CHAPTER 17

Basic Numerical Procedures

This chapter discusses three numerical procedures for valuing derivatives when exact formulas are not available. The first involves representing the asset price movements in the form of a tree and was introduced in Chapter 11. The second involves Monte Carlo simulation, which we encountered briefly in Chapter 12 when explaining stochastic processes. The third involves finite difference methods.

Monte Carlo simulation is usually used for derivatives where the payoff is dependent on the history of the underlying variable or where there are several underlying variables. Trees and finite difference methods are usually used for American options and other derivatives where the holder has early exercise decisions to make prior to maturity. In addition to valuing a derivative, all the procedures can be used to calculate Greek letters such as delta, gamma, and vega.

The basic procedures we discuss in this chapter can be used to handle most of the derivatives valuations problems that are encountered in practice. However, sometimes they have to be adapted to cope with particular situations. We discuss this in Chapter 24.

17.1 BINOMIAL TREES

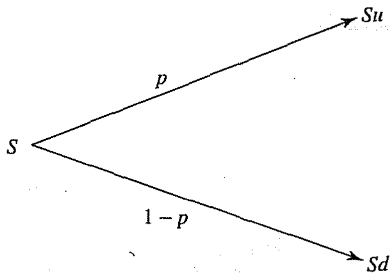
We introduced binomial trees in Chapter 11. They can be used to value either European or American options. The Black–Scholes formulas and their extensions that we presented in Chapters 13 and 14 provide analytic valuations for European options.[1] There are no analytic valuations for American options. Binomial trees are therefore most useful for valuing these types of options.[2]

As explained in Chapter 11, the binomial tree valuation approach involves dividing the life of the option into a large number of small time intervals of length Δt. It assumes that in each time interval the price of the underlying asset moves from its initial value of

[1] The Black–Scholes formulas are based on the same set of assumptions as binomial trees. As one might expect, in the limit as the number of time steps is increased, the price given for a European option by the binomial method converges to the Black–Scholes price.

[2] Some analytic approximations for valuing American options have been suggested. The most well-known one is the quadratic approximation approach. See Technical Note 8 on the author's website for a description of this approach.

Figure 17.1 Asset price movements in time Δt under the binomial model.



S to one of two new values, Su and Sd. The approach is illustrated in Figure 17.1. In general, $u > 1$ and $d < 1$. The movement from S to Su, therefore, is an "up" movement and the movement from S to Sd is a "down" movement. The probability of an up movement will be denoted by p. The probability of a down movement is $1 - p$.

Risk-Neutral Valuation

The risk-neutral valuation principle, explained in Chapters 11 and 13, states that an option (or other derivative) can be valued on the assumption that the world is risk neutral. This means that for valuation purposes we can use the following procedure:

1. Assume that the expected return from all traded assets is the risk-free interest rate.
2. Value payoffs from the derivative by calculating their expected values and discounting at the risk-free interest rate.

This principle is a key element of the ways in which trees are used.

Determination of p, u, and d

The parameters p, u, and d must give correct values for the mean and variance of asset price changes during a time interval of length Δt. Because we are working in a risk-neutral world, the expected return from the asset is the risk-free interest rate, r. Suppose that the asset provides a yield of q. The expected return in the form of capital gains must be $r - q$. This means that the expected value of the asset price at the end of a time interval of length Δt must be $Se^{(r-q)\Delta t}$, where S is the stock price at the beginning of the time interval. It follows that

$$Se^{(r-q)\Delta t} = pSu + (1-p)Sd \tag{17.1}$$

or

$$e^{(r-q)\Delta t} = pu + (1-p)d \tag{17.2}$$

As explained in Section 13.4, the variance of the percentage change in the stock price in a small time interval of length Δt is $\sigma^2 \Delta t$. The variance of a variable Q is defined as $E(Q^2) - [E(Q)]^2$. There is a probability p that the percentage change is u and $1 - p$ that it is d. The expected percentage change is $e^{(r-q)\Delta t}$. It follows that

$$pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t} = \sigma^2 \Delta t$$

Substituting for p from equation (17.2) gives

$$e^{(r-q)\Delta t}(u+d) - ud - e^{2(r-q)\Delta t} = \sigma^2 \Delta t \tag{17.3}$$

Equations (17.2) and (17.3) impose two conditions on p, u, and d. A third condition used by Cox, Ross, and Rubinstein (1979) is[3]

$$u = \frac{1}{d}$$

A solution to equations (17.2) and (17.3) when terms of higher order than Δt are ignored is[4]

$$p = \frac{a-d}{u-d} \tag{17.4}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{17.5}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{17.6}$$

where

$$a = e^{(r-q)\Delta t} \tag{17.7}$$

The variable a is sometimes referred to as the *growth factor*. Equations (17.4) to (17.7) are the same as those in Section 11.9.

Tree of Asset Prices

Figure 17.2 illustrates the complete tree of asset prices that is considered when the binomial model is used. At time zero, the asset price, S_0, is known. At time Δt, there are two possible asset prices, S_0u and S_0d; at time $2\Delta t$, there are three possible asset prices, S_0u^2, S_0, and S_0d^2; and so on. In general, at time $i\,\Delta t$, we consider $i+1$ asset prices. These are

$$S_0u^jd^{i-j}, \quad j = 0, 1, \dots, i$$

Note that the relationship $u = 1/d$ is used in computing the asset price at each node of the tree in Figure 17.2. For example, $S_0u^2d = S_0u$. Note also that the tree recombines in the sense that an up movement followed by a down movement leads to the same asset price as a down movement followed by an up movement.

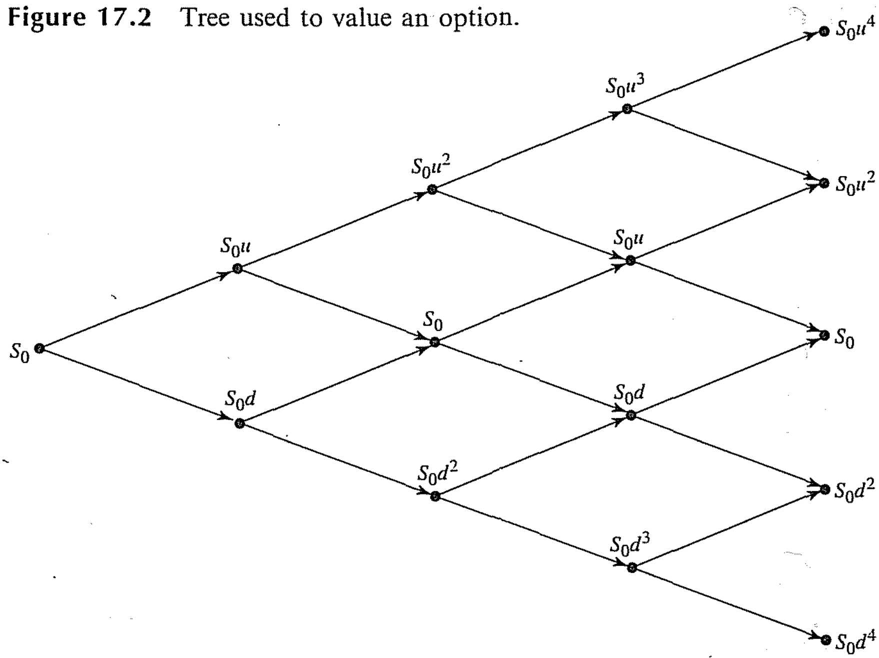
Working Backward through the Tree

Options are evaluated by starting at the end of the tree (time T) and working backward. The value of the option is known at time T. For example, a put option is worth $\max(K - S_T,\ 0)$ and a call option is worth $\max(S_T - K,\ 0)$, where S_T is the asset price at time T and K is the strike price. Because a risk-neutral world is being assumed, the

[3] See J.C. Cox, S.A. Ross, and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7 (October 1979), 229–63.

[4] To see this, we note that equations (17.4) and (17.7) satisfy the condition in equation (17.2) exactly. The exponential function e^x can be expanded as $1 + x + x^2/2 + \cdots$. When terms of higher order than Δt are ignored, equation (17.5) implies that $u = 1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\,\Delta t$ and equation (17.6) implies that $d = 1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\,\Delta t$. Also, $e^{(r-q)\Delta t} = 1 + (r-q)\Delta t$ and $e^{2(r-q)\Delta t} = 1 + 2(r-q)\Delta t$. By substitution we see that equation (17.3) is satisfied when terms of higher order than Δt are ignored.

Figure 17.2 Tree used to value an option.



value at each node at time $T - \Delta t$ can be calculated as the expected value at time T discounted at rate r for a time period Δt. Similarly, the value at each node at time $T - 2\Delta t$ can be calculated as the expected value at time $T - \Delta t$ discounted for a time period Δt at rate r, and so on. If the option is American, it is necessary to check at each node to see whether early exercise is preferable to holding the option for a further time period Δt. Eventually, by working back through all the nodes, we are able to obtain the value of the option at time zero.

Example 17.1

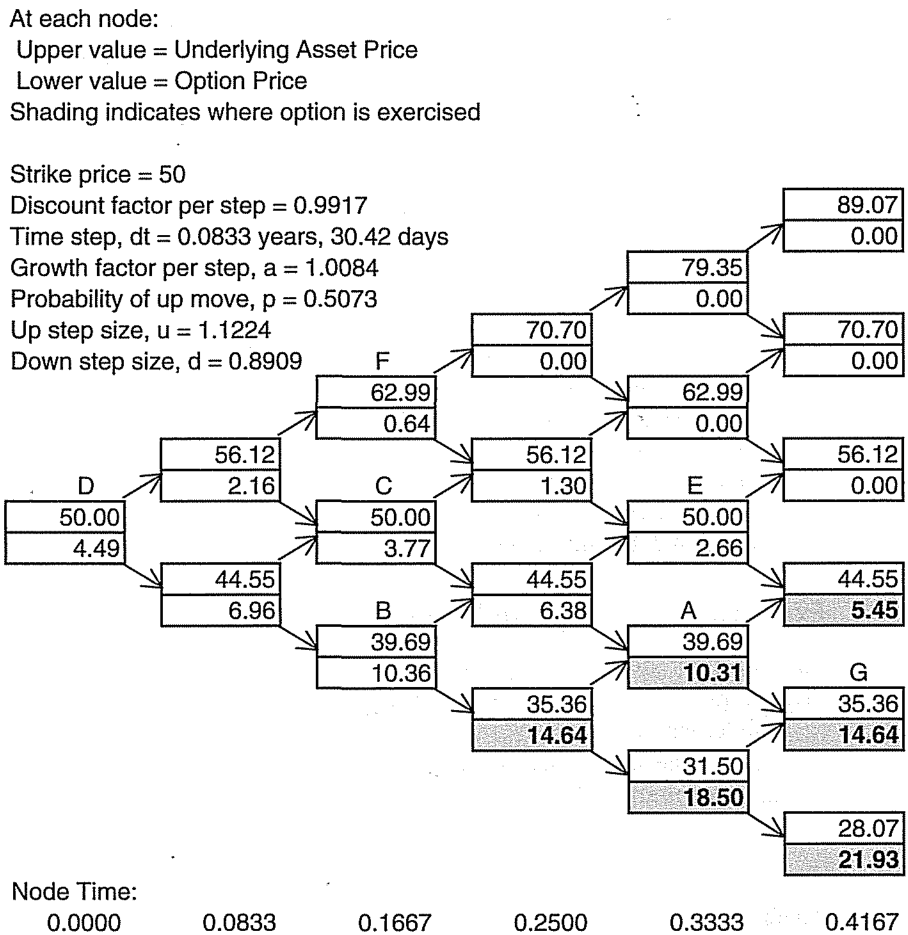
Consider a 5-month American put option on a non-dividend-paying stock when the stock price is \$50, the strike price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 40% per annum. With our usual notation, this means that $S_0 = 50$, $K = 50$, $r = 0.10$, $\sigma = 0.40$, $T = 0.4167$, and $q = 0$. Suppose that we divide the life of the option into five intervals of length 1 month (= 0.0833 year) for the purposes of constructing a binomial tree. Then $\Delta t = 0.0833$ and, using equations (17.4) to (17.7), we have

$$u = e^{\sigma\sqrt{\Delta t}} = 1.1224, \qquad d = e^{-\sigma\sqrt{\Delta t}} = 0.8909, \qquad a = e^{r\Delta t} = 1.0084$$

$$p = \frac{a-d}{u-d} = 0.5073, \qquad 1 - p = 0.4927$$

Figure 17.3 shows the binomial tree produced by DerivaGem. At each node there are two numbers. The top one shows the stock price at the node; the lower one shows the value of the option at the node. The probability of an up movement is always 0.5073; the probability of a down movement is always 0.4927.

Figure 17.3 Binomial tree from DerivaGem for American put on non-dividend-paying stock (Example 17.1).



The stock price at the jth node ($j = 0, 1, \ldots, i$) at time $i\,\Delta t$ ($i = 0, 1, \ldots, 5$) is calculated as $S_0 u^j d^{i-j}$. For example, the stock price at node A ($i = 4$, $j = 1$) (i.e., the second node up at the end of the fourth time step) is $50 \times 1.1224 \times 0.8909^3 =$ \$39.69. The option prices at the final nodes are calculated as $\max(K - S_T,\ 0)$. For example, the option price at node G is $50.00 - 35.36 = 14.64$. The option prices at the penultimate nodes are calculated from the option prices at the final nodes. First, we assume no exercise of the option at the nodes. This means that the option price is calculated as the present value of the expected option price one time step later. For example, at node E, the option price is calculated as

$$(0.5073 \times 0 + 0.4927 \times 5.45)e^{-0.10\times 0.0833} = 2.66$$

whereas at node A it is calculated as

$$(0.5073 \times 5.45 + 0.4927 \times 14.64)e^{-0.10\times 0.0833} = 9.90$$

We then check to see if early exercise is preferable to waiting. At node E, early exercise would give a value for the option of zero because both the stock price and strike price are \$50. Clearly it is best to wait. The correct value for the option at node E, therefore, is \$2.66. At node A, it is a different story. If the option is exercised, it is worth \$50.00 – \$39.69, or \$10.31. This is more than \$9.90. If node A is reached, then the option should be exercised and the correct value for the option at node A is \$10.31.

Option prices at earlier nodes are calculated in a similar way. Note that it is not always best to exercise an option early when it is in the money. Consider node B. If the option is exercised, it is worth \$50.00 – \$39.69, or \$10.31. However, if it is held, it is worth

$$(0.5073 \times 6.38 + 0.4927 \times 14.64)e^{-0.10\times 0.0833} = 10.36$$

The option should, therefore, not be exercised at this node, and the correct option value at the node is \$10.36.

