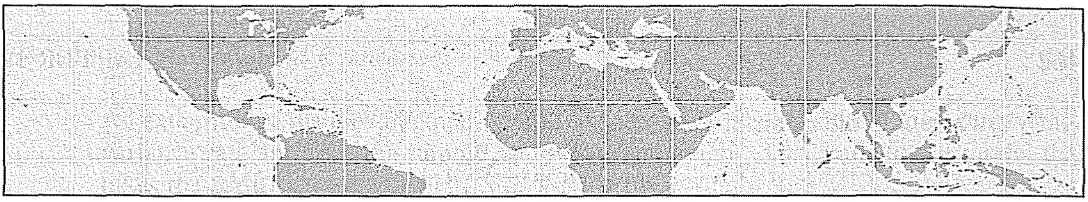

CHAPTER 19

Estimating Volatilities and Correlations

In this chapter we explain how historical data can be used to produce estimates of the current and future levels of volatilities and correlations. The chapter is relevant both to the calculation of value at risk using the model-building approach and to the valuation of derivatives. When calculating value at risk, we are most interested in the current levels of volatilities and correlations because we are assessing possible changes in the value of a portfolio over a very short period of time. When valuing derivatives, forecasts of volatilities and correlations over the whole life of the derivative are usually required.

The chapter considers models with imposing names such as exponentially weighted moving average (EWMA), autoregressive conditional heteroscedasticity (ARCH), and generalized autoregressive conditional heteroscedasticity (GARCH). The distinctive feature of the models is that they recognize that volatilities and correlations are not constant. During some periods, a particular volatility or correlation may be relatively low, whereas during other periods it may be relatively high. The models attempt to keep track of the variations in the volatility or correlation through time.

19.1 ESTIMATING VOLATILITY

Define σ_n as the volatility of a market variable on day n, as estimated at the end of day $n-1$. The square of the volatility, σ_n^2, on day n is the *variance rate*. We described the standard approach to estimating σ_n from historical data in Section 13.4. Suppose that the value of the market variable at the end of day i is S_i. The variable u_i is defined as the continuously compounded return during day i (between the end of day $i-1$ and the end of day i):

$$u_i = \ln\frac{S_i}{S_{i-1}}$$

An unbiased estimate of the variance rate per day, σ_n^2, using the most recent m observations on the u_i is

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2 \tag{19.1}$$

where $\bar{u}$ is the mean of the u_is:

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

For the purposes of monitoring daily volatility, the formula in equation (19.1) is usually changed in a number of ways:

1. u_i is defined as the percentage change in the market variable between the end of day $i-1$ and the end of day i, so that:[1]

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}} \tag{19.2}$$

2. $\bar{u}$ is assumed to be zero.[2]
3. $m-1$ is replaced by m.[3]

These three changes make very little difference to the estimates that are calculated, but they allow us to simplify the formula for the variance rate to

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2 \tag{19.3}$$

where u_i is given by equation (19.2).[4]

Weighting Schemes

Equation (19.3) gives equal weight to $u_{n-1}^2, u_{n-2}^2, \ldots, u_{n-m}^2$. Our objective is to estimate the current level of volatility, σ_n. It therefore makes sense to give more weight to recent data. A model that does this is

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \tag{19.4}$$

The variable α_i is the amount of weight given to the observation i days ago. The α's are positive. If we choose them so that $\alpha_i < \alpha_j$ when $i > j$, less weight is given to older observations. The weights must sum to unity, so we have

$$\sum_{i=1}^{m} \alpha_i = 1$$

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[1] This is consistent with the point made in Section 18.3 about the way that volatility is defined for the purposes of VaR calculations.

[2] As explained in Section 18.3, this assumption usually has very little effect on estimates of the variance because the expected change in a variable in one day is very small when compared with the standard deviation of changes.

[3] Replacing $m-1$ by m moves us from an unbiased estimate of the variance to a maximum likelihood estimate. Maximum likelihood estimates are discussed later in the chapter.

[4] Note that the u's in this chapter play the same role as the Δx's in Chapter 18. Both are daily percentage changes in market variables. In the case of the u's, the subscripts count observations made on different days on the same market variable. In the case of the Δx's, they count observations made on the same day on different market variables. The use of subscripts for σ is similarly different between the two chapters. In this chapter, the subscripts refer to days; in Chapter 18 they referred to market variables.

An extension of the idea in equation (19.4) is to assume that there is a long-run average variance rate and that this should be given some weight. This leads to the model that takes the form

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \tag{19.5}$$

where V_L is the long-run variance rate and γ is the weight assigned to V_L. Because the weights must sum to unity, we have

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

This is known as an ARCH(m) model. It was first suggested by Engle.[5] The estimate of the variance is based on a long-run average variance and m observations. The older an observation, the less weight it is given. Defining $\omega = \gamma V_L$, the model in equation (19.5) can be written

$$\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \tag{19.6}$$

In the next two sections we discuss two important approaches to monitoring volatility using the ideas in equations (19.4) and (19.5).

19.2 THE EXPONENTIALLY WEIGHTED MOVING AVERAGE MODEL

The exponentially weighted moving average (EWMA) model is a particular case of the model in equation (19.4) where the weights α_i decrease exponentially as we move back through time. Specifically, $\alpha_{i+1} = \lambda \alpha_i$, where λ is a constant between 0 and 1.

It turns out that this weighting scheme leads to a particularly simple formula for updating volatility estimates. The formula is

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2 \tag{19.7}$$

The estimate, σ_n, of the volatility for day n (made at the end of day $n-1$) is calculated from σ_{n-1} (the estimate that was made at the end of day $n-2$ of the volatility for day $n-1$) and u_{n-1} (the most recent daily percentage change).

To understand why equation (19.7) corresponds to weights that decrease exponentially, we substitute for σ_{n-1}^2 to get

$$\sigma_n^2 = \lambda[\lambda \sigma_{n-2}^2 + (1 - \lambda) u_{n-2}^2] + (1 - \lambda) u_{n-1}^2$$

or

$$\sigma_n^2 = (1 - \lambda)(u_{n-1}^2 + \lambda u_{n-2}^2) + \lambda^2 \sigma_{n-2}^2$$

Substituting in a similar way for σ_{n-2}^2 gives

$$\sigma_n^2 = (1 - \lambda)(u_{n-1}^2 + \lambda u_{n-2}^2 + \lambda^2 u_{n-3}^2) + \lambda^3 \sigma_{n-3}^2$$

[5] See R. Engle "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of UK Inflation," *Econometrica*, 50 (1982): 987–1008.

