Polynomial Systems Solving

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Abstract

This work is a survey on the subject Polynomial Systems Solving. First , Polynomial Systems definition is studied , then their solution techniques are studied, finally application areas are discussed.

1 Introduction

1.1 Monomials

A monomial in $\mathrm{x_1, x_2, ..., x_n}$ is a product of the form

$$x_1{}^{\alpha_1} \times x_2{}^{\alpha_2} .... \times x_n{}^{\alpha_n} \tag{1}$$

where all of the exponents $\alpha_1, ..., \alpha_\mathrm{n}$ are nonnegative integers. The total degree of this monomial is the sum $\alpha_1 + ... + \alpha_\mathrm{n}$.

1.2 Polynomials

A polynomial f in $\mathrm{x_1, x_2, ..., x_n}$ with coefficients in a field k is a finite linear combination (with coefficients in k) of monomials. We will write a polynomial f in the form :[29]

$$f = \sum_\alpha a_\alpha x^\alpha, a^\alpha \in k, \alpha \in Z, x^\alpha \in C \tag{2}$$

where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, ..., \alpha_\mathrm{n})$. The set of all polynomials in $\mathrm{x_1, x_2, ..., x_n}$ with coefficients in k is denoted $\mathrm{k[x_1, x_2, ..., x_n]}$.

A polynomial of order/degree **d** in one variable ($\mathbf{x} \in \mathbf{C}$) (i.e., a univariate polynomial) with constant coefficients ($\mathbf{a_i} \in \mathbf{Q}$) is given by [25]

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d \tag{3}$$

1.2.1 Properties of Polynomials

Properties of polynomials are as follows :

1. A polynomial may not have a term with negative and fractional exponent ($\mathrm{x^{-n}}$ and $\mathrm{x^{1/n}}$).

2. $\mathrm{x \in C}$ and $\mathrm{a_i}$ may be etiher $\in$ Q or $\in$ R or $\in$ C , that we call field k.

3. Every term ($\mathbf{a_i x^i}$) is composed of a coefficient ($\mathrm{a_i}$) and a monomial ($\mathbf{x^i}$) .

1.2.2 Solution of Polynomials

Polynomials of orders one to four are solvable using only rational operations and finite root extractions. [25]

1. A first-order equation is trivially solvable.
2. A second-order equation is soluble using the quadratic equation.
3. A third-order equation is solvable using the cubic equation.
4. A fourth-order equation is solvable using the quartic equation.
5. It was proved by Abel and Galois using group theory that general equations of fifth and higher order cannot be solved rationally with finite root extractions (Abel's impossibility theorem).

Solutions of the general quintic equation may be given in terms of **Jacobi theta functions** or **hypergeometric functions** in one variable.[25]

1. Hermite and Kronecker proved that higher order polynomials are not soluble in the same manner.
2. Klein showed that the work of Hermite was implicit in the group properties of the icosahedron. Klein's method of solving the quintic in terms of hypergeometric functions in one variable can be extended to the sextic, but for higher order polynomials, either hypergeometric functions in several variables or ”Siegel functions” must be used (Belardinelli 1960, King 1996, Chow 1999).
3. In the 1880s, Poincaré created functions which give the solution to the nth order polynomial equation in finite form. These functions turned out to be ”natural” generalizations of the elliptic functions.

1.3 Polynomial System

A system of (multivariate) polynomial equations is a set of simultaneous equations f_1 = 0, ..., f_m = 0 where the f_i are polynomials in several variables, say x_1, ..., x_n, over some field k (usually $\mathbb{C}$ or $\mathbb{R}$). [24].

In abstract algebra books, a **Polynomial System** is called as **Polynomial Ring** or **Polynomial Algebra**. [25] Abstract algebraic notions are explained in the section 2.

A **Univariate Polynomial System** (coefficients $\mathrm{a}_{\mathrm{ij}} \in$ R and i,j,n $\in$ N) may be given as m equations:

$$\begin{cases} a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4 + \ldots + a_{1d}x^d = 0 \\ a_{20} + a_{21}x + a_{22}x^2 + a_{13}x^3 + a_{14}x^4 + \ldots + a_{2d}x^d = 0 \\ a_{30} + a_{31}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4 + \ldots + a_{3d}x^d = 0 \\ \vdots \\ a_{m0} + a_{m1}x + a_{m2}x^2 + a_{m3}x^3 + a_{m4}x^4 + \ldots + a_{md}x^d = 0 \end{cases} \tag{4}$$

A **Bivariate Polynomial System** (variables $\mathrm{x}, \mathrm{y} \in$ R , coefficients $\mathrm{a}_{\mathrm{ij}} \in$ R and i,j,n $\in$ N) may be given as m equations :

$$\begin{cases} a_{00} + a_{01}xy + a_{02}xy^2 + a_{03}xy^3 + \ldots + a_{0(d\times d)}x^dy^d = 0 \\ a_{10} + a_{11}xy + a_{12}xy^2 + a_{13}xy^3 + \ldots + a_{1(d\times d)}x^dy^d = 0 \\ a_{20} + a_{21}xy + a_{22}xy^2 + a_{23}xy^3 + \ldots + a_{2(d\times d)}x^dy^d = 0 \\ \vdots \\ a_{m00} + a_{m11}xy + a_{m12}xy^2 + a_{m13}xy^3 + \ldots + a_{m(d\times d)}x^dy^d = 0 \end{cases} \tag{5}$$

A **Multivariate Polynomial System** (N variables $\mathrm{x_1 \ldots x_N} \in \mathrm{R}$, coefficients $\mathrm{a_{ij}} \in \mathrm{R}$ and i,j,n $\in$ N) may be given as m equations :

$$\begin{cases} a_{00} + \sum_{i_1,\ldots,i_n}^{d} a_{0i}{x_1}^{i_1}{x_2}^{i_2}\ldots{x_n}^{i_n} = 0 \\ a_{10} + \sum_{i_1,\ldots,i_n}^{d} a_{1i}{x_1}^{i_1}{x_2}^{i_2}\ldots{x_n}^{i_n} = 0 \\ \vdots \\ a_{m0} + \sum_{i_1,\ldots,i_n}^{d} a_{mi}{x_1}^{i_1}{x_2}^{i_2}\ldots{x_n}^{i_n} = 0 \end{cases} \tag{6}$$

2 Abstract Algebraic Notions

The following abstract algebraic notions are explained briefly in order to understand better the polynomials and their relation to the geometry. **At the heart of the group theory, symmetry lies**. All the contents in this section is taken either from [22] or [29] if not cited explicitly.