Working back through the tree, the value of the option at the initial node is \$4.49. This is our numerical estimate for the option's current value. In practice, a smaller value of Δt, and many more nodes, would be used. DerivaGem shows that with 30, 50, 100, and 500 time steps we get values for the option of 4.263, 4.272, 4.278, and 4.283.

Expressing the Approach Algebraically

Suppose that the life of an American put option on a non-dividend-paying stock is divided into N subintervals of length Δt. We will refer to the jth node at time $i\,\Delta t$ as the (i, j) node, where $0 \leqslant i \leqslant N$ and $0 \leqslant j \leqslant i$). Define $f_{i,j}$ as the value of the option at the (i, j) node. The stock price at the (i, j) node is $S_0 u^j d^{i-j}$. Since the value of an American put at its expiration date is $\max(K - S_T, 0)$, we know that

$$f_{N,j} = \max(K - S_0 u^j d^{N-j}, 0), \quad j = 0, 1, \dots, N$$

There is a probability p of moving from the (i, j) node at time $i\,\Delta t$ to the $(i+1, j+1)$ node at time $(i+1)\,\Delta t$, and a probability $1-p$ of moving from the (i, j) node at time $i\,\Delta t$ to the $(i+1, j)$ node at time $(i+1)\,\Delta t$. Assuming no early exercise, risk-neutral valuation gives

$$f_{i,j} = e^{-r\Delta t}[p f_{i+1,j+1} + (1-p) f_{i+1,j}]$$

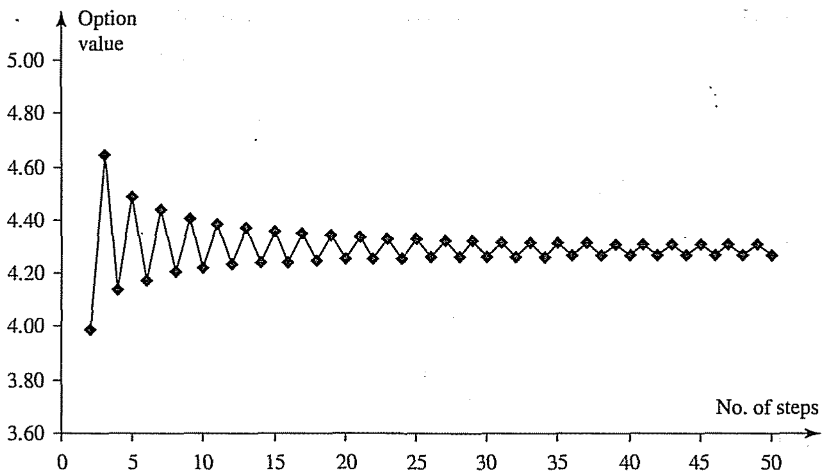
for $0 \leqslant i \leqslant N-1$ and $0 \leqslant j \leqslant i$. When early exercise is taken into account, this value for $f_{i,j}$ must be compared with the option's intrinsic value, and we obtain

$$f_{i,j} = \max\{K - S_0 u^j d^{i-j},\ e^{-r\Delta t}[p f_{i+1,j+1} + (1-p) f_{i+1,j}]\}$$

Note that, because the calculations start at time T and work backward, the value at time $i\,\Delta t$ captures not only the effect of early exercise possibilities at time $i\,\Delta t$, but also the effect of early exercise at subsequent times.

In the limit as Δt tends to zero, an exact value for the American put is obtained. In practice, $N = 30$ usually gives reasonable results. Figure 17.4 shows the convergence of the option price in the example we have been considering. This figure was calculated

Figure 17.4 Convergence of the price of the option in Example 17.1 calculated from the DerivaGem Application Builder functions.



using the Application Builder functions provided with the DerivaGem software (see Sample Application A).

Estimating Delta and Other Greek Letters

It will be recalled that the delta (Δ) of an option is the rate of change of its price with respect to the underlying stock price. It can be calculated as

$$\frac{\Delta f}{\Delta S}$$

where ΔS is a small change in the stock price and Δf is the corresponding small change in the option price. At time Δt, we have an estimate f_{11} for the option price when the stock price is S_0u and an estimate f_{10} for the option price when the stock price is S_0d. In other words, when $\Delta S = S_0u - S_0d$, $\Delta f = f_{11} - f_{10}$. Therefore an estimate of delta at time Δt is

$$\Delta = \frac{f_{11} - f_{10}}{S_0u - S_0d} \tag{17.8}$$

To determine gamma (Γ), note that we have two estimates of Δ at time $2\Delta t$. When $S = (S_0u^2 + S_0)/2$ (halfway between the second and third node), delta is $(f_{22} - f_{21})/(S_0u^2 - S_0)$; when $S = (S_0 + S_0d^2)/2$ (halfway between the first and second node), delta is $(f_{21} - f_{20})/(S_0 - S_0d^2)$. The difference between the two values of S is h, where

$$h = 0.5(S_0u^2 - S_0d^2)$$

Gamma is the change in delta divided by h:

$$\Gamma = \frac{[(f_{22} - f_{21})/(S_0u^2 - S_0)] - [(f_{21} - f_{20})/(S_0 - S_0d^2)]}{h} \tag{17.9}$$

These procedures provide estimates of delta at time Δt and of gamma at time $2\Delta t$. In practice, they are usually used as estimates of delta and gamma at time zero as well.[5]

A further hedge parameter that can be obtained directly from the tree is theta (Θ). This is the rate of change of the option price with time when all else is kept constant. If the tree starts at time zero, an estimate of theta is

$$\Theta = \frac{f_{21} - f_{00}}{2\Delta t} \tag{17.10}$$

Vega can be calculated by making a small change, $\Delta\sigma$, in the volatility and constructing a new tree to obtain a new value of the option. (The time step Δt should be kept the same.) The estimate of vega is

$$\mathcal{V} = \frac{f^* - f}{\Delta\sigma}$$

where f and f^* are the estimates of the option price from the original and the new tree, respectively. Rho can be calculated similarly.

Example 17.2

Consider again Example 17.1. From Figure 17.3, we have $f_{1,0} = 6.96$ and $f_{1,1} = 2.16$. Equation (17.8) gives an estimate for delta of

$$\frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$

From equation (17.9), an estimate of the gamma of the option can be obtained from the values at nodes B, C, and F as

$$\frac{[(0.64 - 3.77)/(62.99 - 50.00)] - [(3.77 - 10.36)/(50.00 - 39.69)]}{11.65} = 0.03$$

From equation (17.10), an estimate of the theta of the option can be obtained from the values at nodes D and C as

$$\frac{3.77 - 4.49}{0.1667} = -4.3 \quad \text{per year}$$

or -0.012 per calendar day. These are only rough estimates. They become progressively better as the number of time steps on the tree is increased. Using 50 time steps, DerivaGem provides estimates of -0.415, 0.034, and -0.0117 for delta, gamma, and theta, respectively. By making small changes to parameters and recomputing values, vega and rho are estimated as 0.123 and -0.072, respectively.

17.2 USING THE BINOMIAL TREE FOR OPTIONS ON INDICES, CURRENCIES, AND FUTURES CONTRACTS

As explained in Chapters 11 and 14, stock indices, currencies, and futures contracts can, for the purposes of option valuation, be considered as assets providing known yields. In

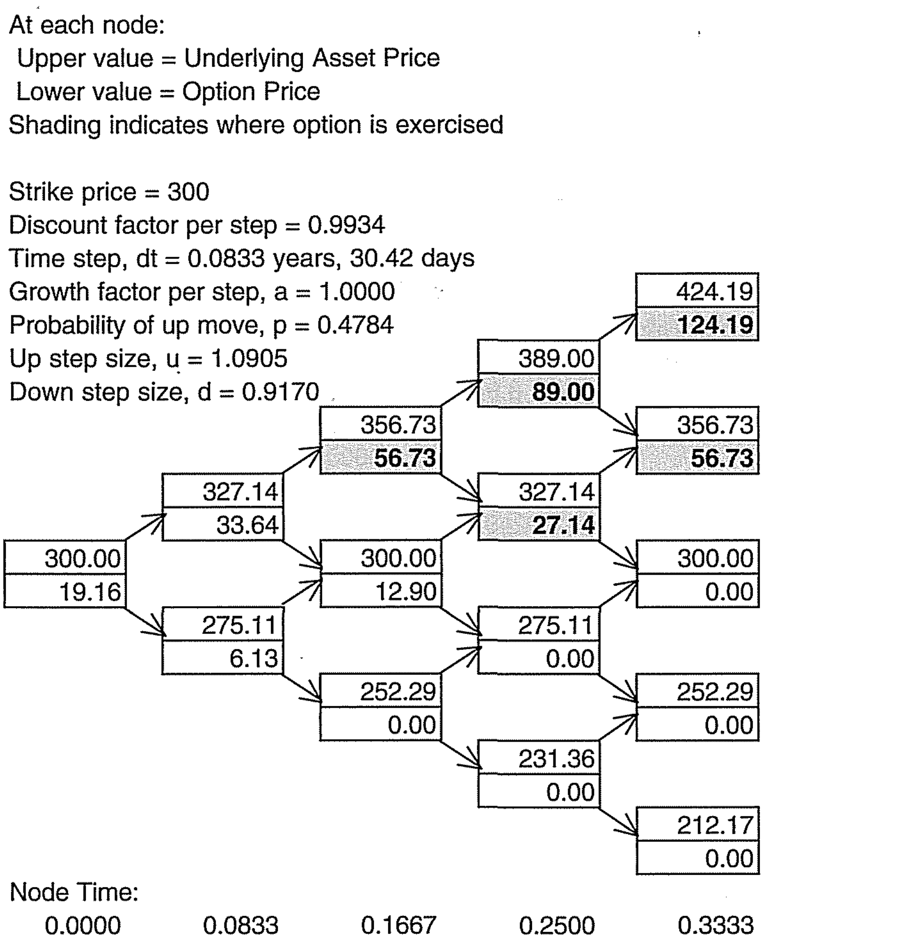
[5] If slightly more accuracy is required for delta and gamma, we can start the binomial tree at time $-2\Delta t$ and assume that the stock price is S_0 at this time. This leads to the option price being calculated for three different stock prices at time zero.

the case of a stock index, the relevant yield is the dividend yield on the stock portfolio underlying the index; in the case of a currency, it is the foreign risk-free interest rate; in the case of a futures contract, it is the domestic risk-free interest rate. The binomial tree approach can therefore be used to value options on stock indices, currencies, and futures contracts provided that q in equation (17.7) is interpreted appropriately.

Example 17.3

Consider a 4-month American call option on index futures where the current futures price is 300, the exercise price is 300, the risk-free interest rate is 8% per annum, and the volatility of the index is 30% per annum. We divide the life of the option into four 1-month periods for the purposes of constructing the tree. In this case, $F_0 = 300$, $K = 300$, $r = 0.08$, $\sigma = 0.3$, $T = 0.3333$, and $\Delta t = 0.0833$. Because a futures contract is analogous to a stock paying dividends at a rate r, q should be set equal to r in equation (17.7). This gives $a = 1$. The other parameters

Figure 17.5 Binomial tree produced by DerivaGem for American call option on an index futures contract (Example 17.3).



necessary to construct the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = 1.0905, \qquad d = \frac{1}{u} = 0.9170$$

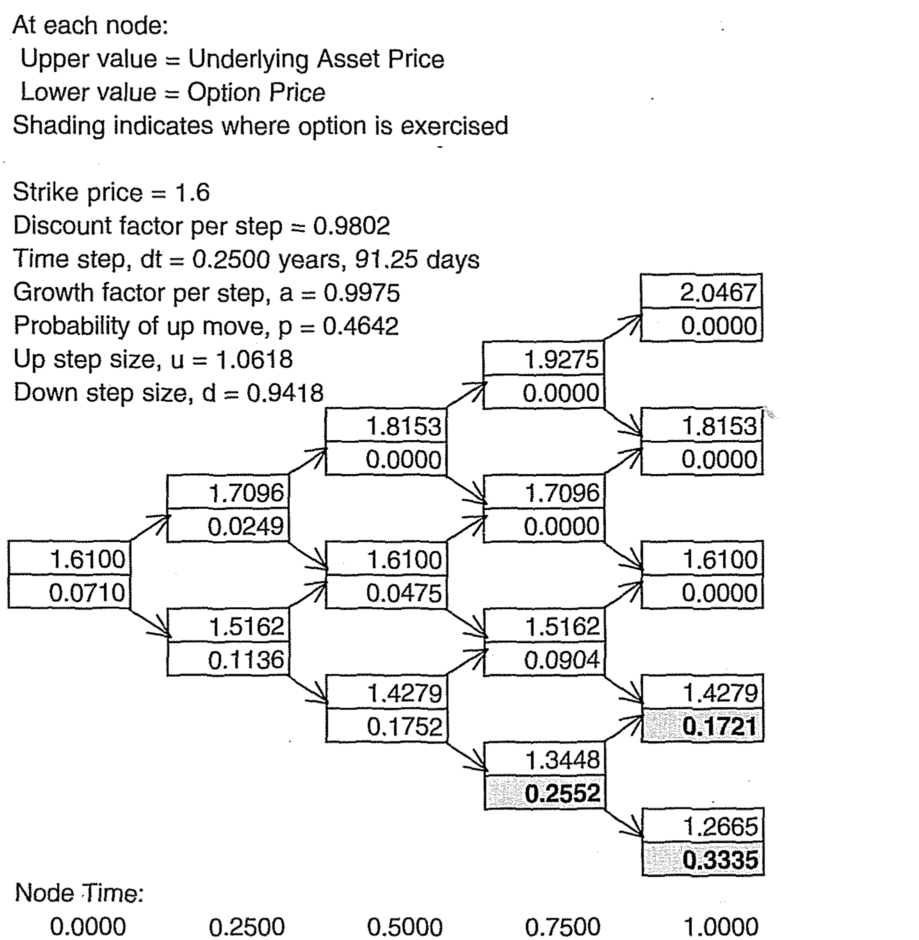
$$p = \frac{a-d}{u-d} = 0.4784, \qquad 1 - p = 0.5216$$

The tree, as produced by DerivaGem, is shown in Figure 17.5. (The upper number is the futures price; the lower number is the option price.) The estimated value of the option is 19.16. More accuracy is obtained using more steps. With 50 time steps, DerivaGem gives a value of 20.18; with 100 time steps it gives 20.22.

