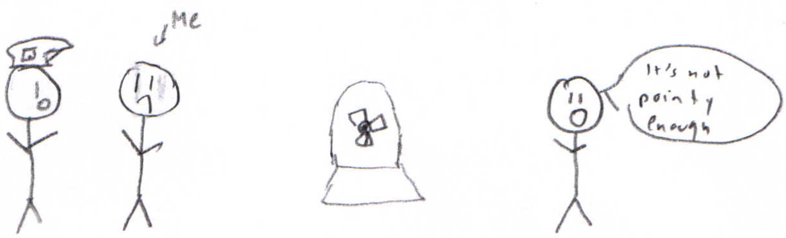
MATH 53 FINAL

FULL NAME: Mehul Nair

ID: 303 846 5432

For full credit, please show all your work and reasoning. Your work and explanations are your only representative when your work is being graded. Illegible, messy, and mysterious work could be perceived as an error that undermines the grader's ability to understand your work. If you add false statements to a correct argument, you will lose points. The exam has 10 questions, and each question is worth 10 points, for a total of 100 points. The exam score is capped at 100 points, and extra credit cannot push your exam score past 100 points.

(Extra Credit): Draw a picture of yourself after college graduation.



(Pls get the reference)

(0)

(1) Let S be the part of the cone with parametric equations $\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), u\rangle$, $0 \le u \le 2$, $0 \le v \le 2\pi$, oriented upwards.

(a) Find the equation of the tangent plane to S at the point $(\sqrt{3}, 1, 2)$.

(b) Calculate $\int\int_S (x^2+y^2)^2 dS$.

~~不会~~

$x = u\cos v$

$y = u\sin v$

$z = u$

a $z - z_0 = f_x(x-x_0) + f_y(y-y_0)$

$z - 2 = \frac{\sqrt{3}}{2}(x-\sqrt{3}) + \frac{1}{2}(y-1)$

$z^2 = x^2+y^2$

$z = \sqrt{x^2+y^2}$

$f_x = \frac{x}{\sqrt{x^2+y^2}}$ $f_y = \frac{y}{\sqrt{x^2+y^2}}$

$f_{x \,|\, x=\sqrt{3},\, y=1} = \frac{\sqrt{3}}{2}$ $f_{y \,|\, x=\sqrt{3},\, y=1} = \frac{1}{2}$

b. $r_u = \langle \cos v, \sin v, 1\rangle$

$r_v = \langle -u\sin v, u\cos v, \cancel{0}\rangle$

$\cancel{\hat{i} \quad \hat{j} \quad \hat{k} \quad \hat{i} \quad \hat{j}}$

$\cancel{\cos v \quad \sin v \quad -u\sin v \quad u\cos v}$

$J =$

$-u\sin v\,\hat{j} + u\cos^2 v\,\hat{k} - u\cos v\,\hat{i} + u\sin^2 v\,\hat{k}$

$\|r_u \times r_v\| = \|\langle -u\cos v, -u\sin v, u\rangle\| = \sqrt{2u^2}$

$\sqrt{u^2+u^2}$

$\int_0^{2\pi}\int_0^2 u^4 \,\|\sqrt{2u^2}\|\, du\, dv$

$\sqrt{2}\int_0^{2\pi}\int_0^2 u^5\, du\, dv = \sqrt{2}\,(2\pi)\left(\frac{u^6}{6}\right)_0^2$

$2\sqrt{2}\,\pi\left(\frac{2^6}{\cancel{6}}\right)$ $\frac{2^6\sqrt{2}\,\pi}{3}$

(1)

(2) True or false. Provide explanations.

(a) Any curve in $\mathbb{R}^3$ has two distinct orientations.

(b) Any surface in $\mathbb{R}^3$ has two distinct orientations.

(c) Suppose a smooth vector field $\mathbf{F}$ is parallel to the z-axis everywhere in $\mathbb{R}^3$. The circulation of $\mathbf{F}$ around any closed curve is 0.

(d) If $\mathbf{F}$ is a smooth conservative vector field on $\mathbb{R}^3$, then its flux through a smooth closed surface is zero.

