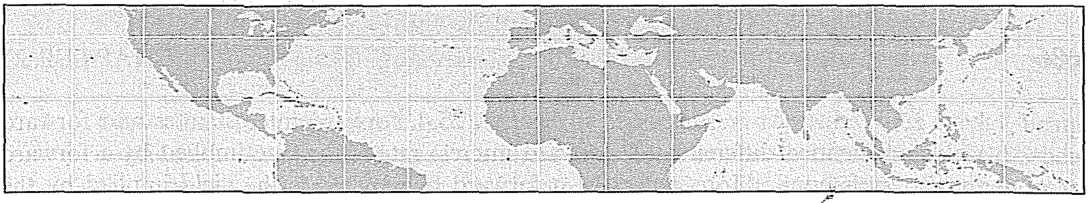


CHAPTER 27

# Convexity, Timing, and Quanto Adjustments

A popular two-step procedure for valuing a European-style derivative is:

1. Calculate the expected payoff by assuming that the expected value of each underlying variable equals its forward value.
2. Discount the expected payoff at the risk-free rate applicable for the time period between the valuation date and the payoff date.

We first used this procedure in Chapter 4 when valuing FRAs. We found that an FRA can be valued by calculating the payoff on the assumption that the forward interest rate will be realized and then discounting the payoff at the risk-free rate. Similarly, in Chapter 7 we found that swaps can be valued by calculating cash flows on the assumptions that forward rates will be realized and discounting them at risk-free rates. In Chapter 26 we found that Black's model provides a general approach to valuing a wide range of European options—and Black's model, as we saw in Section 26.1, is an application of this two-step procedure. The models presented in Chapter 26 for bond options, caps/floors, and swap options are all examples of the two-step procedure.

This raises the issue of whether it always correct to value European-style interest rate derivatives by using the two-step procedure? The answer is no! For nonstandard interest rate derivatives it is sometimes necessary to modify the two-step procedure so that an adjustment is made to the forward value of the variable in the first step. In this chapter we consider three types of adjustments: convexity adjustments, timing adjustments, and quanto adjustments.

## 27.1 CONVEXITY ADJUSTMENTS

We start by discussing the valuation an instrument that provides a payoff dependent on a bond yield observed at the time of the payoff.

Usually the forward value of a variable $S$ is calculated with reference to a forward contract that pays off $S_T - K$ at time $T$. It is the value of $K$ that causes the contract to have zero value. As we discussed in Section 25.4, forward interest rates and forward yields are defined differently. A forward interest rate is the rate implied by a forward zero-coupon bond. More generally, a forward bond yield is the yield implied by the forward bond price.

Suppose that $B_T$ is the price of a bond at time $T$, $y_T$ is its yield, and the (bond pricing) relationship between $B_T$ and $y_T$ is

$$B_T = G(y_T)$$

Define $F_0$ as the forward bond price at time zero for a contract maturing at time $T$ and $y_0$ as the forward bond yield at time zero. The definition of a forward bond yield means that

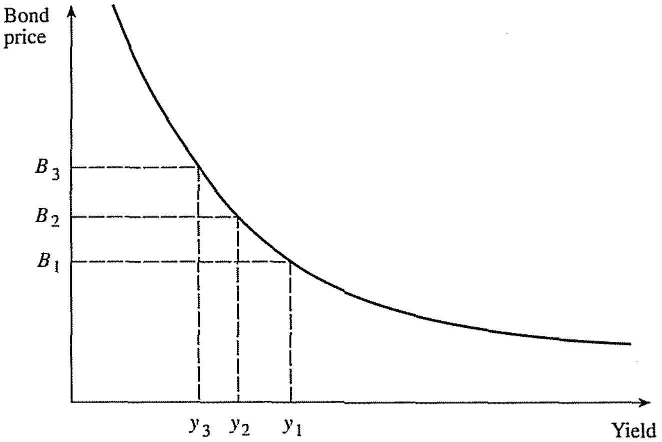
$$F_0 = G(y_0)$$

The function $G$ is nonlinear. This means that, when the expected future bond price equals the forward bond price (so that we are a world that is forward risk neutral with respect to a zero-coupon bond maturing at time $T$), the expected future bond yield does not equal the forward bond yield.