Example 17.4

Consider a 1-year American put option on the British pound. The current exchange rate is 1.6100, the strike price is 1.6000, the US risk-free interest rate is 8%

Figure 17.6 Binomial tree produced by DerivaGem for American put option on a currency (Example 17.4).



per annum, the sterling risk-free interest rate is 9% per annum, and the volatility of the sterling exchange rate is 12% per annum. In this case, $S_0 = 1.61$, $K = 1.60$, $r = 0.08$, $r_f = 0.09$, $\sigma = 0.12$, and $T = 1.0$. We divide the life of the option into four 3-month periods for the purposes of constructing the tree, so that $\Delta t = 0.25$. In this case, $q = r_f$ and equation (17.7) gives

$$a = e^{(0.08-0.09)\times 0.25} = 0.9975$$

The other parameters necessary to construct the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = 1.0618, \qquad d = \frac{1}{u} = 0.9418$$

$$p = \frac{a-d}{u-d} = 0.4642, \qquad 1 - p = 0.5358$$

The tree, as produced by DerivaGem, is shown in Figure 17.6. (The upper number is the exchange rate; the lower number is the option price.) The estimated value of the option is \$0.0710. (Using 50 time steps, DerivaGem gives the value of the option as 0.0738; with 100 time steps it also gives 0.0738.)

17.3 BINOMIAL MODEL FOR A DIVIDEND-PAYING STOCK

We now move on to the more tricky issue of how the binomial model can be used for a dividend-paying stock. As in Chapter 13, the word *dividend* will, for the purposes of our discussion, be used to refer to the reduction in the stock price on the ex-dividend date as a result of the dividend.

Known Dividend Yield

If it is assumed that there is a single dividend, and the dividend yield (i.e., the dividend as a percentage of the stock price) is known, the tree takes the form shown in Figure 17.7 and can be analyzed in similar manner to that just described. If the time $i\,\Delta t$ is prior to the stock going ex-dividend, the nodes on the tree correspond to stock prices

$$S_0 u^j d^{i-j}, \quad j = 0, 1, \dots, i$$

where u and d are defined as in equations (17.5) and (17.6). If the time $i\,\Delta t$ is after the stock goes ex-dividend, the nodes correspond to stock prices

$$S_0(1-\delta)u^j d^{i-j}, \quad j = 0, 1, \dots, i$$

where δ is the dividend yield. Several known dividend yields during the life of an option can be dealt with similarly. If δ_i is the total dividend yield associated with all ex-dividend dates between time zero and time $i\,\Delta t$, the nodes at time $i\,\Delta t$ correspond to stock prices

$$S_0(1-\delta_i)u^j d^{i-j}$$

Known Dollar Dividend

In some situations, the most realistic assumption is that the dollar amount of the dividend rather than the dividend yield is known in advance. If the volatility of the stock, σ, is assumed constant, the tree then takes the form shown in Figure 17.8. It does not recombine, which means that the number of nodes that have to be evaluated, particularly if there are several dividends, is liable to become very large. Suppose that there is only one dividend, that the ex-dividend date, τ, is between $k\,\Delta t$ and $(k+1)\,\Delta t$, and that the dollar amount of the dividend is D. When $i \leqslant k$, the nodes on the tree at time $i\,\Delta t$ correspond to stock prices

$$S_0 u^j d^{i-j}, \quad j = 0, 1, 2, \ldots, i$$

as before. When $i = k+1$, the nodes on the tree correspond to stock prices

$$S_0 u^j d^{i-j} - D, \quad j = 0, 1, 2, \ldots, i$$

When $i = k+2$, the nodes on the tree correspond to stock prices

$$(S_0 u^j d^{i-1-j} - D)u \quad \text{and} \quad (S_0 u^j d^{i-1-j} - D)d$$

for $j = 0, 1, 2, \ldots, i-1$, so that there are $2i$ rather than $i+1$ nodes. When $i = k+m$, there are $m(k+2)$ rather than $k+m+1$ nodes.

The problem can be simplified by assuming, as in the analysis of European options in Section 13.12, that the stock price has two components: a part that is uncertain and a part that is the present value of all future dividends during the life of the option. Suppose, as before, that there is only one ex-dividend date, τ, during the life of the

Figure 17.7 Tree when stock pays a known dividend yield at one particular time.

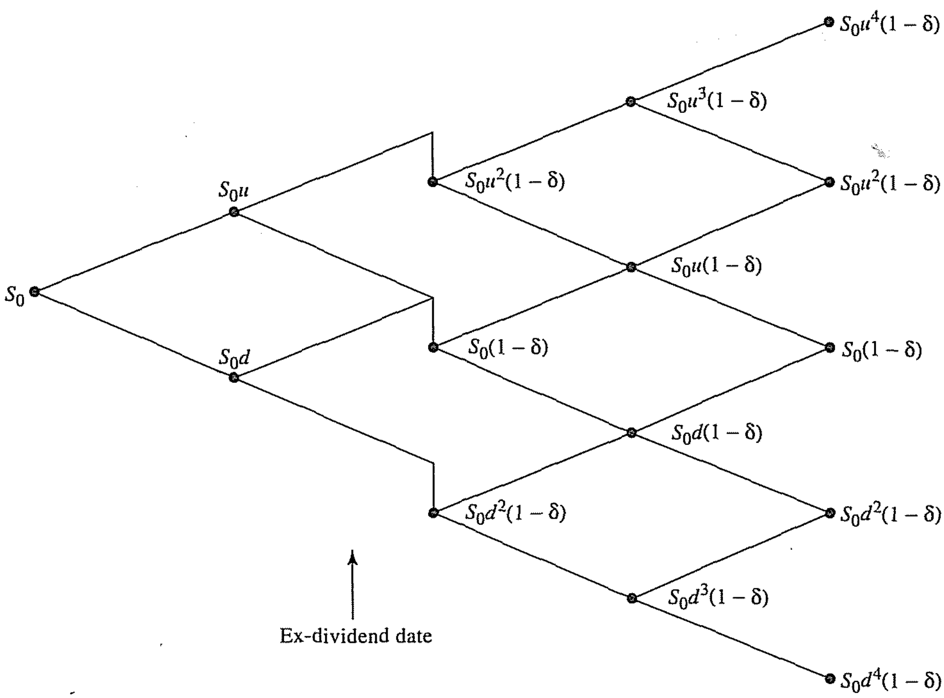
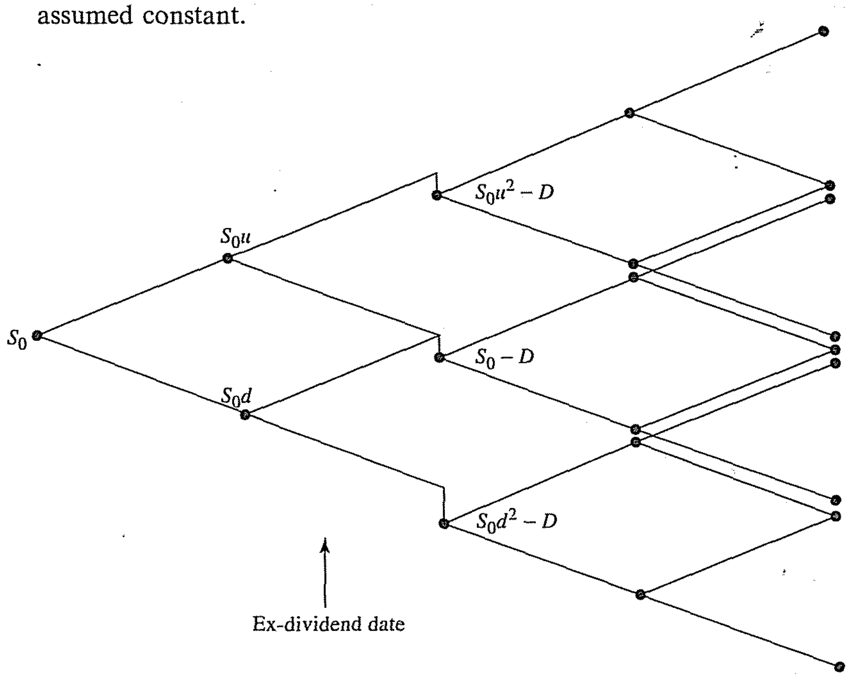


Figure 17.8 Tree when dollar amount of dividend is assumed known and volatility is assumed constant.



option and that $k\,\Delta t \leqslant \tau \leqslant (k+1)\,\Delta t$. The value of the uncertain component, S^*, at time $i\,\Delta t$ is given by

$$S^* = S \quad \text{when } i\,\Delta t > \tau$$

and

$$S^* = S - De^{-r(\tau - i\Delta t)} \quad \text{when } i\,\Delta t \leqslant \tau$$

where D is the dividend. Define σ^* as the volatility of S^* and assume that σ^* is constant.[6] The parameters p, u, and d can be calculated from equations (17.4), (17.5), (17.6), and (17.7) with σ replaced by σ^* and a tree can be constructed in the usual way to model S^*. By adding to the stock price at each node, the present value of future dividends (if any), the tree can be converted into another tree that models S. Suppose that S_0^* is the value of S^* at time zero. At time $i\,\Delta t$, the nodes on this tree correspond to the stock prices

$$S_0^* u^j d^{i-j} + De^{-r(\tau - i\,\Delta t)}, \quad j = 0, 1, \dots, i$$

when $i\,\Delta t < \tau$ and

$$S_0^* u^j d^{i-j}, \quad j = 0, 1, \dots, i$$

when $i\,\Delta t > \tau$. This approach, which has the advantage of being consistent with the approach for European options in Section 13.12, succeeds in achieving a situation where

[6] As mentioned in Section 12.13, σ^* is in theory slightly greater than σ, the volatility of S. In practice, the use of implied volatilities avoids the need for analysts to distinguish between σ and σ^*.

the tree recombines so that there are $i + 1$ nodes at time $i\,\Delta t$. It can be generalized in a straightforward way to deal with the situation where there are several dividends.

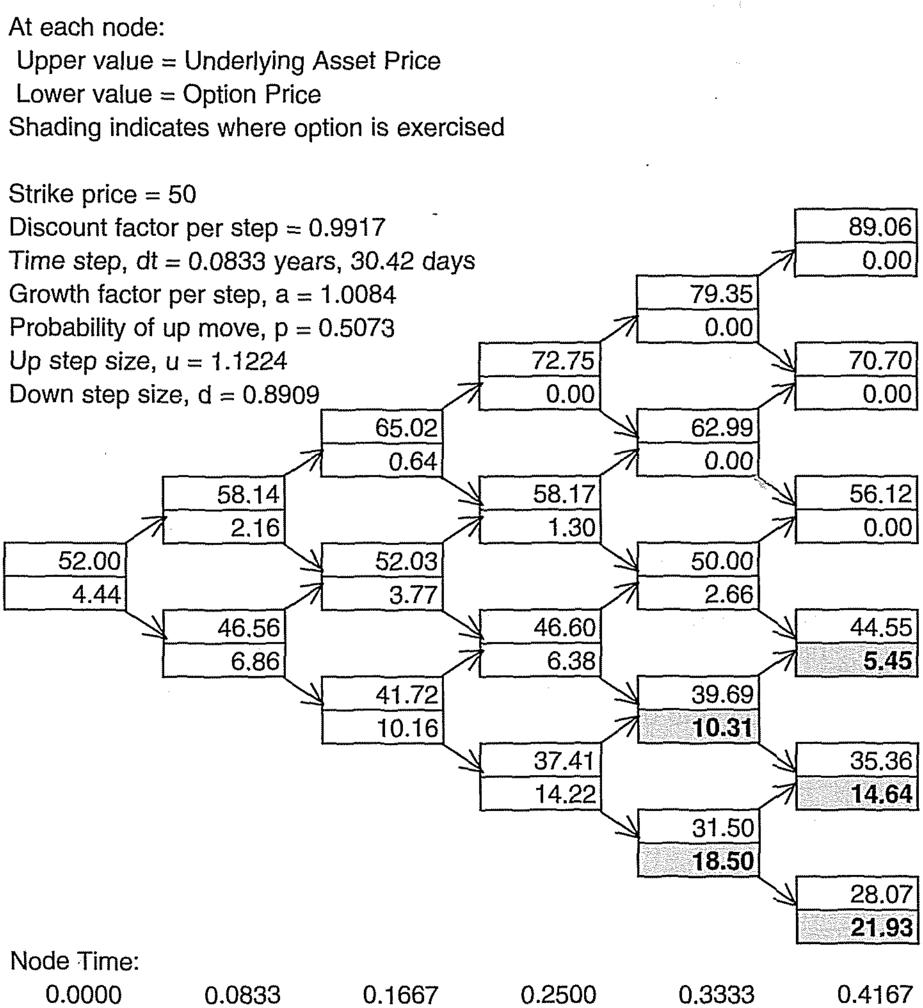
Example 17.5

Consider a 5-month American put option on a stock that is expected to pay a single dividend of \$2.06 during the life of the option. The initial stock price is \$52, the strike price is \$50, the risk-free interest rate is 10% per annum, the volatility is 40% per annum, and the ex-dividend date is in $3\frac{1}{2}$ months.

We first construct a tree to model S^*, the stock price less the present value of future dividends during the life of the option. At time zero, the present value of the dividend is

$$2.06e^{-0.2917\times 0.1} = 2.00$$

Figure 17.9 Tree produced by DerivaGem for Example 17.5.



The initial value of S^* is therefore 50.00. Assuming that the 40% per annum volatility refers to S^*, we find that Figure 17.3 provides a binomial tree for S^*. (This is because S^* has the same initial value and volatility as the stock price that Figure 17.3 was based upon.) Adding the present value of the dividend at each node leads to Figure 17.9, which is a binomial model for S. The probabilities at each node are, as in Figure 17.3, 0.5073 for an up movement and 0.4927 for a down movement. Working back through the tree in the usual way gives the option price as \$4.44. (Using 50 time steps, DerivaGem gives a value for the option of 4.202; using 100 steps it gives 4.212.)

When the option lasts a long time (say, 3 or more years) it is usually more appropriate to assume a known dividend yield rather than a known cash dividend because the latter cannot reasonably be assumed to be the same for all the stock prices that might be encountered in the future.[7] Often for convenience the dividend yield is assumed to be paid continuously. Valuing an option on a dividend paying stock is then similar to valuing an option on a stock index.

Control Variate Technique

A technique known as the *control variate technique* can improve the accuracy of the pricing of an American option.[8] This involves using the same tree to calculate both the value of the American option, f_A, and the value of the corresponding European option, f_E. We also calculate the Black–Scholes price of the European option, f_{BS}. The error given by the tree in the pricing of the European option is assumed equal to that given by the tree in the pricing of the American option. This gives the estimate of the price of the American option as

$$f_A + f_{\mathrm{BS}} - f_E$$

To illustrate this approach, Figure 17.10 values the option in Figure 17.3 on the assumption that it is European. The price obtained is \$4.32. From the Black–Scholes formula, the true European price of the option is \$4.08. The estimate of the American price in Figure 17.3 is \$4.49. The control variate estimate of the American price, therefore, is

$$4.49 + 4.08 - 4.32 = 4.25$$

A good estimate of the American price, calculated using 100 steps, is 4.278. The control variate approach does, therefore, produce a considerable improvement over the basic tree estimate of 4.49 in this case.

