International Conference on Control, Automation and Systems 2008
Oct. 14-17, 2008 in COEX, Seoul, Korea

Hybrid Controller for Swinging up and Stabilizing the Inverted Pendulum on Cart

Ekachai Asa[1], Taworn Benjanarasuth[1], Jongkol Ngamwiwit[1] and Noriyuki Komine[2]

[1]Faculty of Engineering and Research Center for Communications and Information Technology,
King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand
(Tel : +66-2-326-4221, Fax: +66-2-326-4225, E-mail: kbtaworn@kmitl.ac.th)
[2]Dept. of Embedded Technology, School of Information and Telecommunication Engineering,
Tokai University, Hiratsuka-Shi, Kanagawa-Ken 259-1292, Japan
(Tel: +81-463-58-1211, Fax: +81-463-50-2240, E-mail: komine@keyaki.cc.u-tokai.ac.jp)

Abstract: In this paper, a hybrid controller for swinging up and stabilizing an inverted pendulum on cart is presented. The energy control concept is employed to swing the inverted pendulum up to around the upright position within the assigned switching condition. After that, the stabilizing controller is then switched to stabilize the inverted pendulum at the upright position. The stabilizing controller is a linear servo state feedback controller designed by Coefficient Diagram Method. The simulation results show that the designed hybrid controller can be mutually operated with acceptable efficiency.

Keywords: Energy Control Method, Coefficient Diagram Method (CDM), Inverted Pendulum.

1. INTRODUCTION

Control of the inverted pendulum on cart is a classical problem in control theory. The inverted pendulum on cart is the SIMO system where the force applied to the cart is the input and cart position and pendulum angle are the outputs of the system. Its structure, shown as in Fig. 1, composes of a cart and pendulum where the pendulum is hinged in series to the cart via a pivot and only the cart is actuated. These limited travels of the cart can be changed by changing the values of some designed parameters (a_0 and b_0). Although the inverted pendulum on cart is easy to be described, it is not easy to be controlled given its inherent instability and nonlinear characteristics. It is also a more challenging task to autonomously move the pendulum to the upright position from its pendent position and to keep it there.

In control area, there have been many studies on the inverted pendulum on cart. For instance, the method using a PD controller for swinging up the pendulum is described in [1] which controls the cart's position to preassigned values obtained by observation, and the method using the energy control is explained in [2] in which the limitation of the cart travels is not considered. In this paper the controller is designed based on Energy Control Method [3] to swing up the pendulum from its pendent position to around the upright position within a limited travel of the cart without preassigned cart's position and Coefficient Diagram Method (CDM) [4] to stabilize the pendulum in the upright position while maintaining the cart at a certain position.

2. MATHEMATICAL MODEL

The inverted pendulum on cart to be controlled is shown in Fig. 1. Where θ is the pendulum angle (rad), x is the cart position (m), M is the mass of the cart (kg), m is the mass of the pendulum (kg), l is the distance from turning center to center of mass of the pendulum (m), L is the length of the pendulum (m), c is the cart's friction coefficient (kg/s) and F is the force applied to the cart (N).

The mathematical model of inverted pendulum on cart can be derived using Lagrange's equation. First, determine the kinetic energy T and potential energy E of the system's components in terms of generalized coordinates q. Then, apply the Lagrange's equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]-\frac{\partial L}{\partial q}=Q_q \tag{1}$$

where Lagrangian $L=T-E$, Q_q is generalized forces not taken into account in T and E.

For the inverted pendulum on cart, we select q as $q=[x,\theta]^T$. Then, the system dynamic can be described by Eqs. (2) ~ (3).

$$(M+m)\ddot{x}-(ml\cos\theta)\ddot{\theta}=u-c\dot{x}-ml\dot{\theta}^2\sin\theta \tag{2}$$

$$-\cos\theta\ddot{x}+l\ddot{\theta}=g\sin\theta \tag{3}$$

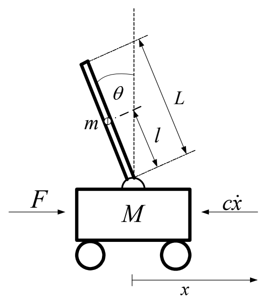

Fig. 1 Inverted pendulum on cart.

