are 13% instead of 15%. What should the 5-year swap rate in 3 years' time be assumed for the purpose of valuing the swap? What is the value of the swap?

30.7. Explain why a plain vanilla interest rate swap and the compounding swap in Section 30.2 can be valued using the "assume forward rates are realized" rule, but a LIBOR-in-arrears swap in Section 30.4 cannot.

30.8. In the accrual swap discussed in the text, the fixed side accrues only when the floating reference rate lies below a certain level. Discuss how the analysis can be extended to cope with a situation where the fixed side accrues only when the floating reference rate is above one level and below another.

Assignment Questions

30.9. LIBOR zero rates are flat at 5% in the United States and flat at 10% in Australia (both annually compounded). In a 4-year swap Australian LIBOR is received and 9% is paid with both being applied to a USD principal of $10 million. Payments are exchanged annually. The volatility of all 1-year forward rates in Australia is estimated to be 25%, the volatility of the forward USD/AUD exchange rate (AUD per USD) is 15% for all maturities, and the correlation between the two is 0.3. What is the value of the swap?

30.10. Estimate the interest rate paid by P&G on the 5/30 swap in Section 30.7 if (a) the CP rate is 6.5% and the Treasury yield curve is flat at 6% and (b) the CP rate is 7.5% and the Treasury yield curve is flat at 7% with semiannual compounding.

30.11. Suppose that you are trading a LIBOR-in-arrears swap with an unsophisticated counterparty who does not make convexity adjustments. To take advantage of the situation, should you be paying fixed or receiving fixed? How should you try to structure the swap as far as its life and payment frequencies?

Consider the situation where the yield curve is flat at 10% per annum with annual compounding. All cap volatilities are 18%. Estimate the difference between the way a sophisticated trader and an unsophisticated trader would value a LIBOR-in-arrears swap where payments are made annually and the life of the swap is (a) 5 years, (b) 10 years, and (c) 20 years. Assume a notional principal of $1 million.

30.12. Suppose that the LIBOR zero rate is flat at 5% with annual compounding. In a 5-year swap, company X pays a fixed rate of 6% and receives LIBOR. The volatility of the 2-year swap rate in 3 years is 20%.
(a) What is the value of the swap?
(b) Use DerivaGem to calculate the value of the swap if company X has the option to cancel after 3 years.
(c) Use DerivaGem to calculate the value of the swap if the counterparty has the option to cancel after 3 years.
(d) What is the value of the swap if either side can cancel at the end of 3 years?

CHAPTER 31

Real Options

Up to now we have been almost entirely concerned with the valuation of financial assets. In this chapter we explore how the ideas we have developed can be extended to assess capital investment opportunities in real assets such as land, buildings, plant, and equipment. Often there are options embedded in these investment opportunities (the option to expand the investment, the option to abandon the investment, the option to defer the investment, and so on.) These options are very difficult to value using traditional capital investment appraisal techniques. The approach known as *real options* attempts to deal with this problem using option-pricing theory.

The chapter starts by explaining the traditional approach to evaluating investments in real assets and shows how difficult it is to correctly value embedded options when this approach is used. It explains how the risk-neutral valuation approach can be extended to handle the valuation of real assets and presents a number of examples illustrating the application of the approach in a range of different situations.

31.1 CAPITAL INVESTMENT APPRAISAL

The traditional approach to valuing a potential capital investment project is known as the "net present value" (NPV) approach. The NPV of a project is the present value of its expected future incremental cash flows. The discount rate used to calculate the present value is a "risk-adjusted" discount rate, chosen to reflect the risk of the project. As the riskiness of the project increases, the discount rate also increases.

As an example, consider an investment that costs \$100 million and will last 5 years. The expected cash inflow in each year is estimated to be \$25 million. If the risk-adjusted discount rate is 12% (with continuous compounding), the net present value of the investment is (in millions of dollars)

$$-100 + 25e^{-0.12\times 1} + 25e^{-0.12\times 2} + 25e^{-0.12\times 3} + 25e^{-0.12\times 4} + 25e^{-0.12\times 5} = -11.53$$

A negative NPV, such as the one we have just calculated, indicates that the project will reduce the value of the company to its shareholders and should not be undertaken. A positive NPV indicates that the project should be undertaken because it will increase shareholder wealth.

