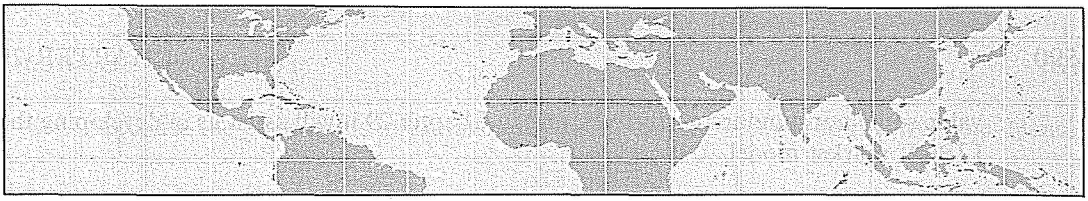

CHAPTER 25 Martingales and Measures

Up to now we have assumed that interest rates are constant when valuing options. In this chapter we relax this assumption in preparation for valuing interest rate derivatives in Chapters 26 to 30.

The risk-neutral valuation principle we have used up to now states that a derivative can be valued by (a) calculating the expected payoff on the assumption that the expected return from the underlying asset equals the risk-free interest rate and (b) discounting the expected payoff at the risk-free interest rate. When interest rates are constant, risk-neutral valuation provides a well-defined and unambiguous valuation tool. When interest rates are stochastic, it is less clear-cut. What does it mean to assume that the expected return on the underlying asset equals to the risk-free rate? Does it mean (a) that each day the expected return is the one-day risk-free rate, or (b) that each year the expected return is the 1-year risk-free rate, or (c) that over a 5-year period the expected return is the 5-year rate at the beginning of the period? What does it mean to discount expected payoffs at the risk-free rate? Can we, for example, discount an expected payoff realized in year 5 at today's 5-year risk-free rate?

In this chapter we explain the theoretical underpinnings of risk-neutral valuation when interest rates are stochastic and show that there are many different risk-neutral worlds that can be assumed in any given situation. We first define a parameter known as the *market price of risk* and show that the excess return over the risk-free interest rate earned by any derivative in a short period of time is linearly related to the market prices of risk of the underlying stochastic variables. What we will refer to as the *traditional risk-neutral world* assumes that all market prices of risk are zero, but we will find that other assumptions about the market price of risk are useful in some situations.

Martingales and *measures* are critical to a full understanding of risk neutral valuation. A martingale is a zero-drift stochastic process. A measure is the unit in which we value security prices. A key result in this chapter will be the *equivalent martingale measure result*. This states that if we use the price of a traded security as the unit of measurement then there is some market price of risk for which all security prices follow martingales.

In this chapter we illustrate the power of the equivalent martingale measure result by using it to value stock options when interest rates are stochastic and to value options to exchange one asset for another. In Chapter 26 we use the result to understand the standard market models for valuing interest rate derivatives, in Chapter 27 we use it to value some nonstandard derivatives, and in Chapter 29 it will assist us in developing the LIBOR market model.

25.1 THE MARKET PRICE OF RISK

We start by considering the properties of derivatives dependent on the value of a single variable θ. We will assume that the process followed by θ is

$$\frac{d\theta}{\theta} = m\,dt + s\,dz \tag{25.1}$$

where dz is a Wiener process. The parameters m and s are the expected growth rate in θ and the volatility of θ, respectively. We assume that they depend only on θ and time t. The variable θ need not be the price of an investment asset. It could be something as far removed from financial markets as the temperature in the center of New Orleans.

Suppose that f_1 and f_2 are the prices of two derivatives dependent only on θ and t. These can be options or other instruments that provide a payoff equal to some function of θ at some future time. We assume that during the time period under consideration f_1 and f_2 provide no income.[1]

Suppose that the processes followed by f_1 and f_2 are

$$\frac{df_1}{f_1} = \mu_1\,dt + \sigma_1\,dz$$

and

$$\frac{df_2}{f_2} = \mu_2\,dt + \sigma_2\,dz$$

where μ_1, μ_2, σ_1, and σ_2 are functions of θ and t. The "dz" in these processes must be the same dz as in equation (25.1) because it is the only source of the uncertainty in the prices of f_1 and f_2.

We now relate the prices f_1 and f_2 using an analysis similar to the Black–Scholes analysis described in Section 13.6. The discrete versions of the processes for f_1 and f_2 are

$$\Delta f_1 = \mu_1 f_1\,\Delta t + \sigma_1 f_1\,\Delta z \tag{25.2}$$

$$\Delta f_2 = \mu_2 f_2\,\Delta t + \sigma_2 f_2\,\Delta z \tag{25.3}$$

We can eliminate the Δz by forming an instantaneously riskless portfolio consisting of $\sigma_2 f_2$ of the first derivative and $-\sigma_1 f_1$ of the second derivative. If Π is the value of the portfolio, then

$$\Pi = (\sigma_2 f_2)f_1 - (\sigma_1 f_1)f_2 \tag{25.4}$$

and

$$\Delta\Pi = \sigma_2 f_2\,\Delta f_1 - \sigma_1 f_1\,\Delta f_2$$

Substituting from equations (25.2) and (25.3), this becomes

$$\Delta\Pi = (\mu_1\sigma_2 f_1 f_2 - \mu_2\sigma_1 f_1 f_2)\,\Delta t \tag{25.5}$$

[1] The analysis can be extended to derivatives that provide income (see Problem 25.7).