This is illustrated in Figure 27.1, which shows the relationship between bond prices and bond yields at time $T$. For simplicity, we suppose that there are only three possible bond prices, $B_1$, $B_2$, and $B_3$ and that they are equally likely in a world that is forward risk neutral with respect to $P(t, T)$. We assume that the bond prices are equally spaced, so that $B_2 - B_1 = B_3 - B_2$. The forward bond price is the expected bond price $B_2$. The bond prices translate into three equally likely bond yields: $y_1$, $y_2$, and $y_3$. These are not equally spaced. The variable $y_2$ is the forward bond yield because it is the yield corresponding to the forward bond price. The expected bond yield is the average of $y_1$, $y_2$, and $y_3$ and is clearly greater than $y_2$.

Consider now a derivative that provides a payoff dependent on the bond yield at time $T$. We know from equation (25.20) that it can be valued by (a) calculating the expected payoff in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time $T$ and (b) discounting at the current risk-free rate for maturity $T$. We

**Figure 27.1** Relationship between bond prices and bond yields at time $T$.



know that the expected bond price equals the forward price in the world being considered. We therefore need to know the value of the expected bond yield when the expected bond price equals the forward bond price. The analysis in the appendix at the end of this chapter shows that an approximate expression for the required expected bond yield is

$$E_T(y_T) = y_0 - \tfrac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)} \tag{27.1}$$

where $G'$ and $G''$ denote the first and second partial derivatives of $G$, $E_T$ denotes expectations in a world that is forward risk neutral with respect to $P(t, T)$, and $\sigma_y$ is the forward yield volatility. It follows that we can discount expected payoffs at the current risk-free rate for maturity $T$ providing we assume that the expected bond yield is

$$y_0 - \tfrac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

rather than $y_0$. The difference between the expected bond yield and the forward bond yield

$$-\tfrac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

is the known as a *convexity adjustment*. It corresponds to the difference between $y_2$ and the expected yield in Figure 27.1. (The convexity adjustment is positive because $G'(y_0) < 0$ and $G''(y_0) > 0$.)

## Application 1: Interest Rates

As our first application of equation (27.1) we consider an instrument that provides a cash flow at time $T$ equal to the interest rate between times $T$ and $T^*$ applied to a principal of $L$. (This example will be useful when we consider LIBOR-in-arrears swaps in Chapter 30.) Note that the interest rate applicable to the time period between times $T$ and $T^*$ is normally paid at time $T^*$; here we are assuming that it is paid early, at time $T$.

The cash flow at time $T$ in the instrument we are considering is $LR_T\tau$, where $\tau = T^* - T$ and $R_T$ is the zero-coupon interest rate applicable to the period between $T$ and $T^*$ (expressed with a compounding period of $\tau$).[1] The variable $R_T$ can be viewed as the yield at time $T$ on a zero-coupon bond maturing at time $T^*$. The relationship between the price of this bond and its yield is

$$G(y) = \frac{1}{1 + y\tau}$$

From equation (27.1),

$$E_T(R_T) = R_0 - \tfrac{1}{2} R_0^2 \sigma_R^2 T \frac{G''(R_0)}{G'(R_0)}$$

or

$$E_T(R_T) = R_0 + \frac{R_0^2 \sigma_R^2 \tau T}{1 + R_0 \tau} \tag{27.2}$$

[1] As usual, for ease of exposition we assume actual/actual day counts in our examples.

where $R_0$ is the forward rate applicable to the period between $T$ and $T^*$ and $\sigma_R$ is the volatility of the forward rate.

The value of the instrument is therefore

$$P(0,T)L\tau\left[R_0+\frac{R_0^2\sigma_R^2\tau T}{1+R_0\tau}\right]$$

***Example 27.1***

Consider a derivative that provides a payoff in 3 years equal to the 1-year zero-coupon rate (annually compounded) at that time multiplied by \$1000. Suppose that the zero rate for all maturities is 10% per annum with annual compounding and the volatility of the forward rate applicable to the time period between year 3 and year 4 is 20%. In this case, $R_0 = 0.10$, $\sigma_R = 0.20$, $T = 3$, $\tau = 1$, and $P(0,3) = 1/1.10^3 = 0.7513$. The value of the derivative is

$$0.7513\times 1000\times 1\times\left[0.10+\frac{0.10^2\times 0.20^2\times 1\times 3}{1+0.10\times 1}\right]$$

or \$75.95. (This compares with a price of \$75.13 when no convexity adjustment is made.)