The risk-adjusted discount rate should be the return required by the company, or the company's shareholders, on the investment. This can be calculated in a number of ways. One approach often recommended involves the capital asset pricing model. The steps are as follows:

1. Take a sample of companies whose main line of business is the same as that of the project being contemplated.
2. Calculate the betas of the companies and average them to obtain a proxy beta for the project.
3. Set the required rate of return equal to the risk-free rate plus the proxy beta times the excess return of the market portfolio over the risk-free rate.

One problem with the traditional NPV approach is that many projects contain embedded options. Consider, for example, a company that is considering building a plant to manufacture a new product. Often the company has the option to abandon the project if things do not work out well. It may also have the option to expand the plant if demand for the output exceeds expectations. These options usually have quite different risk characteristics from the base project and require different discount rates.

To understand the problem here, return to the example at the beginning of Chapter 11. This involved a stock whose current price is \$20. In three months the price will be either \$22 or \$18. Risk-neutral valuation shows that the value of a three-month call option on the stock with a strike price of 21 is 0.633. Footnote 1 of Chapter 11 shows that if the expected return required by investors on the stock in the real world is 16% then the expected return required on the call option is 42.6%. A similar analysis shows that if the option is a put rather than a call the expected return required on the option is −52.5%. These analyses mean that if the traditional NPV approach were used to value the call option the correct discount rate would be 42.6%, and if it were used to value a put option the correct discount rate would be −52.5%. There is no easy way of estimating these discount rates. (We know them only because we are able to value the options another way.) Similarly, there is no easy way of estimating the risk-adjusted discount rates appropriate for cash flows when they arise from abandonment, expansion, and other options. This is the motivation for exploring whether the risk-neutral valuation principle can be applied to options on real assets as well as to options on financial assets.

Another problem with the traditional NPV approach lies in the estimation of the appropriate risk-adjusted discount rate for the base project (i.e., the project without embedded options). The companies that are used to estimate a proxy beta for the project in the three-step procedure above have expansion options and abandonment options of their own. Their betas reflect these options and may not therefore be appropriate for estimating a beta for the base project.

31.2 EXTENSION OF THE RISK-NEUTRAL VALUATION FRAMEWORK

In Section 25.1 we defined the market price of risk for a variable θ as

$$\lambda = \frac{\mu - r}{\sigma} \tag{31.1}$$

where r is the risk-free rate, μ is the return on a traded security dependent only on θ, and σ is its volatility. As we showed in Section 25.1, we get the same market price of risk, λ, regardless of the traded security chosen.

Suppose that a real asset depends on several variables θ_i ($i = 1, 2, \ldots$). Let m_i and s_i be the expected growth rate and volatility of θ_i so that

$$\frac{d\theta_i}{\theta_i} = m_i\, dt + s_i\, dz_i$$

where z_i is a Wiener process. Define λ_i as the market price of risk of θ_i. We can extend risk-neutral valuation to show that any asset dependent on the θ_i can be valued by[1]

1. Reducing the expected growth rate of each θ_i from m to $m - \lambda s$
2. Discounting cash flows at the risk-free rate.

Example 31.1

The cost of renting commercial real estate in a certain city is quoted as the amount that would be paid per square foot per year in a new 5-year rental agreement. The current cost is \$30 per square foot. The expected growth rate of the cost is 12% per annum, its volatility is 20% per annum, and its market price of risk is 0.3. A company has the opportunity to pay \$1 million now for the option to rent 100,000 square feet at \$35 per square foot for a 5-year period starting in 2 years. The risk-free rate is 5% per annum (assumed constant). Define V as the quoted cost per square foot of office space in 2 years. We make the simplifying assumption that rent is paid annually in advance. The payoff from the option is

$$100{,}000A\max(V - 35,\ 0)$$

where A is an annuity factor given by

$$A = 1 + 1 \times e^{-0.05\times 1} + 1 \times e^{-0.05\times 2} + 1 \times e^{-0.05\times 3} + 1 \times e^{-0.05\times 4} = 4.5355$$

