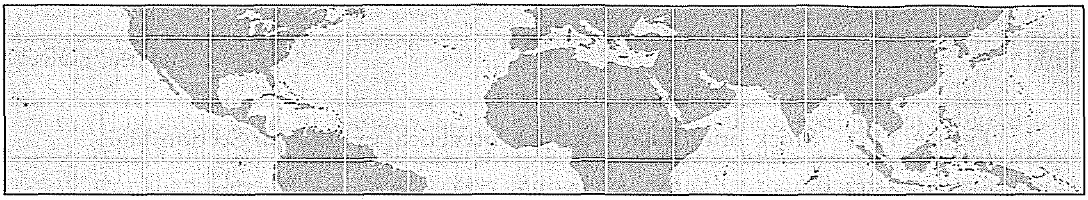


CHAPTER 11 Binomial Trees

A useful and very popular technique for pricing an option involves constructing a *binomial tree*. This is a diagram representing different possible paths that might be followed by the stock price over the life of an option. The underlying assumption is that the stock price follows a *random walk*. In each time step, it has a certain probability of moving up by a certain percentage amount and a certain probability of moving down by a certain percentage amount. In the limit, as the time step becomes smaller, this model leads to the lognormal assumption for stock prices that underlies the Black–Scholes model we will be discussing in Chapter 13.

In this chapter we will take a first look at binomial trees and their relationship to an important principle known as risk-neutral valuation. The general approach adopted here is similar to that in an important paper published by Cox, Ross, and Rubinstein in 1979. More details on numerical procedures involving binomial and trinomial trees are given in Chapter 17.

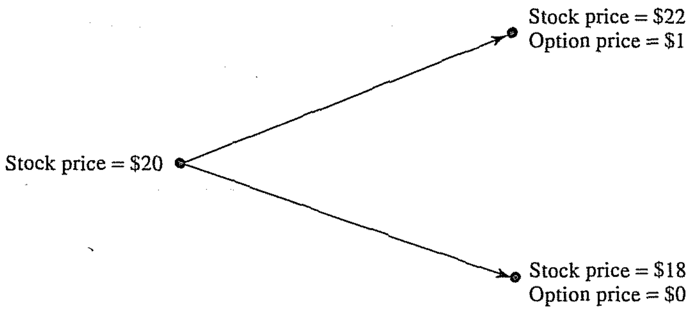
11.1 A ONE-STEP BINOMIAL MODEL

We start by considering a very simple situation. A stock price is currently \$20, and it is known that at the end of 3 months it will be either \$22 or \$18. We are interested in valuing a European call option to buy the stock for \$21 in 3 months. This option will have one of two values at the end of the 3 months. If the stock price turns out to be \$22, the value of the option will be \$1; if the stock price turns out to be \$18, the value of the option will be zero. The situation is illustrated in Figure 11.1.

It turns out that a relatively simple argument can be used to price the option in this example. The only assumption needed is that arbitrage opportunities do not exist. We set up a portfolio of the stock and the option in such a way that there is no uncertainty about the value of the portfolio at the end of the 3 months. We then argue that, because the portfolio has no risk, the return it earns must equal the risk-free interest rate. This enables us to work out the cost of setting up the portfolio and therefore the option's price. Because there are two securities (the stock and the stock option) and only two possible outcomes, it is always possible to set up the riskless portfolio.

Consider a portfolio consisting of a long position in Δ shares of the stock and a short position in one call option. We calculate the value of Δ that makes the portfolio riskless.

Figure 11.1 Stock price movements for numerical example in Section 11.1.



If the stock price moves up from \$20 to \$22, the value of the shares is 22Δ and the value of the option is 1, so that the total value of the portfolio is $22\Delta - 1$. If the stock price moves down from \$20 to \$18, the value of the shares is 18Δ and the value of the option is zero, so that the total value of the portfolio is 18Δ. The portfolio is riskless if the value of Δ is chosen so that the final value of the portfolio is the same for both alternatives. This means that

$$22\Delta - 1 = 18\Delta$$

or

$$\Delta = 0.25$$

A riskless portfolio is therefore

Long: 0.25 shares
Short: 1 option

If the stock price moves up to \$22, the value of the portfolio is

$$22 \times 0.25 - 1 = 4.5$$

If the stock price moves down to \$18, the value of the portfolio is

$$18 \times 0.25 = 4.5$$

Regardless of whether the stock price moves up or down, the value of the portfolio is always 4.5 at the end of the life of the option.

Riskless portfolios must, in the absence of arbitrage opportunities, earn the risk-free rate of interest. Suppose that in this case the risk-free rate is 12% per annum. It follows that the value of the portfolio today must be the present value of 4.5, or

$$4.5e^{-0.12\times 3/12} = 4.367$$

The value of the stock price today is known to be \$20. Suppose the option price is denoted by f. The value of the portfolio today is

$$20 \times 0.25 - f = 5 - f$$

It follows that

$$5 - f = 4.367$$

or

$$f = 0.633$$

This shows that, in the absence of arbitrage opportunities, the current value of the option must be 0.633. If the value of the option were more than 0.633, the portfolio would cost less than 4.367 to set up and would earn more than the risk-free rate. If the value of the option were less than 0.633, shorting the portfolio would provide a way of borrowing money at less than the risk-free rate.

A Generalization

We can generalize the argument just presented by considering a stock whose price is S_0 and an option on the stock whose current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, S_0u, where $u > 1$, or down from S_0 to a new level, S_0d, where $d < 1$. The percentage increase in the stock price when there is an up movement is $u - 1$; the percentage decrease when there is a down movement is $1 - d$. If the stock price moves up to S_0u, we suppose that the payoff from the option is f_u; if the stock price moves down to S_0d, we suppose the payoff from the option is f_d. The situation is illustrated in Figure 11.2.

As before, we imagine a portfolio consisting of a long position in Δ shares and a short position in one option. We calculate the value of Δ that makes the portfolio riskless. If there is an up movement in the stock price, the value of the portfolio at the end of the life of the option is

$$S_0u\Delta - f_u$$

If there is a down movement in the stock price, the value becomes

$$S_0d\Delta - f_d$$

The two are equal when

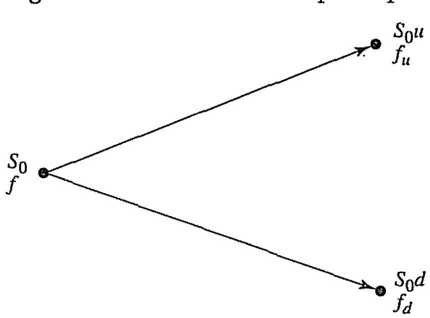
$$S_0u\Delta - f_u = S_0d\Delta - f_d$$

or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d} \tag{11.1}$$

In this case, the portfolio is riskless and must earn the risk-free interest rate. Equation (11.1) shows that Δ is the ratio of the change in the option price to the change in the stock price as we move between the nodes.

