CHAPTER 4 Interest Rates

Interest rates are a factor in the valuation of virtually all derivatives and will feature prominently in much of the material that will be presented in the rest of this book. In this chapter we cover some fundamental issues concerned with the way interest rates are measured and analyzed. We explain the compounding frequency used to define an interest rate and the meaning of continuously compounded interest rates, which are used extensively in the analysis of derivatives. We cover zero rates, par yields, and yield curves, discuss bond pricing, and outline a procedure commonly used by a derivatives trading desk to calculate zero-coupon Treasury interest rates. We cover forward rates and forward rate agreements and review different theories of the term structure of interest rates. Finally we explain the use of duration and convexity measures to determine the sensitivity of bond prices to interest rate changes.

Chapter 6 will cover interest rate futures and show how the duration measure can be used when interest rate exposures are hedged. For ease of exposition we will ignore day count conventions throughout this chapter. The nature of these conventions and their impact on calculations will be discussed in Chapters 6 and 7.

4.1 TYPES OF RATES

An interest rate in a particular situation defines the amount of money a borrower promises to pay the lender. For any given currency, many different types of interest rates are regularly quoted. These include mortgage rates, deposit rates, prime borrowing rates, and so on. The interest rate applicable in a situation depends on the credit risk. This is the risk that there will be a default by the borrower of funds, so that the interest and principal are not paid to the lender as promised. The higher the credit risk, the higher the interest rate that is promised by the borrower.

Treasury Rates

Treasury rates are the rates an investor earns on Treasury bills and Treasury bonds. These are the instruments used by a government to borrow in its own currency. Japanese Treasury rates are the rates at which the Japanese government borrows in yen; US Treasury rates are the rates at which the US government borrows in US dollars; and so on. It is usually assumed that there is no chance that a government will default on an obligation denominated in its own currency.[1] Treasury rates are therefore totally risk-free rates in the sense that an investor who buys a Treasury bill or Treasury bond is certain that interest and principal payments will be made as promised.

Treasury rates are important because they are used to price Treasury bonds and are sometimes used to define the payoff from a derivative. However, derivatives traders (particularly those active in the over-the-counter market) do not usually use Treasury rates as risk-free rates. Instead they use LIBOR rates.

LIBOR

LIBOR is short for *London Interbank Offer Rate*. A LIBOR quote by a particular bank is the rate of interest at which the bank is prepared to make a large wholesale deposit with other banks. Large banks and other financial institutions quote 1-month, 3-month, 6-month, and 12-month LIBOR in all major currencies. Here 1-month LIBOR is the rate at which 1-month deposits are offered, 3-month LIBOR is the rate at which 3-month deposits are offered, and so on.

A deposit with a bank can be regarded as a loan to that bank. A bank must therefore satisfy certain creditworthiness criteria in order to be able to accept a LIBOR quote from another bank and receive deposits from that bank at LIBOR. Typically it must have to have a AA credit rating.[2]

AA-rated financial institutions regard LIBOR as their short-term opportunity cost of capital. They can borrow short-term funds at the LIBOR quotes of other financial institutions. Their own LIBOR quotes determine the rate at which surplus funds are lent to other financial institutions. LIBOR rates are not totally free of credit risk. There is a small chance that a AA-rated financial institution will default on a LIBOR loan. However, they are close to risk-free. Derivatives traders regard LIBOR rates as a better indication of the "true" risk-free rate than Treasury rates, because a number of tax and regulatory issues cause Treasury rates to be artifically low (see Business Snapshot 4.1). To be consistent with the normal practice in derivatives markets, the term "risk-free rate" in this book should be interpreted as the LIBOR rate.[3]

In addition to quoting LIBOR rates, large banks also quote LIBID rates. This is the *London Interbank Bid Rate* and is the rate at which they will accept deposits from other banks. At any specified time, there is usually a small spread between the quoted LIBID and LIBOR rates (with LIBOR higher than LIBID). The rates themselves are determined by active trading between banks and are continually changing so that the supply of funds in the interbank market equals the demand for funds in that market. For example, if more banks want to borrow US dollars for 3 months than lend US dollars for 3 months, the 3-month US LIBID and LIBOR rates quoted by banks will increase. Similarly, if more banks want to lend 3-month funds than borrow these funds, the 3-month LIBID and LIBOR rates will decrease. LIBOR and LIBID trade in what is known as the *Eurocurrency market*. This market is outside the control of any one government.

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[1] The reason for this is that the government can always meet its obligation by printing more money.

[2] The best credit rating given to a company by the rating agency S&P is AAA. The second best is AA. The corresponding ratings from the rival rating agency Moody's are Aaa and Aa, respectively.

[3] As we shall see in Chapter 7, it is more accurate to say that the risk-free rate should be interpreted as the rate derived from LIBOR, swap, and Eurodollar futures quotes.

Business Snapshot 4.1 What Is the Risk-Free Rate?

It is natural to assume that the rates on Treasury bills and Treasury bonds are the correct benchmark risk-free rates for derivative traders working for financial institutions. In fact, these derivative traders usually use LIBOR rates as short-term risk-free rates. This is because they regard LIBOR as their opportunity cost of capital (see Section 4.1). Traders argue that Treasury rates are too low to be used as risk-free rates because:

1. Treasury bills and Treasury bonds must be purchased by financial institutions to fulfill a variety of regulatory requirements. This increases demand for these Treasury instruments driving the price up and the yield down.
2. The amount of capital a bank is required to hold to support an investment in Treasury bills and bonds is substantially smaller than the capital required to support a similar investment in other instruments with very low risk.
3. In the United States, Treasury instruments are given a favorable tax treatment compared with most other fixed-income investments because they are not taxed at the state level.

LIBOR is approximately equal to the short-term borrowing rate of a AA-rated company. It is therefore not a perfect proxy for the risk-free rate. There is a small chance that a AA borrower will default within the life of a LIBOR loan. Nevertheless, traders feel it is the best proxy for them to use. LIBOR rates are quoted out to 12 months. As we shall see in Chapter 7, the Eurodollar futures market and the swap market are used to extend the trader's proxy for the risk-free rate beyond 12 months.

