State Feedback and LQR Controllers for an Inverted Pendulum System

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Abstract—In this paper, a control problem of a nonlinear pendulum is considered. The control problem contains two stages. First, swing up the pendulum by a nonlinear control to bring the pendulum from pendant to a limited region near of upright position. Secondly linear state feedback controller is used to stabilize pendulum at the upright position. Energy based control is used for swinging up task. The stabilizing pendulum at upright position task is achieved by two linear state feedback controllers namely pole placement and linear quadratic regulator LQR. The closed loop stability is achieved by Lyapunov method. The performance of the closed loop system is discussed.

Keywords—; Inverted Pendulum, Nonlinear Control, Linear Control.

I. INTRODUCTION

The inverted pendulum system contains nonlinearity and it has been used to illustrate many of the ideas emerging in the field of nonlinear and control systems [1-5]. The inverted pendulum system is depicted in Fig.1, which shows the rod, mounted on a motor-driven cart, can rotate freely around the fixed point *P*, the angle between the vertical axis and the rod is θ. The control force u compromises two parts and applied to the cart. In Fig. 1, l represents the half length of rod where its mass m assumed to be at its geometric center. The mass of the cart is represented by M. Here we consider only a two dimensional problem in which the pendulum moves only in the plane of the page. The problem task to be solved is swinging up the pendulum and balancing it at the upright position, $\theta=0^0$, with the control signal u, by moving the cart on the direction of displacement z.

The controller u contains two parts;

Controller	Switch condition

$$u = \begin{cases} u_n & if\ |\theta| < 20^0 \\ u_l & if\ |\theta| > 20^0 \end{cases} \qquad (1)$$

where u_n is nonlinear control which is designed to swing up the pendulum from the pendant position, $\theta = \pm 180^0$ and bring it to linear region $\theta \leq \pm 20^0$ which is near the upright position $\theta = 0^0$ by acceleration of the cart. Once the rod entered in the linear region $\theta \leq \pm 20^0$, the nonlinear controller is released by a switching process and then the linear part of the control u_l becomes active.

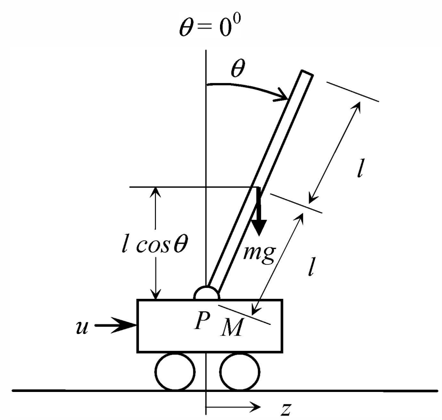

Fig. 1. Inverted pendulum system.

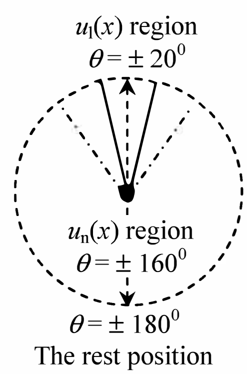

Fig. 2. The controller parts active region: Switching process.

The nonlinear control is an energy-based control witch proposed in[3], model adapted from computed torque control to bring the pendulum from pendant position, $\theta = \pm180^0$ to enter in the limited region, $|\theta| \leq \pm 20^0$ away from the upright position $\theta = 0^0$ as shown in Fig. 2, in this figure the region nearby to the upright position and limited to $|\theta| \leq 20^0$ will be called linear region where the linear controller u_l is active and other region is called nonlinear region that nonlinear controller u_n is active.

