CHAPTER 9

Properties of Stock Options

In this chapter we look at the factors affecting stock option prices. We use a number of different arbitrage arguments to explore the relationships between European option prices, American option prices, and the underlying stock price. The most important of these relationships is put–call parity, which is a relationship between European call option prices and European put option prices.

The chapter examines whether American options should be exercised early. It shows that it is never optimal to exercise an American call option on a non-dividend-paying stock prior to the option's expiration, but that under some circumstances the early exercise of an American put option on such a stock is optimal.

9.1 FACTORS AFFECTING OPTION PRICES

There are six factors affecting the price of a stock option:

1. The current stock price, S_0
2. The strike price, K
3. The time to expiration, T
4. The volatility of the stock price, σ
5. The risk-free interest rate, r
6. The dividends expected during the life of the option

In this section we consider what happens to option prices when one of these factors changes, with all the others remaining fixed. The results are summarized in Table 9.1.

Figures 9.1 and 9.2 show how European call and put prices depend on the first five factors in the situation where $S_0 = 50$, $K = 50$, $r = 5\%$ per annum, $\sigma = 30\%$ per annum, $T = 1$ year, and there are no dividends. In this case the call price is 7.116 and the put price is 4.677.

Stock Price and Strike Price

If a call option is exercised at some future time, the payoff will be the amount by which the stock price exceeds the strike price. Call options therefore become more valuable as

Table 9.1 Summary of the effect on the price of a stock option of increasing one variable while keeping all others fixed.*

Variable	*European call*	*European put*	*American call*	*American put*
Current stock price	+	–	+	–
Strike price	–	+	–	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	–	+	–
Amount of future dividends	–	+	–	+

* + indicates that an increase in the variable causes the option price to increase; – indicates that an increase in the variable causes the option price to decrease; ? indicates that the relationship is uncertain.

the stock price increases and less valuable as the strike price increases. For a put option, the payoff on exercise is the amount by which the strike price exceeds the stock price. Put options therefore behave in the opposite way from call options: they become less valuable as the stock price increases and more valuable as the strike price increases. Figures 9.1(a–d) illustrate the way in which put and call prices depend on the stock price and strike price.

Time to Expiration

Now consider the effect of the expiration date. Both put and call American options become more valuable as the time to expiration increases. Suppose that we have two American options that differ only as far as the expiration date is concerned. The owner of the long-life option has all the exercise opportunities open to the owner of the short-life option—and more. The long-life option must therefore always be worth at least as much as the short-life option.

Although European put and call options usually become more valuable as the time to expiration increases (see, e.g., Figures 9.1(e, f)), this is not always the case. Consider two European call options on a stock: one with an expiration date in 1 month, the other with an expiration date in 2 months. Suppose that a very large dividend is expected in 6 weeks. The dividend will cause the stock price to decline, so that the short-life option could be worth more than the long-life option.

Volatility

The precise way in which volatility is defined is discussed in Chapter 13. Roughly speaking, the *volatility* of a stock price is a measure of how uncertain we are about future stock price movements. As volatility increases, the chance that the stock will do very well or very poorly increases. For the owner of a stock, these two outcomes tend to offset each other. However, this is not so for the owner of a call or put. The owner of a call benefits from price increases but has limited downside risk in the event of price decreases because the most the owner can lose is the price of the option. Similarly, the owner of a put benefits from price decreases, but has limited downside risk in the event of price increases. The values of both calls and puts therefore increase as volatility increases (see Figures 9.2(a, b)).

Risk-Free Interest Rate

The risk-free interest rate affects the price of an option in a less clear-cut way. As interest rates in the economy increase, the expected return required by investors from the stock

Figure 9.1 Effect of changes in stock price, strike price, and expiration date on option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

