CHAPTER 16

Volatility Smiles

How close are the market prices of options to those predicted by Black–Scholes? Do traders really use Black–Scholes when determining a price for an option? Are the probability distributions of asset prices really lognormal? In this chapter we answer these questions. We explain that traders do use the Black–Scholes model—but not in exactly the way that Black and Scholes originally intended. This is because they allow the volatility used to price an option to depend on its strike price and time to maturity.

A plot of the implied volatility of an option as a function of its strike price is known as a *volatility smile*. In this chapter we describe the volatility smiles that traders use in equity and foreign currency markets. We explain the relationship between a volatility smile and the risk-neutral probability distribution being assumed for the future asset price. We also discuss how option traders allow volatility to be a function of option maturity and how they use volatility surfaces as pricing tools.

16.1 PUT–CALL PARITY REVISITED

Put–call parity, which we explained in Chapter 9, provides a good starting point for understanding volatility smiles. It is an important relationship between the price c of a European call and the price p of a European put:

$$p + S_0 e^{-qT} = c + Ke^{-rT} \tag{16.1}$$

The call and the put have the same strike price, K, and time to maturity, T. The variable S_0 is the price of the underlying asset today, r is the risk-free interest rate for maturity T, and q is the yield on the asset.

A key feature of the put–call parity relationship is that it is based on a relatively simple no-arbitrage argument. It does not require any assumption about the probability distribution of the asset price in the future. It is true both when the asset price distribution is lognormal and when it is not lognormal.

Suppose that, for a particular value of the volatility, p_{BS} and c_{BS} are the values of European put and call options calculated using the Black–Scholes model. Suppose further that p_{mkt} and c_{mkt} are the market values of these options. Because put–call parity holds for the Black–Scholes model, we must have

$$p_{\text{BS}} + S_0e^{-qT} = c_{\text{BS}} + Ke^{-rT}$$

In the absence of arbitrage opportunities, it also holds for the market prices, so that

$$p_{\text{mkt}} + S_0e^{-qT} = c_{\text{mkt}} + Ke^{-rT}$$

Subtracting these two equations, we get

$$p_{\text{BS}} - p_{\text{mkt}} = c_{\text{BS}} - c_{\text{mkt}} \tag{16.2}$$

This shows that the dollar pricing error when the Black–Scholes model is used to price a European put option should be exactly the same as the dollar pricing error when it is used to price a European call option with the same strike price and time to maturity.

Suppose that the implied volatility of the put option is 22%. This means that $p_{\text{BS}} = p_{\text{mkt}}$ when a volatility of 22% is used in the Black–Scholes model. From equation (16.2), it follows that $c_{\text{BS}} = c_{\text{mkt}}$ when this volatility is used. The implied volatility of the call is, therefore, also 22%. This argument shows that the implied volatility of a European call option is always the same as the implied volatility of a European put option when the two have the same strike price and maturity date. To put this another way, for a given strike price and maturity, the correct volatility to use in conjunction with the Black–Scholes model to price a European call should always be the same as that used to price a European put. This is also approximately true for American options. It follows that when traders refer to the relationship between implied volatility and strike price, or to the relationship between implied volatility and maturity, they do not need to state whether they are talking about calls or puts. The relationship is the same for both.

Example 16.1

The value of the Australian dollar is $0.60. The risk-free interest rate is 5% per annum in the United States and 10% per annum in Australia. The market price of a European call option on the Australian dollar with a maturity of 1 year and a strike price of $0.59 is 0.0236. DerivaGem shows that the implied volatility of the call is 14.5%. For there to be no arbitrage, the put–call parity relationship in equation (16.1) must apply with q equal to the foreign risk-free rate. The price p of a European put option with a strike price of $0.59 and maturity of 1 year therefore satisfies

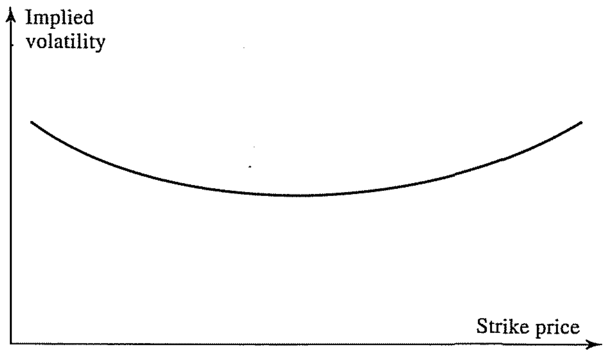
$$p + 0.60e^{-0.10\times 1} = 0.0236 + 0.59e^{-0.05\times 1}$$

so that $p = 0.0419$. DerivaGem shows that, when the put has this price, its implied volatility is also 14.5%. This is what we expect from the analysis just given.

