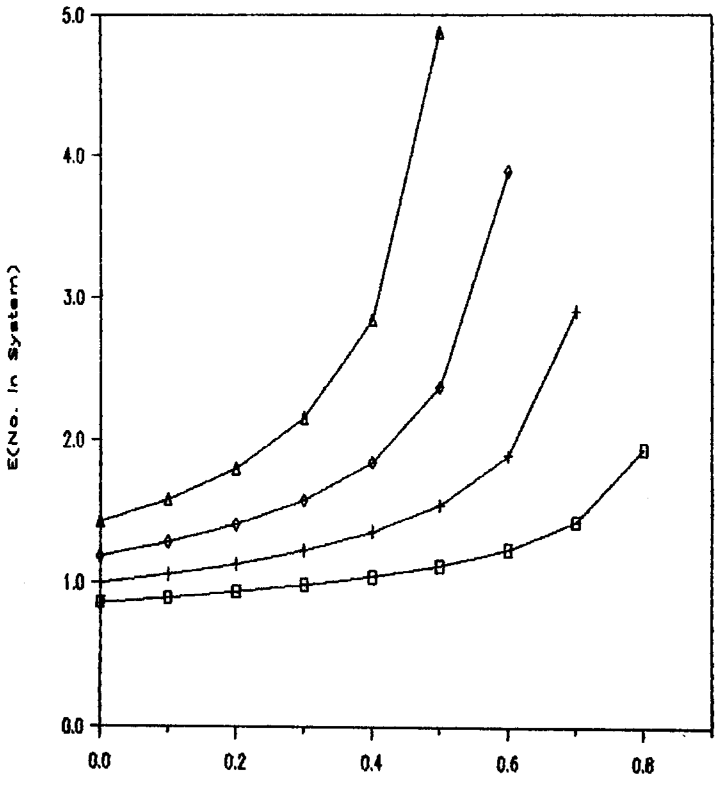
□ $\lambda = 1.0$, $+\lambda = 0.3$, ◇ $\lambda = 0.5$, △ $\lambda = 0.7$

Figure 6. Dep. prob. q vs. E(Time in system). $p = (1-q)0.5$, $r = (1-q)0.5$, $\mu = 40$.



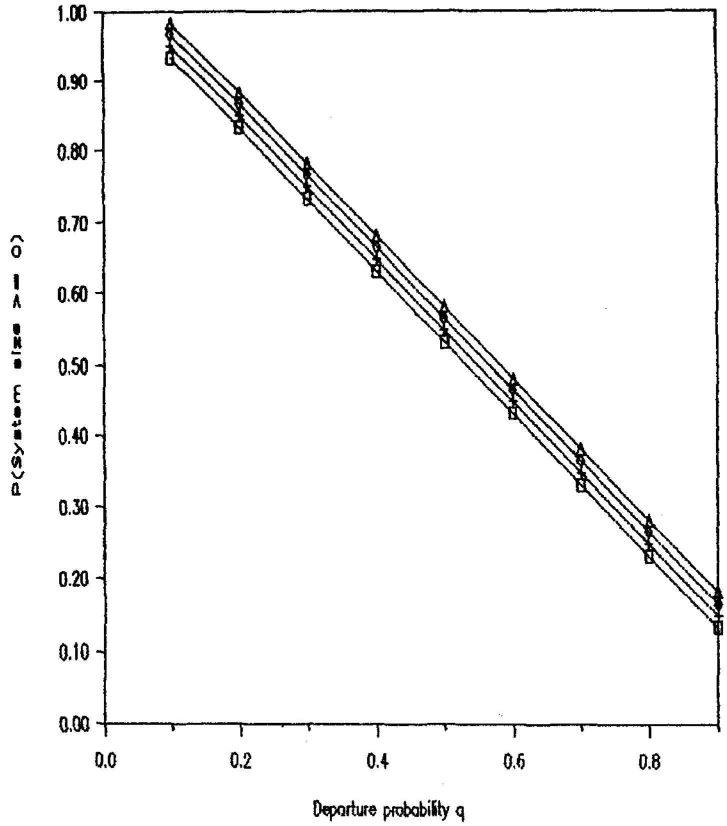
□ $\lambda = 0.2$, $+\lambda = 0.4$, ◇ $\lambda = 0.6$, △ $\lambda = 0.7$

Figure 7. Feedback prob. p vs. E(No. in system). $q = (1-p)0.25$, $r = (1-p)0.75$, $\mu = 40$.



