CHAPTER 26

Interest Rate Derivatives: The Standard Market Models

Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates. In the 1980s and 1990s, the volume of trading in interest rate derivatives in both the over-the-counter and exchange-traded markets increased very quickly. Many new products were developed to meet particular needs of end users. A key challenge for derivatives traders is to find good, robust procedures for pricing and hedging these products.

Interest rate derivatives are more difficult to value than equity and foreign exchange derivatives. There are a number of reasons for this:

1. The behavior of an individual interest rate is more complicated than that of a stock price or an exchange rate.
2. For the valuation of many products it is necessary to develop a model describing the behavior of the entire zero-coupon yield curve.
3. The volatilities of different points on the yield curve are different.
4. Interest rates are used for discounting as well as for defining the payoff from the derivative.

In this chapter we look at the three most popular over-the-counter interest rate option products: bond options, interest rate caps/floors, and swap options. We explain the standard market models for valuing these products and use the analysis from Chapter 25 to show that the models are internally consistent.

26.1 BLACK'S MODEL

Since the Black–Scholes model was first published in 1973, it has become a very popular tool. As explained in Chapter 14, the model has been extended so that it can be used to value options on foreign exchange, options on indices, and options on futures contracts. Traders have become very comfortable with both the lognormal assumption that underlies the model and the volatility measure that describes uncertainty. It is not surprising that there have been attempts to extend the model so that it covers interest rate derivatives.

In this chapter we will discuss three of the most popular interest rate derivatives (bond options, interest rate caps/floors, and swap options) and describe how the lognormal assumption underlying the Black–Scholes model can be used to value these instruments. The model we will use is usually referred to as Black's model because it is structurally similar to the model suggested by Fischer Black for valuing options on commodity futures (see Section 14.8). If, in this model, the futures contract and the option have the same maturity date, then the futures price equals the spot price at the end of the option's life. This means that the model then gives the value of an option on spot as well as an option on futures.

Using Black's Model to Price European Options

Consider a European call option on a variable whose value is V. (The variable V does not have to be the price of a traded security.) Define:

T: Time to maturity of the option

F: Forward price of V for a contract with maturity T

F_0: Value of F at time zero

K: Strike price of the option

$P(t, T)$: Price at time t of a zero-coupon bond paying \$1 at time T

V_T: Value of V at time T

σ: Volatility of F

We value the option by:

1. Assuming $\ln V_T$ is normal with mean F_0 and standard deviation $\sigma\sqrt{T}$
2. Discounting the expected payoff at the T-year rate. (This is equivalent to multiplying the expected payoff by $P(0, T)$.)

The payoff from the option at time T is $\max(V_T - K, 0)$. As shown in the appendix to Chapter 13, the lognormal assumption for V_T implies that the expected payoff is

$$E(V_T)N(d_1) - KN(d_2)$$

where $E(V_T)$ is the expected value of V_T and

$$d_1 = \frac{\ln[E(V_T)/K] + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[E(V_T)/K] - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Because we are assuming that $E(V_T) = F_0$, the value of the option is

$$c = P(0, T)[F_0 N(d_1) - KN(d_2)] \tag{26.1}$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Similarly the value p of the corresponding put option is given by

$$p = P(0, T)[KN(-d_2) - F_0 N(-d_1)] \tag{26.2}$$

This is the model we will refer to as Black's model from now on. An important feature of Black's model is that we do not have to assume geometric Brownian motion for the evolution of either V or F. All that we require is that V_T be lognormal at time T. The parameter σ is usually referred to as the volatility of F or the forward volatility of V. However, its only role is to define the standard deviation of $\ln V_T$ by means of the relationship

$$\text{Standard deviation of } \ln V_T = \sigma\sqrt{T}$$

The volatility parameter does not necessarily say anything about the standard deviation of $\ln V$ at times other than T.

Delayed Payoff

We can extend Black's model to allow for the situation where the payoff is calculated from the value of the variable V at time T, but the payoff is actually made at some later time T^*. The expected payoff is discounted from time T^* instead of time T so that equation (26.1) and (26.2) become

$$c = P(0, T^*)[F_0 N(d_1) - KN(d_2)] \tag{26.3}$$

$$p = P(0, T^*)[KN(-d_2) - F_0 N(-d_1)] \tag{26.4}$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Validity of Black's Model

It is easy to see that Black's model is appropriate when interest rates are assumed to be either constant or deterministic. In this case, as explained in Chapter 5, the forward price of V equals its futures price and from Section 14.7 the $E(S_T) = F_0$ in a risk-neutral world.

When interest rates are stochastic, there are two aspects of the derivation of equations (26.1) to (26.4) that are open to question.

1. Why do we set $E(V_T)$ equal to the forward price F_0 of V? This is not the same as the futures price.
2. Why do we ignore the fact that interest rates are stochastic when discounting?

It turns out that these two assumptions offset each other. As we apply Black's model to bond options, caps/floors, and swap options, we will use the results in Section 25.4 to show that there are no approximations in equations (26.1) to (26.4) when interest rates are stochastic. Black's model therefore has a sounder theoretical basis and wider applicability than sometimes supposed.

26.2 BOND OPTIONS

A bond option is an option to buy or sell a particular bond by a particular date for a particular price. In addition to trading in the over-the-counter market, bond options are frequently embedded in bonds when they are issued to make them more attractive to either the issuer or potential purchasers.

Embedded Bond Options

One example of a bond with an embedded bond option is a *callable bond*. This is a bond that contains provisions allowing the issuing firm to buy back the bond at a predetermined price at certain times in the future. The holder of such a bond has sold a call option to the issuer. The strike price or call price in the option is the predetermined price that must be paid by the issuer to the holder. Callable bonds cannot usually be called for the first few years of their life. (This is known as the lock-out period.) After that the call price is usually a decreasing function of time. For example, in a 10-year callable bond, there might be no call privileges for the first 2 years. After that, the issuer might have the right to buy the bond back at a price of 110 in years 3 and 4 of its life, at a price of 107.5 in years 5 and 6, at a price of 106 in years 7 and 8, and at a price of 103 in years 9 and 10. The value of the call option is reflected in the quoted yields on bonds. Bonds with call features generally offer higher yields than bonds with no call features.

Another type of bond with an embedded option is a *puttable bond*. This contains provisions that allow the holder to demand early redemption at a predetermined price at certain times in the future. The holder of such a bond has purchased a put option on the bond as well as the bond itself. Because the put option increases the value of the bond to the holder, bonds with put features provide lower yields than bonds with no put features. A simple example of a puttable bond is a 10-year bond where the holder has the right to be repaid at the end of 5 years. (This is sometimes referred to as a *retractable bond*.)

Loan and deposit instruments also often contain embedded bond options. For example, a 5-year fixed-rate deposit with a financial institution that can be redeemed without penalty at any time contains an American put option on a bond. (The deposit instrument is a bond that the investor has the right to put back to the financial institution at its face value at any time.) Prepayment privileges on loans and mortgages are similarly call options on bonds.

Finally, we note that a loan commitment made by a bank or other financial institution is a put option on a bond. Consider, for example, the situation where a bank quotes a 5-year interest rate of 5% per annum to a potential borrower and states that the rate is good for the next 2 months. The client has, in effect, obtained the right to sell a 5-year bond with a 5% coupon to the financial institution for its face value any time within the next 2 months.

