CHAPTER 22

Exotic Options

Derivatives such as European and American call and put options are what are termed *plain vanilla products*. They have standard well-defined properties and trade actively. Their prices or implied volatilities are quoted by exchanges or by brokers on a regular basis. One of the exciting aspects of the over-the-counter derivatives market is the number of nonstandard products that have been created by financial engineers. These products are termed *exotic options*, or simply *exotics*. Although they are usually a relatively small part of its portfolio, these exotics are important to an investment bank because they are generally much more profitable than plain vanilla products.

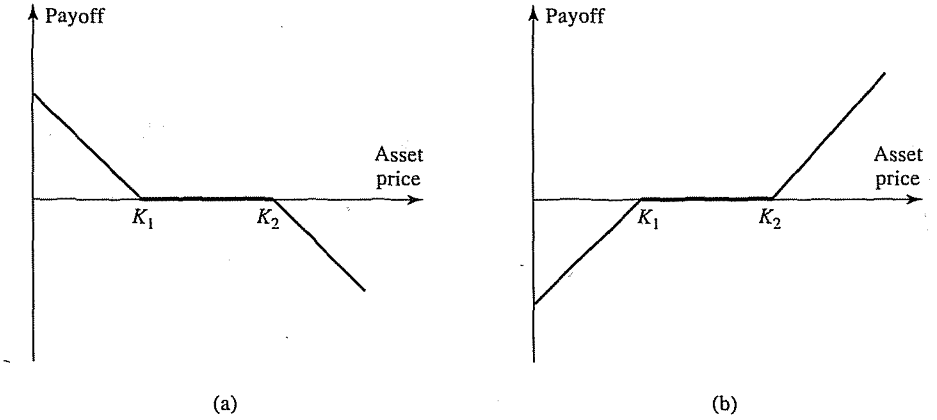
Exotic products are developed for a number of reasons. Sometimes they meet a genuine hedging need in the market; sometimes there are tax, accounting, legal, or regulatory reasons why corporate treasurers, fund managers, and financial institutions find exotic products attractive; sometimes the products are designed to reflect a view on potential future movements in particular market variables; occasionally an exotic product is designed by an investment bank to appear more attractive than it is to an unwary corporate treasurer or fund manager.

In this chapter we describe different types of exotic options and discuss their valuation. We use a categorization of exotic options similar to that in an excellent series of articles written by Eric Reiner and Mark Rubinstein for *Risk* magazine in 1991 and 1992. We assume that the asset provides a yield at rate q. As discussed in Chapter 14, for an option on a stock index we set q equal to the dividend yield on the index; for an option on a currency we set q equal to the foreign risk-free rate; and for an option on a futures contract we set q equal to the domestic risk-free rate. Most of the options discussed in this chapter can be valued using the DerivaGem software.

22.1 PACKAGES

A *package* is a portfolio consisting of standard European calls, standard European puts, forward contracts, cash, and the underlying asset itself. We discussed a number of different types of packages in Chapter 10: bull spreads, bear spreads, butterfly spreads, calendar spreads, straddles, strangles, and so on.

Figure 22.1 Payoffs from (a) short position and (b) long position in a range forward contract.



Often a package is structured by traders so that it has zero cost initially. An example is a *range forward contract*.[1] Figure 22.1 shows the payoff from short and long range forward contracts. A short range forward contract consists of a long position in a put with a low strike price, K_1, and a short position in a call with a high strike price, K_2. It guarantees that the underlying asset can be sold for a price between K_1 and K_2 at the maturity of the options. A long range forward contract consists of a short position in a put with the low strike price, K_1, and a long position in a call with the high strike price, K_2. It guarantees that the underlying asset can be purchased for a price between K_1 and K_2 at the maturity of the options. The value of the call usually equals the value of the put when the contract is initiated. As K_1 and K_2 are moved closer to each other, the price that will be received or paid for the asset at maturity becomes more certain. In the limit when $K_1 = K_2$, the range forward contract becomes a regular forward contract.

It is worth noting that any derivative can be converted into a zero-cost product by deferring payment until maturity. Consider a European call option. If c is the cost of the option when payment is made at time zero, then $A = ce^{rT}$ is the cost when payment is made at time T, the maturity of the option. The payoff is then $\max(S_T - K,\ 0) - A$ or $\max(S_T - K - A,\ -A)$. When the strike price, K, equals the forward price, other names for a deferred payment option are break forward, Boston option, forward with optional exit, and cancelable forward.

22.2 NONSTANDARD AMERICAN OPTIONS

In a standard American option, exercise can take place at any time during the life of the option and the exercise price is always the same. In practice, the American options that are traded in the over-the-counter market sometimes have nonstandard features.

[1] Other names used for a range forward contract are zero-cost collar, flexible forward, cylinder option, option fence, min–max, and forward band.

For example:

1. Early exercise may be restricted to certain dates. The instrument is then known as a *Bermudan option*.
2. Early exercise may be allowed during only part of the life of the option. For example, there may be an initial "lock out" period with no early exercise.
3. The strike price may change during the life of the option.

The warrants issued by corporations on their own stock often have some or all of these features. For example, in a 7-year warrant, exercise might be possible on particular dates during years 3 to 7, with the strike price being \$30 during years 3 and 4, \$32 during the next 2 years, and \$33 during the final year.

Nonstandard American options can usually be valued using a binomial tree. At each node, the test (if any) for early exercise is adjusted to reflect the terms of the option.

22.3 FORWARD START OPTIONS

Forward start options are options that will start at some time in the future. Executive options, which were discussed in Business Snapshot 8.3, can be viewed as a type of forward start option. This is because a company commits (implicitly or explicitly) to granting at-the-money options to employees in the future.

Consider a forward start at-the-money European call option that will start at time T_1 and mature at time T_2. Suppose that the asset price is S_0 at time zero and the S_1 at time T_1. To value the option, we note from the European option pricing formulas in Chapters 13 and 14 that the value of an at-the-money call option is proportional to the asset price. The value of the forward start option at time T_1 is therefore cS_1/S_0, where c is the value at time zero of an at-the-money option that lasts for $T_2 - T_1$. Using risk-neutral valuation, the value of the forward start option at time zero is

$$e^{-rT_1}\hat{E}\left[c\frac{S_1}{S_0}\right]$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Because c and S_0 are known and $\hat{E}[S_1] = S_0 e^{(r-q)T_1}$, it follows that the value of the forward start option is ce^{-qT_1}. For a non-dividend-paying stock, $q = 0$ and the value of the forward start option is exactly the same as the value of a regular at-the-money option with the same life as the forward start option.