16.2 FOREIGN CURRENCY OPTIONS

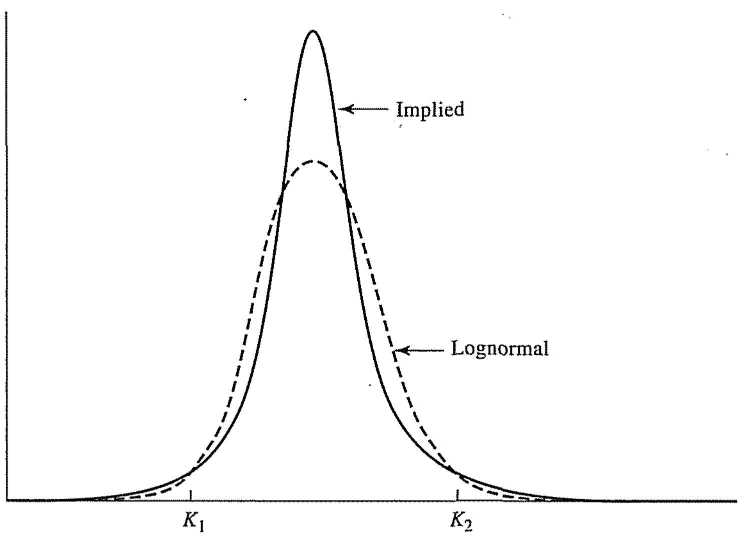
The volatility smile used by traders to price foreign currency options has the general form shown in Figure 16.1. The volatility is relatively low for at-the-money options. It becomes progressively higher as an option moves either into the money or out of the money.

Figure 16.1 Volatility smile for foreign currency options.



In the appendix at the end of this chapter we show how to determine the risk-neutral probability distribution for an asset price at a future time from the volatility smile given by options maturing at that time. We refer to this as the *implied distribution*. The volatility smile in Figure 16.1 corresponds to the probability distribution shown by the solid line in Figure 16.2. A lognormal distribution with the same mean and standard deviation as the implied distribution is shown by the dashed line in Figure 16.2. It can be seen that the implied distribution has heavier tails than the lognormal distribution.[1]

Figure 16.2 Implied and lognormal distribution for foreign currency options.



[1] This is known as *kurtosis*. Note that, in addition to having a heavier tail, the implied distribution is more "peaked". Both small and large movements in the exchange rate are more likely than with the lognormal distribution. Intermediate movements are less likely.

To see that Figures 16.1 and 16.2 are consistent with each other, consider first a deep-out-of-the-money call option with a high strike price of K_2. This option pays off only if the exchange rate proves to be above K_2. Figure 16.2 shows that the probability of this is higher for the implied probability distribution than for the lognormal distribution. We therefore expect the implied distribution to give a relatively high price for the option. A relatively high price leads to a relatively high implied volatility—and this is exactly what we observe in Figure 16.1 for the option. The two figures are therefore consistent with each other for high strike prices. Consider next a deep-out-of-the-money put option with a low strike price of K_1. This option pays off only if the exchange rate proves to be below K_1. Figure 16.2 shows that the probability of this is also higher for implied probability distribution than for the lognormal distribution. We therefore expect the implied distribution to give a relatively high price, and a relatively high implied volatility, for this option as well. Again, this is exactly what we observe in Figure 16.1.

Empirical Results

We have just shown that the smile used by traders for foreign currency options implies that they consider that the lognormal distribution understates the probability of extreme movements in exchange rates. To test whether they are right, Table 16.1 examines the daily movements in 12 different exchange rates over a 10-year period.[2] The first step in the production of the table is to calculate the standard deviation of daily percentage change in each exchange rate. The next stage is to note how often the actual percentage change exceeded one standard deviation, two standard deviations, and so on. The final stage is to calculate how often this would have happened if the percentage changes had been normally distributed. (The lognormal model implies that percentage changes are almost exactly normally distributed over a one-day time period.)

Daily changes exceed three standard deviations on 1.34% of days. The lognormal model predicts that this should happen on only 0.27% of days. Daily changes exceed four, five, and six standard deviations on 0.29%, 0.08%, and 0.03% of days, respectively. The lognormal model predicts that we should hardly ever observe this happening. The table therefore provides evidence to support the existence of heavy tails and the volatility smile used by traders. Business Snapshot 16.1 shows how you

Table 16.1 Percentage of days when daily exchange rate moves are greater than one, two,..., six standard deviations (SD = standard deviation of daily change).