Figure 11.2 Stock and option prices in a general one-step tree.



If we denote the risk-free interest rate by r, the present value of the portfolio is

$$(S_0 u\Delta - f_u)e^{-rT}$$

The cost of setting up the portfolio is

$$S_0\Delta - f$$

It follows that

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT}$$

or

$$f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

Substituting from equation (11.1) for Δ and simplifying, we can reduce this equation to

$$f = e^{-rT}[pf_u + (1-p)f_d] \tag{11.2}$$

where

$$p = \frac{e^{rT} - d}{u - d} \tag{11.3}$$

Equations (11.2) and (11.3) enable an option to be priced when stock price movements are given by a one-step binomial tree.

In the numerical example considered previously (see Figure 11.1), $u = 1.1$, $d = 0.9$, $r = 0.12$, $T = 0.25$, $f_u = 1$, and $f_d = 0$. From equation (11.3), we have

$$p = \frac{e^{0.12\times 3/12} - 0.9}{1.1 - 0.9} = 0.6523$$

and, from equation (11.2), we have

$$f = e^{-0.12\times 0.25}(0.6523 \times 1 + 0.3477 \times 0) = 0.633$$

The result agrees with the answer obtained earlier in this section.

Irrelevance of the Stock's Expected Return

The option pricing formula in equation (11.2) does not involve the probabilities of the stock price moving up or down. For example, we get the same option price when the probability of an upward movement is 0.5 as we do when it is 0.9. This is surprising and seems counterintuitive. It is natural to assume that, as the probability of an upward movement in the stock price increases, the value of a call option on the stock increases and the value of a put option on the stock decreases. This is not the case.

The key reason is that we are not valuing the option in absolute terms. We are calculating its value in terms of the price of the underlying stock. The probabilities of future up or down movements are already incorporated into the stock price: we do not need to take them into account again when valuing the option in terms of the stock price.

11.2 RISK-NEUTRAL VALUATION

Although we do not need to make any assumptions about the probabilities of up and down movements in order to derive equation (11.2), it is natural to interpret the variable p in equation (11.2) as the probability of an up movement in the stock price.

The variable $1 - p$ is then the probability of a down movement, and the expression

$$pf_u + (1 - p)f_d$$

is the expected payoff from the option. With this interpretation of p, equation (11.2) then states that the value of the option today is its expected future payoff discounted at the risk-free rate.

We now investigate the expected return from the stock when the probability of an up movement is p. The expected stock price at time T, $E(S_T)$, is given by

$$E(S_T) = pS_0u + (1 - p)S_0d$$

or

$$E(S_T) = pS_0(u - d) + S_0d$$

Substituting from equation (11.3) for p, we obtain

$$E(S_T) = S_0e^{rT} \tag{11.4}$$

showing that the stock price grows on average at the risk-free rate. Setting the probability of the up movement equal to p is therefore equivalent to assuming that the return on the stock equals the risk-free rate.

In a *risk-neutral world* all individuals are indifferent to risk. In such a world, investors require no compensation for risk, and the expected return on all securities is the risk-free interest rate. Equation (11.4) shows that we are assuming a risk-neutral world when we set the probability of an up movement to p. Equation (11.2) shows that the value of the option is its expected payoff in a risk-neutral world discounted at the risk-free rate.

This result is an example of an important general principle in option pricing known as *risk-neutral valuation*. The principle states that we can with complete impunity assume the world is risk neutral when pricing options. The resulting prices are correct not just in a risk-neutral world, but in other worlds as well.

The One-Step Binomial Example Revisited

We now return to the example in Figure 11.1 and illustrate that risk-neutral valuation gives the same answer as no-arbitrage arguments. In Figure 11.1, the stock price is currently \$20 and will move either up to \$22 or down to \$18 at the end of 3 months. The option considered is a European call option with a strike price of \$21 and an expiration date in 3 months. The risk-free interest rate is 12% per annum.

We define p as the probability of an upward movement in the stock price in a risk-neutral world. We can calculate p from equation (11.3). Alternatively, we can argue that the expected return on the stock in a risk-neutral world must be the risk-free rate of 12%. This means that p must satisfy

$$22p + 18(1 - p) = 20e^{0.12\times3/12}$$

or

$$4p = 20e^{0.12\times3/12} - 18$$

That is, p must be 0.6523.

At the end of the 3 months, the call option has a 0.6523 probability of being worth 1 and a 0.3477 probability of being worth zero. Its expected value is therefore

$$0.6523 \times 1 + 0.3477 \times 0 = 0.6523$$

In a risk-neutral world this should be discounted at the risk-free rate. The value of the option today is therefore

$$0.6523e^{-0.12\times 3/12}$$

or \$0.633. This is the same as the value obtained earlier, demonstrating that no-arbitrage arguments and risk-neutral valuation give the same answer.

Real World vs. Risk-Neutral World

It should be emphasized that p is the probability of an up movement in a risk-neutral world. In general this is not the same as the probability of an up movement in the real world. In our example $p = 0.6523$. When the probability of an up movement is 0.6523, the expected return on both the stock and the option is the risk-free rate of 12%. Suppose that, in the real world, the expected return on the stock is 16% and p^* is the probability of an up movement. It follows that

$$22p^* + 18(1 - p^*) = 20e^{0.16\times 3/12}$$

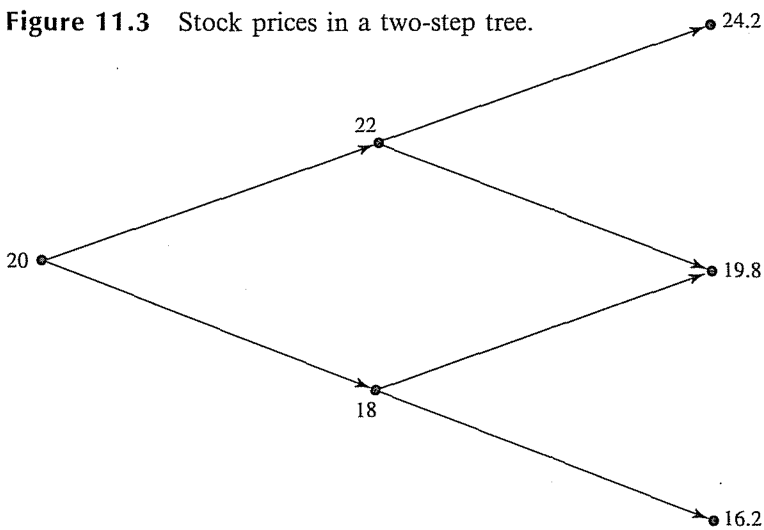
so that $p^* = 0.7041$.