2.1 Group

A group is a pair (G,μ) with a non-empty set G and a "binary operation" (μ) that satisfies :

1. **Closure** : $\mathrm{a, b \in G \implies \mu(a, b) \in G}$
2. **Associativity** : $\mathrm{a, b, c \in G, \mu(\mu(a, b), c) = \mu(a, \mu(b, c))}$
3. **Neutral Element** : $\mathrm{\exists e \in G, \forall a \in G, \mu(a, e) = \mu(e, a) = a}$
4. **Inverse Element** : $\mathrm{\forall a \in G, \exists a^{-1} \in G, \mu(a, a^{-1}) = \mu(a^{-1}, a) = e}$

For Example : (N,+), (R\0,.).

2.1.1 Commutative (Abelien) Group

if a group has the commutativity property, it is called commutative (Abelien) group.

1. **Commutativity** : $\mathrm{\forall a, b \in G, \mu(a, b) = \mu(b, a)}$

2.1.2 Homomorphism

A map(function) between two groups ($\phi : \mathrm{G} \rightarrow \mathrm{H}$) , is called Homomorphism iff

$$\phi(a+b) = \phi(a) * \phi(b) \tag{7}$$

$(\mathrm{a}, \mathrm{b} \in \mathrm{G}, (\mathrm{G}, +), (\mathrm{H}, *))$
For example exponential function (or map) is a Homomorphism because $\mathrm{e}^{(\mathrm{a}+\mathrm{b})} = \mathrm{e}^{\mathrm{a}}.\mathrm{e}^{\mathrm{b}}$,

1. **Isomorphism** : if ϕ is bijective

2. **Automorphism** : Isomorphism with G=H

2.2 Ring

Every ring is a group with one aditional operation. (i.e. (G,+,.). More formally, a ring is a triple (R,α, μ) , with a set R together with two maps(or functions, or operations) :

$$\alpha : R \times R \rightarrow R, (a, b) \rightarrow a + b := \alpha(a, b), addition \tag{8}$$

and

$$\mu : R \times R \rightarrow R, (a, b) \rightarrow a.b := \mu(a, b), multiplication \tag{9}$$

such that :

1. The pair (R, α) is an (additively written) commutative (abelian) group.

2. The multiplication μ is associative : (ab)c=a(bc), a,b,c $\in$ R

3. The multiplication is "distributive" over the addition : a(b + c) = ab + ac , (a + b)c = ac + bc , $\forall$ a, b, c $\in$ R .

For example , (Z,+,.) forms a ring.

2.2.1 Commutative (Abelien) Ring with Unity

A ring is commutative with unity if :

1. There is an element $1 \in \mathrm{R}\backslash 0$, such that $1\mathrm{a} = \mathrm{a} = \mathrm{a}1, \forall \mathrm{a} \in \mathrm{R}$

2. The multiplication is commutative : $\mathrm{ab} = \mathrm{ba}, \forall \mathrm{a}, \mathrm{b} \in \mathrm{R}$.

2.2.2 Polynomial Ring

The sum and product of two polynomials is again a polynomial. We say that a polynomial f divides a polynomial g provided that g = f.h for some polynomial $\mathrm{h} \in \mathrm{k}[\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_\mathrm{n}]$.
One can show that, under addition and multiplication, $\mathrm{k}[\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_\mathrm{n}]$ satisfies all of the field axioms except for the existence of multiplicative inverses (because, for example, 1/x 1 is not a polynomial).
Such a mathematical structure is called a commutative ring and for this reason we will refer to $\mathbf{k}[\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}]$ as a **polynomial ring**.The coefficients of the polynomials are the elements of a Polynomial Ring.(We put the a_i instead of x_i)

2.3 Field

A field is a commutative ring with identity (1), in which every non-zero element has a multiplicative inverse. [1]

For example : the rings Q, R, C are actually fields.

2.4 Affine Space

Given a field k and a positive integer n, we define the n-dimensional affine space over k to be the set

$$k^n = \{(a_1, .., a_n) | a_1, .., a_n \in k\} \tag{10}$$

For an example of affine space, consider the case k = R .Here we get the familiar space R^n from calculus and linear algebra. In general, we call k^1 = k affine line, and k^2 affine plane.

2.4.1 Affine vs Vector Space

[2]An affine space is basically a vector space that doesn't necessarily have an identity vector. Vector spaces and Affine spaces are abstractions of different properties of Euclidean space. Like many abstractions, once abstracted they become more general.

A Vector space abstracts linearity/linear combinations. This involves the concept of a zero, scaling things up and down, and adding them to each other.

An Affine space abstracts the affine combinations. You can think of an affine combination as a weighted average, or a convex hull (if you limit the coefficients to be between 0 and 1). As it turns out, you do not need a zero, nor do you need the concept of "scaling", nor do you need full on addition, in order to have a concept of weighted average and convex hull within a space.

If you look at the Earth, the lines of longitude have a zero point, but that zero point is arbitrary – it has no meaning. The lines of longitude are an affine space. We measure them in degrees (or radians), and we have picked a zero, but other than it being useful to agree where the zero is, it isn't a special line.

The space of rotations around a circle, on the other hand, have a zero that is meaningful – zero means you don't rotate. We measure them as a vector space.

The lines of longitude are measured as rotations away from our arbitrary point we assigned zero. But what matters about them is the ability to say how far apart two longitude are from each other, not any one line's absolute value.