## Application 2: Swap Rates

Consider next a derivative providing a payoff at time $T$ equal to a swap rate observed at that time. A swap rate is a par yield. For the purposes of calculating a convexity adjustment we can make an approximation and assume that the $N$-year swap rate at time $T$ equals the yield at that time on an $N$-year bond with a coupon equal to today's forward swap rate. This enables equation (27.1) to be used.

***Example 27.2***

Consider an instrument that provides a payoff in 3 years equal to the 3-year swap rate at that time multiplied by \$100. Suppose that payments are made annually on the swap, the zero rate for all maturities is 12% per annum with annual compounding, the volatility for the 3-year forward swap rate in 3 years (implied from swap option prices) is 22%. We approximate the swap rate as the yield on a 12% bond, so that the relevant function $G(y)$ is

$$G(y)=\frac{0.12}{1+y}+\frac{0.12}{(1+y)^2}+\frac{1.12}{(1+y)^3}$$

$$G'(y)=-\frac{0.12}{(1+y)^2}-\frac{0.24}{(1+y)^3}-\frac{3.36}{(1+y)^4}$$

$$G''(y)=\frac{0.24}{(1+y)^3}+\frac{0.72}{(1+y)^4}+\frac{13.44}{(1+y)^5}$$

In this case the forward yield $y_0$ is 0.12, so that $G'(y_0) = -2.4018$ and $G''(y_0) = 8.2546$. From equation (27.1),

$$E_T(y_T)=0.12+\tfrac{1}{2}\times 0.12^2\times 0.22^2\times 3\times\frac{8.2546}{2.4018}=0.1236$$

We should therefore assume a forward swap rate of 0.1236 (=12.36%) rather than 0.12 when valuing the instrument. The instrument is worth

$$\frac{100 \times 0.1236}{1.12^3} = 8.80$$

or \$8.80. (This compares with a price of 8.54 obtained without any convexity adjustment.)

## 27.2 TIMING ADJUSTMENTS

In this section we consider the situation where we observe a market variable $V$ at time $T$ and use its value to calculate a payoff that occurs at a later time $T^*$. Define:

$V_T$: Value of $V$ at time $T$

$E_T(V_T)$: Expected value of $V_T$ in a world that is forward risk-neutral with respect to $P(t, T)$

$E_{T^*}(V_T)$: Expected value of $V_T$ in a world that is forward risk-neutral with respect to $P(t, T^*)$

The numeraire ratio when we move from the $P(t, T)$ numeraire to the $P(t, T^*)$ numeraire (see Section 25.7) is

$$W = \frac{P(t, T^*)}{P(t, T)}$$

This is the forward price of a zero-coupon bond lasting between times $T$ and $T^*$. Define:

$\sigma_V$: Volatility of $V$

$\sigma_W$: Volatility of $W$

$\rho_{VW}$: Correlation between $V$ and $W$

From equation (25.35) the change of numeraire increases the growth rate of $V$ by $\alpha_V$, where

$$\alpha_V = \rho_{VW}\sigma_V\sigma_W \tag{27.3}$$

We can express this result in terms of the forward interest rate between times $T$ and $T^*$. Define:

$R$: Forward interest rate for period between $T$ and $T^*$, expressed with a compounding frequency of $m$

$\sigma_R$: Volatility of $R$

The relationship between $W$ and $R$ is

$$W = \frac{1}{(1 + R/m)^{m(T^*-T)}}$$

The relationship between the volatility of $W$ and the volatility of $R$ can be calculated from Itô's lemma as[2]

$$\sigma_W = \frac{\sigma_R R(T^* - T)}{1 + R/m}$$

Hence equation (27.3) becomes

$$\alpha_V = -\frac{\rho_{VR}\sigma_V\sigma_R R(T^* - T)}{1 + R/m}$$

where $\rho_{VR} = -\rho_{VW}$ is the instantaneous correlation between $V$ and $R$. As an approximation we can assume that $R$ remains constant at $R_0$ and that the volatilities and correlation in this expression are constant to get at time zero

$$E_{T^*}(V_T) = E_T(V_T)\exp\left[-\frac{\rho_{VR}\sigma_V\sigma_R R_0(T^* - T)}{1 + R_0/m}T\right] \tag{27.4}$$