Continuing in this way, we see that

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{m} \lambda^{i-1} u_{n-i}^2 + \lambda^m \sigma_{n-m}^2$$

For large m, the term $\lambda^m \sigma_{n-m}^2$ is sufficiently small to be ignored, so that equation (19.7) is the same as equation (19.4) with $\alpha_i = (1-\lambda)\lambda^{i-1}$. The weights for the u_i decline at rate λ as we move back through time. Each weight is λ times the previous weight.

Example 19.1

Suppose that λ is 0.90, the volatility estimated for a market variable for day $n-1$ is 1% per day, and during day $n-1$ the market variable increased by 2%. This means that $\sigma_{n-1}^2 = 0.01^2 = 0.0001$ and $u_{n-1}^2 = 0.02^2 = 0.0004$. Equation (19.7) gives

$$\sigma_n^2 = 0.9 \times 0.0001 + 0.1 \times 0.0004 = 0.00013$$

The estimate of the volatility, σ_n, for day n is therefore $\sqrt{0.00013}$, or 1.14%, per day. Note that the expected value of u_{n-1}^2 is σ_{n-1}^2, or 0.0001. In this example, the realized value of u_{n-1}^2 is greater than the expected value, and as a result our volatility estimate increases. If the realized value of u_{n-1}^2 had been less than its expected value, our estimate of the volatility would have decreased.

The EWMA approach has the attractive feature that relatively little data need to be stored. At any given time, we need to remember only the current estimate of the variance rate and the most recent observation on the value of the market variable. When we get a new observation on the value of the market variable, we calculate a new daily percentage change and use equation (19.7) to update our estimate of the variance rate. The old estimate of the variance rate and the old value of the market variable can then be discarded.

The EWMA approach is designed to track changes in the volatility. Suppose there is a big move in the market variable on day $n-1$, so that u_{n-1}^2 is large. From equation (19.7) this causes our estimate of the current volatility to move upward. The value of λ governs how responsive the estimate of the daily volatility is to the most recent daily percentage change. A low value of λ leads to a great deal of weight being given to the u_{n-1}^2 when σ_n is calculated. In this case, the estimates produced for the volatility on successive days are themselves highly volatile. A high value of λ (i.e., a value close to 1.0) produces estimates of the daily volatility that respond relatively slowly to new information provided by the daily percentage change.

The RiskMetrics database, which was originally created by J. P. Morgan and made publicly available in 1994, uses the EWMA model with $\lambda = 0.94$ for updating daily volatility estimates in its RiskMetrics database. The company found that, across a range of different market variables, this value of λ gives forecasts of the variance rate that come closest to the realized variance rate.[6] The realized variance rate on a particular day was calculated as an equally weighted average of the u_i^2 on the subsequent 25 days (see Problem 19.17).

[6] See J. P. Morgan, *RiskMetrics Monitor*, Fourth Quarter, 1995. We will explain an alternative (maximum likelihood) approach to estimating parameters later in the chapter.

19.3 THE GARCH(1,1) MODEL

We now move on to discuss what is known as the GARCH(1,1) model, proposed by Bollerslev in 1986.[7] The difference between the GARCH(1,1) model and the EWMA model is analogous to the difference between equation (19.4) and equation (19.5). In GARCH(1,1), σ_n^2 is calculated from a long-run average variance rate, V_L, as well as from σ_{n-1} and u_{n-1}. The equation for GARCH(1,1) is

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \tag{19.8}$$

where γ is the weight assigned to V_L, α is the weight assigned to u_{n-1}^2, and β is the weight assigned to σ_{n-1}^2. Because the weights must sum to one, we have

$$\gamma + \alpha + \beta = 1$$

The EWMA model is a particular case of GARCH(1,1) where $\gamma = 0$, $\alpha = 1 - \lambda$, and $\beta = \lambda$.

The "(1,1)" in GARCH(1,1) indicates that σ_n^2 is based on the most recent observation of u^2 and the most recent estimate of the variance rate. The more general GARCH(p,q) model calculates σ_n^2 from the most recent p observations on u^2 and the most recent q estimates of the variance rate.[8] GARCH(1,1) is by far the most popular of the GARCH models.

Setting $\omega = \gamma V_L$, the GARCH(1,1) model can also be written

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \tag{19.9}$$

This is the form of the model that is usually used for the purposes of estimating the parameters. Once ω, α, and β have been estimated, we can calculate γ as $1 - \alpha - \beta$. The long-term variance V_L can then be calculated as ω/γ. For a stable GARCH(1,1) process we require $\alpha + \beta < 1$. Otherwise the weight applied to the long-term variance is negative.

Example 19.2

Suppose that a GARCH(1,1) model is estimated from daily data as

$$\sigma_n^2 = 0.000002 + 0.13 u_{n-1}^2 + 0.86 \sigma_{n-1}^2$$

This corresponds to $\alpha = 0.13$, $\beta = 0.86$, and $\omega = 0.000002$. Because $\gamma = 1 - \alpha - \beta$, it follows that $\gamma = 0.01$. Because $\omega = \gamma V_L$, it follows that $V_L = 0.0002$. In other words, the long-run average variance per day implied by the model is 0.0002. This corresponds to a volatility of $\sqrt{0.0002} = 0.014$, or 1.4%, per day.

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[7] See T. Bollerslev, "Generalized Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics*, 31 (1986): 307–27.

[8] Other GARCH models have been proposed that incorporate asymmetric news. These models are designed so that σ_n depends on the sign of u_{n-1}. Arguably, the models are more appropriate for equities than GARCH(1,1). As mentioned in Chapter 16, the volatility of an equity's price tends to be inversely related to the price so that a negative u_{n-1} should have a bigger effect on σ_n than the same positive u_{n-1}. For a discussion of models for handling asymmetric news, see D. Nelson, "Conditional Heteroscedasticity and Asset Returns: A New Approach," *Econometrica*, 59 (1990): 347–70; R. F. Engle and V. Ng, "Measuring and Testing the Impact of News on Volatility," *Journal of Finance*, 48 (1993): 1749–78.

Suppose that the estimate of the volatility on day $n-1$ is 1.6% per day, so that $\sigma_{n-1}^2 = 0.016^2 = 0.000256$, and that on day $n-1$ the market variable decreased by 1%, so that $u_{n-1}^2 = 0.01^2 = 0.0001$. Then

$$\sigma_n^2 = 0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023516$$

The new estimate of the volatility is therefore $\sqrt{0.00023516} = 0.0153$, or 1.53%, per day.

