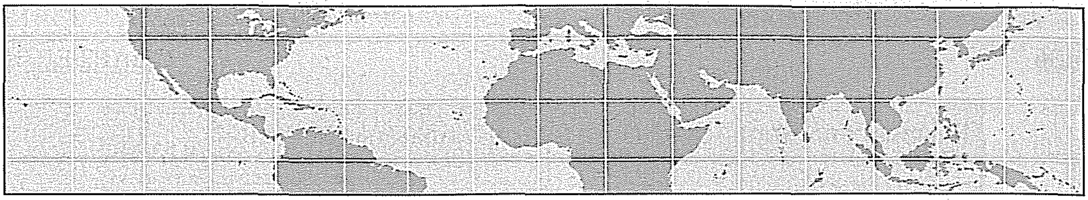


CHAPTER 18

Value at Risk

In Chapter 15 we examined measures such as delta, gamma, and vega for describing different aspects of the risk in a portfolio of derivatives. A financial institution usually calculates each of these measures each day for every market variable to which it is exposed. Often there are hundreds, or even thousands, of these market variables. A delta–gamma–vega analysis, therefore, leads to a huge number of different risk measures being produced each day. These risk measures provide valuable information for a trader who is responsible for managing the part of the financial institution's portfolio that is dependent on the particular market variable. However, they do not provide a way of measuring the total risk to which the financial institution is exposed.

Value at Risk (VaR) is an attempt to provide a single number summarizing the total risk in a portfolio of financial assets. It has become widely used by corporate treasurers and fund managers as well as by financial institutions. Central bank regulators also use VaR in determining the capital a bank is required to keep to reflect the market risks it is bearing.

In this chapter we explain the VaR measure and describe the two main approaches for calculating it. These are known as the *historical simulation* approach and the *model-building* approach.

18.1 THE VaR MEASURE

When using the value-at-risk measure, we are interested in making a statement of the following form:

> We are X percent certain that we will not lose more than V dollars in the next N days.

The variable V is the VaR of the portfolio. It is a function of two parameters: the time horizon (N days) and the confidence level (X%). It is the loss level over N days that we are X% certain will not be exceeded. Bank regulators require banks to calculate VaR with $N = 10$ and $X = 99$ (see the discussion in Business Snapshot 18.1).

In general, when N days is the time horizon and X% is the confidence level, VaR is the loss corresponding to the $(100 - X)$th percentile of the distribution of the change in the value of the portfolio over the next N days. For example, when $N = 5$ and $X = 97$, VaR is the third percentile of the distribution of changes in the value of the portfolio

Business Snapshot 18.1 How Bank Regulators Use VaR

The Basel Committee on Bank Supervision is a committee of the world's bank regulators that meets regularly in Basel, Switzerland. In 1988 it published what has become known as *The 1988 BIS Accord*, or simply *The Accord*. This is an agreement between the regulators on how the capital a bank is required to hold for credit risk should be calculated. Several years later the Basel Committee published *The 1996 Amendment*, which was implemented in 1998 and required banks to hold capital for market risk as well as credit risk. *The Amendment* distinguishes between a bank's trading book and its banking book. The banking book consists primarily of loans and is not usually revalued on a regular basis for managerial and accounting purposes. The trading book consists of the myriad of different instrument that are traded by the bank (stocks, bonds, swaps, forward contracts, options, etc.) and is normally revalued daily.

The 1996 BIS Amendment calculates capital for the trading book using the VaR measure with $N = 10$ and $X = 99$. This means that it focuses on the revaluation loss over a 10-day period that is expected to be exceeded only 1% of the time. The capital it requires the bank to hold is k times this VaR measure (with an adjustment for what are termed specific risks). The multiplier k is chosen on a bank-by-bank basis by the regulators and must be at least 3.0. For a bank with excellent well-tested VaR estimation procedures, it is likely that k will be set equal to the minimum value of 3.0. For other banks it may be higher.

over the next 5 days. VaR is illustrated for the situation where the change in the value of the portfolio is approximately normally distributed in Figure 18.1.

VaR is an attractive measure because it is easy to understand. In essence, it asks the simple question "How bad can things get?" This is the question all senior managers want answered. They are very comfortable with the idea of compressing all the Greek letters for all the market variables underlying a portfolio into a single number.

If we accept that it is useful to have a single number to describe the risk of a portfolio, an interesting question is whether VaR is the best alternative. Some researchers have

Figure 18.1 Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is X%.

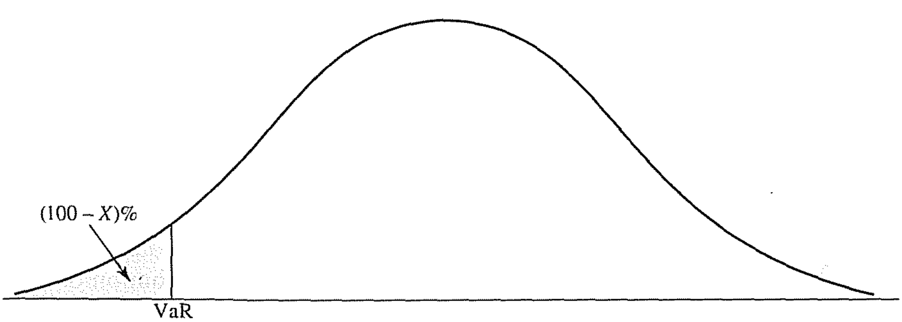
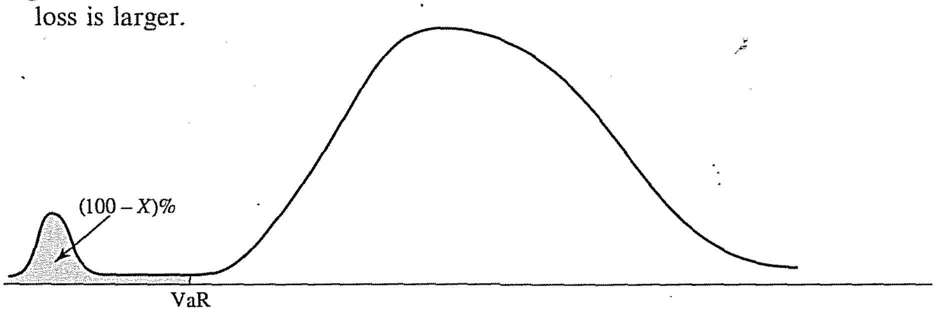


Figure 18.2 Alternative situation to Figure 18.1. VaR is the same, but the potential loss is larger.



argued that VaR may tempt traders to choose a portfolio with a return distribution similar to that in Figure 18.2. The portfolios in Figures 18.1 and 18.2 have the same VaR, but the portfolio in Figure 18.2 is much riskier because potential losses are much larger.

A measure that deals with the problem we have just mentioned is *Conditional VaR* (C-VaR).[1] Whereas VaR asks the question "How bad can things get?", C-VaR asks "If things do get bad, how much can we expect to lose?" C-VaR is the expected loss during an N-day period conditional that we are in the $(100 - X)\%$ left tail of the distribution. For example, with $X = 99$ and $N = 10$, C-VaR is the average amount we lose over a 10-day period assuming that a 1% worst-case event occurs.

In spite of its weaknesses, VaR (not C-VaR) is the most popular measure of risk among both regulators and risk managers. We will therefore devote most of the rest of this chapter to how it can be measured.

The Time Horizon

In theory, VaR has two parameters. These are N, the time horizon measured in days, and X, the confidence interval. In practice analysts almost invariably set $N = 1$ in the first instance. This is because there is not enough data to estimate directly the behavior of market variables over periods of time longer than 1 day. The usual assumption is

$$N\text{-day VaR} = 1\text{-day VaR} \times \sqrt{N}$$

This formula is exactly true when the changes in the value of the portfolio on successive days have independent identical normal distributions with mean zero. In other cases it is an approximation.

As explained in Business Snapshot 18.1, regulators require a bank's capital to be at least three times the 10-day 99% VaR. Given the way a 10-day VaR is calculated, this minimum capital level is, to all intents and purposes $3 \times \sqrt{10} = 9.49$ times the 1-day 99% VaR.