*Example 27.3*

Consider a derivative that provides a payoff in 6 years equal to the value of a stock index observed in 5 years. Suppose that 1,200 is the forward value of the stock index for a contract maturing in 5 years. Suppose that the volatility of the index is 20%, the volatility of the forward interest rate between years 5 and 6 is 18%, and the correlation between the two is –0.4. Suppose further that the zero curve is flat at 8% with annual compounding. We apply the results we have just produced to the situation where $V$ is equal the value of the index. In this case $T = 5$, $T^* = 6$, $m = 1$, $R_0 = 0.08$, $\rho_{VR} = -0.4$, $\sigma_V = 0.20$, and $\sigma_R = 0.18$, so that

$$E_{T^*}(V_T) = E_T(V_T)\exp\left[-\frac{-0.4 \times 0.20 \times 0.18 \times 0.08 \times 1}{1 + 0.08} \times 5\right]$$

or $E_{T^*}(V_T) = 1.00535 E_T(V_T)$. From the arguments in Chapter 25 we know that $E_T(V_T)$ is the forward price of the index, or 1,200. It follows that $E_{T^*}(V_T) = 1{,}200 \times 1.00535 = 1206.42$. Using again the arguments in Chapter 25, it follows from equation (25.20) that the value of the derivative is $1206.42 \times P(0, 6)$. In this case $P(0, 6) = 1/1.08^6 = 0.6302$, so that the value of the derivative is 760.25.

## Application 1 Revisited

The analysis just given provides a different way of producing the result in Application 1 of Section 27.1. Using the notation from that application, we define $R_T$ as the interest rate between $T$ and $T^*$ and $R_0$ as the forward rate for the period between time $T$ and $T^*$. We know from equation (25.22) that

$$E_{T^*}(R_T) = R_0$$

We can apply equation (27.4) with $V$ equal to $R$ to get

$$E_{T^*}(R_T) = E_T(R_T)\exp\left[-\frac{\sigma_R^2 R_0\tau}{1 + R_0\tau}T\right]$$

[2] Instead of Itô's lemma we can use the duration result in Section 26.2

where $\tau = T^* - T$ (note that $m = 1/\tau$). It follows that

$$R_0 = E_T(R_T)\exp\left[-\frac{\sigma_R^2 R_0 T\tau}{1+R_0\tau}\right]$$

or

$$E_T(R_T) = R_0 \exp\left[\frac{\sigma_R^2 R_0 T\tau}{1+R_0\tau}\right]$$

Approximating the exponential function, we see that

$$E_T(R_T) = R_0 + \frac{R_0^2 \sigma_R^2 \tau T}{1+R_0\tau}$$

This is the same result as equation (27.2).

## 27.3 QUANTOS

A *quanto* or *cross-currency derivative* is an instrument where two currencies are involved. The payoff is defined in terms of a variable that is measured in one of the currencies and the payoff is made in the other currency. One example of a quanto is the CME futures contract on the Nikkei discussed in Business Snapshot 5.3. The market variable underlying this contract is the Nikkei 225 index (which is measured in yen), but contract is settled in US dollars.

Consider a quanto that provides a payoff in currency $X$ at time $T$. We assume that the payoff depends on the value $V$ of a variable that is observed in currency $Y$ at time $T$. Define:

$P_X(t, T)$: Value at time $t$ in currency $X$ of a zero-coupon bond paying off 1 unit of currency $X$ at time $T$

$P_Y(t, T)$: Value at time $t$ in currency $Y$ of a zero-coupon bond paying off 1 unit of currency $Y$ at time $T$

$V_T$: Value of $V$ at time $T$

$E_X(V_T)$: Expected value of $V_T$ in a world that is forward risk neutral with respect to $P_X(t, T)$

$E_Y(V_T)$: Expected value of $V_T$ in a world that is forward risk neutral with respect to $P_Y(t, T)$

The numeraire ratio when we move from the $P_Y(t, T)$ numeraire to the $P_X(t, T)$ numeraire is

$$W(t) = \frac{P_X(t, T)}{P_Y(t, T)} S(t)$$

where $S(t)$ is the spot exchange rate (units of $Y$ per unit of $X$) at time $t$. It follows from this that the numeraire ratio $W(t)$ is the forward exchange rate (units of $Y$ per unit of $X$) for a contract maturing at time $T$. Define:

$\sigma_W$ Volatility of $W$

$\sigma_V$: Volatility of $V$

$\rho_{VW}$: Instantaneous correlation between $V$ and $W$.

