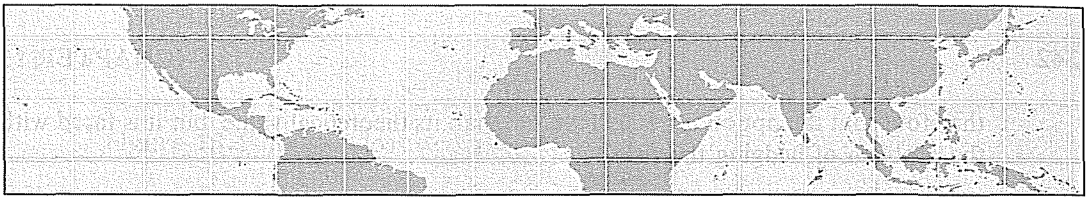


CHAPTER 15 The Greek Letters

A financial institution that sells an option to a client in the over-the-counter markets is faced with the problem of managing its risk. If the option happens to be the same as one that is traded on an exchange, the financial institution can neutralize its exposure by buying on the exchange the same option as it has sold. But when the option has been tailored to the needs of a client and does not correspond to the standardized products traded by exchanges, hedging the exposure is far more difficult.

In this chapter we discuss some of the alternative approaches to this problem. We cover what are commonly referred to as the "Greek letters", or simply the "Greeks". Each Greek letter measures a different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable. The analysis presented in this chapter is applicable to market makers in options on an exchange as well as to traders working in the over-the-counter market for financial institutions.

Toward the end of the chapter, we will consider the creation of options synthetically. This turns out to be very closely related to the hedging of options. Creating an option position synthetically is essentially the same task as hedging the opposite option position. For example, creating a long call option synthetically is the same as hedging a short position in the call option.

15.1 ILLUSTRATION

In the next few sections we use as an example the position of a financial institution that has sold for \$300,000 a European call option on 100,000 shares of a non-dividend-paying stock. We assume that the stock price is \$49, the strike price is \$50, the risk-free interest rate is 5% per annum, the stock price volatility is 20% per annum, the time to maturity is 20 weeks (0.3846 years), and the expected return from the stock is 13% per annum.[1] With our usual notation, this means that

$$S_0 = 49, \quad K = 50, \quad r = 0.05, \quad \sigma = 0.20, \quad T = 0.3846, \quad \mu = 0.13$$

The Black–Scholes price of the option is about \$240,000. The financial institution has

[1] As shown in Chapters 11 and 13, the expected return is irrelevant to the pricing of an option. It is given here because it can have some bearing on the effectiveness of a hedging scheme.

therefore sold the option for \$60,000 more than its theoretical value, but it is faced with the problem of hedging the risks.[2]

15.2 NAKED AND COVERED POSITIONS

One strategy open to the financial institution is to do nothing. This is sometimes referred to as adopting a *naked position*. It is a strategy that works well if the stock price is below \$50 at the end of the 20 weeks. The option then costs the financial institution nothing and it makes a profit of \$300,000. A naked position works less well if the call is exercised because the financial institution then has to buy 100,000 shares at the market price prevailing in 20 weeks to cover the call. The cost to the financial institution is 100,000 times the amount by which the stock price exceeds the strike price. For example, if after 20 weeks the stock price is \$60, the option costs the financial institution \$1,000,000. This is considerably greater than the \$300,000 charged for the option.

As an alternative to a naked position, the financial institution can adopt a *covered position*. This involves buying 100,000 shares as soon as the option has been sold. If the option is exercised, this strategy works well, but in other circumstances it could lead to a significant loss. For example, if the stock price drops to \$40, the financial institution loses \$900,000 on its stock position. This is considerably greater than the \$300,000 charged for the option.[3]

Neither a naked position nor a covered position provides a good hedge. If the assumptions underlying the Black–Scholes formula hold, the cost to the financial institution should always be \$240,000 on average for both approaches.[4] But on any one occasion the cost is liable to range from zero to over \$1,000,000. A good hedge would ensure that the cost is always close to \$240,000.

15.3 A STOP-LOSS STRATEGY

One interesting hedging scheme that is sometimes proposed involves a *stop-loss strategy*. To illustrate the basic idea, consider an institution that has written a call option with strike price K to buy one unit of a stock. The hedging scheme involves buying one unit of the stock as soon as its price rises above K and selling it as soon as its price falls below K. The objective is to hold a naked position whenever the stock price is less than K and a covered position whenever the stock price is greater than K. The scheme is designed to ensure that at time T the institution owns the stock if the option closes in the money and does not own it if the option closes out of the money. The strategy appears to produce payoffs that are the same as the payoffs on the option. In the situation illustrated in Figure 15.1, it involves buying the stock at time t_1, selling it at time t_2, buying it at time t_3, selling it at time t_4, buying it at time t_5, and delivering it at time T.

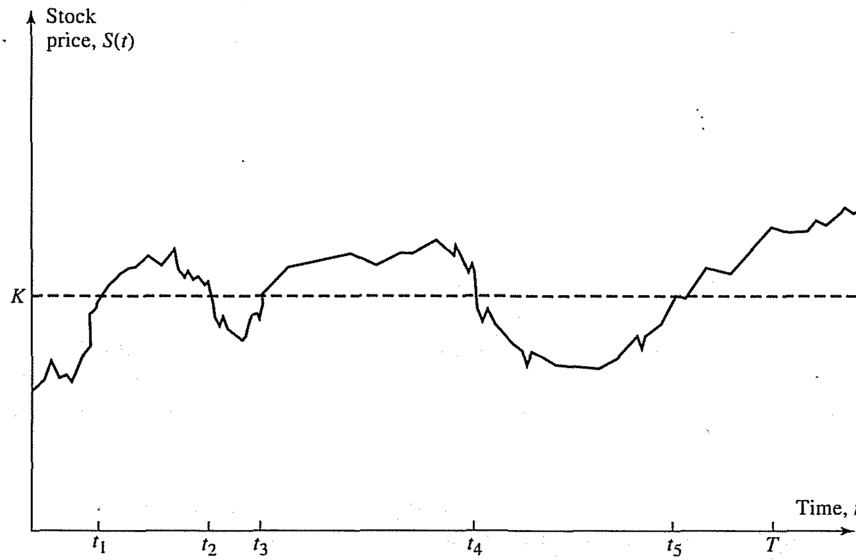
---

[2] A call option on a non-dividend-paying stock is a convenient example with which to develop our ideas. The points that will be made apply to other types of options and to other derivatives.

[3] Put–call parity shows that the exposure from writing a covered call is the same as the exposure from writing a naked put.

[4] More precisely, the present value of the expected cost is \$240,000 for both approaches assuming that appropriate risk-adjusted discount rates are used.

Figure 15.1 A stop-loss strategy.



As usual, we denote the initial stock price by S_0. The cost of setting up the hedge initially is S_0 if $S_0 > K$ and zero otherwise. It seems as though the total cost, Q, of writing and hedging the option is the option's intrinsic value:

$$Q = \max(S_0 - K,\ 0) \tag{15.1}$$

This is because all purchases and sales subsequent to time 0 are made at price K. If this were in fact correct, the hedging scheme would work perfectly in the absence of transactions costs. Furthermore, the cost of hedging the option would always be less than its Black–Scholes price. Thus, an investor could earn riskless profits by writing options and hedging them.

There are two basic reasons why equation (15.1) is incorrect. The first is that the cash flows to the hedger occur at different times and must be discounted. The second is that purchases and sales cannot be made at exactly the same price K. This second point is critical. If we assume a risk-neutral world with zero interest rates, we can justify ignoring the time value of money. But we cannot legitimately assume that both purchases and sales are made at the same price. If markets are efficient, the hedger cannot know whether, when the stock price equals K, it will continue above or below K.

As a practical matter, purchases must be made at a price $K + \epsilon$ and sales must be made at a price $K - \epsilon$, for some small positive number ϵ. Thus, every purchase and subsequent sale involves a cost (apart from transaction costs) of 2ϵ. A natural response on the part of the hedger is to monitor price movements more closely, so that ϵ is reduced. Assuming that stock prices change continuously, ϵ can be made arbitrarily small by monitoring the stock prices closely. But as ϵ is made smaller, trades tend to occur more frequently. Thus, the lower cost per trade is offset by the

Table 15.1 Performance of stop-loss strategy. The performance measure is the ratio of the standard deviation of the cost of writing the option and hedging it to the theoretical price of the option.

Δt (weeks)	5	4	2	1	0.5	0.25
Hedge performance	1.02	0.93	0.82	0.77	0.76	0.76

increased frequency of trading. As $\epsilon \to 0$, the expected number of trades tends to infinity.[5]

A stop-loss strategy, although superficially attractive, does not work particularly well as a hedging scheme. Consider its use for an out-of-the-money option. If the stock price never reaches the strike price K, the hedging scheme costs nothing. If the path of the stock price crosses the strike price level many times, the scheme is quite expensive. Monte Carlo simulation can be used to assess the overall performance of stop-loss hedging. This involves randomly sampling paths for the stock price and observing the results of using the scheme. Table 15.1 shows the results for the option considered earlier. It assumes that the stock price is observed at the end of time intervals of length Δt.[6] The hedge performance measure is the ratio of the standard deviation of the cost of hedging the option to the Black–Scholes option price. Each result is based on 1,000 sample paths for the stock price and has a standard error of about 2%. A perfect hedge would have a hedge performance measure of zero. In this case it appears to be impossible to produce a value for the hedge performance measure below 0.70 regardless of how small Δt is made.

15.4 DELTA HEDGING

Most traders use more sophisticated hedging schemes than those mentioned so far. These involve calculating measures such as delta, gamma, and vega. In this section we consider the rôle played by delta.