The expected payoff from the option in the real world is then given by

$$p^* \times 1 + (1 - p^*) \times 0$$

This is 0.7041. Unfortunately it is not easy to know the correct discount rate to apply to the expected payoff in the real world. A position in a call option is riskier than a position in the stock. As a result the discount rate to be applied to the payoff from a call option is greater than 16%. Without knowing the option's value, we do not know how much greater than 16% it should be.[1] Using risk-neutral valuation is convenient

Figure 11.3 Stock prices in a two-step tree.



[1] Because the correct value of the option is 0.633, we can deduce that the correct discount rate is 42.58%. This is because $0.633 = 0.7041e^{-0.4258\times 3/12}$.

because we know that in a risk-neutral world the expected return on all assets (and therefore the discount rate to use for all expected payoffs) is the risk-free rate.

11.3 TWO-STEP BINOMIAL TREES

We can extend the analysis to a two-step binomial tree such as that shown in Figure 11.3. Here the stock price starts at \$20 and in each of two time steps may go up by 10% or down by 10%. We suppose that each time step is 3 months long and the risk-free interest rate is 12% per annum. As before, we consider an option with a strike price of \$21.

The objective of the analysis is to calculate the option price at the initial node of the tree. This can be done by repeatedly applying the principles established earlier in the chapter. Figure 11.4 shows the same tree as Figure 11.3, but with both the stock price and the option price at each node. (The stock price is the upper number and the option price is the lower number.) The option prices at the final nodes of the tree are easily calculated. They are the payoffs from the option. At node D the stock price is 24.2 and the option price is $24.2 - 21 = 3.2$; at nodes E and F the option is out of the money and its value is zero.

At node C the option price is zero, because node C leads to either node E or node F and at both nodes the option price is zero. We calculate the option price at node B by focusing our attention on the part of the tree shown in Figure 11.5. Using the notation introduced earlier in the chapter, $u = 1.1$, $d = 0.9$, $r = 0.12$, and $T = 0.25$, so that $p = 0.6523$, and equation (11.2) gives the value of the option at node B as

$$e^{-0.12\times 3/12}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$$

Figure 11.4 Stock and option prices in a two-step tree. The upper number at each node is the stock price and the lower number is the option price.

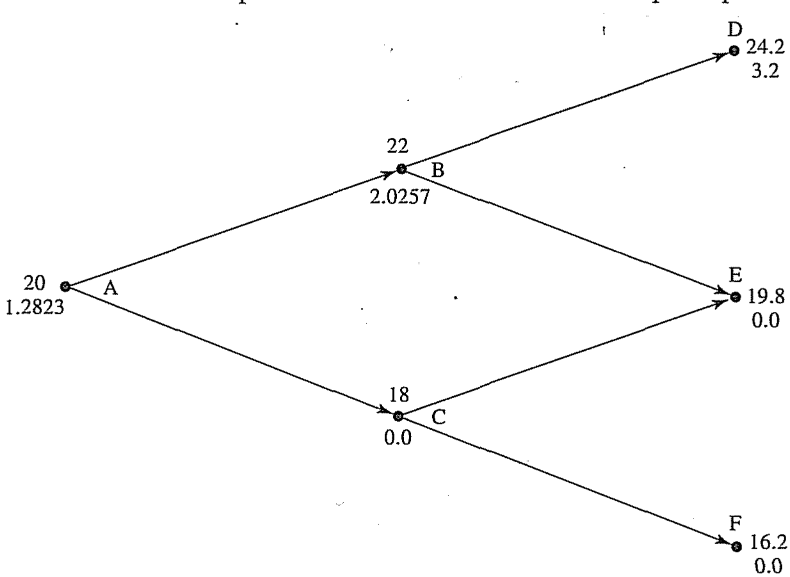
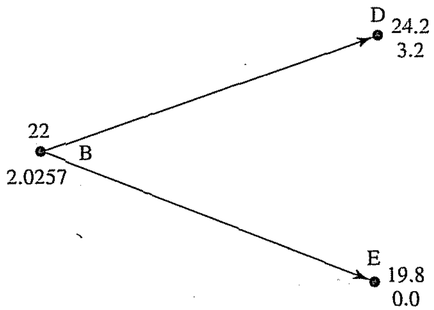


Figure 11.5 Evaluation of option price at node B.



It remains for us to calculate to option price at the initial node A. We do so by focusing on the first step of the tree. We know that the value of the option at node B is 2.0257 and that at node C it is zero. Equation (11.2) therefore gives the value at node A as

$$e^{-0.12\times 3/12}(0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$$

The value of the option is \$1.2823.

Note that this example was constructed so that u and d (the proportional up and down movements) were the same at each node of the tree and so that the time steps were of the same length. As a result, the risk-neutral probability, p, as calculated by equation (11.3) is the same at each node.

A Generalization

We can generalize the case of two time steps by considering the situation in Figure 11.6. The stock price is initially S_0. During each time step, it either moves up to u times its initial value or moves down to d times its initial value. The notation for the value of the option is shown on the tree. (For example, after two up movements the value of the option is f_{uu}.) We suppose that the risk-free interest rate is r and the length of the time step is Δt years.