Repo Rates

Sometimes trading activities are funded with a *repo* or *repurchase agreement*. This is a contract where an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price. The other company is providing a loan to the investment dealer. The difference between the price at which the securities are sold and the price at which they are repurchased is the interest it earns. The interest rate is referred to as the *repo rate*. If structured carefully, the loan involves very little credit risk. If the borrower does not honor the agreement, the lending company simply keeps the securities. If the lending company does not keep to its side of the agreement, the original owner of the securities keeps the cash.

The most common type of repo is an *overnight repo*, in which the agreement is renegotiated each day. However, longer-term arrangements, known as *term repos*, are sometimes used.

4.2 MEASURING INTEREST RATES

A statement by a bank that the interest rate on 1-year deposits is 10% per annum sounds straightforward and unambiguous. In fact, its precise meaning depends on the way the interest rate is measured.

Table 4.1 Effect of the compounding frequency on the value of \$100 at the end of 1 year when the interest rate is 10% per annum.

Compounding frequency	*Value of \$100 at end of year (\$)*
Annually ($m = 1$)	110.00
Semiannually ($m = 2$)	110.25
Quarterly ($m = 4$)	110.38
Monthly ($m = 12$)	110.47
Weekly ($m = 52$)	110.51
Daily ($m = 365$)	110.52

If the interest rate is measured with annual compounding, the bank's statement that the interest rate is 10% means that \$100 grows to

$$\$100 \times 1.1 = \$110$$

at the end of 1 year. When the interest rate is measured with semiannual compounding, it means that we earn 5% every 6 months, with the interest being reinvested. In this case \$100 grows to

$$\$100 \times 1.05 \times 1.05 = \$110.25$$

at the end of 1 year. When the interest rate is measured with quarterly compounding, the bank's statement means that we earn 2.5% every 3 months, with the interest being reinvested. The \$100 then grows to

$$\$100 \times 1.025^4 = \$110.38$$

at the end of 1 year. Table 4.1 shows the effect of increasing the compounding frequency further.

The compounding frequency defines the units in which an interest rate is measured. A rate expressed with one compounding frequency can be converted into an equivalent rate with a different compounding frequency. For example, from Table 4.1 we see that 10.25% with annual compounding is equivalent to 10% with semiannual compounding. We can think of the difference between one compounding frequency and another to be analogous to the difference between kilometers and miles. They are two different units of measurement.

To generalize our results, suppose that an amount A is invested for n years at an interest rate of R per annum. If the rate is compounded once per annum, the terminal value of the investment is

$$A(1+R)^n$$

If the rate is compounded m times per annum, the terminal value of the investment is

$$A\left(1+\frac{R}{m}\right)^{mn} \tag{4.1}$$

When $m = 1$, the rate is sometimes referred to as the *equivalent annual interest rate*.

Continuous Compounding

The limit as the compounding frequency, m, tends to infinity is known as *continuous compounding*.[4] With continuous compounding, it can be shown that an amount A invested for n years at rate R grows to

$$Ae^{Rn} \tag{4.2}$$

where $e = 2.71828$. The exponential function, e^x, is built into most calculators, so the computation of the expression in equation (4.2) presents no problems. In the example in Table 4.1, $A = 100$, $n = 1$, and $R = 0.1$, so that the value to which A grows with continuous compounding is

$$100e^{0.1} = \$110.52$$

This is (to two decimal places) the same as the value with daily compounding. For most practical purposes, continuous compounding can be thought of as being equivalent to daily compounding. Compounding a sum of money at a continuously compounded rate R for n years involves multiplying it by e^{Rn}. Discounting it at a continuously compounded rate R for n years involves multiplying by e^{-Rn}.

In this book, interest rates will be measured with continuous compounding except where stated otherwise. Readers used to working with interest rates that are measured with annual, semiannual, or some other compounding frequency may find this a little strange at first. However, continuously compounded interest rates are used to such a great extent in pricing derivatives that it makes sense to get used to working with them now.

Suppose that R_c is a rate of interest with continuous compounding and R_m is the equivalent rate with compounding m times per annum. From the results in equations (4.1) and (4.2), we have

$$Ae^{R_c n} = A\left(1 + \frac{R_m}{m}\right)^{mn}$$

or

$$e^{R_c} = \left(1 + \frac{R_m}{m}\right)^{m}$$

This means that

$$R_c = m \ln\left(1 + \frac{R_m}{m}\right) \tag{4.3}$$

and

$$R_m = m(e^{R_c/m} - 1) \tag{4.4}$$

These equations can be used to convert a rate with a compounding frequency of m times per annum to a continuously compounded rate and vice versa. The natural logarithm function $\ln x$, which is built into most calculators, is the *inverse* of the exponential function, so that, if $y = \ln x$, then $x = e^y$.

Example 4.1

Consider an interest rate that is quoted as 10% per annum with semiannual compounding. From equation (4.3) with $m = 2$ and $R_m = 0.1$, the equivalent rate

[4] Actuaries sometimes refer to a continuously compounded rate as the *force of interest*.

with continuous compounding is

$$2\ln\left(1+\frac{0.1}{2}\right) = 0.09758$$

or 9.758% per annum.

Example 4.2

Suppose that a lender quotes the interest rate on loans as 8% per annum with continuous compounding, and that interest is actually paid quarterly. From equation (4.4) with $m = 4$ and $R_c = 0.08$, the equivalent rate with quarterly compounding is

$$4(e^{0.08/4} - 1) = 0.0808$$

or 8.08% per annum. This means that on a \$1,000 loan, interest payments of \$20.20 would be required each quarter.

4.3 ZERO RATES

The n-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for n years. All the interest and principal is realized at the end of n years. There are no intermediate payments. The n-year zero-coupon interest rate is sometimes also referred to as the n-year *spot rate*, the n-year *zero rate*, or just the n-year zero. Suppose a 5-year zero rate with continuous compounding is quoted as 5% per annum. This means that \$100, if invested for 5 years, grows to

$$100 \times e^{0.05\times 5} = 128.40$$

Many of the interest rates we observe directly in the market are not pure zero rates. Consider a 5-year government bond that provides a 6% coupon. The price of this bond does not by itself determine the 5-year Treasury zero rate because some of the return on the bond is realized in the form of coupons prior to the end of year 5. Later in this chapter we will discuss how we can determine Treasury zero rates from the market prices of coupon-bearing bonds.