	Real world	*Lognormal model*
>1 SD	25.04	31.73
>2 SD	5.27	4.55
>3 SD	1.34	0.27
>4 SD	0.29	0.01
>5 SD	0.08	0.00
>6 SD	0.03	0.00

[2] This table is taken from J. C. Hull and A. White, "Value at Risk When Daily Changes in Market Variables Are Not Normally Distributed." *Journal of Derivatives*, 5, No. 3 (Spring 1998): 9–19.

Business Snapshot 16.1 Making Money from Foreign Currency Options

Suppose that most market participants think that exchange rates are lognormally distributed. They will be comfortable using the same volatility to value all options on a particular exchange rate. You have just done the analysis in Table 16.1 and know that the lognormal assumption is not a good one for exchange rates. What should you do?

The answer is that you should buy deep-out-the-money call and put options on a variety of different currencies and wait. These options will be relatively inexpensive and more of them will close in the money than the lognormal model predicts. The present value of your payoffs will on average be much greater than the cost of the options.

In the mid-1980s a few traders knew about the heavy tails of foreign exchange probability distributions. Everyone else thought that the lognormal assumption of Black–Scholes was reasonable. The few traders who were well informed followed the strategy we have described—and made lots of money. By the late 1980s everyone realized that foreign currency options should be priced with a volatility smile and the trading opportunity disappeared.

could have made money if you had done the analysis in Table 16.1 ahead of the rest of the market.

Reasons for the Smile in Foreign Currency Options

Why are exchange rates not lognormally distributed? Two of the conditions for an asset price to have a lognormal distribution are:

1. The volatility of the asset is constant.
2. The price of the asset changes smoothly with no jumps.

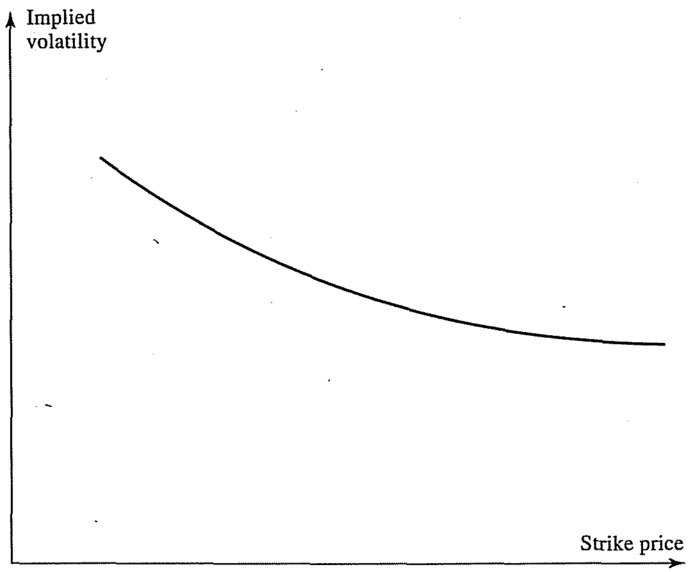
In practice, neither of these conditions is satisfied for an exchange rate. The volatility of an exchange rate is far from constant, and exchange rates frequently exhibit jumps.[3] It turns out that the effect of both a nonconstant volatility and jumps is that extreme outcomes become more likely. The impact of jumps and nonconstant volatility depends on the option maturity. The percentage impact of a nonconstant volatility on prices becomes more pronounced as the maturity of the option is increased, but the volatility smile created by the nonconstant volatility usually becomes less pronounced. The percentage impact of jumps on both prices and the volatility smile becomes less pronounced as the maturity of the option is increased. When we look at sufficiently long-dated options, jumps tend to get "averaged out" so that the stock price distribution when there are jumps is almost indistinguishable from the one obtained when the stock price changes smoothly.

16.3 EQUITY OPTIONS

The volatility smile for equity options has been studied by Rubinstein (1985, 1994) and Jackwerth and Rubinstein (1996). Prior to 1987 there was no marked volatility smile.

[3] Often the jumps are in response to the actions of central banks.

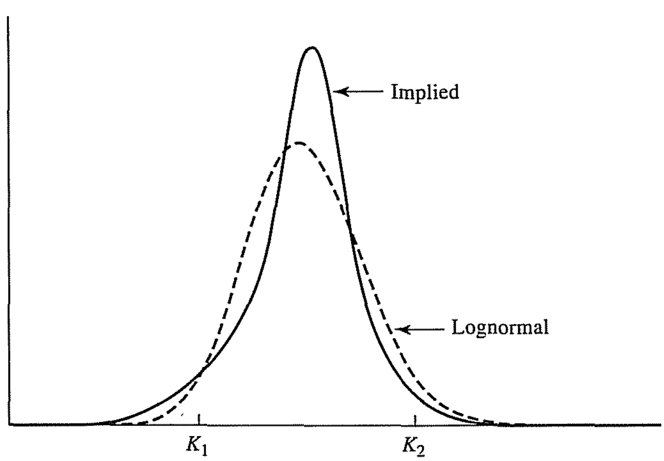
Figure 16.3 Volatility smile for equities.