The control variate technique in effect involves using the tree to calculate the difference between the European and the American price rather than the American price itself. We give a further application of the control variate technique when we discuss Monte Carlo simulation later in the chapter.

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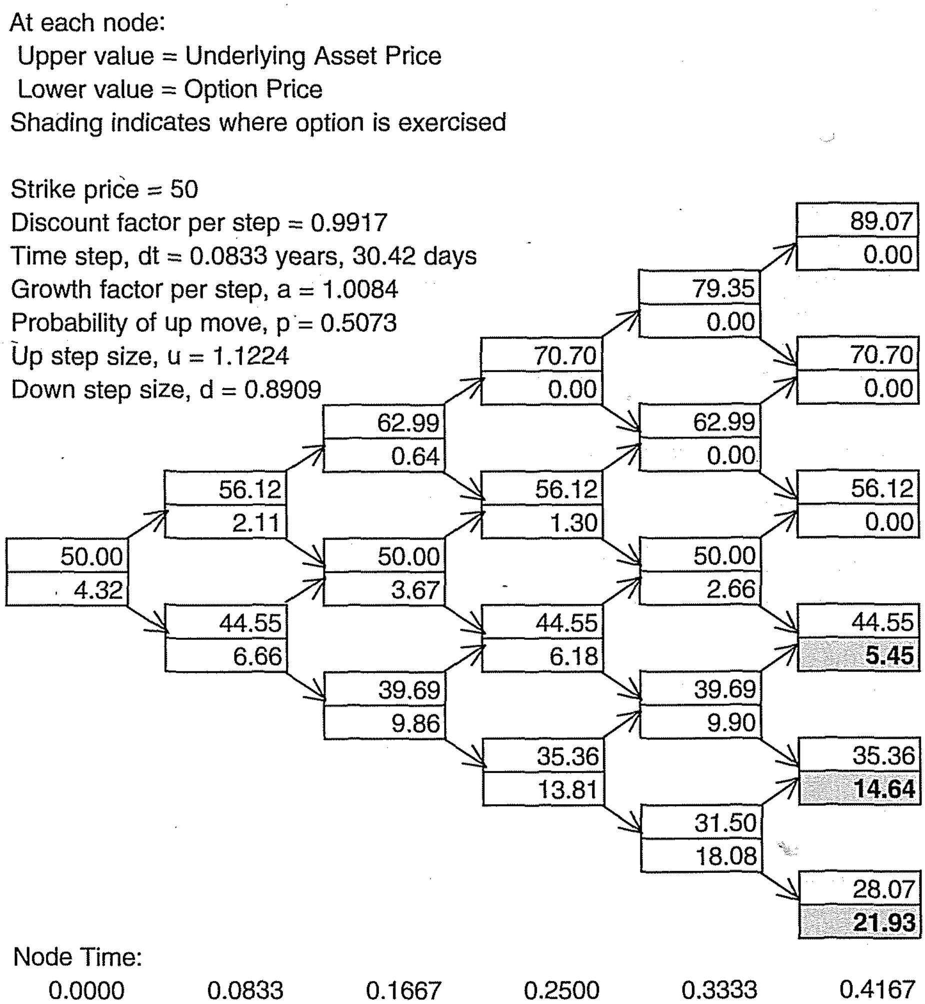
7 Another problem is that, for long-dated options, S^* is significantly less than S_0 and volatility estimates can be very high.

8 See J. Hull and A. White, "The Use of the Control Variate Technique in Option Pricing," *Journal of Financial and Quantitative Analysis*, 23 (September 1988): 237–51.

Figure 17.10 Tree, as produced by DerivaGem, for European version of option in Figure 17.3. At each node, the upper number is the stock price, and the lower number is the option price.



17.4 ALTERNATIVE PROCEDURES FOR CONSTRUCTING TREES

The Cox, Ross, and Rubinstein approach is not the only way of building a binomial tree. Instead of imposing the assumption $u = 1/d$ on equations (17.2) and (17.3), we can set $p = 0.5$. A solution to the equations when terms of higher order than Δt are ignored is then

$$u = e^{(r-q-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}}$$

$$d = e^{(r-q-\sigma^2/2)\Delta t-\sigma\sqrt{\Delta t}}$$

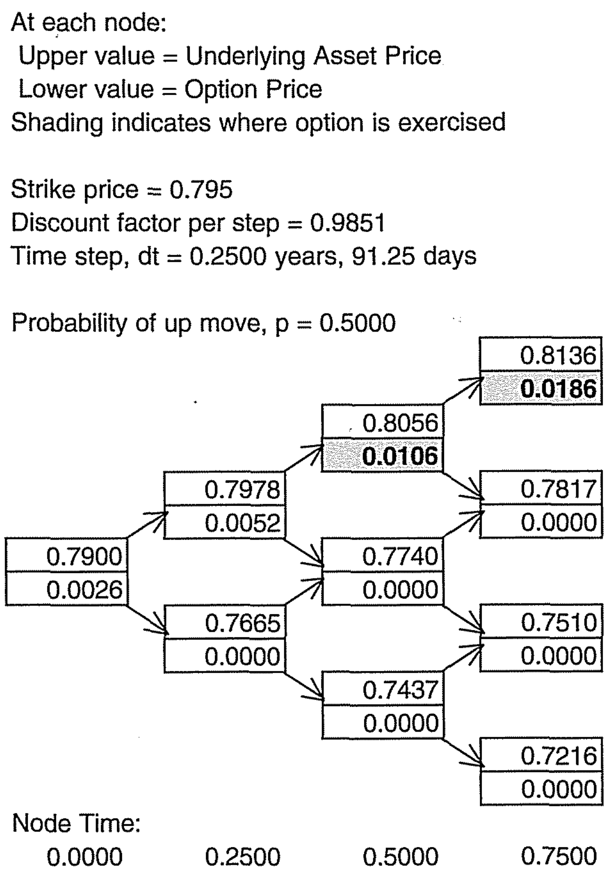
This allows trees with $p = 0.5$ to be built for options on indices, foreign exchange, and futures.

This alternative tree-building procedure has the advantage over the Cox, Ross, and Rubinstein approach that the probabilities are always 0.5 regardless of the value of σ or the number of time steps.[9] Its disadvantage is that it is not as straightforward to calculate delta, gamma, and rho from the tree because the tree is no longer centered at the initial stock price.

Example 17.6

Consider a 9-month American call option on the Canadian dollar. The current exchange rate is 0.7900, the strike price is 0.7950, the US risk-free interest rate is 6% per annum, the Canadian risk-free interest rate is 10% per annum, and the

Figure 17.11 Binomial tree for American call option on the Canadian dollar. At each node, upper number is spot exchange rate and lower number is option price. All probabilities are 0.5.



[9] When time steps are so large that $\sigma < |(r - q)\sqrt{\Delta t}|$, the Cox, Ross, and Rubinstein tree gives negative probabilities. The alternative procedure described here does not have that drawback.

volatility of the exchange rate is 4% per annum. In this case, $S_0 = 0.79$, $K = 0.795$, $r = 0.06$, $r_f = 0.10$, $\sigma = 0.04$, and $T = 0.75$. We divide the life of the option into 3-month periods for the purposes of constructing the tree, so that $\Delta t = 0.25$. We set the probabilities on each branch to 0.5 and

$$u = e^{(0.06-0.10-0.0016/2)0.25+0.04\sqrt{0.25}} = 1.0098$$

$$d = e^{(0.06-0.10-0.0016/2)0.25-0.04\sqrt{0.25}} = 0.9703$$

The tree for the exchange rate is shown in Figure 17.11. The tree gives the value of the option as $0.0026.

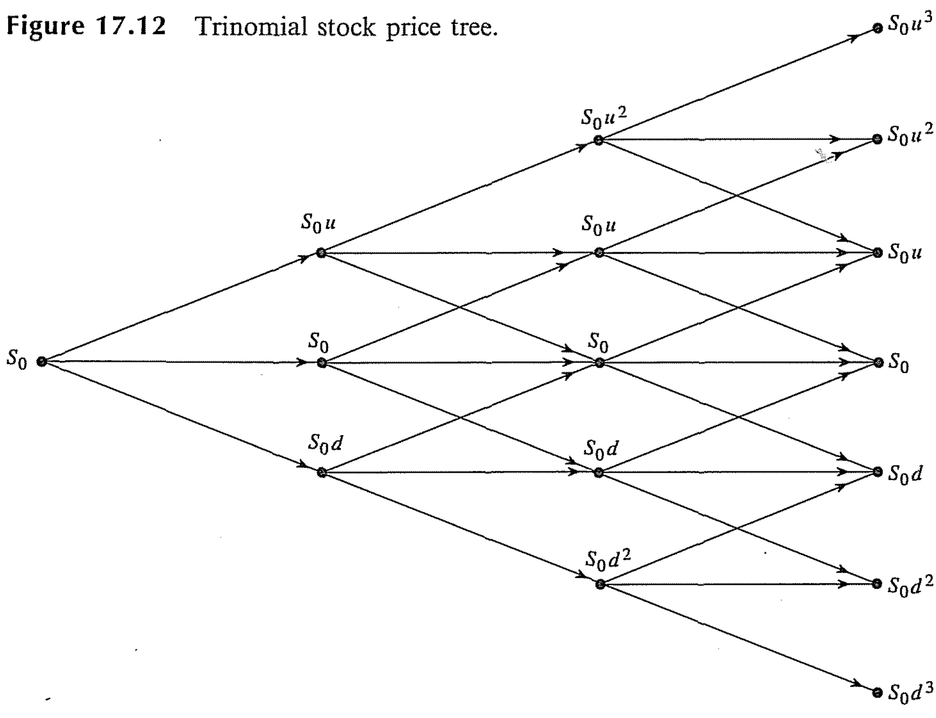
Trinomial Trees

Trinomial trees can be used as an alternative to binomial trees. The general form of the tree is as shown in Figure 17.12. Suppose that p_u, p_m, and p_d are the probabilities of up, middle, and down movements at each node and Δt is the length of the time step. For a non-dividend-paying stock, parameter values that match the mean and standard deviation of price changes when terms of higher order than Δt are ignored are

$$u = e^{\sigma\sqrt{3\Delta t}}, \qquad d = \frac{1}{u}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6}, \qquad p_m = \frac{2}{3}, \qquad p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

Figure 17.12 Trinomial stock price tree.



Calculations for a trinomial tree are analogous to those for a binomial tree. We work from the end of the tree to the beginning. At each node we calculate the value of exercising and the value of continuing. The value of continuing is

$$e^{-r\Delta t}(p_u f_u + p_m f_m + p_d f_d)$$

where f_u, f_m, and f_d are the values of the option at the subsequent up, middle, and down nodes, respectively. The trinomial tree approach proves to be equivalent to the explicit finite difference method, which will be described in Section 17.8.

Figlewski and Gao have proposed an enhancement of the trinomial tree method, which they call the *adaptive mesh model*. In this, a high-resolution (small-Δt) tree is grafted onto a low-resolution (large-Δt) tree.[10] When valuing a regular American option, high resolution is most useful for the parts of the tree close to the strike price at the end of the life of the option.

17.5 TIME-DEPENDENT PARAMETERS

Up to now we have assumed that r, q, r_f, and σ are constants. In practice, they are usually assumed to be time dependent. The values of these variables between times t and $t + \Delta t$ are assumed to be equal to their forward values.[11]

We can make r and q (or r_f) a function of time in a Cox–Ross–Rubinstein binomial tree. We set

$$a = e^{[f(t)-g(t)]\Delta t} \tag{17.11}$$

for nodes at time t, where $f(t)$ is the forward interest rate between times t and $t + \Delta t$ and $g(t)$ is the forward value of q between these times. This does not change the geometry of the tree because u and d do not depend on a. The probabilities on the branches emanating from nodes at time t are:[12]

$$p = \frac{e^{[f(t)-g(t)]\Delta t} - d}{u - d} \tag{17.12}$$

$$1 - p = \frac{u - e^{[f(t)-g(t)]\Delta t}}{u - d}$$

The rest of the way that we use the tree is the same as before, except that when discounting between times t and $t + \Delta t$ we use $f(t)$.

Making σ a function of time in a binomial tree is more challenging. One approach is to make the lengths of time steps inversely proportional to the variance rate. The values of u and d are then always the same and the tree recombines. Suppose that $\sigma(t)$ is the volatility for a maturity t so that $\sigma(t)^2 t$ is the cumulative variance by time t. Define $V = \sigma(T)^2 T$, where T is the life of the tree, and let t_i be the end of the ith time step. If there is a total

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[10] See S. Figlewski and B. Gao, "The Adaptive Mesh Model: A New Approach to Efficient Option Pricing," *Journal of Financial Economics*, 53 (1999): 313–51.

[11] The forward dividend yield and forward variance rate are calculated in the same way as the forward interest rate. (The variance rate is the square of the volatility.)

[12] For a sufficiently large number of time steps, these probabilities are always positive.

Business Snapshot 17.1 Calculating Pi with Monte Carlo Simulation

Suppose the sides of the square in Figure 17.13 are one unit in length. Imagine that you fire darts randomly at the square and calculate the percentage that lie in the circle. What should you find? The square has an area of 1.0 and the circle has a radius of 0.5 The area of the circle is π times the radius squared or $\pi/4$. It follows that the percentage of darts that lie in the circle should be $\pi/4$. We can estimate π by multiplying the percentage that lie in the circle by 4.

We can use an Excel spreadsheet to simulate the dart throwing as illustrated in Table 17.1. We define both cell A1 and cell B1 as =RAND(). A1 and B1 are random numbers between 0 and 1 and define how far to the right and how high up the dart lands in the square in Figure 17.13. We then define cell C1 as

=IF((A1−0.5)^2+(B1−0.5)^2<0.5^2,4,0)

This has the effect of setting C1 equal to 4 if the dart lies in the circle and 0 otherwise.

Define the next 99 rows of the spreadsheet similarly to the first one. (This is a "select and drag" operation in Excel.) Define C102 as =AVERAGE(C1:C100) and C103 as =STDEV(C1:C100). C102 (which is 3.04 in Table 17.1) is an estimate of π calculated from 100 random trials. C103 is the standard deviation of our results and as we will see in Example 17.7 can be used to assess the accuracy of the estimate. Increasing the number of trials improves accuracy—but convergence to the correct value of 3.14162 is slow.

of N time steps, we choose t_i to satisfy $\sigma(t_i)^2 t_i = iV/N$. The variance between times t_{i-1} and t_i is then V/N for all i.