If we where doing some math and it would be useful to move the zero of longitude, we are free to do so. But if we want to move the zero in the space of rotation (to say bending things 90 degrees) we are not nearly as free.

In general, your location is an affine space, as there is no special place, and scaling your location by a factor of 3 makes no sense, and adding two locations makes no sense – but taking the average of two locations makes sense.

The (directed) distance between locations is a vector space. Saying something is twice as far as another distance makes sense, the "same place" (distance zero) makes sense, and adding two directed distances together makes sense.

[1] http://www-history.mcs.st-and.ac.uk/ john/MT4517/Lectures/L4.html

[2] https://math.stackexchange.com/questions/884666/what-are-differences-between-affine-space-and-vector-space

And you can pick a spot and describe locations as the directed distance from that particular spot, but the spot picked was arbitrary, and if it would be useful to pick a different spot, you are free to.

2.4.2 How Polynomials Relate to Affine Space

[29]The key idea is that a polynomial

$$f = \sum_{\alpha} a_\alpha x^\alpha, f \in k, a^\alpha \in k, \alpha \in Z, x^\alpha \in C \tag{11}$$

gives the function

$$f = k^n \to k \tag{12}$$

defined as follows: given $(\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}) \in \mathbf{k^n}$, replace every $\mathrm{x_i}$ by $\mathrm{a_i}$ in the expression for f. Since all of the coefficients also lie in k, this operation gives an element $\mathbf{f}(\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}) \in \mathbf{k}$. The ability to regard a polynomial as a function is what makes it possible to link algebra and geometry.

2.5 Affine Varieties

[29] Affine Variety is a geometric object defined as follows :
Let k be a field, and let $\mathrm{f_1, f_2, ..., f_n}$ be polynomials $\in \mathrm{k[x_1, ..., x_n]}$. Then we set

$$V(f_1, .., f_s) = \{(a_1, ..a_n) \in k^n | f_i(a_1, ...1_n) = 0, \forall i, 1 \leq i \leq s\} \tag{13}$$

We call affine variety $\mathrm{V(f_1, ..., f_s)}$ defined by $\mathrm{f_1, ..., f_s}$.
Thus, an affine variety $\mathrm{V(f_1, ..., f_s) \subseteq k^n}$ is the set of all solutions of the system of equations $\mathrm{f_1(x_1, ..., x_s) = ... = f_s(x_1, ..., x_s) = 0}$. We will use the letters V, W, etc. to denote affine varieties.

2.5.1 Example 1

The variety $\mathrm{V(x^2 + y^2 - 1) \subseteq R^2}$ is the circle of radius 1 centered at the origin :

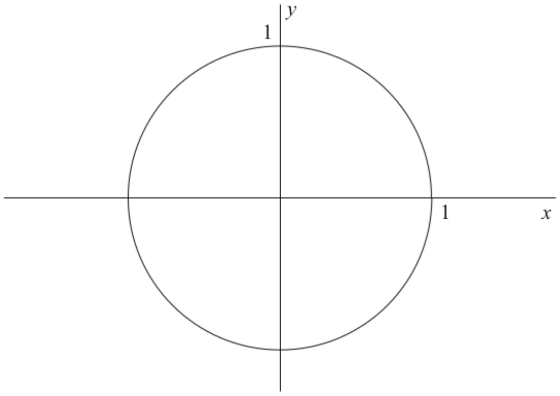


Figure 1

2.5.2 Example 2

The variety $\mathrm{V}((\mathrm{x}^2+\mathrm{y}^2-1)(3\mathrm{x}+6\mathrm{y}-4)) \subseteq \mathrm{R}^2$ defines all the points satisfying the circle of radius 1 centered at the origin and the line defined by the equation (3x+6y-4):

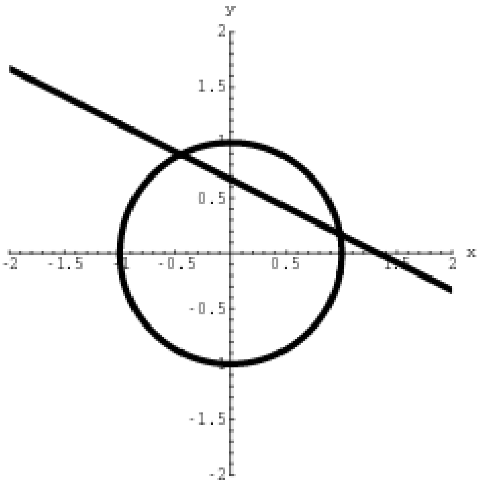


Figure 2[3]

2.6 Ideal

Ideal is an algebraic object. The real importance of ideals is that they will give us a language for computing with affine varieties.In the next section we will define relation between polynomials, varieties (geometric object) and ideals (algebraic objects).

Definition 1 :
R is a commutative ring, a subset $\mathrm{I} \subseteq \mathrm{k}[\mathrm{x}_1, ..., \mathrm{x}_\mathrm{n}]$ $(\mathrm{I} \subseteq \mathrm{R})$ is an **Ideal** if it satisfies :

1. $0 \in \mathrm{I}$
2. $\mathrm{f}, \mathrm{g} \in \mathrm{I} \implies \mathrm{f}+\mathrm{g} \in \mathrm{I}$
3. $\mathrm{f} \in \mathrm{I}, \mathrm{h} \in \mathrm{G}, \implies \mathrm{h.f} \in \mathrm{I}$

Example 1 :
(Z,+,.) is a commutative ring , Even numbers (2Z,+,.) is an ideal.

Definition 2 :
Let $\mathrm{f}_1, ..., \mathrm{f}_\mathrm{s}$ be a polynomial ring , in other words, be polynomials in $\mathrm{k}[\mathrm{x}_1, ..., \mathrm{x}_\mathrm{n}]$ then we set

$$\langle f_1, .., f_s \rangle = \left\{ \sum_{i=1}^{s} h_i.f_i | h_1, ..., h_s \in k[x_1, ..., x_n] \right\} \tag{14}$$

$\langle f_1, .., f_s\rangle$ is an ideal **generated by** $f_1, .., f_s \in k[x_1, ..., x_n]$.

