CHAPTER 29

Interest Rate Derivatives: HJM and LMM

The interest rate models discussed in Chapter 28 are widely used for pricing instruments when the simpler models in Chapter 26 are inappropriate. They are easy to implement and, if used carefully, can ensure that most nonstandard interest rate derivatives are priced consistently with actively traded instruments such as interest rate caps, European swap options, and European bond options. Two limitations of the models are:

1. Most involve only one factor (i.e., one source of uncertainty).
2. They do not give the user complete freedom in choosing the volatility structure.

By making the parameters a and σ functions of time, an analyst can use the models so that they fit the volatilities observed in the market today, but as mentioned in Section 28.8 the volatility term structure is then nonstationary. The volatility structure in the future is liable to be quite different from that observed in the market today.

This chapter discusses some general approaches to building term structure models that give the user more flexibility in specifying the volatility environment and allow several factors to be used. The models require much more computation time than the models in Chapter 28. As a result, they are often used for research and development rather than routine pricing.

This chapter also covers the mortgage-backed security market in the United States and describes how some of the ideas presented in the chapter can be used to price instruments in that market.

29.1 THE HEATH, JARROW, AND MORTON MODEL

In 1990 David Heath, Bob Jarrow, and Andy Morton (HJM) published an important paper describing the no-arbitrage conditions that must be satisfied by a model of the yield curve.[1] To describe their model, we will use the following notation:

$P(t, T)$: Price at time t of a zero-coupon bond with principal \$1 maturing at time T

[1] See D. Heath, R. A. Jarrow, and A. Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology," *Econometrica*, 60, 1 (1992): 77–105.

Ω_t: Vector of past and present values of interest rates and bond prices at time t that are relevant for determining bond price volatilities at that time

$v(t, T, \Omega_t)$: Volatility of $P(t, T)$

$f(t, T_1, T_2)$: Forward rate as seen at time t for the period between time T_1 and time T_2

$F(t, T)$: Instantaneous forward rate as seen at time t for a contract maturing at time T

$r(t)$: Short-term risk-free interest rate at time t

$dz(t)$: Wiener process driving term structure movements

Processes for Zero-Coupon Bond Prices and Forward Rates

We start by assuming there is just one factor and will use the traditional risk-neutral world. A zero-coupon bond is a traded security providing no income. Its return in the traditional risk-neutral world must therefore be r. This means that its stochastic process has the form

$$dP(t, T) = r(t)P(t, T)\,dt + v(t, T, \Omega_t)P(t, T)\,dz(t) \tag{29.1}$$

As the argument Ω_t indicates, the zero-coupon bond's volatility v can be, in the most general form of the model, any well-behaved function of past and present interest rates and bond prices. Because a bond's price volatility declines to zero at maturity, we must have[2]

$$v(t, t, \Omega_t) = 0$$

From equation (4.5), the forward rate $f(t, T_1, T_2)$ can be related to zero-coupon bond prices as follows:

$$f(t, T_1, T_2) = \frac{\ln[P(t, T_1)] - \ln[P(t, T_2)]}{T_2 - T_1} \tag{29.2}$$

From equation (29.1) and Itô's lemma,

$$d\ln[P(t, T_1)] = \left[r(t) - \frac{v(t, T_1, \Omega_t)^2}{2}\right]dt + v(t, T_1, \Omega_t)\,dz(t)$$

and

$$d\ln[P(t, T_2)] = \left[r(t) - \frac{v(t, T_2, \Omega_t)^2}{2}\right]dt + v(t, T_2, \Omega_t)\,dz(t)$$

so that

$$df(t, T_1, T_2) = \frac{v(t, T_2, \Omega_t)^2 - v(t, T_1, \Omega_t)^2}{2(T_2 - T_1)}\,dt + \frac{v(t, T_1, \Omega_t) - v(t, T_2, \Omega_t)}{T_2 - T_1}\,dz(t) \tag{29.3}$$

Equation (29.3) shows that the risk-neutral process for f depends solely on the v's. It depends on r and the P's only to the extent that the v's themselves depend on these variables.

[2] The $v(t, t, \Omega_t) = 0$ condition is equivalent to the assumption that all discount bonds have finite drifts at all times. If the volatility of the bond does not decline to zero at maturity, an infinite drift may be necessary to ensure that the bond's price equals its face value at maturity.

When we put $T_1 = T$ and $T_2 = T + \Delta T$ in equation (29.3) and then take limits as ΔT tends to zero, $f(t, T_1, T_2)$ becomes $F(t, T)$, the coefficient of $dz(t)$ becomes $v_T(t, T, \Omega_t)$, and the coefficient of dt becomes

$$\frac{1}{2}\frac{\partial[v(t, T, \Omega_t)^2]}{\partial T} = v(t, T, \Omega_t)\, v_T(t, T, \Omega_t)$$

where the subscript to v denotes a partial derivative. It follows that

$$dF(t, T) = v(t, T, \Omega_t)\, v_T(t, T, \Omega_t)\, dt - v_T(t, T, \Omega_t)\, dz(t) \tag{29.4}$$

Once the function $v(t, T, \Omega_t)$ has been specified, the risk-neutral processes for the $F(t, T)$'s are known.