With a trinomial tree, a generalized tree-building procedure can be used to match time-dependent interest rates and volatilities (see Technical Note 9 on the author's website).

17.6 MONTE CARLO SIMULATION

We now explain Monte Carlo simulation, a quite different approach for valuing derivatives from binomial trees. Business Snapshot 17.1 illustrates the random sampling idea underlying Monte Carlo simulation by showing how a simple Excel program can be constructed to estimate π.

Figure 17.13 Calculation of π by throwing darts.

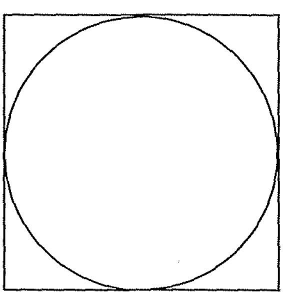


Table 17.1 Sample spreadsheet calculations in Business Snapshot 17.1.

	A	*B*	*C*
1	0.207	0.690	4
2	0.271	0.520	4
3	0.007	0.221	0
⋮	⋮	⋮	⋮
100	0.198	0.403	4
101			
102		Mean:	3.04
103		SD:	1.69

When used to value an option, Monte Carlo simulation uses the risk-neutral valuation result. We sample paths to obtain the expected payoff in a risk-neutral world and then discount this payoff at the risk-free rate. Consider a derivative dependent on a single market variable S that provides a payoff at time T. Assuming that interest rates are constant, we can value the derivative as follows:[13]

1. Sample a random path for S in a risk-neutral world.
2. Calculate the payoff from the derivative.
3. Repeat steps 1 and 2 to get many sample values of the payoff from the derivative in a risk-neutral world.
4. Calculate the mean of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world.
5. Discount the expected payoff at the risk-free rate to get an estimate of the value of the derivative.

Suppose that the process followed by the underlying market variable in a risk-neutral world is

$$dS = \hat{\mu} S\,dt + \sigma S\,dz \tag{17.13}$$

where dz is a Wiener process, $\hat{\mu}$ is the expected return in a risk-neutral world, and σ is the volatility.[14] To simulate the path followed by S, we can divide the life of the derivative into N short intervals of length Δt and approximate equation (17.13) as

$$S(t+\Delta t) - S(t) = \hat{\mu} S(t)\,\Delta t + \sigma S(t)\epsilon\sqrt{\Delta t} \tag{17.14}$$

where $S(t)$ denotes the value of S at time t, ϵ is a random sample from a normal distribution with mean zero and standard deviation of 1.0. This enables the value of S at time Δt to be calculated from the initial value of S, the value at time $2\,\Delta t$ to be calculated from the value at time Δt, and so on. An illustration of the procedure is in Section 12.3. One simulation trial involves constructing a complete path for S using N random samples from a normal distribution.

[13] We discuss how Monte Carlo simulation can be used with stochastic interest rates in Section 25.4.

[14] If S is the price of a non-dividend-paying stock then $\hat{\mu} = r$, if it is an exchange rate then $\hat{\mu} = r - r_f$, and so on. Note that the volatility is the same in a risk-neutral world as in the real world, as shown in Section 11.7.

In practice, it is usually more accurate to simulate $\ln S$ rather than S. From Itô's lemma the process followed by $\ln S$ is

$$d\ln S = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)dt + \sigma\,dz \tag{17.15}$$

so that

$$\ln S(t+\Delta t) - \ln S(t) = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}$$

or equivalently

$$S(t+\Delta t) = S(t)\exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}\right] \tag{17.16}$$

This equation is used to construct a path for S.

The advantage of working with $\ln S$ is that it follows a generalized Wiener process. This means that the equation

$$\ln S(T) - \ln S(0) = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}$$

is true for all T.[15] It follows that

$$S(T) = S(0)\exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}\right] \tag{17.17}$$

This equation can be used to value derivatives that provide a nonstandard payoff at time T. As indicated in Business Snapshot 17.2, it can also be used to check the Black–Scholes formulas.

The key advantage of Monte Carlo simulation is that it can be used when the payoff depends on the path followed by the underlying variable S as well as when it depends only on the final value of S. (For example, is can be used when payoffs depend on the average value of S.) Payoffs can occur at several times during the life of the derivative rather than all at the end. Any stochastic process for S can be accommodated. As will be shown shortly, the procedure can also be extended to accommodate situations where the payoff from the derivative depends on several underlying market variables. The drawbacks of Monte Carlo simulation are that it is computationally very time consuming and cannot easily handle situations where there are early exercise opportunities.[16]

Derivatives Dependent on More than One Market Variable

Consider the situation where the payoff from a derivative depends on n variables θ_i $(1 \leqslant i \leqslant n)$. Define s_i as the volatility of θ_i, $\hat{m}_i$ as the expected growth rate of θ_i in a risk-neutral world, and ρ_{ik} as the instantaneous correlation between θ_i and θ_k.[17] As in the

[15] By contrast, equation (17.14) is true only in the limit as Δt tends to zero.

[16] As we will discuss in Chapter 24, a number of researchers have suggested ways Monte Carlo simulation can be extended to value American options.

[17] Note that s_i, $\hat{m}_i$, and ρ_{ik} are not necessarily constant; they may depend on the θ_i.

Business Snapshot 17.2 Checking Black–Scholes

The Black–Scholes formula for a European call option can be checked by using a binomial tree with a very large number of time steps. An alternative way of checking it is to use Monte Carlo simulation. Table 17.2 shows a spreadsheet that can be constructed. The cells C2, D2, E2, F2, and G2 contain S_0, K, r, σ, and T, respectively. Cells D4, E4, and F4 calculate d_1, d_2, and the Black–Scholes price, respectively. (The Black–Scholes price is 4.817 in the sample spreadsheet.)

NORMSINV is the inverse cumulative function for the standard normal distribution. It follows that NORMSINV(RAND())gives a random sample from a standard normal distribution. We set cell A1 as

=$C$2*EXP(($E$2−$F$2*$F$2/2)*$G$2+$F$2*NORMSINV(RAND())*SQRT($G$2))

This is random sample from the set of all stock prices at time T. We set cell B1 as

=EXP(−$E$2*$G$2)*MAX(A1−$D$2,0)

This is the present value of the payoff from a call option. We define the next 999 rows of the spreadsheet similarly to the first one. (This is a "select and drag" operation in Excel.) Define B1002 as AVERAGE(B1:B1000) and B1003 as STDEV(B1:B1000). B1002 (which is 4.98 in the sample spreadsheet) is an estimate of the value of the option. This should be not too far from the Black–Scholes price. As we shall see in Example 17.8, B1003 can be used to assess the accuracy of the estimate.

single-variable case, the life of the derivative must be divided into N subintervals of length Δt. The discrete version of the process for θ_i is then

$$\theta_i(t+\Delta t)-\theta_i(t)=\hat{m}_i\theta_i(t)\,\Delta t+s_i\theta_i(t)\epsilon_i\sqrt{\Delta t} \tag{17.18}$$

where ϵ_i is a random sample from a standard normal distribution. The coefficient of correlation between ϵ_i and ϵ_k is ρ_{ik} ($1 \leqslant i; k \leqslant n$). One simulation trial involves obtaining N samples of the ϵ_i ($1 \leqslant i \leqslant n$) from a multivariate standardized normal distribution. These are substituted into equation (17.18) to produce simulated paths for each θ_i, thereby enabling a sample value for the derivative to be calculated.

Table 17.2 Monte Carlo simulation to check Black–Scholes

	A	*B*	*C*	*D*	*E*	*F*	*G*
1	45.95	0	S_0	K	r	σ	T
2	54.49	4.38	50	50	0.05	0.3	0.5
3	50.09	0.09		d_1	d_2	BS price	
4	47.46	0		0.2239	0.0118	4.817	
5	44.93	0					
⋮	⋮	⋮					
1000	68.27	17.82					
1001							
1002	Mean:	4.98					
1003	SD:	7.68					

Generating the Random Samples from Normal Distributions

An approximate sample from a univariate standardized normal distribution can be obtained from the formula

$$\epsilon = \sum_{i=1}^{12} R_i - 6 \tag{17.19}$$

where the R_i ($1 \leqslant i \leqslant 12$) are independent random numbers between 0 and 1, and ϵ is the required sample from $\phi(0, 1)$. This approximation is satisfactory for most purposes. An alternative approach in Excel is to use =NORMSINV(RAND()) as in Business Snapshot 17.2.

When two correlated samples ϵ_1 and ϵ_2 from standard normal distributions are required, an appropriate procedure is as follows. Independent samples x_1 and x_2 from a univariate standardized normal distribution are obtained as just described. The required samples ϵ_1 and ϵ_2 are then calculated as follows:

$$\epsilon_1 = x_1$$
$$\epsilon_2 = \rho x_1 + x_2\sqrt{1-\rho^2}$$

where ρ is the coefficient of correlation.

More generally, consider the situation where we require n correlated samples from normal distributions with the correlation between sample i and sample j being ρ_{ij}. We first sample n independent variables x_i ($1 \leqslant i \leqslant n$), from univariate standardized normal distributions. The required samples, ϵ_i ($1 \leqslant i \leqslant n$), are then defined as follows:

$$\epsilon_1 = \alpha_{11}x_1$$
$$\epsilon_2 = \alpha_{21}x_1 + \alpha_{22}x_2$$
$$\epsilon_3 = \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3$$

and so on. We choose the coefficients α_{ij} so that the correlations and variances are correct. This can be done step by step as follows. Set $\alpha_{11} = 1$; choose α_{21} so that $\alpha_{21}\alpha_{11} = \rho_{21}$; choose α_{22} so that $\alpha_{21}^2 + \alpha_{22}^2 = 1$; choose α_{31} so that $\alpha_{31}\alpha_{11} = \rho_{31}$; choose α_{32} so that $\alpha_{31}\alpha_{21} + \alpha_{32}\alpha_{22} = \rho_{32}$; choose α_{33} so that $\alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1$; and so on.[18] This procedure is known as the *Cholesky decomposition*.

Number of Trials

The accuracy of the result given by Monte Carlo simulation depends on the number of trials. It is usual to calculate the standard deviation as well as the mean of the discounted payoffs given by the simulation trials. Denote the mean by μ and the standard deviation by ω. The variable μ is the simulation's estimate of the value of the derivative. The standard error of the estimate is

$$\frac{\omega}{\sqrt{M}}$$

where M is the number of trials. A 95% confidence interval for the price f of the

[18] If the equations for the α's do not have real solutions, the assumed correlation structure is internally inconsistent This will be discussed further in Chapter 19.

derivative is therefore given by

$$\mu - \frac{1.96\omega}{\sqrt{M}} < f < \mu + \frac{1.96\omega}{\sqrt{M}}$$

This shows that our uncertainty about the value of the derivative is inversely proportional to the square root of the number of trials. To double the accuracy of a simulation, we must quadruple the number of trials; to increase the accuracy by a factor of 10, the number of trials must increase by a factor of 100; and so on.

Example 17.7

In Table 17.1, π is calculated as the average of 100 numbers. The standard deviation of the numbers is 1.69. In this case, $\omega = 1.69$ and $M = 100$, so that the standard error of the estimate is $1.69/\sqrt{100} = 0.169$. The spreadsheet therefore gives a 95% confidence interval for π as $(3.04 - 1.96 \times 0.169)$ to $(3.04 + 1.96 \times 0.169)$ or 2.71 to 3.37.

Example 17.8

In Table 17.2, the value of the option is calculated as the average of 1000 numbers. The standard deviation of the numbers is 7.68. In this case, $\omega = 7.68$ and $M = 1000$. The standard error of the estimate is $7.68/\sqrt{1000} = 0.24$. The spreadsheet therefore gives a 95% confidence interval for the option value as $(4.98 - 1.96 \times 0.24)$ to $(4.98 + 1.96 \times 0.24)$, or 4.51 to 5.45.

Applications

Monte Carlo simulation tends to be numerically more efficient than other procedures when there are three or more stochastic variables. This is because the time taken to carry out a Monte Carlo simulation increases approximately linearly with the number of variables, whereas the time taken for most other procedures increases exponentially with the number of variables. One advantage of Monte Carlo simulation is that it can provide a standard error for the estimates that it makes. Another is that it is an approach that can accommodate complex payoffs and complex stochastic processes. Also, it can be used when the payoff depends on some function of the whole path followed by a variable, not just its terminal value.

Calculating the Greek Letters

The Greek letters discussed in Chapter 15 can be calculated using Monte Carlo simulation. Suppose that we are interested in the partial derivative of f with respect to x, where f is the value of the derivative and x is the value of an underlying variable or a parameter. First, Monte Carlo simulation is used in the usual way to calculate an estimate $\hat{f}$ for the value of the derivative. A small increase Δx is then made in the value of x, and a new value for the derivative, $\hat{f}^*$, is calculated in the same way as $\hat{f}$. An estimate for the hedge parameter is given by

$$\frac{\hat{f}^* - \hat{f}}{\Delta x}$$

In order to minimize the standard error of the estimate, the number of time intervals, N, the random number streams, and the number of trials, M, should be the same for calculating both $\hat{f}$ and $\hat{f}^*$.

Sampling through a Tree

Instead of implementing Monte Carlo simulation by randomly sampling from the stochastic process for an underlying variable, we can use an N-step binomial tree and sample from the 2^N paths that are possible. Suppose we have a binomial tree where the probability of an "up" movement is 0.6. The procedure for sampling a random path through the tree is as follows. At each node, we sample a random number between 0 and 1. If the number is less than 0.4, we take the down path. If it is greater than 0.4, we take the up path. Once we have a complete path from the initial node to the end of the tree, we can calculate a payoff. This completes the first trial. A similar procedure is used to complete more trials. The mean of the payoffs is discounted at the risk-free rate to get an estimate of the value of the derivative.[19]

Example 17.9

Suppose that the tree in Figure 17.3 is used to value an option that pays off $\max(S_{ave} - 50, 0)$, where S_{ave} is the average stock price during the 5 months (with the first and last stock price being included in the average). This is known as an Asian option. When ten simulation trials are used one possible result is shown in Table 17.3.

Table 17.3 Monte Carlo simulation to value Asian option from the tree in Figure 17.3. Payoff is amount by which average stock price exceeds \$50. U = up movement; D = down movement.

Trial	*Path*	*Average stock price*	*Option payoff*
1	UUUUD	64.98	14.98
2	UUUDD	59.82	9.82
3	DDDUU	42.31	0.00
4	UUUUU	68.04	18.04
5	UUDDU	55.22	5.22
6	UDUUD	55.22	5.22
7	DDUDD	42.31	0.00
8	UUDDU	55.22	5.22
9	UUUDU	62.25	12.25
10	DDUUD	45.56	0.00
Average			7.08

The value of the option is calculated as the average payoff discounted at the risk-free rate. In this case, the average payoff is \$7.08 and the risk-free rate is 10% and so the calculated value is $7.08e^{-0.1\times 5/12} = 6.79$. (This illustrates the methodology. In practice, we would have to use more time steps on the tree and many more simulation trials to get an accurate answer.)