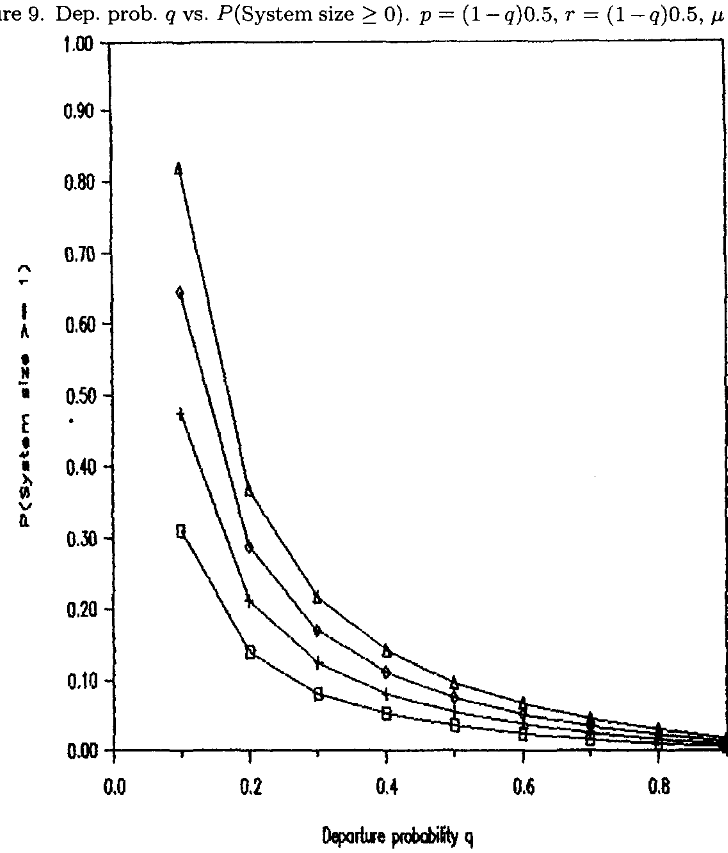
□ $\lambda = 1.0$, $+\lambda = 2.0$, ◇ $\lambda = 3.0$, △ $\lambda = 4.0$

Figure 8. Feedback prob. p vs. E(No. in system). $q = (1-p)0.25$, $r = (1-p)0.75$, $\mu = 40$.



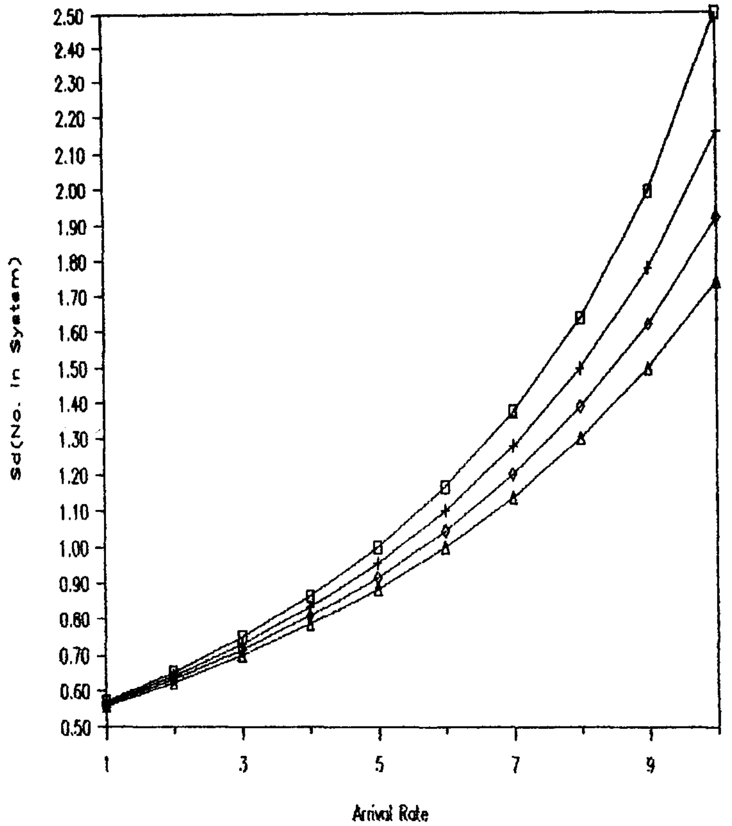
$\square\,\lambda = 1.0$, $+\lambda = 1.5$, $\Diamond\,\lambda = 2.0$, $\triangle\,\lambda = 2.5$

Figure 9. Dep. prob. q vs. P(System size ≥ 0). $p = (1-q)0.5$, $r = (1-q)0.5$, $\mu = 30$.