22.4 COMPOUND OPTIONS

Compound options are options on options. There are four main types of compound options: a call on a call, a put on a call, a call on a put, and a put on a put. Compound options have two strike prices and two exercise dates. Consider, for example, a call on a call. On the first exercise date, T_1, the holder of the compound option is entitled to pay the first strike price, K_1, and receive a call option. The call option gives the holder the right to buy the underlying asset for the second strike price, K_2, on the second exercise date, T_2. The compound option will be exercised on the first exercise date only if the value of the option on that date is greater than the first strike price.

When the usual geometric Brownian motion assumption is made, European-style compound options can be valued analytically in terms of integrals of the bivariate normal distribution.[2] With our usual notation, the value at time zero of a European call option on a call option is

$$S_0 e^{-qT_2} M(a_1, b_1; \sqrt{T_1/T_2}) - K_2 e^{-rT_2} M(a_2, b_2; \sqrt{T_1/T_2}) - e^{-rT_1} K_1 N(a_2)$$

where

$$a_1 = \frac{\ln(S_0/S^*) + (r - q + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \qquad a_2 = a_1 - \sigma\sqrt{T_1}$$

$$b_1 = \frac{\ln(S_0/K_2) + (r - q + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \qquad b_2 = b_1 - \sigma\sqrt{T_2}$$

The function $M(a, b : \rho)$ is the cumulative bivariate normal distribution function that the first variable will be less than a and the second will be less than b when the coefficient of correlation between the two is ρ.[3] The variable S^* is the asset price at time T_1 for which the option price at time T_1 equals K_1. If the actual asset price is above S^* at time T_1, the first option will be exercised; if it is not above S^*, the option expires worthless.

With similar notation, the value of a European put on a call is

$$K_2 e^{-rT_2} M(-a_2, b_2; -\sqrt{T_1/T_2}) - S_0 e^{-qT_2} M(-a_1, b_1; -\sqrt{T_1/T_2}) + e^{-rT_1} K_1 N(-a_2)$$

The value of a European call on a put is

$$K_2 e^{-rT_2} M(-a_2, -b_2; \sqrt{T_1/T_2}) - S_0 e^{-qT_2} M(-a_1, -b_1; \sqrt{T_1/T_2}) - e^{-rT_1} K_1 N(-a_2)$$

The value of a European put on a put is

$$S_0 e^{-qT_2} M(a_1, -b_1; -\sqrt{T_1/T_2}) - K_2 e^{-rT_2} M(a_2, -b_2; -\sqrt{T_1/T_2}) + e^{-rT_1} K_1 N(a_2)$$

22.5 CHOOSER OPTIONS

A *chooser* option (sometimes referred to as an *as you like it* option) has the feature that, after a specified period of time, the holder can choose whether the option is a call or a put. Suppose that the time when the choice is made is T_1. The value of the chooser option at this time is

$$\max(c, p)$$

where c is the value of the call underlying the option and p is the value of the put underlying the option.

If the options underlying the chooser option are both European and have the same strike price, put–call parity can be used to provide a valuation formula. Suppose that S_1

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[2] See R. Geske, "The Valuation of Compound Options," *Journal of Financial Economics*, 7 (1979): 63–81; M. Rubinstein, "Double Trouble," *Risk*, December 1991/January 1992: 53–56.

[3] See Technical Note 5 on the author's website for a numerical procedure for calculating M.

is the asset price at time T_1, K is the strike price, T_2 is the maturity of the options, and r is the risk-free interest rate. Put–call parity implies that

$$\begin{aligned}\max(c, p) &= \max(c,\ c + Ke^{-r(T_2-T_1)} - S_1 e^{-q(T_2-T_1)}) \\ &= c + e^{-q(T_2-T_1)} \max(0,\ Ke^{-(r-q)(T_2-T_1)} - S_1)\end{aligned}$$

This shows that the chooser option is a package consisting of:

1. A call option with strike price K and maturity T_2
2. $e^{-q(T_2-T_1)}$ put options with strike price $Ke^{-(r-q)(T_2-T_1)}$ and maturity T_1

As such, it can readily be valued.

More complex chooser options can be defined where the call and the put do not have the same strike price and time to maturity. They are then not packages and have features that are somewhat similar to compound options.

22.6 BARRIER OPTIONS

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time.

A number of different types of barrier options regularly trade in the over-the-counter market. They are attractive to some market participants because they are less expensive than the corresponding regular options. These barrier options can be classified as either *knock-out options* or *knock-in options*. A knock-out option ceases to exist when the underlying asset price reaches a certain barrier; a knock-in option comes into existence only when the underlying asset price reaches a barrier.

Equations (14.4) and (14.5) show that the values at time zero of a regular call and put option are

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

A *down-and-out call* is one type of knock-out option. It is a regular call option that ceases to exist if the asset price reaches a certain barrier level H. The barrier level is below the initial asset price. The corresponding knock-in option is a *down-and-in call*. This is a regular call that comes into existence only if the asset price reaches the barrier level.

If H is less than or equal to the strike price, K, the value of a down-and-in call at time zero is given by

$$c_{\text{di}} = S_0 e^{-qT} (H/S_0)^{2\lambda} N(y) - Ke^{-rT} (H/S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

where

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

$$y = \frac{\ln[H^2/(S_0 K)]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

Because the value of a regular call equals the value of a down-and-in call plus the value of a down-and-out call, the value of a down-and-out call is given by

$$c_{\text{do}} = c - c_{\text{di}}$$

If $H \geqslant K$, then

$$c_{\text{do}} = S_0 N(x_1) e^{-qT} - K e^{-rT} N(x_1 - \sigma\sqrt{T}) \\ - S_0 e^{-qT} (H/S_0)^{2\lambda} N(y_1) + K e^{-rT} (H/S_0)^{2\lambda-2} N(y_1 - \sigma\sqrt{T})$$

and

$$c_{\text{di}} = c - c_{\text{do}}$$

where

$$x_1 = \frac{\ln(S_0/H)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$y_1 = \frac{\ln(H/S_0)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