The *delta* (Δ) of an option was introduced in Chapter 11. It is defined as the rate of change of the option price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price. Suppose that the delta of a call option on a stock is 0.6. This means that when the stock price changes by a small amount, the option price changes by about 60% of that amount. Figure 15.2 shows the relationship between a call price and the underlying stock price. When the stock price corresponds to point A, the option price corresponds to point B, and Δ is the slope of the line indicated. In general,

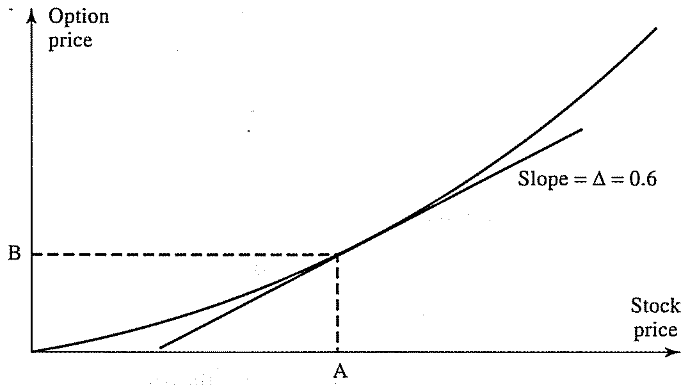
$$\Delta = \frac{\partial c}{\partial S}$$

where c is the price of the call option and S is the stock price.

[5] As mentioned in Section 12.2, the expected number of times a Wiener process equals any particular value in a given time interval is infinite.

[6] The precise hedging rule used was as follows. If the stock price moves from below K to above K in a time interval of length Δt, it is bought at the end of the interval. If it moves from above K to below K in the time interval, it is sold at the end of the interval; otherwise, no action is taken.

Figure 15.2 Calculation of delta.



Suppose that, in Figure 15.2, the stock price is \$100 and the option price is \$10. Imagine an investor who has sold 20 call option contracts—that is, options to buy 2,000 shares. The investor's position could be hedged by buying $0.6 \times 2{,}000 = 1{,}200$ shares. The gain (loss) on the option position would then tend to be offset by the loss (gain) on the stock position. For example, if the stock price goes up by \$1 (producing a gain of \$1,200 on the shares purchased), the option price will tend to go up by $0.6 \times \$1 = \$0.60$ (producing a loss of \$1,200 on the options written); if the stock price goes down by \$1 (producing a loss of \$1,200 on the shares purchased), the option price will tend to go down by \$0.60 (producing a gain of \$1,200 on the options written).

In this example, the delta of the investor's option position is

$$0.6 \times (-2{,}000) = -1{,}200$$

In other words, the investor loses $1{,}200\Delta S$ on the short option position when the stock price increases by ΔS. The delta of the stock is 1.0, so that the long position in 1,200 shares has a delta of +1,200. The delta of the investor's overall position is, therefore, zero. The delta of the stock position offsets the delta of the option position. A position with a delta of zero is referred to as being *delta neutral.*

It is important to realize that, because delta changes, the investor's position remains delta hedged (or delta neutral) for only a relatively short period of time. The hedge has to be adjusted periodically. This is known as *rebalancing*. In our example, at the end of 3 days the stock price might increase to \$110. As indicated by Figure 15.2, an increase in the stock price leads to an increase in delta. Suppose that delta rises from 0.60 to 0.65. An extra $0.05 \times 2{,}000 = 100$ shares would then have to be purchased to maintain the hedge.

The delta-hedging scheme just described is an example of a *dynamic-hedging scheme.* It can be contrasted with *static-hedging schemes*, where the hedge is set up initially and never adjusted. Static hedging schemes are sometimes also referred to as *hedge-and-forget schemes*. Delta is closely related to the Black–Scholes–Merton analysis. As explained in Chapter 13, Black, Scholes, and Merton showed that it is possible to set up a riskless portfolio consisting of a position in an option on a stock and a position in the stock. Expressed in terms of Δ, the Black–Scholes portfolio is

-1: option
$+\Delta$: shares of the stock

Using our new terminology, we can say that Black and Scholes valued options by setting up a delta-neutral position and arguing that the return on the position should be the risk-free interest rate.

Delta of European Stock Options

For a European call option on a non-dividend-paying stock, it can be shown (see Problem 13.17) that

$$\Delta(\text{call}) = N(d_1)$$

where d_1 is defined as in equation (13.20). Using delta hedging for a short position in a European call option therefore involves keeping a long position of $N(d_1)$ shares at any given time. Similarly, using delta hedging for a long position in a European call option involves maintaining a short position of $N(d_1)$ shares at any given time.

For a European put option on a non-dividend-paying stock, delta is given by

$$\Delta(\text{put}) = N(d_1) - 1$$

Delta is negative, which means that a long position in a put option should be hedged with a long position in the underlying stock, and a short position in a put option should be hedged with a short position in the underlying stock. Figure 15.3 shows the variation of the delta of a call option and a put option with the stock price. Figure 15.4 shows the variation of delta with the time to maturity for in-the-money, at-the-money, and out-of-the-money call options.

Figure 15.3 Variation of delta with stock price for (a) a call option and (b) a put option on a non-dividend-paying stock.

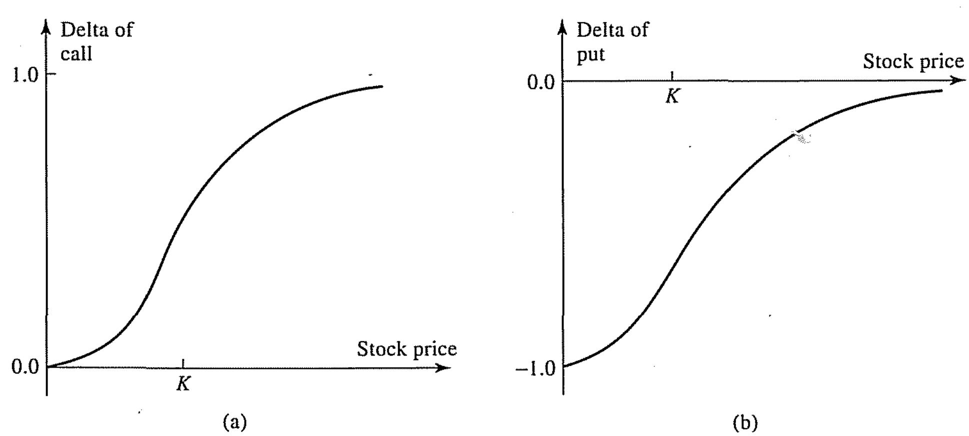
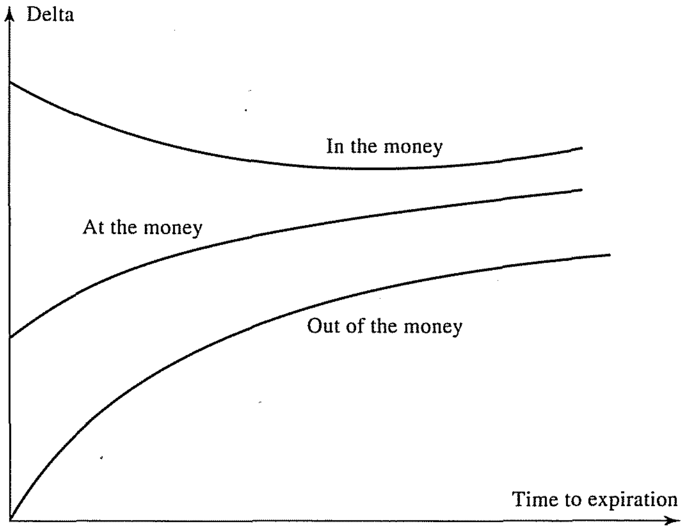


Figure 15.4 Typical patterns for variation of delta with time to maturity for a call option.



Delta of Other European Options

For European call options on an asset paying a yield q,

$$\Delta(\text{call}) = e^{-qT} N(d_1)$$

where d_1 is defined by equation (14.4). For European put options on the asset,

$$\Delta(\text{put}) = e^{-qT}[N(d_1) - 1]$$

When the asset is a stock index, these formulas are correct with q equal to the dividend yield on the index. When the asset is a currency, they are correct with q equal to the foreign risk-free rate, r_f. When the asset is a futures contract, they are correct with q equal to the domestic risk-free rate, r, and $S_0 = F_0$ in the definition of d_1. (In the latter case, delta gives the rate of change of the option price with respect to the futures price.)

Example 15.1

A US bank has sold 6-month put options on £1 million with a strike price of 1.6000 and wishes to make its portfolio delta neutral. Suppose that the current exchange rate is 1.6200, the risk-free interest rate in the United Kingdom is 13% per annum, the risk-free interest rate in the United States is 10% per annum, and the volatility of sterling is 15%. In this case, $S_0 = 1.6200$, $K = 1.6000$, $r = 0.10$, $r_f = 0.13$, $\sigma = 0.15$, and $T = 0.5$. The delta of a put option on a currency is

$$[N(d_1) - 1]e^{-r_f T}$$

where d_1 is given by equation (14.7). It can be shown that

$$d_1 = 0.0287 \quad \text{and} \quad N(d_1) = 0.5115$$

so that the delta of the put option is -0.458. This is the delta of a long position in one put option. (It means that, when the exchange rate increases by ΔS, the price of the put goes down by 45.8% of ΔS.) The delta of the bank's total short option position is +458,000. To make the position delta neutral, we must therefore add a short sterling position of £458,000 to the option position. This short sterling position has a delta of $-458{,}000$ and neutralizes the delta of the option position.