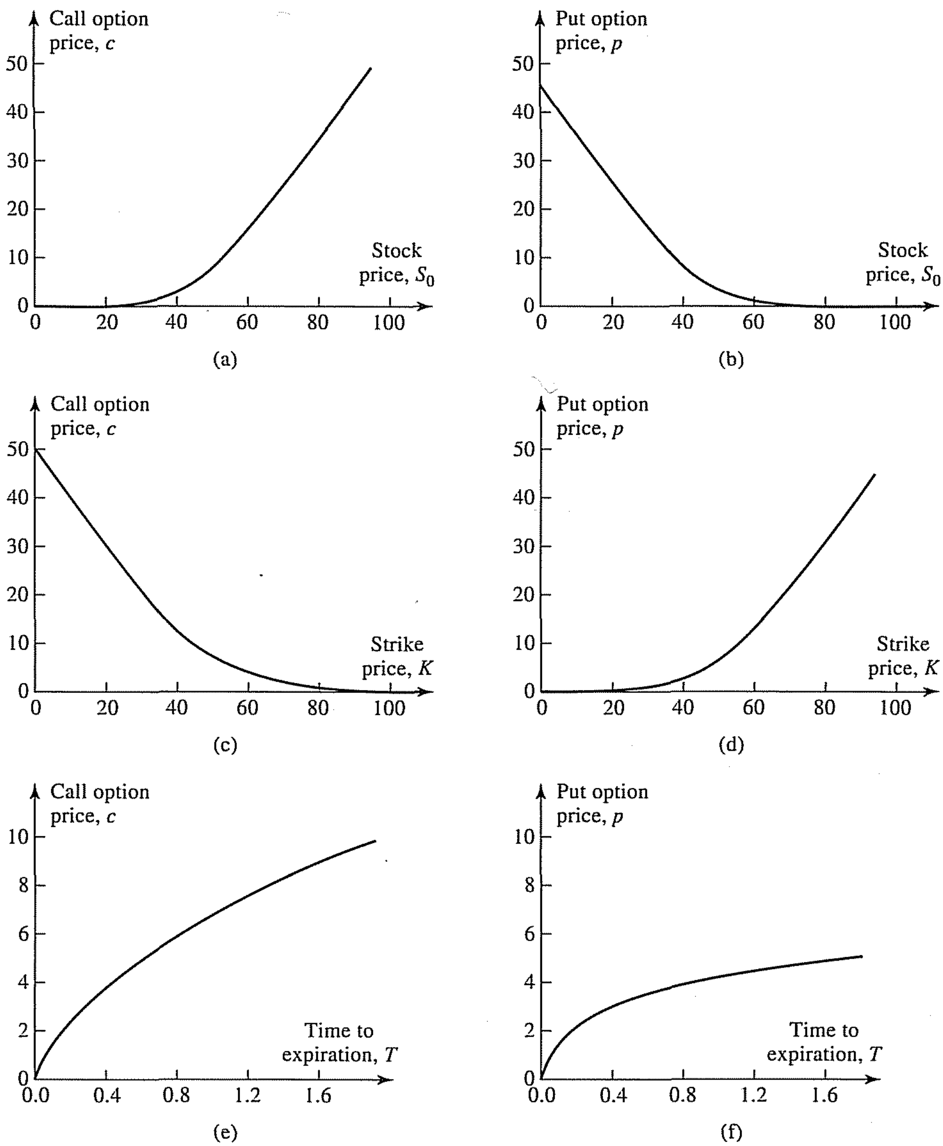
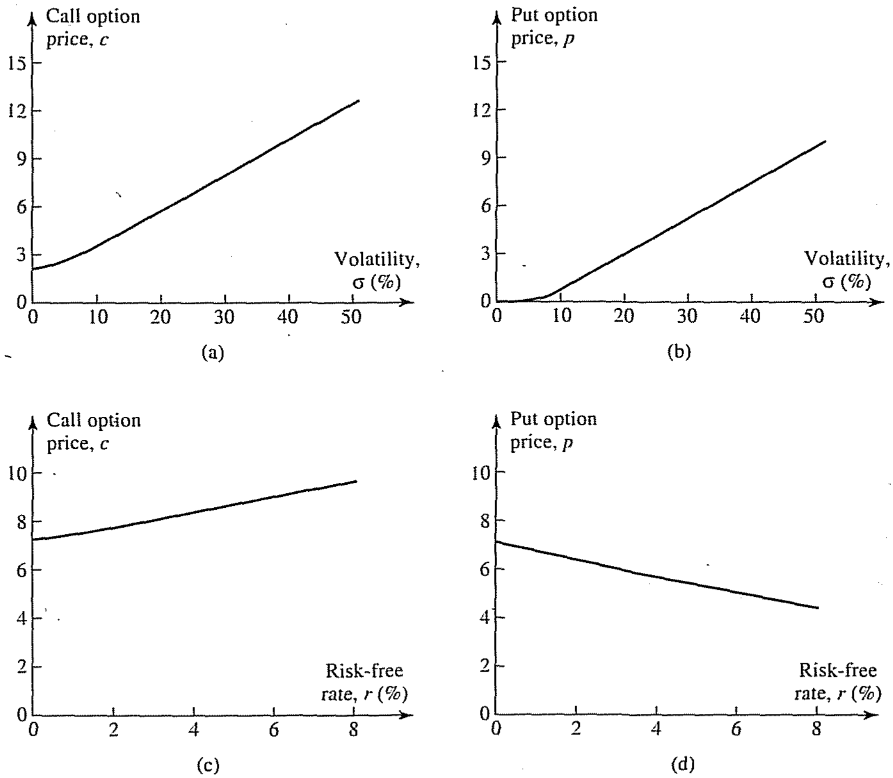


Figure 9.2 Effect of changes in volatility and risk-free interest rate on option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.



tends to increase. In addition, the present value of any future cash flow received by the holder of the option decreases. The combined impact of these two effects is to increase the value of call options and decrease the value of put options (see Figures 9.2(c, d)).

It is important to emphasize that we are assuming that interest rates change while all other variables stay the same. In particular we are assuming that interest rates change while the stock price remains the same. In practice, when interest rates rise (fall), stock prices tend to fall (rise). The net effect of an interest rate increase and the accompanying stock price decrease can be to decrease the value of a call option and increase the value of a put option. Similarly, the net effect of an interest rate decrease and the accompanying stock price increase can be to increase the value of a call option and decrease the value of a put option.

Amount of Future Dividends

Dividends have the effect of reducing the stock price on the ex-dividend date. This is bad news for the value of call options and good news for the value of put options. The value of a call option is therefore negatively related to the size of an anticipated future dividend, and the value of a put option is positively related to the size of an anticipated future dividend.