European Bond Options

Many over-the-counter bond options and some embedded bond options are European. We now consider the standard market models used to value European options.

The assumption usually made is that the bond price at the maturity of the option is lognormal. Equations (26.1) and (26.2) can be used to price the option with F_0 equal to the forward bond price F_B. The variable σ is set equal to the forward bond price volatility, σ_B. As explained in Section 26.1 σ_B is defined so that $\sigma_B\sqrt{T}$ is the standard deviation of the logarithm of the bond price at the maturity of the option. The equations for pricing a European bond option are therefore

$$c = P(0, T)[F_B N(d_1) - KN(d_2)] \tag{26.5}$$

$$p = P(0, T)[KN(-d_2) - F_B N(-d_1)] \tag{26.6}$$

where

$$d_1 = \frac{\ln(F_B/K) + \sigma_B^2 T/2}{\sigma_B\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_B\sqrt{T}$$

From Section 5.5, F_B can be calculated using the formula

$$F_B = \frac{B_0 - I}{P(0, T)} \tag{26.7}$$

where B_0 is the bond price at time zero and I is the present value of the coupons that will be paid during the life of the option. In this formula, both the spot bond price and the forward bond price are cash prices rather than quoted prices. The relationship between cash and quoted bond prices is explained in Section 6.2.

The strike price K in equations (26.5) and (26.6) should be the cash strike price. In choosing the correct value for K, the precise terms of the option are therefore important. If the strike price is defined as the cash amount that is exchanged for the bond when the option is exercised, K should be put equal to this strike price. If, as is more common, the strike price is the quoted price applicable when the option is exercised, K should be set equal to the strike price plus accrued interest at the expiration date of the option. Traders refer to the quoted price of a bond as the *clean price* and the cash price as the *dirty price*.

Example 26.1

Consider a 10-month European call option on a 9.75-year bond with a face value of \$1,000. (When the option matures, the bond will have 8 years and 11 months remaining.) Suppose that the current cash bond price is \$960, the strike price is \$1,000, the 10-month risk-free interest rate is 10% per annum, and the volatility of the forward bond price in 10 months is 9% per annum. The bond pays a semiannual coupon of 10% and coupon payments of \$50 are expected in 3 months and 9 months. (This means that the accrued interest is \$25 and the quoted bond price is \$935.) We suppose that the 3-month and 9-month risk-free interest rates are 9.0% and 9.5% per annum, respectively. The present value of the coupon payments is, therefore,

$$50e^{-0.25\times0.09} + 50e^{-0.75\times0.095} = 95.45$$

or \$95.45. The bond forward price is from equation (26.7) given by

$$F_B = (960 - 95.45)e^{0.1\times0.8333} = 939.68$$

(a) If the strike price is the cash price that would be paid for the bond on exercise, the parameters for equation (26.5) are $F_B = 939.68$, $K = 1000$, $P(0, T) = e^{-0.1\times(10/12)} = 0.9200$, $\sigma_B = 0.09$, and $T = 10/12$. The price of the call option is \$9.49.

(b) If the strike price is the quoted price that would be paid for the bond on exercise, 1 month's accrued interest must be added to K because the maturity of the option is 1 month after a coupon date. This produces a value for K of

$$1{,}000 + 50 \times 0.16667 = 1{,}008.33$$

The values for the other parameters in equation (26.5) are unchanged (i.e., $F_B = 939.68$, $P(0, T) = 0.9200$, $\sigma_B = 0.09$, and $T = 0.8333$). The price of the option is \$7.97.

Figure 26.1 shows how the standard deviation of the logarithm of a bond's price changes as we look further ahead. The standard deviation is zero today because there is no uncertainty about the bond's price today. It is also zero at the bond's maturity because we know that the bond's price will equal its face value at maturity. Between today and the maturity of the bond, the standard deviation first increases and then decreases.

The volatility σ_B that should be used when a European option on the bond is valued is

$$\frac{\text{Standard deviation of logarithm of bond price at maturity of option}}{\sqrt{\text{Time to maturity of option}}}$$

What happens when we keep the underlying bond fixed and increase the life of the

Figure 26.1 Standard deviation of logarithm of bond price at future times.

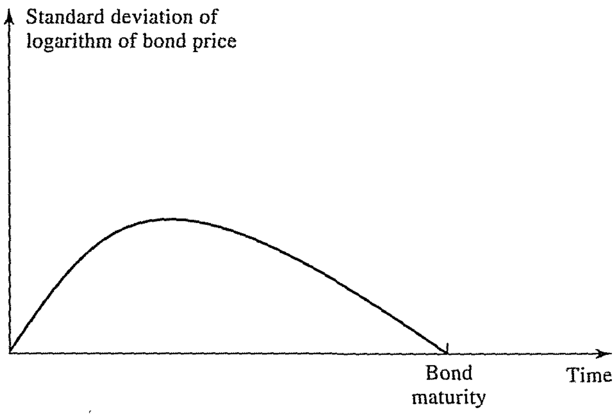
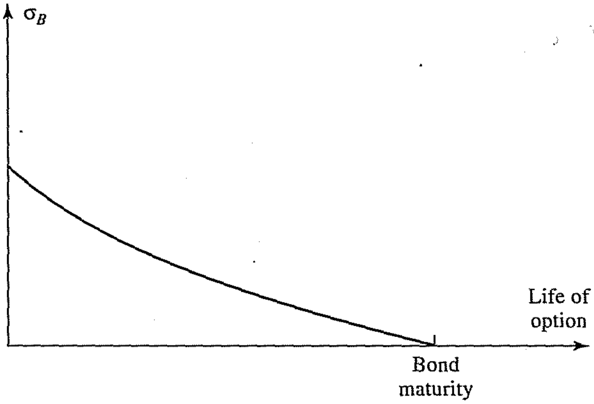


Figure 26.2 Variation of forward bond price volatility σ_B with life of option when bond is kept fixed.



option? Figure 26.2 shows a typical pattern for σ_B as a function of the life of the option. In general, σ_B declines as the life of the option increases.

Yield Volatilities

The volatilities that are quoted for bond options are often yield volatilities rather than price volatilities. The duration concept, introduced in Chapter 4, is used by the market to convert a quoted yield volatility into a price volatility. Suppose that D is the modified duration of the bond underlying the option at the option maturity, as defined in Chapter 4. The relationship between the change ΔF_B in the forward bond price F_B and the change Δy_F in the forward yield y_F is

$$\frac{\Delta F_B}{F_B} \approx -D\Delta y_F$$

or

$$\frac{\Delta F_B}{F_B} \approx -Dy_F\frac{\Delta y_F}{y_F}$$

Volatility is a measure of the standard deviation of percentage changes in the value of a variable. This equation therefore suggests that the volatility of the forward bond price σ_B used in Black's model can be approximately related to the volatility of the forward bond yield σ_y by

$$\sigma_B = Dy_0\sigma_y \tag{26.8}$$

where y_0 is the initial value of y_F. When a yield volatility is quoted for a bond option, the implicit assumption is usually that it will be converted to a price volatility using equation (26.8), and that this volatility will then be used in conjunction with equation (26.5) or (26.6) to obtain a price. Suppose that the bond underlying a call option will have a modified duration of 5 years at option maturity, the forward yield is 8%, and the forward yield volatility quoted by a broker is 20%. This means that the market price of the option corresponding to the broker quote is the price given by equation (26.5) when the volatility variable σ_B is

$$5 \times 0.08 \times 0.2 = 0.08$$

or 8% per annum. Figure 26.2 shows that forward bond volatilities depend on the option considered. Forward yield volatilities as we have just defined them are more constant. This is why traders prefer them.