Delta of Forward Contracts

The concept of delta can be applied to financial instruments other than options. Consider a forward contract on a non-dividend-paying stock. Equation (5.5) shows that the value of a forward contract is $S_0 - Ke^{-rT}$, where K is the delivery price and T is the forward contract's time to maturity. When the price of the stock changes by ΔS, with all else remaining the same, the value of a forward contract on the stock also changes by ΔS. The delta of a forward contract on one share of the stock is therefore always 1.0. This means that a short forward contract on one share can be hedged by purchasing one share; a long forward contract on one share can be hedged by shorting one share.[7]

For an asset providing a dividend yield at rate q, equation (5.7) shows that the forward contract's delta is e^{-qT}. In the case of a stock index, q is set equal to the dividend yield on the index. For a currency, it is set equal to the foreign risk-free rate, r_f.

Delta of a Futures Contract

From equation (5.1), the futures price for a contract on a non-dividend-paying stock is S_0e^{rT}, where T is the time to maturity of the futures contract. This shows that when the price of the stock changes by ΔS, with all else remaining the same, the futures price changes by $\Delta S\, e^{rT}$. Since futures contracts are marked to market daily, the holder of a long futures contract makes an almost immediate gain of this amount. The delta of a futures contract is therefore e^{rT}. For a futures contract on an asset providing a dividend yield at rate q, equation (5.3) shows similarly that delta is $e^{(r-q)T}$. It is interesting that the impact of marking to market is to make the deltas of futures and forward contracts slightly different. This is true even when interest rates are constant and the forward price equals the futures price.

Sometimes a futures contract is used to achieve a delta-neutral position. Define:

T: Maturity of futures contract

H_A: Required position in asset for delta hedging

H_F: Alternative required position in futures contracts for delta hedging

If the underlying asset is a non-dividend-paying stock, the analysis we have just given shows that

$$H_F = e^{-rT} H_A \tag{15.2}$$

[7] These are hedge-and-forget schemes. Since delta is always 1.0, no changes need to be made to the position in the stock during the life of the contract.

When the underlying asset pays a dividend yield q,

$$H_F = e^{-(r-q)T} H_A \tag{15.3}$$

For a stock index, we set q equal to the dividend yield on the index; for a currency we set it equal to the foreign risk-free rate, r_f, so that

$$H_F = e^{-(r-r_f)T} H_A \tag{15.4}$$

Example 15.2

Consider the option in the previous example where hedging using the currency requires a short position of 458,000 pounds sterling. From equation (15.4), hedging using 9-month currency futures requires a short futures position of

$$e^{-(0.10-0.13)\times 9/12} 458{,}000$$

or £468,442. Because each futures contract is for the purchase or sale of £62,500, seven contracts should be shorted (seven being the nearest whole number to 468,442/62,500).

Dynamic Aspects of Delta Hedging

Tables 15.2 and 15.3 provide two examples of the operation of delta hedging for the example in Section 15.1. The hedge is assumed to be adjusted or rebalanced weekly. The initial value of delta can be calculated from the data in Section 15.1 as 0.522. This means that, as soon as the option is written, \$2,557,800 must be borrowed to buy 52,200 shares at a price of \$49. The rate of interest is 5%. An interest cost of approximately \$2,500 is therefore incurred in the first week.

In Table 15.2 the stock price falls by the end of the first week to \$48.12. The delta declines to 0.458, and 6,400 of shares are sold to maintain the hedge. The strategy realizes \$308,000 in cash, and the cumulative borrowings at the end of Week 1 are reduced to \$2,252,300. During the second week, the stock price reduces to \$47.37, delta declines again, and so on. Toward the end of the life of the option, it becomes apparent that the option will be exercised and delta approaches 1.0. By Week 20, therefore, the hedger has a fully covered position. The hedger receives \$5 million for the stock held, so that the total cost of writing the option and hedging it is \$263,300.

Table 15.3 illustrates an alternative sequence of events such that the option closes out of the money. As it becomes clear that the option will not be exercised, delta approaches zero. By Week 20 the hedger has a naked position and has incurred costs totaling \$256,600.

In Tables 15.2 and 15.3, the costs of hedging the option, when discounted to the beginning of the period, are close to but not exactly the same as the Black–Scholes price of \$240,000. If the hedging scheme worked perfectly, the cost of hedging would, after discounting, be exactly equal to the Black–Scholes price for every simulated stock price path. The reason for the variation in the cost of delta hedging is that the hedge is rebalanced only once a week. As rebalancing takes place more frequently, the variation in the cost of hedging is reduced. Of course, the examples in Tables 15.2 and 15.3 are idealized in that they assume that the volatility is constant and there are no transaction costs.

Table 15.2 Simulation of delta hedging. Option closes in the money and cost of hedging is $263,300.

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased ($000)*	*Cumulative cost including interest ($000)*	*Interest cost ($000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

Table 15.4 shows statistics on the performance of delta hedging obtained from 1,000 random stock price paths in our example. As in Table 15.1, the performance measure is the ratio of the standard deviation of the cost of hedging the option to the Black–Scholes price of the option. It is clear that delta hedging is a great improvement over a stop-loss strategy. Unlike a stop-loss strategy, the performance of a delta-hedging strategy gets steadily better as the hedge is monitored more frequently.

Delta hedging aims to keep the value of the financial institution's position as close to unchanged as possible. Initially, the value of the written option is $240,000. In the situation depicted in Table 15.2, the value of the option can be calculated as $414,500 in Week 9. Thus, the financial institution has lost $174,500 on its short option position. Its cash position, as measured by the cumulative cost, is $1,442,900 worse in Week 9 than in Week 0. The value of the shares held has increased from $2,557,800 to $4,171,100. The net effect of all this is that the value of the financial institution's position has changed by only $4,100 during the 9-week period.

Table 15.3 Simulation of delta hedging. Option closes out of the money and cost of hedging is $256,600.

Week	*Stock price*	*Delta*	*Shares purchased*	*Cost of shares purchased ($000)*	*Cumulative cost including interest ($000)*	*Interest cost ($000)*
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
3	50.00	0.579	(12,600)	(630.0)	2,877.7	2.8
4	48.38	0.459	(12,000)	(580.6)	2,299.9	2.2
5	48.25	0.443	(1,600)	(77.2)	2,224.9	2.1
6	48.75	0.475	3,200	156.0	2,383.0	2.3
7	49.63	0.540	6,500	322.6	2,707.9	2.6
8	48.25	0.420	(12,000)	(579.0)	2,131.5	2.1
9	48.25	0.410	(1,000)	(48.2)	2,085.4	2.0
10	51.12	0.658	24,800	1,267.8	3,355.2	3.2
11	51.50	0.692	3,400	175.1	3,533.5	3.4
12	49.88	0.542	(15,000)	(748.2)	2,788.7	2.7
13	49.88	0.538	(400)	(20.0)	2,771.4	2.7
14	48.75	0.400	(13,800)	(672.7)	2,101.4	2.0
15	47.50	0.236	(16,400)	(779.0)	1,324.4	1.3
16	48.00	0.261	2,500	120.0	1,445.7	1.4
17	46.25	0.062	(19,900)	(920.4)	526.7	0.5
18	48.13	0.183	12,100	582.4	1,109.6	1.1
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

Where the Cost Comes From

The delta-hedging scheme in Tables 15.2 and 15.3 in effect creates a long position in the option synthetically. This neutralizes the short position arising from the option that has been written. The scheme generally involves selling stock just after the price has gone down and buying stock just after the price has gone up. It might be termed a buy-high, sell-low scheme! The cost of $240,000 comes from the average difference between the price paid for the stock and the price realized for it.

Table 15.4 Performance of delta hedging. The performance measure is the ratio of the standard deviation of the cost of writing the option and hedging it to the theoretical price of the option.

Time between hedge rebalancing (weeks):	5	4	2	1	0.5	0.25
Performance measure:	0.43	0.39	0.26	0.19	0.14	0.09

Delta of a Portfolio

The delta of a portfolio of options or other derivatives dependent on a single asset whose price is S is

$$\frac{\partial \Pi}{\partial S}$$

where Π is the value of the portfolio.

The delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity w_i of option i ($1 \leqslant i \leqslant n$), the delta of the portfolio is given by

$$\Delta = \sum_{i=1}^{n} w_i \, \Delta_i$$

where Δ_i is the delta of ith option. The formula can be used to calculate the position in the underlying asset or in a futures contract on the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being *delta neutral.*

Suppose a financial institution in the United States has the following three positions in options on the Australian dollar:

1. A long position in 100,000 call options with strike price 0.55 and an expiration date in 3 months. The delta of each option is 0.533.
2. A short position in 200,000 call options with strike price 0.56 and an expiration date in 5 months. The delta of each option is 0.468.
3. A short position in 50,000 put options with strike price 0.56 and an expiration date in 2 months. The delta of each option is −0.508.

The delta of the whole portfolio is

$$100{,}000 \times 0.533 - 200{,}000 \times 0.468 - 50{,}000 \times (-0.508) = -14{,}900$$

This means that the portfolio can be made delta neutral with a long position of 14,900 Australian dollars.

A 6-month forward contract could also be used to achieve delta neutrality here. Suppose that the risk-free rate of interest is 8% per annum in Australia and 5% in the United States ($r = 0.05$ and $r_f = 0.08$). The delta of a forward contract maturing at time T on one Australian dollar is $e^{-r_f T}$ or $e^{-0.08 \times 0.5} = 0.9608$. The long position in Australian dollar forward contracts for delta neutrality is therefore $14{,}900/0.9608 = 15{,}508$.