An *up-and-out call* is a regular call option that ceases to exist if the asset price reaches a barrier level, H, that is higher than the current asset price. An *up-and-in call* is a regular call option that comes into existence only if the barrier is reached. When H is less than or equal to K, the value of the up-and-out call, c_{uo}, is zero and the value of the up-and-in call, c_{ui}, is c. When H is greater than K,

$$c_{\text{ui}} = S_0 N(x_1) e^{-qT} - K e^{-rT} N(x_1 - \sigma\sqrt{T}) - S_0 e^{-qT} (H/S_0)^{2\lambda} [N(-y) - N(-y_1)] \\ + K e^{-rT} (H/S_0)^{2\lambda-2} [N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})]$$

and

$$c_{\text{uo}} = c - c_{\text{ui}}$$

Put barrier options are defined similarly to call barrier options. An *up-and-out put* is a put option that ceases to exist when a barrier, H, that is greater than the current asset price is reached. An *up-and-in put* is a put that comes into existence only if the barrier is reached. When the barrier, H, is greater than or equal to the strike price, K, their prices are

$$p_{\text{ui}} = -S_0 e^{-qT} (H/S_0)^{2\lambda} N(-y) + K e^{-rT} (H/S_0)^{2\lambda-2} N(-y + \sigma\sqrt{T})$$

and

$$p_{\text{uo}} = p - p_{\text{ui}}$$

When H is less than or equal to K,

$$p_{\text{uo}} = -S_0 N(-x_1) e^{-qT} + K e^{-rT} N(-x_1 + \sigma\sqrt{T}) \\ + S_0 e^{-qT} (H/S_0)^{2\lambda} N(-y_1) - K e^{-rT} (H/S_0)^{2\lambda-2} N(-y_1 + \sigma\sqrt{T})$$

and

$$p_{ui} = p - p_{uo}$$

A *down-and-out put* is a put option that ceases to exist when a barrier less than the current asset price is reached. A *down-and-in put* is a put option that comes into existence only when the barrier is reached. When the barrier is greater than the strike price, $p_{do} = 0$ and $p_{di} = p$. When the barrier is less than the strike price,

$$p_{di} = -S_0 N(-x_1)e^{-qT} + Ke^{-rT} N(-x_1 + \sigma\sqrt{T}) + S_0 e^{-qT}(H/S_0)^{2\lambda}[N(y) - N(y_1)] - Ke^{-rT}(H/S_0)^{2\lambda-2}[N(y - \sigma\sqrt{T}) - N(y_1 - \sigma\sqrt{T})]$$

and

$$p_{do} = p - p_{di}$$

All of these valuations make the usual assumption that the probability distribution for the asset price at a future time is lognormal. An important issue for barrier options is the frequency that the asset price, S, is observed for purposes of determining whether the barrier has been reached. The analytic formulas given in this section assume that S is observed continuously and sometimes this is the case.[4] Often, the terms of a contract state that S is observed periodically; for example, once a day at 12 noon. Broadie, Glasserman, and Kou provide a way of adjusting the formulas we have just given for the situation where the price of the underlying is observed discretely.[5] The barrier level H is replaced by $He^{0.5826\sigma\sqrt{T/m}}$ for an up-and-in or up-and-out option and by $He^{-0.5826\sigma\sqrt{T/m}}$ for a down-and-in or down-and-out option, where m is the number of times the asset price is observed (so that T/m is the time interval between observations).

Barrier options often have quite different properties from regular options. For example, sometimes vega is negative. Consider an up-and-out call option when the asset price is close to the barrier level. As volatility increases the probability that the barrier will be hit increases. As a result, a volatility increase can cause the price of the barrier option to decrease in these circumstances.

22.7 BINARY OPTIONS

Binary options are options with discontinuous payoffs. A simple example of a binary option is a *cash-or-nothing call*. This pays off nothing if the asset price ends up below the strike price at time T and pays a fixed amount, Q, if it ends up above the strike price. In a risk-neutral world, the probability of the asset price being above the strike price at the maturity of an option is, with our usual notation, $N(d_2)$. The value of a cash-or-nothing call is therefore $Qe^{-rT}N(d_2)$. A *cash-or-nothing put* is defined analogously to a cash-or-nothing call. It pays off Q if the asset price is below the strike price and nothing if it is above the strike price. The value of a cash-or-nothing put is $Qe^{-rT}N(-d_2)$.

Another type of binary option is an *asset-or-nothing call*. This pays off nothing if the

[4] One way to track whether a barrier has been reached from below (above) is to send a limit order to the exchange to sell (buy) the asset at the barrier price and see whether the order is filled.

[5] M. Broadie, P. Glasserman, and S. G. Kou, "A Continuity Correction for Discrete Barrier Options," *Mathematical Finance* 7, 4 (October 1997): 325–49.

underlying asset price ends up below the strike price and pays an amount equal to the asset price itself if it ends up above the strike price. With our usual notation, the value of an asset-or-nothing call is $S_0e^{-qT}N(d_1)$. An *asset-or-nothing put* pays off nothing if the underlying asset price ends up above the strike price and an amount equal to the asset price if it ends up below the strike price. The value of an asset-or-nothing put is $S_0e^{-qT}N(-d_1)$.

A regular European call option is equivalent to a long position in an asset-or-nothing call and a short position in a cash-or-nothing call where the cash payoff on the cash-or-nothing call equals the strike price. Similarly, a regular European put option is equivalent to a long position in a cash-or-nothing put and a short position in an asset-or-nothing put where the cash payoff on the cash-or-nothing put equals the strike price.

22.8 LOOKBACK OPTIONS

The payoffs from lookback options depend on the maximum or minimum asset price reached during the life of the option. The payoff from a European-style lookback call is the amount that the final asset price exceeds the minimum asset price achieved during the life of the option. The payoff from a European-style lookback put is the amount by which the maximum asset price achieved during the life of the option exceeds the final asset price.

Valuation formulas have been produced for European lookbacks.[6] The value of a European lookback call at time zero is

$$S_0e^{-qT}N(a_1) - S_0e^{-qT}\frac{\sigma^2}{2(r-q)}N(-a_1) - S_{\min}e^{-rT}\left[N(a_2) - \frac{\sigma^2}{2(r-q)}e^{Y_1}N(-a_3)\right]$$

where

$$a_1 = \frac{\ln(S_0/S_{\min}) + (r-q+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$a_2 = a_1 - \sigma\sqrt{T},$$

$$a_3 = \frac{\ln(S_0/S_{\min}) + (-r+q+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$Y_1 = -\frac{2(r-q-\sigma^2/2)\ln(S_0/S_{\min})}{\sigma^2}$$

and $S_{\min}$ is the minimum asset price achieved to date. (If the lookback has just been originated, $S_{\min} = S_0$.) See Problem 22.23 for the $r = q$ case.