The Bond_Options worksheet of the software DerivaGem accompanying this book can be used to price European bond options using Black's model by selecting Black-European as the Pricing Model. The user inputs a yield volatility, which is handled in the way just described. The strike price can be the cash or quoted strike price.

Example 26.2

Consider a European put option on a 10-year bond with a principal of 100. The coupon is 8% per year payable semiannually. The life of the option is 2.25 years and the strike price of the option is 115. The forward yield volatility is 20%. The zero curve is flat at 5% with continuous compounding. DerivaGem shows that the quoted price of the bond is 122.84. The price of the option when the strike price is a quoted price is 2.37. When the strike price is a cash price, the price of the option is \$1.74. (Note that DerivaGem's prices may not exactly agree with manually calculated prices because DerivaGem assumes 365 days per year and rounds times to the nearest whole number of days. See Problem 26.16 for the manual calculation.)

Theoretical Justification for the Model

In Section 25.4, we explored alternatives to the usual risk-neutral valuation assumption for valuing derivatives. One alternative was a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T. We showed that:

1. The current value of any security is its expected value at time T in this world multiplied by the price of a zero-coupon bond maturing at time T (see equation 25.20).
2. The expected value of any variable (except an interest rate) at time T in this world equals its forward value (see equation 25.21).

The first of these results shows that the price of a call option with maturity T years on a bond is

$$c = P(0, T)E_T[\max(B_T - K, 0)] \tag{26.9}$$

where B_T is the bond price at time T and E_T denotes expected value in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T. The second result implies that

$$E_T(B_T) = F_B \tag{26.10}$$

Assuming the bond price is lognormal with the standard deviation of the logarithm of the bond price equal to $\sigma_B\sqrt{T}$, the appendix to Chapter 13 shows that equation (26.9) becomes

$$c = P(0, T)[E_T(B_T)N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln[E_T(B_T)/K] + \sigma_B^2 T/2}{\sigma_B \sqrt{T}}$$

$$d_2 = \frac{\ln[E_T(B_T)/K] - \sigma_B^2 T/2}{\sigma_B \sqrt{T}} = d_1 - \sigma_B \sqrt{T}$$

Using equation (26.10), this reduces to the Black's model formula in equation (26.5). We have shown that we can use today's T-year maturity interest rate for discounting provided that we also set the expected bond price equal to the forward bond price.

26.3 INTEREST RATE CAPS AND FLOORS

A popular interest rate option offered by financial institutions in the over-the-counter market is an *interest rate cap*. Interest rate caps can best be understood by first considering a floating-rate note where the interest rate is reset periodically equal to LIBOR. The time between resets is known as the *tenor*. Suppose the tenor is 3 months. The interest rate on the note for the first 3 months is the initial 3-month LIBOR rate; the interest rate for the next 3 months is set equal to the 3-month LIBOR rate prevailing in the market at the 3-month point; and so on.

An interest rate cap is designed to provide insurance against the rate of interest on the floating-rate note rising above a certain level. This level is known as the *cap rate*. Suppose that the principal amount is \$10 million, the tenor is 3 months, the life of the cap is 3 years, and the cap rate is 4%. (Because the payments are made quarterly, this cap rate is expressed with quarterly compounding.) The cap provides insurance against the interest on the floating rate note rising above 4%.

For the moment we ignore day count issues and assume that there is exactly 0.25 year between each payment date. We will cover day count issues at the end of this section. Suppose that on a particular reset date the 3-month LIBOR interest rate is 5%. The floating rate note would require

$$0.25 \times 0.05 \times \$10{,}000{,}000 = \$125{,}000$$

of interest to be paid 3 months later. With a 3-month LIBOR rate of 4% the interest payment would be

$$0.25 \times 0.04 \times \$10{,}000{,}000 = \$100{,}000$$

The cap therefore provides a payoff of \$25,000. Note that the payoff does not occur on the reset date when the 5% is observed. It occurs 3 months later. This reflects the usual time lag between an interest rate being observed and the corresponding payment being required.

At each reset date during the life of the cap we observe LIBOR. If LIBOR is less than 4%, there is no payoff from the cap three months later. If LIBOR is greater than 4%, the payoff is one quarter of the excess applied to the principal of \$10 million. Note that caps are usually defined so that the initial LIBOR rate, even if it is greater than the cap rate, does not lead to a payoff on the first reset date. In our example, the cap lasts for 5 years. There are, therefore, a total of 19 reset dates (at times 0.25, 0.50, 0.75, ..., 4.75 years) and 19 potential payoffs from the caps (at times 0.50, 0.75, 1.00, ..., 5.00 years).

The Cap as a Portfolio of Interest Rate Options

We now consider a cap with a total life of T, a principal of L, and a cap rate of R_K. Suppose that the reset dates are $t_1, t_2, \dots, t_n$ and define $t_{n+1} = T$. Define R_k as the interest rate for the period between time t_k and t_{k+1} observed at time t_k ($1 \leqslant k \leqslant n$). The cap leads to a payoff at time t_{k+1} ($k = 1, 2, \dots, n$) of

$$L\delta_k \max(R_k - R_K, 0) \tag{26.11}$$

where $\delta_k = t_{k+1} - t_k$.[1] Both R_k and R_K are expressed with a compounding frequency equal to the frequency of resets.

Equation (26.11) is a call option on the LIBOR rate observed at time t_k with the payoff occurring at time t_{k+1}. The cap is a portfolio of n such options. LIBOR rates are observed at times $t_1, t_2, t_3, \dots, t_n$ and the corresponding payoffs occur at times $t_2, t_3, t_4, \dots, t_{n+1}$. The n call options underlying the cap are known as *caplets*.

A Cap as a Portfolio of Bond Options

An interest rate cap can also be characterized as a portfolio of put options on zero-coupon bonds with payoffs on the puts occurring at the time they are calculated. The payoff in equation (26.11) at time t_{k+1} is equivalent to

$$\frac{L\delta_k}{1 + R_k\delta_k} \max(R_k - R_K, 0)$$

at time t_k. A few lines of algebra show that this reduces to

$$\max\left[L - \frac{L(1 + R_K\delta_k)}{1 + R_k\delta_k}, 0\right] \tag{26.12}$$

The expression

$$\frac{L(1 + R_K\delta_k)}{1 + R_k\delta_k}$$

is the value at time t_k of a zero-coupon bond that pays off $L(1 + R_K\delta_k)$ at time t_{k+1}. The expression in equation (26.12) is therefore the payoff from a put option with maturity t_k on a zero-coupon bond with maturity t_{k+1} when the face value of the bond is $L(1 + R_K\delta_k)$ and the strike price is L. It follows that an interest rate cap can be regarded as a portfolio of European put options on zero-coupon bonds.