Another alternative is to use a 6-month futures contract. From equation (15.4), the long position in Australian dollar futures for delta neutrality is

$$14{,}900 e^{-(0.05-0.08)\times 0.5} = 15{,}125$$

Transaction Costs

Maintaining a delta-neutral position in a single option and the underlying asset, in the way that has just been described, is liable to be prohibitively expensive because of the transactions costs incurred on trades. For a large portfolio of options, delta neutrality is more feasible. Only one trade in the underlying asset is necessary to zero out delta for the whole portfolio. The hedging transactions costs are absorbed by the profits on many different trades.

15.5 THETA

The *theta* (Θ) of a portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same. Theta is sometimes referred to as the *time decay* of the portfolio. For a European call option on a non-dividend-paying stock, it can be shown from the Black–Scholes formula that

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where d_1 and d_2 are defined as in equation (13.20) and

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \tag{15.5}$$

For a European put option on the stock (see Problem 13.17),

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

For a European call option on an asset paying a dividend at rate q,

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma e^{-qT}}{2\sqrt{T}} + qS_0 N(d_1)e^{-qT} - rKe^{-rT}N(d_2)$$

where d_1 and d_2 are defined as in equation (14.4), and, for a European put option on the asset,

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1)\sigma e^{-qT}}{2\sqrt{T}} - qS_0 N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2)$$

When the asset is a stock index, these last two equations are true with q equal to the dividend yield on the index. When it is a currency, they are true with q equal to the foreign risk-free rate, r_f. When it is a futures contract, they are true with $S_0 = F_0$ and $q = r$.

In these formulas, time is measured in years. Usually, when theta is quoted, time is measured in days, so that theta is the change in the portfolio value when 1 day passes with all else remaining the same. We can measure theta either "per calendar day" or "per trading day". To obtain the theta per calendar day, the formula for theta must be divided by 365; to obtain theta per trading day, it must be divided by 252. (DerivaGem measures theta per calendar day.)

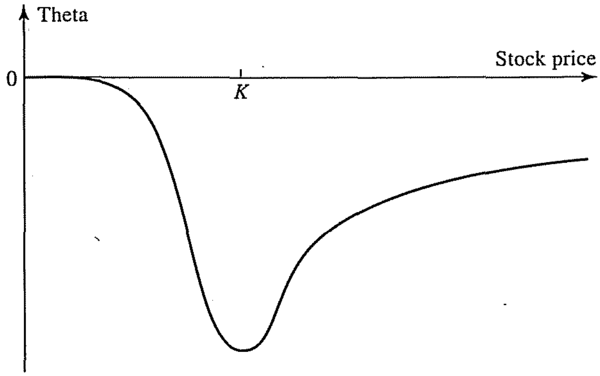
Example 15.3

Consider a 4-month put option on a stock index. The current value of the index is 305, the strike price is 300, the dividend yield is 3% per annum, the risk-free interest rate is 8% per annum, and the volatility of the index is 25% per annum. In this case, $S_0 = 305$, $K = 300$, $q = 0.03$, $r = 0.08$, $\sigma = 0.25$, and $T = 0.3333$. The option's theta is

$$-\frac{S_0 N'(d_1)\sigma e^{-qT}}{2\sqrt{T}} - qS_0 N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2) = -18.15$$

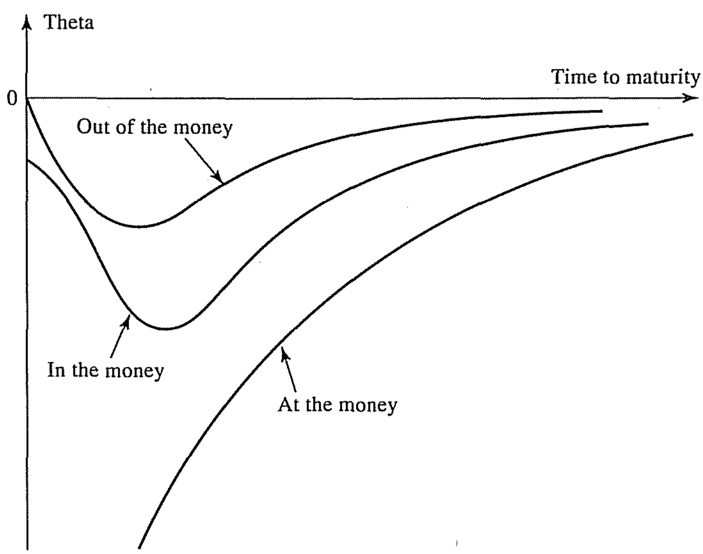
The theta is $-18.15/365 = -0.0497$ per calendar day or $-18.15/252 = -0.0720$ per trading day.

Figure 15.5 Variation of theta of a European call option with stock price.



Theta is usually negative for an option.[8] This is because, as the time to maturity decreases with all else remaining the same, the option tends to become less valuable. The variation of Θ with stock price for a call option on a stock is shown in Figure 15.5. When the stock price is very low, theta is close to zero. For an at-the-money call option, theta is large and negative. As the stock price becomes larger, theta tends to $-rKe^{-rT}$. Figure 15.6 shows typical patterns for the variation of Θ with the time to maturity for in-the-money, at-the-money, and out-of-the-money call options.

Figure 15.6 Typical patterns for variation of theta of a European call option with time to maturity.



[8] An exception to this could be an in-the-money European put option on a non-dividend-paying stock or an in-the-money European call option on a currency with a very high interest rate.

Theta is not the same type of hedge parameter as delta. There is uncertainty about the future stock price, but there is no uncertainty about the passage of time. It makes sense to hedge against changes in the price of the underlying asset, but it does not make any sense to hedge against the effect of the passage of time on an option portfolio. In spite of this, many traders regard theta as a useful descriptive statistic for a portfolio. This is because, as we shall see later, in a delta-neutral portfolio theta is a proxy for gamma.

15.6 GAMMA

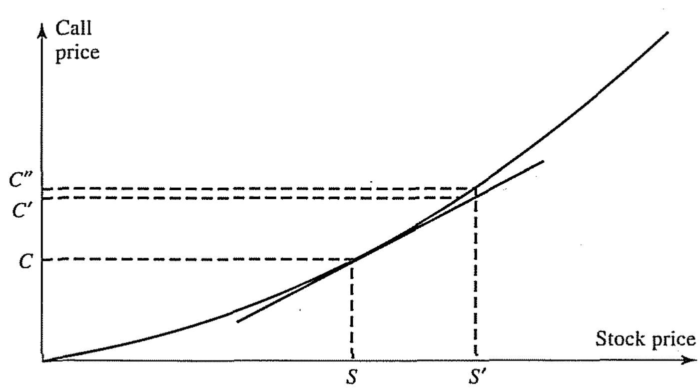
The *gamma* (Γ) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently. However, if gamma is large in absolute terms, delta is highly sensitive to the price of the underlying asset. It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time. Figure 15.7 illustrates this point. When the stock price moves from S to S', delta hedging assumes that the option price moves from C to C', when in fact it moves from C to C''. The difference between C' and C'' leads to a hedging error. This error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature.[9]

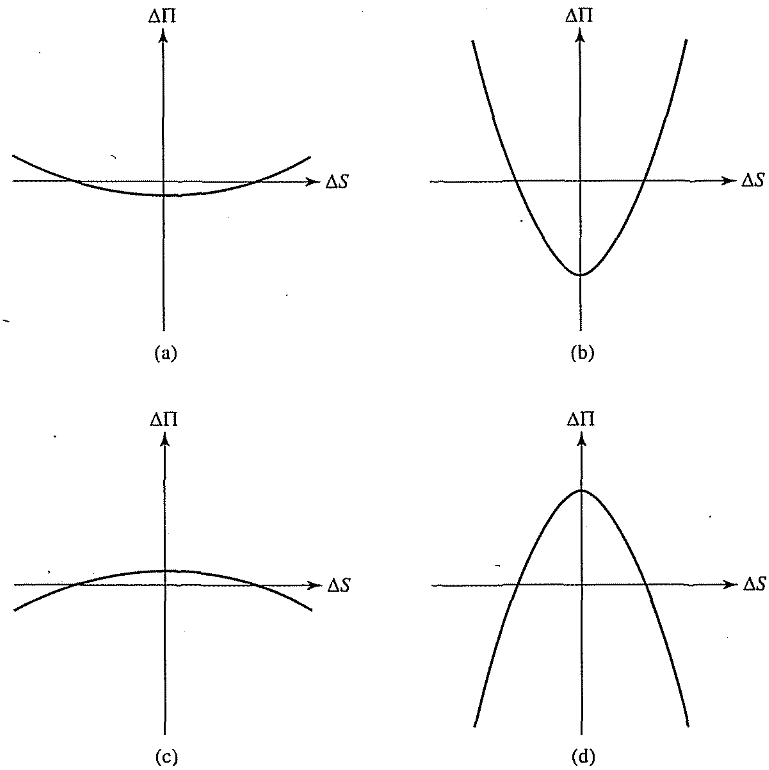
Suppose that ΔS is the price change of an underlying asset during a small interval of time, Δt, and $\Delta \Pi$ is the corresponding price change in the portfolio. The appendix at

Figure 15.7 Hedging error introduced by nonlinearity.



[9] Indeed, the gamma of an option is sometimes referred to as its *curvature* by practitioners.

Figure 15.8 Alternative relationships between $\Delta\Pi$ and ΔS for a delta-neutral portfolio with (a) slightly positive gamma, (b) large positive gamma, (c) slightly negative gamma, and (d) large negative gamma.



the end of this chapter shows that, if terms of order higher than Δt are ignored, we have

$$\Delta\Pi = \Theta\,\Delta t + \tfrac{1}{2}\Gamma\,\Delta S^2 \tag{15.6}$$

for a delta-neutral portfolio, where Θ is the theta of the portfolio. Figure 15.8 shows the nature of this relationship between $\Delta\Pi$ and ΔS. When gamma is positive, theta tends to be negative. The portfolio declines in value if there is no change in S, but increases in value if there is a large positive or negative change in S. When gamma is negative, theta tends to be positive and the reverse is true; the portfolio increases in value if there is no change in S but decreases in value if there is a large positive or negative change in S. As the absolute value of gamma increases, the sensitivity of the value of the portfolio to S increases.