The value of a European lookback put is

$$S_{\max}e^{-rT}\left[N(b_1) - \frac{\sigma^2}{2(r-q)}e^{Y_2}N(-b_3)\right] + S_0e^{-qT}\frac{\sigma^2}{2(r-q)}N(-b_2) - S_0e^{-qT}N(b_2)$$

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[6] See B. Goldman, H. Sosin, and M. A. Gatto, "Path-Dependent Options: Buy at the Low, Sell at the High," *Journal of Finance*, 34 (December 1979): 1111–27.; M. Garman, "Recollection in Tranquility," *Risk*, March (1989): 16–19.

where

$$b_1 = \frac{\ln(S_{max}/S_0) + (-r + q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$b_2 = b_1 - \sigma\sqrt{T}$$

$$b_3 = \frac{\ln(S_{max}/S_0) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$Y_2 = \frac{2(r - q - \sigma^2/2)\ln(S_{max}/S_0)}{\sigma^2}$$

and S_{max} is the maximum asset price achieved to date. (If the lookback has just been originated, then $S_{max} = S_0$.)

Example 22.1

Consider a newly issued lookback put on a non-dividend-paying stock where the stock price is 50, the stock price volatility is 40% per annum, the risk-free rate is 10% per annum, and the time to maturity is 3 months. In this case, $S_{max} = 50$, $S_0 = 50$, $r = 0.1$, $q = 0$, $\sigma = 0.4$, and $T = 0.25$. From the formulas just given, $b_1 = -0.025$, $b_2 = -0.225$, $b_3 = 0.025$, and $Y_2 = 0$, so that the value of the lookback put is 7.79. A newly issued lookback call on the same stock is worth 8.04.

A lookback call is a way that the holder can buy the underlying asset at the lowest price achieved during the life of the option. Similarly, a lookback put is a way that the holder can sell the underlying asset at the highest price achieved during the life of the option. The underlying asset in a lookback option is often a commodity. As with barrier options, the value of a lookback option is liable to be sensitive to the frequency that the asset price is observed for the purposes of computing the maximum or minimum. The formulas above assume that the asset price is observed continuously. Broadie, Glasserman, and Kou provide a way of adjusting the formulas we have just given for the situation where the asset price is observed discretely.[7]

22.9 SHOUT OPTIONS

A *shout option* is a European option where the holder can "shout" to the writer at one time during its life. At the end of the life of the option, the option holder receives either the usual payoff from a European option or the intrinsic value at the time of the shout, whichever is greater. Suppose the strike price is $50 and the holder of a call shouts when the price of the underlying asset is $60. If the final asset price is less than $60, the holder receives a payoff of $10. If it is greater than $60, the holder receives the excess of the asset price over $50.

A shout option has some of the same features as a lookback option, but is considerably less expensive. It can be valued by noting that if the holder shouts at a time τ when the asset price is S_τ the payoff from the option is

$$\max(0,\ S_T - S_\tau) + (S_\tau - K)$$

[7] M. Broadie, P. Glasserman, and S. G. Kou, "Connecting Discrete and Continuous Path-Dependent Options," *Finance and Stochastics*, 2 (1998): 1–28.

where, as usual, K is the strike price and S_T is the asset price at time T. The value at time τ if the holder shouts is therefore the present value of $S_\tau - K$ plus the value of a European option with strike price S_τ. The latter can be calculated using Black–Scholes formulas.

We value a shout option by constructing a binomial or trinomial tree for the underlying asset in the usual way. As we roll back through the tree, we calculate at each node the value of the option if we shout and the value if we do not shout. The option's price at the node is the greater of the two. The procedure for valuing a shout option is therefore similar to the procedure for valuing a regular American option.

22.10 ASIAN OPTIONS

Asian options are options where the payoff depends on the average price of the underlying asset during at least some part of the life of the option. The payoff from an *average price call* is $\max(0, S_{\text{ave}} - K)$ and that from an *average price put* is $\max(0, K - S_{\text{ave}})$, where S_{ave} is the average value of the underlying asset calculated over a predetermined averaging period. Average price options are less expensive than regular options and are arguably more appropriate than regular options for meeting some of the needs of corporate treasurers. Suppose that a US corporate treasurer expects to receive a cash flow of 100 million Australian dollars spread evenly over the next year from the company's Australian subsidiary. The treasurer is likely to be interested in an option that guarantees that the average exchange rate realized during the year is above some level. An average price put option can achieve this more effectively than regular put options.

Another type of Asian option is an average strike option. An *average strike call* pays off $\max(0, S_T - S_{\text{ave}})$ and an *average strike put* pays off $\max(0, S_{\text{ave}} - S_T)$. Average strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Alternatively, it can guarantee that the average price received for an asset in frequent trading over a period of time is not less than the final price.

If the underlying asset price, S, is assumed to be lognormally distributed and S_{ave} is a geometric average of the S's, analytic formulas are available for valuing European average price options.[8] This is because the geometric average of a set of lognormally distributed variables is also lognormal. Consider a newly issued option that will provide a payoff at time T based on the geometric average calculated between time zero and time T. In a risk-neutral world, it can be shown that the probability distribution of the geometric average of a asset price over a certain period is the same as that of the asset price at the end of the period if the asset's expected growth rate is set equal to $(r - q - \sigma^2/6)/2$ (rather than $r - q$) and its volatility is set equal to $\sigma/\sqrt{3}$ (rather than σ). The geometric average price option can, therefore, be treated like a regular option with the volatility set equal to $\sigma/\sqrt{3}$ and the dividend yield equal to

$$r - \tfrac{1}{2}\left(r - q - \frac{\sigma^2}{6}\right) = \tfrac{1}{2}\left(r + q + \frac{\sigma^2}{6}\right)$$

[8] See A. Kemna and A. Vorst, "A Pricing Method for Options Based on Average Asset Values," *Journal of Banking and Finance*, 14 (March 1990): 113–29.

