CHAPTER 24

More on Models and Numerical Procedures

Up to now the models we have used to value options have been based on the geometric Brownian motion model of asset price behavior that underlies the Black–Scholes formulas and the numerical procedures we have used have been relatively straightforward. In this chapter we introduce a number of new models and explain how the numerical procedures can be adapted to cope with particular situations.

In Chapter 16 we explained how traders overcome the weaknesses in the geometric Brownian motion model by using volatility surfaces. A volatility surface determines an appropriate volatility to substitute into Black–Scholes when pricing plain vanilla options. Unfortunately it says little about the volatility that should be used for exotic options when the pricing formulas of Chapter 22 are used. Suppose the volatility surface shows that the correct volatility to use when pricing a 1-year plain vanilla option with a strike price of \$40 is 27%. This is liable to be totally inappropriate for pricing a barrier option (or some other exotic option) that has a strike price of \$40 and a life of 1 year.

The first part of this chapter discusses a number of alternatives to geometric Brownian motion that are designed to deal with the problem of pricing exotic options consistently with plain vanilla options. These alternative asset price processes fit the market prices of plain vanilla options better than geometric Brownian motion. As a result, we can have more confidence in using them to value exotic options.

The second part of the chapter extends our discussion of numerical procedures. We explain how some types of path-dependent derivatives can be valued using trees. We discuss the special problems associated with valuing barrier options numerically and how these problems can be handled. Finally, we outline alternative ways of constructing trees for two correlated variables and show how Monte Carlo simulation can be used to value derivatives when there are early exercise opportunities.

As in earlier chapters our results are presented for derivatives dependent on an asset providing a yield at rate q. For an option on a stock index, q should be set equal to the dividend yield on the index; for an option on a currency, q should be set equal to the foreign risk-free rate; for an option on a futures contract, q should be set equal to the domestic risk-free rate.

24.1 ALTERNATIVES TO BLACK–SCHOLES

The Black–Scholes model assumes that an asset's price changes continously in a way that produces a lognormal distribution for the price at any future time. There are many alternative processes that can be assumed. One possibility is to retain the property that the asset price changes continuously, but assume a process other than geometric Brownian motion. Another alternative is to overlay continuous asset price changes with jumps. Yet another alternative is to assume a process where all the asset price changes that take place are jumps. We will consider examples of all three types of processes in this section. A model where stock prices change continuously is known as a *diffusion model*. A model where continuous changes are overlaid with jumps is known as a *mixed jump–diffusion model*. A model where all stock price changes are jumps is known as a *pure jump model*. These types of processes are known collectively as *Levy processes*.[1]

The Constant Elasticity of Variance Model

One alternative to Black–Scholes is the *constant elasticity of variance* (CEV) model. This is a diffusion model where the risk-neutral process for a stock price S is

$$dS = (r - q)S\,dt + \sigma S^{\alpha}\,dz$$

where r is the risk-free rate, q is the dividend yield, dz is a Wiener process, σ is a volatility parameter, and α is a positive constant.[2]

When $\alpha = 1$, the CEV model is the geometric Brownian motion model we have been using up to now. When $\alpha < 1$, the volatility increases as the stock price decreases. This creates a probability distribution similar to that observed for equities with a heavy left tail and less heavy right tail (see Figure 15.4).[3] When $\alpha > 1$, the volatility increases as the stock price increases. This creates a probability distribution with a heavy right tail and a less heavy left tail. This corresponds to a volatility smile where the implied volatility is an increasing function of the strike price. This type of volatility smile is sometimes observed for options on futures (see Assignment 14.46).

The valuation formulas for European call and put options under the CEV model are

$$c = S_0 e^{-qT}[1 - \chi^2(a, b+2, c)] - Ke^{-rT}\chi^2(c, b, a)$$

$$p = Ke^{-rT}[1 - \chi^2(c, b, a)] - S_0 e^{-qT}\chi^2(a, b+2, c)$$

when $0 < \alpha < 1$, and

$$c = S_0 e^{-qT}[1 - \chi^2(c, -b, a)] - Ke^{-rT}\chi^2(a, 2-b, c)$$

$$p = Ke^{-rT}[1 - \chi^2(a, 2-b, c)] - S_0 e^{-qT}\chi^2(c, -b, a)$$

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[1] Roughly speaking, a Levy process is a continuous-time stochastic process with stationary independent increments.

[2] See J. C. Cox and S. A. Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, 3 (March 1976): 145–66.

[3] The reason is as follows. As the stock price decreases, the volatility increases making even lower stock price more likely; when the stock price increases, the volatility decreases making higher stock prices less likely.

when $\alpha > 1$, with

$$a = \frac{[Ke^{-(r-q)T}]^{2(1-\alpha)}}{(1-\alpha)^2 v}, \qquad b = \frac{1}{1-\alpha}, \qquad c = \frac{S^{2(1-\alpha)}}{(1-\alpha)^2 v}$$

where

$$v = \frac{\sigma^2}{2(r-q)(\alpha-1)}[e^{2(r-q)(\alpha-1)T} - 1]$$

and $\chi^2(z, k, v)$ is the cumulative probability that a variable with a noncentral χ^2 distribution with noncentrality parameter v and k degrees of freedom is less than z. A procedure for computing $\chi^2(z, k, v)$ is provided in Technical Note 12 on the author's website.

The CEV model is particularly useful for valuing exotic equity options. The parameters of the model can be chosen to fit the prices of plain vanilla options as closely as possible by minimizing the sum of the squared differences between model prices and market prices.

Merton's Mixed Jump–Diffusion Model

Merton has suggested a model where jumps are combined with continuous changes.[4] Define:

λ: Average number of jumps per year

k: Average jump size measured as a percentage of the asset price

The percentage jump size is assumed to be drawn from a probability distribution in the model.

The probability of a jump in time Δt is $\lambda\,\Delta t$. The average growth rate in the asset price from the jumps is therefore λk. The risk-neutral process for the asset price is

$$\frac{dS}{S} = (r - q - \lambda k)\,dt + \sigma\,dz + dp$$

where dz is a Wiener process, dp is the Poisson process generating the jumps, and σ is the volatility of the geometric Brownian motion. The processes dz and dp are assumed to be independent.

An important particular case of Merton's model is where the logarithm of the size of the percentage jump is normal. Assume that the standard deviation of the normal distribution is s. Merton shows that a European option price can then be written

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} f_n$$

where $\lambda' = \lambda(1 + k)$. The variable f_n is the Black–Scholes option price when the dividend yield is q, the variance rate is

$$\sigma^2 + \frac{ns^2}{T}$$

[4] See R. C. Merton, "Option Pricing When Underlying Stock Returns Are Discontinuous," *Journal of Financial Economics*, 3 (March 1976): 125–44.

and the risk-free rate is

$$r - \lambda k + \frac{n\gamma}{T}$$

where $\gamma = \ln(1 + k)$.

This model gives rise to heavier left and heavier right tails than Black–Scholes. It can be used for pricing currency options. As in the case of the CEV model we choose the model parameters by minimizing the sum of the squared differences between model prices and market prices.

Another particular case of Merton's model is the model we used in Section 21.9 for valuing convertible bonds. In this case the jump is always down and equal to the current stock price (see Problem 24.5).

The Variance-Gamma Model

An example of a pure jump model that is proving quite popular is the *variance-gamma model*.[5] For this model, we first define a variable g as the change over time T in a variable that follows a gamma process with mean rate of 1 and variance rate of v. A gamma process is a pure jump process where small jumps occur very frequently and large jumps occur only occasionally. The probability density for g is

$$\phi(g) = \frac{g^{T/v-1}e^{-g/v}}{v^{T/v}\Gamma(T/v)}$$

where $\Gamma(\cdot)$ denotes the gamma function. This can be computed in Excel using the GAMMADIST$(\cdot, \cdot, \cdot, \cdot)$ function. The first argument of the function is g, the second is T/v, the third is v, and the fourth is TRUE or FALSE, where TRUE returns the cumulative probability distribution function and FALSE returns the probability density function we have just given.

As usual, we define S_T as the asset price at time T, S_0 as the asset price today, r as the risk-free interest rate, and q as the dividend yield. In a risk-neutral world $\ln S_T$, under the variance-gamma model, has a probability distribution that, conditional on g, is normal. The conditional mean is

$$\ln S_0 + (r - q)T + \omega + \theta g$$

and the conditional standard deviation is

$$\sigma\sqrt{g}$$

where

$$\omega = \frac{T}{v}\ln(1 - \theta v - \sigma^2 v/2)$$

The variance-gamma model has three parameters: v, σ, and θ.[6] The parameter v is the variance rate of the gamma process, σ is the volatility, and θ is a parameter defining skewness. When $\theta = 0$, $\ln S_T$ is symmetric; when $\theta < 0$, it is negatively skewed (as for equities); and when $\theta > 0$, it is positively skewed.