The Weights

Substituting for σ_{n-1}^2 in equation (19.9), we obtain

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta(\omega + \alpha u_{n-2}^2 + \beta\sigma_{n-2}^2)$$

or

$$\sigma_n^2 = \omega + \beta\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \beta^2\sigma_{n-2}^2$$

Substituting for σ_{n-2}^2, we get

$$\sigma_n^2 = \omega + \beta\omega + \beta^2\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \alpha\beta^2 u_{n-3}^2 + \beta^3\sigma_{n-3}^2$$

Continuing in this way, we see that the weight applied to u_{n-i}^2 is $\alpha\beta^{i-1}$. The weights decline exponentially at rate β. The parameter β can be interpreted as a "decay rate". It is similar to λ in the EWMA model. It defines the relative importance of the observations on the u's in determining the current variance rate. For example, if $\beta = 0.9$, then u_{n-2}^2 is only 90% as important as u_{n-1}^2; u_{n-3}^2 is 81% as important as u_{n-1}^2; and so on. The GARCH(1,1) model is similar to the EWMA model except that, in addition to assigning weights that decline exponentially to past u^2, it also assigns some weight to the long-run average volatility.

Mean Reversion

The GARCH (1,1) model recognizes that over time the variance tends to get pulled back to a long-run average level of V_L. The amount of weight assigned to V_L is $\gamma = 1 - \alpha - \beta$. The GARCH(1,1) is equivalent to a model where the variance V follows the stochastic process

$$dV = a(V_L - V)\,dt + \xi V\,dz$$

where time is measured in days, $a = 1 - \alpha - \beta$, and $\xi = \alpha\sqrt{2}$ (see Problem 19.14). This is a mean-reverting model. The variance has a drift that pulls it back to V_L at rate a. When $V > V_L$, the variance has a negative drift; when $V_L < V$, it has a positive drift. Superimposed on the drift is a volatility ξ. We will discuss this type of model further in Chapter 24.

19.4 CHOOSING BETWEEN THE MODELS

In practice, variance rates do tend to be mean reverting. The GARCH(1,1) model incorporates mean reversion, whereas the EWMA model does not. GARCH (1,1) is therefore theoretically more appealing than the EWMA model.

In the next section, we will discuss how best-fit parameters ω, α, and β in GARCH(1,1) can be estimated. When the parameter ω is zero, the GARCH(1, 1) reduces to EWMA. In circumstances where the best-fit value of ω turns out to be negative, the GARCH(1, 1) model is not stable and it makes sense to switch to the EWMA model.

19.5 MAXIMUM LIKELIHOOD METHODS

It is now appropriate to discuss how the parameters in the models we have been considering are estimated from historical data. The approach used is known as the *maximum likelihood method*. It involves choosing values for the parameters that maximize the chance (or likelihood) of the data occurring.

To illustrate the method, we start with a very simple example. Suppose that we sample 10 stocks at random on a certain day and find that the price of one of them declined on that day and the prices of the other nine either remained the same or increased. What is our best estimate of the probability of a price decline? The natural answer is 0.1. Let us see if this is what the maximum likelihood method gives.

Suppose that the probability of a price decline is p. The probability that one particular stock declines in price and the other nine do not is $p(1-p)^9$. (There is a probability p that the stock will decline and $1-p$ that the other nine will not.) Using the maximum likelihood approach, the best estimate of p is the one that maximizes $p(1-p)^9$. Differentiating this expression with respect to p and setting the result equal to zero, we find that $p = 0.1$ maximizes the expression. This shows that the maximum likelihood estimate of p is 0.1, as expected.

Estimating a Constant Variance

Our next example of maximum likelihood methods considers the problem of estimating a variance of a variable X from m observations on X when the underlying distribution is normal with zero mean. We assume that the observations are $u_1, u_2, \dots, u_m$ and that the mean of the underlying distribution is zero. Denote the variance by v. The likelihood of u_i being observed is the probability density function for X when $X = u_i$. This is

$$\frac{1}{\sqrt{2\pi v}} \exp\left(\frac{-u_i^2}{2v}\right)$$

The likelihood of m observations occurring in the order in which they are observed is

$$\prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi v}} \exp\left(\frac{-u_i^2}{2v}\right)\right] \tag{19.10}$$

Using the maximum likelihood method, the best estimate of v is the value that maximizes this expression.

Maximizing an expression is equivalent to maximizing the logarithm of the expression. Taking logarithms of the expression in equation (19.10) and ignoring constant multiplicative factors, it can be seen that we wish to maximize

$$\sum_{i=1}^{m}\left[-\ln(v) - \frac{u_i^2}{v}\right] \tag{19.11}$$

or

$$-m\ln(v) - \sum_{i=1}^{m} \frac{u_i^2}{v}$$

Differentiating this expression with respect to v and setting the result equation to zero, we see that the maximum likelihood estimator of v is[9]

$$\frac{1}{m}\sum_{i=1}^{m} u_i^2$$

Estimating GARCH (1,1) Parameters

We now consider how the maximum likelihood method can be used to estimate the parameters when GARCH (1,1) or some other volatility updating scheme is used. Define $v_i = \sigma_i^2$ as the variance estimated for day i. We assume that the probability distribution of u_i conditional on the variance is normal. A similar analysis to the one just given shows the best parameters are the ones that maximize

$$\prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi v_i}}\exp\left(\frac{-u_i^2}{2v_i}\right)\right]$$

Taking logarithms, we see that this is equivalent to maximizing

$$\sum_{i=1}^{m}\left[-\ln(v_i) - \frac{u_i^2}{v_i}\right] \tag{19.12}$$

This is the same as the expression in equation (19.11), except that v is replaced by v_i. We search iteratively to find the parameters in the model that maximize the expression in equation (19.12).