Equation (29.4) shows that there is a link between the drift and standard deviation of an instantaneous forward rate. This is the key HJM result. Integrating $v_\tau(t, \tau, \Omega_t)$ between $\tau = t$ and $\tau = T$, we obtain

$$v(t, T, \Omega_t) - v(t, t, \Omega_t) = \int_t^T v_\tau(t, \tau, \Omega_t)\, d\tau$$

Because $v(t, t, \Omega_t) = 0$, this becomes

$$v(t, T, \Omega_t) = \int_t^T v_\tau(t, \tau, \Omega_t)\, d\tau$$

If $m(t, T, \Omega_t)$ and $s(t, T, \Omega_t)$ are the instantaneous drift and standard deviation of $F(t, T)$, so that

$$dF(t, T) = m(t, T, \Omega_t)\, dt + s(t, T, \Omega_t)\, dz$$

then it follows from equation (29.4) that

$$m(t, T, \Omega_t) = s(t, T, \Omega_t) \int_t^T s(t, \tau, \Omega_t)\, d\tau \tag{29.5}$$

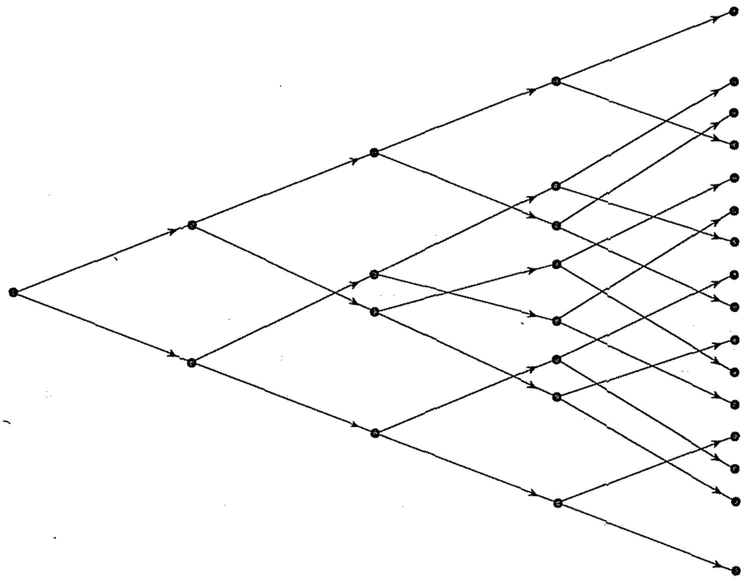
This is the HJM result.

The process for the short rate r in the general HJM model is non-Markov. To understand what this means, suppose that we are at time zero and calculate the process followed by r at a future time T. We would find that the process is liable to depend on the particular path followed by the Wiener process $z(t)$ in equation (29.1) between time 0 and time t.[3]

This highlights the key problem in implementing a general HJM model. We have to use Monte Carlo simulation. Trees create difficulties. When we construct a tree representing term structure movements, it is usually nonrecombining. Assuming the model has one factor and the tree is binomial as in Figure 29.1, there are 2^n nodes after n time steps. If the model has two factors, the tree must be constructed in three dimensions and there are then 4^n nodes after n time steps. Therefore, for $n = 30$, the number of terminal nodes in a one-factor model is about 10^9; in a two-factor model, it is about 10^{18}.

[3] For more details, see Technical Note 17 on the author's website.

Figure 29.1 A nonrecombining tree such as that arising from the general HJM model.



Extension to Several Factors

The HJM result can be extended to the situation where there are several independent factors. Suppose

$$dF(t, T) = m(t, T, \Omega_t)\, dt + \sum_k s_k(t, T, \Omega_t)\, dz_k$$

A similar analysis to that just given (see Problem 29.2) shows that

$$m(t, T, \Omega_t) = \sum_k s_k(t, T, \Omega_t) \int_t^T s_k(t, \tau, \Omega_t)\, d\tau \qquad \textbf{(29.6)}$$

29.2 THE LIBOR MARKET MODEL

One drawback of the HJM model is that it is expressed in terms of instantaneous forward rates and these are not directly observable in the market. Another drawback is that it is difficult to calibrate the model to prices of actively traded instruments. This has led Brace, Gatarek, and Musiela (BGM), Jamshidian, and Miltersen, Sandmann, and Sondermann to propose an alternative.[4] It is known as the *LIBOR market model* (LMM) or the *BGM model* and it is expressed in terms of the forward rates that traders are used to working with.

[4] See A. Brace, D. Gatarek, and M. Musiela "The Market Model of Interest Rate Dynamics," *Mathematical Finance* 7, 2 (1997): 127–55; F. Jamshidian, "LIBOR and Swap Market Models and Measures," *Finance and Stochastics*, 1 (1997): 293–330; and K. Miltersen, K. Sandmann, and D. Sondermann, "Closed Form Solutions for Term Structure Derivatives with LogNormal Interest Rate," *Journal of Finance*, 52, 1 (March 1997): 409–30.

The Model

Define $t_0 = 0$ and let $t_1, t_2, \ldots$ be the reset times for caps that trade in the market today. In the United States, the most popular caps have quarterly resets, so that it is approximately true that $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, and so on. Define $\Delta_k = t_{k+1} - t_k$, and

- $F_k(t)$: Forward rate between times t_k and t_{k+1} as seen at time t, expressed with a compounding period of Δ_k and an actual/actual day count
- $m(t)$: Index for the next reset date at time t; this means that $m(t)$ is the smallest integer such that $t \leqslant t_{m(t)}$
- $\zeta_k(t)$: Volatility of $F_k(t)$ at time t
- $v_k(t)$: Volatility of the zero-coupon bond price $P(t, t_k)$ at time t

Initially, we will assume that there is only one factor. As shown in Section 25.4, in a world that is forward risk neutral with respect to $P(t, t_{k+1})$, $F_k(t)$ is a martingale and follows the process

$$dF_k(t) = \zeta_k(t) F_k(t)\, dz \tag{29.7}$$

where dz is a Wiener process.