Because the portfolio is instantaneously riskless, it must earn the risk-free rate. Hence,

$$\Delta\Pi = r\Pi\,\Delta t$$

Substituting into this equation from equations (25.4) and (25.5) gives

$$\mu_1\sigma_2 - \mu_2\sigma_1 = r\sigma_2 - r\sigma_1$$

or

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \tag{25.6}$$

Note that the left-hand side of equation (25.6) depends only on the parameters of the process followed by f_1 and the right-hand side depends only on the parameters of the process followed by f_2. Define λ as the value of each side in equation (25.6), so that

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda$$

Dropping subscripts, we have shown that if f is the price of a derivative dependent only on θ and t with

$$\frac{df}{f} = \mu\,dt + \sigma\,dz \tag{25.7}$$

then

$$\frac{\mu - r}{\sigma} = \lambda \tag{25.8}$$

The parameter λ is known as the *market price of risk* of θ. It can be dependent on both θ and t, but it is not dependent on the nature of the derivative f. Our analysis shows that, for no arbitrage, $(\mu - r)/\sigma$ must at any given time be the same for all derivatives that are dependent only on θ and t.

It is worth noting that σ, which we are referring to as the volatility of f, is defined as the coefficient of dz in equation (25.7). It can be either positive or negative. If the volatility, s, of θ is positive and f is positively related to θ (so that $\partial f/\partial\theta$ is positive), σ is positive. But if f is negatively related to θ, then σ is negative. The volatility of f, as it is traditionally defined, is $|\sigma|$.

The market price of risk of θ measures the trade-offs between risk and return that are made for securities dependent on θ. Equation (25.8) can be written

$$\mu - r = \lambda\sigma \tag{25.9}$$

For an intuitive understanding of this equation, we note that the variable σ can be loosely interpreted as the quantity of θ-risk present in f. On the right-hand side of the equation we are, therefore, multiplying the quantity of θ-risk by the price of θ-risk. The left-hand side is the expected return in excess of the risk-free interest rate that is required to compensate for this risk. Equation (25.9) is analogous to the capital asset pricing model, which relates the expected excess return on a stock to its risk.

We will not be concerned with the measurement of the market price of risk in this chapter. But this will be discussed in Chapter 31 in the context of the evaluation of real options.

In Chapter 5, we distinguished between investment assets and consumption assets. An investment asset is an asset that is bought or sold purely for investment purposes by a significant number of investors. Consumption assets are held primarily for consumption.

Equation (25.8) is true for all investment assets that provide no income and depend only on θ. If the variable θ itself happens to be such an asset, then

$$\frac{m-r}{s}=\lambda$$

But, in other circumstances, this relationship is not usually true.

Example 25.1

Consider a derivative whose price is positively related to the price of oil and depends on no other stochastic variables. Suppose that it provides an expected return of 12% per annum and has a volatility of 20% per annum. Assume that the risk-free interest rate is 8% per annum. It follows that the market price of risk of oil is

$$\frac{0.12-0.08}{0.2}=0.2$$

Note that oil is a consumption asset rather than an investment asset, so its market price of risk cannot be calculated from equation (25.8) by setting μ equal to the expected return from an investment in oil and σ equal to the volatility of oil prices.

Example 25.2

Consider two securities, both of which are positively dependent on the 90-day interest rate. Suppose that the first one has an expected return of 3% per annum and a volatility of 20% per annum, and the second one has a volatility of 30% per annum. Assume that the instantaneous risk-free rate of interest is 6% per annum. The market price of interest rate risk is, using the expected return and volatility for the first security,

$$\frac{0.03-0.06}{0.2}=-0.15$$

From a rearrangement of equation (25.9), the expected return from the second security is, therefore,

$$0.06-0.15\times 0.3=0.015$$

or 1.5% per annum.

Alternative Worlds

The process followed by derivative price f is

$$df=\mu f\,dt+\sigma f\,dz$$

The value of μ depends on the risk preferences of investors. In a world where the market price of risk is zero, λ equals zero. From equation (25.9) $\mu=r$, so that the process followed by f is

$$df=rf\,dt+\sigma f\,dz$$

We will refer to this as the *traditional risk-neutral world*.

By making other assumptions about the market price of risk, λ, we define other worlds that are internally consistent. In general, we see from equation (25.9) that

$$\mu=r+\lambda\sigma$$

so that

$$df = (r + \lambda\sigma) f\, dt + \sigma f\, dz \tag{25.10}$$

The market price of risk of a variable determines the growth rates of all securities dependent on the variable. As we move from one market price of risk to another, the expected growth rates of security prices change, but their volatilities remain the same. This is a general property of variables following diffusion processes and was illustrated in Section 11.7. Choosing a particular market price of risk is also referred to as defining the *probability measure*. For some value of the market price of risk, we obtain the "real world" and the growth rates of security prices that are observed in practice.

25.2 SEVERAL STATE VARIABLES

Suppose that n variables, $\theta_1, \theta_2, \ldots, \theta_n$, follow stochastic processes of the form

$$\frac{d\theta_i}{\theta_i} = m_i\, dt + s_i\, dz_i \tag{25.11}$$

for $i = 1, 2, \ldots, n$, where the dz_i are Wiener processes. The parameters m_i and s_i are expected growth rates and volatilities and may be functions of the θ_i and time. The appendix at the end of this chapter provides a version of Itô's lemma that covers functions of several variables. It shows that the process for the price, f, of a security that is dependent on the θ_i has the form

$$\frac{df}{f} = \mu\, dt + \sum_{i=1}^{n} \sigma_i\, dz_i \tag{25.12}$$

In this equation, μ is the expected return from the security and $\sigma_i\, dz_i$ is the component of the risk of this return attributable to θ_i.

The appendix at the end of this chapter shows that

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i \tag{25.13}$$

where λ_i is the market price of risk for θ_i. This equation relates the expected excess return that investors require on the security to the λ_i and σ_i. Equation (25.9) is the particular case of this equation when $n = 1$. The term $\lambda_i\sigma_i$ on the right-hand side measures the extent that the excess return required by investors on a security is affected by the dependence of the security on θ_i. If $\lambda_i\sigma_i = 0$, there is no effect; if $\lambda_i\sigma_i > 0$, investors require a higher return to compensate them for the risk arising from θ_i; if $\lambda_i\sigma_i < 0$, the dependence of the security on θ_i causes investors to require a lower return than would otherwise be the case. The $\lambda_i\sigma_i < 0$ situation occurs when the variable has the effect of reducing rather than increasing the risks in the portfolio of a typical investor.