Since 1987 the volatility smile used by traders to price equity options (both on individual stocks and on stock indices) has the general form shown in Figure 16.3. This is sometimes referred to as a *volatility skew*. The volatility decreases as the strike price increases. The volatility used to price a low-strike-price option (i.e., a deep-out-of-the-money put or a deep-in-the-money call) is significantly higher than that used to price a high-strike-price option (i.e., a deep-in-the-money put or a deep-out-of-the-money call).

The volatility smile for equity options corresponds to the implied probability distribution given by the solid line in Figure 16.4. A lognormal distribution with the same

Figure 16.4 Implied distribution and lognormal distribution for equity options.



Business Snapshot 16.2 Crashophobia

It is interesting that the pattern in Figure 16.3 for equities has existed only since the stock market crash of October 1987. Prior to October 1987, implied volatilities were much less dependent on strike price. This has led Mark Rubinstein to suggest that one reason for the equity volatility smile may be "crashophobia". Traders are concerned about the possibility of another crash similar to October 1987, and they price options accordingly.

There is some empirical support for this explanation. Declines in the S&P 500 tend to be accompanied by a steepening of the volatility skew. When the S&P increases, the skew tends to become less steep.

mean and standard deviation as the implied distribution is shown by the dotted line. It can be seen that the implied distribution has a heavier left tail and a less heavy right tail than the lognormal distribution.

To see that Figures 16.3 and 16.4 are consistent with each other, we proceed as for Figures 16.1 and 16.2 and consider options that are deep out of the money. From Figure 16.4 a deep-out-of-the-money call with a strike price of K_2 has a lower price when the implied distribution is used than when the lognormal distribution is used. This is because the option pays off only if the stock price proves to be above K_2, and the probability of this is lower for the implied probability distribution than for the lognormal distribution. Therefore, we expect the implied distribution to give a relatively low price for the option. A relatively low price leads to a relatively low implied volatility—and this is exactly what we observe in Figure 16.4 for the option. Consider next a deep-out-of-the-money put option with a strike price of K_1. This option pays off only if the stock price proves to be below K_1. Figure 16.3 shows that the probability of this is higher for implied probability distribution than for the lognormal distribution. We therefore expect the implied distribution to give a relatively high price, and a relatively high implied volatility, for this option. Again, this is exactly what we observe in Figure 16.3.

The Reason for the Smile in Equity Options

One possible explanation for the smile in equity options concerns leverage. As a company's equity declines in value, the company's leverage increases. This means that the equity becomes more risky and its volatility increases. As a company's equity increases in value, leverage decreases. The equity then becomes less risky and its volatility decreases. This argument shows that we can expect the volatility of equity to be a decreasing function of price and is consistent with Figures 16.3 and 16.4. Another explanation is crashophobia (see Business Snapshot 16.2).

16.4 THE VOLATILITY TERM STRUCTURE AND VOLATILITY SURFACES

In addition to a volatility smile, traders use a volatility term structure when pricing options. This means that the volatility used to price an at-the-money option depends on the maturity of the option. Volatility tends to be an increasing function of maturity when short-dated volatilities are historically low. This is because there is then an expectation that volatilities will increase. Similarly, volatility tends to be an decreasing

Table 16.2 Volatility surface.

	Strike price				
	0.90	*0.95*	*1.00*	*1.05*	*1.10*
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

function of maturity when short-dated volatilities are historically high. This is because there is then an expectation that volatilities will decrease.

Volatility surfaces combine volatility smiles with the volatility term structure to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity. An example of a volatility surface that might be used for foreign currency options is given in Table 16.2.

One dimension of Table 16.2 is strike price; the other is time to maturity. The main body of the table shows implied volatilities calculated from the Black–Scholes model. At any given time, some of the entries in the table are likely to correspond to options for which reliable market data are available. The implied volatilities for these options are calculated directly from their market prices and entered into the table. The rest of the table is determined using linear interpolation.

When a new option has to be valued, financial engineers look up the appropriate volatility in the table. For example, when valuing a 9-month option with a strike price of 1.05, a financial engineer would interpolate between 13.4 and 14.0 in Table 16.2 to obtain a volatility of 13.7%. This is the volatility that would be used in the Black–Scholes formula or a binomial tree.