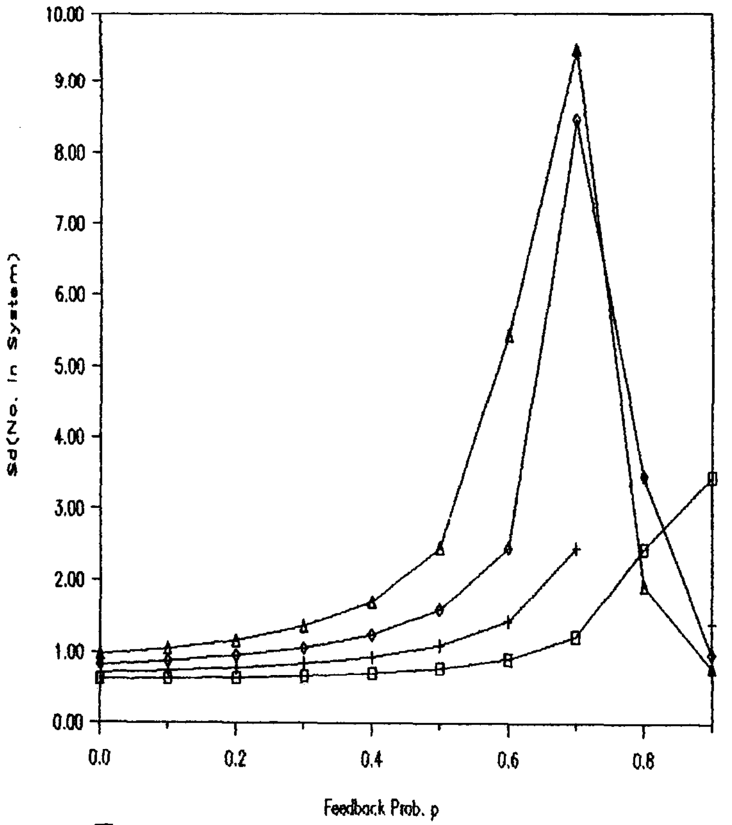
$\square\,\lambda = 1.0$, $+\lambda = 1.5$, $\Diamond\,\lambda = 2.0$, $\triangle\,\lambda = 2.5$

Figure 10. Dep. prob. q vs. P(System size ≥ 1). $p = (1-q)0.5$, $r = (1-q)0.5$, $\mu = 30$.



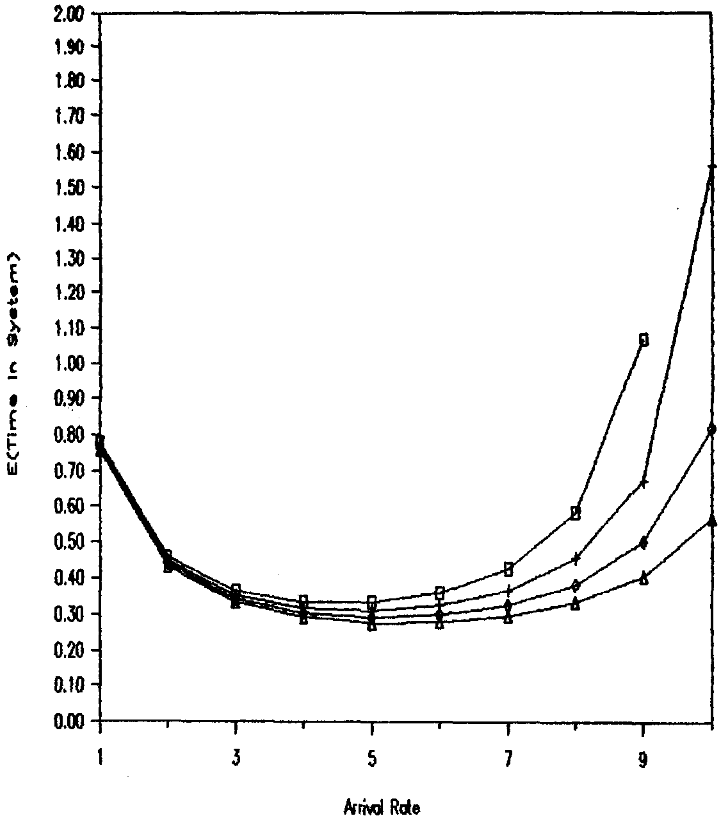
$\square\, \mu = 30$, $+\mu = 32$, $\Diamond\, \mu = 34$, $\triangle\, \mu = 36$