The expected payoff in a risk-neutral world is therefore

$$100{,}000 \times 4.5355 \times \hat{E}[\max(V - 35,\ 0)] = 453{,}550 \times \hat{E}[\max(V - 35,\ 0)]$$

where $\hat{E}$ denotes expectations in a risk-neutral world. Using the result in equation (13A.1), this is

$$453{,}550[\hat{E}(V)N(d_1) - 35N(d_2)]$$

where

$$d_1 = \frac{\ln[\hat{E}(V)/35] + .2^2 \times 2/2}{0.2\sqrt{2}}$$

$$d_2 = \frac{\ln[\hat{E}(V)/35] - .2^2 \times 2/2}{0.2\sqrt{2}}$$

The expected growth rate in the cost of commercial real estate in a risk-neutral

[1] To see that this is consistent with regular risk-neutral valuation, suppose that θ_i is the price of a non-dividend-paying stock. Since this is the price of a traded security, equation (31.1) implies that $(m_i - r)/s_i = \lambda_i$, or $m_i - \lambda_i s_i = r$. The expected growth-rate adjustment is therefore the same as setting the return on the stock equal to the risk-free rate. For a proof of the more general result, see Technical Note 20 on the author's website.

world is $m - \lambda s$, where m is the real-world growth rate, s is the volatility, and λ is the market price of risk. In this case, $m = 0.12$, $s = 0.2$, and $\lambda = 0.3$, so that the expected risk-neutral growth rate is 0.06, or 6%, per year. It follows that $\hat{E}(V) = 30e^{0.06\times 2} = 33.82$. Substituting this in the expression above gives the expected payoff in a risk-neutral world as \$1.5015 million. Discounting at the risk-free rate the value of the option is $1.5015e^{-0.05\times 2} = \$1.3586$ million. This shows that it is worth paying \$1 million for the option.

31.3 ESTIMATING THE MARKET PRICE OF RISK

The real-options approach to evaluating an investment avoids the need to estimate risk-adjusted discount rates in the way described in Section 31.1, but it does require market price of risk parameters for all stochastic variables. When historical data are available for a particular variable, its market price of risk can be estimated using the capital asset pricing model. To show how this is done, we consider an investment asset dependent solely on the variable and define:

μ: Expected return of the investment asset

σ: Volatility of the return of the investment asset

λ: Market price of risk of the variable

ρ: Instantaneous correlation between the percentage changes in the variable and returns on a broad index of stock market prices

μ_m: Expected return on broad index of stock market prices

σ_m: Volatility of return on the broad index of stock market prices

r: Short-term risk-free rate

Because the investment asset is dependent solely on the market variable, the instantaneous correlation between its return and the broad index of stock market prices is also ρ. From the continuous-time version of the capital asset pricing model, we have

$$\mu - r = \frac{\rho\sigma}{\sigma_m}(\mu_m - r)$$

From equation (31.1), another expression for $\mu - r$ is

$$\mu - r = \lambda\sigma$$

It follows that

$$\lambda = \frac{\rho}{\sigma_m}(\mu_m - r) \tag{31.2}$$

This equation can be used to estimate λ.

Example 31.2

A historical analysis of company's sales, quarter by quarter, show that percentage changes in sales have a correlation of 0.3 with returns on the S&P 500 index. The volatility of the S&P 500 is 20% per annum and based on historical data the expected excess return of the S&P 500 over the risk-free rate is 5%. Equation (31.2) estimates the market price of risk for the company's sales as

$$\frac{0.3}{0.2} \times 0.05 = 0.075$$

When no historical data are available for the particular variable under consideration, other similar variables can sometimes be used as proxies. For example, if a plant is being constructed to manufacture a new product, data can be collected on the sales of other similar products. The correlation of the new product with the market index can then be assumed to be the same as that of these other products. In some cases, the estimate of ρ in equation (31.2) must be based on subjective judgment. If an analyst is convinced that a particular variable is unrelated to the performance of a market index, its market price of risk should be set to zero.