2.7 Relation Between Algebra and Geometry

2.7.1 Relation Between Ideal and Variety

if $f_1, ..., f_s$ and $g_1, ..., g_t$ are bases of the same ideal in the ring $k[x_1, ..., x_n]$ so that $\langle f_1, .., f_s\rangle = \langle g_1, .., g_t\rangle$, then we have $V(f_1, ..., f_s) = V(g_1, ..., g_t)$

Example 1 :
As an example, consider the variety $V(2x^2 + 3y^2 - 11, x^2 - y^2 - 3)$.
It is easy to show that
$\langle 2x^2 + 3y^2 - 11, x^2 - y^2 - 3\rangle = \langle x^2 - 4, y^2 - 1\rangle$ so that
$V(2x^2 + 3y^2 - 11, x^2 - y^2 - 3) = V(x^2 - 4, y^2 - 1) = \{(\pm 2, \pm 1)\}$
Thus, by changing the basis of the ideal, we made it easier to determine the variety.

Definition :
Let $V \subseteq k^n$ be an affine variety. Then we set

$$I(V) = \{f \in k[x_1, ..., x_n] | f(a_1, ..., a_n) = 0 \forall (a_1, ..., a_n) \in V\} \tag{15}$$

The crucial observation is that I(V) is an ideal, we call it ideal of V. So we have $f_1, f_2, ..., f_s \in k[x_1, x_2, ..., x_n]$ such that :

$$\begin{array}{lclcl} \textbf{polynomials} & & \textbf{variety} & & \textbf{ideal} \\ f_1, ..., f_s & \longrightarrow & V(f_1, ..., f_s) & \longrightarrow & I(V(f_1, ..., f_s)) \end{array}$$

2.7.2 Relation Between Polynomial and Ring

Let R be a commutative ring with an identity. Then a polynomial with coefficients in R in an indeterminate x is something of the form

$$a_0 + a_1 x^1 + ... + a_n x^n, a_i \in R \tag{16}$$

One adds and multiplies polynomials "in the usual way". The ring of such polynomials is denoted by R[x]. [4]

2.7.3 Greatest Common Divisor (gcd)

Let k be a field and let g be a nonzero polynomial in $k[x_1, ..., x_n]$. Then

$$\forall f, g \in k[x], g \neq 0, \exists q, r \in k[x], f = qg + r \tag{17}$$

Also there is an algorithm to find g,q and r.

Definition :
A greatest common divisor of polynomials $f, g \in k[x]$ is a polynomial h such that :

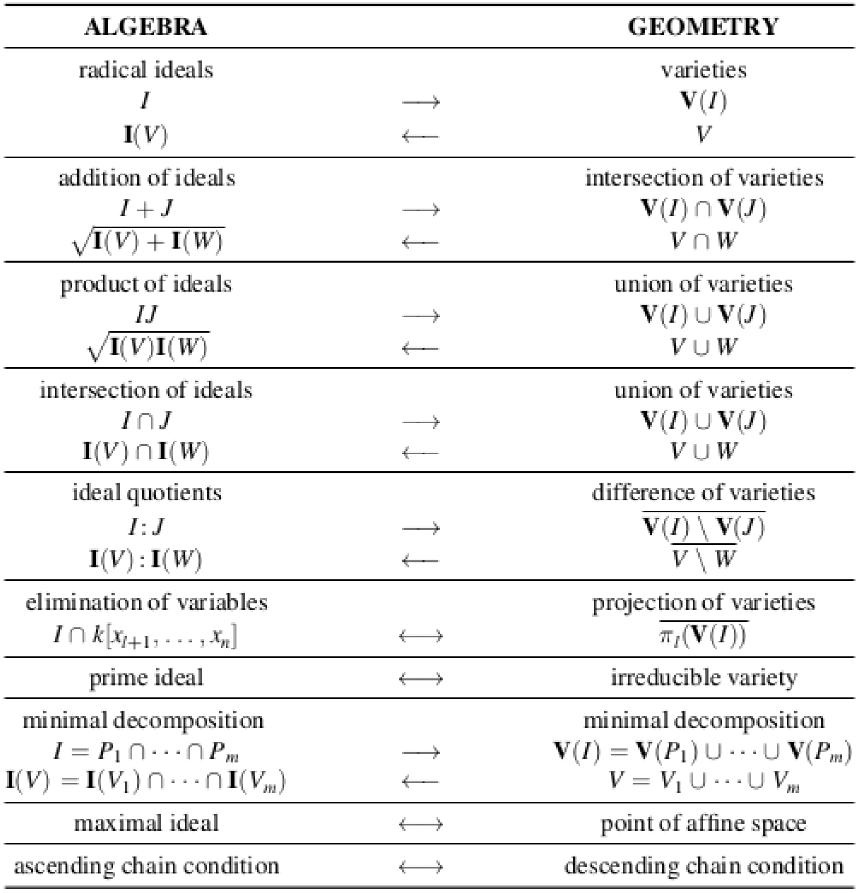
[4] http://wwwhistory.mcs.st-and.ac.uk/ john/MT4517/Lectures/L3.html

1. h divides f and g
2. if p is naother which divides f and g, then p divides h . When h has these properties , we write h= GCD(f,g)

Properties :

1. GCD(f,g) exists and unique up to multiplication by nonzero constant k.
2. GCD(f,g) is a generator of the field $\langle \mathrm{f}, \mathrm{g} \rangle$
3. There is an algorithm for finding GCD(f,g) .