9.2 ASSUMPTIONS AND NOTATION

In this chapter we will make assumptions similar to those made for deriving forward and futures prices in Chapter 5. We assume that there are some market participants, such as large investment banks, for which the following statements are true:

1. There are no transactions costs.
2. All trading profits (net of trading losses) are subject to the same tax rate.
3. Borrowing and lending are possible at the risk-free interest rate.

We assume that these market participants are prepared to take advantage of arbitrage opportunities as they arise. As discussed in Chapters 1 and 5, this means that any available arbitrage opportunities disappear very quickly. For the purposes of our analysis, it is therefore reasonable to assume that there are no arbitrage opportunities.

We will use the following notation:

S_0: Current stock price
K: Strike price of option
T: Time to expiration of option
S_T: Stock price at maturity
r: Continuously compounded risk-free rate of interest for an investment maturing in time T
C: Value of American call option to buy one share
P: Value of American put option to sell one share
c: Value of European call option to buy one share
p: Value of European put option to sell one share

It should be noted that r is the nominal rate of interest, not the real rate of interest. We can assume that $r > 0$. Otherwise, a risk-free investment would provide no advantages over cash. (Indeed, if $r < 0$, cash would be preferable to a risk-free investment.)

9.3 UPPER AND LOWER BOUNDS FOR OPTION PRICES

In this section we derive upper and lower bounds for option prices. These bounds do not depend on any particular assumptions about the factors mentioned in Section 9.1 (except $r > 0$). If an option price is above the upper bound or below the lower bound, then there are profitable opportunities for arbitrageurs.

Upper Bounds

An American or European call option gives the holder the right to buy one share of a stock for a certain price. No matter what happens, the option can never be worth more than the stock. Hence, the stock price is an upper bound to the option price:

$$c \leqslant S_0 \quad \text{and} \quad C \leqslant S_0$$

If these relationships were not true, an arbitrageur could easily make a riskless profit by buying the stock and selling the call option.

An American or European put option gives the holder the right to sell one share of a stock for K. No matter how low the stock price becomes, the option can never be worth more than K. Hence,

$$p \leqslant K \quad \text{and} \quad P \leqslant K$$

For European options, we know that at maturity the option cannot be worth more than K. It follows that it cannot be worth more than the present value of K today:

$$p \leqslant Ke^{-rT}$$

If this were not true, an arbitrageur could make a riskless profit by writing the option and investing the proceeds of the sale at the risk-free interest rate.

Lower Bound for Calls on Non-Dividend-Paying Stocks

A lower bound for the price of a European call option on a non-dividend-paying stock is

$$S_0 - Ke^{-rT}$$

We first look at a numerical example and then consider a more formal argument.

Suppose that $S_0 = \$20$, $K = \$18$, $r = 10\%$ per annum, and $T = 1$ year. In this case,

$$S_0 - Ke^{-rT} = 20 - 18e^{-0.1} = 3.71$$

or \$3.71. Consider the situation where the European call price is \$3.00, which is less than the theoretical minimum of \$3.71. An arbitrageur can short the stock and buy the call to provide a cash inflow of \$20.00 – \$3.00 = \$17.00. If invested for 1 year at 10% per annum, the \$17.00 grows to $17e^{0.1} = \$18.79$. At the end of the year, the option expires. If the stock price is greater than \$18.00, the arbitrageur exercises the option for \$18.00, closes out the short position, and makes a profit of

$$\$18.79 - \$18.00 = \$0.79$$

If the stock price is less than \$18.00, the stock is bought in the market and the short position is closed out. The arbitrageur then makes an even greater profit. For example, if the stock price is \$17.00, the arbitrageur's profit is

$$\$18.79 - \$17.00 = \$1.79$$

For a more formal argument, we consider the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}

Portfolio B: one share

In portfolio A, the cash, if it is invested at the risk-free interest rate, will grow to K in time T. If $S_T > K$, the call option is exercised at maturity and portfolio A is worth S_T. If $S_T < K$, the call option expires worthless and the portfolio is worth K. Hence, at time T, portfolio A is worth

$$\max(S_T,\ K)$$

Portfolio B is worth S_T at time T. Hence, portfolio A is always worth as much as, and can be worth more than, portfolio B at the option's maturity. It follows that in the absence of arbitrage opportunities this must also be true today. Hence,

$$c + Ke^{-rT} \geqslant S_0$$

or

$$c \geqslant S_0 - Ke^{-rT}$$

Because the worst that can happen to a call option is that it expires worthless, its value cannot be negative. This means that $c \geqslant 0$ and therefore

$$c \geqslant \max(S_0 - Ke^{-rT}, 0) \tag{9.1}$$

Example 9.1

Consider a European call option on a non-dividend-paying stock when the stock price is \$51, the strike price is \$50, the time to maturity is 6 months, and the risk-free rate of interest is 12% per annum. In this case, $S_0 = 51$, $K = 50$, $T = 0.5$, and $r = 0.12$. From equation (9.1), a lower bound for the option price is $S_0 - Ke^{-rT}$, or

$$51 - 50e^{-0.12\times0.5} = \$3.91$$

Lower Bound for European Puts on Non-Dividend-Paying Stocks

For a European put option on a non-dividend-paying stock, a lower bound for the price is

$$Ke^{-rT} - S_0$$

Again, we first consider a numerical example and then look at a more formal argument.