For some variables, it is not necessary to estimate the market price of risk because the process followed by a variable in a risk-neutral world can be estimated directly. For example, if the variable is the price of an investment asset, its total return in a risk-neutral world is the risk-free rate. If the variable is the short-term interest rate r, Chapter 28 shows how a risk-neutral process can be estimated from the initial term structure of interest rates. Later in this chapter we will show how the risk-neutral process for a commodity can be estimated from futures prices.

31.4 APPLICATION TO THE VALUATION OF A BUSINESS

Traditional methods of business valuation, such as applying a price/earnings multiplier to current earnings, do not work well for new businesses. Typically a company's earnings are negative during its early years as it attempts to gain market share and establish relationships with customers. The company must be valued by estimating future earnings and cash flows under different scenarios.

The company's future cash flows typically depend on a number of variables such as sales, variable costs as a percent of sales, fixed costs, and so on. Single estimates should be sufficient for some of the variables. For key variables, a risk-neutral stochastic process should be estimated as outlined in the previous two sections. A Monte Carlo simulation can then be carried out to generate alternative scenarios for the net cash flows per year in a risk-neutral world. It is likely that under some of these scenarios the company does very well and under others it becomes bankrupt and ceases operations. (The simulation must have a built in rule for determining when bankruptcy happens.) The value of the company is the present value of the expected cash flow in each year using the risk-free rate for discounting. Business Snapshot 31.1 gives an example of the application of the approach to Amazon.com.

31.5 COMMODITY PRICES

Many investments involve uncertainties related to future commodity prices. Often futures prices can be used to estimate the risk-neutral stochastic process for a commodity price directly. This avoids the need to explicitly estimate a market price of risk for the commodity.

Business Snapshot 31.1 Valuing Amazon.com

One of the earliest published attempts to value a company using the real options approach was Schwartz and Moon (2000), who considered Amazon.com at the end of 1999. They assumed the following stochastic processes for the company's sales revenue R and its revenue growth rate μ:

$$\frac{dR}{R} = \mu\,dt + \sigma(t)\,dz_1$$

$$d\mu = \kappa(\bar{\mu} - \mu)\,dt + \eta(t)\,dz_2$$

They assumed that the two Wiener processes dz_1 and dz_2 were uncorrelated and made reasonable assumptions about $\sigma(t)$, $\eta(t)$, κ, and $\bar{\mu}$ based on available data.

They assumed the cost of goods sold would be 75% of sales, other variable expenses would be 19% of sales, and fixed expenses would be \$75 million per quarter. The initial sales level was \$356 million, the initial tax loss carry forward was \$559 million and the tax rate was assumed to be 35%. The market price of risk for R was estimated from historical data using the approach described in the previous section. The market price of risk for μ was assumed to be zero.

The time horizon for the analysis was 25 years and the terminal value of the company was assumed to be ten times pretax operating profit. The initial cash position was \$906 million and the company was assumed to go bankrupt if the cash balance became negative.

Different future scenarios were generated in a risk-neutral world using Monte Carlo simulation. The evaluation of the scenarios involved taking account of the possible exercise of convertible bonds and the possible exercise of employee stock options. The value of the company to the share holders was calculated as the present value of the net cash flows discounted at the risk-free rate.

Using these assumptions, Schwartz and Moon provided an estimate of the value of Amazon.com's shares at the end of 1999 equal to \$12.42. The market price at the time was \$76.125 (although it declined sharply in 2000). One of the key advantages of the real-options approach is that identifies the key assumptions. Schwartz and Moon found that the estimated share value was very sensitive to $\eta(t)$, the volatility of the growth rate. This was an important source of optionality. A small increase in $\eta(t)$ leads to more optionality and a big increase in the value of Amazon.com shares.