4.4 BOND PRICING

Most bonds provide coupons periodically. The bond's principal (which is also known as its par value or face value) is received at the end of its life. The theoretical price of a bond can be calculated as the present value of all the cash flows that will be received by the owner of the bond. Sometimes bond traders use the same discount rate for all the cash flows underlying a bond, but a more accurate approach is to use a different zero rate for each cash flow.

To illustrate this, consider the situation where Treasury zero rates, measured with continuous compounding, are as in Table 4.2. (We explain later how these can be calculated.) Suppose that a 2-year Treasury bond with a principal of \$100 provides coupons at the rate of 6% per annum semiannually. To calculate the present value of the first coupon of \$3, we discount it at 5.0% for 6 months; to calculate the present

Table 4.2 Treasury zero rates.

Maturity (years)	*Zero rate (%) (continuously compounded)*
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

value of the second coupon of \$3, we discount it at 5.8% for 1 year; and so on. Therefore the theoretical price of the bond is

$$3e^{-0.05\times 0.5}+3e^{-0.058\times 1.0}+3e^{-0.064\times 1.5}+103e^{-0.068\times 2.0}=98.39$$

or \$98.39.

Bond Yield

A bond's yield is the discount rate that, when applied to all cash flows, gives a bond price equal to its market price. Suppose that the theoretical price of the bond we have been considering, \$98.39, is also its market value (i.e., the market's price of the bond is in exact agreement with the data in Table 4.2). If y is the yield on the bond, expressed with continuous compounding, we have

$$3e^{-y\times 0.5}+3e^{-y\times 1.0}+3e^{-y\times 1.5}+103e^{-y\times 2.0}=98.39$$

This equation can be solved using an iterative ("trial and error") procedure to give $y=6.76\%$.[5]

Par Yield

The *par yield* for a certain bond maturity is the coupon rate that causes the bond price to equal its par value. (The par value is the same as the principal value.) Usually the bond is assumed to provide semiannual coupons. Suppose that the coupon on a 2-year bond in our example is c per annum (or $\frac{1}{2}c$ per 6 months). Using the zero rates in Table 4.2, the value of the bond is equal to its par value of 100 when

$$\frac{c}{2}e^{-0.05\times 0.5}+\frac{c}{2}e^{-0.058\times 1.0}+\frac{c}{2}e^{-0.064\times 1.5}+\left(100+\frac{c}{2}\right)e^{-0.068\times 2.0}=100$$

This equation can be solved in a straightforward way to give $c=6.87$. The 2-year par yield is therefore 6.87% per annum with semiannual compounding (or 6.75% with continuous compounding).

More generally, if d is the present value of \$1 received at the maturity of the bond, A is the value of an annuity that pays one dollar on each coupon payment date, and m

[5] One way of solving nonlinear equations of the form $f(y)=0$, such as this one, is to use the Newton–Raphson method. We start with an estimate y_0 of the solution and produce successively better estimates $y_1, y_2, y_3, \dots$ using the formula $y_{i+1}=y_i-f(y_i)/f'(y_i)$, where $f'(y)$ denotes the derivative of f with respect to y.

is the number of coupon payments per year, then the par yield c must satisfy

$$100 = A\frac{c}{m} + 100d$$

so that

$$c = \frac{(100 - 100d)m}{A}$$

In our example, $m = 2$, $d = e^{-0.068\times 2} = 0.87284$, and

$$A = e^{-0.05\times 0.5} + e^{-0.058\times 1.0} + e^{-0.064\times 1.5} + e^{-0.068\times 2.0} = 3.70027$$

The formula confirms that the par yield is 6.87% per annum. Note that this is a rate expressed with semiannual compounding; with continuous compounding, it would be 6.75% per annum.

4.5 DETERMINING TREASURY ZERO RATES

We now discuss how Treasury zero rates can be calculated from the prices of Treasury bonds. The most popular approach is known as the *bootstrap method.* To illustrate the nature of the method, consider the data in Table 4.3 on the prices of five bonds. Because the first three bonds pay no coupons, the zero rates corresponding to the maturities of these bonds can easily be calculated. The 3-month bond provides a return of 2.5 in 3 months on an initial investment of 97.5. With quarterly compounding, the 3-month zero rate is $(4 \times 2.5)/97.5 = 10.256\%$ per annum. Equation (4.3) shows that, when the rate is expressed with continuous compounding, it becomes

$$4\ln\left(1 + \frac{0.10256}{4}\right) = 0.10127$$

or 10.127% per annum. The 6-month bond provides a return of 5.1 in 6 months on an initial investment of 94.9. With semiannual compounding the 6-month rate is $(2 \times 5.1)/94.9 = 10.748\%$ per annum. Equation (4.3) shows that, when the rate is

Table 4.3 Data for bootstrap method.

Bond principal (\$)	*Time to maturity* (*years*)	*Annual coupon** (\$)	*Bond price* (\$)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

* Half the stated coupon is assumed to be paid every 6 months.

expressed with continuous compounding, it becomes

$$2\ln\left(1+\frac{0.10748}{2}\right)=0.10469$$

or 10.469% per annum. Similarly, the 1-year rate with continuous compounding is

$$\ln\left(1+\frac{10}{90}\right)=0.10536$$

or 10.536% per annum.

The fourth bond lasts 1.5 years. The payments are as follows:

6 months: \$4
1 year: \$4
1.5 years: \$104

From our earlier calculations, we know that the discount rate for the payment at the end of 6 months is 10.469% and that the discount rate for the payment at the end of 1 year is 10.536%. We also know that the bond's price, \$96, must equal the present value of all the payments received by the bondholder. Suppose the 1.5-year zero rate is denoted by R. It follows that

$$4e^{-0.10469\times 0.5}+4e^{-0.10536\times 1.0}+104e^{-R\times 1.5}=96$$

This reduces to

$$e^{-1.5R}=0.85196$$

or

$$R=-\frac{\ln(0.85196)}{1.5}=0.10681$$

The 1.5-year zero rate is therefore 10.681%. This is the only zero rate that is consistent with the 6-month rate, 1-year rate, and the data in Table 4.3.