In practice, it is often most convenient to value interest rate derivatives by working in a world that is always forward risk neutral with respect to a bond maturing at the next reset date. We refer to this as a *rolling forward risk-neutral world*.[5] In this world we can discount from time t_{k+1} to time t_k using the zero rate observed at time t_k for a maturity t_{k+1}. We do not have to worry about what happens to interest rates between times t_k and t_{k+1}.

At time t the rolling forward risk-neutral world is a world that is forward risk neutral with respect to the bond price, $P[t, t_{m(t)}]$. Equation (29.7) gives the process followed by $F_k(t)$ in a world that is forward risk neutral with respect to $P(t, t_{k+1})$. From Section 25.7, it follows that the process followed by $F_k(t)$ in the rolling forward risk-neutral world is

$$dF_k(t) = \zeta_k(t)[v_{m(t)}(t) - v_{k+1}(t)] F_k(t)\, dt + \zeta_k(t) F_k(t)\, dz \tag{29.8}$$

The relationship between forward rates and bond prices is

$$\frac{P(t, t_i)}{P(t, t_{i+1})} = 1 + \delta_i F_i(t)$$

or

$$\ln P(t, t_i) - \ln P(t, t_{i+1}) = \ln[1 + \delta_i F_i(t)]$$

Using Itô's lemma we can calculate the process followed the left-hand side and the right-hand side of this equation. Equating the coefficients of dz, we obtain

$$v_i(t) - v_{i+1}(t) = \frac{\delta_i F_i(t) \zeta_i(t)}{1 + \delta_i F_i(t)} \tag{29.9}$$

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[5] In the terminology of Section 25.4, this world corresponds to using a "rolling CD" as the numeraire. A rolling CD (certificate of deposit) is one where we start with \$1, buy a bond maturing at time t_1, reinvest the proceeds at time t_1 in a bond maturing at time t_2, reinvest the proceeds at time t_2 in a bond maturing at time t_3, and so on. Strictly speaking, the interest rate trees we constructed in Chapter 28 are in a rolling forward risk-neutral world rather than the traditional risk-neutral world. The numeraire is a CD rolled over at the end of each time step.

so that from equation (29.8) the process followed by $F_k(t)$ in the rolling forward risk-neutral world is

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)\zeta_k(t)}{1+\delta_i F_i(t)}\,dt + \zeta_k(t)\,dz \tag{29.10}$$

The HJM result in equation (29.4) is the limiting case of this as the δ_i tend to zero (see Problem 29.7).

Forward Rate Volatilities

We now simplify the model by assuming that $\zeta_k(t)$ is a function only of the number of whole accrual periods between the next reset date and time t_k. Define Λ_i as the value of $\zeta_k(t)$ when there are i such accrual periods. This means that $\zeta_k(t) = \Lambda_{k-m(t)}$ is a step function.

The Λ_i can (at least in theory) be estimated from the volatilities used to value caplets in Black's model (i.e., from the spot volatilities in Figure 26.3).[6] Suppose that σ_k is the Black volatility for the caplet that corresponds to the period between times t_k and t_{k+1}. Equating variances, we must have

$$\sigma_k^2 t_k = \sum_{i=1}^{k} \Lambda_{k-i}^2\,\delta_{i-1} \tag{29.11}$$

This equation can be used to obtain the Λ's iteratively.

Example 29.1

Assume that the δ_i are all equal and the Black caplet spot volatilities for the first three caplets are 24%, 22%, and 20%. This means that $\Lambda_0 = 24\%$. Since

$$\Lambda_0^2 + \Lambda_1^2 = 2 \times 0.22^2$$

Λ_1 is 19.80%. Also, since

$$\Lambda_0^2 + \Lambda_1^2 + \Lambda_2^2 = 3 \times 0.20^2$$

Λ_2 is 15.23%.

Example 29.2

Consider the data in Table 29.1 on caplet volatilities σ_k. These exhibit the hump discussed in Section 26.3. The Λ's are shown in the second row. Notice that the hump in the Λ's is more pronounced than the hump in the σ's.

Table 29.1 Volatility data; accrual period = 1 year.

Year, k:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
σ_k (%):	15.50	18.25	17.91	17.74	17.27	16.79	16.30	16.01	15.76	15.54
Λ_{k-1} (%):	15.50	20.64	17.21	17.22	15.25	14.15	12.98	13.81	13.60	13.40

[6] In practice the Λ's are determined using a least-squares calibration that we will discuss later.

Implementation of the Model

The LIBOR market model can be implemented using Monte Carlo simulation. Expressed in terms of the Λ_i's, equation (29.10) is

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\Lambda_{i-m(t)}\Lambda_{k-m(t)}}{1+\delta_i F_i(t)}\,dt + \Lambda_{k-m(t)}\,dz \tag{29.12}$$

or

$$d\ln F_k(t) = \left[\sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\Lambda_{i-m(t)}\Lambda_{k-m(t)}}{1+\delta_i F_i(t)} - \frac{(\Lambda_{k-m(t)})^2}{2}\right]dt + \Lambda_{k-m(t)}\,dz \tag{29.13}$$

If, as an approximation, we assume in the calculation of the drift of $\ln F_k(t)$ that $F_i(t) = F_i(t_j)$ for $t_j < t < t_{j+1}$, then

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j)\Lambda_{i-j-1}\Lambda_{k-j-1}}{1+\delta_i F_i(t_j)} - \frac{\Lambda_{k-j-1}^2}{2}\right)\delta_j + \Lambda_{k-j-1}\epsilon\sqrt{\delta_j}\right] \tag{29.14}$$

where ϵ is a random sample from a normal distribution with mean equal to zero and standard deviation equal to one.