From Section 14.7 the expected future price of a commodity in the traditional risk-neutral world is its futures price. If we assume that the expected growth rate in the commodity price is dependent solely on time and that the volatility of the commodity price is constant, then the risk-neutral process for the commodity price S has the form

$$\frac{dS}{S} = \mu(t)\,dt + \sigma\,dz \tag{31.3}$$

and we have

$$F(t) = \hat{E}[S(t)] = S(0)e^{\int_0^t \mu(\tau)d\tau}$$

where $F(t)$ is the futures price for a contract with maturity t and $\hat{E}$ denotes expected value in a risk-neutral world. It follows that

$$\ln F(t) = \ln S(0) + \int_0^t \mu(\tau)d\tau$$

Differentiating both sides with respect to time gives

$$\mu(t) = \frac{\partial}{\partial t}[\ln F(t)]$$

Example 31.3

Suppose that the futures prices of live cattle at the end of July 2005 are (in cents per pound) as follows:

August 2005	62.20
October 2005	60.60
December 2005	62.70
February 2006	63.37
April 2006	64.42
June 2006	64.40

These can be used to estimate the expected growth rate in live cattle prices in a risk-neutral world. For example, when the model in equation (31.3) is used, the expected growth rate in live cattle prices between October and December 2005, in a risk-neutral world is

$$\ln\left(\frac{62.70}{60.60}\right) = 0.034$$

or 3.4% with continuous compounding. On an annualized basis, this is 20.4% per annum.

Example 31.4

Suppose that the futures prices of live cattle are as in Example 31.3. A certain breeding decision would involve an investment of \$100,000 now and expenditures of \$20,000 in 3 months, 6 months, and 9 months. The result is expected to be that an extra cattle will be available for sale at the end of the year. There are two major uncertainties: the number of pounds of extra cattle that will be available for sale and the price per pound. The expected number of pounds is 300,000. The expected price of cattle in 1 year in a risk-neutral world is, from Example 31.3, 64.40 cents per pound. Assuming that the risk-free rate of interest is 10% per annum, the value of the investment (in thousands of dollars) is

$$-100 - 20e^{-0.1\times0.25} - 20e^{-0.1\times0.50} - 20e^{-0.1\times0.75} + 300 \times 0.644e^{-0.1\times1} = 17.729$$

This assumes that any uncertainty about the extra amount of cattle that will be available for sale has zero systematic risk and that there is no correlation between the amount of cattle that will be available for sale and the price.

A Mean-Reverting Process

It can be argued that the process in equation (31.3) for commodity prices is too simplistic. In practice, most commodity prices follow mean-reverting processes. They tend to get pulled back to a central value. A more realistic process than equation (31.3) for the risk-neutral process followed by the commodity price S is

$$d\ln S = [\theta(t) - a\ln S]\,dt + \sigma\,dz \tag{31.4}$$

This incorporates mean reversion and is analogous to the lognormal process assumed for the short-term interest rate in Chapter 28. The trinomial tree methodology in Section 28.7 can be adapted to construct a tree for S and determine the value of $\theta(t)$ such that $F(t) = \hat{E}[S(t)]$

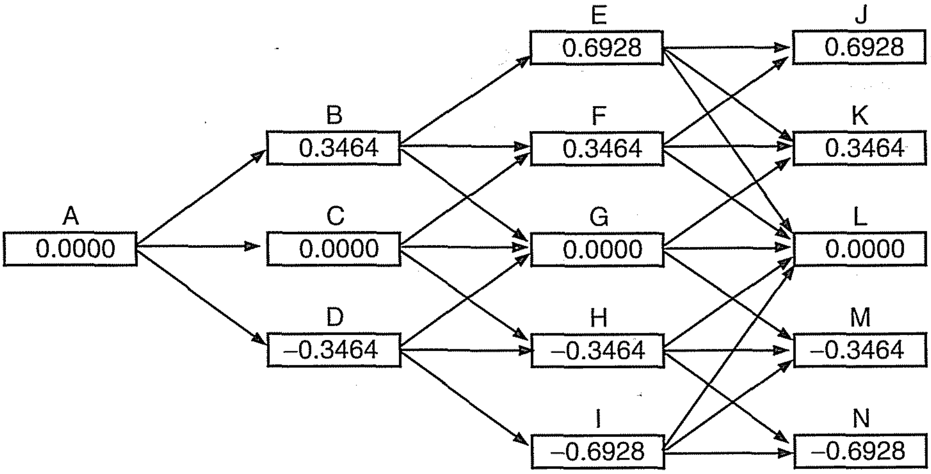
We will illustrate this process by building a three-step tree for oil. Suppose that the spot price of oil is \$20 per barrel and the 1-year, 2-year, and 3-year futures prices are \$22, \$23, and \$24, respectively. Suppose that $a = 0.1$ and $\sigma = 0.2$ in equation (31.4). We first define a variable X that is initially zero and follows the process

$$dX = -a\,dt + \sigma\,dz \tag{31.5}$$

Using the procedure in Section 28.7, a trinomial tree can be constructed for X. This is shown in Figure 31.1.