Suppose that $S_0 = \$37$, $K = \$40$, $r = 5\%$ per annum, and $T = 0.5$ years. In this case,

$$Ke^{-rT} - S_0 = 40e^{-0.05\times0.5} - 37 = \$2.01$$

Consider the situation where the European put price is \$1.00, which is less than the theoretical minimum of \$2.01. An arbitrageur can borrow \$38.00 for 6 months to buy both the put and the stock. At the end of the 6 months, the arbitrageur will be required to repay $38e^{0.05\times0.5} = \$38.96$. If the stock price is below \$40.00, the arbitrageur exercises the option to sell the stock for \$40.00, repays the loan, and makes a profit of

$$\$40.00 - \$38.96 = \$1.04$$

If the stock price is greater than \$40.00, the arbitrageur discards the option, sells the stock, and repays the loan for an even greater profit. For example, if the stock price is \$42.00, the arbitrageur's profit is

$$\$42.00 - \$38.96 = \$3.04$$

For a more formal argument, we consider the following two portfolios:

Portfolio C: one European put option plus one share

Portfolio D: an amount of cash equal to Ke^{-rT}

If $S_T < K$, then the option in portfolio C is exercised at option maturity, and the portfolio becomes worth K. If $S_T > K$, then the put option expires worthless, and the portfolio is worth S_T at this time. Hence, portfolio C is worth

$$\max(S_T,\ K)$$

in time T. Assuming the cash is invested at the risk-free interest rate, portfolio D is worth K in time T. Hence, portfolio C is always worth as much as, and can sometimes be worth more than, portfolio D in time T. It follows that in the absence of arbitrage opportunities portfolio C must be worth at least as much as portfolio D today. Hence,

$$p + S_0 \geqslant Ke^{-rT}$$

or

$$p \geqslant Ke^{-rT} - S_0$$

Because the worst that can happen to a put option is that it expires worthless, its value cannot be negative. This means that

$$p \geqslant \max(Ke^{-rT} - S_0,\ 0) \tag{9.2}$$

Example 9.2

Consider a European put option on a non-dividend-paying stock when the stock price is \$38, the strike price is \$40, the time to maturity is 3 months, and the risk-free rate of interest is 10% per annum. In this case $S_0 = 38$, $K = 40$, $T = 0.25$, and $r = 0.10$. From equation (9.2), a lower bound for the option price is $Ke^{-rT} - S_0$, or

$$40e^{-0.1\times0.25} - 38 = \$1.01$$

9.4 PUT–CALL PARITY

We now derive an important relationship between p and c. Consider the following two portfolios that were used in the previous section:

Portfolio A: one European call option plus an amount of cash equal to Ke^{-rT}

Portfolio C: one European put option plus one share

Both are worth

$$\max(S_T,\ K)$$

at expiration of the options. Because the options are European, they cannot be exercised prior to the expiration date. The portfolios must therefore have identical values today. This means that

$$c + Ke^{-rT} = p + S_0 \tag{9.3}$$

This relationship is known as *put–call parity*. It shows that the value of a European call with a certain strike price and exercise date can be deduced from the value of a European put with the same strike price and exercise date, and vice versa.

If equation (9.3) does not hold, there are arbitrage opportunities. Suppose that the stock price is \$31, the strike price is \$30, the risk-free interest rate is 10% per annum, the price of a 3-month European call option is \$3, and the price of a three-month European put option is \$2.25. In this case,

$$c + Ke^{-rT} = 3 + 30e^{-0.1\times3/12} = \$32.26$$

and

$$p + S_0 = 2.25 + 31 = \$33.25$$

Portfolio C is overpriced relative to portfolio A. The correct arbitrage strategy is to buy the securities in portfolio A and short the securities in portfolio C. The strategy involves buying the call and shorting both the put and the stock, generating a positive cash flow of

$$-3 + 2.25 + 31 = \$30.25$$

up front. When invested at the risk-free interest rate, this amount grows to

$$30.25e^{0.1\times 0.25} = \$31.02$$

in 3 months.

If the stock price at expiration of the option is greater than \$30, the call will be exercised; and if it is less than \$30, the put will be exercised. In either case, the investor ends up buying one share for \$30. This share can be used to close out the short position. The net profit is therefore

$$\$31.02 - \$30.00 = \$1.02$$

For an alternative situation, suppose that the call price is \$3 and the put price is \$1. In this case,

$$c + Ke^{-rT} = 3 + 30e^{-0.1\times 3/12} = \$32.26$$

and

$$p + S_0 = 1 + 31 = \$32.00$$

Portfolio A is overpriced relative to portfolio C. An arbitrageur can short the securities in portfolio A and buy the securities in portfolio C to lock in a profit. The strategy involves

Table 9.2 Arbitrage opportunities when put–call parity does not hold. Stock price = \$31; interest rate = 10%; call price = \$3. Both put and call have a strike price of \$30 and 3 months to maturity.