[1] This measure, which is also known as *expected shortfall* or *tail loss*, was suggested by P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, "Coherent Measures of Risk," *Mathematical Finance*, 9 (1999): 203–28. These authors define certain properties that a good risk measure should have and show that the standard VaR measure does not have all of them.

18.2 HISTORICAL SIMULATION

Historical simulation is one popular way of estimating VaR. It involves using past data in a very direct way as a guide to what might happen in the future. Suppose that we wish to calculate VaR for a portfolio using a 1-day time horizon, a 99% confidence level, and 500 days of data. The first step is to identify the market variables affecting the portfolio. These will typically be exchange rates, equity prices, interest rates, and so on. We then collect data on the movements in these market variables over the most recent 500 days. This provides us with 500 alternative scenarios for what can happen between today and tomorrow. Scenario 1 is where the percentage changes in the values of all variables are the same as they were on the first day for which we have collected data; scenario 2 is where they are the same as on the second day for which we have data; and so on. For each scenario we calculate the dollar change in the value of the portfolio between today and tomorrow. This defines a probability distribution for daily changes in the value of our portfolio. The fifth-worst daily change is the first percentile of the distribution. The estimate of VaR is the loss when we are at this first percentile point. Assuming that the last 500 days are a good guide to what could happen during the next day, we are 99% certain that we will not take a loss greater than our VaR estimate.

The historical simulation methodology is illustrated in Tables 18.1 and 18.2. Table 18.1 shows observations on market variables over the last 500 days. The observations are taken at some particular point in time during the day (usually the close of trading). We denote the first day for which we have data as Day 0; the second as Day 1; and so on. Today is Day 500; tomorrow is Day 501.

Table 18.2 shows the values of the market variables tomorrow if their percentage changes between today and tomorrow are the same as they were between Day $i-1$ and Day i for $1 \leqslant i \leqslant 500$. The first row in Table 18.2 shows the values of market variables tomorrow assuming their percentage changes between today and tomorrow are the same as they were between Day 0 and Day 1; the second row shows the values of market variables tomorrow assuming their percentage changes between Day 1 and Day 2 occur; and so on. The 500 rows in Table 18.2 are the 500 scenarios considered.

Table 18.1 Data for VaR historical simulation calculation.

Day	*Market variable 1*	*Market variable 2*	...	*Market variable n*
0	20.33	0.1132	...	65.37
1	20.78	0.1159	...	64.91
2	21.44	0.1162	...	65.02
3	20.97	0.1184	...	64.90
⋮	⋮	⋮	⋮	⋮
498	25.72	0.1312	...	62.22
499	25.75	0.1323	...	61.99
500	25.85	0.1343	...	62.10

Table 18.2 Scenarios generated for tomorrow (Day 501) using data in Table 18.1.

Scenario number	*Market variable 1*	*Market variable 2*	...	*Market variable n*	*Portfolio value ($ millions)*	*Change in value ($ millions)*
1	26.42	0.1375	...	61.66	23.71	0.21
2	26.67	0.1346	...	62.21	23.12	−0.38
3	25.28	0.1368	...	61.99	22.94	−0.56
⋮	⋮	⋮	⋮	⋮	⋮	⋮
499	25.88	0.1354	...	61.87	23.63	0.13
500	25.95	0.1363	...	62.21	22.87	−0.63

Define v_i as the value of a market variable on Day i and suppose that today is Day m. The ith scenario assumes that the value of the market variable tomorrow will be

$$v_m \frac{v_i}{v_{i-1}}$$

In our example, $m = 500$. For the first variable, the value today, v_{500}, is 25.85. Also $v_0 = 20.33$ and $v_1 = 20.78$. It follows that the value of the first market variable in the first scenario is

$$25.85 \times \frac{20.78}{20.33} = 26.42$$

The penultimate column of Table 18.2 shows the value of the portfolio tomorrow for each of the 500 scenarios. We suppose the value of the portfolio today is $23.50 million. This leads to the numbers in the final column for the change in the value between today and tomorrow for all the different scenarios. For Scenario 1 the change in value is +$210,000, for Scenario 2 it is −$380,000, and so on.

We are interested in the 1-percentile point of the distribution of changes in the portfolio value. Because there are a total of 500 scenarios in Table 18.2 we can estimate this as the fifth worst number in the final column of the table. Alternatively, we can use the techniques of what is known as *extreme value theory* to smooth the numbers in the left tail of the distribution in an attempt to obtain a more accurate estimate of the 1% point of the distribution.[2] As mentioned in the previous section, the N-day VaR for a 99% confidence level is calculated as $\sqrt{N}$ times the 1-day VaR.

Each day the VaR estimate in our example would be updated using the most recent 500 days of data. Consider, for example, what happens on Day 501. We find out new values for all the market variables and are able to calculate a new value for our portfolio.[3] We then go through the procedure we have outlined to calculate a new VaR. We use data on the market variables from Day 1 to Day 501. (This gives us the required 500 observations on the percentage changes in market variables; the Day-0

[2] See P. Embrechts, C. Kluppelberg, and T. Mikosch. *Modeling Extremal Events for Insurance and Finance*. New York: Springer, 1997; A. J. McNeil, "Extreme Value Theory for Risk Managers," in *Internal Modeling and CAD II*. London, Risk Books, 1999, and available from www.math.ethz.ch/~mcneil.

[3] Note that the portfolio's composition may have changed between Day 500 and Day 501.

values of the market variables are no longer used.) Similarly, on Day 502, we use data from Day 2 to Day 502 to determine VaR, and so on.

18.3 MODEL-BUILDING APPROACH

The main alternative to historical simulation is the model-building approach. Before getting into the details of the approach, it is appropriate to mention one issue concerned with the units for measuring volatility.

Daily Volatilities

In option pricing we usually measure time in years, and the volatility of an asset is usually quoted as a "volatility per year". When using the model-building approach to calculate VaR, we usually measure time in days and the volatility of an asset is usually quoted as a "volatility per day".

What is the relationship between the volatility per year used in option pricing and the volatility per day used in VaR calculations? Let us define σ_{year} as the volatility per year of a certain asset and σ_{day} as the equivalent volatility per day of the asset. Assuming 252 trading days in a year, we can use equation (13.2) to write the standard deviation of the continuously compounded return on the asset in 1 year as either σ_{year} or $\sigma_{\text{day}}\sqrt{252}$. It follows that

$$\sigma_{\text{year}} = \sigma_{\text{day}}\sqrt{252}$$

or

$$\sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}}$$

so that daily volatility is about 6% of annual volatility.

As pointed out in Section 13.4, σ_{day} is approximately equal to the standard deviation of the percentage change in the asset price in one day. For the purposes of calculating VaR we assume exact equality. We define the daily volatility of an asset price (or any other variable) as equal to the standard deviation of the percentage change in one day.

Our discussion in the next few sections assumes that we have estimates of daily volatilities and correlations. In Chapter 19, we discuss how the estimates can be produced.

Single-Asset Case

We now consider how VaR is calculated using the model-building approach in a very simple situation where the portfolio consists of a position in a single stock. The portfolio we consider is one consisting of \$10 million in shares of Microsoft. We suppose that $N = 10$ and $X = 99$, so that we are interested in the loss level over 10 days that we are 99% confident will not be exceeded. Initially, we consider a 1-day time horizon.

We assume that the volatility of Microsoft is 2% per day (corresponding to about 32% per year). Because the size of the position is \$10 million, the standard deviation of daily changes in the value of the position is 2% of \$10 million, or \$200,000.

It is customary in the model-building approach to assume that the expected change in a market variable over the time period considered is zero. This is not strictly true, but it is a reasonable assumption. The expected change in the price of a market variable over a short time period is generally small when compared with the standard deviation of the change. Suppose, for example, that Microsoft has an expected return of 20% per annum. Over a 1-day period, the expected return is 0.20/252, or about 0.08%, whereas the standard deviation of the return is 2%. Over a 10-day period, the expected return is 0.08×10, or about 0.8%, whereas the standard deviation of the return is $2\sqrt{10}$, or about 6.3%.