The spreadsheet in Table 19.1 indicates how the calculations could be organized for the GARCH(1,1) model. The table analyzes data on the Japanese yen exchange rate between January 6, 1988, and August 15, 1997. The numbers in the table are based on trial estimates of the three GARCH(1,1) parameters: ω, α, and β. The first column in the table records the date. The second column counts the days. The third column shows the exchange rate, S_i, at the end of day i. The fourth column shows the proportional change in the exchange rate between the end of day $i-1$ and the end of day i. This is $u_i = (S_i - S_{i-1})/S_{i-1}$. The fifth column shows the estimate of the variance rate, $v_i = \sigma_i^2$, for day i made at the end of day $i-1$. On day 3, we start things off by setting the variance equal to u_2^2. On subsequent days, equation (19.9) is used. The sixth column tabulates the likelihood measure, $-\ln(v_i) - u_i^2/v_i$. The values in the fifth and sixth columns are based on the current trial estimates of ω, α, and β. We are interested in choosing ω, α, and β to maximize the sum of the numbers in the sixth column. This involves an iterative search procedure.[10]

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[9] This confirms the point made in footnote 3.

[10] As discussed later, a general purpose algorithm such as Solver in Microsoft's Excel can be used. Alternatively, a special purpose algorithm, such as Levenberg–Marquardt, can be used. See, e.g., W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling. *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press, 1988.

Table 19.1 Estimation of parameters in GARCH(1,1) model.

Date	*Day i*	S_i	u_i	$v_i = \sigma_i^2$	$-\ln(v_i) - u_i^2/v_i$
06-Jan-88	1	0.007728			
07-Jan-88	2	0.007779	0.006599		
08-Jan-88	3	0.007746	−0.004242	0.00004355	9.6283
11-Jan-88	4	0.007816	0.009037	0.00004198	8.1329
12-Jan-88	5	0.007837	0.002687	0.00004455	9.8568
13-Jan-88	6	0.007924	0.011101	0.00004220	7.1529
⋮	⋮	⋮	⋮	⋮	⋮
13-Aug-97	2421	0.008643	0.003374	0.00007626	9.3321
14-Aug-97	2422	0.008493	−0.017309	0.00007092	5.3294
15-Aug-97	2423	0.008495	0.000144	0.00008417	9.3824
					22,063.5763

Trial estimates of GARCH parameters
$\omega = 0.00000176 \quad \alpha = 0.0626 \quad \beta = 0.8976$

In our example, the optimal values of the parameters turn out to be

$$\omega = 0.00000176, \quad \alpha = 0.0626, \quad \beta = 0.8976$$

and the maximum value of the function in equation (19.12) is 22,063.5763. The numbers shown in Table 19.1 were calculated on the final iteration of the search for the optimal ω, α, and β.

The long-term variance rate, V_L, in our example is

$$\frac{\omega}{1 - \alpha - \beta} = \frac{0.00000176}{0.0398} = 0.00004422$$

The long-term volatility is $\sqrt{0.00004422}$, or 0.665%, per day.

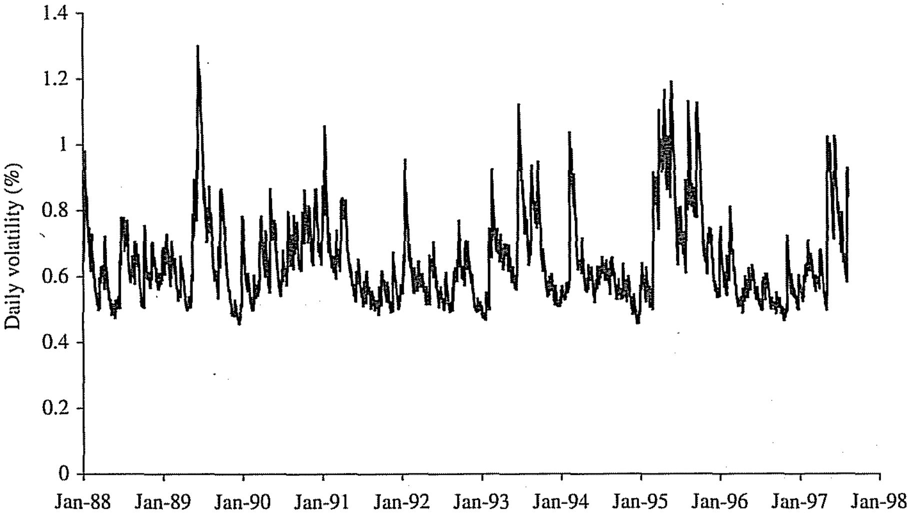
Figure 19.1 shows the way the GARCH (1,1) volatility for the Japanese yen changed over the 10-year period covered by the data. Most of the time, the volatility was between 0.4% and 0.8% per day, but volatilities over 1% were experienced during some periods.

An alternative and more robust approach to estimating parameters in GARCH(1,1) is known as *variance targeting*.[11] This involves setting the long-run average variance rate, V_L, equal to the sample variance calculated from the data (or to some other value that is believed to be reasonable). The value of ω then equals $V_L(1 - \alpha - \beta)$ and only two parameters have to be estimated. For the data in Table 19.1, the sample variance is 0.00004341, which gives a daily volatility of 0.659%. Setting V_L equal to the sample variance, the values of α and β that maximize the objective function in equation (19.12) are 0.0607 and 0.8990, respectively. The value of the objective function is 22,063.5274, only marginally below the value of 22,063.5763 obtained using the earlier procedure.

When the EWMA model is used, the estimation procedure is relatively simple. We set $\omega = 0$, $\alpha = 1 - \lambda$, and $\beta = \lambda$, and only one parameter has to be estimated. In the data in Table 19.1, the value of λ that maximizes the objective function in equation (19.12) is 0.9686 and the value of the objective function is 21,995.8377.

[11] See R. Engle and J. Mezrich, "GARCH for Groups," *Risk*, August 1996: 36–40.

Figure 19.1 Daily volatility of the yen/USD exchange rate, 1988–1997.



Both GARCH (1,1) and the EWMA method can be implemented by using the Solver routine in Excel to search for the values of the parameters that maximize the likelihood function. The routine works well provided that we structure our spreadsheet so that the parameters we are searching for have roughly equal values. For example, in GARCH (1,1) we could let cells A1, A2, and A3 contain $\omega \times 10^5$, α, and 0.1β. We could then set B1=A1/100000, B2=A2, and B3=10∗A3. We would use B1, B2, and B3 to calculate the likelihood function. We would ask Solver to calculate the values of A1, A2, and A3 that maximize the likelihood function.

How Good Is the Model?

The assumption underlying a GARCH model is that volatility changes with the passage of time. During some periods volatility is relatively high; during other periods it is relatively low. To put this another way, when u_i^2 is high, there is a tendency for u_{i+1}^2, $u_{i+2}^2, \ldots$ to be high; when u_i^2 is low, there is a tendency for u_{i+1}^2, $u_{i+2}^2, \ldots$ to be low. We can test how true this is by examining the autocorrelation structure of the u_i^2.