The 2-year zero rate can be calculated similarly from the 6-month, 1-year, and 1.5-year zero rates, and the information on the last bond in Table 4.3. If R is the 2-year zero rate, then

$$6e^{-0.10469\times 0.5}+6e^{-0.10536\times 1.0}+6e^{-0.10681\times 1.5}+106e^{-R\times 2.0}=101.6$$

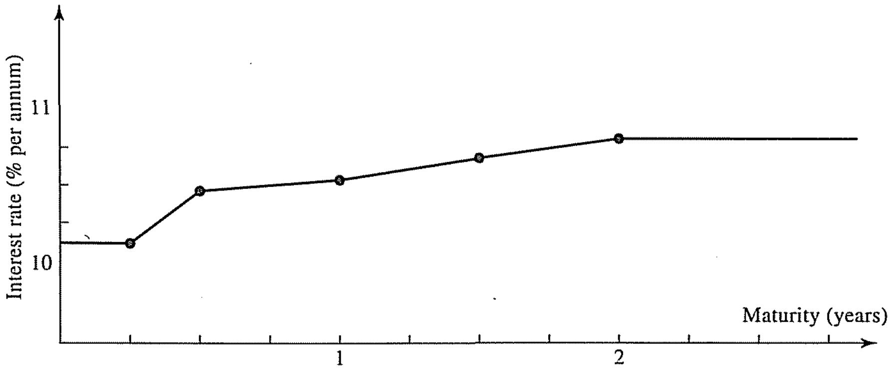
This gives $R=0.10808$, or 10.808%.

The rates we have calculated are summarized in Table 4.4. A chart showing the zero

Table 4.4 Continuously compounded zero rates determined from data in Table 4.3.

Maturity (years)	*Zero rate (%) (continuously compounded)*
0.25	10.127
0.50	10.469
1.00	10.536
1.50	10.681
2.00	10.808

Figure 4.1 Zero rates given by the bootstrap method.



rate as a function of maturity is known as the *zero curve*. A common assumption is that the zero curve is linear between the points determined using the bootstrap method. (This means that the 1.25-year zero rate is $0.5 \times 10.536 + 0.5 \times 10.681 = 10.6085\%$ in our example.) It is also usually assumed that the zero curve is horizontal prior to the first point and horizontal beyond the last point. Figure 4.1 shows the zero curve for our data using these assumptions. By using longer maturity bonds, the zero curve would be more accurately determined beyond 2 years.

In practice, we do not usually have bonds with maturities equal to exactly 1.5 years, 2 years, 2.5 years, and so on. The approach often used by analysts is to interpolate between the bond price data before it is used to calculate the zero curve. For example, if they know that a 2.3-year bond with a coupon of 6% sells for 98 and a 2.7-year bond with a coupon of 6.5% sells for 99, it might be assumed that a 2.5-year bond with a coupon of 6.25% would sell for 98.5.

4.6 FORWARD RATES

Forward interest rates are the rates of interest implied by current zero rates for periods of time in the future. To illustrate how they are calculated, we suppose that a particular set of zero rates are as shown in the second column of Table 4.5. The rates are assumed to be continuously compounded. Thus, the 3% per annum rate for 1 year means that, in return for an investment of \$100 today, an investor receives $100e^{0.03\times 1} = \$103.05$ in 1 year; the 4% per annum rate for 2 years means that, in return for an investment of \$100 today, the investor receives $100e^{0.04\times 2} = \$108.33$ in 2 years; and so on.

The forward interest rate in Table 4.5 for year 2 is 5% per annum. This is the rate of interest that is implied by the zero rates for the period of time between the end of the first year and the end of the second year. It can be calculated from the 1-year zero interest rate of 3% per annum and the 2-year zero interest rate of 4% per annum. It is the rate of interest for year 2 that, when combined with 3% per annum for year 1, gives 4% overall for the 2 years. To show that the correct answer is 5% per annum, suppose that \$100 is invested. A rate of 3% for the first year and 5% for the second year gives

$$100e^{0.03\times 1}e^{0.05\times 1} = \$108.33$$

at the end of the second year. A rate of 4% per annum for 2 years gives

$$100e^{0.04\times 2}$$

which is also \$108.33. This example illustrates the general result that when interest rates are continuously compounded and rates in successive time periods are combined, the overall equivalent rate is simply the average rate during the whole period. In our example, 3% for the first year and 5% for the second year average to 4% over the 2 years. The result is only approximately true when the rates are not continuously compounded.

The forward rate for the year 3 is the rate of interest that is implied by a 4% per annum 2-year zero rate and a 4.6% per annum 3-year zero rate. It is 5.8% per annum. The reason is that an investment for 2 years at 4% per annum combined with an investment for one year at 5.8% per annum gives an overall average return for the three years of 4.6% per annum. The other forward rates can be calculated similarly and are shown in the third column of the table. In general, if R_1 and R_2 are the zero rates for maturities T_1 and T_2, respectively, and R_F is the forward interest rate for the period of time between T_1 and T_2, then

$$R_F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1} \tag{4.5}$$

To illustrate this formula, consider the calculation of the year-4 forward rate from the data in Table 4.5: $T_1 = 3$, $T_2 = 4$, $R_1 = 0.046$, and $R_2 = 0.05$, and the formula gives $R_F = 0.062$.

Equation (4.5) can be written as

$$R_F = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1} \tag{4.6}$$

This shows that if the zero curve is upward sloping between T_1 and T_2, so that $R_2 > R_1$, then $R_F > R_2$ (i.e., the forward rate is greater than both zero rates). Similarly if the zero curve is downward sloping with $R_2 < R_1$, then $R_F < R_2$ (i.e., the forward rate is less

Table 4.5 Calculation of forward rates.

Year (*n*)	*Zero rate for an n-year investment* (*% per annum*)	*Forward rate for nth year* (*% per annum*)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

Business Snapshot 4.2 Orange County's Yield Curve Plays

Suppose an investor can borrow or lend at the rates given in Table 4.5 and thinks that 1-year interest rates will not change much over the next 5 years. The investor can borrow 1-year funds and invest for 5-years. The 1-year borrowings can be rolled over for further 1-year periods at the end of the first, second, third, and fourth years. If interest rates do stay about the same, this strategy will yield a profit of about 2.3% per year, because interest will be received at 5.3% and paid at 3%. This type of trading strategy is known as a *yield curve play*. The investor is speculating that rates in the future will be quite different from the forward rates observed in the market today. (In our example, forward rates observed in the market today for future 1-year periods are 5%, 5.8%, 6.2%, and 6.5%.)

Robert Citron, the Treasurer at Orange County, used yield curve plays similar the one we have just described very successfully in 1992 and 1993. The profit from Mr. Citron's trades became an important contributor Orange County's budget and he was re-elected. (No one listened to his opponent in the election who said his trading strategy was too risky.)