[5] See D. B. Madan, P. P. Carr, and E. C. Chang, "The Variance-Gamma Process and Option Pricing," *European Finance Review*, 2 (1998): 7–105.

[6] Note that all these parameters are liable to change when we move from the real world to the risk-neutral world. This is in contract to pure diffusion models where the volatility remains the same.

Suppose that we are interested in using Excel to obtain 10,000 random samples of the change in an asset price between time 0 and time T using the variance-gamma model. As a preliminary, we set cells E1, E2, E3, E4, E5, E6, and E7 equal to T, v, θ, σ, r, q, and S_0, respectively. We also set E8 equal to ω by defining it as

$$= \$E\$1 * \mathrm{LN}(1 - \$E\$3 * \$E\$2 - \$E\$4 * \$E\$4 * \$E\$2/2)/\$E\$2$$

We then proceed as follows:

1. We sample values for g using the GAMMAINV function. We set the contents of cells A1, A2, ..., A10000 as

$$= \mathrm{GAMMAINV}(\mathrm{RAND}(),\ \$E\$1/\$E\$2,\ \$E\$2)$$

2. For each value of g we sample a value for a variable that is normally distributed with mean θg and standard deviation $\sigma\sqrt{g}$. We do this by defining cell B1 as

$$= \mathrm{A1} * \$E\$3 + \mathrm{SQRT(A1)} * \$E\$4 * \mathrm{NORMSINV(RAND())}$$

and cells B2, B3, ..., B10000 similarly.

3. The stock price S_T is given by

$$S_T = S_0 \exp[(r - q)T + \omega + \theta g]$$

If we define C1 as

$$= \$E\$7 * \mathrm{EXP}((\$E\$5 - \$E\$6) * \$E\$1 + \mathrm{B1} + \$E\$8)$$

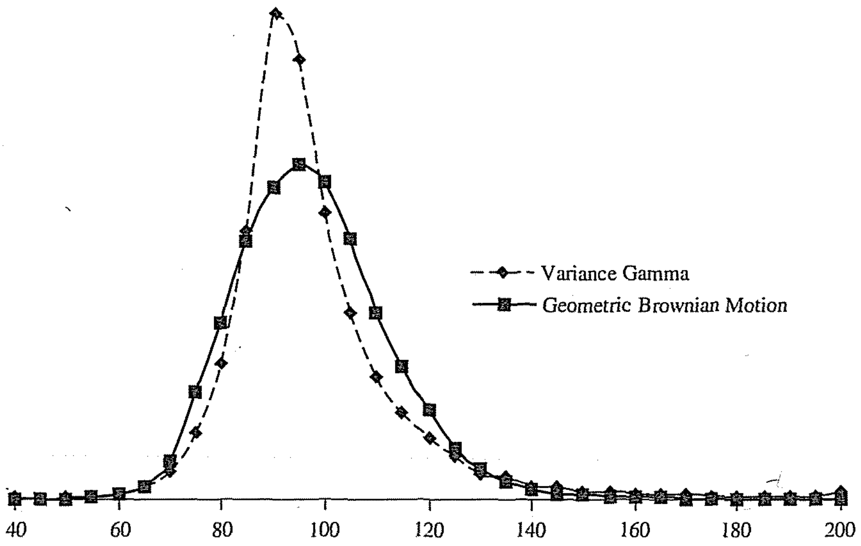
and define C2, C3, ..., C10000 similarly, we have created random samples from the distribution of S_T in C1, C2, ..., C10000.

Figure 24.1 shows the probability distribution that is obtained using the variance-gamma model for S_T when $S_0 = 100$, $T = 0.5$, $v = 0.5$, $\theta = 0.1$, $\sigma = 0.2$, and $r = q = 0$. For comparison it also shows the distribution given by geometric Brownian motion when the volatility, σ is 0.2 (or 20%). Although not clear in Figure 24.1, the variance-gamma distribution has heavier tails than the lognormal distribution given by geometric Brownian motion.

One way of characterizing the variance-gamma distribution is that g defines the rate at which information arrives during time T. If g is large, a great deal of information arrives and the sample we take from a normal distribution in step 2 above has a relatively large mean and variance. If g is small, relatively little information arrives and the sample we take has a relatively small mean and variance. The parameter T is the usual time measure, and g is sometimes referred to as measuring economic time or time adjusted for the flow of information.

Semi-analytic European option valuation formulas are provided by Madan *et al.* (1998). The variance-gamma model tends to produce a U-shaped volatility smile. The smile is not necessarily symmetrical. It is very pronounced for short maturities and "dies away" for long maturities. The model can be fitted to either equity or foreign currency plain vanilla option prices.

Figure 24.1 Distributions obtained with variance-gamma process and geometric Brownian motion.



24.2 STOCHASTIC VOLATILITY MODELS

The Black–Scholes model assumes that volatility is constant. In practice as we saw in Chapter 19 volatility varies through time. The variance-gamma model reflects this with its g parameter. Low values of g correspond to a low arrival rate for information and a low volatility; high values of g correspond to a high arrival rate for information and a high volatility.

An alternative to the variance-gamma model is a model where the process followed by the volatility variable is specified explicitly. Suppose first that we make the volatility parameter in the geometric Brownian motion a known function of time. The risk-neutral process followed by the asset price is then

$$dS = (r - q)S\,dt + \sigma(t)S\,dz \tag{24.1}$$

The Black–Scholes formulas are then correct provided that the variance rate is set equal to the average variance rate during the life of the option (see Problem 24.6). The variance rate is the square of the volatility. Suppose that during a 1-year period the volatility of a stock will be 20% during the first 6 months and 30% during the second 6 months. The average variance rate is

$$0.5 \times 0.20^2 + 0.5 \times 0.30^2 = 0.065$$

It is correct to use Black–Scholes with a variance rate of 0.065. This corresponds to a volatility of $\sqrt{0.065} = 0.255$, or 25.5%.

Equation (24.1) assumes that the instantaneous volatility of an asset is perfectly predictable. In practice volatility varies stochastically. This has led to the development more complex models with two stochastic variables: the stock price and its volatility.

One model that has been used by researchers is

$$\frac{dS}{S} = (r - q)\,dt + \sqrt{V}\,dz_S \tag{24.2}$$

$$dV = a(V_L - V)\,dt + \xi V^{\alpha}\,dz_V \tag{24.3}$$

where a, V_L, ξ, and α are constants, and dz_S and dz_V are Wiener processes. The variable V in this model is the asset's variance rate. The variance rate has a drift that pulls it back to a level V_L at rate a.

Hull and White show that, when volatility is stochastic but uncorrelated with the asset price, the price of a European option is the Black–Scholes price integrated over the probability distribution of the average variance rate during the life of the option.[7] Thus a European call price is

$$\int_0^{\infty} c(\bar{V})g(\bar{V})\,d\bar{V}$$

where $\bar{V}$ is the average value of the variance rate, c is the Black–Scholes price expressed as a function of $\bar{V}$, and g is the probability density function of $\bar{V}$ in a risk-neutral world. This result can be used to show that Black–Scholes overprices options that are at the money or close to the money, and underprices options that are deep in or deep out of the money. The model is consistent with the pattern of implied volatilities observed for currency options (see Section 16.2).

The case where the asset price and volatility are correlated is more complicated. Option prices can be obtained using Monte Carlo simulation. In the particular case where $\alpha = 0.5$, Hull and White provide a series expansion and Heston provides an analytic result.[8] The pattern of implied volatilities obtained when the volatility is negatively correlated with the asset price is similar to that observed for equities (see Section 16.3).[9]

Chapter 19 discusses exponentially weighted moving average (EWMA) and GARCH(1,1) models. These are alternative approaches to characterizing a stochastic volatility model. Duan shows that it is possible to use GARCH(1,1) as the basis for an internally consistent option pricing model.[10] (See Problem 19.14 for the equivalence of GARCH(1,1) and stochastic volatility models.)

Stochastic volatility models can be fitted to the prices of plain vanilla options and then used to price exotic options.[11] For options that last less than a year, the impact of a stochastic volatility on pricing is fairly small in absolute terms (although in percentage

[7] See J. C. Hull and A. White, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42 (June 1987): 281–300. This result is independent of the process followed by the variance rate.

[8] See J. C. Hull and A. White, "An Analysis of the Bias in Option Pricing Caused by a Stochastic Volatility," *Advances in Futures and Options Research*, 3 (1988): 27–61; S. L. Heston, "A Closed Form Solution for Options with Stochastic Volatility with Applications to Bonds and Currency Options," *Review of Financial Studies*, 6, 2 (1993): 327–43.

[9] The reason is given in footnote 3.

[10] See J.-C. Duan, "The GARCH Option Pricing Model," *Mathematical Finance*, vol. 5 (1995), 13–32; and J.-C. Duan, "Cracking the Smile" *RISK*, vol. 9 (December 1996), 55-59.