The parameters of inverted on cart in the laboratory are listed in Table 1.

Table 1 Parameters of inverted pendulum on cart

M (kg)	m (kg)	l (m)	c (kg/s)
0.642	0.123	0.25	2.14

3. DESIGN METHOD

The overall control system structure of the inverted pendulum on cart is shown in Fig. 2.

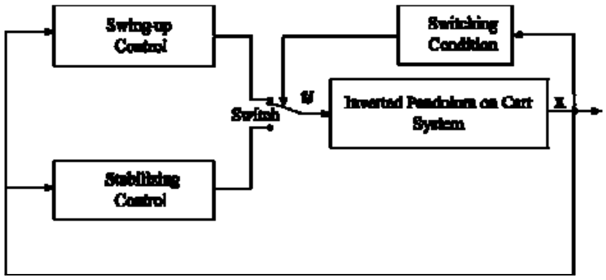

Fig. 2 Structure of the proposed control system.

In this paper, the controller design for the inverted pendulum on cart is separated into two parts. The first part is the swing-up controller design and the second part is the stabilizing controller design.

3.1 Swing-up Controller by Energy Control Method

The total mechanical energy V of the pendulum and its time derivative are as follows.

$$V = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1+\cos\theta) \tag{4}$$

$$\dot{V} = ml\dot{\theta}\cos\theta\ddot{x} \tag{5}$$

From Eq. (5), the value of $\dot{V}$ depends on changing the sign of $\ddot{x}$ and $\dot{\theta}\cos\theta$. The purpose of energy control method is to construct a servo system having a sinusoidal reference input, which is obtained from $(\theta, \dot{\theta})$, and generate $\ddot{x}$ satisfying the sign condition in order to control V to a prescribed value. Apply u into Eq. (2) in order to construct a servo system as

$$\begin{aligned} u &= \left(M + m\sin^2\theta\right)\left\{f_1\left(r_d - x\right) - f_2\dot{x}\right\} + c\dot{x} + ml\dot{\theta}^2 \sin\theta \\ &\quad - mg\cos\theta\sin\theta \end{aligned} \tag{6}$$

where r_d is the reference input of the servo system for x and f_1 and f_2 are given by

$$f_1 = \Omega^2,\ f_2 = 2\zeta\Omega,\ \Omega = \omega_n / c_0,\ \omega_n = \sqrt{g/l}\,. \tag{7}$$

Then the transfer function from r_d to x is represented by

$$G(s) = \frac{\Omega^2}{s^2 + 2\zeta\Omega s + \Omega^2} \tag{8}$$

where ζ and c_0 are the design parameters. Let $g(\omega)$ and $\phi(\omega)$ be the gain and phase lag of $G(j\omega)$.

The reference input r_d is assigned as

$$r_d(t) = \frac{a}{g(\omega_n)} \sin\left(\varphi(t) - \pi + \phi(\omega_n)\right) \tag{9}$$

when $\varphi(t)$ is a monotonically increasing function of time t and is defined, in this paper, as

$$\varphi(t) = \omega_n' t\,. \tag{10}$$

Theoretically, the parameter ω_n' in Eq. (10) is the function of the amplitude of θ, in which ω_n' is nearly equal to ω_n when the amplitude of θ is small, and ω_n' become smaller as the amplitude of θ becomes larger. In this case, the parameter ω_n' is chosen as

$$\omega_n' = \frac{\omega_n}{10\left|\theta - \pi\right|^2 + 1}\,. \tag{11}$$

The parameter a in Eq. (9) determines the amplitude and sign of r_d and is given by

$$a = \begin{cases} a_0 \operatorname{sgn}\left(V - V_d\right) & \text{if } \left|V - V_d\right| \ge b_0 \\ a_0\left(V - V_d\right)/b_0 & \text{if } \left|V - V_d\right| < b_0 \end{cases} \tag{12}$$

where $a_0 > 0$ and $b_0 > 0$ are the design parameters and V_d is the mechanical energy at the upright position ($V_d = 0$) the parameter a_0 is chosen smaller than the desired cart's travel. According to Eq. (12), the amplitude of parameter a is decreased as V approaches V_d. The parameter b_0 is a number that determines the time when the amplitude of parameter a is decreased.