From equation (25.35) the change of numeraire increases the growth rate of $V$ by $\alpha_V$ where

$$\alpha_V = \rho_{VW}\sigma_V\sigma_W \tag{27.5}$$

If as an approximation we assume that the volatilities and correlation are constant, this means that

$$E_X(V_T) = E_Y(V_T)e^{\rho\sigma_V\sigma_W T}$$

or as an approximation

$$E_X(V_T) = E_Y(V_T)(1 + \rho\sigma_V\sigma_W T) \tag{27.6}$$

We will apply this equation to the valuation of what are known as diff swaps in Chapter 30.

***Example 27.4***

Suppose that the current value of the Nikkei stock index for a 1-year contract is 15,000 yen, the 1-year dollar risk-free rate is 5%, the 1-year yen risk-free rate is 2%, and the Nikkei dividend yield is 1%. The forward price of the Nikkei for a contract denominated in yen can be calculated in the usual way from equation (5.8) as

$$15{,}000e^{(0.02-0.01)\times 1} = 15{,}150.75$$

Suppose that the volatility of the index is 20%, the volatility of the 1-year forward yen per dollar exchange rate is 12%, and the correlation between the two is 0.3. In this case $F(0) = 15{,}150.75$, $\sigma_F = 0.20$, $\sigma_W = 0.12$ and $\rho = 0.3$. From equation (27.6), the expected value of the Nikkei in a world that is forward risk neutral with respect to a dollar bond maturing in 1 year is

$$15{,}150.75e^{0.3\times 0.2\times 0.12\times 1} = 15{,}260.23$$

This is the forward price of the Nikkei for a contract that provides a payoff in dollars rather than yen. (As an approximation, it is also the futures price of such a contract.)

## Using Traditional Risk-Neutral Measures

The forward risk-neutral measure we have been using works well when payoffs occur at only one time. In other situations it is sometimes more appropriate to use the traditional risk-neutral measure. Suppose we know the process followed by a variable $V$ in the traditional currency-$Y$ risk-neutral world and we wish to estimate its process in the traditional currency-$X$ risk-neutral world. Define:

$S$: Spot exchange rate (units of $Y$ per unit of $X$)

$\sigma_S$: Volatility of $S$

$\sigma_V$: Volatility of $V$

$\rho$: Instantaneous correlation between $S$ and $V$

In this case, the change of numeraire is from the money market account in currency $Y$ to the money market account in currency $X$ (with both money market accounts being denominated in currency $X$). Define $g_X$ as the value of the money market account in

**Business Snapshot 27.1** Siegel's Paradox

Consider two currencies, $X$ and $Y$. Suppose that the interest rates in the two currencies, $r_X$ and $r_Y$, are constant. Define $S$ as the number of units of currency $Y$ per unit of currency $X$. As explained in Chapter 5, a currency is an asset that provides a yield at the foreign risk-free rate. The traditional risk-neutral process for $S$ is therefore

$$dS = (r_Y - r_X)S\,dt + \sigma_S S\,dz$$

From Itô's lemma, this implies that the process for $1/S$ is

$$d(1/S) = (r_X - r_Y + \sigma_S^2)(1/S)\,dt - \sigma_S(1/S)\,dz$$

This leads to what is known as *Siegel's paradox*. Since the expected growth rate of $S$ is $r_Y - r_X$ in a risk-neutral world, symmetry suggests that the expected growth rate of $1/S$ should be $r_X - r_Y$ rather than $r_X - r_Y + \sigma_S^2$.