Example 15.4

Suppose that the gamma of a delta-neutral portfolio of options on an asset is $-10{,}000$. Equation (15.6) shows that, if a change of $+2$ or -2 in the price of the asset occurs over a short period of time, there is an unexpected decrease in the value of the portfolio of approximately $0.5 \times 10{,}000 \times 2^2 = \$20{,}000$.

Making a Portfolio Gamma Neutral

A position in the underlying asset itself or a forward contract on the underlying asset both have zero gamma and cannot be used to change the gamma of a portfolio. What is required is a position in an instrument such as an option that is not linearly dependent on the underlying asset.

Suppose that a delta-neutral portfolio has a gamma equal to Γ, and a traded option has a gamma equal to Γ_T. If the number of traded options added to the portfolio is w_T, the gamma of the portfolio is

$$w_T \Gamma_T + \Gamma$$

Hence, the position in the traded option necessary to make the portfolio gamma neutral is $-\Gamma/\Gamma_T$. Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset then has to be changed to maintain delta neutrality. Note that the portfolio is gamma neutral only for a short period of time. As time passes, gamma neutrality can be maintained only if the position in the traded option is adjusted so that it is always equal to $-\Gamma/\Gamma_T$.

Making a delta-neutral portfolio gamma neutral can be regarded as a first correction for the fact that the position in the underlying asset cannot be changed continuously when delta hedging is used. Delta neutrality provides protection against relatively small stock price moves between rebalancing. Gamma neutrality provides protection against larger movements in this stock price between hedge rebalancing. Suppose that a portfolio is delta neutral and has a gamma of −3,000. The delta and gamma of a particular traded call option are 0.62 and 1.50, respectively. The portfolio can be made gamma neutral by including in the portfolio a long position of

$$\frac{3{,}000}{1.5} = 2{,}000$$

in the call option. However, the delta of the portfolio will then change from zero to $2{,}000 \times 0.62 = 1{,}240$. A quantity, 1,240, of the underlying asset must therefore be sold from the portfolio to keep it delta neutral.

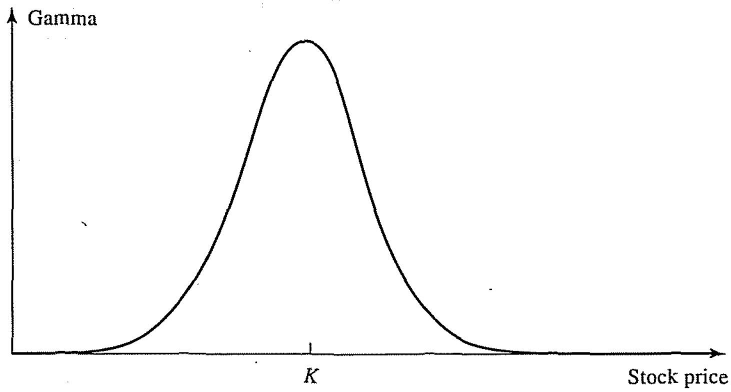
Calculation of Gamma

For a European call or put option on a non-dividend-paying stock, the gamma is given by

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

where d_1 is defined as in equation (13.20) and $N'(x)$ is as given by equation (15.5). The gamma of a long position is always positive and varies with S_0 in the way indicated in Figure 15.9. The variation of gamma with time to maturity for out-of-the-money, at-the-money, and in-the-money options is shown in Figure 15.10. For an at-the-money option, gamma increases as the time to maturity decreases. Short-life at-the-money options have very high gammas, which means that the value of the option holder's position is highly sensitive to jumps in the stock price.

Figure 15.9 Variation of gamma with stock price for an option.

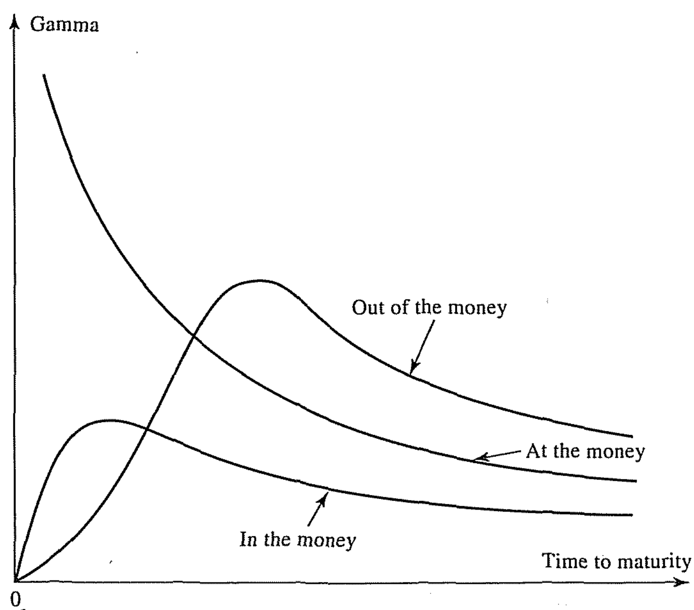


For a European call or put option on an asset paying a continuous dividend at rate q,

$$\Gamma = \frac{N'(d_1)\, e^{-qT}}{S_0 \sigma \sqrt{T}}$$

where d_1 is as in equation (14.4). When the asset is a stock index, q is set equal to the dividend yield on the index. When it is a currency, q is set equal to the foreign risk-free rate, r_f. When it is a futures contract, $S_0 = F_0$ and $q = r$.

Figure 15.10 Variation of gamma with time to maturity for a stock option.



Example 15.5

Consider a 4-month put option on a stock index. The current value of the index is 305, the strike price is 300, the dividend yield is 3% per annum, the risk-free interest rate is 8% per annum, and volatility of the index is 25% per annum. In this case, $S_0 = 305$, $K = 300$, $q = 0.03$, $r = 0.08$, $\sigma = 0.25$, and $T = 4/12$. The gamma of the index option is given by

$$\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}} = 0.00857$$

Thus, an increase of 1 in the index (from 305 to 306) increases the delta of the option by approximately 0.00857.

15.7 RELATIONSHIP BETWEEN DELTA, THETA, AND GAMMA

The price of a single derivative dependent on a non-dividend-paying stock must satisfy the differential equation (13.16). It follows that the value Π of a portfolio of such derivatives also satisfies the differential equation

$$\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Since

$$\Theta = \frac{\partial \Pi}{\partial t}, \qquad \Delta = \frac{\partial \Pi}{\partial S}, \qquad \Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

it follows that

$$\Theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi \tag{15.7}$$

Similar results can be produced for other underlying assets (see Problem 15.19).

For a delta-neutral portfolio, $\Delta = 0$ and

$$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

This shows that, when Θ is large and positive, gamma of a portfolio tends to be large and negative, and vice versa. This is consistent with the way in which Figure 15.8 has been drawn and explains why theta can be regarded as a proxy for gamma in a delta-neutral portfolio.

15.8 VEGA

Up to now we have implicitly assumed that the volatility of the asset underlying a derivative is constant. In practice, volatilities change over time. This means that the value of a derivative is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time.

The *vega* of a portfolio of derivatives, $\mathcal{V}$, is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset.[10]

$$\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}$$

If vega is high in absolute terms, the portfolio's value is very sensitive to small changes in volatility. If vega is low in absolute terms, volatility changes have relatively little impact on the value of the portfolio.

A position in the underlying asset has zero vega. However, the vega of a portfolio can be changed by adding a position in a traded option. If $\mathcal{V}$ is the vega of the portfolio and $\mathcal{V}_T$ is the vega of a traded option, a position of $-\mathcal{V}/\mathcal{V}_T$ in the traded option makes the portfolio instantaneously vega neutral. Unfortunately, a portfolio that is gamma neutral will not in general be vega neutral, and vice versa. If a hedger requires a portfolio to be both gamma and vega neutral, at least two traded derivatives dependent on the underlying asset must usually be used.

Example 15.6

Consider a portfolio that is delta neutral, with a gamma of −5,000 and a vega of −8,000. A traded option has a gamma of 0.5, a vega of 2.0, and a delta of 0.6. The portfolio can be made vega neutral by including a long position in 4,000 traded options. This would increase delta to 2,400 and require that 2,400 units of the asset be sold to maintain delta neutrality. The gamma of the portfolio would change from −5,000 to −3,000.

To make the portfolio gamma and vega neutral, we suppose that there is a second traded option with a gamma of 0.8, a vega of 1.2, and a delta of 0.5. If w_1 and w_2 are the quantities of the two traded options included in the portfolio, we require that

$$-5{,}000 + 0.5w_1 + 0.8w_2 = 0$$

and

$$-8{,}000 + 2.0w_1 + 1.2w_2 = 0$$

The solution to these equations is $w_1 = 400$, $w_2 = 6{,}000$. The portfolio can therefore be made gamma and vega neutral by including 400 of the first traded option and 6,000 of the second traded option. The delta of the portfolio after the addition of the positions in the two traded options is $400 \times 0.6 + 6{,}000 \times 0.5 = 3{,}240$. Hence, 3,240 units of the asset would have to be sold to maintain delta neutrality.