2

a True, For every curve in R_3 C, it can be traversed forwards or backwards (i.e, a ~~fn~~ circle in R^3 can be traversed clockwise or counterclockwise)

b. False, let S be a mobius strip

~~ftbn~~

c. True,

$$\text{curl } F = 0 \qquad \oint F \cdot dr = \iint_S \text{curl } F \cdot dS = 0$$

d. True,

~~IF~~ ~~teth~~ F is conservative, so let $F = \text{curl}(f)$

$$\therefore \text{ flux} = \iint_S \text{curl } f \, dS = \iiint_E \text{div}(\text{curl } f)\, dV = 0$$

(2)

(3) Find and classify the critical points of $f(x.y) = x^4 + y^4 - 4xy + 1$.

$\nabla f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle$

$(1,1) \leftarrow$ minimum

$(-1,-1) \leftarrow$ minimum

$(0,0) \leftarrow$ saddle point

$D = (12x^2)(12y^2) - (-4)^2$

$D = 144x^2y^2 - 16$

(3)

(4) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 1, x + yz, xy - \sqrt{z} \rangle$ and C is parameterized by

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 8\cos(t)\sin(t) \rangle \ , \ 0 \leq t \leq 2\pi.$$

Hint: C lies on the surface $z = 2xy$.

$2xy^2 = 2\sin^2(t)\cos t)$

$r(t) = \langle 2\cos(t), 2\sin t, 4\sin 2t \rangle$

~~$r'(t) = \langle -2\sin(t), 2\cos t, 8\cos 2t \rangle$~~

$F(r(t)) = \langle 1, 2\cos t + 2\sin^2 t\cos t, 2\cos 2t - \sqrt{8\cos t \sin t}$

~~$F \cdot r'(t) = -2\sin t + 2\cos^2 t + 2\sin^2 t\cos^2 t + 16\cos^2 2t - 8\cos 2t\sqrt{4\sin 2t}$~~

$z = 2xy$

$\int_C F \cdot dr = \iint \operatorname{curl} F \cdot dS \quad r(t) = \langle x, y, 2xy \rangle$

$\iint$

$\langle x - y, 0 - y, 1 - 0 \rangle$

$\langle x - y, -y, 1 \rangle$

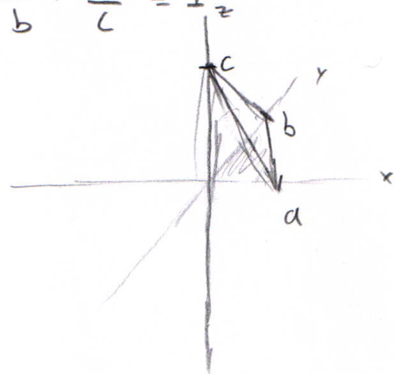
$r(t) = \langle 2\cos(t), 2\sin(t), 8\cos(t)\sin(t) \rangle$

$r'(t) = \langle -2\sin t, 2\cos t, 8\cos 2t \rangle$

(4)

(5) Find constants a, b, c such that the plane $(x/a) + (y/b) + (z/c) = 1$ that passes through the point $(2, 1, 2)$ and cuts off the least volume from the first octant. Hint: the volume of a triangular pyramid with base area B and height h is $V = \frac{1}{3}Bh$.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



$$V = \frac{1}{3} c \left(\frac{1}{2} a b\right)$$

$$\frac{2}{a} + \frac{1}{b} + \frac{2}{c} = 1$$

$$V = \frac{1}{6} abc$$

$$\frac{\cancel{2}^{1}}{\cancel{2}b} + \frac{1}{b} + \frac{\cancel{2}^{1}}{\cancel{2}b} = 1$$

$$F = \frac{1}{6} abc$$

$$\frac{2}{a} + \frac{1}{b} + \frac{2}{c} = G$$

$$\frac{3}{b} = 1$$

$$\boxed{\begin{aligned} b &= 3 \\ a &= 6 \\ c &= 6 \end{aligned}}$$

$$\nabla F = \left\langle \frac{1}{6} bc, \frac{1}{6} ac, \frac{1}{6} ab \right\rangle = \lambda \left\langle \frac{-2}{a^{-2}}, \frac{-1}{b^{-2}}, \frac{-2}{c^{-2}} \right\rangle$$