3.2 Stabilizing Controller by Coefficient Diagram Method

In order to stabilize the inverted pendulum at the upright position, the servo state feedback is employed and its design is based on CDM concept.

<u>CDM concept</u>

In CDM, the monic characteristic polynomial is given in the following form

$$\begin{aligned} P_m(s) &= \frac{\prod_{j=1}^{n-1} \gamma_{n-j}^{j}}{\tau^n} \left[\left\{ \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} \frac{1}{\gamma_{i-j}^{j}} \right) (\tau s)^i \right\} + \tau s + 1 \right] \\ &= s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \end{aligned} \tag{13}$$

where γ_i is the stability index and τ is the equivalent time constant.

The choice of stability index γ_i due to the control design specifications must satisfy the following inequality

$$\gamma_i > 1.5\gamma_i^* \tag{14}$$

where γ_i^* is the stability limit and is defined as

$$\gamma_i^* = (1/\gamma_{i-1}) + (1/\gamma_{i+1}), \tag{15}$$

where $i = 1 \sim n-1, \quad \gamma_1 = \gamma_n = \infty$.

In general, the stability index known as standard stability index is recommended as

$$\gamma_{n-1} = \ldots = \gamma_3 = \gamma_2 = 2 \quad \text{and} \quad \gamma_1 = 2.5 . \tag{16}$$

The equivalent time constant τ normally can be chosen from settling time specification as

$$\tau = \frac{t_s}{(2.5 \sim 3)} . \tag{17}$$

Servo state feedback design

Linearizing Eqs. (2)~(3) around the equilibrium point $\left[x, \theta, \dot{x}, \dot{\theta}\right] \approx [0,0,0,0]$ and substituting the parameter values as in Table 1, the linearized state-space model of the inverted pendulum on cart can be obtained as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{18}$$

where $\mathbf{x}(t) = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^T$ is the state variable and

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.880 & -3.333 & 0 \\ 0 & 46.758 & -13.333 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1.558 \\ 6.231 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T$$

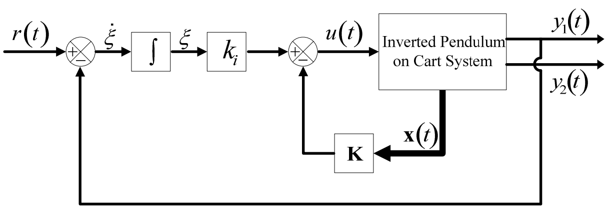

Fig. 3 Servo state feedback system.

In order to design the servo state feedback system, the servo state feedback gain $\mathbf{K}_s = \left[\mathbf{K} \ \vdots \ k_i\right]$ will be described first. Since only the cart's position x is used in servo scheme, the output matrix $\mathbf{C}$ of Eq. (18) is then reduced to $\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. Let $\xi(t)$ be the augmented state variable for the cart's position output, the servo system is then expressed as

$$\dot{\mathbf{x}}_s(t) = \mathbf{A}_s \mathbf{x}_s(t) + \mathbf{B}_s u(t) + \mathbf{F}_s r(t) \tag{19}$$

$$\bar{y}(t) = \mathbf{H}_s \mathbf{x}_s(t) \tag{20}$$

where $r(t)$ is the cart's position reference signal, $\bar{y}(t)$ is the controlled output cart's position $y_1(t)$ and

$$\mathbf{A}_s = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{H} & 0 \end{bmatrix}, \ \mathbf{B}_s = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \ \mathbf{H}_s = \begin{bmatrix} \mathbf{H} & 0 \end{bmatrix},$$

$$\mathbf{F}_s = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \ \mathbf{x}_s(t) = \begin{bmatrix} \mathbf{x}(t) \\ \xi(t) \end{bmatrix}.$$

If the pair $\mathbf{A}_s$ and $\mathbf{B}_s$ is completely controllable, the servo system in Eq. (19) can be transformed into the controllable form by the appropriate linear transformation $\mathbf{x}_s(t) = \mathbf{T}\mathbf{z}(t)$ as [5]

$$\dot{\mathbf{z}}(t) = \mathbf{A}_c \mathbf{z}(t) + \mathbf{B}_c u(t) + \mathbf{F}_c r(t) \tag{21}$$

where $\mathbf{A}_c = \mathbf{T}^{-1}\mathbf{A}_s\mathbf{T}, \quad \mathbf{B}_c = \mathbf{T}^{-1}\mathbf{B}_s, \quad \mathbf{F}_c = \mathbf{T}^{-1}\mathbf{F}_s$.