When, as is nearly always the case, Asian options are defined in terms of arithmetic averages, exact analytic pricing formulas are not available. This is because the distribution of the arithmetic average of a set of lognormal distributions does not have analytically tractable properties. However, the distribution is approximately lognormal and this leads to a good analytic approximation for valuing average price options. We calculate the first two moments of the probability distribution of the arithmetic average in a risk-neutral world exactly and then fit a lognormal distribution to the moments.[9]

Consider a newly issued Asian option that provides a payoff at time T based on the arithmetic average between time zero and time T. The first moment, M_1, and the second moment, M_2, of the average in a risk-neutral world can be shown to be

$$M_1 = \frac{e^{(r-q)T} - 1}{(r-q)T} S_0$$

and

$$M_2 = \frac{2e^{[2(r-q)+\sigma^2]T} S_0^2}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S_0^2}{(r-q)T^2}\left[\frac{1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right]$$

when $q \neq r$ (see Problem 22.23 for the $q = r$ case).

If we assume that the average asset price is lognormal, we can regard an option on the average as like an option on a futures contract and use equations (14.16) and (14.17) with

$$F_0 = M_1 \tag{22.1}$$

and

$$\sigma^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right) \tag{22.2}$$

Example 22.2

Consider a newly issued average price call option on a non-dividend-paying stock where the stock price is 50, the strike price is 50, the stock price volatility is 40% per annum, the risk-free rate is 10% per annum, and the time to maturity is 1 year. In this case, $S_0 = 50$, $K = 50$, $r = 0.1$, $q = 0$, $\sigma = 0.4$, and $T = 1$. If the average is a geometric average, we can value the option as a regular option with the volatility equal to $0.4/\sqrt{3}$, or 23.09%, and dividend yield equal to $(0.1 + 0.4^2/6)/2$, or 6.33%. The value of the option is 5.13. If the average is an arithmetic average, we first calculate $M_1 = 52.59$ and $M_2 = 2{,}922.76$. When we assume the average is lognormal, the option has the same value as an option on a futures contract. From equations (22.1) and (22.2), $F_0 = 52.59$ and $\sigma = 23.54\%$. DerivaGem gives the value of the option as 5.62.

The formulas just given for M_1 and M_2 assume that the average is calculated from continuous observations on the asset price. The appendix to this chapter shows how M_1 and M_2 can be obtained when the average is calculated from observations on the asset price at discrete points in time.

We can modify the analysis to accommodate the situation where the option is not newly issued and some prices used to determine the average have already been observed. Suppose that the averaging period is composed of a period of length t_1 over which

[9] See S. M. Turnbull and L. M. Wakeman, "A Quick Algorithm for Pricing European Average Options," *Journal of Financial and Quantitative Analysis*, 26 (September 1991): 377–89.

prices have already been observed and a future period of length t_2 (the remaining life of the option). Suppose that the average asset price during the first time period is $\bar{S}$. The payoff from an average price call is

$$\max\left(\frac{\bar{S}t_1 + S_{\text{ave}}t_2}{t_1 + t_2} - K,\ 0\right)$$

where S_{ave} is the average asset price during the remaining part of the averaging period. This is the same as

$$\frac{t_2}{t_1 + t_2}\max(S_{\text{ave}} - K^*,\ 0)$$

where

$$K^* = \frac{t_1 + t_2}{t_2}K - \frac{t_1}{t_2}\bar{S}$$

When $K^* > 0$, the option can be valued in the same way as a newly issued Asian option provided that we change the strike price from K to K^* and multiply the result by $t_2/(t_1 + t_2)$. When $K^* < 0$ the option is certain to be exercised and can be valued as a forward contract. The value is

$$\frac{t_2}{t_1 + t_2}[M_1 e^{-rt_2} - K^* e^{-rt_2}]$$

22.11 OPTIONS TO EXCHANGE ONE ASSET FOR ANOTHER

Options to exchange one asset for another (sometimes referred to as *exchange options*) arise in various contexts. An option to buy yen with Australian dollars is, from the point of view of a US investor, an option to exchange one foreign currency asset for another foreign currency asset. A stock tender offer is an option to exchange shares in one stock for shares in another stock.

Consider a European option to give up an asset worth U_T at time T and receive in return an asset worth V_T. The payoff from the option is

$$\max(V_T - U_T,\ 0)$$

A formula for valuing this option was first produced by Margrabe.[10] Suppose that the asset prices, U and V, both follow geometric Brownian motion with volatilities σ_U and σ_V. Suppose further that the instantaneous correlation between U and V is ρ, and the yields provided by U and V are q_U and q_V, respectively. The value of the option at time zero is

$$V_0 e^{-q_V T} N(d_1) - U_0 e^{-q_U T} N(d_2) \qquad \textbf{(22.3)}$$

where

$$d_1 = \frac{\ln(V_0/U_0) + (q_U - q_V + \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}, \qquad d_2 = d_1 - \hat{\sigma}\sqrt{T}$$

and

$$\hat{\sigma} = \sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$$

and U_0 and V_0 are the values of U and V at times zero.

[10] See W. Margrabe, "The Value of an Option to Exchange One Asset for Another," *Journal of Finance*, 33 (March 1978): 177–86.

This result will be proved in Chapter 25. It is interesting to note that equation (22.3) is independent of the risk-free rate r. This is because, as r increases, the growth rate of both asset prices in a risk-neutral world increases, but this is exactly offset by an increase in the discount rate. The variable $\hat{\sigma}$ is the volatility of V/U. Comparisons with equation (14.4) show that the option price is the same as the price of U_0 European call options on an asset worth V/U when the strike price is 1.0, the risk-free interest rate is q_U, and the dividend yield on the asset is q_V. Mark Rubinstein shows that the American version of this option can be characterized similarly for valuation purposes.[11] It can be regarded as U_0 American options to buy an asset worth V/U for 1.0 when the risk-free interest rate is q_U and the dividend yield on the asset is q_V. The option can therefore be valued as described in Chapter 17 using a binomial tree.

An option to obtain the better or worse of two assets can be regarded as a position in one of the assets combined with an option to exchange it for the other asset:

$$\min(U_T, V_T) = V_T - \max(V_T - U_T, 0)$$
$$\max(U_T, V_T) = U_T + \max(V_T - U_T, 0)$$

22.12 OPTIONS INVOLVING SEVERAL ASSETS

Options involving two or more risky assets are sometimes referred to as *rainbow options*. One example is the bond futures contract traded on the CBOT described in Chapter 5. The party with the short position is allowed to choose between a large number of different bonds when making delivery.