So far, we have established that the change in the value of the portfolio of Microsoft shares over a 1-day period has a standard deviation of \$200,000 and (at least approximately) a mean of zero. We assume that the change is normally distributed.[4] From the tables at the end of this book, $N(-2.33) = 0.01$. This means that there is a 1% probability that a normally distributed variable will decrease in value by more than 2.33 standard deviations. Equivalently, it means that we are 99% certain that a normally distributed variable will not decrease in value by more than 2.33 standard deviations. The 1-day 99% VaR for our portfolio consisting of a \$10 million position in Microsoft is therefore

$$2.33 \times 200{,}000 = \$466{,}000$$

As discussed earlier, the N-day VaR is calculated as $\sqrt{N}$ times the 1-day VaR. The 10-day 99% VaR for Microsoft is therefore

$$466{,}000 \times \sqrt{10} = \$1{,}473{,}621$$

Consider next a portfolio consisting of a \$5 million position in AT&T, and suppose the daily volatility of AT&T is 1% (approximately 16% per year). A similar calculation to that for Microsoft shows that the standard deviation of the change in the value of the portfolio in 1 day is

$$5{,}000{,}000 \times 0.01 = 50{,}000$$

Assuming the change is normally distributed, the 1-day 99% VaR is

$$50{,}000 \times 2.33 = \$116{,}500$$

and the 10-day 99% VaR is

$$116{,}500 \times \sqrt{10} = \$368{,}405$$

Two-Asset Case

Now consider a portfolio consisting of both \$10 million of Microsoft shares and \$5 million of AT&T shares. We suppose that the returns on the two shares have a bivariate normal distribution with a correlation of 0.3. A standard result in statistics tells us that, if two variables X and Y have standard deviations equal to σ_X and σ_Y with the coefficient of correlation between them equal to ρ, the standard deviation of $X + Y$ is given by

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

[4] To be consistent with the option pricing assumption in Chapter 13, we could assume that the price of Microsoft is lognormal tomorrow. Because 1 day is such a short period of time, this is almost indistinguishable from the assumption we do make—that the change in the stock price between today and tomorrow is normal.

To apply this result, we set X equal to the change in the value of the position in Microsoft over a 1-day period and Y equal to the change in the value of the position in AT&T over a 1-day period, so that

$$\sigma_X = 200{,}000 \quad \text{and} \quad \sigma_Y = 50{,}000$$

The standard deviation of the change in the value of the portfolio consisting of both stocks over a 1-day period is therefore

$$\sqrt{200{,}000^2 + 50{,}000^2 + 2 \times 0.3 \times 200{,}000 \times 50{,}000} = 220{,}227$$

The mean change is assumed to be zero. The change is normally distributed. So the 1-day 99% VaR is therefore

$$220{,}227 \times 2.33 = \$513{,}129$$

The 10-day 99% VaR is $\sqrt{10}$ times this, or $1,622,657.

The Benefits of Diversification

In the example we have just considered:

1. The 10-day 99% VaR for the portfolio of Microsoft shares is $1,473,621.
2. The 10-day 99% VaR for the portfolio of AT&T shares is $368,405.
3. The 10-day 99% VaR for the portfolio of both Microsoft and AT&T shares is $1,622,657.

The amount

$$(1{,}473{,}621 + 368{,}405) - 1{,}622{,}657 = \$219{,}369$$

represents the benefits of diversification. If Microsoft and AT&T were perfectly correlated, the VaR for the portfolio of both Microsoft and AT&T would equal the VaR for the Microsoft portfolio plus the VaR for the AT&T portfolio. Less than perfect correlation leads to some of the risk being "diversified away".[5]

18.4 LINEAR MODEL

The examples we have just considered are simple illustrations of the use of the linear model for calculating VaR. Suppose that we have a portfolio worth P consisting of n assets with an amount α_i being invested in asset i ($1 \leqslant i \leqslant n$). We define Δx_i as the return on asset i in 1 day. It follows that the dollar change in the value of our investment in asset i in 1 day is $\alpha_i \, \Delta x_i$ and

$$\Delta P = \sum_{i=1}^{n} \alpha_i \, \Delta x_i \tag{18.1}$$

where ΔP is the dollar change in the value of the whole portfolio in 1 day.

[5] Harry Markowitz was one of the first researchers to study the benefits of diversification to a portfolio manager. He was awarded a Nobel prize for this research in 1990. See H. Markowitz, "Portfolio Selection," *Journal of Finance*, 7, 1 (March 1952): 77–91.

In the example considered in the previous section, \$10 million was invested in the first asset (Microsoft) and \$5 million was invested in the second asset (AT&T), so that (in millions of dollars) $\alpha_1 = 10$, $\alpha_2 = 5$, and

$$\Delta P = 10\Delta x_1 + 5\Delta x_2$$

If we assume that the Δx_i in equation (18.1) are multivariate normal, ΔP is normally distributed. To calculate VaR, we therefore need to calculate only the mean and standard deviation of ΔP. We assume, as discussed in the previous section, that the expected value of each Δx_i is zero. This implies that the mean of ΔP is zero.

To calculate the standard deviation of ΔP, we define σ_i as the daily volatility of the ith asset and ρ_{ij} as the coefficient of correlation between returns on asset i and asset j. This means that σ_i is the standard deviation of Δx_i, and ρ_{ij} is the coefficient of correlation between Δx_i and Δx_j. The variance of ΔP, which we will denote by σ_P^2, is given by

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j$$

This equation can also be written

$$\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2\sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j<i} \rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j \tag{18.2}$$

The standard deviation of the change over N days is $\sigma_P\sqrt{N}$, and the 99% VaR for an N-day time horizon is $2.33\sigma_P\sqrt{N}$.

In the example considered in the previous section, $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, and $\rho_{12} = 0.3$. As already noted, $\alpha_1 = 10$ and $\alpha_2 = 5$, so that

$$\sigma_P^2 = 10^2 \times 0.02^2 + 5^2 \times 0.01^2 + 2 \times 10 \times 5 \times 0.3 \times 0.02 \times 0.01 = 0.0485$$

and $\sigma_P = 0.220$. This is the standard deviation of the change in the portfolio value per day (in millions of dollars). The 10-day 99% VaR is $2.33 \times 0.220 \times \sqrt{10} =$ \$1.623 million. This agrees with the calculation in the previous section.

Handling Interest Rates

It is out of the question to define a separate market variable for every single bond price or interest rate to which a company is exposed. Some simplifications are necessary when the model-building approach is used. One possibility is to assume that only parallel shifts in the yield curve occur. It is then necessary to define only one market variable: the size of the parallel shift. The changes in the value of bond portfolio can then be calculated using the duration relationship

$$\Delta P = -DP\Delta y$$

where P is the value of the portfolio, ΔP is the its change in P in one day, D is the modified duration of the portfolio, and Δy is the parallel shift in 1 day.

This approach does not usually give enough accuracy. The procedure usually followed is to choose as market variables the prices of zero-coupon bonds with standard maturities: 1 month, 3 months, 6 months, 1 year, 2 years, 5 years, 7 years, 10 years, and 30 years. For the purposes of calculating VaR, the cash flows from instruments in the portfolio are mapped into cash flows occurring on the standard maturity dates. Consider a \$1 million position in a Treasury bond lasting 1.2 years that pays a coupon of 6% semiannually. Coupons are paid in 0.2, 0.7, and 1.2 years, and the principal is paid in 1.2 years. This bond is, therefore, in the first instance regarded as a \$30,000 position in 0.2-year zero-coupon bond plus a \$30,000 position in a 0.7-year zero-coupon bond plus a \$1.03 million position in a 1.2-year zero-coupon bond. The position in the 0.2-year bond is then replaced by an equivalent position in 1-month and 3-month zero-coupon bonds; the position in the 0.7-year bond is replaced by an equivalent position in 6-month and 1-year zero-coupon bonds; and the position in the 1.2-year bond is replaced by an equivalent position in 1-year and 2-year zero-coupon bonds. The result is that the position in the 1.2-year coupon-bearing bond is for VaR purposes regarded as a position in zero-coupon bonds having maturities of 1 month, 3 months, 6 months, 1 year, and 2 years.