$$\frac{1}{6} a^2 bc = 2\lambda$$

$$\frac{1}{6} a^{\cancel{2}} \cancel{b} \cancel{c} = \frac{1}{6} \cancel{a} \cancel{b} c^{\cancel{2}}$$

$$a = c$$

$$\frac{1}{6} a b^2 c = -\lambda$$

$$\frac{1}{6} abc^2 = -2\lambda$$

$$-\lambda = \frac{1}{12} a^2 bc$$

$$2 \frac{1}{6} \cancel{a} b^{\cancel{2}} \cancel{c} = \frac{1}{12} a^{\cancel{2}} \cancel{b} \cancel{c}$$

$$a = 2b$$

$$c = 2b$$

$$V = \frac{1}{6} abc$$

$$V = \frac{1}{6} (\ )^3$$

(5)

$x^2+y^2+z^2+a^2=\rho^2$

(6) Define $B_n(R)$ to be the n-ball in $\mathbb{R}^n$ to be the set of points that are within distance R from the origin. In $\mathbb{R}^3$ we can find the volume of the 3-ball with radius R using spherical coordinates:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi_1 d\rho\, d\phi_1\, d\phi_2.$$

We can extend the spherical coordinate system to integrate in higher dimensions. The spherical coordinate system in $\mathbb{R}^n$ has a radial coordinate ρ and angular coordinates $\phi_1 \cdots \phi_{n-1}$, where the domain of each ϕ_i for $i = 1, 2, \ldots, n-2$ is $[0, \pi)$ and the domain of ϕ_{n-1} is $[0, 2\pi)$. The volume of the n-ball is can be found by integrating

$$\int \cdots \int_{B_n(R)} dV = \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{R} \rho^{n-1}\sin^{n-2}(\phi_1)\sin^{n-3}(\phi_2)\cdots\sin(\phi_{n-2})\, d\rho\, d\phi_1\, d\phi_2\, \cdots\, d\phi_{n-1}.$$

Find the volume of the 5-ball and 6-ball with radius R.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{R} \rho^4 \sin^3(\phi_1)\sin^2(\phi_2)\sin(\phi_3)\, d\rho\, d\phi_1\, d\phi_2\, d\phi_3\, d\phi_4$$

πr^2 $\quad \frac{4}{3}\pi r^3$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{R} \rho^3 \sin^2(\phi_1)\sin(\phi_2)\, d\rho\, d\phi_1\, d\phi_2\, d\phi_3$$

$$\frac{1-2\cos 2\theta}{2}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \frac{R^4}{4}\sin^2(\phi_1)\sin(\phi_2)\, d\phi_1\, d\phi_2\, d\phi_3$$

$\pi r^2 \quad \frac{4}{3}\pi r^3$

$2\pi r$

$\pi r^2 \quad \pi r^2$

$$\frac{R^4}{4}\left[\frac{\phi_1 - \sin 2\phi_1}{4}\right]_0^{\pi}\left[-\cos(\phi_2)\right]_0^{\pi}\left[2\pi\right.$$

$\frac{1}{3}\pi r^3 \cdot 4$

$$\frac{R^4}{4}\left[\frac{\pi-1}{4}\right]\left[\right.$$

$\frac{1}{2}\pi \cdot \frac{4}{3}\pi r^3$

$\frac{R^5}{5}\pi$

$\frac{2}{3}\pi^2 r^4$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi_1\, d\rho\, d\phi_1\, d\phi_2$$

$\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi \quad \frac{1}{2}\pi$

$$\left[\frac{\rho^3}{3}\right] = \frac{R^3}{3} \qquad \phi = 2\pi$$

$$\frac{2\pi}{3}R^3$$

(6)

(7) Evaluate

$$\int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} \left(\frac{2x-y}{2} + \frac{z}{3}\right)\, dx\, dy\, dz$$

by applying the transfomation $u = (2x-y)/2,\ v = y/2,\ w = z/3$.