Example 25.3

A stock price depends on three underlying variables: the price of oil, the price of gold, and the performance of a stock index. Suppose that the market prices of risk for these variables are 0.2, −0.1, and 0.4, respectively. Suppose also that the σ_i

factors in equation (25.12) corresponding to the three variables have been estimated as 0.05, 0.1, and 0.15, respectively. The excess return on the stock over the risk-free rate is

$$0.2 \times 0.05 - 0.1 \times 0.1 + 0.4 \times 0.15 = 0.06$$

or 6.0% per annum. If variables other than those considered affect the stock price, this result is still true provided that the market price of risk for each of these other variables is zero.

Equation (25.13) is closely related to arbitrage pricing theory, developed by Stephen Ross in 1976.[2] The continuous-time version of the capital asset pricing model (CAPM) can be regarded as a particular case of the equation. CAPM argues that an investor requires excess returns to compensate for any risk that is correlated to the risk in the return from the stock market, but requires no excess return for other risks. Risks that are correlated with the return from the stock market are referred to as *systematic*; other risks are referred to as *nonsystematic*. If CAPM is true, then λ_i is proportional to the correlation between changes in θ_i and the return from the market. When θ_i is uncorrelated with the return from the market, λ_i is zero.

25.3 MARTINGALES

A martingale is a zero-drift stochastic process. A variable θ follows a martingale if its process has the form

$$d\theta = \sigma\, dz$$

where dz is a Wiener process. The variable σ may itself be stochastic. It can depend on θ and other stochastic variables. A martingale has the convenient property that its expected value at any future time is equal to its value today. This means that

$$E(\theta_T) = \theta_0$$

where θ_0 and θ_T denote the values of θ at times zero and T, respectively. To understand this result, we note that over a very small time interval the change in θ is normally distributed with zero mean. The expected change in θ over any very small time interval is therefore zero. The change in θ between time 0 and time T is the sum of its changes over many small time intervals. It follows that the expected change in θ between time 0 and time T must also be zero.

The Equivalent Martingale Measure Result

Suppose that f and g are the prices of traded securities dependent on a single source of uncertainty. We assume that the securities provide no income during the time period under consideration.[3] We define

$$\phi = \frac{f}{g}$$

[2] See S. A. Ross, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13 (December 1976): 343–62.

[3] Problem 25.8 extends the analysis to situations where the securities provide income.

The variable ϕ is the relative price of f with respect to g. It can be thought of as measuring the price of f in units of g rather than dollars. The security price g is referred to as the *numeraire*.

The *equivalent martingale measure* result shows that, when there are no arbitrage opportunities, ϕ is a martingale for some choice of the market price of risk. What is more, for a given numeraire security g, the same choice of the market price of risk makes ϕ a martingale for all securities f. This choice of the market price of risk is the volatility of g. In other words, when the market price of risk is set equal to the volatility of g, the ratio f/g is a martingale for all security prices f.

To prove this result, we suppose that the volatilities of f and g are σ_f and σ_g. From equation (25.10), in a world where the market price of risk is σ_g, we have

$$df = (r + \sigma_g \sigma_f) f\,dt + \sigma_f f\,dz$$

$$dg = (r + \sigma_g^2) g\,dt + \sigma_g g\,dz$$

Using Itô's lemma, we have

$$d \ln f = (r + \sigma_g \sigma_f - \sigma_f^2/2)\,dt + \sigma_f\,dz$$

$$d \ln g = (r + \sigma_g^2/2)\,dt + \sigma_g\,dz$$

so that

$$d(\ln f - \ln g) = (\sigma_g \sigma_f - \sigma_f^2/2 - \sigma_g^2/2)\,dt + (\sigma_f - \sigma_g)\,dz$$

or

$$d\left(\ln \frac{f}{g}\right) = -\frac{(\sigma_f - \sigma_g)^2}{2}\,dt + (\sigma_f - \sigma_g)\,dz$$

Using Itô's lemma to determine the process for f/g from the process for $\ln(f/g)$, we obtain

$$d\left(\frac{f}{g}\right) = (\sigma_f - \sigma_g)\frac{f}{g}\,dz \tag{25.14}$$

showing that f/g is a martingale.

This provides the required result. We refer to a world where the market price of risk is the volatility of g, σ_g, as a world that is *forward risk neutral* with respect to g.

Because f/g is a martingale in a world that is forward risk neutral with respect to g, it follows from the result at the beginning of this section that

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

or

$$f_0 = g_0 E_g\left(\frac{f_T}{g_T}\right) \tag{25.15}$$

where E_g denotes the expected value in a world that is forward risk neutral with respect to g.

25.4 ALTERNATIVE CHOICES FOR THE NUMERAIRE

We now present a number of examples of the equivalent martingale measure result. Our first example shows that it is consistent with the traditional risk-neutral valuation result we have used up to now. The other examples prepare the way for the valuation of bond options, interest rate caps, and swap options in Chapter 26.

Money Market Account as the Numeraire

The dollar money market account is a security that is worth \$1 at time zero and earns the instantaneous risk-free rate r at any given time.[4] The variable r may be stochastic. If we set g equal to the money market account, it grows at rate r so that

$$dg = rg\,dt \tag{25.16}$$

The drift of g is stochastic, but the volatility of g is zero. The world that is forward risk neutral with respect to g is therefore a world where the market price of risk is zero. This is the world we defined earlier as the traditional risk-neutral world. It follows from equation (25.15), that

$$f_0 = g_0 \hat{E}\left(\frac{f_T}{g_T}\right) \tag{25.17}$$

where $\hat{E}$ denotes expectations in the traditional risk-neutral world.