The shape of the volatility smile depends on the option maturity. As illustrated in Table 16.2, the smile tends to become less pronounced as the option maturity increases. Define T as the time to maturity and F_0 as the forward price of the asset. Some financial engineers choose to define the volatility smile as the relationship between implied volatility and

$$\frac{1}{\sqrt{T}}\ln\left(\frac{K}{F_0}\right)$$

rather than as the relationship between the implied volatility and K. The smile is then usually much less dependent on the time to maturity.[4]

The Role of the Model

How important is the pricing model if traders are prepared to use a different volatility for every option? It can be argued that the Black–Scholes model is no more than a sophisticated interpolation tool used by traders for ensuring that an option is priced

[4] For a discussion of this approach, see S. Natenberg *Option Pricing and Volatility: Advanced Trading Strategies and Techniques*, 2nd edn. McGraw-Hill, 1994; R. Tompkins *Options Analysis: A State of the Art Guide to Options Pricing*, Burr Ridge, IL: Irwin, 1994.

consistently with the market prices of other actively traded options. If traders stopped using Black–Scholes and switched to another plausible model, then the volatility surface and the shape of the smile would change, but arguably the dollar prices quoted in the market would not change appreciably.

16.5 GREEK LETTERS

The volatility smile complicates the calculation of Greek letters. Derman describes a number of volatility regimes or rules of thumb that are sometimes assumed by traders.[5] The simplest of these is known as the *sticky strike rule*. This assumes that the implied volatility of an option remains constant from one day to the next. It means that Greek letters calculated using the Black–Scholes assumptions are correct provided that the volatility used for an option is its current implied volatility.

A more complicated rule is known as the *sticky delta* rule. This assumes that the relationship we observe between an option price and S/K today will apply tomorrow. As the price of the underlying asset changes, the implied volatility of the option is assumed to change to reflect the option's "moneyness" (i.e., the extent to which it is in or out of the money). If we use the sticky delta rule, the formulas for Greek letters given in the Chapter 15 are no longer correct. For example, delta of a call option is given by

$$\frac{\partial c_{\text{BS}}}{\partial S} + \frac{\partial c_{\text{BS}}}{\partial \sigma_{\text{imp}}} \frac{\partial \sigma_{\text{imp}}}{\partial S}$$

where c_{BS} is the Black-Scholes price of the option expressed as a function of the asset price S and the implied volatility σ_{imp}. Consider the impact of this formula on the delta of an equity call option. From Figure 16.3, volatility is a decreasing function of the strike price K. Alternatively it can be regarded as an increasing function of S/K. Under the sticky delta model, therefore, the volatility increases as the asset price increases, so that

$$\frac{\partial \sigma_{\text{imp}}}{\partial S} > 0$$

As a result, delta is higher than that given by the Black–Scholes assumptions.

It turns out that the sticky strike and sticky delta rules do not correspond to internally consistent models (except when the volatility smile is flat for all maturities). A model that can be made exactly consistent with the smiles is known as the *implied volatility function* model or the *implied tree model*. We will explain this model in Chapter 24.

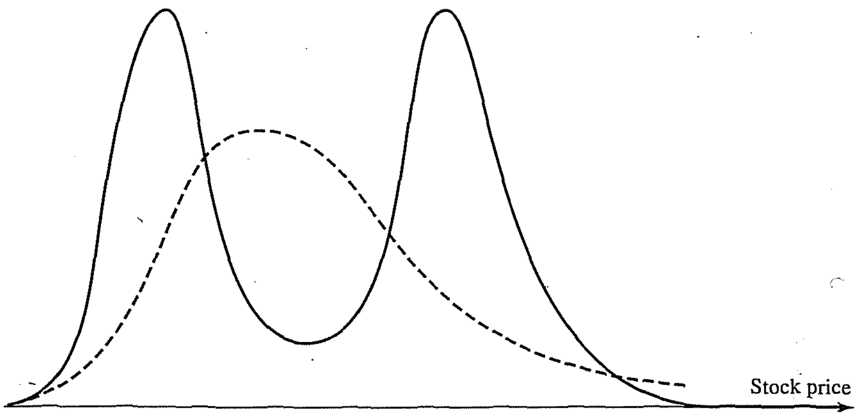
In practice, banks try to ensure that their exposure to the most commonly observed changes in the volatility surface is reasonably small. One technique for identifying these changes is principal components analysis, which we discuss in Chapter 18.