Three-month put price = \$2.25	*Three-month put price = \$1*
Action now:	*Action now*:
Buy call for \$3	Borrow \$29 for 3 months
Short put to realize \$2.25	Short call to realize \$3
Short the stock to realize \$31	Buy put for \$1
Invest \$30.25 for 3 months	Buy the stock for \$31
Action in 3 months if $S_T > 30$:	*Action in 3 months if* $S_T > 30$:
Receive \$31.02 from investment	Call exercised: sell stock for \$30
Exercise call to buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27
Action in 3 months if $S_T < 30$:	*Action in 3 months if* $S_T < 30$:
Receive \$31.02 from investment	Exercise put to sell stock for \$3
Put exercised: buy stock for \$30	Use \$29.73 to repay loan
Net profit = \$1.02	Net profit = \$0.27

Business Snapshot 9.1 Put–Call Parity and Capital Structure

The pioneers of option pricing were Fischer Black, Myron Scholes, and Robert Merton. In the early 1970s, they showed that options can be used to characterize the capital structure of a company. Today this model is widely used by financial institutions to assess a company's credit risk.

To illustrate the model, consider a company that has assets that are financed with zero-coupon bonds and equity. Suppose that the bonds mature in 5 years at which time a principal payment of K is required. The company pays no dividends. If the assets are worth more than K in 5 years, the equity holders choose to repay the bondholders. If the assets are worth less than K, the equity holders choose to declare bankruptcy and the bondholders end up owning the company.

The value of the equity in 5 years is therefore $\max(A_T - K, 0)$, where A_T is the value of the company's assets at that time. This shows that the equity holders have a 5-year European call option on the assets of the company with a strike price of K. What about the bondholders? They get $\min(A_T, K)$ in 5 years. This is the same as $K - \max(K - A_T, 0)$. The bondholders have given the equity holders the right to sell the company's assets to them for K in 5 years. The bonds are therefore worth the present value of K minus the value of a 5-year European put option on the assets with a strike price of K.

To summarize, if c and p are the value of the call and put options, respectively, then

$$\text{Value of equity} = c$$

$$\text{Value of debt} = PV(K) - p$$

Denote the value of the assets of the company today by A_0. The value of the assets must equal the total value of the instruments used to finance the assets. This means that it must equal the sum of the value of the equity and the value of the debt, so that

$$A_0 = c + [PV(K) - p]$$

Rearranging this equation, we have

$$c + PV(K) = p + A_0$$

This is the put–call parity result in equation (9.3) for call and put options on the assets of the company.

shorting the call and buying both the put and the stock with an initial investment of

$$\$31 + \$1 - \$3 = \$29$$

When the investment is financed at the risk-free interest rate, a repayment of $29e^{0.1\times 0.25} =$ \$29.73 is required at the end of the 3 months. As in the previous case, either the call or the put will be exercised. The short call and long put option position therefore leads to the stock being sold for \$30.00. The net profit is therefore

$$\$30.00 - \$29.73 = \$0.27$$

These examples are illustrated in Table 9.2. Business Snapshot 9.1 shows how options and put–call parity can help us understand the positions of the debt and equity holders in a company.

American Options

Put–call parity holds only for European options. However, it is possible to derive some results for American option prices. It can be shown (see Problem 9.18) that, when there are no dividends,

$$S_0 - K \leqslant C - P \leqslant S_0 - Ke^{-rT} \tag{9.4}$$

Example 9.3

An American call option on a non-dividend-paying stock with strike price $20.00 and maturity in 5 months is worth $1.50. Suppose that the current stock price is $19.00 and the risk-free interest rate is 10% per annum. From equation (9.4), we have

$$19 - 20 \leqslant C - P \leqslant 19 - 20e^{-0.1\times 5/12}$$

or

$$1 \geqslant P - C \geqslant 0.18$$

showing that $P - C$ lies between $1.00 and $0.18. With C at $1.50, P must lie between $1.68 and $2.50. In other words, upper and lower bounds for the price of an American put with the same strike price and expiration date as the American call are $2.50 and $1.68.

9.5 EARLY EXERCISE: CALLS ON A NON-DIVIDEND-PAYING STOCK

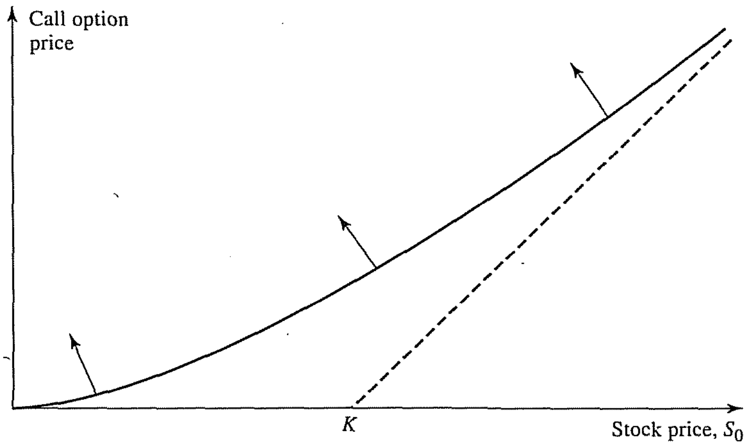
This section demonstrates that it is never optimal to exercise an American call option on a non-dividend-paying stock before the expiration date.

To illustrate the general nature of the argument, consider an American call option on a non-dividend-paying stock with 1 month to expiration when the stock price is $50 and the strike price is $40. The option is deep in the money, and the investor who owns the option might well be tempted to exercise it immediately. However, if the investor plans to hold the stock obtained by exercising the option for more than 1 month, this is not the best strategy. A better course of action is to keep the option and exercise it at the end of the month. The $40 strike price is then paid out 1 month later than it would be if the option were exercised immediately, so that interest is earned on the $40 for 1 month. Because the stock pays no dividends, no income from the stock is sacrificed. A further advantage of waiting rather than exercising immediately is that there is some chance (however remote) that the stock price will fall below $40 in 1 month. In this case, the investor will not exercise in 1 month and will be glad that the decision to exercise early was not taken!