This procedure is known as *cash-flow mapping*. One way of doing it is explained in the appendix at the end of this chapter.

Applications of the Linear Model

The simplest application of the linear model is to a portfolio with no derivatives consisting of positions in stocks, bonds, foreign exchange, and commodities. In this case, the change in the value of the portfolio is linearly dependent on the percentage changes in the prices of the assets comprising the portfolio. Note that, for the purposes of VaR calculations, all asset prices are measured in the domestic currency. The market variables considered by a large bank in the United States are therefore likely to include the value of the Nikkei 225 index measured in dollars, the price of a 10-year sterling zero-coupon bond measured in dollars, and so on.

An example of a derivative that can be handled by the linear model is a forward contract to buy a foreign currency. Suppose the contract matures at time T. It can be regarded as the exchange of a foreign zero-coupon bond maturing at time T for a domestic zero-coupon bond maturing at time T. For the purposes of calculating VaR, the forward contract is therefore treated as a long position in the foreign bond combined with a short position in the domestic bond. Each bond can be handled using a cash-flow mapping procedure.

Consider next an interest rate swap. As explained in Chapter 7, this can be regarded as the exchange of a floating-rate bond for a fixed-rate bond. The fixed-rate bond is a regular coupon-bearing bond. The floating-rate bond is worth par just after the next payment date. It can be regarded as a zero-coupon bond with a maturity date equal to the next payment date. The interest rate swap therefore reduces to a portfolio of long and short positions in bonds and can be handled using a cash-flow mapping procedure.

The Linear Model and Options

We now consider how the linear model can be used when there are options. Consider first a portfolio consisting of options on a single stock whose current price is S. Suppose that the delta of the position (calculated in the way described in Chapter 15) is δ.[6] Since δ

[6] Normally we denote the delta and gamma of a portfolio by Δ and Γ. In this section and the next, we use the lower case Greek letters δ and γ to avoid overworking Δ.

is the rate of change of the value of the portfolio with S, it is approximately true that

$$\delta = \frac{\Delta P}{\Delta S}$$

or

$$\Delta P = \delta\,\Delta S \tag{18.3}$$

where ΔS is the dollar change in the stock price in 1 day and ΔP is, as usual, the dollar change in the portfolio in 1 day. We define Δx as the percentage change in the stock price in 1 day, so that

$$\Delta x = \frac{\Delta S}{S}$$

It follows that an approximate relationship between ΔP and Δx is

$$\Delta P = S\delta\,\Delta x$$

When we have a position in several underlying market variables that includes options, we can derive an approximate linear relationship between ΔP and the Δx_i similarly. This relationship is

$$\Delta P = \sum_{i=1}^{n} S_i\delta_i\,\Delta x_i \tag{18.4}$$

where S_i is the value of the ith market variable and δ_i is the delta of the portfolio with respect to the ith market variable. This corresponds to equation (18.1):

$$\Delta P = \sum_{i=1}^{n} \alpha_i\,\Delta x_i \tag{18.5}$$

with $\alpha_i = S_i\delta_i$. Equation (18.2) can therefore be used to calculate the standard deviation of ΔP.

Example 18.1

A portfolio consists of options on Microsoft and AT&T. The options on Microsoft have a delta of 1,000, and the options on AT&T have a delta of 20,000. The Microsoft share price is \$120, and the AT&T share price is \$30. From equation (18.4), it is approximately true that

$$\Delta P = 120 \times 1{,}000 \times \Delta x_1 + 30 \times 20{,}000 \times \Delta x_2$$

or

$$\Delta P = 120{,}000\Delta x_1 + 600{,}000\Delta x_2$$

where Δx_1 and Δx_2 are the returns from Microsoft and AT&T in 1 day and ΔP is the resultant change in the value of the portfolio. (The portfolio is assumed to be equivalent to an investment of \$120,000 in Microsoft and \$600,000 in AT&T.) Assuming that the daily volatility of Microsoft is 2% and the daily volatility of AT&T is 1% and the correlation between the daily changes is 0.3, the standard deviation of ΔP (in thousands of dollars) is

$$\sqrt{(120 \times 0.02)^2 + (600 \times 0.01)^2 + 2 \times 120 \times 0.02 \times 600 \times 0.01 \times 0.3} = 7.099$$

Since $N(-1.65) = 0.05$, the 5-day 95% VaR is $1.65 \times \sqrt{5} \times 7{,}099 = \$26{,}193$.

18.5 QUADRATIC MODEL

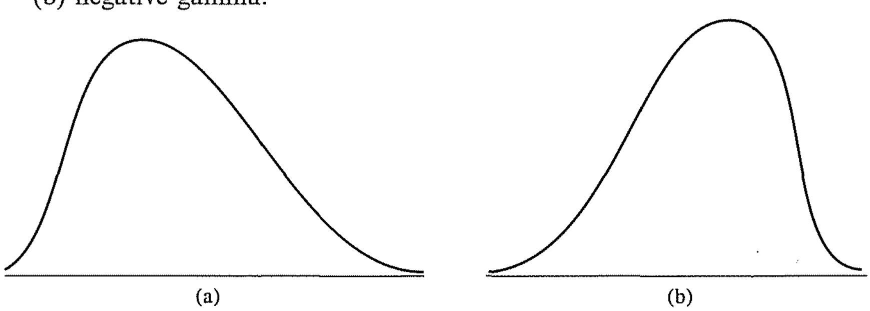
When a portfolio includes options, the linear model is an approximation. It does not take account of the gamma of the portfolio. As discussed in Chapter 15, delta is defined as the rate of change of the portfolio value with respect to an underlying market variable and gamma is defined as the rate of change of the delta with respect to the market variable. Gamma measures the curvature of the relationship between the portfolio value and an underlying market variable.

Figure 18.3 shows the impact of a nonzero gamma on the probability distribution of the value of the portfolio. When gamma is positive, the probability distribution tends to be positively skewed; when gamma is negative, it tends to be negatively skewed. Figures 18.4 and 18.5 illustrate the reason for this result. Figure 18.4 shows the relationship between the value of a long call option and the price of the underlying asset. A long call is an example of an option position with positive gamma. The figure shows that, when the probability distribution for the price of the underlying asset at the end of 1 day is normal, the probability distribution for the option price is positively skewed.[7] Figure 18.5 shows the relationship between the value of a short call position and the price of the underlying asset. A short call position has a negative gamma. In this case, we see that a normal distribution for the price of the underlying asset at the end of 1 day gets mapped into a negatively skewed distribution for the value of the option position.

The VaR for a portfolio is critically dependent on the left tail of the probability distribution of the portfolio value. For example, when the confidence level used is 99%, the VaR is the value in the left tail below which there is only 1% of the distribution. As indicated in Figures 18.3(a) and 18.4, a positive gamma portfolio tends to have a less heavy left tail than the normal distribution. If we assume the distribution is normal, we will tend to calculate a VaR that is too high. Similarly, as indicated in Figures 18.3(b) and 18.5, a negative gamma portfolio tends to have a heavier left tail than the normal distribution. If we assume the distribution is normal, we will tend to calculate a VaR that is too low.

For a more accurate estimate of VaR than that given by the linear model, we can use both delta and gamma measures to relate ΔP to the Δx_i. Consider a portfolio

Figure 18.3 Probability distribution for value of portfolio: (a) positive gamma; (b) negative gamma.



[7] As mentioned in footnote 4, we can use the normal distribution as an approximation to the lognormal distribution in VaR calculations.

Figure 18.4 Translation of normal probability distribution for asset into probability distribution for value of a long call on asset.