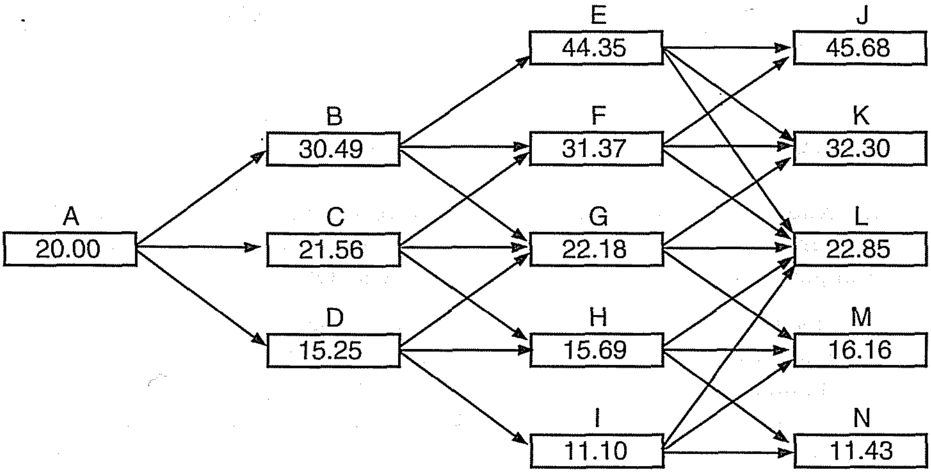
The variable $\ln S$ follows the same process as X except for a time-dependent drift. Analogously to Section 28.7, we can convert the tree for X to a tree for $\ln S$ by displacing the positions of nodes. This tree is shown in Figure 31.2. The initial node corresponds to an oil price of 20, so the displacement for that node is ln 20. Suppose

Figure 31.1 Tree for X. Constructing this tree is the first stage in constructing a tree for the spot price of oil, S. Here p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 31.2 Tree for spot price of oil: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

that the displacements of the nodes at 1 year is α_1. The values of the X at the three nodes at the 1-year point are $+0.3464$, 0, and -0.3464. The corresponding values of $\ln S$ are $0.3464+\alpha_1$, α_1, and $-0.3464+\alpha_1$. The values of S are therefore $e^{0.3464+\alpha_1}$, e^{α_1}, and $e^{-0.3464+\alpha_1}$, respectively. We require the expected value of S to equal the futures price. This means that

$$0.1667e^{0.3464+\alpha_1}+0.6666e^{\alpha_1}+0.1667e^{-0.3464+\alpha_1}=22$$

The solution to this is $\alpha_1 = 3.071$. The values of S at the 1-year point are therefore 30.49, 21.56, and 15.25.

At the 2-year point, we first calculate the probabilities of nodes E, F, G, H, and I being reached from the probabilities of nodes B, C, and D being reached. The probability of reaching node F is the probability of reaching node B times the probability of moving from B to F plus the probability of reaching node C times the probability of moving from C to F. This is

$$0.1667 \times 0.6566 + 0.6666 \times 0.1667 = 0.2206$$

Similarly the probabilities of reaching nodes E, G, H, and I are 0.0203, 0.5183, 0.2206, and 0.0203, respectively. The amount α_2 by which the nodes at time 2 years are displaced must satisfy

$$0.0203e^{0.6928+\alpha_2}+0.2206e^{0.3464+\alpha_2}+0.5183e^{\alpha_2}$$
$$+0.2206e^{-0.3464+\alpha_2}+0.0203e^{-0.6928+\alpha_2}=23$$

The solution to this is $\alpha_2 = 3.099$. This means that the values of S at the 2-year point are 44.35, 31.37, 22.18, 15.69, and 11.10, respectively.