II. INVERTED PENDULUM SYSTEM AND SWING UP CONTROL

The motion equation for the system shown in Fig. 1, can be representing by the following equations:

$$ml\,\ddot{\theta}\cos\theta - ml\,\dot{\theta}^2\sin\theta + (M+m)\ddot{z} = u \qquad (2)$$

$$J\ddot{\theta} - mlg\sin\theta + ml\cos\theta\,\ddot{z} = 0 \qquad (3)$$

where $= (I + ml^2)$, I is pendulum moment of inertia about the center of gravity, and g is universal gravitation constant. Equation (2) and (3) represent cart dynamic and pendulum dynamics respectively. Rewriting (2) and (3) as:

$$\ddot{\theta} = f_\theta(\theta) + b_\theta(\theta)u \qquad (4)$$

$$\ddot{z} = f_z(\theta) + b_z(\theta)u \qquad (5)$$

where

$$f_\theta(\theta) = \frac{(M+m)mlg\sin\theta - m^2l^2\sin\theta\cos\theta\dot{\theta}^2}{J(M+m) - m^2l^2\cos^2\theta}$$

$$b_\theta(\theta) = -\frac{ml\cos\theta}{J(M+m) - m^2l^2\cos^2\theta}$$

$$f_z(\theta) = \frac{Jml\sin\theta\dot{\theta}^2 - m^2l^2g\sin\theta\cos\theta}{J(M+m) - m^2l^2\cos^2\theta}$$

$$b_z(\theta) = \frac{J}{J(M+m) - m^2l^2\cos^2\theta}$$

The control goal in this paper is to swing up the pendulum from the downward (i.e rest position) to the upright position at first and then stabilizing it.

Swing up

There are many control strategies have been introduced and used to swing-up of inverted pendulum [2-6]. Here energy-based control is considered to swing up the pendulum.

The pendulum subsystem's energy equation is given by

$$E = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos\theta - 1) \qquad (6)$$

The time derivative of E is

$$\dot{E} = (J\ddot{\theta} - mlg\sin\theta)\dot{\theta} \qquad (7)$$

Substituting (3) in (7) then

$$\dot{E} = -ml\dot{\theta}\ddot{z}\cos\theta \qquad (8)$$

The swing up controller u_n can be obtained by considering above energy function as a candidate Lyapunov function. By select the Lyapunov function as follows[3];

$$V(\theta) = \frac{1}{2}(E_0 - E)^2 \qquad (9)$$

where E_o is desired energy and E pendulum energy.

The swing up controller u_n is selected to make the derivative of Lyapunov function semi-negative definite. Taking the derivative of Lyapunov function (9) along the system trajectory yields to

$$\dot{V} = -(E_0 - E)\dot{E}$$

by substituting (8) in $\dot{V}$ lead to

$$\dot{V} = ml\dot{\theta}\cos\theta\,(E_0 - E)\ddot{z} \qquad (10)$$

substituting (5) in (10) lead to

$$\dot{V} = \sigma(f_z(\theta) + b_z(\theta)u_n) \qquad (11)$$

where $\sigma = ml\dot{\theta}\cos\theta\,(E_0 - E)$

Let the nonlinear controller u_n is designed as

$$u_n = -\eta sgn(\sigma) \quad \text{where } \eta > 0 \qquad (12)$$

The condition to make (11) become semi-negative definite is

$$\eta \geq \frac{|f_z(\theta)|}{|b_z(\theta)|}$$

Note that $cos\theta$ cannot continue to be zero at $\theta = \pm\pi/2$, because the system does not have an equilibrium point at the horizontal position, then the control law drives the energy to a desired value as a result, the pendulum swung up[1].

Once the swing up is achieved, the pendulum reaches to the linear region$|\theta| \leq \pm20^0$ then the nonlinear controller is replaced with the linear controller by a switch, to stabilize the pendulum at $\theta = 0^0$.

III. LINEARIZATION MODEL AND STATE FEEDBACK CONTROLLER

As a result from nonlinear controller (swing-up) that converged the pendulum to its unstable equilibrium point (upright position $\theta=0^0$), the inverted pendulum system can be considered as a linear system. Hence the linear controller u_l given in (1) is designed to stabilize the pendulum system.

Based on the assumption that the inverted pendulum rod is nearby to the vertical position (equilibrium point $(\theta,\dot{\theta})=(0^0,0)$), the value of θ is decreased to small quantity, based on this we assume that $sin\theta=\theta$, and $cos\theta = 1$. Since θ is a small quantity leads $\dot{\theta}$ is also be small quantities thus any nonlinearity in (4) and (5) that contains $\dot{\theta}$ can be neglected.