[11] For an example of this, see J. C. Hull and W. Suo, "A Methodology for the Assessment of Model Risk and its Application to the Implied Volatility Function Model," *Journal of Financial and Quantitative Analysis*, 37, 2 (June 2002): 297–318.

terms it can be quite large for deep-out-of-the-money options). It becomes progressively larger as the life of the option increases. The impact of a stochastic volatility on the performance of delta hedging is generally quite large. Traders recognize this and, as described in Chapter 15, monitor their exposure to volatility changes by calculating vega.

24.3 THE IVF MODEL

The parameters of the models we have discussed so far can be chosen so that they provide an approximate fit to the prices of plain vanilla options on any given day. Financial institutions sometimes want to go one stage further and use a model that provides an exact fit to the prices of these options.[12] In 1994 Derman and Kani, Dupire, and Rubinstein developed a model that is designed to do this. It has become known as the *implied volatility function* (IVF) model or the *implied tree* model.[13] It provides an exact fit to the European option prices observed on any given day, regardless of the shape of the volatility surface.

The risk-neutral process for the asset price in the model has the form

$$dS = [r(t) - q(t)]S\,dt + \sigma(S, t)S\,dz$$

where $r(t)$ is the instantaneous forward interest rate for a contract maturing at time t and $q(t)$ is the dividend yield as a function of time. The volatility $\sigma(S, t)$ is a function of both S and t and is chosen so that the model prices all European options consistently with the market. It is shown both by Dupire and by Andersen and Brotherton-Ratcliffe that $\sigma(S, t)$ can be calculated analytically:[14]

$$[\sigma(K, T)]^2 = 2\frac{\partial c_{\text{mkt}}/\partial T + q(T)c_{\text{mkt}} + K[r(T) - q(T)]\partial c_{\text{mkt}}/\partial K}{K^2(\partial^2 c_{\text{mkt}}/\partial K^2)} \tag{24.4}$$

where $c_{\text{mkt}}(K, T)$ is the market price of a European call option with strike price K and maturity T. If a sufficiently large number of European call prices are available in the market, this equation can be used to estimate the $\sigma(S, t)$ function.[15]

Andersen and Brotherton-Ratcliffe implement the model by using equation (24.4) together with the implicit finite difference method. An alternative approach, the *implied tree* methodology suggested by Derman and Kani and Rubinstein, involves constructing a tree for the asset price that is consistent with option prices in the market.

When it is used in practice the IVF model is recalibrated daily to the prices of plain vanilla options. It is a tool to price exotic options consistently with plain vanilla options. As discussed in Chapter 16 plain vanilla options define the risk-neutral

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[12] There is a practical reason for this. If the bank does not use a model with this property, there is a danger that traders working for the bank will spend their time arbitraging the bank's internal models.

[13] See B. Dupire, "Pricing with a Smile," *Risk*, February (1994): 18–20; E. Derman and I. Kani, "Riding on a Smile," *Risk*, February (1994): 32–39; M. Rubinstein, "Implied Binomial Trees" *Journal of Finance*, 49, 3 (July 1994), 771–818.

[14] See B. Dupire, "Pricing with a Smile," *Risk*, February (1994), 18–20; L. B. G. Andersen and R. Brotherton-Ratcliffe "The Equity Option Volatility Smile: An Implicit Finite Difference Approach," *Journal of Computation Finance* 1, No. 2 (Winter 1997/98): 5–37. Dupire considers the case where r and q are zero; Andersen and Brotherton-Ratcliffe consider the more general situation.

[15] Some smoothing of the observed volatility surface is typically necessary.

probability distribution of the asset price at all future times. It follows that the IVF model gets the risk-neutral probability distribution of the asset price at all future times correct. This means that options providing payoffs at just one time (e.g., all-or-nothing and asset-or-nothing options) are priced correctly by the IVF model. However, the model does not necessarily get the joint distribution of the asset price at two or more times correct. This means that exotic options such as compound options and barrier options may be priced incorrectly.[16]

24.4 PATH-DEPENDENT DERIVATIVES

We now move on to discuss how the numerical procedures we presented in Chapter 17 can be modified to handle particular valuation problems. We start by considering how trees can be used for path-dependent derivatives.

A path-dependent derivative (or history-dependent derivative) is a derivative where the payoff depends on the path followed by the price of the underlying asset, not just its final value. Asian options and lookback options are examples of path-dependent derivatives. As explained in Chapter 22, the payoff from an Asian option depends on the average price of the underlying asset; the payoff from a lookback option depends on its maximum or minimum price. One approach to valuing path-dependent options when analytic results are not available is Monte Carlo simulation, as discussed in Chapter 17. A sample value of the derivative can be calculated by sampling a random path for the underlying asset in a risk-neutral world, calculating the payoff, and discounting the payoff at the risk-free interest rate. An estimate of the value of the derivative is found by obtaining many sample values of the derivative in this way and calculating their mean.

The main problem with using Monte Carlo simulation is that the computation time necessary to achieve the required level of accuracy can be unacceptably high. Also, American-style path-dependent derivatives (i.e., path-dependent derivatives where one side has exercise opportunities or other decisions to make) cannot easily be handled. In this section, we show how the binomial tree methods presented in Chapter 17 can be extended to cope with some path-dependent derivatives.[17] The procedure can handle American-style path-dependent derivatives and is computationally more efficient than Monte Carlo simulation for European-style path-dependent derivatives.

For the procedure to work, two conditions must be satisfied:

1. The payoff from the derivative must depend on a single function, F, of the path followed by the underlying asset.
2. It must be possible to calculate the value of F at time $\tau + \Delta t$ from the value of F at time τ and the value of the underlying asset at time $\tau + \Delta t$.

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[16] Hull and Suo test the IVF model by assuming that all derivative prices are determined by a stochastic volatility model. They found that the model works reasonably well for compound options, but sometimes gives serious errors for barrier options. See J. C. Hull and W. Suo, "A Methodology for the Assessment of Model Risk and its Application to the Implied Volatility Function Model," *Journal of Financial and Quantitative Analysis*, 37, 2 (June 2002): 297–318

[17] This approach was suggested in J. Hull and A. White, "Efficient Procedures for Valuing European and American Path-Dependent Options," *Journal of Derivatives*, 1, 1 (Fall 1993): 21 31.

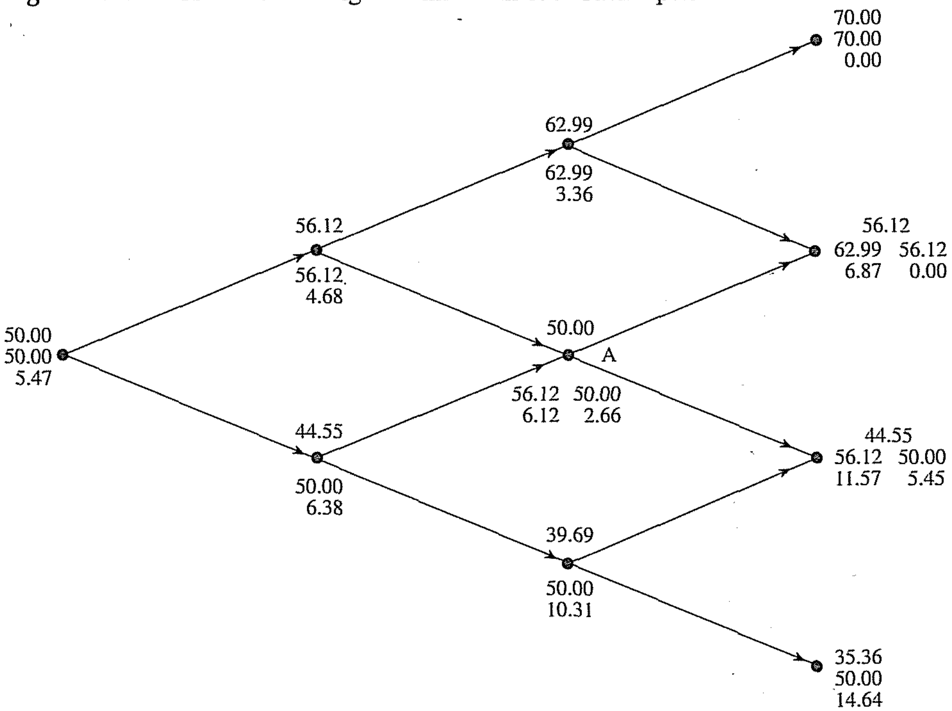
Illustration Using Lookback Options

As a first illustration of the procedure, we consider an American lookback put option on a non-dividend-paying stock.[18] If exercised at time τ, this pays off the amount by which the maximum stock price between time 0 and time τ exceeds the current stock price. We suppose that the initial stock price is \$50, the stock price volatility is 40% per annum, the risk-free interest rate is 10% per annum, the total life of the option is three months, and that stock price movements are represented by a three-step binomial tree. With our usual notation this means that $S_0 = 50$, $\sigma = 0.4$, $r = 0.10$, $\Delta t = 0.08333$, $u = 1.1224$, $d = 0.8909$, $a = 1.0084$, and $p = 0.5073$.