Floors and Collars

Interest rate floors and interest rate collars (sometimes called floor–ceiling agreements) are defined analogously to caps. A *floor* provides a payoff when the interest rate on the underlying floating-rate note falls below a certain rate. With the notation already introduced, a floor provides a payoff at time t_{k+1} ($k = 1, 2, \dots, n$) of

$$L\delta_k \max(R_K - R_k, 0)$$

Analogously to an interest rate cap, an interest rate floor is a portfolio of put options on interest rates or a portfolio of call options on zero-coupon bonds. Each of the

[1] Day count issues are discussed at the end of this section.

Business Snapshot 26.1 Put–Call Parity for Caps and Floors

There is a put–call parity relationship between the prices of caps and floors. This is

$$\text{Value of cap} = \text{Value of floor} + \text{Value of swap}$$

In this relationship, the cap and floor have the same strike price, R_K. The swap is an agreement to receive LIBOR and pay a fixed rate of R_K with no exchange of payments on the first reset date. All three instruments have the same life and the same frequency of payments.

To see that the result is true, consider a long position in the cap combined with a short position in the floor. The cap provides a cash flow of LIBOR $- R_K$ for periods when LIBOR is greater than R_K. The short floor provides a cash flow of $-(R_K - \text{LIBOR}) = \text{LIBOR} - R_K$ for periods when LIBOR is less than R_K. There is therefore a cash flow of LIBOR $- R_K$ in all circumstances. This is the cash flow on the swap. It follows that the value of the cap minus the value of the floor must equal the value of the swap.

Note that swaps are usually structured so that LIBOR at time zero determines a payment on the first reset date. Caps and floors are usually structured so that there is no payoff on the first reset date. This is why the swap has to be defined as one with no payment on the first reset date.

individual options comprising a floor is known as a *floorlet*. A *collar* is an instrument designed to guarantee that the interest rate on the underlying floating-rate note always lies between two levels. A collar is a combination of a long position in a cap and a short position in a floor. It is usually constructed so that the price of the cap is initially equal to the price of the floor. The cost of entering into the collar is then zero.

As explained in Business Snapshot 26.1 there is a put–call parity relationship between caps and floors.

Valuation of Caps and Floors

As shown in equation (26.11), the caplet corresponding to the rate observed at time t_k provides a payoff at time t_{k+1} of

$$L\delta_k \max(R_k - R_K, 0)$$

If the rate R_k is assumed to be lognormal with volatility σ_k, equation (26.3) gives the value of this caplet as

$$L\delta_k P(0, t_{k+1})[F_k N(d_1) - R_K N(d_2)] \tag{26.13}$$

where

$$d_1 = \frac{\ln(F_k/R_K) + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}}$$

$$d_2 = \frac{\ln(F_k/R_K) - \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}} = d_1 - \sigma_k\sqrt{t_k}$$

and F_k is the forward rate for the period between time t_k and t_{k+1}. The value of the corresponding floorlet is, from equation (26.4),

$$L\delta_k P(0, t_{k+1})[R_K N(-d_2) - F_k N(-d_1)] \tag{26.14}$$

Example 26.3

Consider a contract that caps the LIBOR interest rate on \$10,000 at 8% per annum (with quarterly compounding) for 3 months starting in 1 year. This is a caplet and could be one element of a cap. Suppose that the LIBOR/swap zero curve is flat at 7% per annum with quarterly compounding and the volatility of the 3-month forward rate underlying the caplet is 20% per annum. The continuously compounded zero rate for all maturities is 6.9394%. In equation (26.13), $F_k = 0.07$, $\delta_k = 0.25$, $L = 10,000$, $R_K = 0.08$, $t_k = 1.0$, $t_{k+1} = 1.25$, $P(0, t_{k+1}) = e^{-0.069394\times 1.25} = 0.9169$, and $\sigma_k = 0.20$. Also,

$$d_1 = \frac{\ln(0.07/0.08) + 0.2^2 \times 1/2}{0.20 \times 1} = -0.5677$$

$$d_2 = d_1 - 0.20 = -0.7677$$

so that the caplet price is

$$0.25 \times 10,000 \times 0.9169[0.07N(-0.5677) - 0.08N(-0.7677)] = \$5.162$$

(Note that DerivaGem gives \$5.146 for the price of this caplet. This is because it assumes 365 days per year and rounds times to the nearest whole number of days.)

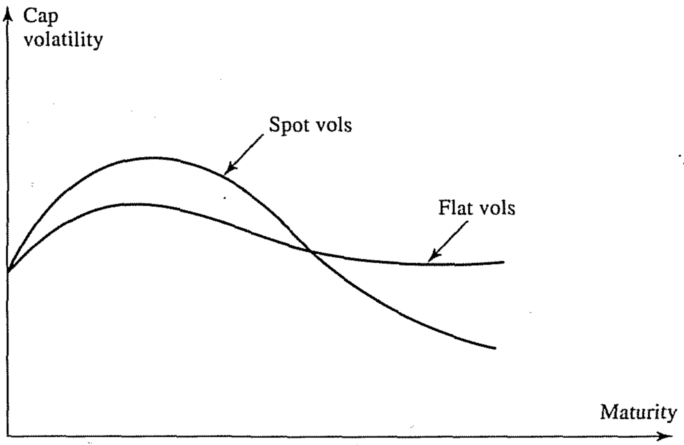
Each caplet of a cap must be valued separately using equation (26.13). One approach is to use a different volatility for each caplet. The volatilities are then referred to as *spot volatilities*. An alternative approach is to use the same volatility for all the caplets comprising any particular cap but to vary this volatility according to the life of the cap. The volatilities used are then referred to as *flat volatilities*.[2] The volatilities quoted in the market are usually flat volatilities. However, many traders like to work with spot volatilities because this allows them to identify underpriced and overpriced caplets. Options on Eurodollar futures are very similar to caplets and the spot volatilities used for caplets on 3-month LIBOR are frequently compared with those calculated from the prices of Eurodollar futures options.

Spot Volatilities vs. Flat Volatilities

Figure 26.3 shows a typical pattern for spot volatilities and flat volatilities as a function of maturity. (In the case of a spot volatility, the maturity is the maturity of a caplet; in the case of a flat volatility, it is the maturity of a cap.) The flat volatilities are akin to cumulative averages of the spot volatilities and therefore exhibit less variability. As indicated by Figure 26.3, we usually observe a "hump" in the volatilities. The peak of the hump is at about the 2- to 3-year point. This hump is observed both when the volatilities are implied from option prices and when they are calculated from historical data. There is no general agreement on the reason for the existence of the hump. One possible explanation is as follows. Rates at the short end of the zero curve are controlled by central banks. By contrast, 2- and 3-year interest rates are determined to a large extent by the activities of traders. These traders may be

[2] Flat volatilities can be calculated from spot volatilities and vice versa (see Problem 26.20).

Figure 26.3 The volatility hump.



overreacting to the changes they observe in the short rate and causing the volatility of these rates to be higher than the volatility of short rates. For maturities beyond 2 to 3 years, the mean reversion of interest rates, which will be discussed in Chapter 28, causes volatilities to decline.

Brokers provide tables of flat implied volatilities for caps and floors. The instruments underlying the quotes are usually at the money. This means that the cap/floor rate equals the swap rate for a swap that has the same payment dates as the cap. Table 26.1 shows typical broker quotes for the US dollar market. The tenor of the cap is 3 months and the cap life varies from 1 to 10 years. The data exhibits the type of "hump" shown in Figure 26.3.