2.7.4 Algebraic Notions and Geometric Notions



ALGEBRA		**GEOMETRY**
radical ideals		varieties
I	$\longrightarrow$	$\mathbf{V}(I)$
$\mathbf{I}(V)$	$\longleftarrow$	V
addition of ideals		intersection of varieties
$I+J$	$\longrightarrow$	$\mathbf{V}(I)\cap\mathbf{V}(J)$
$\sqrt{\mathbf{I}(V)+\mathbf{I}(W)}$	$\longleftarrow$	$V\cap W$
product of ideals		union of varieties
IJ	$\longrightarrow$	$\mathbf{V}(I)\cup\mathbf{V}(J)$
$\sqrt{\mathbf{I}(V)\mathbf{I}(W)}$	$\longleftarrow$	$V\cup W$
intersection of ideals		union of varieties
$I\cap J$	$\longrightarrow$	$\mathbf{V}(I)\cup\mathbf{V}(J)$
$\mathbf{I}(V)\cap\mathbf{I}(W)$	$\longleftarrow$	$V\cup W$
ideal quotients		difference of varieties
$I:J$	$\longrightarrow$	$\overline{\mathbf{V}(I)\setminus\mathbf{V}(J)}$
$\mathbf{I}(V):\mathbf{I}(W)$	$\longleftarrow$	$\overline{V\setminus W}$
elimination of variables		projection of varieties
$I\cap k[x_{l+1},\ldots,x_n]$	$\longleftrightarrow$	$\overline{\pi_l(\mathbf{V}(I))}$
prime ideal	$\longleftrightarrow$	irreducible variety
minimal decomposition		minimal decomposition
$I=P_1\cap\cdots\cap P_m$	$\longrightarrow$	$\mathbf{V}(I)=\mathbf{V}(P_1)\cup\cdots\cup\mathbf{V}(P_m)$
$\mathbf{I}(V)=\mathbf{I}(V_1)\cap\cdots\cap\mathbf{I}(V_m)$	$\longleftarrow$	$V=V_1\cup\cdots\cup V_m$
maximal ideal	$\longleftrightarrow$	point of affine space
ascending chain condition	$\longleftrightarrow$	descending chain condition

Figure 3

3 Geometrical Meaning

An affine variety $\mathrm{V(f_1 \ldots f_s) \subset k^n}$ is the set of all solutions of the system of equations $\mathrm{f_1(x_1 \ldots x_n) = \ldots = f_s(x_1 \ldots x_n) = 0}$. We will use the letters V,W,etc. to denote affine varieties. [29]

The conic sections studied in analytic geometry (circles, ellipses, parabolas) are affine varieties. Likewise, graphs of polynomial functions are affine varieties [the graph of $\mathbf{y = f(x)}$ is $\mathbf{V(y - f(x))}$]. Although not as obvious, graphs of rational functions are also affine varieties. For example, consider the graph of $\mathbf{y = \frac{x^3 - 1}{x}}$:

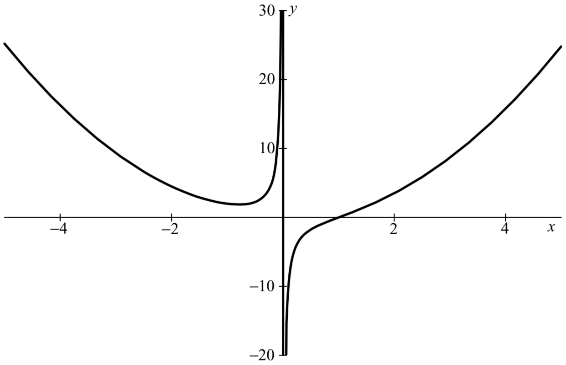


Figure 4

3-dimensional space example. A nice affine variety is given by paraboloid of revolution $\mathbf{V(z - x^2 - y)}$, which is obtained by rotating the parabola $\mathbf{z = x^2}$ about the z-axis. This gives us the picture:

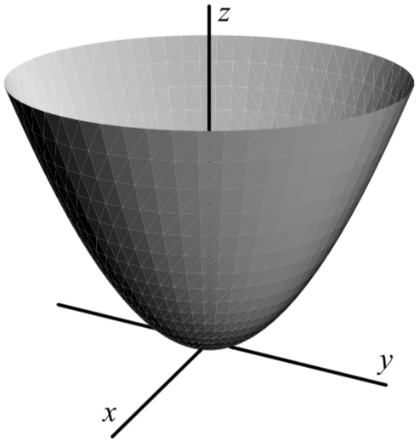


Figure 5

A much more complicated surface is given by $\mathbf{V(x^2 - y^2z^2 + z^3)}$:

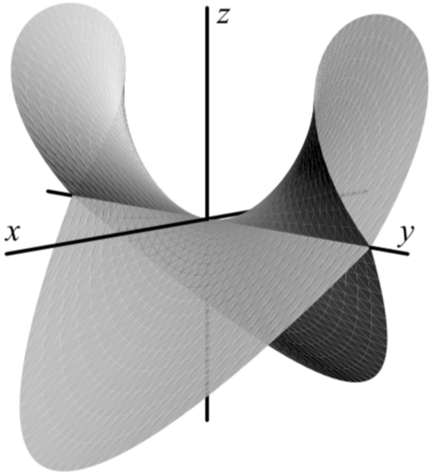


Figure 6

4 Polynomial System Solution with Groebner Bases

Buchberger developed the theory of Groebner Bases in 1965 via his thesis. The source of the develpment of Groebner Bases goes back to solving some thoretical problems of polynomial ring theory. These problems can be named as below:

1- The Ideal Description Problem: Does every ideal $\mathrm{I} \subset \mathrm{k}[\mathrm{x}_1....\mathrm{x}_\mathrm{n}]$ have a finite generating set?

2- The Ideal Membership Problem: Given $\mathrm{f} \in \mathrm{k}[\mathrm{x}_1....\mathrm{x}_\mathrm{n}]$ and an ideal $\mathrm{I} =< \mathrm{f}_1....\mathrm{f}_\mathrm{n} >$ determine if $\mathrm{f} \in \mathrm{I}$.

3- The Problem of Solving Polynomial Equations: Find all common solutions in k^n of a system of polynomial equations. $\mathrm{f}_1(\mathrm{x}_1 \ldots \mathrm{x}_\mathrm{n}) = \ldots = \mathrm{f}_\mathrm{s}(\mathrm{x}_1 \ldots \mathrm{x}_\mathrm{n}) = 0$.