TABLA I. SYSTEM PARAMETERS

Table1 System Parameters		
Symbol	***Parameters name***	***Value***
M	Mass of Cart	2 kg
m	Mss of Pendulum	0.1 kg
l	Half of pendulum length	0.25 m
G	Universal Gravitation Constant	9.8 m/s

As a result the (4) and (5) can be linearized as:

$$\ddot{\theta} = \frac{3(M+m)g}{4Ml+ml}\theta - \frac{3}{4Ml+ml}u$$

$$\ddot{z} = -\frac{3mg}{4M+m}\theta + \frac{4}{4M+m}u$$

By defining new state variables $[x_1, x_2, x_3, x_4] = [\theta, \dot{\theta}, z, \dot{z}]$, then the linear model of the pendulum system can be represented by the following state-space model:

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{x_3}\\ \dot{x_4}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{3(M+m)g}{4Ml+ml} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -\frac{3mg}{4M+m} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ -\frac{3}{4Ml+ml}\\ 0\\ \frac{4}{4M+m}\end{bmatrix}u \qquad (13)$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

Once the pendulum is swung up and enters in the linear region then the linear control can be designed based on the assumption that the linearized system (13) needs to be stabilized by the linear controller. For the linear controller to stabilize the linearized model given in (13), there are two methods has been considered; one is pole placement and other is LQR. The results of both controllers will be discussed.

A. Design state feedback ul(x) based on pole-placement approach

Pole placement is a method used for state feedback control system to put the system's closed-loop poles in desired locations (i.e designing for specific behavior) in $s=\sigma \pm j\omega$ complex plain by designing the state feedback gain matrix K in linear control law $u = -Kx$. The system given in the form of state space model by (13) and let the system output is to be x_3. The following steps are requiring designing the controller. The system model given in (13) is completely state controllable so pole placement technique is possible[7]. Let define the desired closed loop poles as;

$\mu_i = [-1+j \quad -1-j \quad -4 \quad -4]$ where i=1,2,3 and 4. The desired closed loop poles μ_1 and μ_2 are a pair of dominant closed-loop poles, the other two poles μ_3 and μ_4 are located to their left and far from both. Since the system given in (13) has single input then Ackermann's formula namely known *acker* in MATLAB Control toolbox is used[8]. The feed-back gain matrix is obtained equal to:

$$K = [k_1\ k_2\ k_3\ k_4] = [-44.3 \quad -7.8 \quad -2.2 \quad -3.3]\ .$$

The simulation results depicted in Fig. 3, shows the system closed loop time responses where nonlinear controller's parameters are ($E_0 = 0$, $\eta = 2$) and linear controller is designed based on pole placement. Fig. 3(a), shows the rod angular position time response. The nonlinear controller brings pendulum from rest position, $\theta(0)=-180^o$, to nearby the upright position (i.e linear region). Thus the pendulum reaches the linear region approximately at t=8.9 *sec*, the nonlinear controller became inactivated by a switch. After that the linear controller based on the pole placement stabilizes the system and converge the states to zero. The time response of the cart position is depicted in Fig. 3(b), which starts to move during the nonlinear controller without limit and reach about 2.4 *m* away from the initial position. When the linear controller started at t=8.9 *sec,* then it converged the cart position to zero. Fig.3(c), illustrates the control signal which contains nonlinear and linear parts. Nonlinear control is active and has bang-bang property within $0 \le t \le 8.9$*sec*. However the linear control is active for $t \ge 8.9$ *sec*, because of the limit of the linear region the control signal starts with large gain.