Extension to Several Factors

The LIBOR market model can be extended to incorporate several independent factors. Suppose that there are p factors and $\zeta_{k,q}$ is the component of the volatility of $F_k(t)$ attributable to the qth factor. Equation (29.10) becomes (see Problem 29.11)

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\sum_{q=1}^{p}\zeta_{i,q}(t)\zeta_{k,q}(t)}{1+\delta_i F_i(t)}\,dt + \sum_{q=1}^{p}\zeta_{k,q}(t)\,dz_q \tag{29.15}$$

Define $\lambda_{i,q}$ as the qth component of the volatility when there are i accrual periods between the next reset date and the maturity of the forward contract. Equation (29.14) then becomes

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j)\sum_{q=1}^{p}\lambda_{i-j-1,q}\lambda_{k-j-1,q}}{1+\delta_i F_i(t_j)} - \frac{\sum_{q=1}^{p}\lambda_{k-j-1,q}^2}{2}\right)\delta_j + \sum_{q=1}^{p}\lambda_{k-j-1,q}\,\epsilon_q\sqrt{\delta_j}\right] \tag{29.16}$$

where the ϵ_q are random samples from a normal distribution with mean equal to zero and standard deviation equal to one.

The approximation that the drift of a forward rate remains constant within each accrual period allows us to jump from one reset date to the next in the simulation. This is convenient because as already mentioned the rolling forward risk-neutral world allows us to discount from one reset date to the next. Suppose that we wish to simulate a zero curve for N accrual periods. On each trial we start with the forward rates at time zero. These are $F_0(0), F_1(0), \ldots, F_{N-1}(0)$ and are calculated from the initial zero curve. We use equation (29.16) to calculate $F_1(t_1), F_2(t_1), \ldots, F_{N-1}(t_1)$. We then use equation (29.16) again to calculate $F_2(t_2), F_3(t_2), \ldots, F_{N-1}(t_2)$; and so on until $F_{N-1}(t_{N-1})$ is obtained. Note that as we move through time the zero curve gets shorter and shorter. For example, suppose each accrual period is 3 months and $N = 40$. We start with a 10-year zero curve. At the 6-year point (at time t_{24}), the simulation gives us information on a 4-year zero curve.

We can test the drift approximation by valuing caplets using equation (29.16) and comparing the prices to those given by Black's model. The value of $F_k(t_k)$ is the realized rate for the time period between t_k and t_{k+1} and enables the caplet payoff at time t_{k+1} to be calculated. This payoff is discounted back to time zero, one accrual period at a time. The caplet value is the average of the discounted payoffs. The results of this type of analysis show that the cap values from Monte Carlo simulation are not significantly different from those given by Black's model. This is true even when the accrual periods are 1 year in length and a very large number of trials is used.[7] This suggests that the drift assumption is innocuous in most situations.

Ratchet Caps, Sticky Caps, and Flexi Caps

The LIBOR market model can be used to value some types of nonstandard caps. Consider ratchet caps and sticky caps. These incorporate rules for determining how the cap rate for each caplet is set. In a *ratchet cap* it equals the LIBOR rate at the previous reset date plus a spread. In a *sticky cap* it equals the previous capped rate plus a spread. Suppose that the cap rate at time t_j is K_j, the LIBOR rate at time t_j is R_j, and the spread is s. In a ratchet cap, $K_{j+1} = R_j + s$. In a sticky cap, $K_{j+1} = \min(R_j, K_j) + s$.

Tables 29.2 and 29.3 provides valuations of a ratchet cap and sticky cap using the LIBOR market model with one, two, and three factors. The principal is \$100. The term

Table 29.2 Valuation of ratchet caplets.

Caplet start time (years)	*One factor*	*Two factors*	*Three factors*
1	0.196	0.194	0.195
2	0.207	0.207	0.209
3	0.201	0.205	0.210
4	0.194	0.198	0.205
5	0.187	0.193	0.201
6	0.180	0.189	0.193
7	0.172	0.180	0.188
8	0.167	0.174	0.182
9	0.160	0.168	0.175
10	0.153	0.162	0.169

[7] See J. C. Hull and A. White, "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model," *Journal of Fixed Income*, 10, 2 (September 2000): 46–62. The only exception is when the cap volatilities are very high.

Table 29.3 Valuation of sticky caplets.

Caplet start time (years)	*One factor*	*Two factors*	*Three factors*
1	0.196	0.194	0.195
2	0.336	0.334	0.336
3	0.412	0.413	0.418
4	0.458	0.462	0.472
5	0.484	0.492	0.506
6	0.498	0.512	0.524
7	0.502	0.520	0.533
8	0.501	0.523	0.537
9	0.497	0.523	0.537
10	0.488	0.519	0.534

structure is assumed to be flat at 5% per annum and the caplet volatilities are as in Table 29.1. The interest rate is reset annually. The spread is 25 basis points. Tables 29.4 and 29.5 show how the volatility was split into components when two- and three-factor models were used. The results are based on 100,000 Monte Carlo simulations incorporating the antithetic variable technique described in Section 17.7. The standard error of each price is about 0.001.

A third type of nonstandard cap is a *flexi cap*. This is like a regular cap except that there is a limit on the total number of caplets that can be exercised. Consider an annual-pay flexi cap when the principal is \$100, the term structure is flat at 5%, and the cap volatilities are as in Tables 29.1, 29.4, and 29.5. Suppose that all in-the-money caplets are exercised up to a maximum of five. With one, two, and three factors, the LIBOR market model gives the price of the instrument as 3.43, 3.58, and 3.61, respectively (see Problem 29.15 for other types of flexi caps).