4- The Implicitization Problem: Let V be a subset of k^n given parametrically as

$$\begin{aligned} \mathrm{x}_1 &= \mathrm{g}_1(\mathrm{t}_1 \ldots \mathrm{t}_\mathrm{n}) \\ &\vdots \\ \mathrm{x}_\mathrm{n} &= \mathrm{g}_\mathrm{n}(\mathrm{t}_1 \ldots \mathrm{t}_\mathrm{n}) \end{aligned}$$

If g_i is polynomials or rational functions in the variables t_j when V will be affine variety or part of one. Find a system of polynomial equations (in the x_i) that defines the variety.

Although we are interested in the problem 3 all of the problems are closely interrelated. Groebner basis allows computations in multivariate polynomial rings similar to ones in single variable. Groebner bases can also be seen as generalization of Gaussian elimination of a linear system, which yields the row-echelon form in linear algebra.

4.1 Monomial Order

If we examine the polynomial division algorithm in k[x] and the row-reduction (Gaussian elimination) algorithm for systems of linear equations(or matrices) we see that ordering of terms is very important part of these operations.

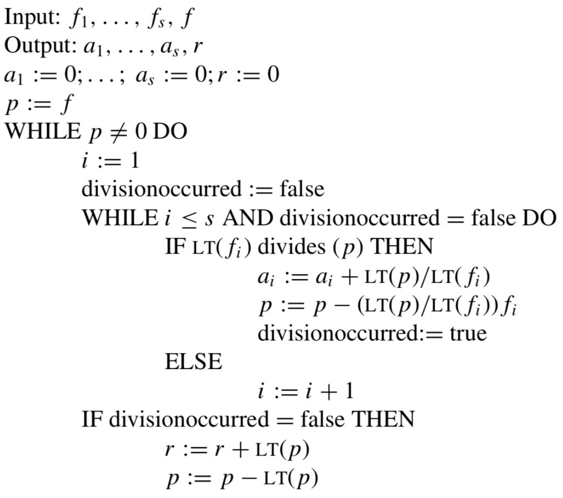
A monomial ordering on $\mathrm{k}[\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_\mathrm{n}]$ is any relation ¿ on the set of monomials $\mathrm{x}^\alpha = \mathrm{x}_1^{\alpha_1} \ldots \mathrm{x}_\mathrm{n}^{\alpha_\mathrm{n}}, \alpha \in \mathrm{Z}^\mathrm{n}$ satisfying: 1- $>$ is a total (or linear) order on Z^n.
2- If $\alpha > \beta$ and $\gamma \in \mathrm{Z}^\mathrm{n}$, then $\alpha + \gamma > \beta + \gamma$
3- $>$ is a well-ordering on Z^n. This means that every nonempty subset of Z^n has a smallest element under $>$.

The Lexicographic ("Dictionary") Order: Let $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_\mathrm{n})\mathrm{and}\beta = (\beta_1, \beta_2 \ldots \beta_\mathrm{n}) \in \mathrm{Z}^\mathrm{n}$ we say $\alpha >_{\mathrm{lex}} \beta$ if in the vector difference $\alpha - \beta \in \mathrm{Z}^\mathrm{n}$ the leftmost nonzero entry is positive. $\mathrm{x}^\alpha >_{\mathrm{lex}} \mathrm{x}^\beta$ if $\alpha >_{\mathrm{lex}} \beta$

The Graded Lexicographic Order: Let $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_\mathrm{n})\mathrm{and}\beta = (\beta_1, \beta_2 \ldots \beta_\mathrm{n}) \in \mathrm{Z}^\mathrm{n}$ we say $\alpha >_{\mathrm{grlex}} \beta$ if,
$\mid \alpha \mid= \sum_{\mathrm{i}=1}^{\mathrm{n}} \alpha_\mathrm{i} >\mid \beta \mid= \sum_{\mathrm{i}=1}^{\mathrm{n}} \beta_\mathrm{i}$ or $\mid \mid \alpha \mid > \mid \beta \mid$ and $, \alpha >_{\mathrm{lex}} \beta$

Definitions: Let $\mathrm{f} = \sum_\alpha \alpha_\alpha \mathrm{x}^\alpha$ be a nonzero polynomial in $\mathrm{k}[x_1, x_2 \ldots x_n]$ and let $>$ be a monomial order:
(i) The **multidegree** of f is $\mathrm{multideg(f)} = \max(\alpha \in \mathrm{Z}^\mathrm{n} : \alpha_\alpha \neq 0)$
(ii) The **leading coefficient** of f is $\mathrm{LC(f)} = \alpha_{multideg(f)} \in k$
(iii) The **leading monomial** of f is $\mathrm{LM(f)} = x^{multideg(f)} \in k$
(iv) The **leading term** of f is LT(f) = LC(f) * LM(f)

4.2 The Multivariate Division Algorithm



```
Input: $f_1, \ldots, f_s, f$
Output: $a_1, \ldots, a_s, r$
$a_1 := 0; \ldots ; a_s := 0; r := 0$
$p := f$
WHILE $p \neq 0$ DO
        $i := 1$
        divisionoccurred := false
        WHILE $i \leq s$ AND divisionoccurred = false DO
                IF LT($f_i$) divides ($p$) THEN
                        $a_i := a_i + \mathrm{LT}(p)/\mathrm{LT}(f_i)$
                        $p := p - (\mathrm{LT}(p)/\mathrm{LT}(f_i))f_i$
                        divisionoccurred:= true
                ELSE
                        $i := i + 1$
        IF divisionoccurred = false THEN
                $r := r + \mathrm{LT}(p)$
                $p := p - \mathrm{LT}(p)$
```

Figure 7

Division of multivariate polynomials can be done with algorithm above. This algorithm does not guarantee that given polynomial is in the ideal if the remainder is 0. Or vice verse. That is because the result and remainder changes when the order of the dividents change. That means if the type of monomial ordering changes the remainder changes. So, that is why we can not solve the problems related to ideals in polynomial rings with this algorithm.