16.6 WHEN A SINGLE LARGE JUMP IS ANTICIPATED

Let us now consider an example of how an unusual volatility smile might arise in equity markets. Suppose that a stock price is currently \$50 and an important news

[5] See E. Derman, "Regimes of Volatility," *Risk*, April 1999, 54–59

Figure 16.5 Effect of a single large jump. The solid line is the true distribution; the dashed line is the lognormal distribution.



announcement due in a few days is expected either to increase the stock price by \$8 or to reduce it by \$8. (This announcement could concern the outcome of a takeover attempt or the verdict in an important lawsuit.) The probability distribution of the stock price in, say, 1 month might then consist of a mixture of two lognormal distributions, the first corresponding to favorable news, the second to unfavorable news. The situation is illustrated in Figure 16.5. The solid line shows the mixtures-of-lognormals distribution for the stock price in 1 month; the dashed line shows a lognormal distribution with the same mean and standard deviation as this distribution.

The true probability distribution is bimodal (certainly not lognormal). One easy way to investigate the general effect of a bimodal stock price distribution is to consider the extreme case where the distribution is binomial. This is what we will now do.

Suppose that the stock price is currently \$50 and that it is known that in 1 month it will be either \$42 or \$58. Suppose further that the risk-free rate is 12% per annum. The situation is illustrated in Figure 16.6 . Options can be valued using the binomial model from Chapter 11. In this case $u = 1.16, d = 0.84, a = 1.0101$, and $p = 0.5314$. The results from valuing a range of different options are shown in Table 16.3. The first column shows alternative strike prices; the second column shows prices of 1-month European call options; the third column shows the prices of one-month European put option prices;

Figure 16.6 Change in stock price in 1 month.

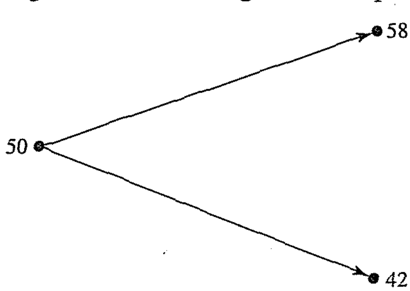


Table 16.3 Implied volatilities in situation where true distribution is binomial.

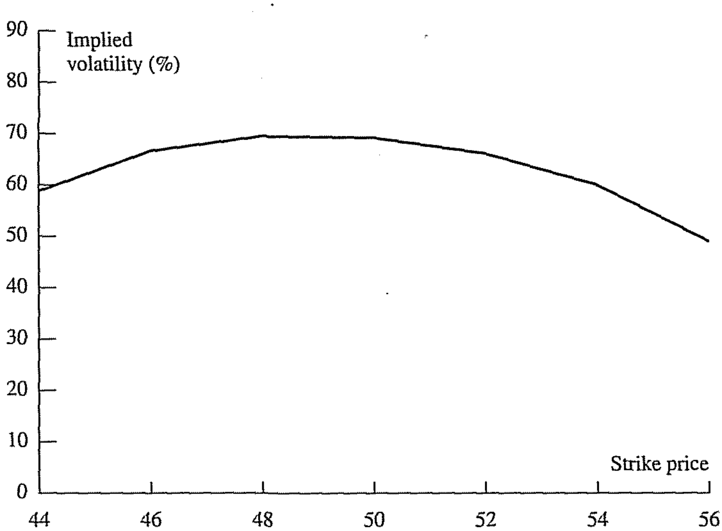
Strike price ($)	*Call price* ($)	*Put price* ($)	*Implied volatility* (%)
42	8.42	0.00	0.0
44	7.37	0.93	58.8
46	6.31	1.86	66.6
48	5.26	2.78	69.5
50	4.21	3.71	69.2
52	3.16	4.64	66.1
54	2.10	5.57	60.0
56	1.05	6.50	49.0
58	0.00	7.42	0.0

the fourth column shows implied volatilities. (As shown in Section 16.1, the implied volatility of a European put option is the same as that of a European call option when they have the same strike price and maturity.) Figure 16.7 shows the volatility smile. It is actually a "frown" (the opposite of that observed for currencies) with volatilities declining as we move out of or into the money. The volatility implied from an option with a strike price of 50 will overprice an option with a strike price of 44 or 56.

SUMMARY

The Black–Scholes model and its extensions assume that the probability distribution of the underlying asset at any given future time is lognormal. This assumption is not the

Figure 16.7 Volatility smile for situation in Table 16.3.



one made by traders. They assume the probability distribution of an equity price has a heavier left tail and a less heavy right tail than the lognormal distribution. They also assume that the probability distribution of an exchange rate has a heavier right tail and a heavier left tail than the lognormal distribution.

Traders use volatility smiles to allow for nonlognormality. The volatility smile defines the relationship between the implied volatility of an option and its strike price. For equity options, the volatility smile tends to be downward sloping. This means that out-of-the-money puts and in-the-money calls tend to have high implied volatilities whereas out-of-the-money calls and in-the-money puts tend to have low implied volatilities. For foreign currency options, the volatility smile is U-shaped. Both out-of-the-money and in-the-money options have higher implied volatilities than at-the-money options.