Theoretical Justification for the Model

We can show that Black's model for a caplet is internally consistent by considering a world that is forward risk neutral with respect to a zero-coupon bond maturing at

Table 26.1 Typical broker flat volatility quotes for US dollar caps and floors (% per annum).

Life	*Cap bid*	*Cap offer*	*Floor bid*	*Floor offer*
1 year	18.00	20.00	18.00	20.00
2 years	23.25	24.25	23.75	24.75
3 years	24.00	25.00	24.50	25.50
4 years	23.75	24.75	24.25	25.25
5 years	23.50	24.50	24.00	25.00
7 years	21.75	22.75	22.00	23.00
10 years	20.00	21.00	20.25	21.25

time t_{k+1}. The analysis in Section 25.4 shows that:

1. The current value of any security is its expected value at time t_{k+1} in this world multiplied by the price of a zero-coupon bond maturing at time t_{k+1} (see equation 25.20).
2. The expected value of an interest rate lasting between times t_k and t_{k+1} equals the forward interest rate in this world (see equation 25.22).

The first of these results shows that, with the notation introduced earlier, the price of a caplet that provides a payoff at time t_{k+1} is

$$L\delta_k P(0, t_{k+1}) E_{k+1}[\max(R_k - R_K, 0)]$$

where E_{k+1} denotes expected value in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time t_{k+1}. From the appendix at the end of Chapter 13, when R_k is assumed to be lognormal, this becomes

$$L\delta_k P(0, t_{k+1})[E_{k+1}(R_k)N(d_1) - R_K N(d_2)]$$

where

$$d_1 = \frac{\ln[E_{k+1}(R_k)/R_K] + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}}$$

$$d_2 = \frac{\ln[E_{k+1}(R_k)/R_K] - \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}} = d_1 - \sigma\sqrt{t_k}$$

The second result implies that

$$E_{k+1}(R_k) = F_k$$

Together the results lead to the cap pricing model in equation (26.13). They show that we can discount at the t_{k+1}-maturity interest rate observed in the market today providing we set the expected interest rate equal to the forward interest rate.

Use of DerivaGem

The software DerivaGem accompanying this book can be used to price interest rate caps and floors using Black's model. In the Cap_and_Swap_Option worksheet select Cap/Floor as the Underlying Type and Black-European as the Pricing Model. The zero curve is input using continuously compounded rates. The inputs include the start and end date of the period covered by the cap, the flat volatility, and the cap settlement frequency (i.e., the tenor). The software calculates the payment dates by working back from the end of period covered by the cap to the beginning. The initial caplet/floorlet is assumed to cover a period of length between 0.5 and 1.5 times a regular period. Suppose, for example, that the period covered by the cap is 1.22 years to 2.80 years and the settlement frequency is quarterly. There are six caplets covering the periods 2.55 to 2.80 years, 2.30 to 2.55 years, 2.05 to 2.30 years, 1.80 to 2.05 years, 1.55 to 1.80 years, and 1.22 to 1.55 years.

The Impact of Day Count Conventions

The formulas we have presented so far in this section do not reflect day count conventions (see Section 6.1 for an explanation of day count conventions). Suppose that the cap rate R_K is expressed with an actual/360 day count (as would be normal in the United States). This means that the time interval δ_k in our formulas should be replaced by a_k, the *accrual fraction* for the time period between t_k and t_{k+1}. Suppose, for example, that t_k is May 1 and t_{k+1} is August 1. Under actual/360 there are 92 days between these payment dates so that $a_k = 92/360 = 0.2521$. We must also express the forward rate F_k with an actual/360 day count. This means that we must set it by solving

$$1 + a_k F_k = \frac{P(0, t_k)}{P(0, t_{k+1})}$$

The impact of all this is much the same as converting R_K from actual/360 to actual/actual and calculating F_k on an actual/actual basis. In other words it is much the same as multiplying the quoted cap rate by 365/360 or 366/360 to get R_K and calculating F_k by solving

$$1 + \delta_k F_k = \frac{P(0, t_k)}{P(0, t_{k+1})}$$

26.4 EUROPEAN SWAP OPTIONS

Swap options, or *swaptions*, are options on interest rate swaps and are another popular type of interest rate option. They give the holder the right to enter into a certain interest rate swap at a certain time in the future. (The holder does not, of course, have to exercise this right.) Many large financial institutions that offer interest rate swap contracts to their corporate clients are also prepared to sell them swaptions or buy swaptions from them. As shown in Business Snapshot 26.2, a swaption can be viewed as a type of bond option. To give an example of how a swaption might be used, consider a company that knows that in 6 months it will enter into a 5-year floating-rate loan agreement and knows that it will wish to swap the floating interest payments for fixed interest payments to convert the loan into a fixed-rate loan (see Chapter 7 for a discussion of how swaps can be used in this way). At a cost, the company could enter into a swaption giving it the right to receive 6-month LIBOR and pay a certain fixed rate of interest, say 8% per annum, for a 5-year period starting in 6 months. If the fixed rate exchanged for floating on a regular 5-year swap in 6 months turns out to be less than 8% per annum, the company will choose not to exercise the swaption and will enter into a swap agreement in the usual way. However, if it turns out to be greater than 8% per annum, the company will choose to exercise the swaption and will obtain a swap at more favorable terms than those available in the market.

Swaptions, when used in the way just described, provide companies with a guarantee that the fixed rate of interest they will pay on a loan at some future time will not exceed some level. They are an alternative to forward swaps (sometimes called *deferred swaps*). Forward swaps involve no up-front cost but have the disadvantage of obligating the company to enter into a swap agreement. With a swaption, the company is able to benefit from favorable interest rate movements while acquiring protection from unfavorable interest rate movements. The difference between a swaption and a forward swap is analogous to the difference between an option on a foreign currency and a forward contract on the currency.

Business Snapshot 26.2 Swaptions and Bond Options

As explained in Chapter 7, an interest rate swap can be regarded as an agreement to exchange a fixed-rate bond for a floating-rate bond. At the start of a swap, the value of the floating-rate bond always equals the principal amount of the swap. A swaption can therefore be regarded as an option to exchange a fixed-rate bond for the principal amount of the swap—that is, a type of bond option.

If a swaption gives the holder the right to pay fixed and receive floating, it is a put option on the fixed-rate bond with strike price equal to the principal. If a swaption gives the holder the right to pay floating and receive fixed, it is a call option on the fixed-rate bond with a strike price equal to the principal.

Valuation of European Swap Options

As explained in Chapter 7 the swap rate for a particular maturity at a particular time is the (mid-market) fixed rate that would be exchanged for LIBOR in a newly issued swap with that maturity. The model usually used to value a European option on a swap assumes that the underlying swap rate at the maturity of the option is lognormal. Consider a swaption where we have the right to pay a rate s_K and receive LIBOR on a swap that will last n years starting in T years. We suppose that there are m payments per year under the swap and that the notional principal is L.

We saw in Chapter 7 that day count conventions may lead to the fixed payments under a swap being slightly different on each payment date. For now we will ignore the effect of day count conventions and assume that each fixed payment on the swap is the fixed rate times L/m. We consider the impact of day count conventions at the end of this section.