To understand Siegel's paradox it is necessary to appreciate that the process we have given for $S$ is the risk-neutral process for $S$ in a world where the numeraire is the money market account in currency $Y$. The process for $1/S$, because it is deduced from the process for $S$, therefore also assumes that this is the numeraire. Because $1/S$ is the number of units of $X$ per unit of $Y$, to be symmetrical we should measure the process for $1/S$ in a world where the numeraire is the money market account in currency $X$. Equation (27.7) shows that when we change the numeraire, from the money market account in currency $Y$ to the money market account in currency $X$, the growth rate of a variable $V$ increases by $\rho\sigma_V\sigma_S$, where $\rho$ is the correlation between $S$ and $V$. In this case, $V = 1/S$, so that $\rho = -1$ and $\sigma_V = \sigma_S$. It follows that the change of numeraire causes the growth rate of $1/S$ to increase by $-\sigma_S^2$. This neutralizes the $+\sigma_S^2$ in the process given above for $1/S$. The process for $1/S$ in a world where the numeraire is the money market account in currency $X$ is therefore

$$d(1/S) = (r_X - r_Y)(1/S)\,dt - \sigma_S(1/S)\,dz$$

This is symmetrical with the process we started with for $S$. The paradox has been resolved!

currency $X$ and $g_Y$ as the value of the money market account in currency $Y$. The numeraire ratio is

$$\frac{g_X}{g_Y}S$$

The variables $g_X(t)$ and $g_Y(t)$ have a stochastic drift but zero volatility as explained in Section 25.4. From Itô's lemma it follows that the volatility of the numeraire ratio is $\sigma_S$. The change of numeraire therefore involves increasing the expected growth rate of $V$ by

$$\rho\sigma_V\sigma_S \tag{27.7}$$

The market price of risk changes from zero to $\rho\sigma_S$. An application of this result is to Siegel's paradox (see Business Snapshot 27.1).

### *Example 27.5*

A 2-year American option provides a payoff of $S - K$ pounds sterling where $S$ is the level of the S&P 500 at the time of exercise and $K$ is the strike price. The current level of the S&P 500 is 1,200. The risk-free interest rates in sterling and dollars are both constant at 5% and 3%, respectively, the correlation between the dollars/sterling exchange rate and the S&P 500 is 0.2, the volatility of the S&P 500 is 25%, and the volatility of the exchange rate is 12%. The dividend yield on the S&P 500 is 1.5%.

We can value this option by constructing a binomial tree for the S&P 500 using as the numeraire the money market account in the UK (i.e., using the traditional risk-neutral world as seen from the perspective of a UK investor). From equation (27.7), the change in numeraire from the US to UK money market account leads to an increase in the expected growth rate in the S&P 500 of

$$0.2 \times 0.25 \times 0.12 = 0.006$$

or 0.6%. The growth rate of the S&P 500 using a US dollar numeraire is $3\% - 1.5\% = 1.5\%$. The growth rate using the sterling numeraire is therefore 2.1%. The risk-free interest rate in sterling is 5%. The S&P 500 therefore behaves like an asset providing a dividend yield of $5\% - 2.1\% = 2.9\%$ under the sterling numeraire. Using the parameter values of $S = 1{,}200$, $K = 1{,}200$, $r = 0.05$, $q = 0.029$, $\sigma = 0.25$, and $T = 2$ with 100 time steps, DerivaGem estimates the value of the option as £179.83.

## SUMMARY

When valuing a derivative providing a payoff at a particular future time it is natural to assume that the variables underlying the derivative equal their forward values and discount at the rate of interest applicable from the valuation date to the payoff date. This chapter has shown that this is not always the correct procedure.

When a payoff depends on a bond yield $y$ observed at time $T$ the expected yield should be assumed to be higher than the forward yield as indicated by equation (27.1). This result can be adapted for situations where a payoff depends on a swap rate. When a variable is observed at time $T$ but the payoff occurs at a later time $T^*$ the forward value of the variable should be adjusted as indicated by equation (27.4). When a variable is observed in one currency but leads to a payoff in another currency the forward value of the variable should also be adjusted. In this case the adjustment is shown in equation (27.6).

We will use these results when we look at nonstandard swaps in Chapter 30.

## FURTHER READING

Brotherton-Ratcliffe, R., and B. Iben, "Yield Curve Applications of Swap Products," in *Advanced Strategies in Financial Risk Management* (R. Schwartz and C. Smith, eds.). New York Institute of Finance, 1993.

Jamshidian, F., "Corralling Quantos," *Risk*, March (1994): 71–75.

Reiner, E., "Quanto Mechanics," *Risk*, March (1992), 59–63.