[19] See D. Mintz, "Less is More," *Risk*, July 1997: 42–45, for a discussion of how sampling through a tree can be made efficient.

17.7 VARIANCE REDUCTION PROCEDURES

If the simulation is carried out as described so far, a very large number of trials is usually necessary to estimate f with reasonable accuracy. This is very expensive in terms of computation time. In this section, we examine a number of variance reduction procedures that can lead to dramatic savings in computation time.

Antithetic Variable Technique

In the antithetic variable technique, a simulation trial involves calculating two values of the derivative. The first value f_1 is calculated in the usual way; the second value f_2 is calculated by changing the sign of all the random samples from standard normal distributions. (If ϵ is a sample used to calculate f_1, then $-\epsilon$ is the corresponding sample used to calculate f_2.) The sample value of the derivative calculated from a simulation trial is the average of f_1 and f_2. This works well because when one value is above the true value, the other tends to be below, and vice versa.

Denote $\bar{f}$ as the average of f_1 and f_2:

$$\bar{f} = \frac{f_1 + f_2}{2}$$

The final estimate of the value of the derivative is the average of the $\bar{f}$'s. If $\bar{\omega}$ is the standard deviation of the $\bar{f}$'s, and M is the number of simulation trials (i.e., the number of pairs of values calculated), then the standard error of the estimate is

$$\bar{\omega}/\sqrt{M}$$

This is usually much less than the standard error calculated using $2M$ random trials.

Control Variate Technique

We have already given one example of the control variate technique in connection with the use of trees to value American options (see Section 17.3). The control variate technique is applicable when there are two similar derivatives, A and B. Derivative A is the security being valued; derivative B is similar to derivative A and has an analytic solution available. Two simulations using the same random number streams and the same Δt are carried out in parallel. The first is used to obtain an estimate f_A^* of the value of A; the second is used to obtain an estimate f_B^*, of the value of B. A better estimate f_A of the value of A is then obtained using the formula

$$f_A = f_A^* - f_B^* + f_B \tag{17.20}$$

where f_B is the known true value of B calculated analytically. Hull and White provide an example of the use of the control variate technique when evaluating the effect of stochastic volatility on the price of a European call option.[20] In this case, f_A is the estimated value of the option assuming stochastic volatility and f_B is its Black–Scholes value assuming constant volatility.

[20] See J. Hull and A. White, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42 (June 1987): 281–300.

Importance Sampling

Importance sampling is best explained with an example. Suppose that we wish to calculate the price of a deep-out-of-the-money European call option with strike price K and maturity T. If we sample values for the underlying asset price at time T in the usual way, most of the paths will lead to zero payoff. This is a waste of computation time because the zero-payoff paths contribute very little to the determination of the value of the option. We therefore try to choose only important paths, that is, paths where the stock price is above K at maturity.

Suppose F is the unconditional probability distribution function for the stock price at time T and q, the probability of the stock price being greater than K at maturity, is known analytically. Then $G = F/q$ is the probability distribution of the stock price conditional on the stock price being greater than K. To implement importance sampling, we sample from G rather than F. The estimate of the value of the option is the average discounted payoff multiplied by q.

Stratified Sampling

Sampling representative values rather than random values from a probability distribution usually gives more accuracy. Stratified sampling is a way of doing this. Suppose we wish to take 1000 samples from a probability distribution we would divide the distribution into 1000 equally likely intervals and choose a representative value (typically the mean or median) for each interval.

In the case of a standard normal distribution when there are n intervals, we can calculate the representative value for the ith interval as

$$N^{-1}\left(\frac{i-0.5}{n}\right)$$

where N^{-1} is the inverse cumulative normal distribution. For example, when $n = 4$ the representative values corresponding to the four intervals are $N^{-1}(0.125)$, $N^{-1}(0.375)$, $N^{-1}(0.625)$, $N^{-1}(0.875)$. The function N^{-1} can be calculated using the NORMSINV function in Excel.

Moment Matching

Moment matching involves adjusting the samples taken from a standardized normal distribution so that the first, second, and possibly higher moments are matched. Suppose that we sample from a normal distribution with mean 0 and standard deviation 1 to calculate the change in the value of a particular variable over a particular time period. Suppose that the samples are ϵ_i $(1 \leqslant i \leqslant n)$. To match the first two moments, we calculate the mean of the samples, m, and the standard deviation of the samples, s. We then define adjusted samples ϵ_i^* $(1 \leqslant i \leqslant n)$ as

$$\epsilon_i^* = \frac{\epsilon_i - m}{s}$$

These adjusted samples have the correct mean of 0 and the correct standard deviation of 1.0. We use the adjusted samples for all calculations.

Moment matching saves computation time, but can lead to memory problems because every number sampled must be stored until the end of the simulation. Moment matching is sometimes termed *quadratic resampling*. It is often used in conjunction with the antithetic variable technique. Because the latter automatically matches all odd moments, the goal of moment matching then becomes that of matching the second moment and, possibly, the fourth moment.

Using Quasi-Random Sequences

A quasi-random sequence (also called a *low-discrepancy* sequence) is a sequence of representative samples from a probability distribution.[21] Descriptions of the use of quasi-random sequences appear in Brotherton-Ratcliffe, and Press *et al.*[22] Quasi-random sequences can have the desirable property that they lead to the standard error of an estimate being proportional to $1/M$ rather than $1/\sqrt{M}$, where M is the sample size.

Quasi-random sampling is similar to stratified sampling. The objective is to sample representative values for the underlying variables. In stratified sampling, it is assumed that we know in advance how many samples will be taken. A quasi-random sampling scheme is more flexible. The samples are taken in such a way that we are always "filling in" gaps between existing samples. At each stage of the simulation, the sampled points are roughly evenly spaced throughout the probability space.

Figure 17.14 shows points generated in two dimensions using a procedure suggested by Sobol'.[23] It can be seen that successive points do tend to fill in the gaps left by previous points.

17.8 FINITE DIFFERENCE METHODS

Finite difference methods value a derivative by solving the differential equation that the derivative satisfies. The differential equation is converted into a set of difference equations, and the difference equations are solved iteratively.

To illustrate the approach, we consider how it might be used to value an American put option on a stock paying a dividend yield of q. The differential equation that the option must satisfy is, from equation (14.6),

$$\frac{\partial f}{\partial t}+(r-q)S\frac{\partial f}{\partial S}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2}=rf \tag{17.21}$$

Suppose that the life of the option is T. We divide this into N equally spaced intervals of length $\Delta t = T/N$. A total of $N+1$ times are therefore considered

$$0,\ \Delta t,\ 2\,\Delta t,\ \ldots,\ T$$

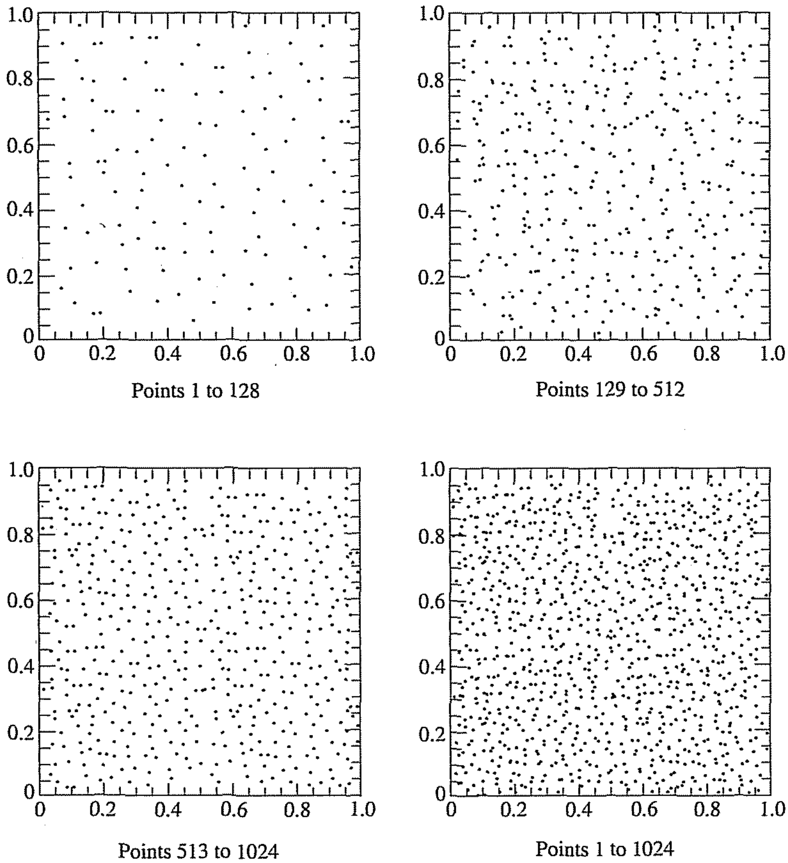
---

[21] The term *quasi-random* is a misnomer. A quasi-random sequence is totally deterministic.

[22] See R. Brotherton-Ratcliffe, "Monte Carlo Motoring," *Risk*, December 1994: 53–58; W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edn. Cambridge University Press, 1992.

[23] See I.M. Sobol', *USSR Computational Mathematics and Mathematical Physics*, 7, 4 (1967): 86–112. A description of Sobol's procedure is in W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd edn. Cambridge University Press, 1992.

Figure 17.14 First 1024 points of a Sobol' sequence.



Suppose that S_{max} is a stock price sufficiently high that, when it is reached, the put has virtually no value. We define $\Delta S = S_{max}/M$ and consider a total of $M + 1$ equally spaced stock prices:

$$0, \ \Delta S, \ 2\,\Delta S, \ \ldots, \ S_{max}$$

The level S_{max} is chosen so that one of these is the current stock price.

The time points and stock price points define a grid consisting of a total of $(M+1)(N+1)$ points, as shown in Figure 17.15. The (i, j) point on the grid is the point that corresponds to time $i\,\Delta t$ and stock price $j\,\Delta S$. We will use the variable $f_{i,j}$ to denote the value of the option at the (i, j) point.

Implicit Finite Difference Method

For an interior point (i, j) on the grid, $\partial f/\partial S$ can be approximated as

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j}}{\Delta S} \tag{17.22}$$

or as

$$\frac{\partial f}{\partial S} = \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \tag{17.23}$$

Equation (17.22) is known as the *forward difference approximation*; equation (17.23) is known as the *backward difference approximation*. We use a more symmetrical approximation by averaging the two:

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\,\Delta S} \tag{17.24}$$

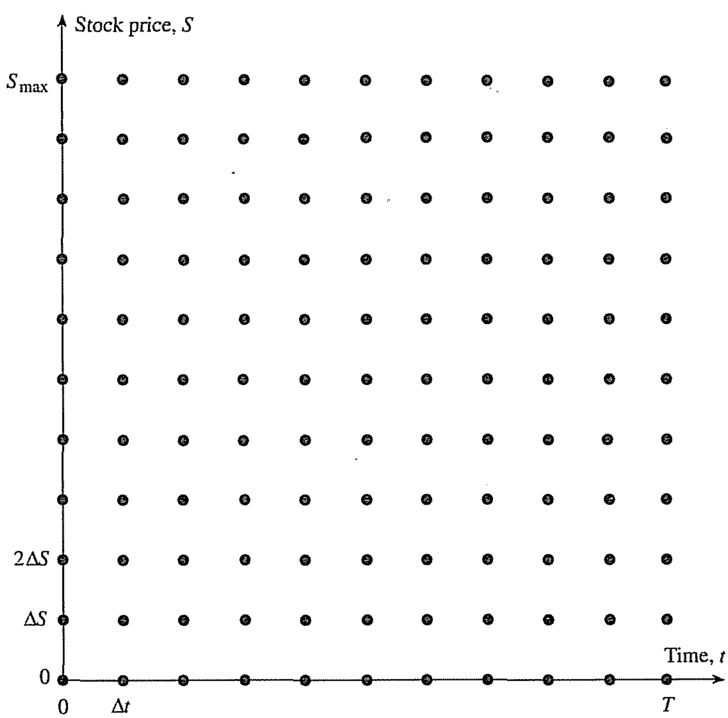
For $\partial f/\partial t$, we will use a forward difference approximation so that the value at time $i\,\Delta t$ is related to the value at time $(i+1)\,\Delta t$:

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \tag{17.25}$$

The backward difference approximation for $\partial f/\partial S$ at the (i, j) point is given by equation (17.23). The backward difference at the $(i, j+1)$ point is

$$\frac{f_{i,j+1} - f_{i,j}}{\Delta S}$$

Figure 17.15 Grid for finite difference approach.



Hence a finite difference approximation for $\partial^2 f/\partial S^2$ at the (i, j) point is

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S}\right) \Big/ \Delta S$$

or

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2} \tag{17.26}$$

Substituting equations (17.24), (17.25), and (17.26) into the differential equation (17.21) and noting that $S = j\,\Delta S$ gives

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r - q)j\,\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\,\Delta S} + \tfrac{1}{2}\sigma^2 j^2 \Delta S^2 \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2} = r f_{i,j}$$

for $j = 1, 2, \ldots, M-1$ and $i = 0, 1 \ldots, N-1$. Rearranging terms, we obtain

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \tag{17.27}$$

where

$$a_j = \tfrac{1}{2}(r - q)j\,\Delta t - \tfrac{1}{2}\sigma^2 j^2 \Delta t$$

$$b_j = 1 + \sigma^2 j^2 \Delta t + r\,\Delta t$$

$$c_j = -\tfrac{1}{2}(r - q)j\,\Delta t - \tfrac{1}{2}\sigma^2 j^2 \Delta t$$

The value of the put at time T is $\max(K - S_T, 0)$, where S_T is the stock price at time T. Hence,

$$f_{N,j} = \max(K - j\,\Delta S, 0), \quad j = 0, 1, \ldots, M \tag{17.28}$$

The value of the put option when the stock price is zero is K. Hence,

$$f_{i,0} = K, \quad i = 0, 1, \ldots, N \tag{17.29}$$

We assume that the put option is worth zero when $S = S_{\max}$, so that

$$f_{i,M} = 0, \quad i = 0, 1, \ldots, N \tag{17.30}$$

Equations (17.28), (17.29), and (17.30) define the value of the put option along the three edges of the grid in Figure 17.15, where $S = 0$, $S = S_{\max}$, and $t = T$. It remains to use equation (17.27) to arrive at the value of f at all other points. First the points corresponding to time $T - \Delta t$ are tackled. Equation (17.27) with $i = N - 1$ gives

$$a_j\, f_{N-1,j-1} + b_j\, f_{N-1,j} + c_j\, f_{N-1,j+1} = f_{N,j} \tag{17.31}$$

for $j = 1, 2, \ldots, M-1$. The right-hand sides of these equations are known from equation (17.28). Furthermore, from equations (17.29) and (17.30),

$$f_{N-1,0} = K \tag{17.32}$$

$$f_{N-1,M} = 0 \tag{17.33}$$

Equations (17.31) are therefore $M-1$ simultaneous equations that can be solved for the $M-1$ unknowns: $f_{N-1,1}, f_{N-1,2}, \dots, f_{N-1,M-1}$.[24] After this has been done, each value of $f_{N-1,j}$ is compared with $K - j\,\Delta S$. If $f_{N-1,j} < K - j\,\Delta S$, early exercise at time $T - \Delta t$ is optimal and $f_{N-1,j}$ is set equal to $K - j\,\Delta S$. The nodes corresponding to time $T - 2\,\Delta t$ are handled in a similar way, and so on. Eventually, $f_{0,1}, f_{0,2}, f_{0,3}, \dots, f_{0,M-1}$ are obtained. One of these is the option price of interest.