The pricing of a plain vanilla cap depends only on the total volatility and is independent of the number of factors. This is because the price of a plain vanilla caplet depends the behavior of only one forward rate. The prices of caplets in the nonstandard instruments we have looked at are different in that they depend on the joint probability distribution of several different forward rates. As a result they do depend on the number of factors.

Table 29.4 Volatility components in two-factor model.

Year, k:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\lambda_{k-1,1}$ (%):	14.10	19.52	16.78	17.11	15.25	14.06	12.65	13.06	12.36	11.63
$\lambda_{k-1,2}$ (%):	−6.45	−6.70	−3.84	−1.96	0.00	1.61	2.89	4.48	5.65	6.65
Total volatility (%):	15.50	20.64	17.21	17.22	15.25	14.15	12.98	13.81	13.60	13.40

Table 29.5 Volatility components in a three-factor model.

Year, k:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
$\lambda_{k-1,1}$ (%):	13.65	19.28	16.72	16.98	14.85	13.95	12.61	12.90	11.97	10.97
$\lambda_{k-1,2}$ (%):	−6.62	−7.02	−4.06	−2.06	0.00	1.69	3.06	4.70	5.81	6.66
$\lambda_{k-1,3}$ (%):	3.19	2.25	0.00	−1.98	−3.47	−1.63	0.00	1.51	2.80	3.84
Total volatility (%):	15.50	20.64	17.21	17.22	15.25	14.15	12.98	13.81	13.60	13.40

Valuing European Swap Options

As shown by Hull and White, there is an analytic approximation for valuing European swap options in the LIBOR market model.[8] Let T_0 be the maturity of the swap option and assume that the payment dates for the swap are $T_1, T_2, \dots, T_N$. Define $\tau_i = T_{i+1} - T_i$. From equation (25.23), the swap rate at time t is given by

$$s(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=0}^{N-1} \tau_i P(t, T_{i+1})}$$

It is also true that

$$\frac{P(t, T_i)}{P(t, T_0)} = \prod_{j=0}^{i-1} \frac{1}{1 + \tau_j G_j(t)}$$

for $1 \leqslant i \leqslant N$, where $G_j(t)$ is the forward rate at time t for the period between T_j and T_{j+1}. These two equations together define a relationship between $s(t)$ and the $G_j(t)$. Applying Itô's lemma (see Problem 29.12), the variance $V(t)$ of the swap rate $s(t)$ is given by

$$V(t) = \sum_{q=1}^{p} \left[\sum_{k=0}^{N-1} \frac{\tau_k \beta_{k,q}(t) G_k(t) \gamma_k(t)}{1 + \tau_k G_k(t)} \right]^2 \tag{29.17}$$

where

$$\gamma_k(t) = \frac{\prod_{j=0}^{N-1} [1 + \tau_j G_j(t)]}{\prod_{j=0}^{N-1} [1 + \tau_j G_j(t)] - 1} - \frac{\sum_{i=0}^{k-1} \tau_i \prod_{j=i+1}^{N-1} [1 + \tau_j G_j(t)]}{\sum_{i=0}^{N-1} \tau_i \prod_{j=i+1}^{N} [1 + \tau_j G_j(t)]}$$

and $\beta_{j,q}(t)$ is the qth component of the volatility of $G_j(t)$. We approximate $V(t)$ by setting $G_j(t) = G_j(0)$ for all j and t. The swap volatility that is substituted into the

[8] See J. C. Hull and A. White, "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model," *Journal of Fixed Income*, 10, 2 (September 2000): 46–62. Other analytic approximations have been suggested by A. Brace, D. Gatarek, and M. Musiela "The Market Model of Interest Rate Dynamics," *Mathematical Finance*, 7, 2 (1997): 127–55 and L. Andersen and J. Andreasen, "Volatility Skews and Extensions of the LIBOR Market Model," *Applied Mathematical Finance*, 7, 1 (March 2000), 1–32.

standard market model for valuing a swaption is then

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0} V(t)\,dt}$$

or

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0}\sum_{q=1}^{p}\left[\sum_{k=0}^{N-1}\frac{\tau_k\beta_{k,q}(t)G_k(0)\gamma_k(0)}{1+\tau_k G_k(0)}\right]^2 dt} \tag{29.18}$$

In the situation where the length of the accrual period for swap underlying the swaption is the same as the length of the accrual period for a cap, $\beta_{k,q}(t)$ is the qth component of volatility of a cap forward rate when the time to maturity is $T_k - t$. This can be looked up in a table such as Table 29.5

The accrual periods for the swaps underlying broker quotes for European swap options do not always match the accrual periods for the caps and floors underlying broker quotes. For example, in the United States, the benchmark caps and floors have quarterly resets, while the swaps underlying the benchmark European swap options have semiannual resets on the fixed side. Fortunately, the valuation result for European swap options can be extended to the situation where each swap accrual period includes M subperiods that could be accrual periods in a typical cap. Define $\tau_{j,m}$ as the length of the mth subperiod in the jth accrual period so that

$$\tau_j = \sum_{m=1}^{M}\tau_{j,m}$$

Define $G_{j,m}(t)$ as the forward rate observed at time t for the $\tau_{j,m}$ accrual period. Because

$$1+\tau_j G_j(t) = \prod_{m=1}^{M}[1+\tau_{j,m}G_{j,m}(t)]$$

we can modify the analysis leading to equation (29.18) so that the volatility of $s(t)$ is obtained in terms of the volatilities of the $G_{j,m}(t)$ rather than the volatilities of the $G_j(t)$. The swap volatility to be substituted into the standard market model for valuing a swap option proves to be (see Problem 29.13)

$$\sqrt{\frac{1}{T_0}\int_{t=0}^{T_0}\sum_{q=1}^{p}\left[\sum_{k=n}^{N-1}\sum_{m=1}^{M}\frac{\tau_{k,m}\beta_{k,m,q}(t)G_{k,m}(0)\gamma_k(0)}{1+\tau_{k,m}G_{k,m}(0)}\right]^2 dt} \tag{29.19}$$

Here $\beta_{j,m,q}(t)$ is the qth component of the volatility of $G_{j,m}(t)$. It is the qth component of the volatility of a cap forward rate when the time to maturity is from t to the beginning of the mth subperiod in the (T_j, T_{j+1}) swap accrual period.