Often traders also use a volatility term structure. The implied volatility of an option then depends on its life. When volatility smiles and volatility term structures are combined, they produce a volatility surface. This defines implied volatility as a function of both the strike price and the time to maturity.

FURTHER READING

Bakshi, G., C. Cao, and Z. Chen. "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 52, No. 5 (December 1997): 2004–49.

Bates, D.S. "Post-'87 Crash Fears in the S&P Futures Market," *Journal of Econometrics*, 94 (January/February 2000): 181–238.

Derman, E. "Regimes of Volatility," *Risk*, April 1999: 55–59.

Ederington, L.H., and W. Guan. "Why Are Those Options Smiling," *Journal of Derivatives*, 10, 2 (2002): 9–34.

Jackwerth, J.C., and M. Rubinstein. "Recovering Probability Distributions from Option Prices," *Journal of Finance*, 51 (December 1996): 1611–31.

Lauterbach, B., and P. Schultz. "Pricing Warrants: An Empirical Study of the Black–Scholes Model and Its Alternatives," *Journal of Finance*, 4, No. 4 (September 1990): 1181–1210.

Melick, W.R., and C.P. Thomas. "Recovering an Asset's Implied Probability Density Function from Option Prices: An Application to Crude Oil during the Gulf Crisis," *Journal of Financial and Quantitative Analysis*, 32, 1 (March 1997): 91–115.

Rubinstein, M. "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976, through August 31, 1978," *Journal of Finance*, 40 (June 1985): 455–80.

Rubinstein, M. "Implied Binomial Trees," *Journal of Finance*, 49, 3 (July 1994): 771–818.

Xu, X., and S.J. Taylor. "The Term Structure of Volatility Implied by Foreign Exchange Options," *Journal of Financial and Quantitative Analysis*, 29 (1994): 57–74.

Questions and Problems (Answers in Solutions Manual)

16.1. What volatility smile is likely to be observed when:

(a) Both tails of the stock price distribution are less heavy than those of the lognormal distribution?

(b) The right tail is heavier, and the left tail is less heavy, than that of a lognormal distribution?

16.2. What volatility smile is observed for equities?

16.3. What volatility smile is likely to be caused by jumps in the underlying asset price? Is the pattern likely to be more pronounced for a 2-year option than for a 3-month option?

16.4. A European call and put option have the same strike price and time to maturity. The call has an implied volatility of 30% and the put has an implied volatility of 25%. What trades would you do?

16.5. Explain carefully why a distribution with a heavier left tail and less heavy right tail than the lognormal distribution gives rise to a downward sloping volatility smile.

16.6. The market price of a European call is \$3.00 and its price given by Black–Scholes model with a volatility of 30% is \$3.50. The price given by this Black–Scholes model for a European put option with the same strike price and time to maturity is \$1.00. What should the market price of the put option be? Explain the reasons for your answer.

16.7. Explain what is meant by "crashophobia".

16.8. A stock price is currently \$20. Tomorrow, news is expected to be announced that will either increase the price by \$5 or decrease the price by \$5. What are the problems in using Black–Scholes to value 1-month options on the stock?

16.9. What volatility smile is likely to be observed for 6-month options when the volatility is uncertain and positively correlated to the stock price?

16.10. What problems do you think would be encountered in testing a stock option pricing model empirically?

16.11. Suppose that a central bank's policy is to allow an exchange rate to fluctuate between 0.97 and 1.03. What pattern of implied volatilities for options on the exchange rate would you expect to see?

16.12. Option traders sometimes refer to deep-out-of-the-money options as being options on volatility. Why do you think they do this?

16.13. A European call option on a certain stock has a strike price of \$30, a time to maturity of 1 year, and an implied volatility of 30%. A European put option on the same stock has a strike price of \$30, a time to maturity of 1 year, and an implied volatility of 33%. What is the arbitrage opportunity open to a trader? Does the arbitrage work only when the lognormal assumption underlying Black–Scholes holds? Explain carefully the reasons for your answer.

16.14. Suppose that the result of a major lawsuit affecting Microsoft is due to be announced tomorrow. Microsoft's stock price is currently \$60. If the ruling is favorable to Microsoft, the stock price is expected to jump to \$75. If it is unfavorable, the stock is expected to jump to \$50. What is the risk-neutral probability of a favorable ruling? Assume that the volatility of Microsoft's stock will be 25% for 6 months after the ruling if the ruling is favorable and 40% if it is unfavorable. Use DerivaGem to calculate the relationship between implied volatility and strike price for 6-month European options on Microsoft today. Microsoft does not pay dividends. Assume that the 6-month risk-free rate is 6%. Consider call options with strike prices of 30, 40, 50, 60, 70, and 80.