Let us assume the u_i^2 do exhibit autocorrelation. If a GARCH model is working well, it should remove the autocorrelation. We can test whether it has done so by considering the autocorrelation structure for the variables u_i^2/σ_i^2. If these show very little autocorrelation, our model for σ_i has succeeded in explaining autocorrelations in the u_i^2.

Table 19.2 shows results for the yen/dollar exchange rate data referred to above. The first column shows the lags considered when the autocorrelation is calculated. The second shows autocorrelations for u_i^2; the third shows autocorrelations for u_i^2/σ_i^2.[12] The table shows that the autocorrelations are positive for u_i^2 for all lags between 1 and 15. In the case of u_i^2/σ_i^2, some of the autocorrelations are positive and some are negative. They are all much smaller in magnitude than the autocorrelations for u_i^2.

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[12] For a series x_i, the autocorrelation with a lag of k is the coefficient of correlation between x_i and x_{i+k}.

Table 19.2 Autocorrelations before and after the use of a GARCH model.

Time lag	*Autocorrelation for u_i^2*	*Autocorrelation for u_i^2/σ_i^2*
1	0.072	0.004
2	0.041	−0.005
3	0.057	0.008
4	0.107	0.003
5	0.075	0.016
6	0.066	0.008
7	0.019	−0.033
8	0.085	0.012
9	0.054	0.010
10	0.030	−0.023
11	0.038	−0.004
12	0.038	−0.021
13	0.057	−0.001
14	0.040	0.002
15	0.007	−0.028

The GARCH model appears to have done a good job in explaining the data. For a more scientific test, we can use what is known as the Ljung–Box statistic.[13] If a certain series has m observations the Ljung–Box statistic is

$$m \sum_{k=1}^{K} w_k \eta_k^2$$

where η_k is the autocorrelation for a lag of k, K is the number of lags considered, and

$$w_k = \frac{m+2}{m-k}$$

For $K = 15$, zero autocorrelation can be rejected with 95% confidence when the Ljung–Box statistic is greater than 25.

From Table 19.2, the Ljung–Box Statistic for the u_i^2 series is about 123. This is strong evidence of autocorrelation. For the u_i^2/σ_i^2 series, the Ljung–Box statistic is 8.2, suggesting that the autocorrelation has been largely removed by the GARCH model.

19.6 USING GARCH(1,1) TO FORECAST FUTURE VOLATILITY

The variance rate estimated at the end of day $n-1$ for day n, when GARCH(1,1) is used, is

$$\sigma_n^2 = (1 - \alpha - \beta)V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

[13] See G. M. Ljung and G. E. P. Box, "On a Measure of Lack of Fit in Time Series Models," *Biometrica*, 65 (1978): 297–303.

so that

$$\sigma_n^2 - V_L = \alpha(u_{n-1}^2 - V_L) + \beta(\sigma_{n-1}^2 - V_L)$$

On day $n+t$ in the future, we have

$$\sigma_{n+t}^2 - V_L = \alpha(u_{n+t-1}^2 - V_L) + \beta(\sigma_{n+t-1}^2 - V_L)$$

The expected value of u_{n+t-1}^2 is σ_{n+t-1}^2. Hence,

$$E[\sigma_{n+t}^2 - V_L] = (\alpha + \beta)E[\sigma_{n+t-1}^2 - V_L]$$

where E denotes expected value. Using this equation repeatedly yields

$$E[\sigma_{n+t}^2 - V_L] = (\alpha + \beta)^t(\sigma_n^2 - V_L)$$

or

$$E[\sigma_{n+t}^2] = V_L + (\alpha + \beta)^t(\sigma_n^2 - V_L) \tag{19.13}$$

This equation forecasts the volatility on day $n+t$ using the information available at the end of day $n-1$. In the EWMA model, $\alpha + \beta = 1$ and equation (19.13) shows that the expected future variance rate equals the current variance rate. When $\alpha + \beta < 1$, the final term in the equation becomes progressively smaller as t increases. Figure 19.2 shows the expected path followed by the variance rate for situations where the current variance rate is different from V_L. As mentioned earlier, the variance rate exhibits mean reversion with a reversion level of V_L and a reversion rate of $1 - \alpha - \beta$. Our forecast of the future variance rate tends towards V_L as we look further and further ahead. This analysis emphasizes the point that we must have $\alpha + \beta < 1$ for a stable GARCH(1,1) process. When $\alpha + \beta > 1$, the weight given to the long-term average variance is negative and the process is "mean fleeing" rather than "mean reverting".

In the yen-dollar exchange rate example considered earlier $\alpha + \beta = 0.9602$ and $V_L = 0.00004422$. Suppose that our estimate of the current variance rate per day is 0.00006. (This corresponds to a volatility of 0.77% per day.) In 10 days the expected variance rate is

$$0.00004422 + 0.9602^{10}(0.00006 - 0.00004422) = 0.00005473$$

The expected volatility per day is 0.74%, still well above the long-term volatility of 0.665% per day. However, the expected variance rate in 100 days is

$$0.00004422 + 0.9602^{100}(0.00006 - 0.00004422) = 0.00004449$$

and the expected volatility per day is 0.667%, very close to the long-term volatility.

Volatility Term Structures

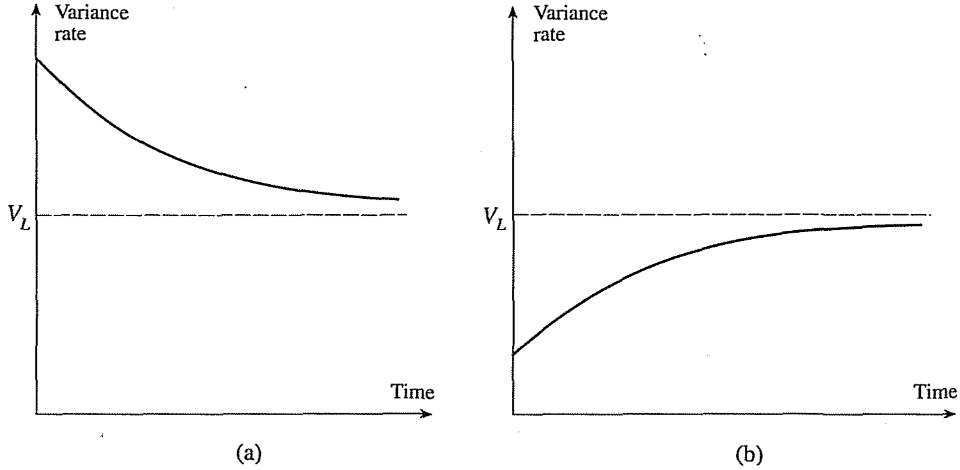
Suppose it is day n. Define:

$$V(t) = E(\sigma_{n+t}^2)$$

and

$$a = \ln\frac{1}{\alpha + \beta}$$

Figure 19.2 Expected path for the variance rate when (a) current variance rate is above long-term variance rate and (b) current variance rate is below long-term variance rate.