A similar calculation can be carried out at time 3 years. Figure 31.2 shows the resulting tree for S. We will illustrate how the tree can be used in the valuation of a real option in the next section.

31.6 EVALUATING OPTIONS IN AN INVESTMENT OPPORTUNITY

As already mentioned, most investment projects involve options. These options can add considerable value to the project and are often either ignored or valued incorrectly. Examples of the options embedded in projects are:

1. *Abandonment options*. This is an option to sell or close down a project. It is an American put option on the project's value. The strike price of the option is the liquidation (or resale) value of the project less any closing-down costs. When the liquidation value is low, the strike price can be negative. Abandonment options mitigate the impact of very poor investment outcomes and increase the initial valuation of a project.
2. *Expansion options*. This is the option to make further investments and increase the output if conditions are favorable. It is an American call option on the value of additional capacity. The strike price of the call option is the cost of creating this additional capacity discounted to the time of option exercise. The strike price often depends on the initial investment. If management initially choose to build capacity in excess of the expected level of output, the strike price can be relatively small.
3. *Contraction options*. This is the option to reduce the scale of a project's operation. It is an American put option on the value of the lost capacity. The strike price is the present value of the future expenditures saved as seen at the time of exercise of the option.
4. *Options to defer*. One of the most important options open to a manager is the option to defer a project. This is an American call option on the value of the project.
5. *Options to extend*. Sometimes it is possible to extend the life of an asset by paying a fixed amount. This is a European call option on the asset's future value.

As a simple example of the evaluation of an investment with an embedded option, consider a company that has to decide whether to invest \$15 million to obtain 6 million barrels of oil from a certain source at the rate of 2 million barrels per year for 3 years. The fixed costs of operating the equipment are \$6 million per year and the variable costs are \$17 per barrel. We assume that the risk-free interest rate is 10% per annum for all maturities, that the spot price of oil is \$20 per barrel, and that the 1-, 2-, and 3-year futures prices are \$22, \$23, and \$24 per barrel, respectively. We assume that the stochastic process for oil prices has been estimated as equation (31.4) with $a = 0.1$ and $\sigma = 0.2$. This means that the tree in Figure 31.2 describes the behavior of oil prices in a risk-neutral world.

First assume that the project has no embedded options. The expected prices of oil in 1, 2, and 3 years' time in a risk-neutral world are \$22, \$23, and \$24, respectively. The expected payoff from the project (in millions of dollars) in a risk-neutral world can be calculated from the cost data as 4.0, 6.0, and 8.0 in years 1, 2, and 3, respectively. The value of the project is therefore

$$-15.0 + 4.0e^{-0.1\times 1} + 6.0e^{-0.1\times 2} + 8.0e^{0.1\times 3} = -0.54$$

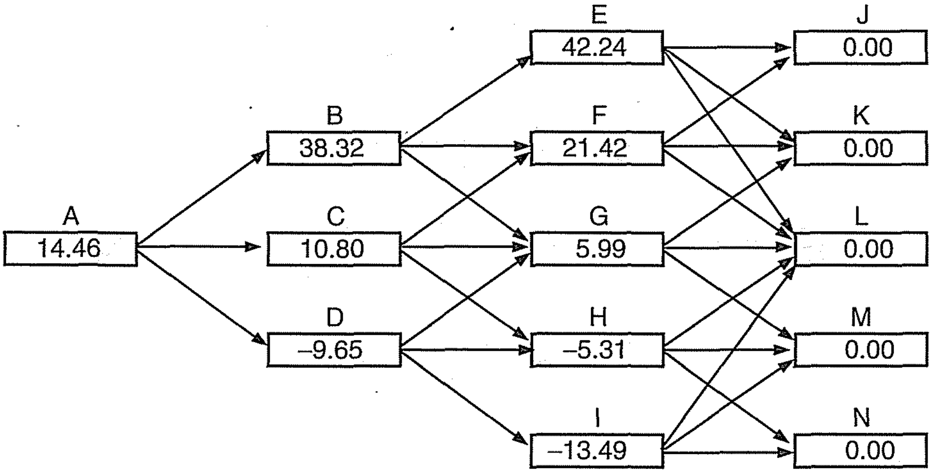
This analysis indicates that the project should not be undertaken because it would reduce shareholder wealth by 0.54 million.