Because the length of a time step is now Δt rather than T, equations (11.2) and (11.3) become

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d] \tag{11.5}$$

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{11.6}$$

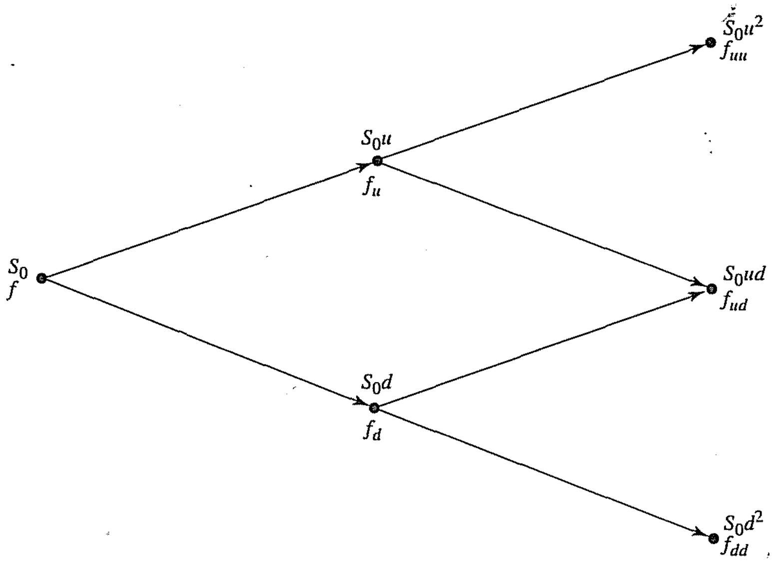
Repeated application of equation (11.5) gives

$$f_u = e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}] \tag{11.7}$$

$$f_d = e^{-r\Delta t}[pf_{ud} + (1-p)f_{dd}] \tag{11.8}$$

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d] \tag{11.9}$$

Figure 11.6 Stock and option prices in general two-step tree.



Substituting from equations (11.7) and (11.8) into (11.9), we get

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}] \tag{11.10}$$

This is consistent with the principle of risk-neutral valuation mentioned earlier. The variables p^2, $2p(1-p)$, and $(1-p)^2$ are the probabilities that the upper, middle, and lower final nodes will be reached. The option price is equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate.

As we add more steps to the binomial tree, the risk-neutral valuation principle continues to hold. The option price is always equal to its expected payoff in a risk-neutral world discounted at the risk-free interest rate.

11.4 A PUT EXAMPLE

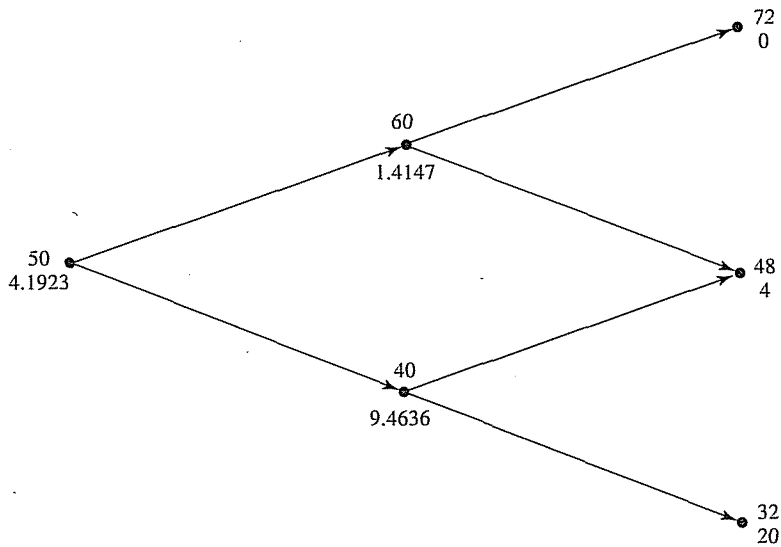
The procedures described in this chapter can be used to price puts as well as calls. Consider a 2-year European put with a strike price of \$52 on a stock whose current price is \$50. We suppose that there are two time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%. We also suppose that the risk-free interest rate is 5%.

The tree is shown in Figure 11.7. In this case $u = 1.2$, $d = 0.8$, $\Delta t = 1$, and $r = 0.05$. From equation (11.6) the value of the risk-neutral probability, p, is given by

$$p = \frac{e^{0.05\times 1} - 0.8}{1.2 - 0.8} = 0.6282$$

The possible final stock prices are: \$72, \$48, and \$32. In this case, $f_{uu} = 0$, $f_{ud} = 4$,

Figure 11.7 Using a two-step tree to value a European put option. At each node, the upper number is the stock price and the lower number is the option price.



and $f_{dd} = 20$. From equation (11.10), we have

$$f = e^{-2\times0.05\times1}(0.6282^2 \times 0 + 2 \times 0.6282 \times 0.3718 \times 4 + 0.3718^2 \times 20) = 4.1923$$

The value of the put is \$4.1923. This result can also be obtained using equation (11.5) and working back through the tree one step at a time. Figure 11.7 shows the intermediate option prices that are calculated.

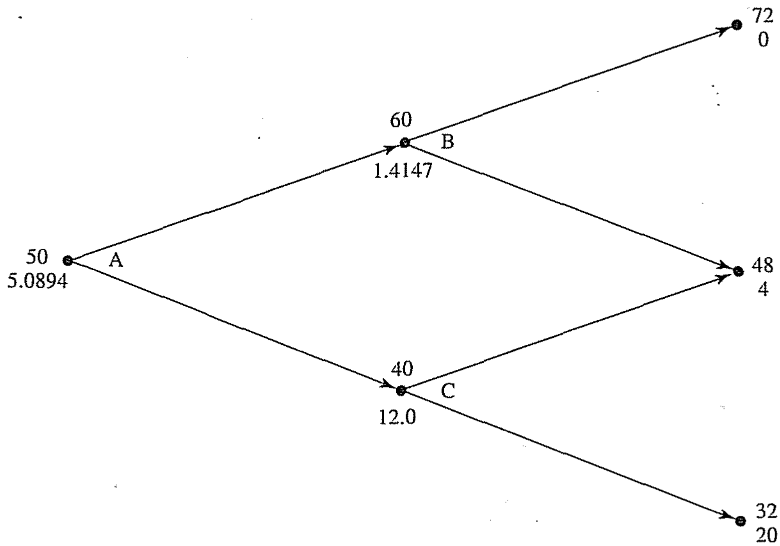
11.5 AMERICAN OPTIONS

Up to now all the options we have considered have been European. We now move on to consider how American options can be valued using a binomial tree such as that in Figure 11.4 or 11.7. The procedure is to work back through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for the European option. At earlier nodes the value of the option is the greater of

1. The value given by equation (11.5)
2. The payoff from early exercise

Figure 11.8 shows how Figure 11.7 is affected if the option under consideration is American rather than European. The stock prices and their probabilities are unchanged. The values for the option at the final nodes are also unchanged. At node B, equation (11.5) gives the value of the option as 1.4147, whereas the payoff from early exercise is negative (= −8). Clearly early exercise is not optimal at node B, and the value of the option at this node is 1.4147. At node C, equation (11.5) gives the value of the

Figure 11.8 Using a two-step tree to value an American put option. At each node, the upper number is the stock price and the lower number is the option price.



option as 9.4636, whereas the payoff from early exercise is 12. In this case, early exercise is optimal and the value of the option at the node is 12. At the initial node A, the value given by equation (11.5) is

$$e^{-0.05\times 1}(0.6282 \times 1.4147 + 0.3718 \times 12.0) = 5.0894$$

and the payoff from early exercise is 2. In this case early exercise is not optimal. The value of the option is therefore \$5.0894.