## Questions and Problems (Answers in Solutions Manual)

27.1. Explain how you would value a derivative that pays off $100R$ in 5 years, where $R$ is the 1-year interest rate (annually compounded) observed in 4 years. What difference would it make if the payoff were in 4 years? What difference would it make if the payoff were in 5 years?

27.2. Explain whether any convexity or timing adjustments are necessary when:
(a) We wish to value a spread option that pays off every quarter the excess (if any) of the 5-year swap rate over the 3-month LIBOR rate applied to a principal of \$100. The payoff occurs 90 days after the rates are observed.
(b) We wish to value a derivative that pays off every quarter the 3-month LIBOR rate minus the 3-month Treasury bill rate. The payoff occurs 90 days after the rates are observed.

27.3. Suppose that in Example 26.3 of Section 26.3 the payoff occurs after 1 year (i.e., when the interest rate is observed) rather than in 15 months. What difference does this make to the inputs to Black's models?

27.4. The yield curve is flat at 10% per annum with annual compounding. Calculate the value of an instrument where, in 5 years' time, the 2-year swap rate (with annual compounding) is received and a fixed rate of 10% is paid. Both are applied to a notional principal of \$100. Assume that the volatility of the swap rate is 20% per annum. Explain why the value of the instrument is different from zero.

27.5. What difference does it make in Problem 27.4 if the swap rate is observed in 5 years, but the exchange of payments takes place in (a) 6 years, and (b) 7 years? Assume that the volatilities of all forward rates are 20%. Assume also that the forward swap rate for the period between years 5 and 7 has a correlation of 0.8 with the forward interest rate between years 5 and 6 and a correlation of 0.95 with the forward interest rate between years 5 and 7.

27.6. The price of a bond at time $T$, measured in terms of its yield, is $G(y_T)$. Assume geometric Brownian motion for the forward bond yield $y$ in a world that is forward risk neutral with respect to a bond maturing at time $T$. Suppose that the growth rate of the forward bond yield is $\alpha$ and its volatility $\sigma_y$.
(a) Use Itô's lemma to calculate the process for the forward bond price in terms of $\alpha$, $\sigma_y$, $y$, and $G(y)$.
(b) The forward bond price should follow a martingale in the world considered. Use this fact to calculate an expression for $\alpha$.
(c) Show that the expression for $\alpha$ is, to a first approximation, consistent with equation (27.1).

27.7. The variable $S$ is an investment asset providing income at rate $q$ measured in currency A. It follows the process

$$dS = \mu_S S\,dt + \sigma_S S\,dz$$

in the real world. Defining new variables as necessary, give the process followed by $S$, and the corresponding market price of risk, in:
(a) A world that is the traditional risk-neutral world for currency A
(b) A world that is the traditional risk-neutral world for currency B

(c) A world that is forward risk neutral with respect to a zero-coupon currency A bond maturing at time $T$

(d) A world that is forward risk neutral with respect to a zero-coupon currency B bond maturing at time $T$

27.8. A call option provides a payoff at time $T$ of $\max(S_T - K, 0)$ yen, where $S_T$ is the dollar price of gold at time $T$ and $K$ is the strike price. Assuming that the storage costs of gold are zero and defining other variables as necessary, calculate the value of the contract.

27.9. Suppose that an index of Canadian stocks currently stands at 400. The Canadian dollar is currently worth 0.70 US dollars. The risk-free interest rates in Canada and the US are constant at 6% and 4%, respectively. The dividend yield on the index is 3%. Define $Q$ as the number of Canadian dollars per U.S dollar and $S$ as the value of the index. The volatility of $S$ is 20%, the volatility of $Q$ is 6%, and the correlation between $S$ and $Q$ is 0.4. Use DerivaGem to determine the value of a 2-year American-style call option on the index if:

(a) It pays off in Canadian dollars the amount by which the index exceeds 400.

(b) It pays off in US dollars the amount by which the index exceeds 400.

## Assignment Questions

27.10. Consider an instrument that will pay off $S$ dollars in 2 years, where $S$ is the value of the Nikkei index. The index is currently 20,000. The yen/dollar exchange rate is 100 (yen per dollar). The correlation between the exchange rate and the index is 0.3 and the dividend yield on the index is 1% per annum. The volatility of the Nikkei index is 20% and the volatility of the yen/dollar exchange rate is 12%. The interest rates (assumed constant) in the US and Japan are 4% and 2%, respectively.