The state feedback control law $u(t)$ is assigned as

$$u(t) = -\mathbf{K}_c \mathbf{z}(t) = -\mathbf{K}_s \mathbf{x}_s(t) \tag{22}$$

when $\mathbf{K}_c = \left[\delta_1 \ \delta_2 \cdots \delta_n\right]$ is the servo state feedback gain for the system in Eq. (21) and $\mathbf{K}_s = \mathbf{K}_c\mathbf{T}^{-1}$ is the servo state feedback gain for the original system in Eq. (19).

The corresponding characteristic polynomial of the closed-loop system will become

$$\begin{aligned} P_c(s) &= \left|s\mathbf{I} - \left(\mathbf{A}_c - \mathbf{B}_c\mathbf{K}_c\right)\right| = \left|s\mathbf{I} - \left(\mathbf{A}_s - \mathbf{B}_s\mathbf{K}_s\right)\right| \\ &= s^n + \left(\sigma_{n-1} + \delta_n\right)s^{n-1} + \cdots + \left(\sigma_1 + \delta_2\right)s + \left(\sigma_0 + \delta_1\right) \end{aligned} \tag{23}$$

where σ_i, $i = 1 \sim n$ are the coefficients of the open-loop characteristic polynomial of the system in Eq. (19).

The design procedure for assigning the servo state feedback gain matrix $\mathbf{K}_s$ by CDM can be summarized as follows

1. Find the open-loop characteristic polynomial of system in Eq. (19) as

$$P_{ol}(s) = \left|s\mathbf{I} - \mathbf{A}_s\right| = s^n + \sigma_{n-1}s^{n-1} + \cdots + \sigma_1 s + \sigma_0 \tag{24}$$

and find the transformation matrix $\mathbf{T}$.

2. Choose the equivalent time constant τ and the stability index γ_i and derive the desired monic CDM characteristic polynomial as in Eq. (13).

3. Calculate the servo state feedback gain by

$$\mathbf{K}_s = \left[\left(\alpha_0 - \sigma_0\right) \ \left(\alpha_1 - \sigma_1\right) \ \cdots \ \left(\alpha_{n-1} - \sigma_{n-1}\right)\right]\mathbf{T}^{-1} \tag{25}$$

3.3 Switching Condition

Since the pair $\left(\theta, \dot{\theta}\right)$ that makes $V = V_d$ is not unique, the upright equilibrium point cannot be stabilized with the energy control. Thus, the switching condition switch the control law from swing-up control to stabilizing control when $(\theta, \dot{\theta})$ approaches the equilibrium point. Use the following condition as a criterion for switch:

$$|V - V_d| < \varepsilon_1 \text{ and } 1 + \cos\theta < \varepsilon_2 \tag{26}$$

where ε_1 and ε_2 are small, positive constants, which are determined by observation from simulation so that the switch is done smoothly.

4. SIMULATION RESULTS

The objective task of this paper is to swing up pendulum from its pendent position to the upright position. The controller is firstly designed based on the methods described in section 3. For swing-up control, the design parameters for swing-up controller are selected as ζ = 1.2, c_0 = 0.9 while the parameter a_0 = 0.14, 0.16, 0.18, and 0.21 will be employed to observe its effect on the cart's travel length and the parameter b_0 will be twice for each value of a_0. For stabilizing control, the servo state feedback gain $\mathbf{K}_s$ is designed based on CDM when choosing $\tau = 1$ second, γ_1 = 2.5 and γ_2 = γ_3 = γ_4 = γ_5 = 2. This yields $\mathbf{K}_s = [\mathbf{K} \vdots k_i] = [-40.90\ 49.83\ -19.54\ 7.56 \vdots 40.90]$.