Figure 11. Arrival rate vs. Sd(No. in system). $q = 0.5$, $p = 0.1$, $r = 0.4$.



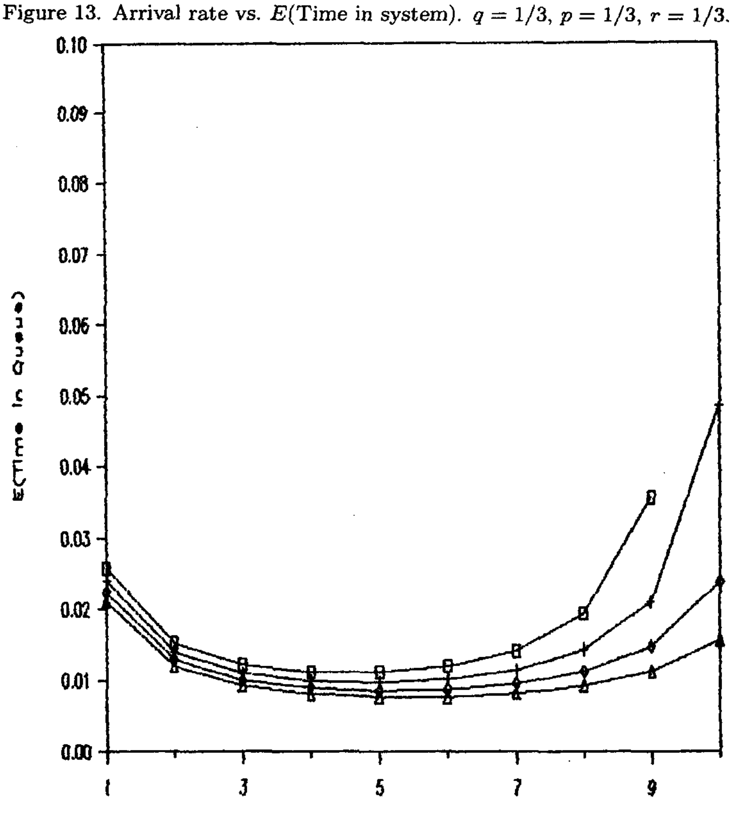
$\square\, \lambda = 1.0$, $+\lambda = 1.5$, $\Diamond\, \lambda = 2.0$, $\triangle\, \lambda = 2.5$

Figure 12. Feedback prob. p vs. Sd(No. in system). $q = (1-p)0.25$, $r = (1-p)0.75$, $\mu = 30$.



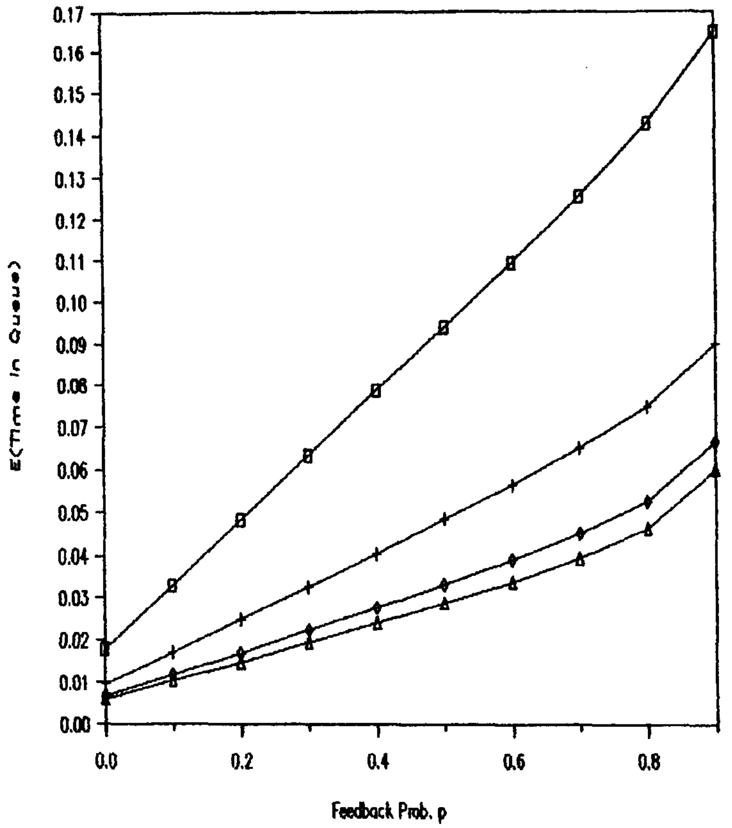
□ $\mu = 30$, $+\mu = 32$, ◊ $\mu = 34$, △ $\mu = 36$