Probably the most popular option involving several assets is a *basket option*. This is an option where the payoff is dependent on the value of a portfolio (or basket) of assets. The assets are usually either individual stocks or stock indices or currencies. A European basket option can be valued with Monte Carlo simulation, by assuming that the assets follow correlated geometric Brownian motion processes. A much faster approach is to calculate the first two moments of the basket at the maturity of the option in a risk-neutral world, and then assume that value of the basket is lognormally distributed at that time. The option can then be regarded as an option on a futures contract with the parameters shown in equations (22.1) and (22.2). The appendix to this chapter shows how the moments of the value of the basket at a future time can be calculated from the volatilities of, and correlations between, the assets. Correlations are typically estimated from historical data.

22.13 STATIC OPTIONS REPLICATION

If we try using the techniques described in Chapter 15 for hedging exotic options, we find that some are easy to handle while others are very difficult because of discontinuities (see Business Snapshot 22.1). For the difficult cases, a technique known as static options replication is sometimes useful.[12] This involves searching for a portfolio of

[11] See M. Rubinstein, "One for Another," *Risk*, July/August 1991: 30–32

[12] See E. Derman, D. Ergener, and I. Kani, "Static Options Replication," *Journal of Derivatives* 2, 4 (Summer 1995): 78–95.

Business Snapshot 22.1 Is Delta Hedging Easier or More Difficult for Exotics?

As described in Chapter 15 we can approach the hedging of exotic options by creating a delta neutral position and rebalancing frequently to maintain delta neutrality. When we do this we find some exotic options are easier to hedge than plain vanilla options and some are more difficult.

An example of an exotic option that is relatively easy to hedge is an average price option where the averaging period is the whole life of the option. As time passes, we observe more of the asset prices that will be used in calculating the final average. This means that our uncertainty about the payoff decreases with the passage of time. As a result, the option becomes progressively easier to hedge. In the final few days, the delta of the option always approaches zero because price movements during this time have very little impact on the payoff.

By contrast barrier options are relatively difficult to hedge. Consider a down-and-out call option on a currency when the exchange rate is 0.0005 above the barrier. If the barrier is hit, the option is worth nothing. If the barrier is not hit, the option may prove to be quite valuable. The delta of the option is discontinuous at the barrier making conventional hedging very difficult.

actively traded options that approximately replicates the exotic option. Shorting this position provides the hedge. The basic principle underlying static options replication is as follows. If two portfolios are worth the same on a certain boundary, they are also worth the same at all interior points of the boundary.

Consider as an example a 9-month up-and-out call option on a non-dividend-paying stock where the stock price is 50, the strike price is 50, the barrier is 60, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum. Suppose that $f(S, t)$ is the value of the option at time t for a stock price of S. We can use any boundary in (S, t) space for the purposes of producing the replicating portfolio. A convenient one to choose is shown in Figure 22.2. It is defined by $S = 60$ and $t = 0.75$. The values of the up-and-out option on the boundary are given by

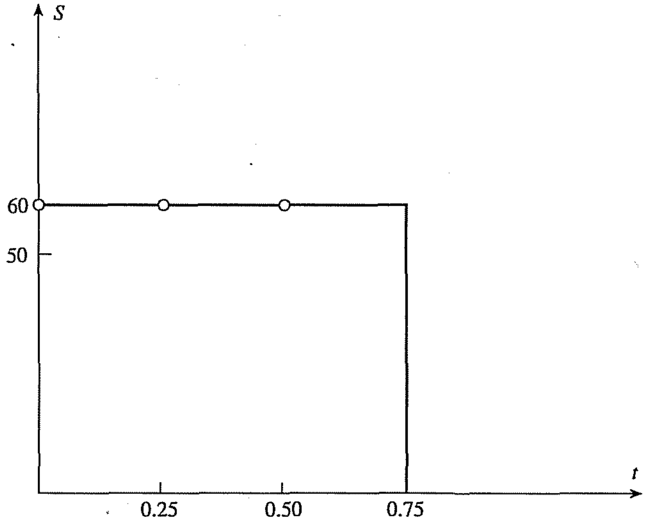
$$f(S, 0.75) = \max(S - 50, 0) \quad \text{when } S < 60$$

$$f(60, t) = 0 \quad \text{when } 0 \leqslant t \leqslant 0.75$$

There are many ways that we can approximately match these boundary values using regular options. The natural instrument to match the first boundary is a regular 9-month European call option with a strike price of 50. The first instrument introduced into the replicating portfolio is therefore likely to be one unit of this option. (We refer to this option as option A.) One way of then proceeding is as follows. We divide the life of the option into a number of time steps and choose options that satisfy the second boundary condition at the beginning of each time step.

Suppose that we choose time steps of 3 months. The next instrument we choose should lead to the second boundary being matched at $t = 0.5$. In other words, it should lead to the value of the complete replicating portfolio being zero when $t = 0.5$ and $S = 60$. The option should have the property that it has zero value on the first boundary since this has already been matched. One possibility is a regular 9-month European call

Figure 22.2 Boundary points used for static options replication example.



option with a strike price of 60. (We will refer to this as option B.) Black–Scholes formulas show that this is worth 4.33 at the 6-month point when $S = 60$. They also show that the position in option A is worth 11.54 at this point. The position we require in option B is therefore $-11.54/4.33 = -2.66$.

We next move on to matching the second boundary condition at $t = 0.25$. The option used should have the property that it has zero value on all boundaries that have been matched thus far. One possibility is a regular 6-month European call option with a strike price of 60. (We refer to this as option C.) This is worth 4.33 at the 3-month point when $S = 60$. Our position in options A and B is worth -4.21 at this point. The position we require in option C is therefore $4.21/4.33 = 0.97$.

Finally, we match the second boundary condition at $t = 0$. For this we use a regular 3-month European option with a strike price of 60. (We refer to this as option D.) Calculations similar to those above show that the required position in option D is 0.28.

The portfolio chosen is summarized in Table 22.1. (See also Sample Application F of the DerivaGem Application Builder.) It is worth 0.73 initially (i.e., at time zero when

Table 22.1 The portfolio of European call options used to replicate an up-and-out option.