Suppose that the swap rate for an n-year swap starting at time T proves to be s_T. By comparing the cash flows on a swap where the fixed rate is s_T to the cash flows on a swap where the fixed rate is s_K, we see that the payoff from the swaption consists of a series of cash flows equal to

$$\frac{L}{m}\max(s_T - s_K,\ 0)$$

The cash flows are received m times per year for the n years of the life of the swap. Suppose that the swap payment dates are $T_1, T_2, \dots, T_{mn}$, measured in years from today. (It is approximately true that $T_k = T + k/m$.) Each cash flow is the payoff from a call option on s_T with strike price s_K.

Using equation (26.3), the value of the cash flow received at time T_i is

$$\frac{L}{m}P(0, T_i)[s_0 N(d_1) - s_K N(d_2)]$$

where

$$d_1 = \frac{\ln(s_0/s_K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(s_0/s_K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

s_0 is the forward swap rate at time zero calculated as indicated in equation (25.23), and σ is the volatility of the forward swap rate (so that $\sigma\sqrt{T}$ is the standard deviation of $\ln s_T$).

The total value of the swaption is

$$\sum_{i=1}^{mn} \frac{L}{m} P(0, T_i)[s_0 N(d_1) - s_K N(d_2)]$$

Defining A as the value of a contract that pays $1/m$ at times T_i $(1 \leqslant i \leqslant mn)$, the value of the swaption becomes

$$LA[s_0 N(d_1) - s_K N(d_2)] \tag{26.15}$$

where

$$A = \frac{1}{m} \sum_{i=1}^{mn} P(0, T_i)$$

If the swaption gives the holder the right to receive a fixed rate of s_K instead of paying it, the payoff from the swaption is

$$\frac{L}{m} \max(s_K - s_T, 0)$$

This is a put option on s_T. As before, the payoffs are received at times T_i $(1 \leqslant i \leqslant mn)$. Equation (26.4) gives the value of the swaption as

$$LA[s_K N(-d_2) - s_0 N(-d_1)] \tag{26.16}$$

Example 26.4

Suppose that the LIBOR yield curve is flat at 6% per annum with continuous compounding. Consider a swaption that gives the holder the right to pay 6.2% in a 3-year swap starting in 5 years. The volatility of the forward swap rate is 20%. Payments are made semiannually and the principal is \$100. In this case,

$$A = \tfrac{1}{2}[e^{-0.06\times5.5} + e^{-0.06\times6} + e^{-0.06\times6.5} + e^{-0.06\times7} + e^{-0.06\times7.5} + e^{-0.06\times8}] = 2.0035$$

A rate of 6% per annum with continuous compounding translates into 6.09% with semiannual compounding. It follows that, in this example, $s_0 = 0.0609$, $s_K = 0.062$, $T = 5$, and $\sigma = 0.2$, so that

$$d_1 = \frac{\ln(0.0609/0.062) + 0.2^2 \times 5/2}{0.2\sqrt{5}} = 0.1836 \quad \text{and} \quad d_2 = d_1 - 0.2\sqrt{5} = -0.2636$$

From equation (26.15), the value of the swaption is

$$100 \times 2.0035[0.0609 \times N(0.1836) - 0.062 \times N(-0.2636)] = 2.07$$

or \$2.07. (This is in agreement with the price given by DerivaGem.)

Broker Quotes

Brokers provide tables of implied volatilities for European swap options. The instruments underlying the quotes are usually at the money. This means that the strike swap rate equals the forward swap rate. Table 26.2 shows typical broker quotes provided for the US dollar market. The life of the option is shown on the vertical scale. This varies from 1 month to 5 years. The life of the underlying swap at the maturity of the option is shown on the horizontal scale. This varies from 1 to 10 years. The volatilities in the 1-year column of the table exhibit a hump similar to that discussed for caps earlier. As

Table 26.2 Typical broker quotes for US European swap options (mid-market volatilities percent per annum).

Expiration	*Swap length (years)*						
	1	*2*	*3*	*4*	*5*	*7*	*10*
1 month	17.75	17.75	17.75	17.50	17.00	17.00	16.00
3 months	19.50	19.00	19.00	18.00	17.50	17.00	16.00
6 months	20.00	20.00	19.25	18.50	18.75	17.75	16.75
1 year	22.50	21.75	20.50	20.00	19.50	18.25	16.75
2 years	22.00	22.00	20.75	19.50	19.75	18.25	16.75
3 years	21.50	21.00	20.00	19.25	19.00	17.75	16.50
4 years	20.75	20.25	19.25	18.50	18.25	17.50	16.00
5 years	20.00	19.50	18.50	17.75	17.50	17.00	15.50

we move to the columns corresponding to options on longer-lived swaps, the hump persists but it becomes less pronounced.

Theoretical Justification for the Swap Option Model

We can show that Black's model for swap options is internally consistent by considering a world that is forward risk neutral with respect to the annuity A. The analysis in Section 25.4 shows that:

1. The current value of any security is the current value of the annuity multiplied by the expected value of
$$\frac{\text{Security price at time } T}{\text{Value of the annuity at time } T}$$
in this world (see equation (25.25)).
2. The expected value of the swap rate at time T in this world equals the forward swap rate (see equation (25.24)).

The first result shows that the value of the swaption is

$$LAE_A[\max(s_T - s_K, 0)]$$

From the appendix at the end of Chapter 13, this is

$$LA[E_A(s_T)N(d_1) - s_K N(d_2)]$$

where

$$d_1 = \frac{\ln[E_A(s_T)/s_K] + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[E_A(s_T)/s_K] - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The second result shows that $E_A(s_T)$ equals s_0. Taken together, the results lead to the swap option pricing formula in equation (26.15). They show that we are entitled to treat interest rates as constant for the purposes of discounting provided that we also set the expected swap rate equal to the forward swap rate.

The Impact of Day Count Conventions

We now make the above formulas more precise by considering day count conventions. The fixed rate for the swap underlying the swap option is expressed with a day count convention such as actual/365 or 30/360. Suppose that $T_0 = T$ and that, for the applicable day count convention, the accrual fraction corresponding to the time period between T_{i-1} and T_i is a_i. (For example, if T_{i-1} corresponds to March 1 and T_i corresponds to September 1 and the day count is actual/365, $a_i = 184/365 = 0.5041$.) The formulas we have presented are then correct with the annuity factor A being defined as

$$A = \sum_{i=1}^{mn} a_i P(0, T_i)$$

As indicated by equation (25.23) the forward swap rate s_0 is given by solving

$$s_0 A = P(0, T) - P(0, T_{mn})$$

26.5 GENERALIZATIONS

We have presented three different versions of Black's model: one for bond options, one for caps, and one for swap options. Each of the models is internally consistent, but they are not consistent with each other. For example, when future bond prices are lognormal, future zero rates and swap rates are not lognormal; when future zero rates are lognormal, future bond prices and swap rates are not lognormal.

The results we have produced can be generalized:

1. Consider any instrument that provides a payoff at time T dependent on the value of a variable observed at time T. Its current value is $P(0, T)$ times the expected payoff provided that expectations are calculated in a world where the expected value of the underlying variable equals its forward price.
2. Consider any instrument that provides a payoff at time T^* dependent on the interest rate observed at time T for the period between T and T^*. Its current value is $P(0, T^*)$ times the expected payoff provided that expectations are calculated in a world where the expected value of the underlying interest rate equals the forward interest rate.
3. Consider any instrument that provides a payoff in the form of an annuity. We suppose that the size of the annuity is determined at time T as a function of the n-year swap rate at time T. We also suppose that annuity lasts for n years and payment dates for the annuity are the same as those for the swap. The value of the instrument is A times the expected payoff per year where (a) A is current value of the annuity when payments are at the rate \$1 per year and (b) expectations are taken in a world where the expected future swap rate equals the forward swap rate.