In this case, $g_0 = 1$ and

$$g_T = e^{\int_0^T r\,dt}$$

so that equation (25.17) reduces to

$$f_0 = \hat{E}\left(e^{-\int_0^T r\,dt} f_T\right) \tag{25.18}$$

or

$$f_0 = \hat{E}\left(e^{-\bar{r}T} f_T\right) \tag{25.19}$$

where $\bar{r}$ is the average value of r between time 0 and time T. This equation shows that one way of valuing an interest rate derivative is to simulate the short-term interest rate r in the traditional risk-neutral world. On each trial we calculate the expected payoff and discount at the average value of the short rate on the sampled path.

When the short-term interest rate r is assumed to be constant, equation (25.19) reduces to

$$f_0 = e^{-rT}\hat{E}(f_T)$$

or the risk-neutral valuation relationship we used in earlier chapters.

Zero-Coupon Bond Price as the Numeraire

Define $P(t, T)$ as the price at time t of a zero-coupon bond that pays off \$1 at time T. We now explore the implications of setting g equal to $P(t, T)$. We use E_T to denote

[4] The money account is the limit as Δt approaches zero of the following security. For the first short period of time of length Δt, it is invested at the initial Δt period rate; at time Δt, it is reinvested for a further period of time Δt at the new Δt period rate; at time $2\Delta t$, it is again reinvested for a further period of time Δt at the new Δt period rate; and so on. The money market accounts in other currencies are defined analogously to the dollar money market account.

expectations in a world that is forward risk neutral with respect to $P(t, T)$. Because $g_T = P(T, T) = 1$ and $g_0 = P(0, T)$, equation (25.15) gives

$$f_0 = P(0, T)E_T(f_T) \tag{25.20}$$

Notice the difference between equations (25.20) and (25.19). In equation (25.19), the discounting is inside the expectations operator. In equation (25.20) the discounting, as represented by the $P(0, T)$ term, is outside the expectations operator. By using a world that is forward risk neutral with respect to $P(t, T)$, we considerably simplify things for a security that provides a payoff solely at time T.

Consider any variable θ that is not an interest rate.[5] A forward contract on θ with maturity T is defined as a contract that pays off $\theta_T - K$ at time T, where θ_T is the value θ at time T. Define f as the value of this forward contract. From equation (25.20), we have

$$f_0 = P(0, T)[E_T(\theta_T) - K]$$

The forward price, F, of θ is the value of K for which f_0 equals zero. It therefore follows that

$$P(0, T)[E_T(\theta_T) - F] = 0$$

or

$$F = E_T(\theta_T) \tag{25.21}$$

Equation (25.21) shows that the forward price of any variable (except an interest rate) is its expected future spot price in a world that is forward risk neutral with respect to $P(t, T)$. Note the difference here between forward prices and futures prices. The argument in Section 14.7 shows that the futures price of a variable is the expected future spot price in the traditional risk-neutral world.

Equation (25.20) shows that we can value any security that provides a payoff at time T by calculating its expected payoff in a world that is forward risk neutral with respect to a bond maturing at time T and discounting at the risk-free rate for maturity T. Equation (25.21) shows that it is correct to assume that the expected value of the underlying variables equal their forward values when computing the expected payoff. These results will be critical to our understanding of the standard market model for bond options in the next chapter.

Interest Rates when a Bond Price is the Numeraire

For our next result, we define $R(t, T, T^*)$ as the forward interest rate as seen at time t for the period between T and T^* expressed with a compounding period of $T^* - T$. (For example, if $T^* - T = 0.5$, the interest rate is expressed with semiannual compounding; if $T^* - T = 0.25$, it is expressed with quarterly compounding; and so on.) The forward price, as seen at time t, of a zero-coupon bond lasting between times T and T^* is

$$\frac{P(t, T^*)}{P(t, T)}$$

We define a forward interest rate differently from the forward value of most variables.

[5] As we shall see later on, forward contracts for interest rates are defined differently from forward contracts for other variables.

A forward interest rate is the interest rate implied by the corresponding forward bond price. It follows that

$$\frac{1}{[1+(T^*-T)R(t,T,T^*)]}=\frac{P(t,T^*)}{P(t,T)}$$

so that

$$R(t,T,T^*)=\frac{1}{T^*-T}\left[\frac{P(t,T)}{P(t,T^*)}-1\right]$$

or

$$R(t,T,T^*)=\frac{1}{T^*-T}\left[\frac{P(t,T)-P(t,T^*)}{P(t,T^*)}\right]$$

Setting

$$f=\frac{1}{T^*-T}[P(t,T)-P(t,T^*)]$$

and $g=P(t,T^*)$, the equivalent martingale measure result shows that $R(t,T,T^*)$ is a martingale in a world that is forward risk neutral with respect to $P(t,T^*)$. This means that

$$R(0,T,T^*)=E_{T^*}[R(T,T,T^*)] \tag{25.22}$$

where E_{T^*} denotes expectations in a world that is forward risk neutral with respect to $P(t,T^*)$.

We have shown that the forward interest rate equals the expected future interest rate in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T^*. This result, when combined with that in equation (25.20), will be critical to our understanding of the standard market model for interest rate caps in the next chapter.

Annuity Factor as the Numeraire

For our next application of equivalent martingale measure arguments we consider a swap starting at a future time T with payment dates at times $T_1, T_2, \ldots, T_N$. Define $T_0=T$. Assume that the principal underlying the swap is \$1. Suppose that the forward swap rate (i.e., the interest rate on the fixed side that makes the swap have a value of zero) is $s(t)$ at time t $(t\leqslant T)$. The value of the fixed side of the swap is

$$s(t)A(t)$$

where

$$A(t)=\sum_{i=0}^{N-1}(T_{i+1}-T_i)P(t,T_{i+1})$$

We showed in Chapter 7 that, when the principal is added to the payment on the last payment date swap, the value of the floating side of the swap on the initiation date equals the underlying principal. It follows that if we add \$1 at time T_N, the floating side is worth \$1 at time T. The value of \$1 received at time T_N is $P(t,T_N)$. The value of \$1 at time T_0 is $P(t,T_0)$. The value of the floating side at time t is, therefore,

$$P(t,T_0)-P(t,T_N)$$

Equating the values of the fixed and floating sides, we obtain

$$s(t)A(t)=P(t,T_0)-P(t,T_N)$$

or

$$s(t) = \frac{P(t, T_0) - P(t, T_N)}{A(t)} \tag{25.23}$$

We can apply the equivalent martingale measure result by setting f equal to $P(t, T_0) - P(t, T_N)$ and g equal to $A(t)$. This leads to

$$s(t) = E_A[s(T)] \tag{25.24}$$

where E_A denotes expectations in a world that is forward risk neutral with respect to $A(t)$. Therefore, in a world that is forward risk neutral with respect to $A(t)$, the expected future swap rate is the current swap rate.