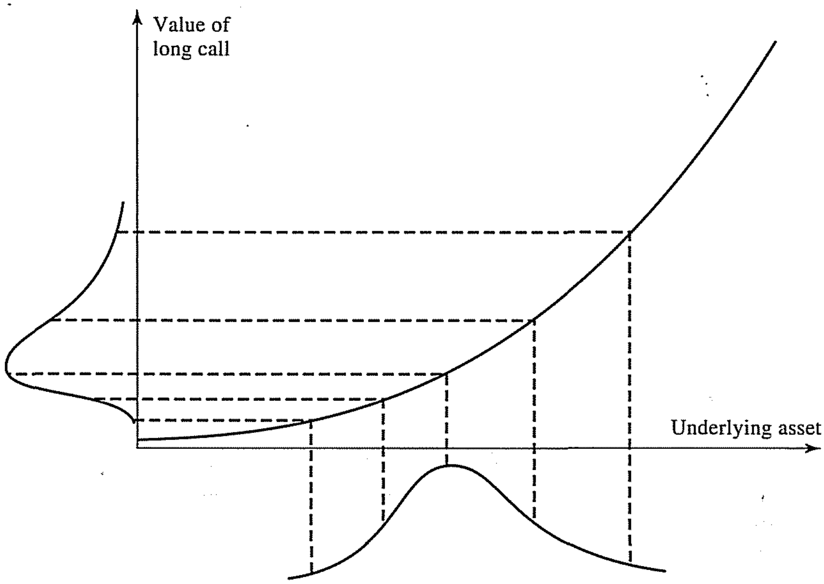
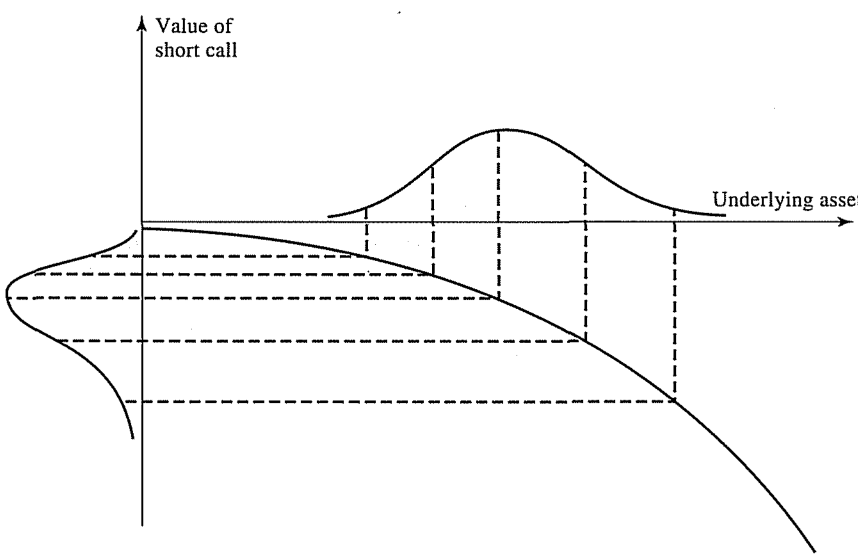


Figure 18.5 Translation of normal probability distribution for asset into probability distribution for value of a short call on asset.



dependent on a single asset whose price is S. Suppose δ and γ are the delta and gamma of the portfolio. From the appendix to Chapter 15, the equation

$$\Delta P = \delta\,\Delta S + \tfrac{1}{2}\gamma(\Delta S)^2$$

is an improvement over the approximation in equation (18.3).[8] Setting

$$\Delta x = \frac{\Delta S}{S}$$

reduces this to

$$\Delta P = S\delta\,\Delta x + \tfrac{1}{2}S^2\gamma(\Delta x)^2 \tag{18.6}$$

More generally for a portfolio with n underlying market variables, with each instrument in the portfolio being dependent on only one of the market variables, equation (18.6) becomes

$$\Delta P = \sum_{i=1}^{n} S_i\delta_i\,\Delta x_i + \sum_{i=1}^{n} \tfrac{1}{2}S_i^2\gamma_i\,(\Delta x_i)^2$$

where S_i is the value of the ith market variable, and δ_i and γ_i are the delta and gamma of the portfolio with respect to the ith market variable. When individual instruments in the portfolio may be dependent on more than one market variable, this equation takes the more general form

$$\Delta P = \sum_{i=1}^{n} S_i\delta_i\,\Delta x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij}\,\Delta x_i\,\Delta x_j \tag{18.7}$$

where γ_{ij} is a "cross gamma" defined as

$$\gamma_{ij} = \frac{\partial^2 P}{\partial S_i\,\partial S_j}$$

Equation (18.7) is not as easy to work with as equation (18.5), but it can be used to calculate moments for ΔP. A result in statistics known as the Cornish–Fisher expansion can be used to estimate percentiles of the probability distribution from the moments.[9]

18.6 MONTE CARLO SIMULATION

As an alternative to the approaches described so far, we can implement the model-building approach using Monte Carlo simulation to generate the probability distribution

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[8] The Taylor series expansion in the appendix to Chapter 15 suggests the approximation

$$\Delta P = \Theta\,\Delta t + \delta\,\Delta S + \tfrac{1}{2}\gamma(\Delta S)^2$$

when terms of higher order than Δt are ignored. In practice, the $\Theta\,\Delta t$ term is so small that it is usually ignored.

[9] See Technical Note 10 on the author's website for details of the calculation of moments and the use of Cornish–Fisher expansions. When there is a single underlying variable, $E(\Delta P) = 0.5S^2\Gamma\sigma^2$, $E(\Delta P^2) = S^2\Delta^2\sigma^2 + 0.75S^4\Gamma^2\sigma^4$, and $E(\Delta P^3) = 4.5S^4\Delta^2\Gamma\sigma^4 + 1.875S^6\Gamma^3\sigma^6$, where S is the value of the variable and σ is its daily volatility. Sample Application E in the DerivaGem Application Builder implements the Cornish–Fisher expansion method for this case.

for ΔP. Suppose we wish to calculate a 1-day VaR for a portfolio. The procedure is as follows:

1. Value the portfolio today in the usual way using the current values of market variables.
2. Sample once from the multivariate normal probability distribution of the Δx_i.[10]
3. Use the values of the Δx_i that are sampled to determine the value of each market variable at the end of one day.
4. Revalue the portfolio at the end of the day in the usual way.
5. Subtract the value calculated in Step 1 from the value in Step 4 to determine a sample ΔP.
6. Repeat Steps 2 to 5 many times to build up a probability distribution for ΔP.

The VaR is calculated as the appropriate percentile of the probability distribution of ΔP. Suppose, for example, that we calculate 5,000 different sample values of ΔP in the way just described. The 1-day 99% VaR is the value of ΔP for the 50th worst outcome; the 1-day VaR 95% is the value of ΔP for the 250th worst outcome; and so on.[11] The N-day VaR is usually assumed to be the 1-day VaR multiplied by $\sqrt{N}$.[12]

The drawback of Monte Carlo simulation is that it tends to be slow because a company's complete portfolio (which might consist of hundreds of thousands of different instruments) has to be revalued many times.[13] One way of speeding things up is to assume that equation (18.7) describes the relationship between ΔP and the Δx_i. We can then jump straight from Step 2 to Step 5 in the Monte Carlo simulation and avoid the need for a complete revaluation of the portfolio. This is sometimes referred to as the *partial simulation approach*.

18.7 COMPARISON OF APPROACHES

We have discussed two methods for estimating VaR: the historical simulation approach and the model-building approach. The advantages of the model-building approach are that results can be produced very quickly and it can be used in conjunction with volatility updating schemes such as those we will describe in the next chapter. The main disadvantage of the model-building approach is that it assumes that the market variables have a multivariate normal distribution. In practice, daily changes in market variables often have distributions that are quite different from normal (see, e.g., Table 16.1).

The historical simulation approach has the advantage that historical data determine the joint probability distribution of the market variables. It also avoids the need for cash-flow mapping (see Problem 18.2). The main disadvantages of historical simulation

[10] One way of doing so is given in Chapter 17.

[11] As in the case of historical simulation, extreme value theory can be used to "smooth the tails" so that better estimates of extreme percentiles are obtained.

[12] This is only approximately true when the portfolio includes options, but it is the assumption that is made in practice for most VaR calculation methods.

[13] An approach for limiting the number of portfolio revaluations is proposed in F. Jamshidian and Y. Zhu "Scenario simulation model: theory and methodology," *Finance and Stochastics*, 1 (1997), 43–67.

are that it is computationally slow and does not easily allow volatility updating schemes to be used.[14]

One disadvantage of the model-building approach is that it tends to give poor results for low-delta portfolios (see Problem 18.21).