11.6 DELTA

At this stage it is appropriate to introduce *delta*, an important parameter in the pricing and hedging of options.

The delta of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock. It is the number of units of the stock we should hold for each option shorted in order to create a riskless hedge. It is the same as the Δ introduced earlier in this chapter. The construction of a riskless hedge is sometimes referred to as *delta hedging*. The delta of a call option is positive, whereas the delta of a put option is negative.

From Figure 11.1, we can calculate the value of the delta of the call option being considered as

$$\frac{1-0}{22-18} = 0.25$$

This is because when the stock price changes from \$18 to \$22, the option price changes from \$0 to \$1.

In Figure 11.4 the delta corresponding to stock price movements over the first time step is

$$\frac{2.0257 - 0}{22 - 18} = 0.5064$$

The delta for stock price movements over the second time step is

$$\frac{3.2 - 0}{24.2 - 19.8} = 0.7273$$

if there is an upward movement over the first time step, and

$$\frac{0 - 0}{19.8 - 16.2} = 0$$

if there is a downward movement over the first time step.

From Figure 11.7, delta is

$$\frac{1.4147 - 9.4636}{60 - 40} = -0.4024$$

at the end of the first time step, and either

$$\frac{0 - 4}{72 - 48} = -0.1667 \quad \text{or} \quad \frac{4 - 20}{48 - 32} = -1.0000$$

at the end of the second time step.

The two-step examples show that delta changes over time. (In Figure 11.4, delta changes from 0.5064 to either 0.7273 or 0; and, in Figure 11.7, it changes from −0.4024 to either −0.1667 or −1.0000.) Thus, in order to maintain a riskless hedge using an option and the underlying stock, we need to adjust our holdings in the stock periodically. This is a feature of options that we will return to in Chapter 15.

11.7 MATCHING VOLATILITY WITH u AND d

In practice, when constructing a binomial tree to represent the movements in a stock price, we choose the parameters u and d to match the volatility of the stock price. To see how this is done, we suppose that the expected return on a stock (in the real world) is μ and its volatility is σ. Figure 11.9(a) shows stock price movements over the first step of a binomial tree. The step is of length Δt. The stock price starts at S_0 and moves either up to S_0u or down to S_0d. The probability of an up movement (in the real world) is assumed to be p^*.

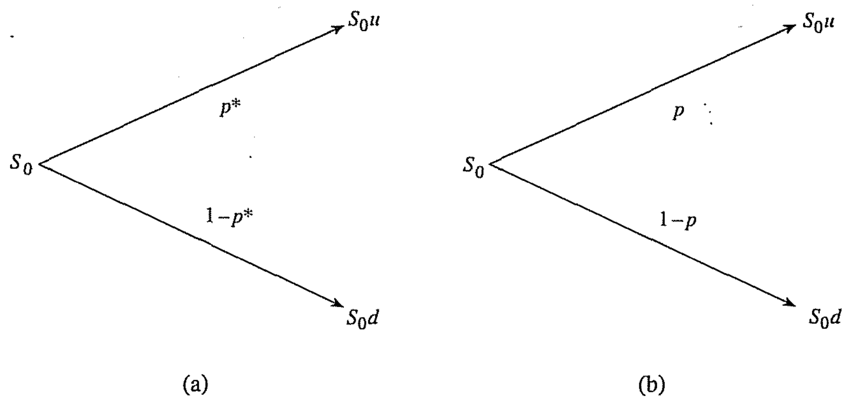
The expected stock price at the end of the first time step is $S_0e^{\mu\Delta t}$. On the tree the expected stock price at this time is

$$p^*S_0u + (1 - p^*)S_0d$$

In order to match the expected return on the stock with the tree's parameters, we must therefore have

$$p^*S_0u + (1 - p^*)S_0d = S_0e^{\mu\Delta t}$$

Figure 11.9 Change in stock price in time Δt in (a) the real world and (b) the risk-neutral world.



or

$$p^* = \frac{e^{\mu\Delta t} - d}{u - d} \tag{11.11}$$

As we will explain in Chapter 13, the volatility σ of a stock price is defined so that $\sigma\sqrt{\Delta t}$ is the standard deviation of the return on the stock price in a short period of time of length Δt. Equivalently, the variance of the return is $\sigma^2\Delta t$. On the tree in Figure 11.9(a), the variance of the stock price return is[2]

$$p^*u^2 + (1 - p^*)d^2 - [p^*u + (1 - p^*)d]^2$$

In order to match the stock price volatility with the tree's parameters, we must therefore have

$$p^*u^2 + (1 - p^*)d^2 - [p^*u + (1 - p^*)d]^2 = \sigma^2\Delta t \tag{11.12}$$

Substituting from equation (11.11) into equation (11.12), we get

$$e^{\mu\Delta t}(u + d) - ud - e^{2\mu\Delta t} = \sigma^2\Delta t$$

When terms in Δt^2 and higher powers of Δt are ignored, one solution to this equation is[3]

$$u = e^{\sigma\sqrt{\Delta t}} \tag{11.13}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{11.14}$$

These are the values of u and d proposed by Cox, Ross, and Rubinstein (1979) for matching u and d.

[2] This uses the result that the variance of a variable X equals $E(X^2) - [E(X)]^2$, where E denotes expected value.