so that equation (19.13) becomes

$$V(t) = V_L + e^{-at}[V(0) - V_L]$$

Here, $V(t)$ is an estimate of the instantaneous variance rate in t days. The average variance rate per day between today and time T is given by

$$\frac{1}{T}\int_0^T V(t)\,dt = V_L + \frac{1 - e^{-aT}}{aT}[V(0) - V_L]$$

The longer the life of the option, the closer this is to V_L. Define $\sigma(T)$ as the volatility per annum that should be used to price a T-day option under GARCH(1, 1). Assuming 252 days per year, $\sigma(T)^2$ is 252 times the average variance rate per day, so that

$$\sigma(T)^2 = 252\left(V_L + \frac{1 - e^{-aT}}{aT}[V(0) - V_L]\right) \tag{19.14}$$

As we discussed in Chapter 16, the market prices of different options on the same asset are often used to calculate a *volatility term structure*. This is the relationship between the implied volatilities of the options and their maturities. Equation (19.14) can be used to estimate a volatility term structure based on the GARCH(1, 1) model. The estimated volatility term structure is not usually the same as the actual volatility term structure. However, as we will show, it is often used to predict the way that the actual volatility term structure will respond to volatility changes.

When the current volatility is above the long-term volatility, the GARCH(1, 1) model estimates a downward-sloping volatility term structure. When the current volatility is below the long-term volatility, it estimates an upward-sloping volatility

Table 19.3 Yen/dollar volatility term structure predicted from GARCH(1, 1).

Option life (days)	10	30	50	100	500
Option volatility (% per annum)	12.00	11.59	11.33	11.00	10.65

term structure. In the case of the yen/dollar exchange rate, $a = \ln(1/0.9602) = 0.0406$ and $V_L = 0.00004422$. Suppose that the current variance rate per day, $V(0)$, is estimated as 0.00006 per day. It follows from equation (19.14) that

$$\sigma(T)^2 = 252\left(0.00004422 + \frac{1 - e^{-0.0406T}}{0.0406T}(0.00006 - 0.000044220)\right)$$

where T is measured in days. Table 19.3 shows the volatility per year for different values of T.

Impact of Volatility Changes

Equation (19.14) can be written

$$\sigma(T)^2 = 252\left[V_L + \frac{1 - e^{-aT}}{aT}\left(\frac{\sigma(0)^2}{252} - V_L\right)\right]$$

When $\sigma(0)$ changes by $\Delta\sigma(0)$, $\sigma(T)$ changes by

$$\frac{1 - e^{-aT}}{aT}\frac{\sigma(0)}{\sigma(T)}\Delta\sigma(0) \tag{19.15}$$

Table 19.4 shows the effect of a volatility change on options of varying maturities for our yen/dollar exchange rate example. We assume as before that $V(0) = 0.00006$, so that $\sigma(0) = 12.30\%$. The table considers a 100-basis-point change in the instantaneous volatility from 12.30% per year to 13.30% per year. This means that $\Delta\sigma(0) = 0.01$, or 1%.

Many financial institutions use analyses such as this when determining the exposure of their books to volatility changes. Rather than consider an across-the-board increase of 1% in implied volatilities when calculating vega, they relate the size of the volatility increase that is considered to the maturity of the option. Based on Table 19.4, a 0.84% volatility increase would be considered for a 10-day option, a 0.61% increase for a 30-day option, a 0.46% increase for a 50-day option, and so on.

Table 19.4 Impact of 1% change in the instantaneous volatility predicted from GARCH(1, 1).

Option life (days)	10	30	50	100	500
Increase in volatility (%)	0.84	0.61	0.46	0.27	0.06

19.7 CORRELATIONS

The discussion so far has centered on the estimation and forecasting of volatility. As explained in Chapter 18, correlations also play a key role in the calculation of VaR. In this section, we show how correlation estimates can be updated in a similar way to volatility estimates.

The correlation between two variables X and Y can be defined as

$$\frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

where σ_X and σ_Y are the standard deviations of X and Y and $\text{cov}(X, Y)$ is the covariance between X and Y. The covariance between X and Y is defined as

$$E[(X - \mu_X)(Y - \mu_Y)]$$

where μ_X and μ_Y are the means of X and Y, and E denotes the expected value. Although it is easier to develop intuition about the meaning of a correlation than it is for a covariance, it is covariances that are the fundamental variables of our analysis.[14]

Define x_i and y_i as the percentage changes in X and Y between the end of day $i - 1$ and the end of day i:

$$x_i = \frac{X_i - X_{i-1}}{X_{i-1}}, \qquad y_i = \frac{Y_i - Y_{i-1}}{Y_{i-1}}$$

where X_i and Y_i are the values of X and Y at the end of day i. We also define the following:

$\sigma_{x,n}$: Daily volatility of variable X, estimated for day n

$\sigma_{y,n}$: Daily volatility of variable Y, estimated for day n

cov_n: Estimate of covariance between daily changes in X and Y, calculated on day n.

Our estimate of the correlation between X and Y on day n is

$$\frac{\text{cov}_n}{\sigma_{x,n}\,\sigma_{y,n}}$$

Using an equal-weighting scheme and assuming that the means of x_i and y_i are zero, equation (19.3) shows that we can estimate the variance rates of X and Y from the most recent m observations as

$$\sigma_{x,n}^2 = \frac{1}{m}\sum_{i=1}^{m} x_{n-i}^2, \qquad \sigma_{y,n}^2 = \frac{1}{m}\sum_{i=1}^{m} y_{n-i}^2$$

A similar estimate for the covariance between X and Y is

$$\text{cov}_n = \frac{1}{m}\sum_{i=1}^{m} x_{n-i}\, y_{n-i} \tag{19.16}$$

[14] An analogy here is that variance rates were the fundamental variables for the EWMA and GARCH schemes in first part of this chapter, even though volatilities are easier to understand.