18.8 STRESS TESTING AND BACK TESTING

In addition to calculating VaR, many companies carry out what is known as a *stress test* of their portfolio. Stress testing involves estimating how the portfolio would have performed under some of the most extreme market moves seen in the last 10 to 20 years.

For example, to test the impact of an extreme movement in US equity prices, a company might set the percentage changes in all market variables equal to those on October 19, 1987 (when the S&P 500 moved by 22.3 standard deviations). If this is considered to be too extreme, the company might choose January 8, 1988 (when the S&P 500 moved by 6.8 standard deviations). To test the effect of extreme movements in UK interest rates, the company might set the percentage changes in all market variables equal to those on April 10, 1992 (when 10-year bond yields moved by 7.7 standard deviations).

The scenarios used in stress testing are also sometimes generated by senior management. One technique sometimes used is to ask senior management to meet periodically and "brainstorm" to develop extreme scenarios that might occur given the current economic environment and global uncertainties.

Stress testing can be considered as a way of taking into account extreme events that do occur from time to time but that are virtually impossible according to the probability distributions assumed for market variables. A 5-standard-deviation daily move in a market variable is one such extreme event. Under the assumption of a normal distribution, it happens about once every 7,000 years, but, in practice, it is not uncommon to see a 5-standard-deviation daily move once or twice every 10 years.

Whatever the method used for calculating VaR, an important reality check is *back testing*. It involves testing how well the VaR estimates would have performed in the past. Suppose that we are calculating a 1-day 99% VaR. Back testing would involve looking at how often the loss in a day exceeded the 1-day 99% VaR that would have been calculated for that day. If this happened on about 1% of the days, we can feel reasonably comfortable with the methodology for calculating VaR. If it happened on, say, 7% of days, the methodology is suspect.

18.9 PRINCIPAL COMPONENTS ANALYSIS

One approach to handling the risk arising from groups of highly correlated market variables is principal components analysis. This takes historical data on movements in the market variables and attempts to define a set of components or factors that explain the movements.

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[14] For a way of adapting the historical simulation approach to incorporate volatility updating, see J. Hull and A. White. "Incorporating volatility updating into the historical simulation method for value-at-risk," *Journal of Risk* 1, No. 1 (1998): 5–19.

Table 18.3 Factor loadings for US Treasury data.

	PC1	*PC2*	*PC3*	*PC4*	*PC5*	*PC6*	*PC7*	*PC8*	*PC9*	*PC10*
3m	0.21	−0.57	0.50	0.47	−0.39	−0.02	0.01	0.00	0.01	0.00
6m	0.26	−0.49	0.23	−0.37	0.70	0.01	−0.04	−0.02	−0.01	0.00
12m	0.32	−0.32	−0.37	−0.58	−0.52	−0.23	−0.04	−0.05	0.00	0.01
2y	0.35	−0.10	−0.38	0.17	0.04	0.59	0.56	0.12	−0.12	−0.05
3y	0.36	0.02	−0.30	0.27	0.07	0.24	−0.79	0.00	−0.09	−0.00
4y	0.36	0.14	−0.12	0.25	0.16	−0.63	0.15	0.55	−0.14	−0.08
5y	0.36	0.17	−0.04	0.14	0.08	−0.10	0.09	−0.26	0.71	0.48
7y	0.34	0.27	0.15	0.01	0.00	−0.12	0.13	−0.54	0.00	−0.68
10y	0.31	0.30	0.28	−0.10	−0.06	0.01	0.03	−0.23	−0.63	0.52
30y	0.25	0.33	0.46	−0.34	−0.18	0.33	−0.09	0.52	0.26	−0.13

The approach is best illustrated with an example. The market variables we will consider are 10 US Treasury rates with maturities between 3 months and 30 years. Tables 18.3 and 18.4 shows results produced by Frye for these market variables using 1,543 daily observations between 1989 and 1995.[15] The first column in Table 18.3 shows the maturities of the rates that were considered. The remaining 10 columns in the table show the 10 factors (or principal components) describing the rate moves. The first factor, shown in the column labeled PC1, corresponds to a roughly parallel shift in the yield curve. When we have one unit of that factor, the 3-month rate increases by 0.21 basis points, the 6-month rate increases by 0.26 basis points, and so on. The second factor is shown in the column labeled PC2. It corresponds to a "twist" or "steepening" of the yield curve. Rates between 3 months and 2 years move in one direction; rates between 3 years and 30 years move in the other direction. The third factor corresponds to a "bowing" of the yield curve. Rates at the short end and long end of the yield curve move in one direction; rates in the middle move in the other direction. The interest rate move for a particular factor is known as *factor loading*. In our example, the first factor's loading for the three-month rate is 0.21.[16]

Because there are 10 rates and 10 factors, the interest rate changes observed on any given day can always be expressed as a linear sum of the factors by solving a set of 10 simultaneous equations. The quantity of a particular factor in the interest rate changes on a particular day is known as the *factor score* for that day.

The importance of a factor is measured by the standard deviation of its factor score. The standard deviations of the factor scores in our example are shown in Table 18.4 and

Table 18.4 Standard deviation of factor scores (basis points).

PC1	*PC2*	*PC3*	*PC4*	*PC5*	*PC6*	*PC7*	*PC8*	*PC9*	*PC10*
17.49	6.05	3.10	2.17	1.97	1.69	1.27	1.24	0.80	0.79

[15] See J. Frye, "Principals of Risk: Finding VAR through Factor-Based Interest Rate Scenarios," in *VAR: Understanding and Applying Value at Risk*, pp. 275–88. London: Risk Publications, 1997.

[16] The factor loadings have the property that the sum of their squares for each factor is 1.0.

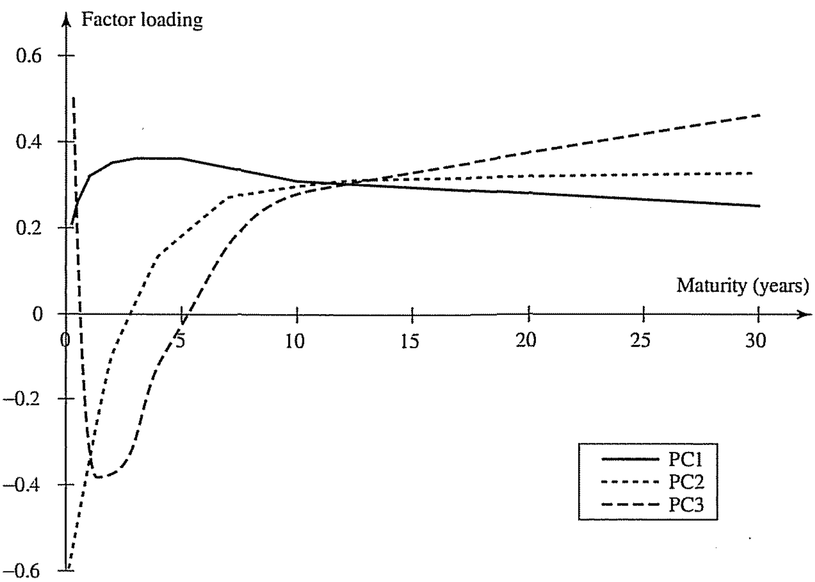
the factors are listed in order of their importance. The numbers in Table 18.4 are measured in basis points. A quantity of the first factor equal to one standard deviation, therefore, corresponds to the 3-month rate moving by $0.21 \times 17.49 = 3.67$ basis points, the 6-month rate moving by $0.26 \times 17.49 = 4.55$ basis points, and so on.