(a) What is the value of the instrument?

(b) Suppose that the exchange rate at some point during the life of the instrument is $Q$ and the level of the index is $S$. Show that a US investor can create a portfolio that changes in value by approximately $\Delta S$ dollar when the index changes in value by $\Delta S$ yen by investing $S$ dollars in the Nikkei and shorting $SQ$ yen.

(c) Confirm that this is correct by supposing that the index changes from 20,000 to 20,050 and the exchange rate changes from 100 to 99.7.

(d) How would you delta hedge the instrument under consideration?

27.11. Suppose that the LIBOR yield curve is flat at 8% (with continuous compounding). The payoff from a derivative occurs in 4 years. It is equal to the 5-year rate minus the 2-year rate at this time, applied to a principal of \$100 with both rates being continuously compounded. (The payoff can be positive or negative.) Calculate the value of the derivative. Assume that the volatility for all rates is 25%. What difference does it make if the payoff occurs in 5 years instead of 4 years? Assume all rates are perfectly correlated.

27.12. Suppose that the payoff from a derivative will occur in 10 years and will equal the 3-year US dollar swap rate for a semiannual-pay swap observed at that time applied to a certain principal. Assume that the yield curve is flat at 8% (semiannually compounded) per annum in dollars and 3% (semiannually compounded) in yen. The forward swap rate volatility is 18%, the volatility of the 10-year "yen per dollar" forward exchange rate is 12%, and the correlation between this exchange rate and US dollar interest rates is 0.25.

(a) What is the value of the derivative if the swap rate is applied to a principal of \$100 million so that the payoff is in dollars?

(b) What is its value of the derivative if the swap rate is applied to a principal of 100 million yen so that the payoff is in yen?

27.13. The payoff from a derivative will occur in 8 years. It will equal the average of the 1-year interest rates observed at times 5, 6, 7, and 8 years applied to a principal of \$1,000. The yield curve is flat at 6% with annual compounding and the volatilities of all rates are 16%. Assume perfect correlation between all rates. What is the value of the derivative?

# APPENDIX

## PROOF OF THE CONVEXITY ADJUSTMENT FORMULA

This appendix calculates a convexity adjustment for forward bond yields. Suppose that the payoff from a derivative at time $T$ depends on a bond yield observed at that time. Define:

$y_0$: Forward bond yield observed today for a forward contract with maturity $T$

$y_T$: Bond yield at time $T$

$B_T$: Price of the bond at time $T$

$\sigma_y$: Volatility of the forward bond yield

We suppose that

$$B_T = G(y_T)$$

Expanding $G(y_T)$ in a Taylor series about $y_T = y_0$ yields the following approximation:

$$B_T = G(y_0) + (y_T - y_0)G'(y_0) + 0.5(y_T - y_0)^2 G''(y_0)$$

where $G'$ and $G''$ are the first and second partial derivatives of $G$. Taking expectations in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time $T$, we get

$$E_T(B_T) = G(y_0) + E_T(y_T - y_0)G'(y_0) + \tfrac{1}{2}E_T[(y_T - y_0)^2]G''(y_0)$$

where $E_T$ denotes expectations in this world. The expression $G(y_0)$ is by definition the forward bond price. Also, because of the particular world we are working in, $E_T(B_T)$ equals the forward bond price. Hence $E_T(B_T) = G(y_0)$, so that

$$E_T(y_T - y_0)G'(y_0) + \tfrac{1}{2}E_T[(y_T - y_0)^2]G''(y_0) = 0$$

The expression $E_T[(y_T - y_0)^2]$ is approximately $\sigma_y^2 y_0^2 T$. Hence it is approximately true that

$$E_T(y_T) = y_0 - \tfrac{1}{2}y_0^2\sigma_y^2 T\frac{G''(y_0)}{G'(y_0)}$$

This shows that, to obtain the expected bond yield in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time $T$, we should add

$$-\tfrac{1}{2}y_0^2\sigma_y^2 T\frac{G''(y_0)}{G'(y_0)}$$

to the forward bond yield. This is the result in equation (27.1). For an alternative proof, see Problem 27.6.