[3] We are here using the series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The analysis in Section 11.2 shows that we can replace the tree in Figure 11.9(a) by the tree in Figure 11.9(b), where the probability of an up movement is p, and then behave as though the world is risk neutral. The variable p is given by equation (11.6) as

$$p = \frac{a - d}{u - d} \tag{11.15}$$

where

$$a = e^{r\Delta t} \tag{11.16}$$

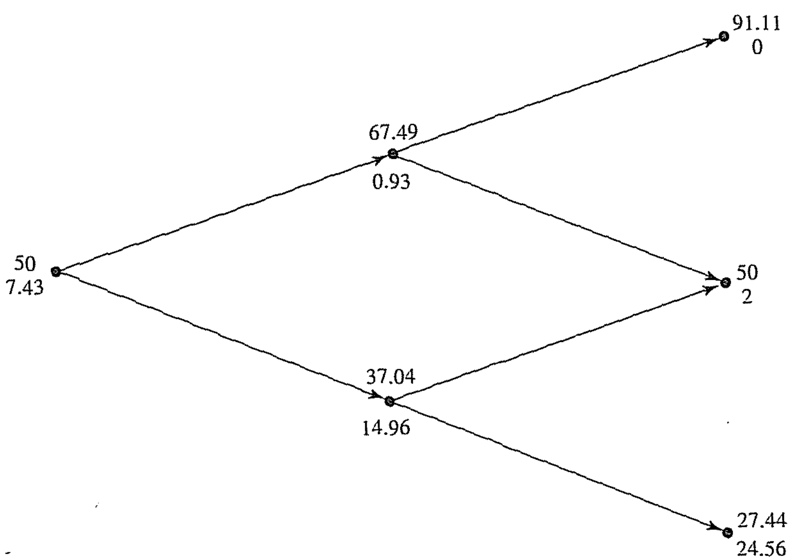
It is the risk-neutral probability of an up movement. In Figure 11.9(b), the expected stock price at the end of the time step is $S_0 e^{r\Delta t}$, as shown in equation (11.4). The variance of the stock price return is

$$pu^2 + (1 - p)d^2 - [pu + (1 - p)d]^2 = [e^{r\Delta t}(u + d) - ud - e^{2r\Delta t}]$$

Substituting for u and d from equations (11.13) and (11.14), we find this equals $\sigma^2 \Delta t$ when terms in Δt^2 and higher powers of Δt are ignored.

This analysis shows that when we move from the real world to the risk-neutral world the expected return on the stock changes, but its volatility remains the same (at least in the limit as Δt tends to zero). This is an illustration of an important general result known as *Girsanov's theorem*. When we move from a world with one set of risk preferences to a world with another set of risk preferences, the expected growth rates in variables change, but their volatilities remain the same. We will examine the impact of risk preferences on the behavior of market variables in more detail in Chapter 25. Moving from one set of risk preferences to another is sometimes referred to as *changing*

Figure 11.10 Two-step tree to value an American 2-year put option when the stock price is 50, strike price is 52, risk-free rate is 5%, and volatility is 30%.



the measure. The real-world measure is sometimes referred to as the *P-measure*, while the risk-neutral world measure is referred to as the *Q-measure*.[4]

Consider again the American put option in Figures 11.8, where the stock price is $50, the strike price is $52, the risk-free rate is 5%, the life of the option is 2 years, and there are two time steps. In this case, $\Delta t = 1$. Suppose that the volatility σ is 30%. Then, from equations (11.13) to (11.16), we have

$$u = e^{0.3\times 1} = 1.3499, \qquad d = \frac{1}{1.3499} = 0.7408, \qquad a = e^{0.05\times 1} = 1.0513$$

and

$$p = \frac{1.053 - 0.7408}{1.3499 - 0.7408} = 0.5097$$

The tree is shown in Figure 11.10. The value of the put option is 7.43. This is different from the value obtained in Figure 11.8 by assuming $u = 1.2$ and $d = 0.8$.

11.8 INCREASING THE NUMBER OF STEPS

The binomial model presented above is unrealistically simple. Clearly, an analyst can expect to obtain only a very rough approximation to an option price by assuming that stock price movements during the life of the option consist of one or two binomial steps.

When binomial trees are used in practice, the life of the option is typically divided into 30 or more time steps. In each time step there is a binomial stock price movement. With 30 time steps there are 31 terminal stock prices and 2^{30}, or about 1 billion, possible stock price paths are considered.

The equations defining the tree are equations (11.13) to (11.16), regardless of the number of time steps. Suppose, for example, that there are five steps instead of two in the example we considered in Figure 11.10. The parameters would be $\Delta t = 2/5 = 0.4$, $r = 0.05$, and $\sigma = 0.3$. These values give $u = e^{0.3\times\sqrt{0.4}} = 1.2089$, $d = 1/1.2089 = 0.8272$, $a = e^{0.05\times 0.4} = 1.0202$, and $p = (1.0202 - 0.8272)/(1.2089 - 0.8272) = 0.5056$.

Using DerivaGem

The software accompanying this book, DerivaGem, is a useful tool for becoming comfortable with binomial trees. After loading the software in the way described at the end of this book, go to the Equity_FX_Index_Futures_Options worksheet. Choose Equity as the Underlying Type and select Binomial American as the Option Type. Enter the stock price, volatility, risk-free rate, time to expiration, exercise price, and tree steps, as 50, 30%, 5%, 2, 52, and 2, respectively. Click on the *Put* button and then on *Calculate*. The price of the option is shown as 7.428 in the box labeled Price. Now click on *Display Tree* and you will see the equivalent of Figure 11.10. (The red numbers in the software indicate the nodes where the option is exercised.)

Return to the Equity_FX_Index_Futures_Options worksheet and change the number of time steps to 5. Hit *Enter* and click on *Calculate*. You will find that the value of the option changes to 7.671. By clicking on *Display Tree* the five-step tree is displayed, together with the values of u, d, a, and p calculated above.

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[4] With the notation we have been using, p is the probability under the Q-measure, while p^* is the probability under the P-measure.

DerivaGem can display trees that have up to 10 steps, but the calculations can be done for up to 500 steps. In our example, 500 steps gives the option price (to two decimal places) as 7.47. This is an accurate answer. By changing the Option Type to Binomial European we can use the tree to value a European option. Using 500 time steps the value of a European option with the same parameters as the American option is 6.76. (By changing the option type to Analytic European we can display the value the option using the Black–Scholes formula that will be presented in Chapter 13. This is also 6.76.)

By changing the Underlying Type, we can consider options on assets other than stocks. These will now be discussed.

11.9 OPTIONS ON OTHER ASSETS

We introduced options on indices, currencies, and futures contracts in Chapter 8 and will cover them in more detail in Chapter 14. It turns out that we can construct and use binomial trees for these options in exactly the same way as for options on stocks except that the equations for p change. As in the case of options on stocks, equation (11.2) applies so that the value at a node (before the possibility of early exercise is considered) is p times the value if there is an up movement plus $1-p$ times the value if there is a down movement, discounted at the risk-free rate.