The tree is shown in Figure 24.2. The top number at each node is the stock price. The next level of numbers at each node shows the possible maximum stock prices achievable on paths leading to the node. The final level of numbers shows the values of the derivative corresponding to each of the possible maximum stock prices.

The values of the derivative at the final nodes of the tree are calculated as the maximum stock price minus the actual stock price. To illustrate the rollback procedure, suppose that we are at node A, where the stock price is \$50. The maximum stock price achieved thus far is either 56.12 or 50. Consider first the situation where it is equal to 50. If there is an up movement, the maximum stock price becomes 56.12 and the value

Figure 24.2 Tree for valuing an American lookback option.



[18] This example is used as a first illustration of the general procedure for handling path dependence. For a more efficient approach to valuing American-style lookback options, see Technical Note 13 on the author's website.

of the derivative is zero. If there is a down movement, the maximum stock price stays at 50 and the value of the derivative is 5.45. Assuming no early exercise, the value of the derivative at A when the maximum achieved so far is 50 is, therefore,

$$(0 \times 0.5073 + 5.45 \times 0.4927)e^{-0.1\times 0.08333} = 2.66$$

Clearly, it is not worth exercising at node A in these circumstances because the payoff from doing so is zero. A similar calculation for the situation where the maximum value at node A is 56.12 gives the value of the derivative at node A, without early exercise, to be

$$(0 \times 0.5073 + 11.57 \times 0.4927)e^{-0.1\times 0.08333} = 5.65$$

In this case, early exercise gives a value of 6.12 and is the optimal strategy. Rolling back through the tree in the way we have indicated gives the value of the American lookback as \$5.47.

Generalization

The approach just described is computationally feasible when the number of alternative values of the path function, F, at each node does not grow too fast as the number of time steps is increased. The example we used, a lookback option, presents no problems because the number of alternative values for the maximum asset price at a node in a binomial tree with n time steps is never greater than n.

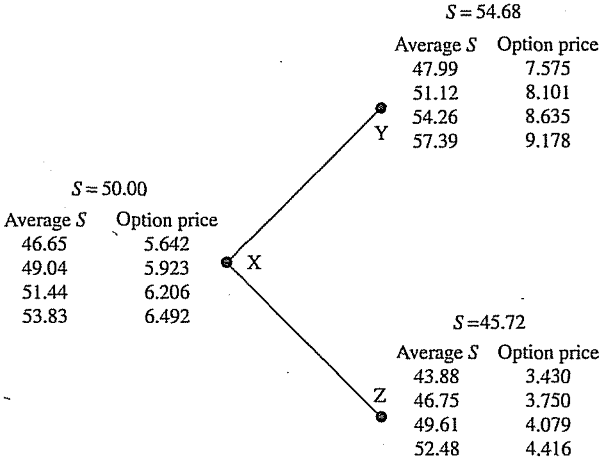
Luckily, the approach can be extended to cope with situations where there are a very large number of different possible values of the path function at each node. The basic idea is as follows. At a node, we carry out calculations for a small number of representative values of F. When the value of the derivative is required for other values of the path function, we calculate it from the known values using interpolation.

The first stage is to work forward through the tree establishing the maximum and minimum values of the path function at each node. Assuming the value of the path function at time $\tau + \Delta t$ depends only on the value of the path function at time τ and the value of the underlying variable at time $\tau + \Delta t$, the maximum and minimum values of the path function for the nodes at time $\tau + \Delta t$ can be calculated in a straightforward way from those for the nodes at time τ. The second stage is to choose representative values of the path function at each node. There are a number of approaches. A simple rule is to choose the representative values as the maximum value, the minimum value, and a number of other values that are equally spaced between the maximum and the minimum. As we roll back through the tree, we value the derivative for each of representative values of the path function.

We illustrate the nature of the calculation by considering the problem of valuing the average price call option in Example 22.2. We examine the case where the payoff depends on the arithmetic average stock price. The initial stock price is 50, the strike price is 50, the risk-free interest rate is 10%, the stock price volatility is 40%, and the time to maturity is 1 year. We use a tree with 20 time steps. The binomial tree parameters are $\Delta t = 0.05$, $u = 1.0936$, $d = 0.9144$, $p = 0.5056$, and $1 - p = 0.4944$. The path function is the arithmetic average of the stock price.

Figure 24.3 shows the calculations that are carried out in one small part of the tree. Node X is the central node at time 0.2 year (at the end of the fourth time step). Nodes Y and Z are the two nodes at time 0.25 year that are reachable from node X. The stock

Figure 24.3 Part of tree for valuing option on the arithmetic average.



price at node X is 50. Forward induction shows that the maximum average stock price that is achievable in reaching node X is 53.83. The minimum is 46.65. (We include both the initial and final stock prices when calculating the average.) From node X we branch to one of the two nodes, Y and Z. At node Y, the stock price is 54.68 and the bounds for the average are 47.99 and 57.39. At node Z, the stock price is 45.72 and the bounds for the average stock price are 43.88 and 52.48.

Suppose that we have chosen the representative values of the average to be four equally spaced values at each node. This means that, at node X, we consider the averages 46.65, 49.04, 51.44, and 53.83. At node Y, we consider the averages 47.99, 51.12, 54.26, and 57.39. At node Z, we consider the averages 43.88, 46.75, 49.61, and 52.48. We assume that backward induction has already been used to calculate the value of the option for each of the alternative values of the average at nodes Y and Z. Values are shown in Figure 24.3 (e.g., at node Y when the average is 51.12, the value of the option is 8.101).

Consider the calculations at node X for the case where the average is 51.44. If the stock price moves up to node Y, the new average will be

$$\frac{5 \times 51.44 + 54.68}{6} = 51.98$$

The value of the derivative at node Y for this average can be found by interpolating between the values when the average is 51.12 and when it is 54.26. It is

$$\frac{(51.98 - 51.12) \times 8.635 + (54.26 - 51.98) \times 8.101}{54.26 - 51.12} = 8.247$$

Similarly, if the stock price moves down to node Z, the new average will be

$$\frac{5 \times 51.44 + 45.72}{6} = 50.49$$

and by interpolation the value of the derivative is 4.182.

The value of the derivative at node X when the average is 51.44 is, therefore,

$$(0.5056 \times 8.247 + 0.4944 \times 4.182)e^{-0.1\times 0.05} = 6.206$$

The other values at node X are calculated similarly. Once the values at all nodes at time 0.2 year have been calculated, we can move on to the nodes at time 0.15 year.

The value given by the full tree for the option at time zero is 7.17. As the number of time steps and the number of averages considered at each node is increased, the value of the option converges to the correct answer. With 60 time steps and 100 averages at each node, the value of the option is 5.58. The analytic approximation for the value of the option calculated in Example 22.2 is 5.62.

A key advantage of the method described here is that it can handle American options. The calculations are as we have described them except that we test for early exercise at each node for each of the alternative values of the path function at the node. (In practice, the early exercise decision is liable to depend on both the value of the path function and the value of the underlying asset.) Consider the American version of the average price call considered here. The value calculated using the 20-step tree and four averages at each node is 7.77; with 60 time steps and 100 averages, the value is 6.17.

The approach just described can be used in a wide range of different situations. The two conditions that must be satisfied were listed at the beginning of this section. Efficiency is improved somewhat if quadratic rather than linear interpolation is used at each node.