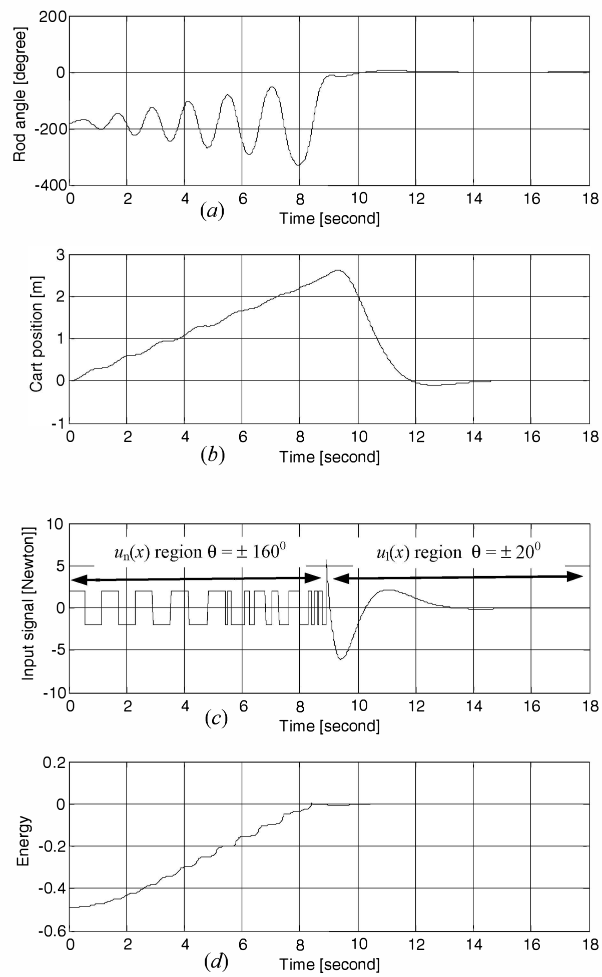

Fig. 3. System closed loop time response where nonlinear controller's parameters are($E_0 = 0$, $\eta = 2$) and linear controller is based on pole placement.

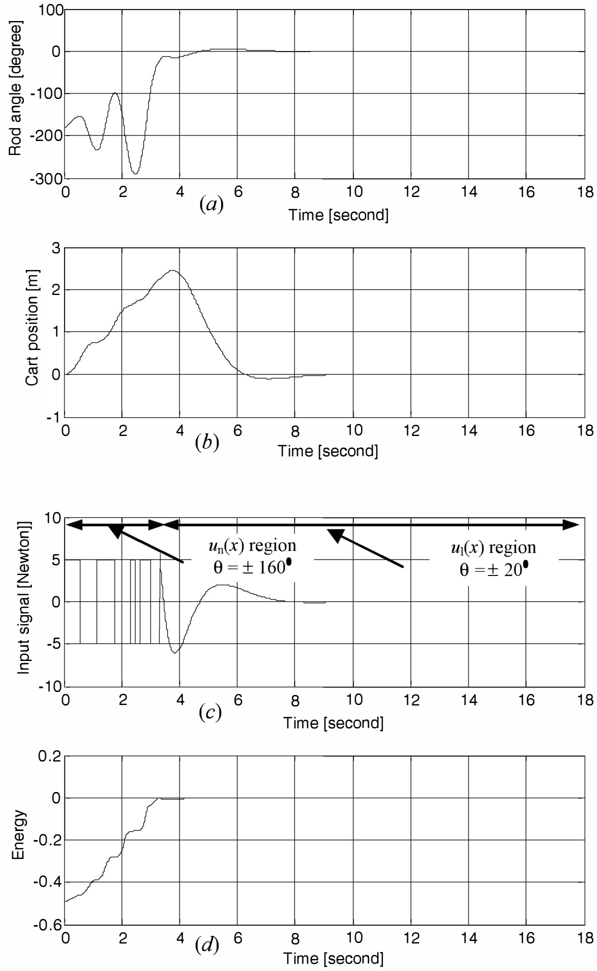

Fig. 4. System closed loop time response where nonlinear controller's parameters are(E_0= 0, η = 5) and linear controller is based on pole placement.

Fig. 3(d), show that the controller brings the energy from its initial value to the desired value (E_0= 0).By increasing the nonlinear parameter η to 5, the swing-up's time speeded up and the control signal during the nonlinear controller is increased as shown by Fig. (4).

B. *Design state feedback $u_l(x)$ based on Linear Quadratic Regulator (LQR)*

The Linear Quadratic Regulator (LQR) is an optimal control problem where the performance index J is given in (14) which is quadratic[9], and is determined for regulator case, x_d=0.

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt \tag{14}$$

where x^T and u^T are respectively represent the transpose of the state vector x and input vector u. Q and R are positive definite (or positive-semi definite) real symmetric matrixes. Q and R are usually chosen to be diagonal.