For a European call or put option on a non-dividend-paying stock, vega is given by

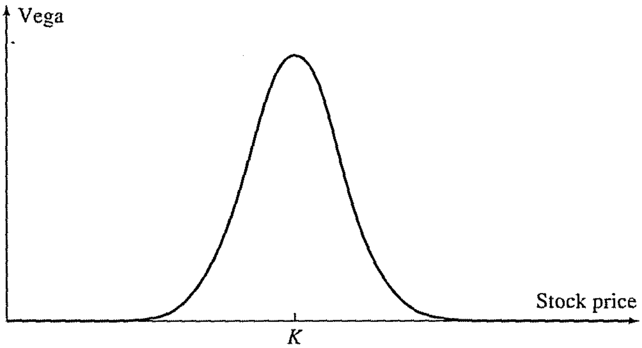
$$\mathcal{V} = S_0\sqrt{T}\,N'(d_1)$$

where d_1 is defined as in equation (13.20). The formula for $N'(x)$ is given in equation (15.5). For a European call or put option on an asset providing a dividend yield at rate q,

$$\mathcal{V} = S_0\sqrt{T}\,N'(d_1)\,e^{-qT}$$

[10] Vega is the name given to one of the "Greek letters" in option pricing, but it is not one of the letters in the Greek alphabet.

Figure 15.11 Variation of vega with stock price for an option.



where d_1 is defined as in equation (14.4). When the asset is a stock index, q is set equal to the dividend yield on the index. When it is a currency, q is set equal to the foreign risk-free rate, r_f. When it is a futures contract, $S_0 = F_0$ and $q = r$.

The vega of a long position in a regular European or American option is always positive. The general way in which vega varies with S_0 is shown in Figure 15.11.

Example 15.7

Consider a 4-month put option on a stock index. The current value of the index is 305, the strike price is 300, the dividend yield is 3% per annum, the risk-free interest rate is 8% per annum, and the volatility of the index is 25% per annum. In this case $S_0 = 305$, $K = 300$, $q = 0.03$, $r = 0.08$, $\sigma = 0.25$, and $T = 4/12$. The option's vega is given by

$$S_0\sqrt{T}\,N'(d_1)\,e^{-qT} = 66.44$$

Thus a 1% (0.01) increase in volatility (from 25% to 26%) increases the value of the option by approximately 0.6644 ($= 0.01 \times 66.44$).

Calculating vega from the Black–Scholes model and its extensions may seem strange because one of the assumptions underlying Black–Scholes is that volatility is constant. It would be theoretically more correct to calculate vega from a model in which volatility is assumed to be stochastic. However, it turns out that the vega calculated from a stochastic volatility model is very similar to the Black–Scholes vega, so the practice of calculating vega from a model in which volatility is constant works reasonably well.[11]

Gamma neutrality protects against large changes in the price of the underlying asset between hedge rebalancing. Vega neutrality protects for a variable σ. As might be expected, whether it is best to use an available traded option for vega or gamma hedging depends on the time between hedge rebalancing and the volatility of the volatility.[12]

---

[11] See J. C. Hull and A. White, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance* 42 (June 1987): 281–300; J. C. Hull and A. White, "An Analysis of the Bias in Option Pricing Caused by a Stochastic Volatility," *Advances in Futures and Options Research* 3 (1988): 27–61.

[12] For a discussion of this issue, see J. C. Hull and A. White, "Hedging the Risks from Writing Foreign Currency Options," *Journal of International Money and Finance* 6 (June 1987): 131–52.

When volatilities change, the implied volatilities of short-dated options tend to change by more than the implied volatilities of long-dated options. The vega of a portfolio is therefore often calculated by changing the volatilities of long-dated options by less than that of short-dated options. One way of doing this is discussed in Section 19.6.

15.9 RHO

The *rho* of a portfolio of options is the rate of change of the value of the portfolio with respect to the interest rate:

$$\text{rho (call)} = \frac{\partial \Pi}{\partial r}$$

It measures the sensitivity of the value of a portfolio to interest rates. For a European call option on a non-dividend-paying stock,

$$\text{rho (put)} = KTe^{-rT}N(d_2)$$

where d_2 is defined as in equation (13.20). For a European put option,

$$\text{rho} = -KTe^{-rT}N(-d_2)$$

These same formulas apply to European call and put options on stocks and stock indices paying known dividend yields when d_2 is as in equation (14.4).

Example 15.8

Consider a 4-month put option on a stock index. The current value of the index is 305, the strike price is 300, the dividend yield is 3% per annum, the risk-free interest rate is 8% per annum, and the volatility of the index is 25% per annum. In this case, $S_0 = 305$, $K = 300$, $q = 0.03$, $r = 0.08$, $\sigma = 0.25$, and $T = 4/12$. The option's rho is

$$-KTe^{-rT}N(-d_2) = -42.6$$

This means that for a 1% (0.01) change in the risk-free interest rate (from 8% to 9%) the value of the option decreases by 0.426 ($= 0.01 \times 42.6$).

In the case of currency options, there are two rhos corresponding to the two interest rates. The rho corresponding to the domestic interest rate is given by the formulas already presented with d_2 as in equation (14.7). The rho corresponding to the foreign interest rate for a European call on a currency is

$$-Te^{-r_f T}S_0 N(d_1)$$

and for a European put it is

$$Te^{-r_f T}S_0 N(-d_1)$$

where d_1 is given by equation (14.7).

For a European call futures option, rho is $-cT$ and for a European put futures option rho is $-pT$, where c and p are the European call and put option prices, respectively.

Business Snapshot 15.1 Dynamic Hedging in Practice

In a typical arrangement at a financial institution, the responsibility for a portfolio of derivatives dependent on a particular underlying asset is assigned to one trader or to a group of traders working together. For example, one trader at Goldman Sachs might be assigned responsibility for all derivatives dependent on the value of the Australian dollar. A computer system calculates the value of the portfolio and Greek letters for the portfolio. Limits are defined for each Greek letter and special permission is required if a trader wants to exceed a limit at the end of a trading day.

The delta limit is often expressed as the equivalent maximum position in the underlying asset. For example, the delta limit of Goldman Sachs on Microsoft might be \$10 million. If the Microsoft stock price is \$50 this means that the absolute value of delta as we have calculated it can be no more that 200,000. The vega limit is usually expressed as a maximum dollar exposure per 1% change in the volatility.

As a matter of course, options traders make themselves delta neutral—or close to delta neutral—at the end of each day. Gamma and vega are monitored, but are not usually managed on a daily basis. Financial institutions often find that their business with clients involves writing options and that as a result they accumulate negative gamma and vega. They are then always looking out for opportunities to manage their gamma and vega risks by buying options at competitive prices.

There is one aspect of an options portfolio that mitigates problems of managing gamma and vega somewhat. Options are often close to the money when they are first sold, so that they have relatively high gammas and vegas. But after some time has elapsed, the underlying asset price has often changed enough for them to become deep out of the money or deep in the money. Their gammas and vegas are then very small and of little consequence. The nightmare scenario for an options trader is where written options remain very close to the money as the maturity date is approached.

15.10 THE REALITIES OF HEDGING

In an ideal world, traders working for financial institutions would be able to rebalance their portfolios very frequently in order to maintain a zero delta, a zero gamma, and a zero vega. In practice, this is not possible. When managing a large portfolio dependent on a single underlying asset, traders usually make delta zero, or close to zero, at least once a day by trading the underlying asset. Unfortunately, a zero gamma and a zero vega are less easy to achieve because it is difficult to find options or other nonlinear derivatives that can be traded in the volume required at competitive prices (see discussion of dynamic hedging in Business Snapshot 15.1)

There are big economies of scale in being an options trader. As noted earlier, maintaining delta neutrality for an individual option on, say, the S&P 500 by trading daily would be prohibitively expensive. But it is realistic to do this for a portfolio of several hundred options on the S&P 500. This is because the cost of daily rebalancing (either by trading the stocks underlying the index or by trading index futures) is covered by the profit on many different trades.

15.11 SCENARIO ANALYSIS

In addition to monitoring risks such as delta, gamma, and vega, option traders often also carry out a scenario analysis. The analysis involves calculating the gain or loss on their portfolio over a specified period under a variety of different scenarios. The time period chosen is likely to depend on the liquidity of the instruments. The scenarios can be either chosen by management or generated by a model.

Consider a bank with a portfolio of options on a foreign currency. There are two main variables on which the value of the portfolio depends. These are the exchange rate and the exchange-rate volatility. Suppose that the exchange rate is currently 1.0000 and its volatility is 10% per annum. The bank could calculate a table such as Table 15.5 showing the profit or loss experienced during a 2-week period under different scenarios. This table considers seven different exchange rates and three different volatilities. Because a one-standard-deviation move in the exchange rate during a 2-week period is about 0.02, the exchange rate moves considered are approximately one, two, and three standard deviations.

In Table 15.5, the greatest loss is in the lower right corner of the table. The loss corresponds to the volatility increasing to 12% and the exchange rate moving up to 1.06. Usually the greatest loss in a table such as 15.5 occurs at one of the corners, but this is not always so. Consider, for example, the situation where a bank's portfolio consists of a short position in a butterfly spread (see Section 10.2). The greatest loss will be experienced if the exchange rate stays where it is.

15.12 PORTFOLIO INSURANCE

A portfolio manager is often interested in acquiring a put option on his or her portfolio. This provides protection against market declines while preserving the potential for a gain if the market does well. One approach (discussed in Section 14.3) is to buy put options on a market index such as the S&P 500. An alternative is to create the options synthetically.

Creating an option synthetically involves maintaining a position in the underlying asset (or futures on the underlying asset) so that the delta of the position is equal to the delta of the required option. The position necessary to create an option synthetically is the reverse of that necessary to hedge it. This is because the procedure for hedging an option involves the creation of an equal and opposite option synthetically.

Table 15.5 Profit or loss realized in 2 weeks under different scenarios ($ million).