This argument shows that there are no advantages to exercising early if the investor plans to keep the stock for the remaining life of the option (1 month, in this case). What if the investor thinks the stock is currently overpriced and is wondering whether to exercise the option and sell the stock? In this case, the investor is better off selling the option than exercising it.[1] The option will be bought by another investor who does want to hold the stock. Such investors must exist: otherwise the current stock price would not be $50. The price obtained for the option will be greater than its intrinsic value of $10, for the reasons mentioned earlier.

[1] As an alternative strategy, the investor can keep the option and short the stock to lock in a better profit than $10.

Figure 9.3 Variation of price of an American or European call option on a non-dividend-paying stock with the stock price, S_0.



For a more formal argument, we can use equation (9.1):

$$c \geqslant S_0 - Ke^{-rT}$$

Because the owner of an American call has all the exercise opportunities open to the owner of the corresponding European call, we must have

$$C \geqslant c$$

Hence,

$$C \geqslant S_0 - Ke^{-rT}$$

Given $r > 0$, it follows that $C > S_0 - K$. If it were optimal to exercise early, C would equal $S_0 - K$. We deduce that it can never be optimal to exercise early.

Figure 9.3 shows the general way in which the call price varies with S_0. It indicates that the call price is always above its intrinsic value of $\max(S_0 - K,\ 0)$. As r or T or the volatility increases, the line relating the call price to the stock price moves in the direction indicated by the arrows (i.e., farther away from the intrinsic value).

To summarize, there are two reasons an American call on a non-dividend-paying stock should not be exercised early. One relates to the insurance that it provides. A call option, when held instead of the stock itself, in effect insures the holder against the stock price falling below the strike price. Once the option has been exercised and the strike price has been exchanged for the stock price, this insurance vanishes. The other reason concerns the time value of money. From the perspective of the option holder, the later the strike price is paid out, the better.

9.6 EARLY EXERCISE: PUTS ON A NON-DIVIDEND-PAYING STOCK

It can be optimal to exercise an American put option on a non-dividend-paying stock early. Indeed, at any given time during its life, a put option should always be exercised early if it is sufficiently deep in the money.

To illustrate this, consider an extreme situation. Suppose that the strike price is \$10 and the stock price is virtually zero. By exercising immediately, an investor makes an immediate gain of \$10. If the investor waits, the gain from exercise might be less than \$10, but it cannot be more than \$10 because negative stock prices are impossible. Furthermore, receiving \$10 now is preferable to receiving \$10 in the future. It follows that the option should be exercised immediately.

Like a call option, a put option can be viewed as providing insurance. A put option, when held in conjunction with the stock, insures the holder against the stock price falling below a certain level. However, a put option is different from a call option in that it may be optimal for an investor to forgo this insurance and exercise early in order to realize the strike price immediately. In general, the early exercise of a put option becomes more attractive as S_0 decreases, as r increases, and as the volatility decreases.

It will be recalled from equation (9.2) that

$$p \geqslant Ke^{-rT} - S_0$$

For an American put with price P, the stronger condition

$$P \geqslant K - S_0$$

must always hold because immediate exercise is always possible.

Figure 9.4 shows the general way in which the price of an American put varies with S_0. Provided that $r > 0$, it is always optimal to exercise an American put immediately when the stock price is sufficiently low. When early exercise is optimal, the value of the option is $K - S_0$. The curve representing the value of the put therefore merges into the put's intrinsic value, $K - S_0$, for a sufficiently small value of S_0. In Figure 9.4, this value of S_0 is shown as point A. The line relating the put price to the stock price moves in the direction indicated by the arrows when r decreases, when the volatility increases, and when T increases.

Because there are some circumstances when it is desirable to exercise an American put option early, it follows that an American put option is always worth more than the

Figure 9.4 Variation of price of an American put option with stock price, S_0.

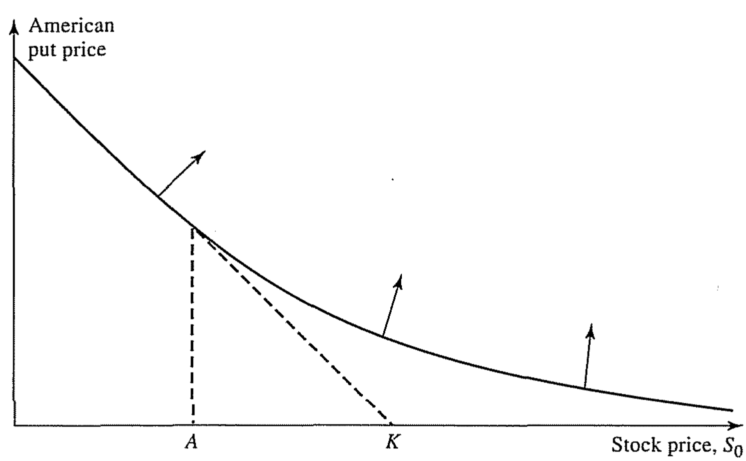
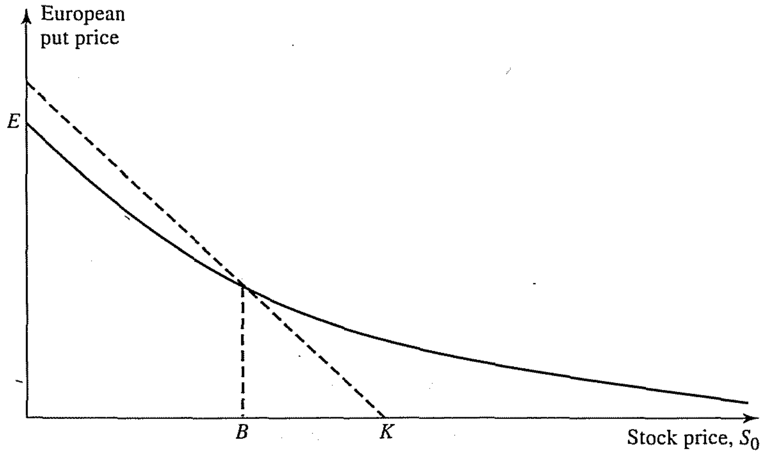