24.5 BARRIER OPTIONS

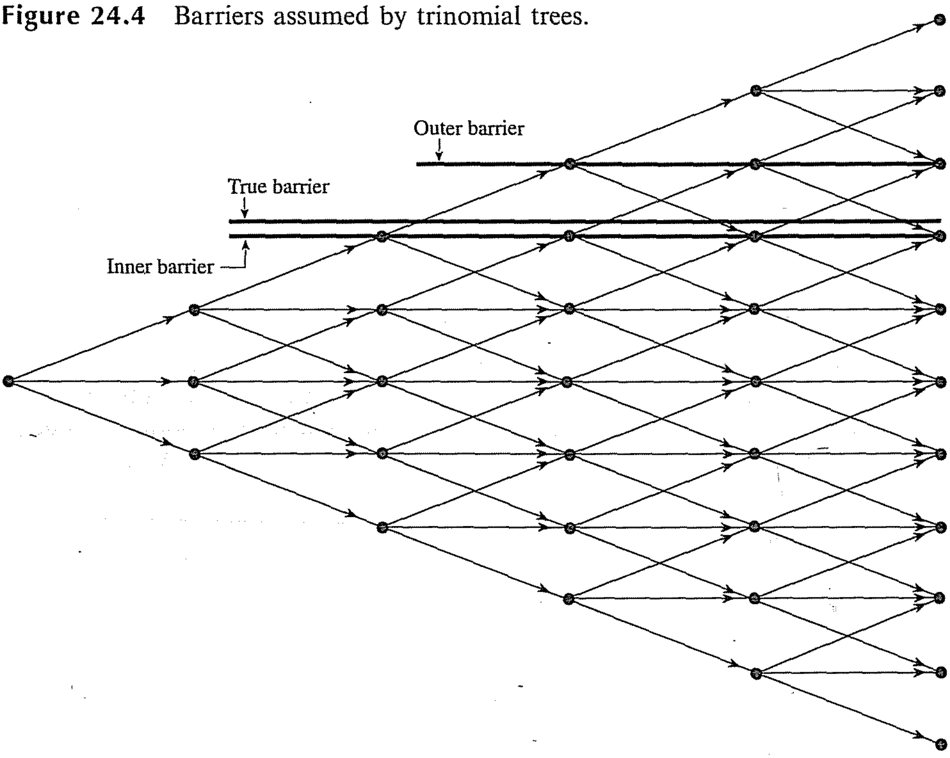
In Chapter 22 we presented analytic results for standard barrier options. Here we consider the numerical procedures that can be used for barrier options when there are no analytic results.

In principle, a barrier option can be valued using the binomial and trinomial trees discussed in Chapter 17. Consider an up-and-out option. We can value this in the same way as a regular option except that, when we encounter a node above the barrier, we set the value of the option equal to zero.

Trinomial trees work better than binomial trees but even for them convergence is very slow when this approach is used. A large number of time steps are required to obtain a reasonably accurate result. The reason for this is that the barrier being assumed by the tree is different from the true barrier.[19] Define the *inner barrier* as the barrier formed by nodes just on the inside of the true barrier (i.e., closer to the center of the tree) and *the outer barrier* as the barrier formed by nodes just outside the true barrier (i.e., farther away from the center of the tree). Figure 24.4 shows the inner and outer barrier for a trinomial tree on the assumption that the true barrier is horizontal. The usual tree calculations implicitly assume that the outer barrier is the true barrier because the barrier conditions are first used at nodes on this barrier. When the time step is Δt, the vertical spacing between the nodes is of order $\sqrt{\Delta t}$. This means that errors created by the difference between the true barrier and the outer barrier also tend to be of order $\sqrt{\Delta t}$.

[19] For a discussion of this, see P.P. Boyle and S.H. Lau, "Bumping Up Against the Barrier with the Binomial Method," *Journal of Derivatives*, 1, 4 (Summer 1994): 6–14.

Figure 24.4 Barriers assumed by trinomial trees.



One approach to overcoming this problem is to

1. Calculate the price of the derivative on the assumption that the inner barrier is the true barrier.
2. Calculate the value of the derivative on the assumption that the outer barrier is the true barrier.
3. Interpolate between the two prices.

Another approach is to ensure that nodes lie on the barrier. Suppose that the initial stock price is S_0 and that the barrier is at H. In a trinomial tree, there are three possible movements in the asset's price at each node: up by a proportional amount u; stay the same; and down by a proportional amount d, where $d = 1/u$. We can always choose u so that nodes lie on the barrier. The condition that must be satisfied by u is

$$H = S_0 u^N$$

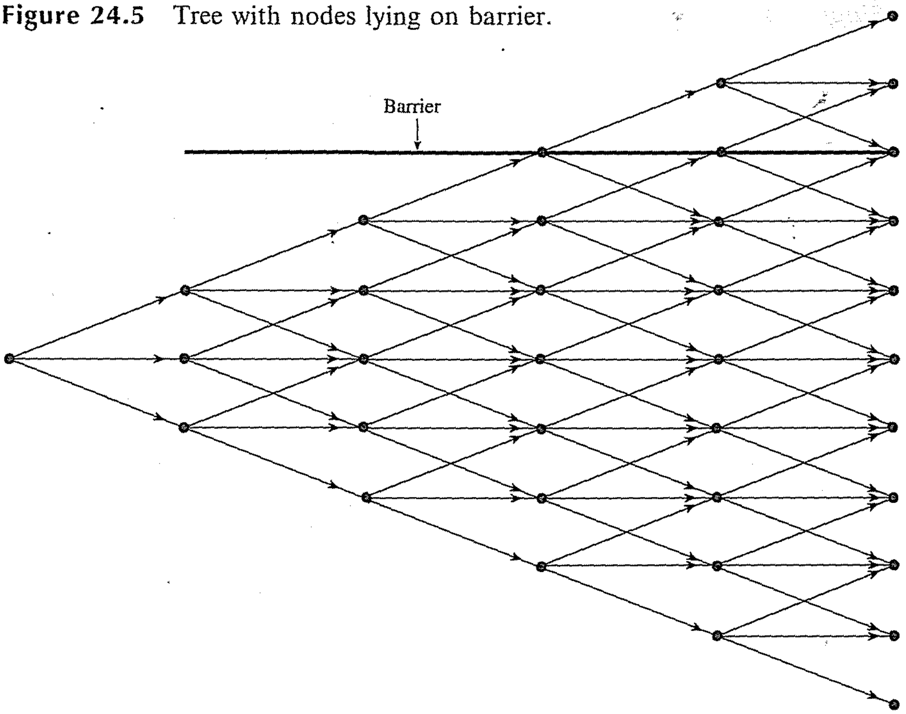
or

$$\ln H = \ln S_0 + N \ln u$$

for some positive or negative N.

When discussing trinomial trees in Section 17.4, the value suggested for u was $e^{\sigma\sqrt{3\Delta t}}$, so that $\ln u = \sigma\sqrt{3\Delta t}$. In the situation considered here, a good rule is to choose $\ln u$ as close as possible to this value, consistent with the condition given above. This means

Figure 24.5 Tree with nodes lying on barrier.



that we set

$$\ln u = \frac{\ln H - \ln S_0}{N}$$

where

$$N = \text{int}\left[\frac{\ln H - \ln S_0}{\sigma\sqrt{3\Delta t}} + 0.5\right]$$

and int(x) is the integral part of x.

This leads to a tree of the form shown in Figure 24.5. The probabilities p_u, p_m, and p_d on the upper, middle, and lower branches of the tree are chosen to match the first two moments of the return, so that

$$p_d = -\frac{(r-q-\sigma^2/2)\,\Delta t}{2\ln u} + \frac{\sigma^2\Delta t}{2(\ln u)^2}, \quad p_m = \frac{\sigma^2\Delta t}{(\ln u)^2}, \quad p_u = \frac{(r-q-\sigma^2/2)\,\Delta t}{2\ln u} + \frac{\sigma^2\Delta t}{2(\ln u)^2}$$

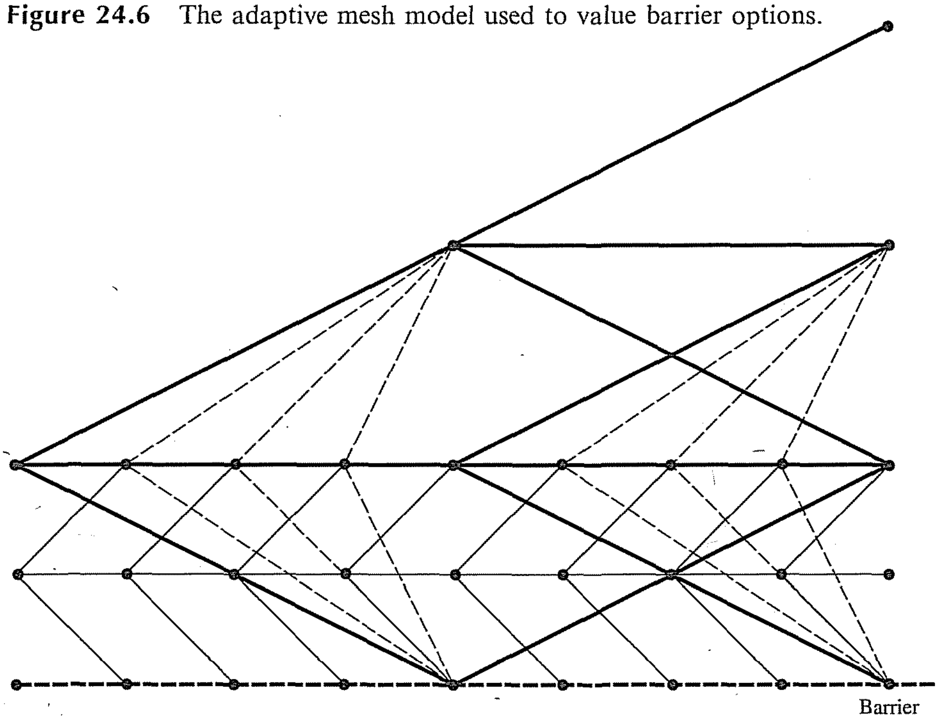
where p_u, p_m, and p_d are the probabilities on the upper, middle, and lower branches.