Figure 13. Arrival rate vs. E(Time in system). $q = 1/3$, $p = 1/3$, $r = 1/3$.



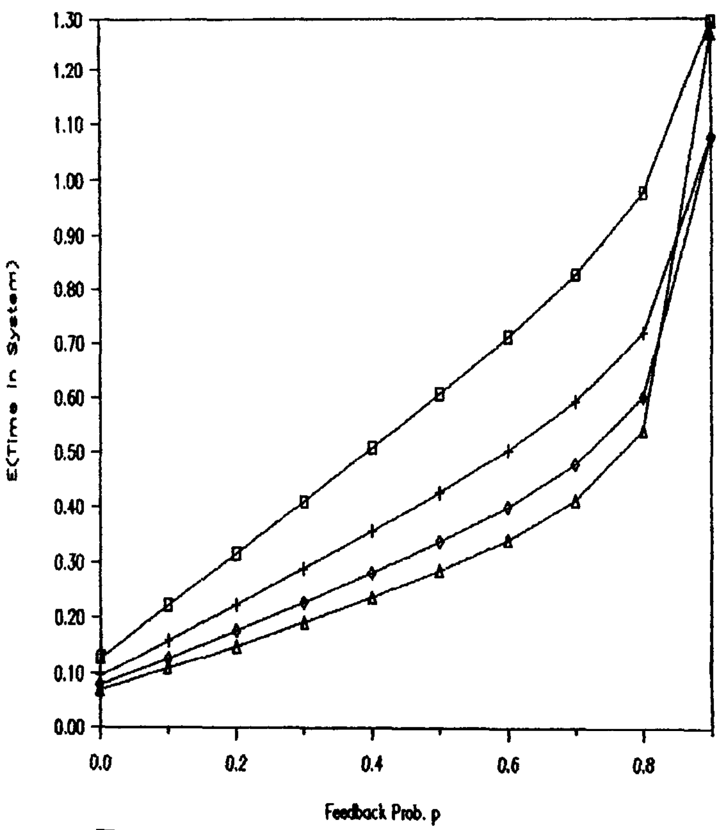
□ $\mu = 30$, $+\mu = 32$, ◊ $\mu = 34$, △ $\mu = 36$

Figure 14. Arrival rate vs. E(Time in queue). $q = 1/3$, $p = 1/3$, $r = 1/3$.



□ $\lambda = 0.2$, $+\lambda = 0.4$, ◇ $\lambda = 0.6$, △ $\lambda = 0.7$

Figure 15. Feedback prob. p vs. E(Time in queue). $q = (1-p)^{\ddagger}0.9$, $r = (1-p)^{\ddagger}0.1$, $\mu = 30$.



□ $\lambda = 1.0$, $+\lambda = 1.5$, ◇ $\lambda = 2.0$, △ $\lambda = 2.5$

Figure 16. Feedback prob. p vs. E(Time in system). $q = (1-p)^{\ddagger}0.9$, $r = (1-p)^{\ddagger}0.1$, $\mu = 40$.

Thus, we infer from the above selected numerical study that the feedback mechanism makes substantial and interesting changes in the operating characteristics of the queueing systems.

In Section 8, we have derived a formula in Theorem 3 for optimum service rate for our queueing model. Numerical examples are given below in Table 1 and Table 2.

Table 1.