16.15. An exchange rate is currently 0.8000. The volatility of the exchange rate is quoted as 12% and interest rates in the two countries are the same. Using the lognormal assumption, estimate the probability that the exchange rate in 3 months will be (a) less than 0.7000, (b) between 0.7000 and 0.7500, (c) between 0.7500 and 0.8000, (d) between

0.8000 and 0.8500, (e) between 0.8500 and 0.9000, and (f) greater than 0.9000. Based on the volatility smile usually observed in the market for exchange rates, which of these estimates would you expect to be too low and which would you expect to be too high?

16.16. A stock price is \$40. A 6-month European call option on the stock with a strike price of \$30 has an implied volatility of 35%. A 6-month European call option on the stock with a strike price of \$50 has an implied volatility of 28%. The 6-month risk-free rate is 5% and no dividends are expected. Explain why the two implied volatilities are different. Use DerivaGem to calculate the prices of the two options. Use put–call parity to calculate the prices of 6-month European put options with strike prices of \$30 and \$50. Use DerivaGem to calculate the implied volatilities of these two put options.

16.17. "The Black–Scholes model is used by traders as an interpolation tool." Discuss this view.

Assignment Questions

16.18. A company's stock is selling for \$4. The company has no outstanding debt. Analysts consider the liquidation value of the company to be at least \$300,000 and there are 100,000 shares outstanding. What volatility smile would you expect to see?

16.19. A company is currently awaiting the outcome of a major lawsuit. This is expected to be known within 1 month. The stock price is currently \$20. If the outcome is positive, the stock price is expected to be \$24 at the end of 1 month. If the outcome is negative, it is expected to be \$18 at this time. The 1-month risk-free interest rate is 8% per annum.
(a) What is the risk-neutral probability of a positive outcome?
(b) What are the values of 1-month call options with strike prices of \$19, \$20, \$21, \$22, and \$23?
(c) Use DerivaGem to calculate a volatility smile for 1-month call options.
(d) Verify that the same volatility smile is obtained for 1-month put options.

16.20. A futures price is currently \$40. The risk-free interest rate is 5%. Some news is expected tomorrow that will cause the volatility over the next 3 months to be either 10% or 30%. There is a 60% chance of the first outcome and a 40% chance of the second outcome. Use DerivaGem to calculate a volatility smile for 3-month options.

16.21. Data for a number of foreign currencies are provided on the author's website:

http://www.rotman.utoronto.ca/~hull

Choose a currency and use the data to produce a table similar to Table 16.1.

16.22. Data for a number of stock indices are provided on the author's website:

http://www.rotman.utoronto.ca/~hull

Choose an index and test whether a three-standard-deviation down movement happens more often than a three-standard-deviation up movement.

16.23. Consider a European call and a European put with the same strike price and time to maturity. Show that they change in value by the same amount when the volatility increases from a level σ_1 to a new level σ_2 within a short period of time. (*Hint*: Use put–call parity.)

APPENDIX

DETERMINING IMPLIED RISK-NEUTRAL DISTRIBUTIONS FROM VOLATILITY SMILES

The price of a European call option on an asset with strike price K and maturity T is given by

$$c = e^{-rT} \int_{S_T=K}^{\infty} (S_T - K)\, g(S_T)\, dS_T$$

where r is the interest rate (assumed constant), S_T is the asset price at time T, and g is the risk-neutral probability density function of S_T. Differentiating once with respect to K, we obtain

$$\frac{\partial c}{\partial K} = -e^{-rT} \int_{S_T=K}^{\infty} g(S_T)\, dS_T$$

Differentiating again with respect to K, we have

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT}\, g(K)$$

This shows that the probability density function g is given by

$$g(K) = e^{rT}\, \frac{\partial^2 c}{\partial K^2}$$

This result, which is from Breeden and Litzenberger (1978), allows risk-neutral probability distributions to be estimated from volatility smiles.[6] Suppose that c_1, c_2, and c_3 are the prices of T-year European call options with strike prices of $K - \delta$, K, and $K + \delta$, respectively. Assuming δ is small, an estimate of $g(K)$ is

$$e^{rT}\, \frac{c_1 + c_3 - 2c_2}{\delta^2}$$

[6] See D. T. Breeden and R. H. Litzenberger, "Prices of State-Contingent Claims Implicit in Option Prices," *Jounal of Business*, 51 (1978), 621–51.