$y = 2v$

$u = \dfrac{2x - y}{2}$

$2u + 2v = 2x$

$x = u + v$

$z = 3w$

$$6\int_0^1\int_0^2\int_v^{1+v} u + w \, du\, dv\, dw$$

$\dfrac{y}{2}$

$$J = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$6\int_0^1\int_0^2 \left[\frac{u^2}{2} + uw\right]_v^{1+v} dv\, dw$$

$$\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \end{array}$$

$$6\int_0^1\int_0^2 \frac{(1+v)^2}{2} + w(1+v) - \frac{v^2}{2} - vw \, dv\, dw$$

$J = 6$

$$6\int_0^1 \left[\frac{(1+v)^3}{6} + \frac{(1+v)^2}{2} w - \frac{v^3}{6} - \frac{v^2}{2} w\right]_0^2 dw$$

$$6\int_0^1 \left[\frac{3^3}{6} + \frac{3^2}{2} w - \frac{4}{3} - 2w - \frac{1}{6} - \frac{1}{2}\right] dw$$

$$\frac{9}{2} w - \frac{4}{2} w + \frac{27}{6} - \frac{8}{6} - \frac{1}{6} - \frac{3}{6}$$

$$6\int_0^1 \frac{5}{2} w + \frac{15}{6} \, dw$$

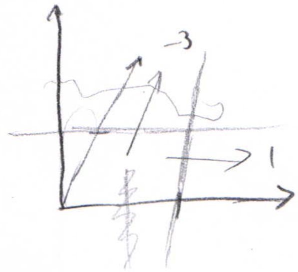
$$6\left[\frac{5}{4} w^2 + \frac{15}{6} w\right]_0^1$$

$$6\left[\frac{5}{4} + \frac{15}{6}\right]$$

(7)

(8) Suppose the function $f(x, y)$ is nonnegative on all of $\mathbb{R}^2$. Consider the closed surface bounded below by $z = 0$, above by the graph of $f(x, y)$, and on four sides by $x = 0,\ x = 1,\ y = 0,\ y = 1$. Let $\mathbf{F} = \langle x, -2y, z + 3 \rangle$. Suppose the outward flux through the face bounded by $x = 1$ is 1 and through the face bounded by $y = 1$ is -3. Can you conclude anything about the flux through the top? If yes, what is the flux through the top?

$\langle x, -2y, z+3 \rangle$



$f(1, y) = 0$

(8)

(9) Let $\mathbf{F} = \langle yz, xz + \frac{1}{y}, xy + 3z^2 \rangle$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the straight line segment from $(2, 1, 3)$ to $(1, 1, 5)$.

$$\int_0^1 F(r(t)) \cdot r'(t)\, dt$$

$$\langle 1, 0, -2 \rangle$$

$$r(t) = (1-t)\langle 2,1,3 \rangle + t(1,1,5)$$

$$\langle 2-2t, 1-t, 3-3t \rangle + (t, t, 5t)$$

$$r(t) = \langle 2-t, 1, 3+2t \rangle$$

$$r'(t) = \langle -1, 0, 2 \rangle$$

$$F(r(t)) = \langle 3+2t, \;\sim\sim,\; 2-t + 3(3+2t)^2 \rangle$$

$$F \cdot r'(t) = -3-2t + 4-2t + 6(3+2t)^2$$

$$1 - 4t + 6(3+2t)^2$$

$$\int_0^1 1-4t+6(3+2t)^2\, dt$$

$u = 3+2t$

$\frac{du}{2} = dt$

$$\int_0^1 1-4t\, dt + \frac{6}{2}\int_3^5 u^2\, du$$

$$\left[t - 2t^2\right]_0^1 + 3\left[\frac{u^3}{3}\right]_3^5$$

$$\begin{array}{r} 125 \\ -27 \\ \hline 98 \end{array}$$

$$-1 + 98 = \boxed{97}$$

(9)

(10) (a) If we think of a two-dimensional field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ as a three-dimensional field whose $\mathbf{k}$-component is zero, explain why the form of Green's theorem that measures flux is a special case of the Divergence Theorem.