The control variate technique can be used in conjunction with finite difference methods. The same grid is used to value an option similar to the one under consideration but for which an analytic valuation is available. Equation (17.20) is then used.

Example 17.10

Table 17.4 shows the result of using the implicit finite difference method as just described for pricing the American put option in Example 17.1. Values of 20, 10, and 5 were chosen for M, N, and ΔS, respectively. Thus, the option price is evaluated at \$5 stock price intervals between \$0 and \$100 and at half-month time intervals throughout the life of the option. The option price given by the grid is \$4.07. The same grid gives the price of the corresponding European option as \$3.91. The true European price given by the Black–Scholes formula is \$4.08. The control variate estimate of the American price is therefore

$$4.07 + 4.08 - 3.91 = \$4.24$$

Explicit Finite Difference Method

The implicit finite difference method has the advantage of being very robust. It always converges to the solution of the differential equation as ΔS and Δt approach zero.[25] One of the disadvantages of the implicit finite difference method is that $M-1$ simultaneous equations have to be solved in order to calculate the $f_{i,j}$ from the $f_{i+1,j}$. The method can be simplified if the values of $\partial f/\partial S$ and $\partial^2 f/\partial S^2$ at point (i, j) on the grid are assumed to be the same as at point $(i+1, j)$. Equations (17.24) and (17.26) then become

$$\frac{\partial f}{\partial S} = \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\,\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta S^2}$$

The difference equation is

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r-q)j\,\Delta S\,\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S} + \tfrac{1}{2}\sigma^2 j^2 \Delta S^2\,\frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta S^2} = r f_{i,j}$$

[24] This does not involve inverting a matrix. The $j=1$ equation in (17.31) can be used to express $f_{N-1,2}$ in terms of $f_{N-1,1}$; the $j=2$ equation, when combined with the $j=1$ equation, can be used to express $f_{N-1,3}$ in terms of $f_{N-1,1}$; and so on. The $j = M-2$ equation, together with earlier equations, enables $f_{N-1,M-1}$ to be expressed in terms of $f_{N-1,1}$. The final $j = M-1$ equation can then be solved for $f_{N-1,1}$, which can then be used to determine the other $f_{N-1,j}$.

[25] A general rule in finite difference methods is that ΔS should be kept proportional to $\sqrt{\Delta t}$ as they approach zero.

Table 17.4 Grid to value American option in Example 17.1 using implicit finite difference methods.

Stock price (dollars)	*Time to maturity (months)*										
	5	*4.5*	*4*	*3.5*	*3*	*2.5*	*2*	*1.5*	*1*	*0.5*	*0*
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
95	0.02	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	0.05	0.04	0.03	0.02	0.01	0.01	0.00	0.00	0.00	0.00	0.00
85	0.09	0.07	0.05	0.03	0.02	0.01	0.01	0.00	0.00	0.00	0.00
80	0.16	0.12	0.09	0.07	0.04	0.03	0.02	0.01	0.00	0.00	0.00
75	0.27	0.22	0.17	0.13	0.09	0.06	0.03	0.02	0.01	0.00	0.00
70	0.47	0.39	0.32	0.25	0.18	0.13	0.08	0.04	0.02	0.00	0.00
65	0.82	0.71	0.60	0.49	0.38	0.28	0.19	0.11	0.05	0.02	0.00
60	1.42	1.27	1.11	0.95	0.78	0.62	0.45	0.30	0.16	0.05	0.00
55	2.43	2.24	2.05	1.83	1.61	1.36	1.09	0.81	0.51	0.22	0.00
50	4.07	3.88	3.67	3.45	3.19	2.91	2.57	2.17	1.66	0.99	0.00
45	6.58	6.44	6.29	6.13	5.96	5.77	5.57	5.36	5.17	5.02	5.00
40	10.15	10.10	10.05	10.01	10.00	10.00	10.00	10.00	10.00	10.00	10.00
35	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00
30	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
25	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00
20	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00
15	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00
10	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00
5	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00
0	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00

or

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1} \tag{17.34}$$

where

$$a_j^* = \frac{1}{1+r\,\Delta t}\left(-\tfrac{1}{2}(r-q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t\right)$$

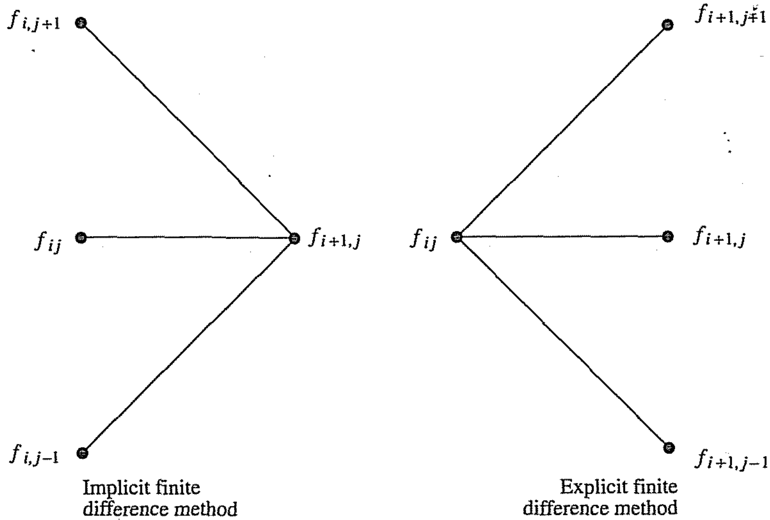
$$b_j^* = \frac{1}{1+r\,\Delta t}(1 - \sigma^2 j^2 \Delta t)$$

$$c_j^* = \frac{1}{1+r\,\Delta t}\left(\tfrac{1}{2}(r-q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t\right)$$

This creates what is known as the *explicit finite difference method.*[26] Figure 17.16 shows the difference between the implicit and explicit methods. The implicit method leads to equation (17.27), which gives a relationship between three different values of the option at time $i\,\Delta t$ (i.e., $f_{i,j-1}$, $f_{i,j}$, and $f_{i,j+1}$) and one value of the option at time $(i+1)\,\Delta t$

[26] We also obtain the explicit finite difference method if we use the backward difference approximation instead of the forward difference approximation for $\partial f/\partial t$.

Figure 17.16 Difference between implicit and explicit finite difference methods.



(i.e., $f_{i+1,j}$). The explicit method leads to equation (17.34), which gives a relationship between one value of the option at time $i\,\Delta t$ (i.e., $f_{i,j}$) and three different values of the option at time $(i+1)\,\Delta t$ (i.e., $f_{i+1,j-1}$, $f_{i+1,j}$, $f_{i+1,j+1}$).

Example 17.11

Table 17.5 shows the result of using the explicit version of the finite difference method for pricing the American put option in Example 17.1. As in Example 17.10, values of 20, 10, and 5 were chosen for M, N, and ΔS, respectively. The option price given by the grid is \$4.26.[27]

Change of Variable

It is computationally more efficient to use finite difference methods with ln S rather than S as the underlying variable. Define $Z = \ln S$. Equation (17.21) becomes

$$\frac{\partial f}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial Z} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial Z^2} = rf$$

The grid then evaluates the derivative for equally spaced values of Z rather than for equally spaced values of S. The difference equation for the implicit method becomes

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r - q - \sigma^2/2)\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta Z} + \tfrac{1}{2}\sigma^2 \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta Z^2} = rf_{i,j}$$

or

$$\alpha_j\, f_{i,j-1} + \beta_j\, f_{i,j} + \gamma_j\, f_{i,j+1} = f_{i+1,j} \tag{17.35}$$

[27] The negative numbers and other inconsistencies in the top left-hand part of the grid will be explained later.

Table 17.5 Grid to value American option in Example 17.1 using explicit finite difference method.

Stock price (dollars)	*Time to maturity (months)*										
	5	*4.5*	*4*	*3.5*	*3*	*2.5*	*2*	*1.5*	*1*	*0.5*	*0*
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
95	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
90	−0.11	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
85	0.28	−0.05	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
80	−0.13	0.20	0.00	0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00
75	0.46	0.06	0.20	0.04	0.06	0.00	0.00	0.00	0.00	0.00	0.00
70	0.32	0.46	0.23	0.25	0.10	0.09	0.00	0.00	0.00	0.00	0.00
65	0.91	0.68	0.63	0.44	0.37	0.21	0.14	0.00	0.00	0.00	0.00
60	1.48	1.37	1.17	1.02	0.81	0.65	0.42	0.27	0.00	0.00	0.00
55	2.59	2.39	2.21	1.99	1.77	1.50	1.24	0.90	0.59	0.00	0.00
50	4.26	4.08	3.89	3.68	3.44	3.18	2.87	2.53	2.07	1.56	0.00
45	6.76	6.61	6.47	6.31	6.15	5.96	5.75	5.50	5.24	5.00	5.00
40	10.28	10.20	10.13	10.06	10.01	10.00	10.00	10.00	10.00	10.00	10.00
35	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00	15.00
30	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
25	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00
20	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00
15	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00	35.00
10	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00
5	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00	45.00
0	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00	50.00

where

$$\alpha_j = \frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) - \frac{\Delta t}{2\Delta Z^2}\sigma^2$$

$$\beta_j = 1 + \frac{\Delta t}{\Delta Z^2}\sigma^2 + r\,\Delta t$$

$$\gamma_j = -\frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) - \frac{\Delta t}{2\Delta Z^2}\sigma^2$$

The difference equation for the explicit method becomes

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + (r - q - \sigma^2/2)\,\frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta Z} + \tfrac{1}{2}\sigma^2\,\frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta Z^2} = r f_{i,j}$$

or

$$\alpha_j^*\, f_{i+1,j-1} + \beta_j^*\, f_{i+1,j} + \gamma_j^*\, f_{i+1,j+1} = f_{i,j} \tag{17.36}$$

where

$$\alpha_j^* = \frac{1}{1+r\,\Delta t}\left[-\frac{\Delta t}{2\Delta Z}(r-q-\sigma^2/2)+\frac{\Delta t}{2\Delta Z^2}\sigma^2\right] \tag{17.37}$$

$$\beta_j^* = \frac{1}{1+r\,\Delta t}\left(1-\frac{\Delta t}{\Delta Z^2}\sigma^2\right) \tag{17.38}$$

$$\gamma_j^* = \frac{1}{1+r\,\Delta t}\left[\frac{\Delta t}{2\Delta Z}(r-q-\sigma^2/2)+\frac{\Delta t}{2\Delta Z^2}\sigma^2\right] \tag{17.39}$$

The change of variable approach has the property that α_j, β_j, and γ_j as well as α_j^*, β_j^*, and γ_j^* are independent of j. It can be shown that it is numerically most efficient to set $\Delta Z = \sigma\sqrt{3\Delta t}$.

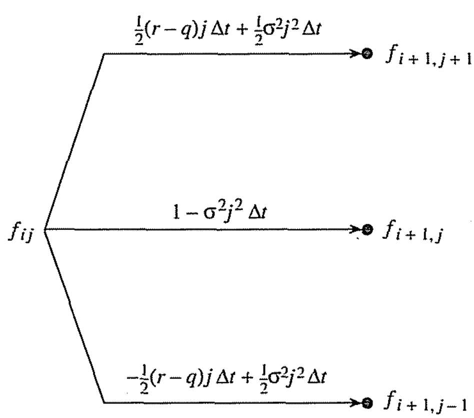
Relation to Trinomial Tree Approaches

The explicit finite difference method is equivalent to the trinomial tree approach.[28] In the expressions for a_j^*, b_j^*, and c_j^* in equation (17.34), we can interpret terms as follows:

$-\frac{1}{2}(r-q)j\,\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t$:	Probability of stock price decreasing from $j\,\Delta S$ to $(j-1)\Delta S$ in time Δt.
$1-\sigma^2 j^2 \Delta t$:	Probability of stock price remaining unchanged at $j\,\Delta S$ in time Δt.
$\frac{1}{2}(r-q)j\,\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t$:	Probability of stock price increasing from $j\,\Delta S$ to $(j+1)\Delta S$ in time Δt.

This interpretation is illustrated in Figure 17.17. The three probabilities sum to unity. They give the expected increase in the stock price in time Δt as $(r-q)j\,\Delta S\,\Delta t = (r-q)S\,\Delta t$. This is the expected increase in a risk-neutral world. For small values

Figure 17.17 Interpretation of explicit finite difference method as a trinomial tree.



[28] It can also be shown that the implicit finite difference method is equivalent to a multinomial tree approach where there are $M+1$ branches emanating from each node.

of Δt, they also give the variance of the change in the stock price in time Δt as $\sigma^2 j^2 \Delta S^2 \Delta t = \sigma^2 S^2 \Delta t$. This corresponds to the stochastic process followed by S. The value of f at time $i\,\Delta t$ is calculated as the expected value of f at time $(i+1)\,\Delta t$ in a risk-neutral world discounted at the risk-free rate.