Volatility	*Exchange rate*						
	0.94	*0.96*	*0.98*	*1.00*	*1.02*	*1.04*	*1.06*
8%	+102	+55	+25	+6	−10	−34	−80
10%	+80	+40	+17	+2	−14	−38	−85
12%	+60	+25	+9	−2	−18	−42	−90

There are two reasons why it may be more attractive for the portfolio manager to create the required put option synthetically than to buy it in the market. The first is that options markets do not always have the liquidity to absorb the trades that managers of large funds would like to carry out. The second is that fund managers often require strike prices and exercise dates that are different from those available in exchange-traded options markets.

The synthetic option can be created from trading the portfolio or from trading in index futures contracts. We first examine the creation of a put option by trading the portfolio. Recall that the delta of a European put on the portfolio is

$$\Delta = e^{-qT}[N(d_1) - 1] \tag{15.8}$$

where, with our usual notation,

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

S_0 is the value of the portfolio, K is the strike price, r is the risk-free rate, q is the dividend yield on the portfolio, σ is the volatility of the portfolio, and T is the life of the option. The volatility of the portfolio can usually be assumed to be its beta times the volatility of a well-diversified market index.

To create the put option synthetically, the fund manager should ensure that at any given time a proportion

$$e^{-qT}[1 - N(d_1)]$$

of the stocks in the original portfolio has been sold and the proceeds invested in riskless assets. As the value of the original portfolio declines, the delta of the put given by equation (15.8) becomes more negative and the proportion of the original portfolio sold must be increased. As the value of the original portfolio increases, the delta of the put becomes less negative and the proportion of the original portfolio sold must be decreased (i.e., some of the original portfolio must be repurchased).

Using this strategy to create portfolio insurance means that at any given time funds are divided between the stock portfolio on which insurance is required and riskless assets. As the value of the stock portfolio increases, riskless assets are sold and the position in the stock portfolio is increased. As the value of the stock portfolio declines, the position in the stock portfolio is decreased and riskless assets are purchased. The cost of the insurance arises from the fact that the portfolio manager is always selling after a decline in the market and buying after a rise in the market.

Example 15.9

A portfolio is worth \$90 million. To protect against market downturns the managers of the portfolio require a 6-month European put option on the portfolio with a strike price of \$87 million. The risk-free rate is 9% per annum, the dividend yield is 3% per annum, and the volatility of the portfolio is 25% per annum. The S&P 500 index stands at 900. As the portfolio is considered to mimic the S&P 500 fairly closely, one alternative is to buy 1,000 put option contracts on the S&P 500 with a strike price of 870. Another alternative is to create the required option synthetically. In this case, $S_0 = 90$ million, $K = 87$ million,

$r = 0.09$, $q = 0.03$, $\sigma = 0.25$, and $T = 0.5$, so that

$$d_1 = \frac{\ln(90/87) + (0.09 - 0.03 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = 0.4499$$

and the delta of the required option is initially

$$e^{-qT}[N(d_1) - 1] = -0.3215$$

This shows that 32.15% of the portfolio should be sold initially to match the delta of the required option. The amount of the portfolio sold must be monitored frequently. For example, if the value of the portfolio reduces to $88 million after 1 day, the delta of the required option changes to 0.3679 and a further 4.64% of the original portfolio should be sold. If the value of the portfolio increases to $92 million, the delta of the required option changes to −0.2787 and 4.28% of the original portfolio should be repurchased.

Use of Index Futures

Using index futures to create options synthetically can be preferable to using the underlying stocks because the transaction costs associated with trades in index futures are generally lower than those associated with the corresponding trades in the underlying stocks. The dollar amount of the futures contracts shorted as a proportion of the value of the portfolio should from equations (15.3) and (15.8) be

$$e^{-qT}e^{-(r-q)T^*}[1 - N(d_1)] = e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]$$

where T^* is the maturity of the futures contract. If the portfolio is worth A_1 times the index and each index futures contract is on A_2 times the index, the number of futures contracts shorted at any given time should be

$$e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]\frac{A_1}{A_2}$$

Example 15.10

Suppose that in the previous example futures contracts on the S&P 500 maturing in 9 months are used to create the option synthetically. In this case initially $T = 0.5$, $T^* = 0.75$, $A_1 = 100{,}000$, $A_2 = 250$, and $d_1 = 0.4499$, so that the number of futures contracts shorted should be

$$e^{q(T^*-T)}e^{-rT^*}[1 - N(d_1)]\frac{A_1}{A_2} = 122.96$$

or 123, rounding to the nearest whole number. As time passes and the index changes, the position in futures contracts must be adjusted.

This analysis assumes that the portfolio mirrors the index. When this is not the case, it is necessary to (a) calculate the portfolio's beta, (b) find the position in options on the index that gives the required protection, and (c) choose a position in index futures to create the options synthetically. As discussed in Section 14.3, the strike price for the options should be the expected level of the market index when the portfolio reaches its insured value. The number of options required is beta times the number that would be required if the portfolio had a beta of 1.0.

Business Snapshot 15.2 Was Portfolio Insurance to Blame for the Crash of 1987?

On Monday, October 19, 1987, the Dow Jones Industrial Average dropped by more than 20%. Many people feel that portfolio insurance played a major role in this crash. In October 1987 between $60 billion and $90 billion of equity assets were subject to portfolio insurance schemes where put options were created synthetically in the way discussed in Section 15.12. During the period Wednesday, October 14, 1987, to Friday, October 16, 1987, the market declined by about 10%, with much of this decline taking place on Friday afternoon. The portfolio insurance schemes should have generated at least $12 billion of equity or index futures sales as a result of this decline. In fact, portfolio insurers had time to sell only $4 billion and they approached the following week with huge amounts of selling already dictated by their models. It is estimated that on Monday, October 19, sell programs by three portfolio insurers accounted for almost 10% of the sales on the New York Stock Exchange, and that portfolio insurance sales amounted to 21.3% of all sales in index futures markets. It is likely that the decline in equity prices was exacerbated by investors other than portfolio insurers selling heavily because they anticipated the actions of portfolio insurers.

Because the market declined so fast and the stock exchange systems were overloaded, many portfolio insurers were unable to execute the trades generated by their models and failed to obtain the protection they required. Needless to say, the popularity of portfolio insurance schemes has declined significantly since 1987. One of the morals of this story is that it is dangerous to follow a particular trading strategy—even a hedging strategy—when many other market participants are doing the same thing.

15.13 STOCK MARKET VOLATILITY

We discussed in Chapter 13 the issue of whether volatility is caused solely by the arrival of new information or whether trading itself generates volatility. Portfolio insurance schemes such as those just described have the potential to increase volatility. When the market declines, they cause portfolio managers either to sell stock or to sell index futures contracts. Either action may accentuate the decline (see Business Snapshot 15.2). The sale of stock is liable to drive down the market index further in a direct way. The sale of index futures contracts is liable to drive down futures prices. This creates selling pressure on stocks via the mechanism of index arbitrage (see Chapter 5), so that the market index is liable to be driven down in this case as well. Similarly, when the market rises, the portfolio insurance schemes cause portfolio managers either to buy stock or to buy futures contracts. This may accentuate the rise.

In addition to formal portfolio insurance schemes, we can speculate that many investors consciously or subconsciously follow portfolio insurance schemes of their own. For example, an investor may be inclined to enter the market when it is rising but will sell when it is falling to limit the downside risk.

Whether portfolio insurance schemes (formal or informal) affect volatility depends on how easily the market can absorb the trades that are generated by portfolio insurance. If portfolio insurance trades are a very small fraction of all trades, there is likely to be no effect. As portfolio insurance becomes more popular, it is liable to have a destabilizing effect on the market.

SUMMARY

Financial institutions offer a variety of option products to their clients. Often the options do not correspond to the standardized products traded by exchanges. The financial institutions are then faced with the problem of hedging their exposure. Naked and covered positions leave them subject to an unacceptable level of risk. One course of action that is sometimes proposed is a stop-loss strategy. This involves holding a naked position when an option is out of the money and converting it to a covered position as soon as the option moves into the money. Although superficially attractive, the strategy does not provide a good hedge.

The delta (Δ) of an option is the rate of change of its price with respect to the price of the underlying asset. Delta hedging involves creating a position with zero delta (sometimes referred to as a delta-neutral position). Because the delta of the underlying asset is 1.0, one way of hedging is to take a position of $-\Delta$ in the underlying asset for each long option being hedged. The delta of an option changes over time. This means that the position in the underlying asset has to be frequently adjusted.

Once an option position has been made delta neutral, the next stage is often to look at its gamma (Γ). The gamma of an option is the rate of change of its delta with respect to the price of the underlying asset. It is a measure of the curvature of the relationship between the option price and the asset price. The impact of this curvature on the performance of delta hedging can be reduced by making an option position gamma neutral. If Γ is the gamma of the position being hedged, this reduction is usually achieved by taking a position in a traded option that has a gamma of $-\Gamma$.

Delta and gamma hedging are both based on the assumption that the volatility of the underlying asset is constant. In practice, volatilities do change over time. The vega of an option or an option portfolio measures the rate of change of its value with respect to volatility. A trader who wishes to hedge an option position against volatility changes can make the position vega neutral. As with the procedure for creating gamma neutrality, this usually involves taking an offsetting position in a traded option. If the trader wishes to achieve both gamma and vega neutrality, two traded options are usually required.

Two other measures of the risk of an option position are theta and rho. Theta measures the rate of change of the value of the position with respect to the passage of time, with all else remaining constant. Rho measures the rate of change of the value of the position with respect to the interest rate, with all else remaining constant.

In practice, option traders usually rebalance their portfolios at least once a day to maintain delta neutrality. It is usually not feasible to maintain gamma and vega neutrality on a regular basis. Typically a trader monitors these measures. If they get too large, either corrective action is taken or trading is curtailed.