(b) Recall the Fundamental Theorem of Calculus: if $f(x)$ is differentiable on (a,b) and continuous on $[a,b]$, then

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a).$$

If we think of a one-dimensional field $\mathbf{F} = f(x)\mathbf{i}$ as a three-dimensional field whose $\mathbf{j}$ and $\mathbf{k}$-components are zero, explain why the Fundamental Theorem of Calculus is a special case of the Divergence Theorem. Hint: what is the normal vector for the interval $[a,b]$?

$\operatorname{div}(\vec{F}) = M_x + N_y + 0$ $\quad \langle 0,0,1\rangle$

a. $\iint_S F \cdot dS = \iiint_E M_x + N_y \, dx\,dy\,dz$ $\qquad \oint_C F\cdot n \, ds = \iint_R \operatorname{div} F \, dA$

$\operatorname{curl} F = N_x - M_y$

b.

$F = \langle f(x), 0, 0\rangle$ $\qquad \operatorname{div} F = \langle \frac{\partial f}{\partial x}, 0, 0\rangle$

$\iiint_E \int_a^b \frac{\partial f}{\partial x} \, dx\,dy\,dz$

b

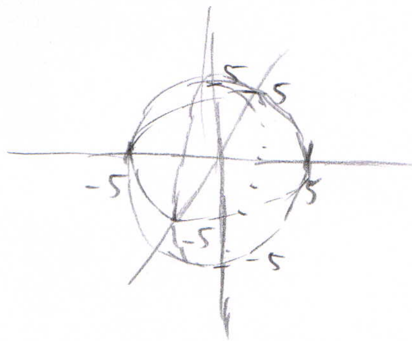
(10)

(11) (Extra credit): Consider the surface $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 25,\ x \leq 4,\ y \leq 4,\ z \leq 4\}$, oriented inwards toward the origin.

(a) Sktech a picture of the surface and find a parameterization for each boundary component of S.

(b) If $\mathbf{F} = \langle 1, 1, 1\rangle$, find the flux of $\mathbf{F}$ through the S, oriented inward toward the origin.

11



$(x, y, z) \to (\theta, \phi) \Rightarrow \langle 5\sin\phi\cos\theta, 5\sin\phi\sin\theta, 5\cos\phi\rangle$

b

$r_\theta = \langle -5\sin\phi\sin\theta, 5\sin\phi\cos\theta, 0\rangle$

$r_\phi = \langle 5\cos\phi\cos\theta, 5\cos\phi\sin\theta, -5\sin\phi\rangle$

$\cancel{\hat{i}\quad \hat{j}\quad \hat{k}\quad \hat{i}\quad \hat{j}}$

$\cancel{-5\sin\phi\sin\theta\quad 5\sin\phi\cos\theta\quad 0\quad -5\sin\phi\sin\theta\quad 5\sin\phi\cos\theta}$

$\cancel{5\cos\phi\cos\theta\quad 5\cos\phi\sin\theta\quad -5\sin\phi\quad 5\cos\phi\cos\theta\quad 5\cos\phi\sin\theta}$

$5^2\sin^2\phi\sin\theta\,\hat{i} - 5^2\cos\phi\sin\phi\sin^2\theta\,\hat{k} - 5^2\sin^2\phi\sin\theta\,\hat{j}$

$- 5^2\sin\phi\cos\phi\cos^2\,\hat{k}$

$r_{\theta\times\phi} = \langle 5^2\sin^2\phi\sin\theta, -5^2\sin^2\phi\sin\theta, -5^2\sin 2\phi\rangle$

$F\cdot r_{\theta\times\phi} = -5^2\sin 2\phi$

$-25\int_0^{\pi}\int_0^{2\pi}\sin 2\phi\, d\theta\, d\phi$

$-50\pi\int_0^{\pi}\sin 2\phi\, d\phi$

$-50\pi\left[-\frac{\cos 2\phi}{2}\right]_0^{\pi}$

$-50\pi\left[-\frac{1}{2}-\frac{1}{2}\right]$

50π

$\boxed{-50\pi}$