The swing-up and stabilizing responses are obtained by simulation using MATLAB/Simulink. The responses of the cart position and responses of pendulum angle are shown by Fig. 4 and Fig. 5 respectively with varying values of parameter a_0. While the values of a_0 = 0.14, 0.16, 0.18, and 0.21, the switching occurred at 3.2, 2.6, 2 and 2 seconds respectively. As seen from the simulation results, the swing-up controller is able to control the cart to travel within an assigned cart's travel length. However, in a smaller cart's travel length, the control takes more time than in a bigger cart's travel length before the pendulum approach the upright position.

5. CONCLUSIONS

The controller designed by energy control method and CDM for swinging up and stabilizing the inverted pendulum on cart has been proposed. The simulation results show that the controller is able to swing up the pendulum from its pendent position to the upright position within the limitation of the cart travels and stabilize the pendulum in a short period of time. The proposed controller is effective and yields the desired system performance despite its simplicity in design.

REFERENCES

[1] S. Nundrakwang, T. Benjanarasuth, J. Ngamwiwit and N. Komine, "Hybrid PD–Servo State Feedback Control Algorithm for Swing up Inverted Pendulum System", *Proc. of ICCAS2005*, 2005.

[2] K.J. Astrom and K. Furuta, "Swinging up a Pendulum by Energy Control", *Proc. of 13th IFAC World congress*, 1996.

[3] K. Yoshida, "Swing-up Control of an Inverted Pendulum by Energy-Based Methods", *Proc. of American Control Conference*, 1999.

[4] S. Manabe, "Coefficient Diagram Method", *Proc. of 14th IFAC Symposium in Aerospace*, 1998.

[5] K. Ogata, *Modern Control Engineering*, Second Edition, Prentice-Hall International, Inc., 1990.

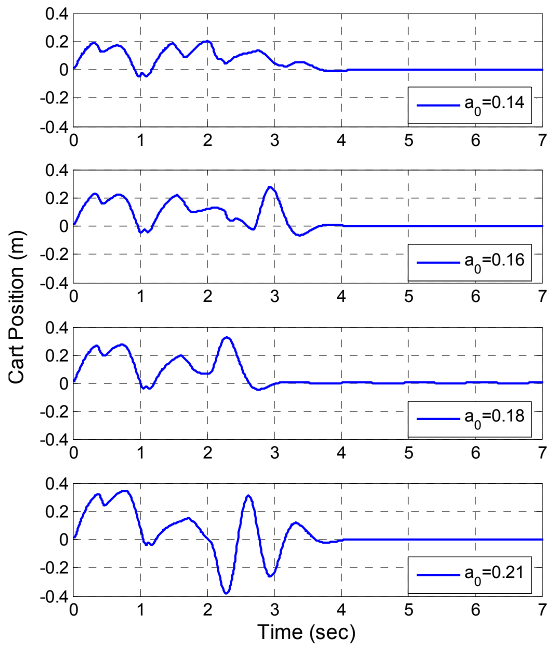

Fig. 4 Responses of cart position (m) when varying the values of parameter a_0.

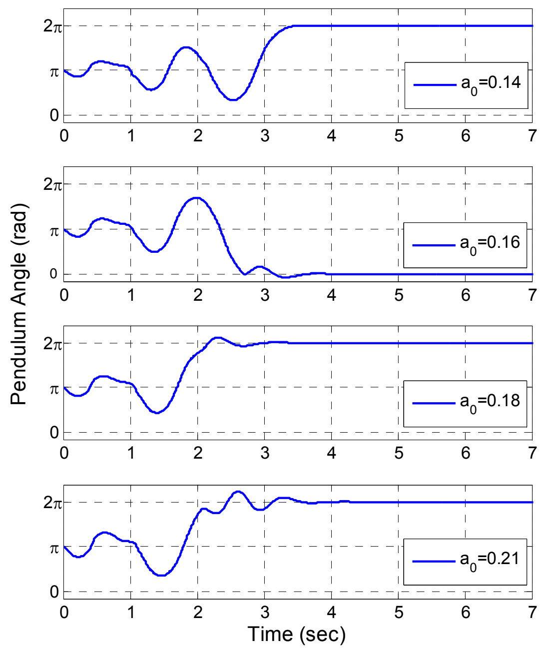

Fig. 5 Responses of pendulum angle (rad) when varying the values of parameter a_0.