The expressions in equations (29.18) and (29.19) for the swap volatility do involve the approximations that $G_j(t) = G_j(0)$ and $G_{j,m}(t) = G_{j,m}(0)$. Hull and White compared the prices of European swap options calculated using equations (29.18) and (29.19) with the prices calculated from a Monte Carlo simulation and found the two to be very close. Once the LIBOR market model has been calibrated, equations (29.18) and (29.19) therefore provide a quick way of valuing European swap options. Analysts can determine whether European swap options are overpriced or underpriced relative to caps. As we will see shortly, they can also use the results to calibrate the model to the market prices of swap options.

Calibrating the Model

To calibrate the LIBOR market model, we must determine the Λ_j and how they are split into $\lambda_{j,q}$. The first step is usually to use a principal components analysis such as that in Section 18.9 to determine the way the Λ's are split into λ's. The principal components model is

$$\Delta F_j = \sum_{q=1}^{M} \alpha_{j,q} x_q$$

where M is the total number of factors, ΔF_j is the change in the forward rate for a forward contract maturing in j accrual periods, $\alpha_{j,q}$ is the factor loading for the jth forward rate and the qth factor, x_q is the factor score for the qth factor, and

$$\sum_{j=1}^{M} \alpha_{j,q_1} \alpha_{j,q_2}$$

equals 1 when $q_1 = q_2$ and 0 when $q_1 \neq q_2$. Define s_q as the standard deviation of the qth factor score. If the number of factors used in the LIBOR market model, p, is equal to the total number of factors, M, it is correct to set

$$\lambda_{j,q} = \alpha_{j,q}\, s_q$$

for $1 \leqslant j, q \leqslant M$. When $p < M$, the $\lambda_{j,q}$ must be scaled so that

$$\Lambda_j = \sqrt{\sum_{q=1}^{p} \lambda_{j,q}^2}$$

This involves setting

$$\lambda_{j,q} = \frac{\Lambda_j\, s_q\, \alpha_{j,q}}{\sqrt{\sum_{q=1}^{p} s_q^2\, \alpha_{i,q}^2}} \tag{29.20}$$

Equation (29.11) provides one way to determine the Λ's so that they are consistent with caplet prices. In practice it is not usually used because it often leads to wild swings in the Λ's.[9] Also, although the LIBOR market model is designed to be consistent with the prices of caplets, analysts sometimes like to calibrate it to European swaptions.

The most commonly used calibration procedure for the LIBOR market model is similar to that described for one-factor models in Section 28.8. Suppose that U_i is the market price of the ith calibrating instrument and V_i is the model price. We choose the Λ's to minimize

$$\sum_i (U_i - V_i)^2 + P$$

[9] Sometimes there is no set of Λ's consistent with a set of cap quotes.

where P is a penalty function chosen to ensure that the Λ's are "well behaved". Similarly to Section 28.8, P has the form

$$P = \sum_i w_{1,i}(\Lambda_{i+1} - \Lambda_i)^2 + \sum_i w_{2,i}(\Lambda_{i+1} + \Lambda_{i-1} - 2\Lambda_i)^2$$

When some calibrating instruments are European swaptions the formulas in equations (29.18) and (29.19) make the minimization feasible using the Levenberg–Marquardt procedure. Equation (29.20) is used to determine the λ's from the Λ's.

Volatility Skews

Brokers provide quotes on caps that are not at the money as well as on caps that are at the money. In some markets a volatility skew is observed, that is, the quoted (Black) volatility for a cap or a floor is a declining function of the strike price. This can be handled using the CEV model. (See Section 24.1 for the application of the CEV model to equities.) The model is

$$dF_i(t) = \cdots + \sum_{q=1}^{p} \zeta_{i,q}(t) F_i(t)^{\alpha}\, dz_q \tag{29.21}$$

where α is a constant $(0 < \alpha < 1)$. It turns out that this model can be handled very similarly to the lognormal model. Caps and floors can be valued analytically using the cumulative noncentral χ^2 distribution. There are similar analytic approximations to those given above for the prices of European swap options.[10]

Bermudan Swap Options

A popular interest rate derivative is a Bermudan swap option. This is a swap option that can be exercised on some or all of the payment dates of the underlying swap. Bermudan swap options are difficult to value using the LIBOR market model because the LIBOR market model relies on Monte Carlo simulation and it is difficult to evaluate early exercise decisions when Monte Carlo simulation is used. Fortunately, the procedures described in Section 24.7 can be used. Longstaff and Schwartz apply the least-squares approach when there are a large number of factors. The value of not exercising on a particular payment date is assumed to be a polynomial function of the values of the factors.[11] Andersen shows that the optimal early exercise boundary approach can be used. He experiments with a number of ways of parameterizing the early exercise boundary and finds that good results are obtained when the early exercise decision is assumed to depend only on the intrinsic value of the option.[12]. Most traders value Bermudan options using one of the one-factor no-arbitrage models discussed in

[10] For details, see L. Andersen and J. Andreasen, "Volatility Skews and Extensions of the LIBOR Market Model," *Applied Mathematical Finance*, 7, 1 (2000): 1–32; J.C. Hull and A. White, "Forward Rate Volatilities, Swap Rate Volatilities, and the Implementation of the LIBOR Market Model," *Journal of Fixed Income*, 10, 2 (September 2000): 46–62.