In 1994 Mr. Citron expanded his yield curve plays. He invested heavily in *inverse floaters*. These pay a rate of interest equal to a fixed rate of interest minus a floating rate. He also leveraged his position by borrowing in the repo market. If short-term interest rates had remained the same or declined he would have continued to do well. As it happened, interest rates rose sharply during 1994. On December 1, 1994, Orange County announced that its investment portfolio had lost \$1.5 billion and several days later it filed for bankruptcy protection.

than both zero rates). Taking limits as T_2 approaches T_1 in equation (4.6) and letting the common value of the two be T, we obtain

$$R_F = R + T\frac{\partial R}{\partial T}$$

where R is the zero rate for a maturity of T. The value of R_F obtained in this way is known as the *instantaneous forward rate* for a maturity of T. This is the forward rate that is applicable to a very short future time period that begins at time T.

Assuming that the zero rates for borrowing and investing are the same (which is close to the truth for a large financial institution), an investor can lock in the forward rate for a future time period. Suppose, for example, that the zero rates are as in Table 4.5. If an investor borrows \$100 at 3% for 1 year and then invests the money at 4% for 2 years, the result is a cash outflow of $100e^{0.03\times 1} = \$103.05$ at the end of year 1 and an inflow of $100e^{0.04\times 2} = \$108.33$ at the end of year 2. Because $108.33 = 103.05e^{0.05}$, a return equal to the forward rate (5%) is earned on \$103.05 during the second year. Suppose next that the investor borrows \$100 for four years at 5% and invests it for three years at 4.6%. The result is a cash inflow of $100e^{0.046\times 3} = \$114.80$ at the end of the third year and a cash outflow of $100e^{0.05\times 4} = \$122.14$ at the end of the fourth year. Because $122.14 = 114.80e^{0.062}$, money is being borrowed for the fourth year at the forward rate of 6.2%.

If an investor thinks that rates in the future will be different from today's forward rates there are many trading strategies that the investor will find attractive (see Business

Snapshot 4.2). One of these involves entering into a contract known as a *forward rate agreement*. We will now discuss how this contract works and how it is valued.

4.7 FORWARD RATE AGREEMENTS

A forward rate agreement (FRA) is an over-the-counter agreement that a certain interest rate will apply to either borrowing or lending a certain principal during a specified future period of time. The assumption underlying the contract is that the borrowing or lending would normally be done at LIBOR.

Consider a forward rate agreement where a company X is agreeing to lend money to company Y for the period of time between T_1 and T_2. Define:

R_K: The rate of interest agreed to in the FRA

R_F: The forward LIBOR interest rate for the period between times T_1 and T_2, calculated today[6]

R_M: The actual LIBOR interest rate observed in the market at time T_1 for the period between times T_1 and T_2

L: The principal underlying the contract

We will depart from our usual assumption of continuous compounding and assume that the rates R_K, R_F, and R_M are all measured with a compounding frequency reflecting their maturity. This means that if $T_2 - T_1 = 0.5$, they are expressed with semiannual compounding; if $T_2 - T_1 = 0.25$, they are expressed with quarterly compounding; and so on.

Normally company X would earn R_M from the LIBOR loan. The FRA means that it will earn R_K. The extra interest rate (which may be negative) that it earns as a result of entering into the FRA is $R_K - R_M$. The interest rate is set at time T_1 and paid at time T_2. The extra interest rate therefore leads to a cash flow to company X at time T_2 of

$$L(R_K - R_M)(T_2 - T_1) \tag{4.7}$$

Similarly there is a cash flow to company Y at time T_2 of

$$L(R_M - R_K)(T_2 - T_1) \tag{4.8}$$

From equations (4.7) and (4.8), we see that there is another interpretation of the FRA. It is an agreement where company X will receive interest on the principal between T_1 and T_2 at the fixed rate of R_K and pay interest at the realized market rate of R_M. Company Y will pay interest on the principal between T_1 and T_2 at the fixed rate of R_K and receive interest at R_M.

Usually FRAs are settled at time T_1 rather than T_2. The payoff must then be discounted from time T_2 to T_1. For company X, the time T_1 payoff is

$$\frac{L(R_K - R_M)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

[6] LIBOR forward rates are calculated as described in Section 4.6 from the LIBOR/swap zero curve, which is calculated as described in Section 7.6.

and, for company Y, the time T_1 payoff is

$$\frac{L(R_M - R_K)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

Example 4.3

Suppose that a company enters into an FRA that specifies it will receive a fixed rate of 4% on a principal of \$1 million for a 3-month period starting in 3 years. If 3-month LIBOR proves to be 4.5% for the 3-month period the cash flow to the lender will be

$$1{,}000{,}000 \times (0.04 - 0.045) \times 0.25 = -\$1{,}250$$

at the 3.25-year point. This is equivalent to a cash flow of

$$-\frac{1{,}250}{1 + 0.045 \times 0.25} = -\$1{,}236.09$$

at the 3-year point. The cash flow to the party on the opposite side of the transaction will be +\$1,250 at the 3.25-year point or +\$1,236.09 at the 3-year point. (All interest rates in this example are expressed with quarterly compounding.)

Valuation

To value an FRA we first note that it is always worth zero when $R_K = R_F$.[7] This is because, as noted in Section 4.6, a large financial institution can at no cost lock in the forward rate for a future time period. For example, it can ensure that it earns the forward rate for the time period between years 2 and 3 by borrowing for a certain amount of money for 2 years and investing it for 3 years. Similarly, it can ensure that it pays the forward rate for the time period between years 2 and 3 by borrowing for a certain amount of money for 3 years and investing it for 2 years.

Compare two FRAs. The first promises that the LIBOR forward rate R_F will be earned on a principal of L between times T_1 and T_2; the second promises that R_K will be earned on the same principal between the same two dates. The two contracts are the same except for the interest payments received at time T_2. The excess of the value of the second contract over the first is, therefore, the present value of the difference between these interest payments, or

$$L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

where R_2 is the continously compounded riskless zero rate for a maturity T_2.[8] Because the value of the FRA where R_F is earned is zero, the value of the FRA where R_K is earned is

$$V_{\text{FRA}} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2} \tag{4.9}$$

Similarly, the value of an FRA where R_K is paid is

$$V_{\text{FRA}} = L(R_F - R_K)(T_2 - T_1)e^{-R_2 T_2} \tag{4.10}$$

[7] It is usually the case that R_K is set equal to R_F when the FRA is first initiated.