Option	*Strike price*	*Maturity (years)*	*Position*	*Initial value*
A	50	0.75	1.00	+6.99
B	60	0.75	−2.66	−8.21
C	60	0.50	0.97	+1.78
D	60	0.25	0.28	+0.17

the stock price is 50). This compares with 0.31 given by the analytic formula for the up-and-out call earlier in this chapter. The replicating portfolio is not exactly the same as the up-and-out option because it matches the latter at only three points on the second boundary. If we use the same scheme, but match at 18 points on the second boundary (using options that mature every half month), the value of the replicating portfolio reduces to 0.38. If 100 points are matched, the value reduces further to 0.32.

To hedge a derivative, we short the portfolio that replicates its boundary conditions. This has the advantage over delta hedging that it does not require frequent rebalancing. The static replication approach can be used for a wide range of derivatives. The user has a great deal of flexibility in choosing the boundary that is to be matched and the options that are to be used. The portfolio must be unwound when any part of the boundary is reached.

SUMMARY

Exotic options are options with rules governing the payoff that are more complicated than standard options. We have discussed 12 different types of exotic options: packages, nonstandard American options, forward start options, compound options, chooser options, barrier options, binary options, lookback options, shout options, Asian options, options to exchange one asset for another, and options involving several assets. We have discussed how these can be valued using the same assumptions as those used to derive the Black–Scholes model in Chapter 13. Some can be valued analytically, but using much more complicated formulas than those for regular European calls and puts, some can be handled using analytic approximations, and some can be valued using extensions of the numerical procedures in Chapter 17. We will present more numerical procedures for valuing exotic options in Chapter 24.

Some exotic options are easier to hedge than the corresponding regular options; others are more difficult. In general, Asian options are easier to hedge because the payoff becomes progressively more certain as we approach maturity. Barrier options can be more difficult to hedge because delta is discontinuous at the barrier. One approach to hedging an exotic option, known as static options replication, is to find a portfolio of regular options whose value matches the value of the exotic option on some boundary. The exotic option is hedged by shorting this portfolio.

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Questions and Problems (Answers in Solutions Manual)

22.1. Explain the difference between a forward start option and a chooser option.

22.2. Describe the payoff from a portfolio consisting of a lookback call and a lookback put with the same maturity.

22.3. Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a 2-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is it ever optimal to make the choice before the end of the 2-year period? Explain your answer.

22.4. Suppose that c_1 and p_1 are the prices of a European average price call and a European average price put with strike price K and maturity T, c_2 and p_2 are the prices of a European average strike call and European average strike put with maturity T, and c_3 and p_3 are the prices of a regular European call and a regular European put with strike price K and maturity T. Show that

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

22.5. The text derives a decomposition of a particular type of chooser option into a call maturing at time T_2 and a put maturing at time T_1. Derive an alternative decomposition into a call maturing at time T_1 and a put maturing at time T_2.

22.6. Section 22.6 gives two formulas for a down-and-out call. The first applies to the situation where the barrier, H, is less than or equal to the strike price, K. The second applies to the situation where $H \geqslant K$. Show that the two formulas are the same when $H = K$.

22.7. Explain why a down-and-out put is worth zero when the barrier is greater than the strike price.

22.8. Suppose that the strike price of an American call option on a non-dividend-paying stock grows at rate g. Show that if g is less than the risk-free rate, r, it is never optimal to exercise the call early.

22.9. How can the value of a forward start put option on a non-dividend-paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

22.10. If a stock price follows geometric Brownian motion, what process does $A(t)$ follow where $A(t)$ is the arithmetic average stock price between time zero and time t?

22.11. Explain why delta hedging is easier for Asian options than for regular options.

22.12. Calculate the price of a 1-year European option to give up 100 ounces of silver in exchange for 1 ounce of gold. The current prices of gold and silver are \$380 and \$4, respectively; the risk-free interest rate is 10% per annum; the volatility of each commodity price is 20%; and the correlation between the two prices is 0.7. Ignore storage costs.

22.13. Is a European down-and-out option on an asset worth the same as a European down-and-out option on the asset's futures price for a futures contract maturing at the same time as the option?

22.14. Answer the following questions about compound options:
(a) What put–call parity relationship exists between the price of a European call on a call and a European put on a call? Show that the formulas given in the text satisfy the relationship.
(b) What put–call parity relationship exists between the price of a European call on a put and a European put on a put? Show that the formulas given in the text satisfy the relationship.

22.15. Does a lookback call become more valuable or less valuable as we increase the frequency with which we observe the asset price in calculating the minimum?

22.16. Does a down-and-out call become more valuable or less valuable as we increase the frequency with which we observe the asset price in determining whether the barrier has been crossed? What is the answer to the same question for a down-and-in call?

22.17. Explain why a regular European call option is the sum of a down-and-out European call and a down-and-in European call. Is the same true for American call options?

22.18. What is the value of a derivative that pays off \$100 in 6 months if the S&P 500 index is greater than 1,000 and zero otherwise? Assume that the current level of the index is 960, the risk-free rate is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 20%.

22.19. In a 3-month down-and-out call option on silver futures the strike price is \$20 per ounce and the barrier is \$18. The current futures price is \$19, the risk-free interest rate is 5%, and the volatility of silver futures is 40% per annum. Explain how the option works and calculate its value. What is the value of a regular call option on silver futures with the same terms? What is the value of a down-and-in call option on silver futures with the same terms?

22.20. A new European-style lookback call option on a stock index has a maturity of 9 months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use DerivaGem to value the option.

22.21. Estimate the value of a new 6-month European-style average price call option on a non-dividend-paying stock. The initial stock price is \$30, the strike price is \$30, the risk-free interest rate is 5%, and the stock price volatility is 30%.

22.22. Use DerivaGem to calculate the value of:
(a) A regular European call option on a non-dividend-paying stock where the stock price is \$50, the strike price is \$50, the risk-free rate is 5% per annum, the volatility is 30%, and the time to maturity is one year
(b) A down-and-out European call which is as in (a) with the barrier at \$45
(c) A down-and-in European call which is as in (a) with the barrier at \$45.

Show that the option in (a) is worth the sum of the values of the options in (b) and (c).

22.23. Explain adjustments that has to be made when $r = q$ for (a) the valuation formulas for lookback call options in Section 22.8 and (b) the formulas for M_1 and M_2 in Section 22.10.