The first of these results is a generalization of the European bond option model; the second is a generalization of the cap/floor model; the third is a generalization of the swap option model.

26.6 HEDGING INTEREST RATE DERIVATIVES

This section discusses how the material on Greek letters in Chapter 15 can be extended to cover interest rate derivatives. In the context of interest rate derivatives, delta risk is the risk associated with a shift in the zero curve. Because there are many ways in which the zero curve can shift, many deltas can be calculated. Some alternatives are:

1. Calculate the impact of a 1-basis-point parallel shift in the zero curve. This is sometimes termed a DV01.
2. Calculate the impact of small changes in the quotes for each of the instruments used to construct the zero curve.
3. Divide the zero curve (or the forward curve) into a number of sections (or buckets). Calculate the impact of shifting the rates in one bucket by 1 basis point, keeping the rest of the initial term structure unchanged. (This is described in Business Snapshot 6.3.)
4. Carry out a principal components analysis as outlined in Section 18.9. Calculate a delta with respect to the changes in each of the first few factors. The first delta then measures the impact of a small, approximately parallel, shift in the zero curve; the second delta measures the impact of a small twist in the zero curve; and so on.

In practice, traders tend to prefer the second approach. They argue that the only way the zero curve can change is if the quote for one of the instruments used to compute the zero curve changes. They therefore feel that it makes sense to focus on the exposures arising from changes in the prices of these instruments.

When several delta measures are calculated, there are many possible gamma measures. Suppose that 10 instruments are used to compute the zero curve and that we measure deltas with respect to changes in the quotes for each of these. Gamma is a second partial derivative of the form $\partial^2\Pi/\partial x_i\,\partial x_j$, where Π is the portfolio value. We have 10 choices for x_i and 10 choices for x_j and a total of 55 different gamma measures. This may be "information overload". One approach is ignore cross-gammas and focus on the 10 partial derivatives where $i = j$. Another is to calculate a single gamma measure as the second partial derivative of the value of the portfolio with respect to a parallel shift in the zero curve. A further possibility is to calculate gammas with respect to the first two factors in a principal components analysis.

The vega of a portfolio of interest rate derivatives measures its exposure to volatility changes. One approach is to calculate the impact on the portfolio of the making the same small change to the Black volatilities of all caps and European swap options. However, this assumes that one factor drives all volatilities and may be too simplistic. A better idea is to carry out a principal components analysis on the volatilities of caps and swap options and calculate vega measures corresponding to the first 2 or 3 factors.

SUMMARY

Black's model provides a popular approach for valuing European-style interest rate options. The essence of Black's model is that the value of the variable underlying the option is assumed to be lognormal at the maturity of the option. In the case of a European bond option, Black's model assumes that the underlying bond price is lognormal at the option's maturity. For a cap, the model assumes that the interest rate underlying each of the constituent caplets is lognormally distributed. In the case of a swap option, the model assumes that the underlying swap rate is lognormally distributed. Each of these models is internally consistent, but they are not consistent with each other.

Black's model involves calculating the expected payoff based on the assumption that the expected value of a variable equals its forward value and then discounting the expected payoff at the zero rate observed in the market today. This is the correct procedure for the "plain vanilla" instruments we have considered in this chapter. However, as we will see in the next chapter it is not correct in all situations.

FURTHER READING

Black, F., "The Pricing of Commodity Contracts," *Journal of Financial Economics*, 3 (March 1976): 167–79.

Questions and Problems (Answers in Solutions Manual)

26.1. A company caps 3-month LIBOR at 10% per annum. The principal amount is \$20 million. On a reset date, 3-month LIBOR is 12% per annum. What payment would this lead to under the cap? When would the payment be made?

26.2. Explain why a swap option can be regarded as a type of bond option.

26.3. Use the Black's model to value a 1-year European put option on a 10-year bond. Assume that the current value of the bond is \$125, the strike price is \$110, the 1-year interest rate is 10% per annum, the bond's forward price volatility is 8% per annum, and the present value of the coupons to be paid during the life of the option is \$10.

26.4. Explain carefully how you would use (a) spot volatilities and (b) flat volatilities to value a 5-year cap.

26.5. Calculate the price of an option that caps the 3-month rate, starting in 15 months' time, at 13% (quoted with quarterly compounding) on a principal amount of \$1,000. The forward interest rate for the period in question is 12% per annum (quoted with quarterly compounding), the 18-month risk-free interest rate (continuously compounded) is 11.5% per annum, and the volatility of the forward rate is 12% per annum.

26.6. A bank uses Black's model to price European bond options. Suppose that an implied price volatility for a 5-year option on a bond maturing in 10 years is used to price a 9-year option on the bond. Would you expect the resultant price to be too high or too low? Explain.

26.7. Calculate the value of a 4-year European call option on bond that will mature 5 years from today using Black's model. The 5-year cash bond price is \$105, the cash price of a 4-year bond with the same coupon is \$102, the strike price is \$100, the 4-year risk-free interest rate is 10% per annum with continuous compounding, and the volatility for the bond price in 4 years is 2% per annum.

26.8. If the yield volatility for a 5-year put option on a bond maturing in 10 years time is specified as 22%, how should the option be valued? Assume that, based on today's interest rates the modified duration of the bond at the maturity of the option will be 4.2 years and the forward yield on the bond is 7%.

26.6 HEDGING INTEREST RATE DERIVATIVES

This section discusses how the material on Greek letters in Chapter 15 can be extended to cover interest rate derivatives. In the context of interest rate derivatives, delta risk is *the risk associated with a shift in the zero curve*. Because there are many ways in which the zero curve can shift, many deltas can be calculated. Some alternatives are:

1. Calculate the impact of a 1-basis-point parallel shift in the zero curve. This is sometimes termed a DV01.
2. Calculate the impact of small changes in the quotes for each of the instruments used to construct the zero curve.
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In practice, traders tend to prefer the second approach. They argue that the only way the zero curve can change is if the quote for one of the instruments used to compute the zero curve changes. They therefore feel that it makes sense to focus on the exposures arising from changes in the prices of these instruments.

When several delta measures are calculated, there are many possible gamma measures. Suppose that 10 instruments are used to compute the zero curve and that we measure deltas with respect to changes in the quotes for each of these. Gamma is a second partial derivative of the form $\partial^2\Pi/\partial x_i\,\partial x_j$, where Π is the portfolio value. We have 10 choices for x_i and 10 choices for x_j and a total of 55 different gamma measures. This may be "information overload". One approach is ignore cross-gammas and focus on the 10 partial derivatives where $i = j$. Another is to calculate a single gamma measure as the second partial derivative of the value of the portfolio with respect to a parallel shift in the zero curve. A further possibility is to calculate gammas with respect to the first two factors in a principal components analysis.