Options on Stocks Paying a Continuous Dividend Yield

Consider a stock paying a known dividend yield at rate q. The total return from dividends and capital gains in a risk-neutral world is r. The dividends provide a return of q. Capital gains must therefore provide a return of $r-q$. If the stock starts at S_0, its expected value after one time step of length Δt must be $S_0 e^{(r-q)\Delta t}$. This means that

$$pS_0 u + (1-p)S_0 d = S_0 e^{(r-q)\Delta t}$$

so that

$$p = \frac{e^{(r-q)\Delta t} - d}{u-d}$$

As in the case of options on non-dividend-paying stocks, we match volatility by setting $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u$. This means that we can use equations (11.13) to (11.16) except that we set $a = e^{(r-q)\Delta t}$.

Options on Stock Indices

When calculating a futures price for a stock index in Chapter 5 we assumed that the stocks underlying the index provided a dividend yield at rate q. We make a similar assumption here. The valuation of an option on a stock index is therefore very similar to the valuation of an option on a stock paying a known dividend yield.

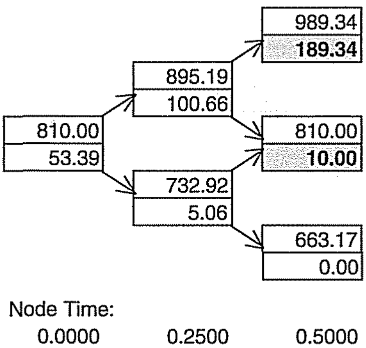
Example 11.1

A stock index is currently 810 and has a volatility of 20% and a dividend yield of 2%. The risk-free rate is 5%. Figure 11.11 shows the output from DerivaGem for valuing a European 6-month call option with a strike price of 800 using a two-step tree.

Figure 11.11 Two-step tree to value an European 6-month call option on an index when the index level is 810, strike price is 800, risk-free rate is 5%, volatility is 20%, and dividend yield is 2%.

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 800
Discount factor per step = 0.9876
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 1.0075
Probability of up move, p = 0.5126
Up step size, u = 1.1052
Down step size, d = 0.9048



In this case,

$$\Delta t = 0.25, \qquad u = e^{0.20\times\sqrt{0.25}} = 1.1052,$$
$$d = 1/u = 0.9048, \qquad a = e^{(0.05-0.02)\times 0.25} = 1.0075$$
$$p = (1.0075 - 0.9048)/(1.1052 - 0.9048) = 0.5126$$

The value of the option is 53.39.

Options on Currencies

As pointed out in Section 5.10, a foreign currency can be regarded as an asset providing a yield at the foreign risk-free rate of interest, r_f. By analogy with the stock index case we can construct a tree for options on a currency by using equations (11.13) to (11.16) and setting $a = e^{(r-r_f)\Delta t}$.

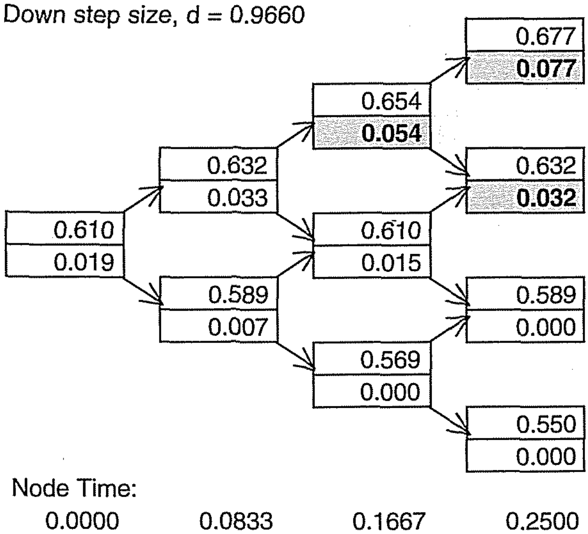
Example 11.2

The Australian dollar is currently worth 0.6100 U.S. dollars and this exchange rate has a volatility of 12%. The Australian risk-free rate is 7% and the U.S. risk-free rate is 5%. Figure 11.12 shows the output from DerivaGem for valuing a 3-month American call option with a strike price of 0.6000 using a three-step tree.

Figure 11.12 Three-step tree to value an American 3-month call option on a currency when the value of the currency is 0.6100, strike price is 0.6000, risk-free rate is 5%, volatility is 12%, and foreign risk-free rate is 7%.

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 0.6
Discount factor per step = 0.9958
Time step, dt = 0.0833 years, 30.42 days
Growth factor per step, a = 0.9983
Probability of up move, p = 0.4673
Up step size, u = 1.0352
Down step size, d = 0.9660



In this case,

$$\Delta t = 0.08333, \quad u = e^{0.12\times\sqrt{0.08333}} = 1.0352$$
$$d = 1/u = 0.9660, \quad a = e^{(0.05-0.07)\times 0.08333} = 0.9983$$
$$p = (0.9983 - 0.9660)/(1.0352 - 0.9660) = 0.4673$$

The value of the option is 0.019.

Options on Futures

It costs nothing to take a long or a short position in a futures contract. It follows that in a risk-neutral world a futures price should have an expected growth rate of zero. (We discuss this point in more detail in Section 14.7.) Similarly to above, we define p as the probability of an up movement in the futures price, u as the percentage up movement, and d as the percentage down movement. If F_0 is the initial futures price, the expected futures price at the end of one time step of length Δt should also be F_0. This means that

$$pF_0u + (1 - p)F_0d = F_0$$

so that

$$p = \frac{1 - d}{u - d}$$

and we can use equations (11.13) to (11.16) with $a = 1$.

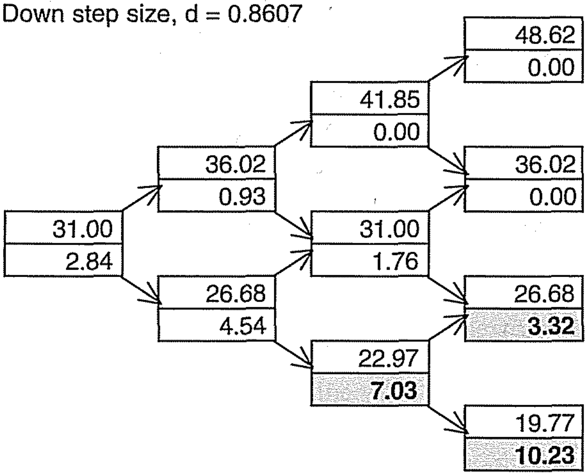
Example 11.3

A futures price is currently 31 and has a volatility of 30%. The risk-free rate is 5%. Figure 11.13 shows the output from DerivaGem for valuing a 9-month American put option with a strike price of 30 using a three-step tree.