Assignment Questions

22.24. What is the value in dollars of a derivative that pays off £10,000 in 1 year provided that the dollar/sterling exchange rate is greater than 1.5000 at that time? The current exchange rate is 1.4800. The dollar and sterling interest rates are 4% and 8% per annum, respectively. The volatility of the exchange rate is 12% per annum.

22.25. Consider an up-and-out barrier call option on a non-dividend-paying stock when the stock price is 50, the strike price is 50, the volatility is 30%, the risk-free rate is 5%, the time to maturity is 1 year, and the barrier at \$80. Use the software to value the option and graph the relationship between (a) the option price and the stock price, (b) the delta and the option price, (c) the option price and the time to maturity, and (d) the option price and the volatility. Provide an intuitive explanation for the results you get. Show that the delta, gamma, theta, and vega for an up-and-out barrier call option can be either positive or negative.

22.26. Sample Application F in the DerivaGem Application Builder Software considers the static options replication example in Section 22.13. It shows the way a hedge can be constructed using four options (as in Section 22.13) and two ways a hedge can be constructed using 16 options.
(a) Explain the difference between the two ways a hedge can be constructed using 16 options. Explain intuitively why the second method works better.
(b) Improve on the four-option hedge by changing Tmat for the third and fourth options.
(c) Check how well the 16-option portfolios match the delta, gamma, and vega of the barrier option.

22.27. Consider a down-and-out call option on a foreign currency. The initial exchange rate is 0.90, the time to maturity is 2 years, the strike price is 1.00, the barrier is 0.80, the domestic risk-free interest rate is 5%, the foreign risk-free interest rate is 6%, and the volatility is 25% per annum. Use DerivaGem to develop a static option replication strategy involving five options.

22.28. Suppose that a stock index is currently 900. The dividend yield is 2%, the risk-free rate is 5%, and the volatility is 40%. Use the results in Appendix 19A to calculate the value of a 1-year average price call where the strike price is 900 and the index level is observed at the end of each quarter for the purposes of the averaging. Compare this with the price calculated by DerivaGem for a 1-year average price option where the price is observed continuously. Provide an intuitive explanation for any differences between the prices.

22.29. Use the DerivaGem Application Builder software to compare the effectiveness of daily delta hedging for (a) the option considered in Tables 15.2 and 15.3 and (b) an average price call with the same parameters. Use Sample Application C. For the average price option you will find it necessary to change the calculation of the option price in cell C16, the payoffs in cells H15 and H16, and the deltas (cells G46 to G186 and N46 to N186). Carry out 20 Monte Carlo simulation runs for each option by repeatedly pressing F9. On each run record the cost of writing and hedging the option, the volume of trading over the whole 20 weeks and the volume of trading between weeks 11 and 20. Comment on the results.

22.30. In the DerivaGem Application Builder Software modify Sample Application D to test the effectiveness of delta and gamma hedging for a call on call compound option on a 100,000 units of a foreign currency where the exchange rate is 0.67, the domestic risk-free rate is 5%, the foreign risk-free rate is 6%, the volatility is 12%. The time to maturity of the first option is 20 weeks, and the strike price of the first option is 0.015. The second option matures 40 weeks from today and has a strike price of 0.68. Explain how you modified the cells. Comment on hedge effectiveness.

APPENDIX

CALCULATION OF MOMENTS FOR VALUATION OF BASKET OPTIONS AND ASIAN OPTIONS

Consider first the problem of calculating the first two moments of the value of a basket of assets at a future time, T, in a risk-neutral world. The price of each asset in the basket is assumed to be lognormal. Define:

n: The number of assets

S_i: The value of the ith asset at time T[13]

F_i: The forward price of the ith asset for a contract maturing at time T

σ_i: The volatility of the ith asset between time zero and time T

ρ_{ij}: Correlation between returns from the ith and jth asset

P: Value of basket at time T

M_1: First moment of P in a risk-neutral world

M_2: Second moment of P in a risk-neutral world

Because $P = \sum_{i=1}^{n} S_i$, $\hat{E}(S_i) = F_i$, $M_1 = \hat{E}(P)$ and $M_2 = \hat{E}(P^2)$, where $\hat{E}$ denotes expected value in a risk-neutral world, it follows that

$$M_1 = \sum_{i=1}^{n} F_i$$

Also,

$$P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} S_i S_j$$

From the properties of lognormal distributions,

$$\hat{E}(S_i S_j) = F_i F_j e^{\rho_{ij}\sigma_i\sigma_j T}$$

Hence

$$M_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j e^{\rho_{ij}\sigma_i\sigma_j T}$$

Asian Options

We now move on to the related problem of calculating the first two moments of the arithmetic average price of an asset in a risk-neutral world when the average is calculated from discrete observations. Suppose that the asset price is observed at times T_i $(1 \leqslant i \leqslant m)$. We redefine variables as follows:

S_i: The value of the asset at time T_i

F_i: The forward price of the asset for a contract maturing at time T_i

σ_i: The implied volatility for an option on the asset with maturity T_i

[13] If the ith asset is a certain stock and there are, say, 200 shares of the stock in the basket, then (for the purposes of the first part of the appendix) the ith "asset" is defined as 200 shares of the stock and S_i is the value of 200 shares of the stock.

ρ_{ij}: Correlation between return on asset up to time T_i and the return on the asset up to time T_j

P: Value of the arithmetic average

M_1: First moment of P in a risk-neutral world

M_2: Second moment of P in a risk-neutral world

In this case,

$$M_1 = \frac{1}{m}\sum_{i=1}^{m} F_i$$

Also,

$$P^2 = \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} S_i S_j$$

In this case,

$$\hat{E}(S_i S_j) = F_i F_j e^{\rho_{ij}\sigma_i\sigma_j\sqrt{T_i T_j}}$$

It can be shown that, when $i < j$,

$$\rho_{ij} = \frac{\sigma_i\sqrt{T_i}}{\sigma_j\sqrt{T_j}}$$

so that

$$\hat{E}(S_i S_j) = F_i F_j e^{\sigma_i^2 T_i}$$

and

$$M_2 = \frac{1}{m^2}\left[\sum_{i=1}^{m} F_i^2 e^{\sigma_i^2 T_i} + 2\sum_{j=1}^{m}\sum_{i=1}^{j-1} F_i F_j e^{\sigma_i^2 T_i}\right]$$