[8] Note that R_K, R_M, and R_F are expressed with a compounding frequency corresponding to $T_2 - T_1$, whereas R_2 is expressed with continuous compounding.

Example 4.4

Suppose that LIBOR zero and forward rates are as in Table 4.5. Consider an FRA where we will receive a rate of 6%, measured with annual compounding, on a principal of $1 million between the end of year 1 and the end of year 2. In this case, the forward rate is 5% with continuous compounding or 5.127% with annual compounding. From equation (4.9), it follows that the value of the FRA is

$$1{,}000{,}000(0.06 - 0.05127)e^{-0.04\times 2} = \$8{,}058$$

By comparing equations (4.7) and (4.9), we see that an FRA can be valued if we

1. Calculate the payoff on the assumption that forward rates are realized, that is, on the assumption that $R_M = R_F$.
2. Discount this payoff at the risk-free rate.

4.8 DURATION

The *duration* of a bond, as its name implies, is a measure of how long on average the holder of the bond has to wait before receiving cash payments. A zero-coupon bond that lasts n years has a duration of n years. However, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n.

Suppose that a bond provides the holder with cash flows c_i at time t_i ($1 \leqslant i \leqslant n$). The price, B, and yield, y (continuously compounded), are related by

$$B = \sum_{i=1}^{n} c_i e^{-yt_i} \tag{4.11}$$

The duration, D, of the bond is defined as

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i}}{B} \tag{4.12}$$

This can be written

$$D = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-yt_i}}{B}\right]$$

The term in square brackets is the ratio of the present value of the cash flow at time t_i to the bond price. The bond price is the present value of all payments. The duration is therefore a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i. The sum of the weights is 1.0.

When a small change Δy in the yield is considered, it is approximately true that

$$\Delta B = \frac{dB}{dy}\Delta y \tag{4.13}$$

Table 4.6 Calculation of duration.

Time (years)	*Cash flow ($)*	*Present value*	*Weight*	*Time × weight*
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total:	130	94.213	1.000	2.653

From equation (4.11), this becomes

$$\Delta B = -\Delta y \sum_{i=1}^{n} c_i t_i e^{-yt_i} \tag{4.14}$$

(Note that there is a negative relationship between B and y. When bond yields increase, bond prices decrease. When bond yields decrease, bond prices increase.) From equations (4.12) and (4.14), we can derive the key duration relationship

$$\Delta B = -BD\,\Delta y \tag{4.15}$$

This can be written

$$\frac{\Delta B}{B} = -D\,\Delta y \tag{4.16}$$

Equation (4.16) is an approximate relationship between percentage changes in a bond price and changes in its yield. It is easy to use and is the reason why duration, which was first suggested by Macaulay in 1938, has become such a popular measure.

Consider a 3-year 10% coupon bond with a face value of $100. Suppose that the yield on the bond is 12% per annum with continuous compounding. This means that $y = 0.12$. Coupon payments of $5 are made every 6 months. Table 4.6 shows the calculations necessary to determine the bond's duration. The present values of the bond's cash flows, using the yield as the discount rate, are shown in column 3 (e.g., the present value of the first cash flow is $5e^{-0.12\times 0.5} = 4.709$). The sum of the numbers in column 3 gives the bond's price as 94.213. The weights are calculated by dividing the numbers in column 3 by 94.213. The sum of the numbers in column 5 gives the duration as 2.653 years.

Small changes in interest rates are often measured in *basis points*. As mentioned earlier, a basis point is 0.01% per annum. The following example investigates the accuracy of the duration relationship in equation (4.15).

Example 4.5

For the bond in Table 4.6, the bond price, B, is 94.213 and the duration, D, is 2.653, so that equation (4.15) gives

$$\Delta B = -94.213 \times 2.653\,\Delta y$$

or

$$\Delta B = -249.95\,\Delta y$$

When the yield on the bond increases by 10 basis points (= 0.1%), it follows that $\Delta y = +0.001$. The duration relationship predicts that $\Delta B = -249.95 \times 0.001 = -0.250$, so that the bond price goes down to $94.213 - 0.250 = 93.963$. How accurate is this? When the bond yield increases by 10 basis points to 12.1%, the bond price is

$$5e^{-0.121\times 0.5} + 5e^{-0.121\times 1.0} + 5e^{-0.121\times 1.5} + 5e^{-0.121\times 2.0} + 5e^{-0.121\times 2.5} + 105e^{-0.121\times 3.0} = 93.963$$

which is (to three decimal places) the same as that predicted by the duration relationship.

Modified Duration

The preceding analysis is based on the assumption that y is expressed with continuous compounding. If y is expressed with annual compounding, it can be shown that the approximate relationship in equation (4.15) becomes

$$\Delta B = -\frac{BD\,\Delta y}{1+y}$$

More generally, if y is expressed with a compounding frequency of m times per year, then

$$\Delta B = -\frac{BD\,\Delta y}{1+y/m}$$

A variable D^*, defined by

$$D^* = \frac{D}{1+y/m}$$

is sometimes referred to as the bond's *modified duration*. It allows the duration relationship to be simplified to

$$\Delta B = -BD^*\Delta y \tag{4.17}$$

when y is expressed with a compounding frequency of m times per year. The following example investigates the accuracy of the modified duration relationship.

Example 4.6

The bond in Table 4.6 has a price of 94.213 and a duration of 2.653. The yield, expressed with semiannual compounding is 12.3673%. The modified duration, D^*, is given by

$$D^* = \frac{2.653}{1+0.123673/2} = 2.499$$

From equation (4.17),

$$\Delta B = -94.213 \times 2.4985\,\Delta y$$

or

$$\Delta B = -235.39\,\Delta y$$

When the yield (semiannually compounded) increases by 10 basis points (= 0.1%), we have $\Delta y = +0.001$. The duration relationship predicts that we expect ΔB to be $-235.39 \times 0.001 = -0.235$, so that the bond price goes down to $94.213 - 0.235 = 93.978$. How accurate is this? When the bond yield (semiannually compounded) increases by 10 basis points to 12.4673% (or to 12.0941% with continuous compounding), an exact calculation similar to that in the previous example shows that the bond price becomes 93.978. This shows that the modified duration calculation gives good accuracy.