	$c_0 = 1.000000$ $c_1 = 1.000000$ $q = 0.333333$		$c_0 = 2.000000$ $c_1 = 3.000000$ $q = 0.333333$		$c_0 = 2.000000$ $c_1 = 1.000000$ $q = 0.333333$		$c_0 = 3.000000$ $c_1 = 1.000000$ $q = 0.333333$	
λ	μ^0	T^0	μ^0	T^0	μ^0	T^0	μ^0	T^0
1.000000	4.732051	7.130768	5.121320	16.485281	4.224745	11.565646	4.000000	15.666667
2.000000	8.449490	11.565646	9.000000	26.000000	7.732051	19.594870	7.414214	27.151948
3.000000	12.000000	15.666667	12.674235	34.696938	44.454350	27.151948	10.732051	38.058972
4.000000	15.464102	19.594870	16.242641	42.970563	14.449490	34.464626	14.000000	48.666667
5.000000	18.872983	23.412633	19.743416	50.973666	17.738613	41.621118	17.236068	59.083075
6.000000	22.242641	27.151948	23.196152	58.784610	21.000000	48.666667	20.449490	69.363605
7.000000	25.582576	30.831818	26.612486	66.449944	24.240370	55.628148	23.645751	79.541175
8.000000	28.898979	34.464626	30.000000	74.000000	27.464102	62.523073	26.828427	89.637229
9.000000	32.196152	38.058972	33.363961	81.455844	30.674235	69.363605	30.000000	99.666667
10.000000	35.477226	41.621118	36.708204	88.832816	33.872983	76.158600	33.162278	109.640333

Table 2.

	$c_0 = 1.000000$ $c_1 = 1.000000$ $\lambda = 1.000000$		$c_0 = 2.000000$ $c_1 = 3.000000$ $\lambda = 2.000000$		$c_0 = 2.000000$ $c_1 = 1.000000$ $\lambda = 1.000000$		$c_0 = 3.000000$ $c_1 = 1.000000$ $\lambda = 2.000000$	
q	μ^0	T^0	μ^0	T^0	μ^0	T^0	μ^0	T^0
0.050000	24.472136	29.894272	47.745967	113.833867	23.162278	53.599111	43.651484	142.858902
0.070000	18.065359	22.775004	35.117965	86.119004	16.958327	40.191878	31.657496	105.160688
0.080000	16.035534	20.491068	31.123724	77.254897	15.000000	35.920000	27.886751	93.240508
0.100000	13.162278	17.224555	25.477226	64.608902	12.236068	29.844272	22.581989	76.391933
0.150000	9.248656	12.680644	17.805469	47.105210	8.492409	21.486301	15.441518	53.499111
0.200000	7.236068	10.272136	13.872983	37.891933	6.581139	17.124555	11.825742	41.754451
0.300000	5.159075	7.684817	9.828944	28.082444	4.624328	12.530644	8.157379	29.644272
0.400000	4.081139	6.262278	7.738613	22.754451	3.618034	10.072136	6.290994	23.345967
0.500000	3.414214	5.328427	6.449490	19.297959	3.000000	8.500000	5.154701	19.428203
0.600000	2.957661	4.648656	5.569401	16.810939	2.579538	7.384817	4.387426	16.724555
0.800000	2.368034	3.686068	4.436492	13.345967	2.040569	5.862278	3.412871	13.177226
0.900000	2.165204	3.319296	4.047964	12.047412	1.856467	5.303646	3.082885	11.930644

REFERENCES

1. L. Takács, A single server queue with feedback, *The Bell System Tech. Journal* **42**, 505–519 (1963).
2. R.L. Disney, D.C. McNickle and B. Simon, The $M/G/1$ queue with instantaneous Bernoulli feedback, *Naval Res. Logis. Quart.* **27**, 635–644 (1980).
3. G.R. D'Avignon and R.L. Disney, Single server queues with state dependent feedback, *INFOR* **14**, 71–85 (1976).
4. G.R. D'Avignon and R.L. Disney, Queues with instantaneous feedback, *Management Science* **24** (2), 168–179 (1977).
5. R.L. Disney, R.L. Farrell and P.R. Demorais, Characterization of $M/G/1$ queues with renewal departure process, *Management Science* **19**, 1222–1228 (1973).
6. V. Thangaraj and A. Santhakumaran, A queue with a pair of instantaneous independent Bernoulli feedback processes, *Optimization* **27**, 259–281 (1993).
7. R.B. Cooper, *Introduction to Queueing Theory*, MacMillan, New York, (1981).