[11] See F.A. Longstaff and E.S. Schwartz, "Valuing American Options by Simulation: A Simple Least Squares Approach," *Review of Financial Studies*, 14, 1 (2001): 113–47.

[12] L. Andersen, "A simple Approach to the Pricing of Bermudan Swaptions in the Multifactor LIBOR Market Model," *Journal of Computational Finance*, 3, 2 (Winter 2000): 5–32.

Chapter 28. However, the accuracy of one-factor models for pricing Bermudan swap options has become a controversial issue.[13]

29.3 MORTGAGE-BACKED SECURITIES

One application of the models presented in this chapter is to the mortgage-backed security (MBS) market in the United States. A mortgage-backed security is created when a financial institution decides to sell part of its residential mortgage portfolio to investors. The mortgages sold are put into a pool and investors acquire a stake in the pool by buying units. The units are known as mortgage-backed securities. A secondary market is usually created for the units so that investors can sell them to other investors as desired. An investor who owns units representing $X\%$ of a certain pool is entitled to $X\%$ of the principal and interest cash flows received from the mortgages in the pool.

The mortgages in a pool are generally guaranteed by a government-related agency such as the Government National Mortgage Association (GNMA) or the Federal National Mortgage Association (FNMA) so that investors are protected against defaults. This makes an MBS sound like a regular fixed-income security issued by the government. In fact, there is a critical difference between an MBS and a regular fixed-income investment. This difference is that the mortgages in an MBS pool have prepayment privileges. These prepayment privileges can be quite valuable to the householder. In the United States, mortgages typically last for 25 years and can be prepaid at any time. This means that the householder has a 25-year American-style option to put the mortgage back to the lender at its face value.

In practice, prepayments on mortgages occur for a variety of reasons. Sometimes interest rates fall and the owner of the house decides to refinance at a lower rate. On other occasions, a mortgage is prepaid simply because the house is being sold. A critical element in valuing an MBS is the determination of what is known as the *prepayment function*. This is a function describing expected prepayments on the underlying pool of mortgages at a time t in terms of the yield curve at time t and other relevant variables.

A prepayment function is very unreliable as a predictor of actual prepayment experience for an individual mortgage. When many similar mortgage loans are combined in the same pool, there is a "law of large numbers" effect at work and prepayments can be predicted more accurately from an analysis of historical data. As mentioned, prepayments are not always motivated by pure interest rate considerations. Nevertheless, there is a tendency for prepayments to be more likely when interest rates are low than when they are high. This means that investors require a higher rate of interest on an MBS than on other fixed-income securities to compensate for the prepayment options they have written.

Collateralized Mortgage Obligations

The MBSs we have described so far are sometimes referred to as *pass-throughs*. All investors receive the same return and bear the same prepayment risk. Not all mortgage-

[13] For opposing viewpoints, see "Factor Dependence of Bermudan Swaptions: Fact or Fiction," by L. Andersen and J. Andreasen, and "Throwing Away a Billion Dollars: The Cost of Suboptimal Exercise Strategies in the Swaption Market," by F. A. Longstaff, P. Santa-Clara, and E. S. Schwartz. Both articles are in *Journal of Financial Economics*, 62, 1 (October 2001).

Business Snapshot 29.1 IOs and POs

In what is known as a *stripped MBS*, principal payments are separated from interest payments. All principal payments are channeled to one class of security, known as a *principal only* (PO). All interest payments are channeled to another class of security known as an *interest only* (IO). Both IOs and POs are risky investments. As prepayment rates increase, a PO becomes more valuable and an IO becomes less valuable. As prepayment rates decrease, the reverse happens. In a PO, a fixed amount of principal is returned to the investor, but the timing is uncertain. A high rate of prepayments on the underlying pool leads to the principal being received early (which is, of course, good news for the holder of the PO). A low rate of prepayments on the underlying pool delays the return of the principal and reduces the yield provided by the PO. In the case of an IO, the total of the cash flows received by the investor is uncertain. The higher the rate of prepayments, the lower the total cash flows received by the investor, and vice versa.

backed securities work in this way. In a *collateralized mortgage obligation* (CMO) the investors are divided into a number of classes and rules are developed for determining how principal repayments are channeled to different classes.

As an example of a CMO, consider an MBS where investors are divided into three classes: class A, class B, and class C. All the principal repayments (both those that are scheduled and those that are prepayments) are channeled to class A investors until investors in this class have been completely paid off. Principal repayments are then channeled to class B investors until these investors have been completely paid off. Finally, principal repayments are channeled to class C investors. In this situation, class A investors bear the most prepayment risk. The class A securities can be expected to last for a shorter time than the class B securities, and these, in turn, can be expected to last less long than the class C securities.

The objective of this type of structure is to create classes of securities that are more attractive to institutional investors than those created by the simpler pass-through MBS. The prepayment risks assumed by the different classes depend on the par value in each class. For example, class C bears very little prepayment risk if the par values in classes A, B, and C are 400, 300, and 100, respectively. Class C bears rather more prepayment risk in the situation where the par values in the classes are 100, 200, and 500.