4.3 Groebner Bases

The Groebner basis is a special generating set for ideals $(\mathrm{f_1, f_2, \ldots, f_n})$ for which the multivariate division algorithm for a given f returns remainder 0 if and only if f $\in <\mathrm{g_1, g_2, ..., g_n}>$. Every nonzero ideal I $\in \mathrm{k[x_1, x_2, \ldots, x_n]}$ has the Groebner Bases.

Moreover, due to multivariate division algorithm if $f \in I$ we have $\mathrm{f = q_1g_1 + q_2g_2 + ... + q_sg_s + r}$ and no term of r is divisible by any of $LT(g_1), LT(g_2), \ldots, LT(g_n)$. Because none of the $LT(g_i)$ divides r by definition we must have r=0. This shows that $f \in < LT(g_1), LT(g_2) \ldots LT(g_n) >$. And it is proven that r is unique for a Groebner basis.

Testing whether a basis has a high connection with S-polynomial for a given pair of polynomials. Let $f, g \in k[x_1, x_2, \ldots, x_n]$ be nonzero polynomials. Find the least common multiple of their leading monomials: $x^y = LCM(LM(f), LM(g))$. Then the S-polynomial of f and g is defined by:

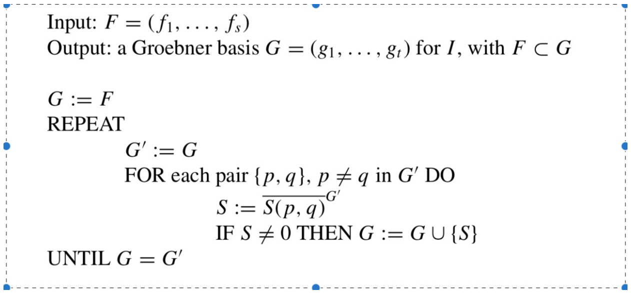
$$S(f,g) = \frac{x^y}{LT(f)} \times f - \frac{x^y}{LT(g)} \times f \tag{18}$$

S-polynomials porvide cancellation of leading terms.

4.4 Buchberger‘s Algorithm

Buchberger’s Criterion: Let I be a polynmial ideal. Then a basis $G = g_1, g_2, ldots, g_n$ for I is a Groebner basis for I if and only if for all pairs $i \neq j$, the remainder on division of $S(g_i, g_j)$ by G is 0.

Buchberger’s algorithm which produces a Groebner bases for nonzero polynomial ideal, is based on the createrion above.



```
Input: $F = (f_1, \ldots, f_s)$
Output: a Groebner basis $G = (g_1, \ldots, g_t)$ for $I$, with $F \subset G$

$G := F$
REPEAT
        $G' := G$
        FOR each pair $\{p, q\}$, $p \neq q$ in $G'$ DO
                $S := \overline{S(p, q)}^{G'}$
                IF $S \neq 0$ THEN $G := G \cup \{S\}$
UNTIL $G = G'$
```

Figure 8

5 Solver Software

There are many solving softwares available for computing polynomial systems.

M4GB [27] is an efficient algorithm for computing Grobner-Bases written in C. It's an extension of Buchberger's algorithm. It stores already computed (tail-)reduced multiplies of basis polynomials to prevent redundant work in reduction step. And it exploits efficient linear algebra for the reduction step. They run a benchmark test with four libraries, OpenF4, FGb, Magma and M4GB. Others are different libraries to compute Grobner-Bases.