For any security, f, the result in equation (25.15) shows that

$$f_0 = A(0)E_A\left[\frac{f_T}{A(T)}\right] \tag{25.25}$$

This result, when combined with the result in equation (25.24), will be critical to our understanding of the standard market model for European swap options in the next chapter.

25.5 EXTENSION TO SEVERAL FACTORS

The results presented in Sections 25.3 and 25.4 can be extended to cover the situation when there are many independent factors.[6] Assume that there are n independent factors and that the processes for f and g in the traditional risk-neutral world are

$$df = rf\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

and

$$dg = rg\,dt + \sum_{i=1}^{n} \sigma_{g,i} g\,dz_i$$

It follows from Section 25.2 that we can define other worlds that are internally consistent by setting

$$df = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

and

$$dg = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{g,i}\right] g\,dt + \sum_{i=1}^{n} \sigma_{g,i} g\,dz_i$$

where the λ_i $(1 \leqslant i \leqslant n)$ are the n market prices of risk. One of these other worlds is the real world.

We define a world that is forward risk neutral with respect to g as a world, where $\lambda_i = \sigma_{g,i}$. It can be shown from Itô's lemma, using the fact that the dz_i are uncorrelated,

[6] The independence condition is not critical. If factors are not independent they can be orthogonalized.

that the process followed by f/g in this world has zero drift (see Problem 25.12). The rest of the results in the last two sections (from equation 25.15 onward) are therefore still true.

25.6 APPLICATIONS

In this section we provide two applications of the forward risk-neutral valuation argument. Several others are given in Chapters 26, 27, 29, and 30.

The Black–Scholes Result

We can use forward risk-neutral arguments to extend the Black–Scholes result to situations where interest rates are stochastic. Consider a European call option maturing at time T on a non-dividend-paying stock. From equation (25.20), the call option's price is given by

$$c = P(0, T)E_T[\max(S_T - K, 0)] \tag{25.26}$$

where S_T is the stock price at time T, K is the strike price, and E_T denotes expectations in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T. Define R as the zero rate for maturity T, so that

$$P(0, T) = e^{-RT}$$

and equation (25.26) becomes

$$c = e^{-RT}E_T[\max(S_T - K, 0)] \tag{25.27}$$

If we assume that S_T is lognormal in the forward risk-neutral world, we are considering with the standard deviation of $\ln(S_T)$ equal to w, the appendix at the end of Chapter 13 shows that

$$E_T[\max(S_T - K, 0)] = E_T(S_T)N(d_1) - KN(d_2) \tag{25.28}$$

where

$$d_1 = \frac{\ln[E_T(S_T)/K] + w^2/2}{w}$$

$$d_2 = \frac{\ln[E_T(S_T)/K] - w^2/2}{w}$$

From equation (25.21), $E_T(S_T)$ is the forward stock price for a contract maturing at time T. From the no-arbitrage arguments in Chapter 5, we have

$$E_T(S_T) = S_0e^{RT} \tag{25.29}$$

Equations (25.27), (25.28), and (25.29) give

$$c = S_0N(d_1) - Ke^{-RT}N(d_2)$$

where

$$d_1 = \frac{\ln[S_0/K] + RT + w^2/2}{w}$$

$$d_2 = \frac{\ln[S_0/K] + RT - w^2/2}{w}$$

If the stock price volatility σ is defined so that $\sigma\sqrt{T} = w$, the expressions for d_1 and d_2 become

$$d_1 = \frac{\ln[S_0/K] + (R + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[S_0/K] + (R - \sigma^2/2)T}{\sigma\sqrt{T}}$$

showing that the call price is given by the Black–Scholes formula with r replaced by R. A similar result can be produced for European put options.

Option to Exchange One Asset for Another

Consider next an option to exchange an investment asset worth U for an investment asset worth V. This has already been discussed in Section 22.11. We suppose that the volatilities of U and V are σ_U and σ_V and the coefficient of correlation between them is ρ.

Suppose first that the assets provide no income. We choose the numeraire security g to be U. Setting $f = V$ in equation (25.15), we obtain

$$V_0 = U_0 E_U\left(\frac{V_T}{U_T}\right) \tag{25.30}$$

where E_U denotes expectations in a world that is forward risk neutral with respect to U.

Next we set f in equation (25.15) as the value the option under consideration, so that $f_T = \max(V_T - U_T, 0)$. It follows that

$$f_0 = U_0\, E_U\left[\frac{\max(V_T - U_T, 0)}{U_T}\right]$$

or

$$f_0 = U_0\, E_U\left[\max\left(\frac{V_T}{U_T} - 1, 0\right)\right] \tag{25.31}$$

The volatility of V/U is (see Problem 25.14) $\hat{\sigma}$, where

$$\hat{\sigma}^2 = \sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V$$

From the appendix at the end of Chapter 13, equation (25.31) becomes

$$f_0 = U_0\left[E_U\left(\frac{V_T}{U_T}\right)N(d_1) - N(d_2)\right]$$

where

$$d_1 = \frac{\ln(V_0/U_0) + \hat{\sigma}^2 T/2}{\hat{\sigma}\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \hat{\sigma}\sqrt{T}$$