The vega of a portfolio of interest rate derivatives measures its exposure to volatility changes. One approach is to calculate the impact on the portfolio of the making the same small change to the Black volatilities of all caps and European swap options. However, this assumes that one factor drives all volatilities and may be too simplistic. A better idea is to carry out a principal components analysis on the volatilities of caps and swap options and calculate vega measures corresponding to the first 2 or 3 factors.

SUMMARY

Black's model provides a popular approach for valuing European-style interest rate options. The essence of Black's model is that the value of the variable underlying the option is assumed to be lognormal at the maturity of the option. In the case of a European bond option, Black's model assumes that the underlying bond price is lognormal at the option's maturity. For a cap, the model assumes that the interest rate underlying each of the constituent caplets is lognormally distributed. In the case of a swap option, the model assumes that the underlying swap rate is lognormally distributed. Each of these models is internally consistent, but they are not consistent with each other.

Black's model involves calculating the expected payoff based on the assumption that the expected value of a variable equals its forward value and then discounting the expected payoff at the zero rate observed in the market today. This is the correct procedure for the "plain vanilla" instruments we have considered in this chapter. However, as we will see in the next chapter it is not correct in all situations.

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26.3. Use the Black's model to value a 1-year European put option on a 10-year bond. Assume that the current value of the bond is \$125, the strike price is \$110, the 1-year interest rate is 10% per annum, the bond's forward price volatility is 8% per annum, and the present value of the coupons to be paid during the life of the option is \$10.

26.4. Explain carefully how you would use (a) spot volatilities and (b) flat volatilities to value a 5-year cap.

26.5. Calculate the price of an option that caps the 3-month rate, starting in 15 months' time, at 13% (quoted with quarterly compounding) on a principal amount of \$1,000. The forward interest rate for the period in question is 12% per annum (quoted with quarterly compounding), the 18-month risk-free interest rate (continuously compounded) is 11.5% per annum, and the volatility of the forward rate is 12% per annum.

26.6. A bank uses Black's model to price European bond options. Suppose that an implied price volatility for a 5-year option on a bond maturing in 10 years is used to price a 9-year option on the bond. Would you expect the resultant price to be too high or too low? Explain.

26.7. Calculate the value of a 4-year European call option on bond that will mature 5 years from today using Black's model. The 5-year cash bond price is \$105, the cash price of a 4-year bond with the same coupon is \$102, the strike price is \$100, the 4-year risk-free interest rate is 10% per annum with continuous compounding, and the volatility for the bond price in 4 years is 2% per annum.

26.8. If the yield volatility for a 5-year put option on a bond maturing in 10 years time is specified as 22%, how should the option be valued? Assume that, based on today's interest rates the modified duration of the bond at the maturity of the option will be 4.2 years and the forward yield on the bond is 7%.

26.9. What other instrument is the same as a 5-year zero-cost collar where the strike price of the cap equals the strike price of the floor? What does the common strike price equal?

26.10. Derive a put–call parity relationship for European bond options.

26.11. Derive a put–call parity relationship for European swap options.

26.12. Explain why there is an arbitrage opportunity if the implied Black (flat) volatility of a cap is different from that of a floor. Do the broker quotes in Table 26.1 present an arbitrage opportunity?

26.13. When a bond's price is lognormal can the bond's yield be negative? Explain your answer.

26.14. What is the value of a European swap option that gives the holder the right to enter into a 3-year annual-pay swap in 4 years where a fixed rate of 5% is paid and LIBOR is received? The swap principal is \$10 million. Assume that the yield curve is flat at 5% per annum with annual compounding and the volatility of the swap rate is 20%. Compare your answer with that given by DerivaGem.

26.15. Suppose that the yield R on a zero-coupon bond follows the process

$$dR = \mu\, dt + \sigma dz$$

where μ and σ are functions of R and t, and dz is a Wiener process. Use Itô's lemma to show that the volatility of the zero-coupon bond price declines to zero as it approaches maturity.

26.16. Carry out a manual calculation to verify the option prices in Example 26.2.

26.17. Suppose that the 1-year, 2-year, 3-year, 4-year, and 5-year zero rates are 6%, 6.4%, 6.7%, 6.9%, and 7%. The price of a 5-year semiannual cap with a principal of \$100 at a cap rate of 8% is \$3. Use DerivaGem to determine:
(a) The 5-year flat volatility for caps and floors
(b) The floor rate in a zero-cost 5-year collar when the cap rate is 8%

26.18. Show that $V_1 + f = V_2$, where V_1 is the value of a swap option to pay a fixed rate of s_K and receive LIBOR between times T_1 and T_2, f is the value of a forward swap to receive a fixed rate of s_K and pay LIBOR between times T_1 and T_2, and V_2 is the value of a swap option to receive a fixed rate of s_K between times T_1 and T_2. Deduce that $V_1 = V_2$ when s_K equals the current forward swap rate.

26.19. Suppose that zero rates are as in Problem 26.17. Use DerivaGem to determine the value of an option to pay a fixed rate of 6% and receive LIBOR on a 5-year swap starting in 1 year. Assume that the principal is \$100 million, payments are exchanged semiannually, and the swap rate volatility is 21%.

26.20. Describe how you would (a) calculate cap flat volatilities from cap spot volatilities and (b) calculate cap spot volatilities from cap flat volatilities.

Assignment Questions

26.21. Consider an 8-month European put option on a Treasury bond that currently has 14.25 years to maturity. The current cash bond price is \$910, the exercise price is \$900, and the volatility for the bond price is 10% per annum. A coupon of \$35 will be paid by the bond in 3 months. The risk-free interest rate is 8% for all maturities up to 1 year. Use Black's model to determine the price of the option. Consider both the case where the strike price corresponds to the cash price of the bond and the case where it corresponds to the quoted price.

26.22. Calculate the price of a cap on the 90-day LIBOR rate in 9 months' time when the principal amount is \$1,000. Use Black's model and the following information:
(a) The quoted 9-month Eurodollar futures price = 92. (Ignore differences between futures and forward rates.)
(b) The interest rate volatility implied by a 9-month Eurodollar option = 15% per annum.
(c) The current 12-month interest rate with continuous compounding = 7.5% per annum.
(d) The cap rate = 8% per annum. (Assume an actual/360 day count.)

26.23. Suppose that the LIBOR yield curve is flat at 8% with annual compounding. A swaption gives the holder the right to receive 7.6% in a 5-year swap starting in 4 years. Payments are made annually. The volatility of the forward swap rate is 25% per annum and the principal is \$1 million. Use Black's model to price the swaption. Compare your answer with that given by DerivaGem.

26.24. Use the DerivaGem software to value a 5-year collar that guarantees that the maximum and minimum interest rates on a LIBOR-based loan (with quarterly resets) are 5% and 7%, respectively. The LIBOR zero curve (continuously compounded) is currently flat at 6%. Use a flat volatility of 20%. Assume that the principal is \$100.

26.25. Use the DerivaGem software to value a European swap option that gives you the right in 2 years to enter into a 5-year swap in which you pay a fixed rate of 6% and receive floating. Cash flows are exchanged semiannually on the swap. The 1-year, 2-year, 5-year, and 10-year zero-coupon interest rates (continuously compounded) are 5%, 6%, 6.5%, and 7%, respectively. Assume a principal of \$100 and a volatility of 15% per annum. Give an example of how the swap option might be used by a corporation. What bond option is equivalent to the swap option?