Figure 9.5 Variation of price of a European put option with the stock price, S_0.



corresponding European put option. Furthermore, because an American put is sometimes worth its intrinsic value (see Figure 9.4), it follows that a European put option must sometimes be worth less than its intrinsic value. Figure 9.5 shows the variation of the European put price with the stock price. Note that point B in Figure 9.5, at which the price of the option is equal to its intrinsic value, must represent a higher value of the stock price than point A in Figure 9.4. Point E in Figure 9.5 is where $S_0 = 0$ and the European put price is Ke^{-rT}.

9.7 EFFECT OF DIVIDENDS

The results produced so far in this chapter have assumed that we are dealing with options on a non-dividend-paying stock. In this section we examine the impact of dividends. In the United States most exchange-traded stock options have a life of less than 1 year and dividends payable during the life of the option can usually be predicted with reasonable accuracy. We will use D to denote the present value of the dividends during the life of the option. In the calculation of D, a dividend is assumed to occur at the time of its ex-dividend date.

Lower Bound for Calls and Puts

We can redefine portfolios A and B as follows:

Portfolio A: one European call option plus an amount of cash equal to $D + Ke^{-rT}$
Portfolio B: one share

A similar argument to the one used to derive equation (9.1) shows that

$$c \geqslant S_0 - D - Ke^{-rT} \tag{9.5}$$

We can also redefine portfolios C and D as follows:

Portfolio C: one European put option plus one share
Portfolio D: an amount of cash equal to $D + Ke^{-rT}$

A similar argument to the one used to derive equation (9.2) shows that

$$p \geqslant D + Ke^{-rT} - S_0 \tag{9.6}$$

Early Exercise

When dividends are expected, we can no longer assert than an American call option will not be exercised early. Sometimes it is optimal to exercise an American call immediately prior to an ex-dividend date. It is never optimal to exercise a call at other times. This point is discussed further in the appendix to Chapter 13.

Put–Call Parity

Comparing the value at option maturity of the redefined portfolios A and C shows that, with dividends, the put–call parity result in equation (9.3) becomes

$$c + D + Ke^{-rT} = p + S_0 \tag{9.7}$$

Dividends cause equation (9.4) to be modified (see Problem 9.19) to

$$S_0 - D - K \leqslant C - P \leqslant S_0 - Ke^{-rT} \tag{9.8}$$

SUMMARY

There are six factors affecting the value of a stock option: the current stock price, the strike price, the expiration date, the stock price volatility, the risk-free interest rate, and the dividends expected during the life of the option. The value of a call generally increases as the current stock price, the time to expiration, the volatility, and the risk-free interest rate increase. The value of a call decreases as the strike price and expected dividends increase. The value of a put generally increases as the strike price, the time to expiration, the volatility, and the expected dividends increase. The value of a put decreases as the current stock price and the risk-free interest rate increase.

It is possible to reach some conclusions about the value of stock options without making any assumptions about the volatility of stock prices. For example, the price of a call option on a stock must always be worth less than the price of the stock itself. Similarly, the price of a put option on a stock must always be worth less than the option's strike price.

A European call option on a non-dividend-paying stock must be worth more than

$$\max(S_0 - Ke^{-rT}, 0)$$

where S_0 is the stock price, K is the strike price, r is the risk-free interest rate, and T is the time to expiration. A European put option on a non-dividend-paying stock must be worth more than

$$\max(Ke^{-rT} - S_0, 0)$$

When dividends with present value D will be paid, the lower bound for a European call option becomes

$$\max(S_0 - D - Ke^{-rT}, 0)$$

and the lower bound for a European put option becomes

$$\max(Ke^{-rT} + D - S_0, 0)$$

Put–call parity is a relationship between the price, c, of a European call option on a stock and the price, p, of a European put option on a stock. For a non-dividend-paying stock, it is

$$c + Ke^{-rT} = p + S_0$$

For a dividend-paying stock, the put–call parity relationship is

$$c + D + Ke^{-rT} = p + S_0$$

Put–call parity does not hold for American options. However, it is possible to use arbitrage arguments to obtain upper and lower bounds for the difference between the price of an American call and the price of an American put.