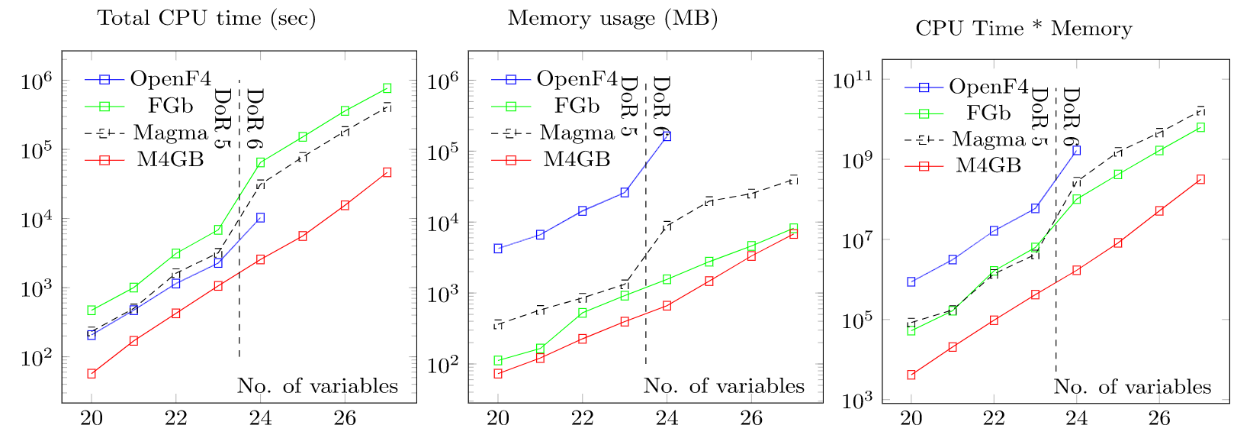


Figure 9: results for equations = variables * 2

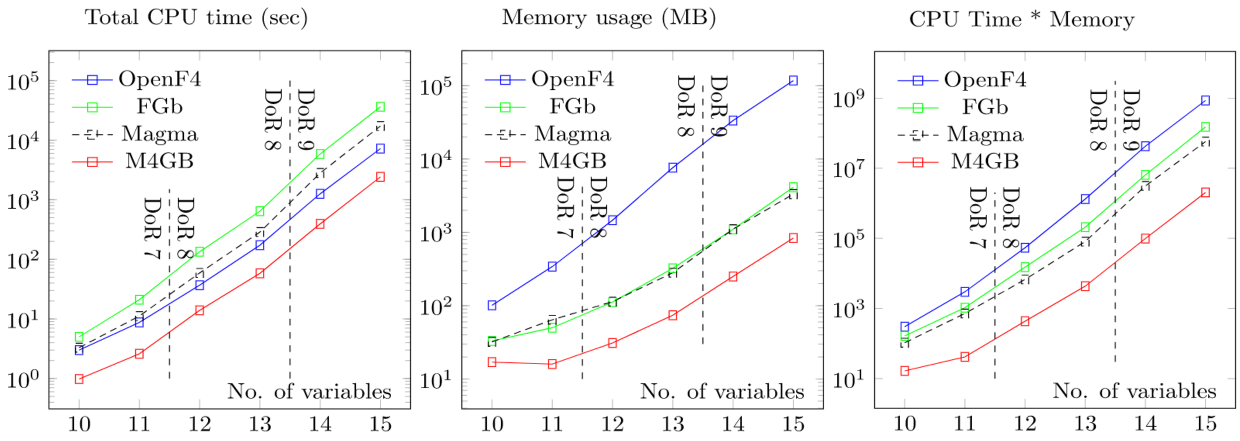


Figure 10: results for equations = variables + 1

SymPy [28] is a Python open-source library for symbolic mathematics. For groebnar-bases calculation it has sympy.polys.polytools.groebner(F, *gens, **args) method. it uses order argument to set the monomial ordering that will be used to compute the basis. Order argument can be "lex", "grlex" and "grevlex". Default order is "lex" example usages are below.

```
>>> groebner([x*y - 2*y, 2*y**2 - x**2], x, y, order='lex')
GroebnerBasis([x**2 - 2*y**2, x*y - 2*y, y**3 - 2*y], x, y, domain='ZZ', order='lex')
```

```
>>> groebner([x*y - 2*y, 2*y**2 - x**2], x, y, order='grlex')
GroebnerBasis([y**3 - 2*y, x**2 - 2*y**2, x*y - 2*y], x, y, domain='ZZ', order='grlex')

>>> groebner([x*y - 2*y, 2*y**2 - x**2], x, y, order='grevlex')
GroebnerBasis([y**3 - 2*y, x**2 - 2*y**2, x*y - 2*y], x, y, domain='ZZ', order='grevlex')
```

Groebner Algebra is a Matlab package which can compute a reduced Groebner-basis of the ideal generated by the polynomials in the list polys. Syntax

```
groebner::gbasis(polys, <order>, options)
```

Examples

```
>>>groebner::gbasis([x^2 - y^2, x^2 + y], LexOrder)
```

$[\mathrm{x}^2 + \mathrm{y}, \mathrm{x}^4 - \mathrm{x}^2]$

```
>>>groebner::gbasis([poly(x^2 - y^2, [y, x]), poly(x^2 + y, [y, x])], LexOrder)
```

$[\mathrm{poly}(\mathrm{y} + \mathrm{x}^2, [\mathrm{y}, \mathrm{x}]), \mathrm{poly}(\mathrm{x}^4 - \mathrm{x}^2, [\mathrm{y}, \mathrm{x}])]$

6 Applications

Systems of polynomial equations show up in many applications areas such as robotics (kinematics,motion planning, collision detection, etc.), computer vision (object modeling, surface fitting,recognition, etc.), graphics, geometric modeling (curve and surface intersections), computer-aided design, mechanical design, and chemical equilibrium systems. [26]

6.1 Robotics

System of polynomial equations may use to model a robot arm. If $(\mathbf{x_i}, \mathbf{y_i})$ is the coordinates of some joint, $(\mathbf{x_i + 1}, \mathbf{y_i + 1})$ is the coordinates of the next joint (or the hand) and the segment has length Li then we get the equation [23]

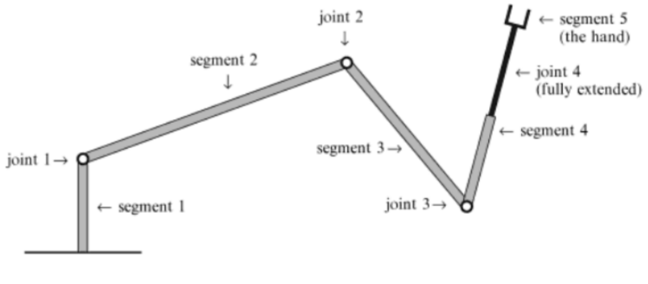


Figure 11: Robot Arm

$$(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = L_i^2 \tag{19}$$

Note that the angle at the i-th joint can be computed from the equations

$$x_{i+1} - x_i = L_i \cos(\theta_i) \tag{20}$$

$$y_{i+1} - y_i = L_i \sin(\theta_i) \tag{21}$$

6.2 Geometry

Many statements in Euclidean geometry can be formulated as result about systems of polynomial equations.[23]

Example: Euclidean geometry says that *the bisectors of the three sides of a triangle meet in one point.*

One vertex, A, of the triangle can be assumed to be the origin and another, B, to have coordinates (c, 0), without loss of generality. The third vertex, C, has coordinates (a,b). The bisectors of AB and BC meet in some point, with coordinates (x1, y1). We get the following equations

$$x_1 - \frac{c}{2} = 0 \tag{22}$$

$$\frac{c-a}{b} \cdot x_1 - y_1 + \frac{a^2 + b^2 - c^2}{2b} = 0 \tag{23}$$

Similarly, if (x2,y2) are the coordinates of the intersection of the bisectors of AB and AC we have

$$x_2 - \frac{c}{2} = 0 \tag{24}$$

$$\frac{a}{b} \cdot x_2 + y_2 - \frac{b^2 + a^2}{2b} = 0 \tag{25}$$

7 Conclusion

Polynomial Systems are solved using both algebraic and geometric structures. These structures are related through the ”Ideals” and ”Varieties”. Groebner basis and its algorithmic solution ”Buchberger” is used to find the greated common divisor of the polynomials. Once we found gcd of the polynomials , it is straight forward to solve the polynomial system. Varieties and Ideals (mixture of the geometry and algebra) not only used to solve the polynomial systems, it is also used to visualize the equations as geometrical objects.

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