Substituting from equation (25.30), we get

$$f_0 = V_0 N(d_1) - U_0 N(d_2) \tag{25.32}$$

Problem 25.8 shows that, when f and g provide income at rate q_f and q_g, equation (25.15) becomes

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

This means that equations (25.30) and (25.31) become

$$E_U\left(\frac{V_T}{U_T}\right) = e^{(q_U - q_V)T}\frac{V_0}{U_0}$$

and

$$f_0 = e^{-q_U T} U_0 E_U\left[\max\left(\frac{V_T}{U_T} - 1,\ 0\right)\right]$$

and equation (25.32) becomes

$$f_0 = e^{-q_V T} V_0 N(d_1) - e^{-q_U T} U_0 N(d_2)$$

with d_1 and d_2 being redefined as

$$d_1 = \frac{\ln(V_0/U_0) + (q_U - q_V + \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \hat{\sigma}\sqrt{T}$$

This is the result given in equation (22.3).

25.7 CHANGE OF NUMERAIRE

In this section we consider the impact of a change in numeraire on the process followed by a market variable. In a world that is forward risk neutral with respect to g, the process followed by a traded security f is

$$df = \left[r + \sum_{i=1}^{n} \sigma_{g,i}\sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

Similarly, in a world that is forward risk neutral with respect to another security h, the process followed by f is

$$df = \left[r + \sum_{i=1}^{n} \sigma_{h,i}\sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

where $\sigma_{h,i}$ is the ith component of the volatility of h.

The effect of moving from a world that is forward risk neutral with respect to g to one that is forward risk neutral with respect to h (i.e., of changing the numeraire from g to h) is therefore to increase the expected growth rate of the price of any traded security f by

$$\sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i})\sigma_{f,i}$$

Consider next a variable v that is a function of the prices of traded securities (where v is not necessarily the price of a traded security itself). Define $\sigma_{v,i}$ as the ith component of the volatility of v. From Itô's lemma in the appendix at the end of this chapter, we can calculate what happens to the process followed by v when there is a change in numeraire causing the expected growth rate of the underlying traded securities to change. It turns out that the expected growth rate of v responds to a change in numeraire in the same way as the expected growth rate of the prices of traded securities (see Problem 12.6 for the situation where there is only one stochastic variable and Problem 25.13 for the general case). It increases by

$$\alpha_v = \sum_{i=1}^{n}(\sigma_{h,i} - \sigma_{g,i})\sigma_{v,i} \tag{25.33}$$

Define $w = h/g$ and $\sigma_{w,i}$ as the ith component of the volatility of w. From Itô's lemma (see Problem 25.14), we have

$$\sigma_{w,i} = \sigma_{h,i} - \sigma_{g,i}$$

so that equation (25.33) becomes

$$\alpha_v = \sum_{i=1}^{n} \sigma_{w,i}\,\sigma_{v,i} \tag{25.34}$$

We will refer to w as the *numeraire ratio*. Equation (25.34) is equivalent to

$$\alpha_v = \rho\sigma_v\sigma_w \tag{25.35}$$

where σ_v is the total volatility of v, σ_w is the total volatility of w, and ρ is the instantaneous correlation between v and w.[7]

This is a surprisingly simple result. The adjustment to the expected growth rate of a variable v when we change from one numeraire to another is the instantaneous covariance between the percentage change in v and the percentage change in the numeraire ratio. We will use this result when considering timing and quanto adjustments in Chapter 27.

SUMMARY

The market price of risk of a variable defines the trade-offs between risk and return for traded securities dependent on the variable. When there is one underlying variable, a derivative's excess return over the risk-free rate equals the market price of risk multiplied by the variable's volatility. When there are many underlying variables, the excess return is the sum of the market price of risk multiplied by the volatility for each variable.

A powerful tool in the valuation of derivatives is risk-neutral valuation. This was introduced in Chapters 11 and 13. The principle of risk-neutral valuation shows that, if we assume that the world is risk neutral when valuing derivatives, we get the right

[7] To see this, note that the changes Δv and Δw in v and w in a short period of time Δt are given by

$$\Delta v = \cdots + \sum \sigma_{v,i}\, v\epsilon_i\sqrt{\Delta t}$$
$$\Delta w = \cdots + \sum \sigma_{w,i}\, w\epsilon_i\sqrt{\Delta t}$$

Since the dz_i are uncorrelated, it follows that $E(\epsilon_i\epsilon_j) = 0$ when $i \neq j$. Also, from the definition of ρ, we have

$$\rho v\sigma_v w\sigma_w = E(\Delta v\,\Delta w) - E(\Delta v)\,E(\Delta w)$$

When terms of higher order than Δt are ignored this leads to

$$\rho\sigma_v\sigma_w = \sum \sigma_{w,i}\,\sigma_{v,i}$$

answer—not just in a risk-neutral world, but in all other worlds as well. In the traditional risk-neutral world, the market price of risk of all variables is zero. Furthermore, the expected price of any asset in this world is its futures price.

In this chapter we have extended the principle of risk-neutral valuation. We have shown that, when interest rates are stochastic, there are many interesting and useful alternatives to the traditional risk-neutral world. When there is only one stochastic variable, a world is defined as forward risk neutral with respect to a security price if the market price of risk for the variable is set equal to the volatility of the security price. A similar definition applies when there are many stochastic variables.

A martingale is a zero drift stochastic process. Any variable following a martingale has the simplifying property that its expected value at any future time equals its value today. We have shown that in a world that is forward risk neutral with respect to a security price g, the ratio f/g is a martingale for all security prices f. It turns out that, by appropriately choosing the numeraire security g, we can simplify the valuation of many interest rate dependent derivatives.

In this chapter we have shown how our extensions of risk-neutral valuation enable European options to exchange one asset for another to be valued when interest rates are stochastic. In Chapters 26, 27, 29, and 30, the extensions will be useful in valuing interest rate derivatives.