The Adaptive Mesh Model

The methods we have presented so far work reasonably well when the initial asset price is not close to the barrier. When the initial asset price is close to a barrier, the adaptive mesh model, which we introduced in Section 17.4, can be used.[20] The idea behind the

[20] See S. Figlewski and B. Gao, "The Adaptive Mesh Model: A New Approach to Efficient Option Pricing," *Journal of Financial Economics*, 53 (1999): 313–51.

Figure 24.6 The adaptive mesh model used to value barrier options.



model is that computational efficiency can be improved by grafting a fine tree onto a coarse tree to achieve a more detailed modeling of the asset price in the regions of the tree where it is needed most.

To value a barrier option, it is useful to have a fine tree close to barriers. Figure 24.6 illustrates the design of the tree. The geometry of the tree is arranged so that nodes lie on the barriers. The probabilities on branches are chosen, as usual, to match the first two moments of the process followed by the underlying asset. The heavy lines in Figure 24.6 are the branches of the coarse tree. The light solid line are the fine tree. We first roll back through the coarse tree in the usual way. We then calculate the value at additional nodes using the branches indicated by the dotted lines. Finally we roll back through the fine tree.

24.6 OPTIONS ON TWO CORRELATED ASSETS

Another tricky numerical problem is that of valuing American options dependent on two assets whose prices are correlated. A number of alternative approaches have been suggested. We will explain three of these.

Transforming Variables

It is relatively easy to construct a tree in three dimensions to represent the movements of two *uncorrelated* variables. The procedure is as follows. First, we construct a two-dimensional tree for each variable. We then combine these trees into a single three-dimensional tree. The probabilities on the branches of the three-dimensional tree are the product of the corresponding probabilities on the two-dimensional trees. Suppose, for example, that the variables are stock prices, S_1 and S_2. Each can be represented in two dimensions by a Cox, Ross, and Rubinstein binomial tree. Assume that S_1 has a probability p_1 of moving up by a proportional amount u_1 and a probability $1 - p_1$ of moving down by a proportional amount d_1. Suppose further that S_2 has a probability p_2 of moving up by a proportional amount u_2 and a probability $1 - p_2$ of moving down by a proportional amount d_2. In the three-dimensional tree there are four branches emanating from each node. The probabilities are:

$$
\begin{aligned}
p_1 p_2&: \ S_1 \text{ increases; } S_2 \text{ increases.}\\
p_1(1-p_2)&: \ S_1 \text{ increases; } S_2 \text{ decreases.}\\
(1-p_1)p_2&: \ S_1 \text{ decreases; } S_2 \text{ increases.}\\
(1-p_1)(1-p_2)&: \ S_1 \text{ decreases; } S_2 \text{ decreases.}
\end{aligned}
$$

Consider next the situation where S_1 and S_2 are correlated. We suppose that the risk-neutral processes are:

$$dS_1 = (r - q_1)S_1\,dt + \sigma_1 S_1\,dz_1$$

$$dS_2 = (r - q_2)S_2\,dt + \sigma_2 S_2\,dz_2$$

and the instantaneous correlation between the Wiener processes, dz_1 and dz_2, is ρ. This means that

$$d\ln S_1 = (r - q_1 - \sigma_1^2/2)\,dt + \sigma_1\,dz_1$$

$$d\ln S_2 = (r - q_2 - \sigma_2^2/2)\,dt + \sigma_2\,dz_2$$

We define two new uncorrelated variables:[21]

$$x_1 = \sigma_2 \ln S_1 + \sigma_1 \ln S_2$$

$$x_2 = \sigma_2 \ln S_1 - \sigma_1 \ln S_2$$

These variables follow the processes

$$dx_1 = [\sigma_2(r - q_1 - \sigma_1^2/2) + \sigma_1(r - q_2 - \sigma_2^2/2)]\,dt + \sigma_1\sigma_2\sqrt{2(1+\rho)}\,dz_A$$

$$dx_2 = [\sigma_2(r - q_1 - \sigma_1^2/2) - \sigma_1(r - q_2 - \sigma_2^2/2)]\,dt + \sigma_1\sigma_2\sqrt{2(1-\rho)}\,dz_B$$

where dz_A and dz_B are uncorrelated Wiener processes.

The variables x_1 and x_2 can be modeled using two separate binomial trees. In time Δt, x_i has a probability p_i of increasing by h_i and a probability $1 - p_i$ of decreasing by h_i. The variables h_i and p_i are chosen so that the tree gives correct values for the first two moments of the distribution of x_1 and x_2. Because they are uncorrelated, the two trees can be combined into a single three-dimensional tree, as already described. At each node of the tree, S_1 and S_2 can be calculated from x_1 and x_2 using the inverse

[21] This idea was suggested in J. Hull and A. White, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis*, 25 (1990): 87–100.

relationships

$$S_1 = \exp\left[\frac{x_1 + x_2}{2\sigma_2}\right] \quad \text{and} \quad S_2 = \exp\left[\frac{x_1 - x_2}{2\sigma_1}\right]$$

The procedure for rolling back through a three-dimensional tree to value a derivative is analogous to that for a two-dimensional tree.

Using a Nonrectangular Tree

Rubinstein has suggested a way of building a three-dimensional tree for two correlated stock prices by using a nonrectangular arrangement of the nodes.[22] From a node (S_1, S_2), where the first stock price is S_1 and the second stock price is S_2, we have a 0.25 chance of moving to each of the following:

$$(S_1 u_1, S_2 A), \quad (S_1 u_1, S_2 B), \quad (S_1 d_1, S_2 C), \quad (S_2 d_1, S_2 D)$$

where

$$u_1 = \exp[(r - q_1 - \sigma_1^2/2)\,\Delta t + \sigma_1\sqrt{\Delta t}]$$
$$d_1 = \exp[(r - q_1 - \sigma_1^2/2)\,\Delta t - \sigma_1\sqrt{\Delta t}]$$

and

$$A = \exp[(r - q_2 - \sigma_2^2/2)\,\Delta t + \sigma_2\sqrt{\Delta t}\,(\rho + \sqrt{1 - \rho^2})]$$
$$B = \exp[(r - q_2 - \sigma_2^2/2)\,\Delta t + \sigma_2\sqrt{\Delta t}\,(\rho - \sqrt{1 - \rho^2})]$$
$$C = \exp[(r - q_2 - \sigma_2^2/2)\,\Delta t - \sigma_2\sqrt{\Delta t}\,(\rho - \sqrt{1 - \rho^2})]$$
$$D = \exp[(r - q_2 - \sigma_2^2/2)\,\Delta t - \sigma_2\sqrt{\Delta t}\,(\rho + \sqrt{1 - \rho^2})]$$

When the correlation is zero, this method is equivalent to constructing separate trees for S_1 and S_2 using the alternative binomial tree construction method in Section 17.4.

Adjusting the Probabilities

A third approach to building a three-dimensional tree for S_1 and S_2 involves first assuming no correlation and then adjusting the probabilities at each node to reflect the correlation.[23] We use the alternative binomial tree construction method for each of S_1 and S_2 in Section 17.4. This method has the property that all probabilities are 0.5.

Table 24.1 Combination of binomials assuming no correlation.

S_2-move	*S_1-move*	
	Down	*Up*
Up	0.25	0.25
Down	0.25	0.25

[22] See M. Rubinstein, "Return to Oz," *Risk*, November (1994): 67–70.

[23] This approach was suggested in the context of interest rate trees in J. Hull and A. White, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, Winter (1994): 37–48.

Table 24.2 Combination of binomials assuming correlation of ρ.

S_2-move	*S_1-move*	
	Down	*Up*
Up	$0.25(1 - \rho)$	$0.25(1 + \rho)$
Down	$0.25(1 + \rho)$	$0.25(1 - \rho)$

When the two binomial trees are combined on the assumption that there is no correlation, the probabilities are as shown in Table 24.1. When we adjust these probabilities to reflect the correlation they become as shown in Table 24.2.

24.7 MONTE CARLO SIMULATION AND AMERICAN OPTIONS

Monte Carlo simulation is well suited to valuing path-dependent options and options where there are many stochastic variables. Trees and finite difference methods are well suited to valuing American-style options. What happens if an option is both path dependent and American? What happens if an American option depends on several stochastic variables? In Section 24.4 we explained a way in which the binomial tree approach can be modified to value path-dependent options in some situations. A number of researchers have adopted a different approach by searching for a way in which Monte Carlo simulation can be used to value American-style options.[24] Here we explain two alternative ways of proceeding.