One alternative for updating covariances is an EWMA model similar to equation (19.7). The formula for updating the covariance estimate is then

$$\text{cov}_n = \lambda\, \text{cov}_{n-1} + (1-\lambda)x_{n-1}\, y_{n-1}$$

A similar analysis to that presented for the EWMA volatility model shows that the weights given to observations on the $x_i\, y_i$ decline as we move back through time. The lower the value of λ, the greater the weight that is given to recent observations.

Example 19.3

Suppose that $\lambda = 0.95$ and that the estimate of the correlation between two variables X and Y on day $n-1$ is 0.6. Suppose further that the estimate of the volatilities for the X and Y on day $n-1$ are 1% and 2%, respectively. From the relationship between correlation and covariance, the estimate of the covariance between the X and Y on day $n-1$ is

$$0.6 \times 0.01 \times 0.02 = 0.00012$$

Suppose that the percentage changes in X and Y on day $n-1$ are 0.5% and 2.5%, respectively. The variance and covariance for day n would be updated as follows:

$$\sigma_{x,n}^2 = 0.95 \times 0.01^2 + 0.05 \times 0.005^2 = 0.00009625$$

$$\sigma_{y,n}^2 = 0.95 \times 0.02^2 + 0.05 \times 0.025^2 = 0.00041125$$

$$\text{cov}_n = 0.95 \times 0.00012 + 0.05 \times 0.005 \times 0.025 = 0.00012025$$

The new volatility of X is $\sqrt{0.00009625} = 0.981\%$ and the new volatility of Y is $\sqrt{0.00041125} = 2.028\%$. The new coefficient of correlation between X and Y is

$$\frac{0.00012025}{0.00981 \times 0.02028} = 0.6044$$

GARCH models can also be used for updating covariance estimates and forecasting the future level of covariances. For example, the GARCH(1,1) model for updating a covariance is

$$\text{cov}_n = \omega + \alpha x_{n-1}\, y_{n-1} + \beta\, \text{cov}_{n-1}$$

and the long-term average covariance is $\omega/(1-\alpha-\beta)$. Formulas similar to those in equations (19.13) and (19.14) can be developed for forecasting future covariances and calculating the average covariance during the life of an option.[15]

Consistency Condition for Covariances

Once all the variances and covariances have been calculated, a variance–covariance matrix can be constructed. When $i \neq j$, the (i, j) element of this matrix shows the covariance between variable i and variable j. When $i = j$, it shows the variance of variable i.

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[15] The ideas in this chapter can be extended to multivariate GARCH models, where an entire variance–covariance matrix is updated in a consistent way. For a discussion of alternative approaches, see R. Engle and J. Mezrich, "GARCH for Groups," *Risk*, August 1996: 36–40.

Not all variance–covariance matrices are internally consistent. The condition for an $N \times N$ variance–covariance matrix Ω to be internally consistent is

$$w^{\mathrm{T}} \Omega w \geqslant 0 \tag{19.17}$$

for all $N \times 1$ vectors w, where w^{T} is the transpose of w. A matrix that satisfies this property is known as *positive-semidefinite.*

To understand why the condition in equation (19.17) must hold, suppose that w^{T} is $[w_1, w_2, \dots, w_n]$. The expression $w^{\mathrm{T}} \Omega w$ is the variance of $w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$, where x_i is the value of variable i. As such, it cannot be negative.

To ensure that a positive-semidefinite matrix is produced, variances and covariances should be calculated consistently. For example, if variances are calculated by giving equal weight to the last m data items, the same should be done for covariances. If variances are updated using an EWMA model with $\lambda = 0.94$, the same should be done for covariances.

An example of a variance–covariance matrix that is not internally consistent is

$$\begin{bmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{bmatrix}$$

The variance of each variable is 1.0, and so the covariances are also coefficients of correlation. The first variable is highly correlated with the third variable and the second variable is highly correlated with the third variable. However, there is no correlation at all between the first and second variables. This seems strange. When we set w equal to $(1, 1, -1)$, we find that the condition in equation (19.17) is not satisfied, proving that the matrix is not positive-semidefinite.[16]

SUMMARY

Most popular option pricing models, such as Black–Scholes, assume that the volatility of the underlying asset is constant. This assumption is far from perfect. In practice, the volatility of an asset, like the asset's price, is a stochastic variable. Unlike the asset price, it is not directly observable. This chapter has discussed schemes for attempting to keep track of the current level of volatility.

We define u_i as the percentage change in a market variable between the end of day $i-1$ and the end of day i. The variance rate of the market variable (that is, the square of its volatility) is calculated as a weighted average of the u_i^2. The key feature of the schemes that have been discussed here is that they do not give equal weight to the observations on the u_i^2. The more recent an observation, the greater the weight assigned to it. In the EWMA and the GARCH(1, 1) models, the weights assigned to observations decrease exponentially as the observations become older. The GARCH(1, 1) model differs from the EWMA model in that some weight is also assigned to the long-run average variance rate. Both the EWMA and GARCH(1, 1) models have structures that enable forecasts of the future level of variance rate to be produced relatively easily.

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16 It can be shown that the condition for a 3×3 matrix of correlations to be internally consistent is

$$\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} \leqslant 1$$

where ρ_{ij} is the coefficient of correlation between variables i and j.

Maximum likelihood methods are usually used to estimate parameters from historical data in GARCH(1,1) and similar models. These methods involve using an iterative procedure to determine the parameter values that maximize the chance or likelihood that the historical data will occur. Once its parameters have been determined, a model can be judged by how well it removes autocorrelation from the u_i^2.

For every model that is developed to track variances, there is a corresponding model that can be developed to track covariances. The procedures described here can therefore be used to update the complete variance–covariance matrix used in value at risk calculations.

FURTHER READING

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Questions and Problems (Answers in Solutions Manual)

19.1. Explain the exponentially weighted moving average (EWMA) model for estimating volatility from historical data.

19.2. What is the difference between the exponentially weighted moving average model and the GARCH(1,1) model for updating volatilities?

19.3. The most recent estimate of the daily volatility of an asset is 1.5% and the price of the asset at the close of trading yesterday was \$30.00. The parameter λ in the EWMA model is 0.94. Suppose that the price of the asset at the close of trading today is \$30.50. How will this cause the volatility to be updated by the EWMA model?