Bond Portfolios

The duration, D, of a bond portfolio can be defined as a weighted average of the durations of the individual bonds in the portfolio, with the weights being proportional to the bond prices. Equations (4.15) to (4.17) then apply, with B being defined as the value of the bond portfolio. They estimate the change in the value of the bond portfolio for a small change Δy in the yields of all the bonds.

It is important to realize that, when duration is used for bond portfolios, there is an implicit assumption that the yields of all bonds will change by the same amount. When the bonds have widely differing maturities, this happens only when there is a parallel shift in the zero-coupon yield curve. We should therefore interpret equations (4.15) to (4.17) as providing estimates of the impact on the price of a bond portfolio of a small parallel shift, Δy, in the zero curve.

4.9 CONVEXITY

The duration relationship applies only to small changes in yields. This is illustrated in Figure 4.2, which shows the relationship between the percentage change in value and change in yield for two bond portfolios having the same duration. The gradients of the two curves are the same at the origin. This means that both bond portfolios change in value by the same percentage for small yield changes and is consistent with equation (4.16). For large yield changes, the portfolios behave differently. Portfolio X has more curvature in its relationship with yields than portfolio Y. A factor known as *convexity* measures this curvature and can be used to improve the relationship in equation (4.16).

A measure of convexity is

$$C = \frac{1}{B}\frac{d^2B}{dy^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B}$$

From Taylor series expansions, we obtain a more accurate expression than equation (4.13), given by

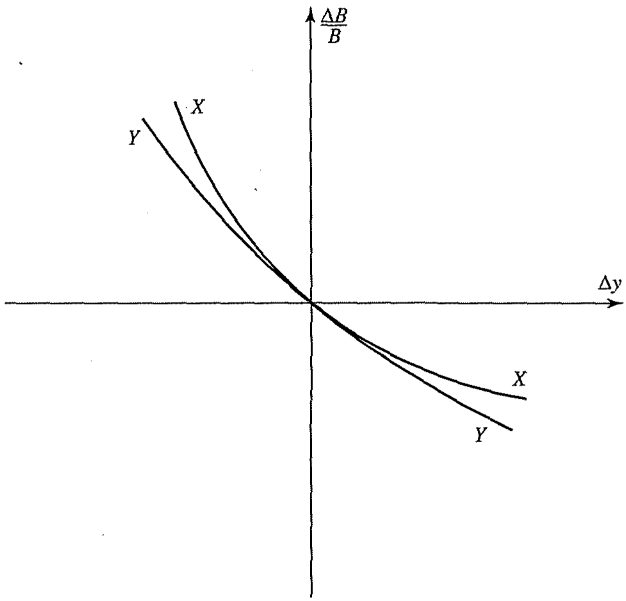
$$\Delta B = \frac{dB}{dy}\Delta y + \tfrac{1}{2}\frac{d^2B}{dy^2}\Delta y^2$$

This leads to

$$\frac{\Delta B}{B} = -D\,\Delta y + \tfrac{1}{2}C(\Delta y)^2$$

The convexity of a bond portfolio tends to be greatest when the portfolio provides payments evenly over a long period of time. It is least when the payments are

Figure 4.2 Two bond portfolios with the same duration.



concentrated around one particular point in time. By matching convexity as well as duration, a company can make itself immune to relatively large parallel shifts in the zero curve. However, it is still exposed to nonparallel shifts.

4.10 THEORIES OF THE TERM STRUCTURE OF INTEREST RATES

It is natural to ask what determines the shape of the zero curve. Why is it sometimes downward-sloping, sometimes upward-sloping, and sometimes partly upward-sloping and partly downward-sloping? A number of different theories have been proposed. The simplest is *expectations theory*, which conjectures that long-term interest rates should reflect expected future short-term interest rates. More precisely, it argues that a forward interest rate corresponding to a certain future period is equal to the expected future zero interest rate for that period. Another idea, *market segmentation theory*, conjectures that there need be no relationship between short-, medium-, and long-term interest rates. Under the theory, a major investor such as a large pension fund invests in bonds of a certain maturity and does not readily switch from one maturity to another. The short-term interest rate is determined by supply and demand in the short-term bond market; the medium-term interest rate is determined by supply and demand in the medium-term bond market; and so on.

The theory that is in some ways most appealing is *liquidity preference theory*, which argues that forward rates should always be higher than expected future zero rates. The basic assumption underlying the theory is that investors prefer to preserve their liquidity and invest funds for short periods of time. Borrowers, on the other hand, usually prefer to borrow at fixed rates for long periods of time. If the interest rates offered by banks and other financial intermediaries corresponded to expectations theory, long-term interest rates would equal the average of expected future short-term interest rates. In the absence of any incentive to do otherwise, investors would tend to deposit their funds for short time periods, and borrowers would tend to choose to borrow for long time periods. Financial intermediaries would then find themselves financing substantial amounts of long-term fixed-rate loans with short-term deposits. Excessive interest rate risk would result. In practice, in order to match depositors with borrowers and avoid interest rate risk, financial intermediaries raise long-term interest rates relative to expected future short-term interest rates. This strategy reduces the demand for long-term fixed-rate borrowing and encourages investors to deposit their funds for long terms.

Liquidity preference theory leads to a situation in which forward rates are greater than expected future zero rates. It is also consistent with the empirical result that yield curves tend to be upward-sloping more often than they are downward-sloping.

SUMMARY

Two important interest rates for derivative traders are Treasury rates and LIBOR rates. Treasury rates are the rates paid by a government on borrowings in its own currency. LIBOR rates are short-term lending rates offered by banks in the interbank market.

The compounding frequency used for an interest rate defines the units in which it is measured. The difference between an annually compounded rate and a quarterly compounded rate is analogous to the difference between a distance measured in miles and a distance measured in kilometers. Traders frequently use continuous compounding when analyzing the value of derivatives.

Many different types of interest rates are quoted in financial markets and calculated by analysts. The n-year zero rate or n-year spot rate is the rate applicable to an investment lasting for n years when all of the return is realized at the end. The par yield on a bond of a certain maturity is the coupon rate that causes the bond to sell for its par value. Forward rates are the rates applicable to future periods of time implied by today's zero rates.