Figure 11.13 Three-step tree to value an American 9-month put option on a futures contract when the futures price is 31, strike price is 30, risk-free rate is 5%, and volatility is 30%.

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 30
Discount factor per step = 0.9876
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 1.000
Probability of up move, p = 0.4626
Up step size, u = 1.1618
Down step size, d = 0.8607



Node Time:
0.0000 0.2500 0.5000 0.7500

In this case,

$$\Delta t = 0.25, \quad u = e^{0.3\sqrt{0.25}} = 1.1618$$
$$d = 1/u = 1/1.1618 = 0.8607, \quad a = 1,$$
$$p = (1 - 0.8607)/(1.1618 - 0.8607) = 0.4626$$

The value of the option is 2.84.

SUMMARY

This chapter has provided a first look at the valuation of options on stocks and other assets. In the simple situation where movements in the price of a stock during the life of an option are governed by a one-step binomial tree, it is possible to set up a portfolio consisting of a stock option and the stock that is riskless. In a world with no arbitrage opportunities, riskless portfolios must earn the risk-free interest. This enables the stock option to be priced in terms of the stock. It is interesting to note that no assumptions are required about the probabilities of up and down movements in the stock price at each node of the tree.

When stock price movements are governed by a multistep binomial tree, we can treat each binomial step separately and work back from the end of the life of the option to the beginning to obtain the current value of the option. Again only no-arbitrage arguments are used, and no assumptions are required about the probabilities of up and down movements in the stock price at each node.

A very important principle states that we can assume the world is risk-neutral when valuing an option. This chapter has shown, through both numerical examples and algebra, that no-arbitrage arguments and risk-neutral valuation are equivalent and lead to the same option prices.

The delta of a stock option, Δ, considers the effect of a small change in the underlying stock price on the change in the option price. It is the ratio of the change in the option price to the change in the stock price. For a riskless position, an investor should buy Δ shares for each option sold. An inspection of a typical binomial tree shows that delta changes during the life of an option. This means that to hedge a particular option position, we must change our holding in the underlying stock periodically.

Constructing binomial trees for valuing options on stock indices, currencies, and futures contracts is very similar to doing so for valuing options on stocks. In Chapter 17, we will return to binomial trees and give a more details on how they can be used in practice.

FURTHER READING

Coval, J. E. and T. Shumway. "Expected Option Returns," *Journal of Finance*, 56, 3 (2001): 983–1009.

Cox, J. C., S. A. Ross, and M. Rubinstein. "Option Pricing: A Simplified Approach." *Journal of Financial Economics* 7 (October 1979): 229–64.

Rendleman, R., and B. Bartter. "Two State Option Pricing." *Journal of Finance* 34 (1979): 1092–1110.

Questions and Problems (Answers in Solutions Manual)

11.1. A stock price is currently \$40. It is known that at the end of 1 month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-month European call option with a strike price of \$39?

11.2. Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

11.3. What is meant by the "delta" of a stock option?

11.4. A stock price is currently \$50. It is known that at the end of 6 months it will be either \$45 or \$55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 6-month European put option with a strike price of \$50?

11.5. A stock price is currently \$100. Over each of the next two 6-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-year European call option with a strike price of \$100?

11.6. For the situation considered in Problem 11.5, what is the value of a 1-year European put option with a strike price of \$100? Verify that the European call and European put prices satisfy put–call parity.

11.7. What are the formulas for u and d in terms of volatility?

11.8. Consider the situation in which stock price movements during the life of a European option are governed by a two-step binomial tree. Explain why it is not possible to set up a position in the stock and the option that remains riskless for the whole of the life of the option.

11.9. A stock price is currently \$50. It is known that at the end of 2 months it will be either \$53 or \$48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2-month European call option with a strike price of \$49? Use no-arbitrage arguments.

11.10. A stock price is currently \$80. It is known that at the end of 4 months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 4-month European put option with a strike price of \$80? Use no-arbitrage arguments.

11.11. A stock price is currently \$40. It is known that at the end of 3 months it will be either \$45 or \$35. The risk-free rate of interest with quarterly compounding is 8% per annum. Calculate the value of a 3-month European put option on the stock with an exercise price of \$40. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

11.12. A stock price is currently \$50. Over each of the next two 3-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 6-month European call option with a strike price of \$51?

11.13. For the situation considered in Problem 11.12, what is the value of a 6-month European put option with a strike price of \$51? Verify that the European call and European put prices satisfy put–call parity. If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the tree?

11.14. A stock price is currently \$25. It is known that at the end of 2 months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of 2 months. What is the value of a derivative that pays off S_T^2 at this time?

11.15. Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is 1 month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

Assignment Questions

11.16. A stock price is currently \$50. It is known that at the end of 6 months it will be either \$60 or \$42. The risk-free rate of interest with continuous compounding is 12% per annum. Calculate the value of a 6-month European call option on the stock with an exercise price of \$48. Verify that no-arbitrage arguments and risk-neutral valuation arguments give the same answers.

11.17. A stock price is currently \$40. Over each of the next two 3-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.
(a) What is the value of a 6-month European put option with a strike price of \$42?
(b) What is the value of a 6-month American put option with a strike price of \$42?

11.18. Using a "trial-and-error" approach, estimate how high the strike price has to be in Problem 11.17 for it to be optimal to exercise the option immediately.

11.19. A stock price is currently \$30. During each 2-month period for the next 4 months it will increase by 8% or reduce by 10%. The risk-free interest rate is 5%. Use a two-step tree to calculate the value of a derivative that pays off $\max[(30 - S_T), 0]^2$, where S_T is the stock price in 4 months. If the derivative is American-style, should it be exercised early?

11.20. Consider a European call option on a non-dividend-paying stock where the stock price is \$40, the strike price is \$40, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is 6 months.
(a) Calculate u, d, and p for a two-step tree.
(b) Value the option using a two-step tree.
(c) Verify that DerivaGem gives the same answer.
(d) Use DerivaGem to value the option with 5, 50, 100, and 500 time steps.

11.21. Repeat Problem 11.20 for an American put option on a futures contract. The strike price and the futures price are \$50, the risk-free rate is 10%, the time to maturity is 6 months, and the volatility is 40% per annum.

11.22. Footnote 1 shows that the correct discount rate to use for the real-world expected payoff in the case of the call option considered in Figure 11.1 is 42.6%. Show that if the option is a put rather than a call the discount rate is –52.5%. Explain why the two real-world discount rates are so different.