19.4. A company uses an EWMA model for forecasting volatility. It decides to change the parameter λ from 0.95 to 0.85. Explain the likely impact on the forecasts.

19.5. The volatility of a certain market variable is 30% per annum. Calculate a 99% confidence interval for the size of the percentage daily change in the variable.

19.6. A company uses the GARCH(1,1) model for updating volatility. The three parameters are ω, α, and β. Describe the impact of making a small increase in each of the parameters while keeping the others fixed.

19.7. The most recent estimate of the daily volatility of the US dollar/sterling exchange rate is 0.6% and the exchange rate at 4 p.m. yesterday was 1.5000. The parameter λ in the EWMA model is 0.9. Suppose that the exchange rate at 4 p.m. today proves to be 1.4950. How would the estimate of the daily volatility be updated?

19.8. Assume that S&P 500 at close of trading yesterday was 1,040 and the daily volatility of the index was estimated as 1% per day at that time. The parameters in a GARCH(1,1) model are $\omega = 0.000002$, $\alpha = 0.06$, and $\beta = 0.92$. If the level of the index at close of trading today is 1,060, what is the new volatility estimate?

19.9. Suppose that the daily volatilities of asset A and asset B, calculated at the close of trading yesterday, are 1.6% and 2.5%, respectively. The prices of the assets at close of trading yesterday were \$20 and \$40 and the estimate of the coefficient of correlation between the returns on the two assets was 0.25. The parameter λ used in the EWMA model is 0.95.
(a) Calculate the current estimate of the covariance between the assets.
(b) On the assumption that the prices of the assets at close of trading today are \$20.5 and \$40.5, update the correlation estimate.

19.10. The parameters of a GARCH(1,1) model are estimated as $\omega = 0.000004$, $\alpha = 0.05$, and $\beta = 0.92$. What is the long-run average volatility and what is the equation describing the way that the variance rate reverts to its long-run average? If the current volatility is 20% per year, what is the expected volatility in 20 days?

19.11. Suppose that the current daily volatilities of asset X and asset Y are 1.0% and 1.2%, respectively. The prices of the assets at close of trading yesterday were \$30 and \$50 and the estimate of the coefficient of correlation between the returns on the two assets made at this time was 0.50. Correlations and volatilities are updated using a GARCH(1,1) model. The estimates of the model's parameters are $\alpha = 0.04$ and $\beta = 0.94$. For the correlation $\omega = 0.000001$, and for the volatilities $\omega = 0.000003$. If the prices of the two assets at close of trading today are \$31 and \$51, how is the correlation estimate updated?

19.12. Suppose that the daily volatility of the FTSE 100 stock index (measured in pounds sterling) is 1.8% and the daily volatility of the dollar/sterling exchange rate is 0.9%. Suppose further that the correlation between the FTSE 100 and the dollar/sterling exchange rate is 0.4. What is the volatility of the FTSE 100 when it is translated to US dollars? Assume that the dollar/sterling exchange rate is expressed as the number of US dollars per pound sterling. (*Hint*: When $Z = XY$, the percentage daily change in Z is approximately equal to the percentage daily change in X plus the percentage daily change in Y.)

19.13. Suppose that in Problem 19.12 the correlation between the S&P 500 Index (measured in dollars) and the FTSE 100 Index (measured in sterling) is 0.7, the correlation between the S&P 500 Index (measured in dollars) and the dollar/sterling exchange rate is 0.3, and the daily volatility of the S&P 500 index is 1.6%. What is the correlation between the S&P 500 index (measured in dollars) and the FTSE 100 index when it is translated to dollars? (*Hint*: For three variables X, Y, and Z, the covariance between $X + Y$ and Z equals the covariance between X and Z plus the covariance between Y and Z.)

19.14. Show that the GARCH (1,1) model $\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$ in equation (19.9) is equivalent to the stochastic volatility model $dV = a(V_L - V)\,dt + \xi V\,dz$, where time is measured in days, V is the square of the volatility of the asset price, and

$$a = 1 - \alpha - \beta, \qquad V_L = \frac{\omega}{1 - \alpha - \beta}, \qquad \xi = \alpha\sqrt{2}$$

What is the stochastic volatility model when time is measured in years? (*Hint*: The variable u_{n-1} is the return on the asset price in time Δt. It can be assumed to be normally distributed with mean zero and standard deviation σ_{n-1}. It follows that the mean of u_{n-1}^2 and u_{n-1}^4 are σ_{n-1}^2 and $3\sigma_{n-1}^4$, respectively.)

Assignment Questions

19.15. Suppose that the price of gold at close of trading yesterday was \$300 and its volatility was estimated as 1.3% per day. The price at the close of trading today is \$298. Update the volatility estimate using
(a) The EWMA model with $\lambda = 0.94$
(b) The GARCH(1,1) model with $\omega = 0.000002$, $\alpha = 0.04$, and $\beta = 0.94$

19.16. Suppose that in Problem 19.15 the price of silver at the close of trading yesterday was \$8, its volatility was estimated as 1.5% per day, and its correlation with gold was estimated as 0.8. The price of silver at the close of trading today is unchanged at \$8. Update the volatility of silver and the correlation between silver and gold using the two models in Problem 19.15. In practice, is the ω parameter likely to be the same for gold and silver?

19.17. An Excel spreadsheet containing over 900 days of daily data on a number of different exchange rates and stock indices can be downloaded from the author's website:

http://www.rotman.utoronto.ca/~hull

Choose one exchange rate and one stock index. Estimate the value of λ in the EWMA model that minimizes the value of $\sum_i (v_i - \beta_i)^2$, where v_i is the variance forecast made at the end of day $i-1$ and β_i is the variance calculated from data between day i and day $i+25$. Use the Solver tool in Excel. Set the variance forecast at the end of the first day equal to the square of the return on that day to start the EWMA calculations.

19.18. Suppose that the parameters in a GARCH (1,1) model are $\alpha = 0.03$, $\beta = 0.95$, and $\omega = 0.000002$.
(a) What is the long-run average volatility?
(b) If the current volatility is 1.5% per day, what is your estimate of the volatility in 20, 40, and 60 days?
(c) What volatility should be used to price 20-, 40-, and 60-day options?
(d) Suppose that there is an event that increases the current volatility by 0.5% to 2% per day. Estimate the effect on the volatility in 20, 40, and 60 days.
(e) Estimate by how much the event increases the volatilities used to price 20-, 40-, and 60-day options? Make estimates using both equation (19.14) and equation (19.15).