The Least-Squares Approach

In order to value an American-style option it is necessary to choose between exercising and continuing at each early exercise point. The value of exercising is normally easy to determine. A number of researchers including Longstaff and Schwartz provide a way of determining the value of continuing when Monte Carlo simulation is used.[25] Their approach involves using a least-squares analysis to determine the best-fit relationship between the value of continuing and the values of relevant variables at each time an early exercise decision has to be made. The approach is best illustrated with a numerical example. We use the one in the Longstaff–Schwartz paper.

Consider a 3-year American put option on a non-dividend-paying stock that can be exercised at the end of year 1, the end of year 2, and the end of year 3. The risk-free rate is 6% per annum (continuously compounded). The current stock price is 1.00 and the strike price is 1.10. Assume that we sample the eight paths for the stock price shown in Table 24.3. (This example is for illustration only; in practice many more paths would be sampled.) If the option can be exercised only at the 3-year point, it provides a cash flow equal to its intrinsic value at that point. This is shown in the last column of Table 24.4.

[24] Tilley was the first researcher to publish a solution to the problem. See J. A. Tilley, "Valuing American Options in a Path Simulation Model," *Transactions of the Society of Actuaries*, 45 (1993): 83–104.

[25] See F. A. Longstaff and E. S. Schwartz, "Valuing American Options by Simulation: A Simple Least-Squares Approach," *Review of Financial Studies*, 14, 1 (Spring 2001): 113–47.

Table 24.3 Sample paths for put option example.

Path	$t=0$	$t=1$	$t=2$	$t=3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

If the put option is in the money at the 2-year point the option holder must decide whether to exercise. From Table 24.3, we see that the option is in the money at the 2-year point for paths 1, 3, 4, 6, and 7. For these paths, we assume an approximate relationship:

$$V = a + bS + cS^2$$

where S is the stock price at the 2-year point and V is the value of continuing, discounted back to the 2-year point. Our five observations on S are: 1.08, 1.07, 0.97, 0.77, and 0.84. From Table 24.4 the corresponding values for V are: 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, $0.20e^{-0.06\times 1}$, and $0.09e^{-0.06\times 1}$. We use this data to calculate the values of a, b, and c that minimize

$$\sum_{i=1}^{5}(V_i - a - bS_i - cS_i^2)^2$$

where S_i and V_i are the ith observation on S and V, respectively. It turns out that $a = -1.070$, $b = 2.983$ and $c = -1.813$, so that the best-fit relationship is

$$V = -1.070 + 2.983S - 1.813S^2$$

This gives the value at the 2-year point of continuing for paths 1, 3, 4, 6, and 7 of 0.0369, 0.0461, 0.1176, 0.1520, and 0.1565, respectively. From Table 24.3 the value of exercising is 0.02, 0.03, 0.13, 0.33, and 0.26. This means that we should exercise at

Table 24.4 Cash flows if exercise only at the 3-year point.

Path	$t=1$	$t=2$	$t=3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.00	0.00	0.18
5	0.00	0.00	0.00
6	0.00	0.00	0.20
7	0.00	0.00	0.09
8	0.00	0.00	0.00

Table 24.5 Cash flows if exercise only possible at 2- and 3-year point.

Path	$t=1$	$t=2$	$t=3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.00	0.13	0.00
5	0.00	0.00	0.00
6	0.00	0.33	0.00
7	0.00	0.26	0.00
8	0.00	0.00	0.00

the 2-year point for paths 4, 6, and 7. Table 24.5 summarizes the cash flow assuming exercise at either the 2-year point or the 3-year point for the eight paths.

We next consider the paths that are in the money at the 1-year point. These are paths 1, 4, 6, 7, and 8. From Table 24.3 the values of S for the paths are 1.09, 0.93, 0.76, 0.92, and 0.88, respectively. From Table 24.5, the corresponding continuation values discounted back to $t=1$ are 0.00, $0.13e^{-0.06\times 1}$, $0.33e^{-0.06\times 1}$, $0.26e^{-0.06\times 1}$, and 0.00, respectively. The least-squares relationship is

$$V = 2.038 - 3.335S + 1.356S^2$$

This gives the value of continuing at the 1-year point for paths 1, 4, 6, 7, 8 as 0.0139, 0.1092, 0.2866, 0.1175, and 0.1533, respectively. From Table 24.3 the value of exercising is 0.01, 0.17, 0.34, 0.18, and 0.22, respectively. This means that we should exercise at the 1-year point for paths 4, 6, 7, and 8. Table 24.6 summarizes the cash flows assuming that early exercise is possible at all three times. The value of the option is determined by discounting each cash flow back to time zero at the risk-free rate and calculating the mean of the results. It is

$$\tfrac{1}{8}(0.07e^{-0.06\times 3} + 0.17e^{-0.06\times 1} + 0.34e^{-0.06\times 1} + 0.18e^{-0.06\times 1} + 0.22e^{-0.06\times 1}) = 0.1144$$

Because this is greater than 0.10, it is not optimal to exercise the option immediately.

Table 24.6 Cash flows from option.

Path	$t=1$	$t=2$	$t=3$
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.17	0.00	0.00
5	0.00	0.00	0.00
6	0.34	0.00	0.00
7	0.18	0.00	0.00
8	0.22	0.00	0.00

This method can be extended in a number of ways. If the option can be exercised at any time we can approximate its value by considering a large number of exercise points (just as a binomial tree does). The relationship between V and S can be assumed to be more complicated. For example we could assume that V is a cubic rather than a quadratic function of S. When the early exercise decision depends on several state variables, we proceed as we did in the example just considered. A functional form for the relationship between V and the variables is assumed and the parameters are estimated using the least-squares approach.

The Exercise Boundary Parameterization Approach

A number of researchers, such as Andersen, have proposed an alternative approach where the early exercise boundary is parameterized and the optimal values of the parameters are determined iteratively by starting at the end of the life of the option and working backward.[26] To illustrate the approach, we continue with the put option example and assume that the eight paths shown in Table 24.3 have been sampled. In this case, the early exercise boundary at time t can be parameterized by a critical value of S, $S^*(t)$. If the asset price at time t is below $S^*(t)$ we exercise at time t; if it is above $S^*(t)$ we do not exercise at time t. The value of $S^*(3)$ is 1.10. If the stock price is above 1.10 when $t = 3$ (the end of the option's life) we do not exercise; if it is below 1.10 we exercise. We now consider the determination of $S^*(2)$.

Suppose that we choose a value of $S^*(2)$ less than 0.77. The option is not exercised at the 2-year point for any of the paths. The value of the option at the 2-year point for the eight paths is then 0.00, 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, 0.00, $0.20e^{-0.06\times 1}$, $0.09e^{-0.06\times 1}$, and 0.00, respectively. The average of these is 0.0636. Suppose next that $S^*(2) = 0.77$. The value of the option at the 2-year point for the eight paths is then 0.00, 0.00, $0.07e^{-0.06\times 1}$, $0.18e^{-0.06\times 1}$, 0.00, 0.33, $0.09e^{-0.06\times 1}$, and 0.00, respectively. The average of these is 0.0813. Similarly when $S^*(2)$ equals 0.84, 0.97, 1.07, and 1.08, the average value of the option at the 2-year point is 0.1032, 0.0982, 0.0938, and 0.0963, respectively. This analysis shows that the optimal value of $S^*(2)$ (i.e., the one that maximizes the average value of the option) is 0.84. (More precisely, it is optimal to choose $0.84 \leqslant S^*(2) < 0.97$.) When we choose this optimal value for $S^*(2)$, the value of the option at the 2-year point for the eight paths is 0.00, 0.00, 0.0659, 0.1695, 0.00, 0.33, 0.26, and 0.00, respectively. The average value is 0.1032.

We now move on to calculate $S^*(1)$. If $S^*(1) < 0.76$ the option is not exercised at the 1-year point for any of the paths and the value at the option at the 1-year point is $0.1032e^{-0.06\times 1} = 0.0972$. If $S^*(1) = 0.76$, the value of the option for each of the eight paths at the 1-year point is 0.00, 0.00, $0.0659e^{-0.06\times 1}$, $0.1695e^{-0.06\times 1}$, 0.0, 0.34, $0.26e^{-0.06\times 1}$, and 0.00, respectively. The average value of the option is 0.1008. Similarly when $S^*(1)$ equals 0.88, 0.92, 0.93, and 1.09 the average value of the option is 0.1283, 0.1202, 0.1215, and 0.1228, respectively. The analysis therefore shows that the optimal value of $S^*(1)$ is 0.88. (More precisely, it is optimal to choose $0.88 \leqslant S^*(1) < 0.92$.) The value of the option at time zero with no early exercise is $0.1283e^{-0.06\times 1} = 0.1208$. This is greater than the value of 0.10 obtained by exercising at time zero.