The technical details of how the factors are determined are not covered here. It is sufficient for us to note that the factors are chosen so that the factor scores are uncorrelated. For instance, in our example, the first factor score (amount of parallel shift) is uncorrelated with the second factor score (amount of twist) across the 1,543 days. The variances of the factor scores (i.e., the squares of the standard deviations) have the property that they add up to the total variance of the data. From Table 18.4, the total variance of the original data (i.e., sum of the variance of the observations on the 3-month rate, the variance of the observations on the 6-month rate, and so on) is

$$17.49^2 + 6.05^2 + 3.10^2 + \cdots + 0.79^2 = 367.9$$

From this it can be seen that the first factor accounts for $17.49^2/367.9 = 83.1\%$ of the variance in the original data; the first two factors account for $(17.49^2 + 6.05^2)/367.9 = 93.1\%$ of the variance in the data; the third factor accounts for a further 2.8% of the variance. This shows most of the risk in interest rate moves is accounted for by the first two or three factors. It suggests that we can relate the risks in a portfolio of interest rate dependent instruments to movements in these factors instead of considering all 10 interest rates. The three most important factors from Table 18.3 are plotted in Figure 18.6.[17]

Figure 18.6 The three most important factors driving yield curve movements.



[17] Similar results to those described here, in respect of the nature of the factors and the amount of the total risk they account for, are obtained when a principal components analysis is used to explain the movements in almost any yield curve in any country.

Using Principal Components Analysis to Calculate VaR

To illustrate how a principal components analysis can be used to calculate VaR, suppose we have a portfolio with the exposures to interest rate moves shown in Table 18.5. A 1-basis-point change in the 1-year rate causes the portfolio value to increase by \$10 million, a 1-basis-point change in the 2-year rate causes it to increase by \$4 million, and so on. We use the first two factors to model rate moves. (As mentioned above, this captures over 90% of the uncertainty in rate moves.) Using the data in Table 18.3, our exposure to the first factor (measured in millions of dollars per factor score basis point) is

$$10 \times 0.32 + 4 \times 0.35 - 8 \times 0.36 - 7 \times 0.36 + 2 \times 0.36 = -0.08$$

and our exposure to the second factor is

$$10 \times (-0.32) + 4 \times (-0.10) - 8 \times 0.02 - 7 \times 0.14 + 2 \times 0.17 = -4.40$$

Suppose that f_1 and f_2 are the factor scores (measured in basis points). The change in the portfolio value is, to a good approximation, given by

$$\Delta P = -0.08 f_1 - 4.40 f_2$$

The factor scores are uncorrelated and have the standard deviations given in Table 18.4. The standard deviation of ΔP is therefore

$$\sqrt{0.08^2 \times 17.49^2 + 4.40^2 \times 6.05^2} = 26.66$$

Hence, the 1-day 99% VaR is $26.66 \times 2.33 = 62.12$. Note that the data in Table 18.5 are such that we have very little exposure to the first factor and significant exposure to the second factor. Using only one factor would significantly understate VaR (see Problem 18.13). The duration-based method for handling interest rates, mentioned in Section 18.4, would also significantly understate VaR as it considers only parallel shifts in the yield curve.

A principal components analysis can in theory be used for market variables other than interest rates. Suppose that a financial institution has exposures to a number of different stock indices. A principal components analysis can be used to identify factors describing movements in the indices and the most important of these can be used to replace the market indices in a VaR analysis. How effective a principal components analysis is for a group of market variables depends on how closely correlated they are.

As explained earlier in the chapter, VaR is usually calculated by relating the actual changes in a portfolio to percentage changes in market variables (the Δx_i). For a VaR calculation, it may therefore be most appropriate to carry out a principal components analysis on percentage changes in market variables rather than actual changes.

Table 18.5 Change in portfolio value for a 1-basis-point rate move (\$ millions).

1-year rate	*2-year rate*	*3-year rate*	*4-year rate*	*5-year rate*
+10	+4	−8	−7	+2

SUMMARY

A value at risk (VaR) calculation is aimed at making a statement of the form: "We are X percent certain that we will not lose more than V dollars in the next N days." The variable V is the VaR, $X\%$ is the confidence level, and N days is the time horizon.

One approach to calculating VaR is historical simulation. This involves creating a database consisting of the daily movements in all market variables over a period of time. The first simulation trial assumes that the percentage changes in each market variable are the same as those on the first day covered by the database; the second simulation trial assumes that the percentage changes are the same as those on the second day; and so on. The change in the portfolio value, ΔP, is calculated for each simulation trial, and the VaR is calculated as the appropriate percentile of the probability distribution of ΔP.

An alternative is the model-building approach. This is relatively straightforward if two assumptions can be made:

1. The change in the value of the portfolio (ΔP) is linearly dependent on percentage changes in market variables.
2. The percentage changes in market variables are multivariate normally distributed.

The probability distribution of ΔP is then normal, and there are analytic formulas for relating the standard deviation of ΔP to the volatilities and correlations of the underlying market variables. The VaR can be calculated from well-known properties of the normal distribution.

When a portfolio includes options, ΔP is not linearly related to the percentage changes in market variables. From knowledge of the gamma of the portfolio, we can derive an approximate quadratic relationship between ΔP and percentage changes in market variables. Monte Carlo simulation can then be used to estimate VaR.

In the next chapter we discuss how volatilities and correlations can be estimated and monitored.

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Questions and Problems (Answers in Solutions Manual)

18.1. Consider a position consisting of a \$100,000 investment in asset A and a \$100,000 investment in asset B. Assume that the daily volatilities of both assets are 1% and that the coefficient of correlation between their returns is 0.3. What is the 5-day 99% VaR for the portfolio?

18.2. Describe three ways of handling instruments that are dependent on interest rates when the model-building approach is used to calculate VaR. How would you handle these instruments when historical simulation is used to calculate VaR?

18.3. A financial institution owns a portfolio of options on the US dollar–sterling exchange rate. The delta of the portfolio is 56.0. The current exchange rate is 1.5000. Derive an approximate linear relationship between the change in the portfolio value and the percentage change in the exchange rate. If the daily volatility of the exchange rate is 0.7%, estimate the 10-day 99% VaR.

18.4. Suppose you know that the gamma of the portfolio in the previous question is 16.2. How does this change your estimate of the relationship between the change in the portfolio value and the percentage change in the exchange rate?

18.5. Suppose that the daily change in the value of a portfolio is, to a good approximation, linearly dependent on two factors, calculated from a principal components analysis. The delta of a portfolio with respect to the first factor is 6 and the delta with respect to the second factor is −4. The standard deviations of the factor are 20 and 8, respectively. What is the 5-day 90% VaR?

18.6. Suppose that a company has a portfolio consisting of positions in stocks, bonds, foreign exchange, and commodities. Assume that there are no derivatives. Explain the assumptions underlying (a) the linear model and (b) the historical simulation model for calculating VaR.

18.7. Explain how an interest rate swap is mapped into a portfolio of zero-coupon bonds with standard maturities for the purposes of a VaR calculation.

18.8. Explain the difference between value at risk and conditional value at risk.

18.9. Explain why the linear model can provide only approximate estimates of VaR for a portfolio containing options.

18.10. Verify that the 0.3-year zero-coupon bond in the cash-flow mapping example in the appendix to this chapter is mapped into a $37,397 position in a 3-month bond and a $11,793 position in a 6-month bond.

18.11. Suppose that the 5-year rate is 6%, the 7-year rate is 7% (both expressed with annual compounding), the daily volatility of a 5-year zero-coupon bond is 0.5%, and the daily volatility of a 7-year zero-coupon bond is 0.58%. The correlation between daily returns on the two bonds is 0.6. Map a cash flow of $1,000 received at time 6.5 years into a position in a 5-year bond and a position in a 7-year bond using the approach in the appendix. What cash flows in 5 and 7 years are equivalent to the 6.5-year cash flow?

18.12. Some time ago a company entered into a forward contract to buy £1 million for $1.5 million. The contract now has 6 months to maturity. The daily volatility of a 6-month zero-coupon sterling bond (when its price is translated to dollars) is 0.06% and the daily volatility of a 6-month zero-coupon dollar bond is 0.05%. The correlation between returns from the two bonds is 0.8. The current exchange rate is 1.53. Calculate the standard deviation of the change in the dollar value of the forward contract in 1 day. What is the 10-day 99% VaR? Assume that the 6-month interest rate in both sterling and dollars is 5% per annum with continuous compounding.