Portfolio managers are sometimes interested in creating put options synthetically for the purposes of insuring an equity portfolio. They can do so either by trading the portfolio or by trading index futures on the portfolio. Trading the portfolio involves splitting the portfolio between equities and risk-free securities. As the market declines, more is invested in risk-free securities. As the market increases, more is invested in equities. Trading index futures involves keeping the equity portfolio intact and selling index futures. As the market declines, more index futures are sold; as it rises, fewer are sold. This type of portfolio insurance works well in normal market conditions. On Monday, October 19, 1987, when the Dow Jones Industrial Average dropped very sharply, it worked badly. Portfolio insurers were unable to sell either stocks or index futures fast enough to protect their positions.

FURTHER READING

Taleb, N. N., *Dynamic Hedging: Managing Vanilla and Exotic Options*. New York: Wiley, 1996.

Questions and Problems (Answers in Solutions Manual)

15.1. Explain how a stop-loss hedging scheme can be implemented for the writer of an out-of-the-money call option. Why does it provide a relatively poor hedge?

15.2. What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

15.3. Calculate the delta of an at-the-money 6-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

15.4. What does it mean to assert that the theta of an option position is −0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?

15.5. What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?

15.6. "The procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position." Explain this statement.

15.7. Why did portfolio insurance not work well on October 19, 1987?

15.8. The Black–Scholes price of an out-of-the-money call option with an exercise price of $40 is $4. A trader who has written the option plans to use a stop-loss strategy. The trader's plan is to buy at $40.10 and to sell at $39.90. Estimate the expected number of times the stock will be bought or sold.

15.9. Suppose that a stock price is currently $20 and that a call option with an exercise price of $25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios:
(a) Stock price increases steadily from $20 to $35 during the life of the option.
(b) Stock price oscillates wildly, ending up at $35.
Which scenario would make the synthetically created option more expensive? Explain your answer.

15.10. What is the delta of a short position in 1,000 European call options on silver futures? The options mature in 8 months, and the futures contract underlying the option matures in 9 months. The current 9-month futures price is $8 per ounce, the exercise price of the options is $8, the risk-free interest rate is 12% per annum, and the volatility of silver is 18% per annum.

15.11. In Problem 15.10, what initial position in 9-month silver futures is necessary for delta hedging? If silver itself is used, what is the initial position? If 1-year silver futures are used, what is the initial position? Assume no storage costs for silver.

15.12. A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?
(a) A virtually constant spot rate
(b) Wild movements in the spot rate
Explain your answer.

15.13. Repeat Problem 15.12 for a financial institution with a portfolio of short positions in put and call options on a currency.

15.14. A financial institution has just sold 1,000 7-month European call options on the Japanese yen. Suppose that the spot exchange rate is 0.80 cent per yen, the exercise price is 0.81 cent per yen, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Japan is 5% per annum, and the volatility of the yen is 15% per annum. Calculate the delta, gamma, vega, theta, and rho of the financial institution's position. Interpret each number.

15.15. Under what circumstances is it possible to make a European option on a stock index both gamma neutral and vega neutral by adding a position in one other European option?

15.16. A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth $360 million. The value of the S&P 500 is 1,200, and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next 6 months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum.
(a) If the fund manager buys traded European put options, how much would the insurance cost?
(b) Explain carefully alternative strategies open to the fund manager involving traded European call options, and show that they lead to the same result.
(c) If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?
(d) If the fund manager decides to provide insurance by using 9-month index futures, what should the initial position be?

15.17. Repeat Problem 15.16 on the assumption that the portfolio has a beta of 1.5. Assume that the dividend yield on the portfolio is 4% per annum.

15.18. Show by substituting for the various terms in equation (15.7) that the equation is true for:
(a) A single European call option on a non-dividend-paying stock
(b) A single European put option on a non-dividend-paying stock
(c) Any portfolio of European put and call options on a non-dividend-paying stock

15.19. What is the equation corresponding to equation (15.7) for (a) a portfolio of derivatives on a currency and (b) a portfolio of derivatives on a futures contract?

15.20. Suppose that $70 billion of equity assets are the subject of portfolio insurance schemes. Assume that the schemes are designed to provide insurance against the value of the assets declining by more than 5% within 1 year. Making whatever estimates you find necessary, use the DerivaGem software to calculate the value of the stock or futures contracts that the administrators of the portfolio insurance schemes will attempt to sell if the market falls by 23% in a single day.

15.21. Does a forward contract on a stock index have the same delta as the corresponding futures contract? Explain your answer.

15.22. A bank's position in options on the dollar/euro exchange rate has a delta of 30,000 and a gamma of −80,000. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

15.23. Use the put–call parity relationship to derive, for a non-dividend-paying stock, the relationship between:
(a) The delta of a European call and the delta of a European put
(b) The gamma of a European call and the gamma of a European put
(c) The vega of a European call and the vega of a European put
(d) The theta of a European call and the theta of a European put

Assignment Questions

15.24. Consider a 1-year European call option on a stock when the stock price is \$30, the strike price is \$30, the risk-free rate is 5%, and the volatility is 25% per annum. Use the DerivaGem software to calculate the price, delta, gamma, vega, theta, and rho of the option. Verify that delta is correct by changing the stock price to \$30.1 and recomputing the option price. Verify that gamma is correct by recomputing the delta for the situation where the stock price is \$30.1. Carry out similar calculations to verify that vega, theta, and rho are correct. Use the DerivaGem Applications Builder functions to plot the option price, delta, gamma, vega, theta, and rho against the stock price for the stock option.

15.25. A financial institution has the following portfolio of over-the-counter options on sterling:

Type	*Position*	*Delta of option*	*Gamma of option*	*Vega of option*
Call	−1,000	0.50	2.2	1.8
Call	−500	0.80	0.6	0.2
Put	−2,000	−0.40	1.3	0.7
Call	−500	0.70	1.8	1.4

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.
(a) What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?
(b) What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

15.26. Consider again the situation in Problem 15.25. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

15.27. A deposit instrument offered by a bank guarantees that investors will receive a return during a 6-month period that is the greater of (a) zero and (b) 40% of the return provided by a market index. An investor is planning to put \$100,000 in the instrument. Describe the payoff as an option on the index. Assuming that the risk-free rate of interest is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum, is the product a good deal for the investor?

15.28. The formula for the price c of a European call futures option in terms of the futures price F_0 is given in Chapter 14 as

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

and K, r, T, and σ are the strike price, interest rate, time to maturity, and volatility, respectively.

(a) Prove that $F_0 N'(d_1) = KN'(d_2)$.

(b) Prove that the delta of the call price with respect to the futures price is $e^{-rT}N(d_1)$.

(c) Prove that the vega of the call price is $F_0\sqrt{T}N'(d_1)e^{-rT}$.

(d) Prove the formula for the rho of a call futures option given at the end of Section 15.9.

The delta, gamma, theta, and vega of a call futures option are the same as those for a call option on a stock paying dividends at rate q, with q replaced by r and S_0 replaced by F_0. Explain why the same is not true of the rho of a call futures option.

15.29. Use DerivaGem to check that equation (15.7) is satisfied for the option considered in Section 15.1. (*Note*: DerivaGem produces a value of theta "per calendar day". The theta in equation (15.7) is "per year".)

15.30. Use the DerivaGem Application Builder functions to reproduce Table 15.2. (In Table 15.2 the stock position is rounded to the nearest 100 shares.) Calculate the gamma and theta of the position each week. Calculate the change in the value of the portfolio each week and check whether equation (15.6) is approximately satisfied. (*Note*: DerivaGem produces a value of theta "per calendar day". The theta in equation (15.6) is "per year".)

APPENDIX

TAYLOR SERIES EXPANSIONS AND HEDGE PARAMETERS

A Taylor series expansion of the change in the portfolio value in a short period of time shows the role played by different Greek letters. If the volatility of the underlying asset is assumed to be constant, the value Π of the portfolio is a function of the asset price S, and time t. The Taylor series expansion gives

$$\Delta\Pi = \frac{\partial \Pi}{\partial S}\Delta S + \frac{\partial \Pi}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\Delta S^2 + \tfrac{1}{2}\frac{\partial^2 \Pi}{\partial t^2}\Delta t^2 + \frac{\partial^2 \Pi}{\partial S\,\partial t}\Delta S\,\Delta t + \cdots \tag{15A.1}$$

where $\Delta\Pi$ and ΔS are the change in Π and S in a small time interval Δt. Delta hedging eliminates the first term on the right-hand side. The second term is nonstochastic. The third term (which is of order Δt) can be made zero by ensuring that the portfolio is gamma neutral as well as delta neutral. Other terms are of order higher than Δt.

For a delta-neutral portfolio, the first term on the right-hand side of equation (15A.1) is zero, so that

$$\Delta\Pi = \Theta\,\Delta t + \tfrac{1}{2}\Gamma\,\Delta S^2$$

when terms of order higher than Δt are ignored. This is equation (15.6).

When the volatility of the underlying asset is uncertain, Π is a function of σ, S, and t. Equation (15A.1) then becomes

$$\Delta\Pi = \frac{\partial \Pi}{\partial S}\Delta S + \frac{\partial \Pi}{\partial \sigma}\Delta \sigma + \frac{\partial \Pi}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\Delta S^2 + \tfrac{1}{2}\frac{\partial^2 \Pi}{\partial \sigma^2}\Delta \sigma^2 + \cdots$$

where $\Delta\sigma$ is the change in σ in time Δt. In this case, delta hedging eliminates the first term on the right-hand side. The second term is eliminated by making the portfolio vega neutral. The third term is nonstochastic. The fourth term is eliminated by making the portfolio gamma neutral. Traders sometimes define other Greek letters to correspond to higher-order terms in the expansion.