FURTHER READING

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Questions and Problems (Answers in the Solutions Manual)

25.1. How is the market price of risk defined for a variable that is not the price of an investment asset?

25.2. Suppose that the market price of risk for gold is zero. If the storage costs are 1% per annum and the risk-free rate of interest is 6% per annum, what is the expected growth rate in the price of gold? Assume that gold provides no income.

25.3. Consider two securities both of which are dependent on the same market variable. The expected returns from the securities are 8% and 12%. The volatility of the first security is 15%. The instantaneous risk-free rate is 4%. What is the volatility of the second security?

25.4. An oil company is set up solely for the purpose of exploring for oil in a certain small area of Texas. Its value depends primarily on two stochastic variables: the price of oil and the quantity of proven oil reserves. Discuss whether the market price of risk for the second of these two variables is likely to be positive, negative, or zero.

25.5. Deduce the differential equation for a derivative dependent on the prices of two non-dividend-paying traded securities by forming a riskless portfolio consisting of the derivative and the two traded securities.

25.6. Suppose that an interest rate x follows the process

$$dx = a(x_0 - x)\,dt + c\sqrt{x}\,dz$$

where a, x_0, and c are positive constants. Suppose further that the market price of risk for x is λ. What is the process for x in the traditional risk-neutral world?

25.7. Prove that, when the security f provides income at rate q, equation (25.9) becomes $\mu + q - r = \lambda\sigma$. (*Hint*: Form a new security f^* that provides no income by assuming that all the income from f is reinvested in f.)

25.8. Show that when f and g provide income at rates q_f and q_g, respectively, equation (25.15) becomes

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

(*Hint*: Form new securities f^* and g^* that provide no income by assuming that all the income from f is reinvested in f and all the income in g is reinvested in g.)

25.9. "The expected future value of an interest rate in a risk-neutral world is greater than it is in the real world." What does this statement imply about the market price of risk for (a) an interest rate and (b) a bond price. Do you think the statement is likely to be true? Give reasons.

25.10. The variable S is an investment asset providing income at rate q measured in currency A. It follows the process

$$dS = \mu_S S\,dt + \sigma_S S\,dz$$

in the real world. Defining new variables as necessary, give the process followed by S, and the corresponding market price of risk, in:

(a) A world that is the traditional risk-neutral world for currency A
(b) A world that is the traditional risk-neutral world for currency B
(c) A world that is forward risk neutral with respect to a zero-coupon currency A bond maturing at time T
(d) A world that is forward risk neutral with respect to a zero coupon currency B bond maturing at time T.

25.11. Explain the difference between the way a forward interest rate is defined and the way the forward values of other variables such as stock prices, commodity prices, and exchange rates are defined.

25.12. Prove the result in Section 25.5 that when

$$df = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n} \sigma_{f,i} f\,dz_i$$

and

$$dg = \left[r + \sum_{i=1}^{n} \lambda_i \sigma_{g,i}\right] g\,dt + \sum_{i=1}^{n} \sigma_{g,i} g\,dz_i$$

with the dz_i uncorrelated, f/g is a martingale for $\lambda_i = \sigma_{g,i}$.

25.13. Prove equation (25.33) in Section 25.7.

25.14. Show that when $w = h/g$ and h and g are each dependent on n Wiener processes, the ith component of the volatility of w is the ith component of the volatility of h minus the ith component of the volatility of g. Use this to prove the result that if σ_U is the volatility of U and σ_V is the volatility of V then the volatility of U/V is $\sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$. (*Hint*: Use the result in Footnote 7.)

Assignment Questions

25.15. A security's price is positively dependent on two variables: the price of copper and the yen/dollar exchange rate. Suppose that the market price of risk for these variables is 0.5 and 0.1, respectively. If the price of copper were held fixed, the volatility of the security would be 8% per annum; if the yen/dollar exchange rate were held fixed, the volatility of the security would be 12% per annum. The risk-free interest rate is 7% per annum. What is the expected rate of return from the security? If the two variables are uncorrelated with each other, what is the volatility of the security?

25.16. Suppose that the price of a zero-coupon bond maturing at time T follows the process

$$dP(t, T) = \mu_P P(t, T)\, dt + \sigma_P P(t, T)\, dz$$

and the price of a derivative dependent on the bond follows the process

$$df = \mu_f f\, dt + \sigma_f f\, dz$$

Assume only one source of uncertainty and that f provides no income.

(a) What is the forward price F of f for a contract maturing at time T?

(b) What is the process followed by F in a world that is forward risk neutral with respect to $P(t, T)$?

(c) What is the process followed by F in the traditional risk-neutral world?

(d) What is the process followed by f in a world that is forward risk neutral with respect to a bond maturing at time T^*, where $T^* \neq T$? Assume that σ_P^* is the volatility of this bond.

25.17. Consider a variable that is not an interest rate:

(a) In what world is the futures price of the variable a martingale?

(b) In what world is the forward price of the variable a martingale?

(c) Defining variables as necessary, derive an expression for the difference between the drift of the futures price and the drift of the forward price in the traditional risk-neutral world.

(d) Show that your result is consistent with the points made in Section 5.8 about the circumstances when the futures price is above the forward price.

APPENDIX

HANDLING MULTIPLE SOURCES OF UNCERTAINTY

In this appendix we extend Itô's lemma to cover situations where there are multiple sources of uncertainty and prove the result in equation (25.13) relating the excess return to market prices of risk when there are multiple sources of uncertainty.

Itô's Lemma for a Function of Several Variables

Itô's lemma, as presented in the appendix to Chapter 12, provides the process followed by a function of a single stochastic variable. Here we present a generalized version of Itô's lemma for the process followed by a function of several stochastic variables.