In practice tens of thousands of simulations are carried out to determine the early

[26] See L. Andersen, "A Simple Approach to the Pricing of Bermudan Swaptions in the Multifactor LIBOR Market Model," *Journal of Computational Finance*, 3, 2 (Winter 2000): 1–32.

exercise boundary in the way we have described. Once we have obtained the early exercise boundary, we discard the paths for the variables and carry out a new Monte Carlo simulation using the early exercise boundary to value the option. Our American put option example is simple in that we know the early exercise boundary at a time can be defined entirely in terms of the value of the stock price at that time. In more complicated situations it is necessary to make assumptions about how the early exercise boundary should be parameterized.

Upper Bounds

The two approaches we have outlined tend to underprice American-style options because they assume a suboptimal early exercise boundary. This has led Andersen and Broadie to propose a procedure that provides an upper bound to the price.[27] This procedure can be used in conjunction with any algorithm that generates a lower bound and pinpoints the true value of an American-style option more precisely than the algorithm does by itself.

SUMMARY

A number of models have been developed to fit the volatility smiles that are observed in practice. The constant elasticity of variance model leads to a volatility smile similar to that observed for equity options. The jump–diffusion model leads to a volatility smile similar to that observed for currency options. Variance-gamma and stochastic volatility models are more flexible in that they can lead to either the type of volatility smile observed for equity options or the type of volatility smile observed for currency options. The implied volatility function model provides even more flexibility than this. It is designed to provide an exact fit to any pattern of European option prices observed in the market.

The natural technique to use for valuing path-dependent options is Monte Carlo simulation. This has the disadvantage that it is fairly slow and unable to handle American-style derivatives easily. Luckily, trees can be used to value many types of path-dependent derivatives. The approach is to choose representative values for the underlying path function at each node of the tree and calculate the value of the derivative for each alternative value of the path function as we roll back through the tree.

Trees can be used to value many types of barrier options, but the convergence of the option value to the correct value as the number of time steps is increased tends to be slow. One approach for improving convergence is to arrange the geometry of the tree so that nodes always lie on the barriers. Another is to use an interpolation scheme to adjust for the fact that the barrier being assumed by the tree is different from the true barrier. A third is to design the tree so that it provides a finer representation of movements in the underlying asset price near the barrier.

One way of valuing options dependent on the prices of two correlated assets is to apply a transformation to the asset price to create two new uncorrelated variables. These two variables are each modeled with trees and the trees are then combined to

[27] See L. Andersen and M. Broadie, "A Primal-Dual Simulation Algorithm for Pricing Multi-Dimensional American Options," *Management Science*, 50, 9 (2004), 1222–34.

form a single three-dimensional tree. At each node of the tree, the inverse of the transformation gives the asset prices. A second approach is to arrange the positions of nodes on the three-dimensional tree to reflect the correlation. A third approach is to start with a tree that assumes no correlation between the variables and then adjust the probabilities on the tree to reflect the correlation.

Monte Carlo simulation is not naturally suited to valuing American-style options, but there are two ways it can be adapted to handle them. The first involves using a least-squares analysis to relate the value of continuing (i.e, not exercising) to the values of relevant variables. The second involves parameterizing the early exercise boundary and determining it iteratively by working back from the end of the life of the option to the beginning.

FURTHER READING

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Questions and Problems (Answers in Solutions Manual)

24.1. Confirm that the CEV model formulas satisfy put–call parity.

24.2. Explain how you would use Monte Carlo simulation to sample paths for the asset price when Merton's jump–diffusion model is used.

24.3. Confirm that Merton's jump–diffusion model satisfies put–call parity when the jump size is lognormal.

24.4. Suppose that the volatility of an asset will be 20% from month 0 to month 6, 22% from month 6 to month 12, and 24% from month 12 to month 24. What volatility should be used in Black–Scholes to value a 2-year option?

24.5. Consider the case of Merton's jump–diffusion model where jumps always reduce the asset price to zero. Assume that the average number of jumps per year is λ. Show that the price of a European call option is the same as in a world with no jumps except that the risk-free rate is $r+\lambda$ rather than r. Does the possibility of jumps increase or reduce the value of the call option in this case? (*Hint*: Value the option assuming no jumps and assuming one or more jumps. The probability of no jumps in time T is $e^{-\lambda T}$).

24.6. At time zero the price of a non-dividend-paying stock is S_0. Suppose that the time interval between 0 and T is divided into two subintervals of length t_1 and t_2. During the first subinterval, the risk-free interest rate and volatility are r_1 and σ_1, respectively. During the second subinterval, they are r_2 and σ_2, respectively. Assume that the world is risk neutral.

(a) Use the results in Chapter 13 to determine the stock price distribution at time T in terms of r_1, r_2, σ_1, σ_2, t_1, t_2, and S_0.

(b) Suppose that $\bar{r}$ is the average interest rate between time zero and T and that $\bar{V}$ is the average variance rate between times zero and T. What is the stock price distribution as a function of T in terms of $\bar{r}$, $\bar{V}$, T, and S_0?

(c) What are the results corresponding to (a) and (b) when there are three subintervals with different interest rates and volatilities?

(d) Show that if the risk-free rate, r, and the volatility, σ, are known functions of time, the stock price distribution at time T in a risk-neutral world is

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\bar{r} - \frac{\bar{V}}{2}\right)T,\ \sqrt{\bar{V}T}\right]$$

where $\bar{r}$ is the average value of r, $\bar{V}$ is equal to the average value of σ^2, and S_0 is the stock price today.

24.7. Write down the equations for simulating the path followed by the asset price in the stochastic volatility model in equations (24.2) and (24.3).

24.8. "The IVF model does not necessarily get the evolution of the volatility surface correct." Explain this statement.

24.9. "When interest rates are constant the IVF model correctly values any derivative whose payoff depends on the value of the underlying asset at only one time." Explain this statement.

24.10. Use a three-time-step tree to value an American lookback call option on a currency when the initial exchange rate is 1.6, the domestic risk-free rate is 5% per annum, the foreign risk-free interest rate is 8% per annum, the exchange rate volatility is 15%, and the time to maturity is 18 months. Use the approach in Section 24.4.

24.11. What happens to the variance-gamma model as the parameter v tends to zero?

24.12. Use a three-time-step tree to value an American put option on the geometric average of the price of a non-dividend-paying stock when the stock price is \$40, the strike price is \$40, the risk-free interest rate is 10% per annum, the volatility is 35% per annum, and the time to maturity is three months. The geometric average is measured from today until the option matures.

24.13. Can the approach for valuing path-dependent options in Section 24.4 be used for a 2-year American-style option that provides a payoff equal to $\max(S_{\text{ave}} - K, 0)$, where S_{ave} is the average asset price over the three months preceding exercise? Explain your answer.

24.14. Verify that the 6.492 number in Figure 24.3 is correct.

24.15. Examine the early exercise policy for the eight paths considered in the example in Section 24.7. What is the difference between the early exercise policy given by the least squares approach and the exercise boundary parameterization approach? Which gives a higher option price for the paths sampled?

24.16. Consider a European put option on a non-dividend paying stock when the stock price is \$100, the strike price is \$110, the risk-free rate is 5% per annum, and the time to maturity is one year. Suppose that the average variance rate during the life of an option has a 0.20 probability of being 0.06, a 0.5 probability of being 0.09, and a 0.3 probability of being 0.12. The volatility is uncorrelated with the stock price. Estimate the value of the option. Use DerivaGem.

24.17. When there are two barriers how can a tree be designed so that nodes lie on both barriers?

Assignment Questions

24.18. A new European-style lookback call option on a stock index has a maturity of 9 months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use the approach in Section 24.4 to value the option and compare your answer to the result given by DerivaGem using the analytic valuation formula.

24.19. Suppose that the volatilities used to price a 6-month currency option are as in Table 16.2. Assume that the domestic and foreign risk-free rates are 5% per annum and the current exchange rate is 1.00. Consider a bull spread that consists of a long position in a 6-month call option with strike price 1.05 and a short position in a 6-month call option with a strike price 1.10.

(a) What is the value of the spread?

(b) What single volatility if used for both options gives the correct value of the bull spread? (Use the DerivaGem Application Builder in conjunction with Goal Seek or Solver.)

(c) Does your answer support the assertion at the beginning of the chapter that the correct volatility to use when pricing exotic options can be counterintuitive?

(d) Does the IVF model give the correct price for the bull spread?

24.20. Repeat the analysis in Section 24.7 for the put option example on the assumption that the strike price is 1.13. Use both the least squares approach and the exercise boundary parameterization approach.

24.21. Consider the situation in Merton's jump–diffusion model where the underlying asset is a non-dividend-paying stock. The average frequency of jumps is one per year. The average percentage jump size is 2% and the standard deviation of the logarithm of the percentage jump size is 20%. The stock price is 100, the risk-free rate is 5%, the volatility, σ provided by the diffusion part of the process is 15%, and the time to maturity is six months. Use the DerivaGem Application Builder to calculate an implied volatility when the strike price is 80, 90, 100, 110, and 120. What does the volatility smile or skew that you obtain imply about the probability distribution of the stock price.