The desired closed loop poles location is determined by the weighting functions Q and R of (14)[9-10]. Here Q and R matrices respectively represent the cost functions placed upon reducing x and the cost placed upon saving energy by limiting control signal. Selection the weighting matrices Q and R in the cost function J is based on criterion given in [7-8]. The physics of the problem may suggest terms in the cost function. Another procedure to select weighting matrices is by trial and error that the designer first specifies which outputs are important to drive to zero.

Consider the linearized model of the inverted pendulum in (13). Since the system has one input then the control gain K is a vector the optimal control law obtained based on the minimizing the performance index of (14) is

$$u_l(x) = -Kx \tag{16}$$

To obtain the controller gain K the following reduced-matrix Riccati equation need to be solved.

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{15}$$

where P is a positive-definite (or real symmetric matrix). For the numerical results of (15) the *lqr* command developed in MATLAB Control toolbox is used[8]. From the solution of (15) the optimal controller gain matrix is

$$K = R^{-1}B^T P \tag{16}$$

Here two different set of the Q and R matrices will be select.

case1

Let

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R = 1$$

by using the *lqr* command in MATLAB Control tool box K=lqr(A,B,Q,R) leads to;

$K = [k_1\ k_2\ k_3\ k_4]$

=[-56.8 -10.3 -3.1 -4.9]

case2

Let

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R = 10$$

by using the *lqr* command in MATLAB Control tool box K=lqr(A,B,Q,R) leads to;

$K = [k_1\ k_2\ k_3\ k_4]$

=[-49.2 -8.9 -1.0 -2.4]

Fig. 5, shows the simulation results of the closed loop system where nonlinear controller's parameters are (E_0 = 0, η= 2) and linear controller is designed based on LQR ***case1***. From Fig. 5(a), shown that the rod is at rest position, $\theta(0)$=–180°, and is swung up by nonlinear controller and then stabilized by

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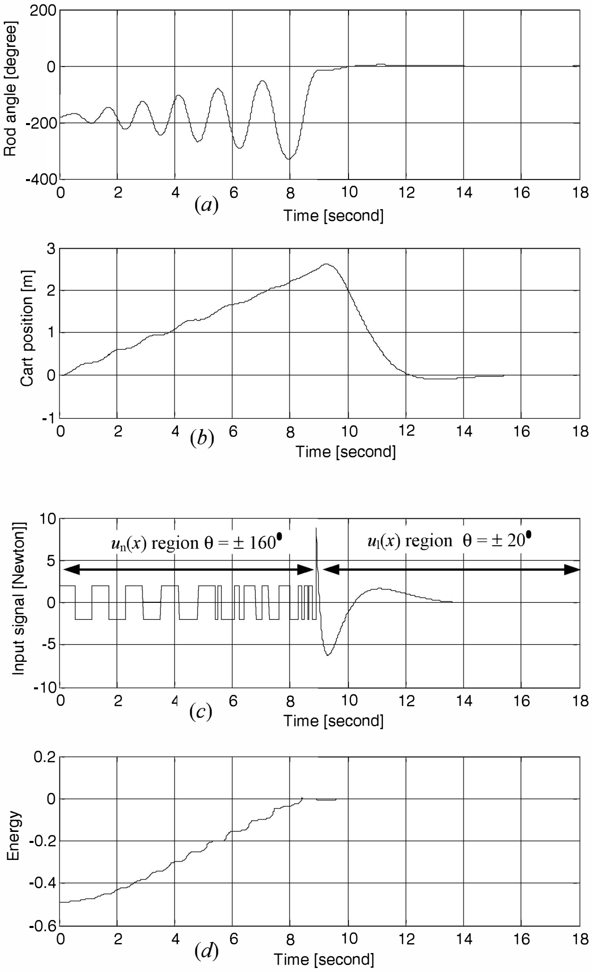

Fig. 5. System closed loop time response where nonlinear controller's parameters are(E_0 = 0, η = 2) and linear controller is based on LQR ***case1***.