In Chapter 13, we will carry the analyses in this chapter further by making specific assumptions about the probabilistic behavior of stock prices. The analysis will enable us to derive exact pricing formulas for European stock options. In Chapters 11 and 17, we will see how numerical procedures can be used to price American options.

FURTHER READING

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81 (May/June 1973): 637–59.

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Merton, R. C. "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973): 141–83.

Merton, R. C. "The Relationship between Put and Call Prices: Comment," *Journal of Finance*, 28 (March 1973): 183–84.

Stoll, H. R. "The Relationship between Put and Call Option Prices," *Journal of Finance*, 31 (May 1969): 319–32.

Questions and Problems (Answers in Solutions Manual)

9.1. List the six factors that affect stock option prices.

9.2. What is a lower bound for the price of a 4-month call option on a non-dividend-paying stock when the stock price is \$28, the strike price is \$25, and the risk-free interest rate is 8% per annum?

9.3. What is a lower bound for the price of a 1-month European put option on a non-dividend-paying stock when the stock price is \$12, the strike price is \$15, and the risk-free interest rate is 6% per annum?

9.4. Give two reasons why the early exercise of an American call option on a non-dividend-paying stock is not optimal. The first reason should involve the time value of money. The second should apply even if interest rates are zero.

9.5. "The early exercise of an American put is a trade-off between the time value of money and the insurance value of a put." Explain this statement.

9.6. Explain why an American call option on a dividend-paying stock is always worth at least as much as its intrinsic value. Is the same true of a European call option? Explain your answer.

9.7. The price of a non-dividend-paying stock is \$19 and the price of a 3-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a 3-month European put option with a strike price of \$20?

9.8. Explain why the arguments leading to put–call parity for European options cannot be used to give a similar result for American options.

9.9. What is a lower bound for the price of a 6-month call option on a non-dividend-paying stock when the stock price is \$80, the strike price is \$75, and the risk-free interest rate is 10% per annum?

9.10. What is a lower bound for the price of a 2-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk-free interest rate is 5% per annum?

9.11. A 4-month European call option on a dividend-paying stock is currently selling for \$5. The stock price is \$64, the strike price is \$60, and a dividend of \$0.80 is expected in 1 month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur?

9.12. A 1-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

9.13. Give an intuitive explanation of why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.

9.14. The price of a European call that expires in 6 months and has a strike price of \$30 is \$2. The underlying stock price is \$29, and a dividend of \$0.50 is expected in 2 months and again in 5 months. The term structure is flat, with all risk-free interest rates being 10%. What is the price of a European put option that expires in 6 months and has a strike price of \$30?

9.15. Explain carefully the arbitrage opportunities in Problem 9.14 if the European put price is \$3.

9.16. The price of an American call on a non-dividend-paying stock is \$4. The stock price is \$31, the strike price is \$30, and the expiration date is in 3 months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

9.17. Explain carefully the arbitrage opportunities in Problem 9.16 if the American put price is greater than the calculated upper bound.

9.18. Prove the result in equation (9.4). (*Hint*: For the first part of the relationship, consider (a) a portfolio consisting of a European call plus an amount of cash equal to K, and (b) a portfolio consisting of an American put option plus one share.)

9.19. Prove the result in equation (9.8). (*Hint*: For the first part of the relationship, consider (a) a portfolio consisting of a European call plus an amount of cash equal to $D + K$, and (b) a portfolio consisting of an American put option plus one share.)

9.20. Regular call options on non-dividend-paying stocks should not be exercised early. However, there is a tendency for executive stock options to be exercised early even when the company pays no dividends (see Business Snapshot 8.3 for a discussion of executive stock options). Give a possible reason for this.

9.21. Use the software DerivaGem to verify that Figures 9.1 and 9.2 are correct.

Assignment Questions

9.22. A European call option and put option on a stock both have a strike price of \$20 and an expiration date in 3 months. Both sell for \$3. The risk-free interest rate is 10% per annum, the current stock price is \$19, and a \$1 dividend is expected in 1 month. Identify the arbitrage opportunity open to a trader.

9.23. Suppose that c_1, c_2, and c_3 are the prices of European call options with strike prices K_1, K_2, and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c_2 \leqslant 0.5(c_1 + c_3)$$

(*Hint*: Consider a portfolio that is long one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2.)

9.24. What is the result corresponding to that in Problem 9.23 for European put options?

9.25. Suppose that you are the manager and sole owner of a highly leveraged company. All the debt will mature in 1 year. If at that time the value of the company is greater than the face value of the debt, you will pay off the debt. If the value of the company is less than the face value of the debt, you will declare bankruptcy and the debt holders will own the company.

(a) Express your position as an option on the value of the company.

(b) Express the position of the debt holders in terms of options on the value of the company.

(c) What can you do to increase the value of your position?

9.26. Consider an option on a stock when the stock price is \$41, the strike price is \$40, the risk-free rate is 6%, the volatility is 35%, and the time to maturity is 1 year. Assume that a dividend of \$0.50 is expected after 6 months.

(a) Use DerivaGem to value the option assuming it is a European call.

(b) Use DerivaGem to value the option assuming it is a European put.

(c) Verify that put–call parity holds.

(d) Explore using DerivaGem what happens to the price of the options as the time to maturity becomes very large. For this purpose, assume there are no dividends. Explain the results you get.