For the explicit version of the finite difference method to work well, the three "probabilities"

$$-\tfrac{1}{2}(r-q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t,$$

$$1 - \sigma^2 j^2 \Delta t$$

$$\tfrac{1}{2}(r-q)j\,\Delta t + \tfrac{1}{2}\sigma^2 j^2 \Delta t$$

should all be positive. In Example 17.11, $1 - \sigma^2 j^2 \Delta t$ is negative when $j \geqslant 13$ (i.e., when $S \geqslant 65$). This explains the negative option prices and other inconsistencies in the top left-hand part of Table 17.5. This example illustrates the main problem associated with the explicit finite difference method. Because the probabilities in the associated tree may be negative, it does not necessarily produce results that converge to the solution of the differential equation.[29]

When the change-of-variable approach is used (see equations (17.36) to (17.39)), the probability that $Z = \ln S$ will decrease by ΔZ, stay the same, and increase by ΔZ are

$$-\frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) + \frac{\Delta t}{2\Delta Z^2}\sigma^2$$

$$1 - \frac{\Delta t}{\Delta z^2}\sigma^2$$

$$\frac{\Delta t}{2\Delta Z}(r - q - \sigma^2/2) + \frac{\Delta t}{2\Delta Z^2}\sigma^2$$

respectively. These movements in Z correspond to the stock price changing from S to $Se^{-\Delta Z}$, S, and $Se^{\Delta Z}$, respectively. If we set $\Delta Z = \sigma\sqrt{3\Delta t}$, then the tree and the probabilities are identical to those for the trinomial tree approach discussed in Section 17.4.

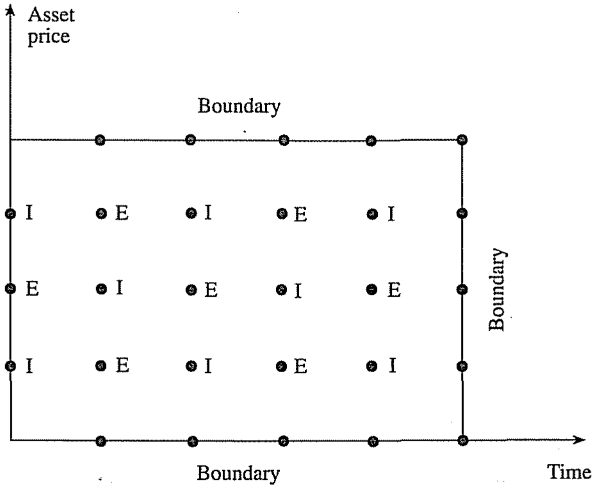
Other Finite Difference Methods

Many of the other finite difference methods that have been proposed have some of the features of the explicit finite difference method and some features of the implicit finite difference method.

In what is known as the *hopscotch method*, we alternate between the explicit and implicit calculations as we move from node to node. This is illustrated in Figure 17.18. At each time, we first do all the calculations at the "explicit nodes" in the usual way. We can then deal with the "implicit nodes" without solving a set of simultaneous equations because the values at the adjacent nodes have already been calculated.

[29] J. Hull and A. White, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis*, 25 (March 1990): 87–100, show how this problem can be overcome. In the situation considered here it is sufficient to construct the grid in $\ln S$ rather than S to ensure convergence.

Figure 17.18 The hopscotch method. I indicates node at which implicit calculations are done; E indicates node at which explicit calculations are done.



The *Crank–Nicolson* scheme is an average of the explicit and implicit methods. For the implicit method, equation (17.27) gives

$$f_{i,j} = a_j f_{i-1,j-1} + b_j f_{i-1,j} + c_j f_{i-1,j+1}$$

For the explicit method, equation (17.34) gives

$$f_{i-1,j} = a_j^* f_{i,j-1} + b_j^* f_{i,j} + c_j^* f_{i,j+1}$$

The Crank–Nicolson method averages these two equations to obtain

$$f_{i,j} + f_{i-1,j} = a_j f_{i-1,j-1} + b_j f_{i-1,j} + c_j f_{i-1,j+1} + a_j^* f_{i,j-1} + b_j^* f_{i,j} + c_j^* f_{i,j+1}$$

Putting

$$g_{i,j} = f_{i,j} - a_j^* f_{i,j-1} - b_j^* f_{i,j} - c_j^* f_{i,j+1}$$

we obtain

$$g_{i,j} = a_j f_{i-1,j-1} + b_j f_{i-1,j} + c_j f_{i-1,j+1} - f_{i-1,j}$$

This shows that implementing the Crank–Nicolson method is similar to implementing the implicit finite difference method. The advantage of the Crank–Nicolson method is that it has faster convergence than either the explicit or implicit method.

Applications of Finite Difference Methods

Finite difference methods can be used for the same types of derivative pricing problems as tree approaches. They can handle American-style as well as European-style derivatives but cannot easily be used in situations where the payoff from a derivative depends on the past history of the underlying variable. Finite difference methods can, at the expense of a considerable increase in computer time, be used when there are several state variables. The grid in Figure 17.15 then becomes multidimensional.

The method for calculating Greek letters is similar to that used for trees. Delta, gamma, and theta can be calculated directly from the $f_{i,j}$ values on the grid. For vega, it is necessary to make a small change to volatility and recalculate the value of the derivative using the same grid.

SUMMARY

We have presented three different numerical procedures for valuing derivatives when no analytic solution is available. These involve the use of trees, Monte Carlo simulation, and finite difference methods.

Binomial trees assume that, in each short interval of time Δt, a stock price either moves up by a multiplicative amount u or down by a multiplicative amount d. The sizes of u and d and their associated probabilities are chosen so that the change in the stock price has the correct mean and standard deviation in a risk-neutral world. Derivative prices are calculated by starting at the end of the tree and working backwards. For an American option, the value at a node is the greater of (a) the value if it is exercised immediately and (b) the discounted expected value if it is held for a further period of time Δt.

Monte Carlo simulation involves using random numbers to sample many different paths that the variables underlying the derivative could follow in a risk-neutral world. For each path, the payoff is calculated and discounted at the risk-free interest rate. The arithmetic average of the discounted payoffs is the estimated value of the derivative.

Finite difference methods solve the underlying differential equation by converting it to a difference equation. They are similar to tree approaches in that the computations work back from the end of the life of the derivative to the beginning. The explicit method is functionally the same as using a trinomial tree. The implicit finite difference method is more complicated but has the advantage that the user does not have to take any special precautions to ensure convergence.

In practice, the method that is chosen is likely to depend on the characteristics of the derivative being evaluated and the accuracy required. Monte Carlo simulation works forward from the beginning to the end of the life of a derivative. It can be used for European-style derivatives and can cope with a great deal of complexity as far as the payoffs are concerned. It becomes relatively more efficient as the number of underlying variables increases. Tree approaches and finite difference methods work from the end of the life of a security to the beginning and can accommodate American-style as well as European-style derivatives. However, they are difficult to apply when the payoffs depend on the past history of the state variables as well as on their current values. Also, they are liable to become computationally very time consuming when three or more variables are involved.

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Questions and Problems (Answers in Solutions Manual)

17.1. Which of the following can be estimated for an American option by constructing a single binomial tree: delta, gamma, vega, theta, rho?

17.2. Calculate the price of a 3-month American put option on a non-dividend-paying stock when the stock price is \$60, the strike price is \$60, the risk-free interest rate is 10% per annum, and the volatility is 45% per annum. Use a binomial tree with a time interval of 1 month.

17.3. Explain how the control variate technique is implemented when a tree is used to value American options.

17.4. Calculate the price of a 9-month American call option on corn futures when the current futures price is 198 cents, the strike price is 200 cents, the risk-free interest rate is 8% per annum, and the volatility is 30% per annum. Use a binomial tree with a time interval of 3 months.

17.5. Consider an option that pays off the amount by which the final stock price exceeds the average stock price achieved during the life of the option. Can this be valued using the binomial tree approach? Explain your answer.

17.6. "For a dividend-paying stock, the tree for the stock price does not recombine; but the tree for the stock price less the present value of future dividends does recombine." Explain this statement.

17.7. Show that the probabilities in a Cox, Ross, and Rubinstein binomial tree are negative when the condition in footnote 9 holds.

17.8. Use stratified sampling with 100 trials to improve the estimate of π in Business Snapshot 17.1 and Table 17.1.

The method for calculating Greek letters is similar to that used for trees. Delta, gamma, and theta can be calculated directly from the $f_{i,j}$ values on the grid. For vega, it is necessary to make a small change to volatility and recalculate the value of the derivative using the same grid.

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17.7. Show that the probabilities in a Cox, Ross, and Rubinstein binomial tree are negative when the condition in footnote 9 holds.

17.8. Use stratified sampling with 100 trials to improve the estimate of π in Business Snapshot 17.1 and Table 17.1.

17.9. Explain why the Monte Carlo simulation approach cannot easily be used for American-style derivatives.

17.10. A 9-month American put option on a non-dividend-paying stock has a strike price of \$49. The stock price is \$50, the risk-free rate is 5% per annum, and the volatility is 30% per annum. Use a three-step binomial tree to calculate the option price.

17.11. Use a three-time-step tree to value a 9-month American call option on wheat futures. The current futures price is 400 cents, the strike price is 420 cents, the risk-free rate is 6%, and the volatility is 35% per annum. Estimate the delta of the option from your tree.

17.12. A 3-month American call option on a stock has a strike price of \$20. The stock price is \$20, the risk-free rate is 3% per annum, and the volatility is 25% per annum. A dividend of \$2 is expected in 1.5 months. Use a three-step binomial tree to calculate the option price.

17.13. A 1-year American put option on a non-dividend-paying stock has an exercise price of \$18. The current stock price is \$20, the risk-free interest rate is 15% per annum, and the volatility of the stock price is 40% per annum. Use the DerivaGem software with four 3-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

17.14. A 2-month American put option on a stock index has an exercise price of 480. The current level of the index is 484, the risk-free interest rate is 10% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum. Divide the life of the option into four half-month periods and use the tree approach to estimate the value of the option.

17.15. How can the control variate approach improve the estimate of the delta of an American option when the tree approach is used?

17.16. Suppose that Monte Carlo simulation is being used to evaluate a European call option on a non-dividend-paying stock when the volatility is stochastic. How could the control variate and antithetic variable technique be used to improve numerical efficiency? Explain why it is necessary to calculate six values of the option in each simulation trial when both the control variate and the antithetic variable technique are used.

17.17. Explain how equations (17.27) to (17.30) change when the implicit finite difference method is being used to evaluate an American call option on a currency.

17.18. An American put option on a non-dividend-paying stock has 4 months to maturity. The exercise price is \$21, the stock price is \$20, the risk-free rate of interest is 10% per annum, and the volatility is 30% per annum. Use the explicit version of the finite difference approach to value the option. Use stock price intervals of \$4 and time intervals of 1 month.

17.19. The spot price of copper is \$0.60 per pound. Suppose that the futures prices (dollars per pound) are as follows:

3 months	0.59
6 months	0.57
9 months	0.54
12 months	0.50

The volatility of the price of copper is 40% per annum and the risk-free rate is 6% per annum. Use a binomial tree to value an American call option on copper with an exercise price of \$0.60 and a time to maturity of 1 year. Divide the life of the option into four 3-month periods for the purposes of constructing the tree. (*Hint*: As explained in Section 14.7, the futures price of a variable is its expected future price in a risk-neutral world.)

17.20. Use the binomial tree in Problem 17.19 to value a security that pays off x^2 in 1 year where x is the price of copper.

17.21. When do the boundary conditions for $S = 0$ and $S \to \infty$ affect the estimates of derivative prices in the explicit finite difference method?

17.22. How would you use the antithetic variable method to improve the estimate of the European option in Business Snapshot 17.2 and Table 17.2?

17.23. A company has issued a 3-year convertible bond that has a face value of \$25 and can be exchanged for two of the company's shares at any time. The company can call the issue when the share price is greater than or equal to \$18. Assuming that the company will force conversion at the earliest opportunity, what are the boundary conditions for the price of the convertible? Describe how you would use finite difference methods to value the convertible assuming constant interest rates. Assume there is no risk of the company defaulting.

17.24. Provide formulas that can be used for obtaining three random samples from standard normal distributions when the correlation between sample i and sample j is $\rho_{i,j}$.

Assignment Questions

17.25. An American put option to sell a Swiss franc for dollars has a strike price of \$0.80 and a time to maturity of 1 year. The volatility of the Swiss franc is 10%, the dollar interest rate is 6%, the Swiss franc interest rate is 3%, and the current exchange rate is 0.81. Use a three-time-step tree to value the option. Estimate the delta of the option from your tree.

17.26. A 1-year American call option on silver futures has an exercise price of \$9.00. The current futures price is \$8.50, the risk-free rate of interest is 12% per annum, and the volatility of the futures price is 25% per annum. Use the DerivaGem software with four 3-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

17.27. A 6-month American call option on a stock is expected to pay dividends of \$1 per share at the end of the second month and the fifth month. The current stock price is \$30, the exercise price is \$34, the risk-free interest rate is 10% per annum, and the volatility of the part of the stock price that will not be used to pay the dividends is 30% per annum. Use the DerivaGem software with the life of the option divided into six time steps to estimate the value of the option. Compare your answer with that given by Black's approximation (see Section 13.12).

17.28. The current value of the British pound is \$1.60 and the volatility of the pound/dollar exchange rate is 15% per annum. An American call option has an exercise price of \$1.62 and a time to maturity of 1 year. The risk-free rates of interest in the United States and the United Kingdom are 6% per annum and 9% per annum, respectively. Use the explicit finite difference method to value the option. Consider exchange rates at intervals of 0.20 between 0.80 and 2.40 and time intervals of 3 months.

17.29. Answer the following questions concerned with the alternative procedures for constructing trees in Section 17.4:

(a) Show that the binomial model in Section 17.4 is exactly consistent with the mean and variance of the change in the logarithm of the stock price in time Δt.

(b) Show that the trinomial model in Section 17.4 is consistent with the mean and variance of the change in the logarithm of the stock price in time Δt when terms of order $(\Delta t)^2$ and higher are ignored.

(c) Construct an alternative to the trinomial model in Section 17.4 so that the probabilities are 1/6, 2/3, and 1/6 on the upper, middle, and lower branches emanating from each node. Assume that the branching is from S to Su, Sm, or Sd with $m^2 = ud$. Match the mean and variance of the change in the logarithm of the stock price exactly.

17.30. The DerivaGem Application Buider functions enable you to investigate how the prices of options calculated from a binomial tree converge to the correct value as the number of time steps increases. (See Figure 17.4 and Sample Application A in DerivaGem.) Consider a put option on a stock index where the index level is 900, the strike price is 900, the risk-free rate is 5%, the dividend yield is 2%, and the time to maturity is 2 years.

(a) Produce results similar to Sample Application A on convergence for the situation where the option is European and the volatility of the index is 20%.

(b) Produce results similar to Sample Application A on convergence for the situation where the option is American and the volatility of the index is 20%.

(c) Produce a chart showing the pricing of the American option when the volatility is 20% as a function of the number of time steps when the control variate technique is used.

(d) Suppose that the price of the American option in the market is 85.0. Produce a chart showing the implied volatility estimate as a function of the number of time steps.