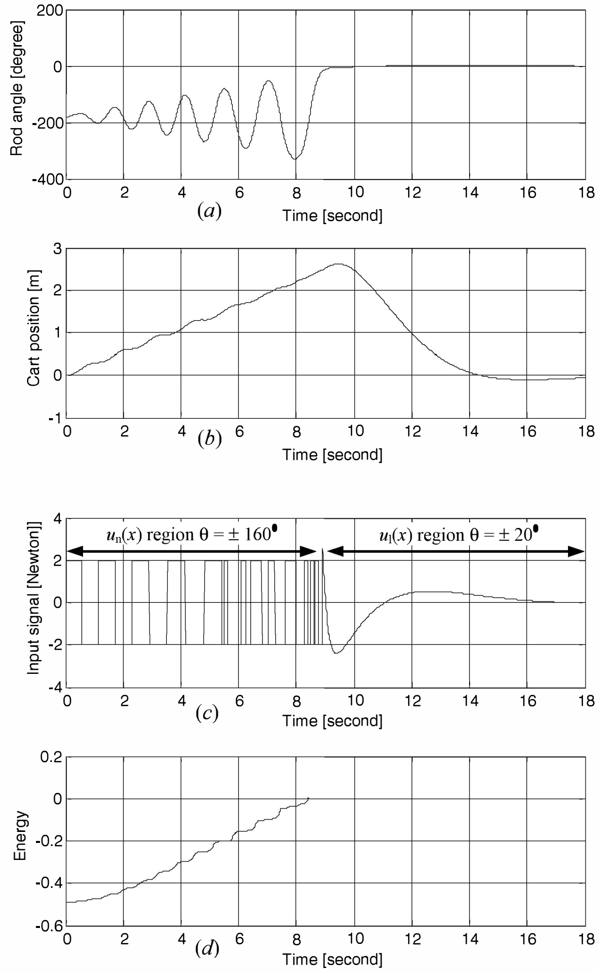

Fig. 6. System closed loop time response where nonlinear controller's parameters are(E_0 = 0, η = 2) and linear controller is based on LQR ***cas2***.

LQR technique, the nonlinear controller inactivated by a switch when the pendulum reach the linear region approximately at t=8.9 *sec*, after that the linear controller (LQR) converges the states to zero when time reaches 15 *sec* . Fig. 5(b), illustrate the cart position time response where it started move from its origin during the nonlinear controller without limit and reach about 2.4 *m*, then it converged to zero by linear controller. Input signal time response depicted in Fig. 5(c), shows that the control signal, during the linear controller based on LQR, is started with a big gain, because in ***case1*** the value of R is small. From Fig. 5(d), can be notes that the energy reaches its desired value during the nonlinear controller.

The simulation results for ***case2*** depicted in Fig. 6, that shows the closed loop system time response where nonlinear controller's parameters are (E_0 = 0, η= 2) and linear controller is designed based on LQR ***case1***. Fig. 6(a), shows that the rod is at rest position, $\theta(0)=-180^{o}$, and is swung up by nonlinear controller and then stabilized by LQR technique, the nonlinear controller inactivated by a switch when the pendulum reach the linear region approximately at t=8.9 *sec*, and then the linear controller (LQR) converges the states to zero when time reaches 17 *sec* with small oscillation. The cart position time response illustrated in Fig. 6 (b), where it started move from its origin during the nonlinear controller without limit and reach about 2.4 *m*, and then it converged to zero by linear controller. The control signal time response depicted in Fig 6(c), shows that the control signal, during the linear controller based on LQR ***case2***, is started with a smaller gain than it in ***case1.*** From Fig. 6(d), can be notes that the system energy is same as previous case.

IV. CONCLUSIONS

The problem of swing up and stabilizing an inverted pendulum has been studied in this paper. Swinging up controller was designed based on energy equation and the nonlinear controller stability was checked by Lyapunov function.

From the results obtained in the simulation, we concluded that:

a) By increasing the nonlinear parameter η, the control signal during the swinging up was increased and the swing up time was decreased.

b) Controller based on the pole placement technique tries to achieve the desired performance but it may produce a large control signal, that causes saturation which lead the system behave a nonlinear system. This is the natural result from pole placement that the desired closed loop selected without considering the control energy.

c) In case of *LQR* method when the value of *R* matrix is increased for the same value of *Q* matrix, the control signal during linear controller starts with smaller gain. Fixing *R* constant and increasing one element of the *Q* matrix, the time response speeds up.

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