The method most commonly used to calculate zero rates is known as the bootstrap method. It involves starting with short-term instruments and moving progressively to longer-term instruments, making sure that the zero rates calculated at each stage are consistent with the prices of the instruments. It is used daily by trading desks to calculate a Treasury zero-rate curve.

A forward rate agreement (FRA) is an over-the-counter agreement that a certain interest rate will apply for either borrowing or lending a certain principal at LIBOR during a specified future period of time. An FRA can be valued by assuming that forward rates are realized and discounting the resulting payoff.

An important concept in interest rate markets is *duration*. Duration measures the sensitivity of the value of a bond portfolio to a small parallel shift in the zero-coupon yield curve. Specifically,

$$\Delta B = -BD\,\Delta y$$

where B is the value of the bond portfolio, D is the duration of the portfolio, Δy is the size of a small parallel shift in the zero curve, and ΔB is the resultant effect on the value of the bond portfolio.

FURTHER READING

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Questions and Problems (Answers in Solutions Manual)

4.1. A bank quotes you an interest rate of 14% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

4.2. What is meant by LIBOR and LIBID. Which is higher?

4.3. The 6-month and 1-year zero rates are both 10% per annum. For a bond that has a life of 18 months and pays a coupon of 8% per annum (with semiannual payments and one having just been made), the yield is 10.4% per annum. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.

4.4. An investor receives $1,100 in one year in return for an investment of $1,000 now. Calculate the percentage return per annum with:
(a) Annual compounding
(b) Semiannual compounding
(c) Monthly compounding
(d) Continuous compounding

4.5. Suppose that zero interest rates with continuous compounding are as follows:

Maturity (months)	*Rate (% per annum)*
3	8.0
6	8.2
9	8.4
12	8.5
15	8.6
18	8.7

Calculate forward interest rates for the second, third, fourth, fifth, and sixth quarters.

4.6. Assuming that zero rates are as in Problem 4.5, what is the value of an FRA that enables the holder to earn 9.5% for a 3-month period starting in 1 year on a principal of $1,000,000? The interest rate is expressed with quarterly compounding.

4.7. The term structure of interest rates is upward-sloping. Put the following in order of magnitude:
(a) The 5-year zero rate
(b) The yield on a 5-year coupon-bearing bond
(c) The forward rate corresponding to the period between 5 and 5.25 years in the future
What is the answer to this question when the term structure of interest rates is downward-sloping?

4.8. What does duration tell you about the sensitivity of a bond portfolio to interest rates. What are the limitations of the duration measure?

4.9. What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

4.10. A deposit account pays 12% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a $10,000 deposit?

4.11. Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are, respectively, 4%, 4.2%, 4.4%, 4.6%, and 4.8% per annum, with continuous compounding. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months and pays a coupon of 4% per annum semiannually.

4.12. A 3-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond's yield?

4.13. Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5%, and 7%, respectively. What is the 2-year par yield?

4.14. Suppose that zero interest rates with continuous compounding are as follows:

Maturity (years)	*Rate (% per annum)*
1	2.0
2	3.0
3	3.7
4	4.2
5	4.5

Calculate forward interest rates for the second, third, fourth, and fifth years.

4.15. Use the rates in Problem 4.14 to value an FRA where you will pay 5% for the third year on $1 million.

4.16. A 10-year 8% coupon bond currently sells for $90. A 10-year 4% coupon bond currently sells for $80. What is the 10-year zero rate? (*Hint*: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.)

4.17. Explain carefully why liquidity preference theory is consistent with the observation that the term structure of interest rates tends to be upward-sloping more often than it is downward-sloping.

4.18. "When the zero curve is upward-sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward-sloping the reverse is true." Explain why this is so.

4.19. Why are US Treasury rates significantly lower that other rates that are close to risk-free?

4.20. Why does a loan in the repo market involve very little credit risk?

4.21. Explain why an FRA is equivalent to the exchange of a floating rate of interest for a fixed rate of interest.

4.22. A 5-year bond with a yield of 11% (continuously compounded) pays an 8% coupon at the end of each year.
(a) What is the bond's price?
(b) What is the bond's duration?
(c) Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.
(d) Recalculate the bond's price on the basis of a 10.8% per annum yield and verify that the result is in agreement with your answer to (c).

4.23. The cash prices of 6-month and 1-year Treasury bills are 94.0 and 89.0. A 1.5-year bond that will pay coupons of $4 every 6 months currently sells for $94.84. A 2-year bond that will pay coupons of $5 every 6 months currently sells for $97.12. Calculate the 6-month, 1-year, 1.5-year, and 2-year zero rates.

Assignment Questions

4.24. An interest rate is quoted as 5% per annum with semiannual compounding. What is the equivalent rate with (a) annual compounding, (b) monthly compounding, and (c) continuous compounding.

4.25. The 6-month, 12-month, 18-month, and 24-month zero rates are 4%, 4.5%, 4.75%, and 5%, with semiannual compounding.
(a) What are the rates with continuous compounding?
(b) What is the forward rate for the 6-month period beginning in 18 months?
(c) What is the value of an FRA that promises to pay you 6% (compounded semi-annually) on a principal of $1 million for the 6-month period starting in 18 months?

4.26. What is the 2-year par yield when the zero rates are as in Problem 4.25? What is the yield on a 2-year bond that pays a coupon equal to the par yield?

4.27. The following table gives the prices of bonds:

Bond principal (\$)	*Time to maturity* (*years*)	*Annual coupon** (\$)	*Bond price* (\$)
100	0.50	0.0	98
100	1.00	0.0	95
100	1.50	6.2	101
100	2.00	8.0	104

* Half the stated coupon is assumed to be paid every six months.

(a) Calculate zero rates for maturities of 6 months, 12 months, 18 months, and 24 months.

(b) What are the forward rates for the following periods: 6 months to 12 months, 12 months to 18 months, and 18 months to 24 months?

(c) What are the 6-month, 12-month, 18-month, and 24-month par yields for bonds that provide semiannual coupon payments?

(d) Estimate the price and yield of a 2-year bond providing a semiannual coupon of 7% per annum.

4.28. Portfolio A consists of a 1-year zero-coupon bond with a face value of \$2,000 and a 10-year zero-coupon bond with a face value of \$6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of \$5,000. The current yield on all bonds is 10% per annum.

(a) Show that both portfolios have the same duration.

(b) Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.

(c) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?