Suppose that a function f depends on the n variables $x_1, x_2, \ldots, x_n$ and time t. Suppose further that x_i follows an Itô process with instantaneous drift a_i and instantaneous variance b_i^2 $(1 \leqslant i \leqslant n)$, that is,

$$dx_i = a_i\, dt + b_i\, dz_i \tag{25A.1}$$

where dz_i $(1 \leqslant i \leqslant n)$ is a Wiener process. Each a_i and b_i may be any function of all the x_i and t. A Taylor series expansion of Δf gives

$$\Delta f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i + \frac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\, \partial x_j}\Delta x_i\, \Delta x_j + \tfrac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i\, \partial t}\Delta x_i\, \Delta t + \cdots \tag{25A.2}$$

Equation (25A.1) can be discretized as

$$\Delta x_i = a_i\, \Delta t + b_i \epsilon_i \sqrt{\Delta t}$$

where ϵ_i is a random sample from a standardized normal distribution. The correlation ρ_{ij} between dz_i and dz_j is defined as the correlation between ϵ_i and ϵ_j. In the appendix to Chapter 12 it was argued that

$$\lim_{\Delta t \to 0} \Delta x_i^2 = b_i^2\, dt$$

Similarly,

$$\lim_{\Delta t \to 0} \Delta x_i\, \Delta x_j = b_i b_j \rho_{ij}\, dt$$

As $\Delta t \to 0$, the first three terms in the expansion of Δf in equation (25A.2) are of order Δt. All other terms are of higher order. Hence,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial t} dt + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\, \partial x_j} b_i b_j \rho_{ij}\, dt$$

This is the generalized version of Itô's lemma. Substituting for dx_i from equation (25A.1) gives

$$df = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial t} + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\, \partial x_j} b_i b_j \rho_{ij}\right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} b_i\, dz_i \tag{25A.3}$$

For an alternative generalization of Itô's lemma, suppose that f depends on a single variable x and that the process for x involves more than one Wiener process:

$$dx = a\,dt + \sum_{i=1}^{m} b_i\,dz_i$$

In this case,

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}\Delta x^2 + \tfrac{1}{2}\frac{\partial^2 f}{\partial x\,\partial t}\Delta x\,\Delta t + \cdots$$

$$\Delta x = a\,\Delta t + \sum_{i=1}^{m} b_i \epsilon_i \sqrt{\Delta t}$$

and

$$\lim_{\Delta t \to 0} \Delta x_i^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} b_i b_j \rho_{ij}\,dt$$

where, as before, ρ_{ij} is the correlation between dz_i and dz_j This leads to

$$df = \left(\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial t} + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}\sum_{i=1}^{m}\sum_{j=1}^{m} b_i b_j \rho_{ij}\right) dt + \frac{\partial f}{\partial x}\sum_{i=1}^{m} b_i\,dz_i \tag{25.4}$$

Finally, consider the more general case where f depends on variables x_i $(1 \leqslant i \leqslant n)$ and

$$dx_i = a_i\,dt + \sum_{k=1}^{m} b_{ik}\,dz_k$$

A similar analysis shows that

$$df = \left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}a_i + \frac{\partial f}{\partial t} + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial dx_j}\sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik} b_{jl} \rho_{kl}\right) dt + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\sum_{k=1}^{m} b_{ik}\,dz_k \tag{25A.5}$$

The Return for a Security Dependent on Multiple Sources of Uncertainty

In Section 25.1 we prove a result relating return to risk when there is one source of uncertainty. We now prove the result in equation (25.13) for the situation where there are multiple sources of uncertainty.

Suppose that there are n stochastic variables following Wiener processes. Consider $n+1$ traded securities whose prices depend on some or all of the n stochastic variables. Define f_j as the price of the jth security $(1 \leqslant j \leqslant n+1)$. We assume that no dividends or other income is paid by the $n+1$ traded securities.[8] It follows from the previous

[8] This is not restrictive. A non-dividend-paying security can always be obtained from a dividend-paying security by reinvesting the dividends in the security.

section that the securities follow processes of the form

$$df_j = \mu_j f_j \, dt + \sum_{i=1}^{n} \sigma_{ij} f_j \, dz_i \tag{25A.6}$$

Since there are $n+1$ traded securities and n Wiener processes, it is possible to form an instantaneously riskless portfolio Π using the securities. Define k_j as the amount of the jth security in the portfolio, so that

$$\Pi = \sum_{j=1}^{n+1} k_j f_j \tag{25A.7}$$

The k_j must be chosen so that the stochastic components of the returns from the securities are eliminated. From equation (25A.6), this means that

$$\sum_{j=1}^{n+1} k_j \sigma_{ij} f_j = 0 \tag{25A.8}$$

for $1 \leqslant i \leqslant n$. The return from the portfolio is then given by

$$d\Pi = \sum_{j=1}^{n+1} k_j \mu_j f_j \, dt$$

The cost of setting up the portfolio is

$$\sum_{j=1}^{n+1} k_j f_j$$

If there are no arbitrage opportunities, the portfolio must earn the risk-free interest rate, so that

$$\sum_{j=1}^{n+1} k_j \mu_j f_j = r \sum_{j=1}^{n+1} k_j f_j \tag{25A.9}$$

or

$$\sum_{j=1}^{n+1} k_j f_j (\mu_j - r) = 0 \tag{25A.10}$$

Equations (25A.8) and (25A.10) can be regarded as $n+1$ homogeneous linear equations in the k_j. The k_j are not all zero. From a well-known theorem in linear algebra, equations (25A.8) and (25A.10) can be consistent only if, for all j,

$$f_j(\mu_j - r) = \sum_{i=1}^{n} \lambda_i \sigma_{ij} f_j \tag{25A.11}$$

or

$$\mu_j - r = \sum_{i=1}^{n} \lambda_i \sigma_{ij} \tag{25A.12}$$

for some λ_i $(1 \leqslant i \leqslant n)$ that are dependent only on the state variables and time. Dropping the j subscript, this shows that, for any security f dependent on the n

stochastic variables,

$$df = \mu f\,dt + \sum_{i=1}^{n} \sigma_i f\,dz_i$$

where

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

This proves the result in equation (25.13).