18.13. The text calculates a VaR estimate for the example in Table 18.5 assuming two factors. How does the estimate change if you assume (a) one factor and (b) three factors.

18.14. A bank has a portfolio of options on an asset. The delta of the options is −30 and the gamma is −5. Explain how these numbers can be interpreted. The asset price is 20 and its volatility is 1% per day. Adapt Sample Application E in the DerivaGem Application Builder software to calculate VaR.

18.15. Suppose that in Problem 18.14 the vega of the portfolio is −2 per 1% change in the annual volatility. Derive a model relating the change in the portfolio value in 1 day to delta, gamma, and vega. Explain without doing detailed calculations how you would use the model to calculate a VaR estimate.

Assignment Questions

18.16. A company has a position in bonds worth $6 million. The modified duration of the portfolio is 5.2 years. Assume that only parallel shifts in the yield curve can take place and that the standard deviation of the daily yield change (when yield is measured in percent) is 0.09. Use the duration model to estimate the 20-day 90% VaR for the portfolio. Explain carefully the weaknesses of this approach to calculating VaR. Explain two alternatives that give more accuracy.

18.17. Consider a position consisting of a \$300,000 investment in gold and a \$500,000 investment in silver. Suppose that the daily volatilities of these two assets are 1.8% and 1.2%, respectively, and that the coefficient of correlation between their returns is 0.6. What is the 10-day 97.5% VaR for the portfolio? By how much does diversification reduce the VaR?

18.18. Consider a portfolio of options on a single asset. Suppose that the delta of the portfolio is 12, the value of the asset is \$10, and the daily volatility of the asset is 2%. Estimate the 1-day 95% VaR for the portfolio from the delta. Suppose next that the gamma of the portfolio is −2.6. Derive a quadratic relationship between the change in the portfolio value and the percentage change in the underlying asset price in one day. How would you use this in a Monte Carlo simulation?

18.19. A company has a long position in a 2-year bond and a 3-year bond, as well as a short position in a 5-year bond. Each bond has a principal of \$100 and pays a 5% coupon annually. Calculate the company's exposure to the 1-year, 2-year, 3-year, 4-year, and 5-year rates. Use the data in Tables 18.3 and 18.4 to calculate a 20-day 95% VaR on the assumption that rate changes are explained by (a) one factor, (b) two factors, and (c) three factors. Assume that the zero-coupon yield curve is flat at 5%.

18.20. A bank has written a call option on one stock and a put option on another stock. For the first option the stock price is 50, the strike price is 51, the volatility is 28% per annum, and the time to maturity is 9 months. For the second option the stock price is 20, the strike price is 19, the volatility is 25% per annum, and the time to maturity is 1 year. Neither stock pays a dividend, the risk-free rate is 6% per annum, and the correlation between stock price returns is 0.4. Calculate a 10-day 99% VaR:

(a) Using only deltas
(b) Using the partial simulation approach
(c) Using the full simulation approach

18.21. A common complaint of risk managers is that the model-building approach (either linear or quadratic) does not work well when delta is close to zero. Test what happens when delta is close to zero by using Sample Application E in the DerivaGem Application Builder software. (You can do this by experimenting with different option positions and adjusting the position in the underlying to give a delta of zero.) Explain the results you get.

APPENDIX

CASH-FLOW MAPPING

In this appendix we explain one procedure for mapping cash flows to standard maturity dates. We will illustrate the procedure by considering a simple example of a portfolio consisting of a long position in a single Treasury bond with a principal of \$1 million maturing in 0.8 years. We suppose that the bond provides a coupon of 10% per annum payable semiannually. This means that the bond provides coupon payments of \$50,000 in 0.3 years and 0.8 years. It also provides a principal payment of \$1 million in 0.8 years. The Treasury bond can therefore be regarded as a position in a 0.3-year zero-coupon bond with a principal of \$50,000 and a position in a 0.8-year zero-coupon bond with a principal of \$1,050,000.

The position in the 0.3-year zero-coupon bond is mapped into an equivalent position in 3-month and 6-month zero-coupon bonds. The position in the 0.8-year zero-coupon bond is mapped into an equivalent position in 6-month and 1-year zero-coupon bonds. The result is that the position in the 0.8-year coupon-bearing bond is, for VaR purposes, regarded as a position in zero-coupon bonds having maturities of 3 months, 6 months, and 1 year.

The Mapping Procedure

Consider the \$1,050,000 that will be received in 0.8 years. We suppose that zero rates, daily bond price volatilities, and correlations between bond returns are as shown in Table 18.6.

The first stage is to interpolate between the 6-month rate of 6.0% and the 1-year rate of 7.0% to obtain a 0.8-year rate of 6.6%. (Annual compounding is assumed for all rates.) The present value of the \$1,050,000 cash flow to be received in 0.8 years is

$$\frac{1{,}050{,}000}{1.066^{0.8}} = 997{,}662$$

We also interpolate between the 0.1% volatility for the 6-month bond and the 0.2% volatility for the 1-year bond to get a 0.16% volatility for the 0.8-year bond.

Table 18.6 Data to illustrate mapping procedure.

Maturity:	*3-month*	*6-month*	*1-year*
Zero rate (% with annual compounding):	5.50	6.00	7.00
Bond price volatility (% per day):	0.06	0.10	0.20

Correlation between daily returns	*3-month bond*	*6-month bond*	*1-year bond*
3-month bond	1.0	0.9	0.6
6-month bond	0.9	1.0	0.7
1-year bond	0.6	0.7	1.0

Table 18.7 The cash-flow mapping result.

	\$50,000 received in 0.3 years	*\$1,050,000 received in 0.8 years*	*Total*
Position in 3-month bond (\$):	37,397		37,397
Position in 6-month bond (\$):	11,793	319,589	331,382
Position in 1-year bond (\$):		678,074	678,074

Suppose we allocate α of the present value to the 6-month bond and $1-\alpha$ of the present value to the 1-year bond. Using equation (18.2) and matching variances, we obtain

$$0.0016^2 = 0.001^2\alpha^2 + 0.002^2(1-\alpha)^2 + 2 \times 0.7 \times 0.001 \times 0.002\alpha(1-\alpha)$$

This is a quadratic equation that can be solved in the usual way to give $\alpha = 0.320337$. This means that 32.0337% of the value should be allocated to a 6-month zero-coupon bond and 67.9663% of the value should be allocated to a 1-year zero coupon bond. The 0.8-year bond worth \$997,662 is therefore replaced by a 6-month bond worth

$$997{,}662 \times 0.320337 = \$319{,}589$$

and a 1-year bond worth

$$997{,}662 \times 0.679663 = \$678{,}074$$

This cash-flow mapping scheme has the advantage that it preserves both the value and the variance of the cash flow. Also, it can be shown that the weights assigned to the two adjacent zero-coupon bonds are always positive.

For the \$50,000 cash flow received at time 0.3 years, we can carry out similar calculations (see Problem 18.10). It turns out that the present value of the cash flow is \$49,189. It can be mapped into a position worth \$37,397 in a 3-month bond and a position worth \$11,793 in a 6-month bond.

The results of the calculations are summarized in Table 18.7. The 0.8-year coupon-bearing bond is mapped into a position worth \$37,397 in a 3-month bond, a position worth \$331,382 in a 6-month bond, and a position worth \$678,074 in a 1-year bond. Using the volatilities and correlations in Table 18.6, equation (18.2) gives the variance of the change in the price of the 0.8-year bond with $n = 3$, $\alpha_1 = 37{,}397$, $\alpha_2 = 331{,}382$, $\alpha_3 = 678{,}074$, $\sigma_1 = 0.0006$, $\sigma_2 = 0.001$, $\sigma_3 = 0.002$, and $\rho_{12} = 0.9$, $\rho_{13} = 0.6$, $\rho_{23} = 0.7$. This variance is 2,628,518. The standard deviation of the change in the price of the bond is therefore $\sqrt{2{,}628{,}518} = 1{,}621.3$. Because we are assuming that the bond is the only instrument in the portfolio, the 10-day 99% VaR is

$$1621.3 \times \sqrt{10} \times 2.33 = 11{,}946$$

or about \$11,950.