The creators of mortgage-backed securities have created many more exotic structures than the one we have just described. Business Snapshot 29.1 gives an example.

Valuing Mortgage-Backed Securities

Mortgage-backed securities are usually valued using Monte Carlo simulation. Either the HJM or LIBOR market models can be used to simulate the behavior of interest rates month by month throughout the life of an MBS. Consider what happens on one simulation trial. Each month, expected prepayments are calculated from the current yield curve and the history of yield curve movements. These prepayments determine the expected cash flows to the holder of the MBS and the cash flows are discounted to time zero to obtain a sample value for the MBS. An estimate of the value of the MBS is the average of the sample values over many simulation trials.

Option-Adjusted Spread

In addition to calculating theoretical prices for mortgage-backed securities and other bonds with embedded options, traders also like to compute what is known as the *option-adjusted spread* (OAS). This is a measure of the spread over the yields on government Treasury bonds provided by the instrument when all options have been taken into account.

An input to any term structure model is the initial zero-coupon yield curve. Usually this is the LIBOR zero curve. However, to calculate an OAS for an instrument, we first price it using the zero-coupon government Treasury curve. The price of the instrument given by the model is compared to the price in the market. A series of iterations is then used to determine the parallel shift to the input Treasury curve that causes the model price to be equal to the market price. This parallel shift is the OAS.

To illustrate the nature of the calculations, suppose that the market price is \$102.00 and that the price calculated using the Treasury curve is \$103.27. As a first trial we might choose to try a 60-basis-point parallel shift to the Treasury zero curve. Suppose that this gives a price of \$101.20 for the instrument. This is less than the market price of \$102.00 and means that a parallel shift somewhere between 0 and 60 basis points will lead to the model price being equal to the market price. We could use linear interpolation to calculate

$$60 \times \frac{103.27 - 102.00}{103.27 - 101.20} = 36.81$$

or 36.81 basis points as the next trial shift. Suppose that this gives a price of \$101.95. This indicates that the OAS is slightly less than 36.81 basis points. Linear interpolation suggests that the next trial shift be

$$36.81 \times \frac{103.27 - 102.00}{103.27 - 101.95} = 35.41$$

or 35.41 basis points; and so on.

SUMMARY

The HJM and LMM models provide approaches to valuing interest rate derivatives that give the user complete freedom in choosing the volatility term structure. The LMM model has two key advantages over the HJM model. First, it is developed in terms of the forward rates that determine the pricing of caps, rather than in terms of instantaneous forward rates. Second, it is relatively easy to calibrate to the price of caps or European swap options. The HJM and LMM models both have the disadvantage that they cannot be represented as recombining trees. In practice, this means that they must be implemented using Monte Carlo simulation.

The mortgage-backed security market in the United States has given birth to many exotic interest rate derivatives: CMOs, IOs, POs, and so on. These instruments provide cash flows to the holder that depend on the prepayments on a pool of mortgages. These prepayments depend on, among other things, the level of interest rates. Because they are heavily path dependent, mortgage-backed securities usually have to be valued using

Monte Carlo simulation. These are, therefore, ideal candidates for applications of the HJM and LMM models.

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Questions and Problems (Answers in Solutions Manual)

29.1. Explain the difference between a Markov and a non-Markov model of the short rate.

29.2. Prove the relationship between the drift and volatility of the forward rate for the multifactor version of HJM in equation (29.6).

29.3. "When the forward rate volatility $s(t, T)$ in LMM is constant, the Ho–Lee model results." Verify that this is true by showing that LMM gives a process for bond prices that is consistent with the Ho–Lee model in Chapter 28.

29.4. "When the forward rate volatility, $s(t, T)$, in LMM is $\sigma e^{-a(T-t)}$, the Hull–White model results." Verify that this is true by showing that LMM gives a process for bond prices that is consistent with the Hull–White model in Chapter 28.

29.5. What is the advantage of LMM over BGM?

29.6. Provide an intuitive explanation of why a ratchet cap increases in value as the number of factors increase.

29.7. Show that equation (29.10) reduces to (29.4) as the δ_i tend to zero.

29.8. Explain why a sticky cap is more expensive than a similar ratchet cap.

29.9. Explain why IOs and POs have opposite sensitivities to the rate of prepayments.

29.10. "An option adjusted spread is analogous to the yield on a bond." Explain this statement.

29.11. Prove equation (29.15).

29.12. Prove the formula for the variance $V(T)$ of the swap rate in equation (29.17).

29.13. Prove equation (29.19).

Assignment Questions

29.14. In an annual-pay cap, the Black volatilities for caplets with maturities 1, 2, 3, and 5 years are 18%, 20%, 22%, and 20%, respectively. Estimate the volatility of a 1-year forward rate in the LIBOR Market Model when the time to maturity is (a) 0 to 1 year, (b) 1 to 2 years, (c) 2 to 3 years, and (d) 3 to 5 years. Assume that the zero curve is flat at 5% per annum (annually compounded). Use DerivaGem to estimate flat volatilities for 2-, 3-, 4-, 5-, and 6-year caps.

29.15. In the flexi cap considered in Section 29.2 the holder is obligated to exercise the first N in-the-money caplets. After that no further caplets can be exercised. (In the example, $N = 5$.) Two other ways that flexi caps are sometimes defined are:

(a) The holder can choose whether any caplet is exercised, but there is a limit of N on the total number of caplets that can be exercised.

(b) Once the holder chooses to exercise a caplet all subsequent in-the-money caplets must be exercised up to a maximum of N.

Discuss the problems in valuing these types of flexi caps. Of the three types of flexi caps, which would you expect to be most expensive? Which would you expect to be least expensive?