Figure 31.3 shows the value of the project at each node of Figure 31.2. This is calculated from Figure 31.2. Consider, for example, node H. There is a 0.2217 probability that the price of oil at the end of the third year is 22.85, so that the third-year profit is $2 \times 22.85 - 2 \times 17 - 6 = 5.70$. Similarly, there is a 0.6566 probability that the price of oil at the end of the third year is 16.16, so that the profit is −7.68 and there is a 0.1217 probability that the price of oil at the end of the third year is 11.43, so that the profit is −17.14. The value of the project at node H in Figure 31.3 is therefore

$$[0.2217 \times 5.70 + 0.6566 \times (-7.68) + 0.1217 \times (-17.14)]e^{-0.1\times 1} = -5.31$$

As another example, consider node C. There is a 0.1667 chance of moving to node F where the oil price is 31.37. The second year cash flow is then $2 \times 31.37 - 2 \times 17 - 6 = 22.74$. The value of subsequent cash flows at node F is 21.42. The total value of the project if we move to node F is therefore $21.42 + 22.74 = 44.16$. Similarly the total value of the project if we move to nodes G and H are 10.35 and −13.93, respectively. The value

Figure 31.3 Valuation of base project with no embedded options: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

of the project at node C is therefore

$$[0.1667 \times 44.16 + 0.6666 \times 10.35 + 0.1667 \times (-13.93)]e^{-0.1\times 1} = 10.80$$

Figure 31.3 shows that the value of the project at the initial node A is 14.46. When the initial investment is taken into account the value of the project is therefore −0.54. This is in agreement with our earlier calculations.

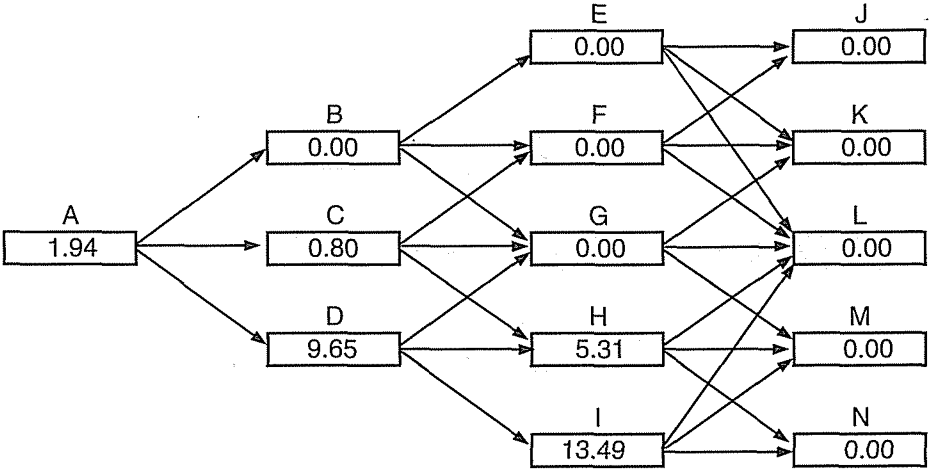
Suppose now that the company has the option to abandon the project at any time. We suppose that there is no salvage value and no further payments are required once the project has been abandoned. Abandonment is an American put option with a strike price of zero and is valued in Figure 31.4. The put option should not be exercised at nodes E, F, and G because the value of the project is positive at these nodes. It should be exercised at nodes H and I. The value of the put option is 5.31 and 13.49 at nodes H and I, respectively. Rolling back through the tree, the value of the abandonment put option at node D if it is not exercised is

$$(0.1217 \times 13.49 + 0.6566 \times 5.31 + 0.2217 \times 0)e^{-0.1\times 1} = 4.64$$

The value of exercising the put option at node D is 9.65. This is greater than 4.64, and so the put should be exercised at node D. The value of the put option at node C is

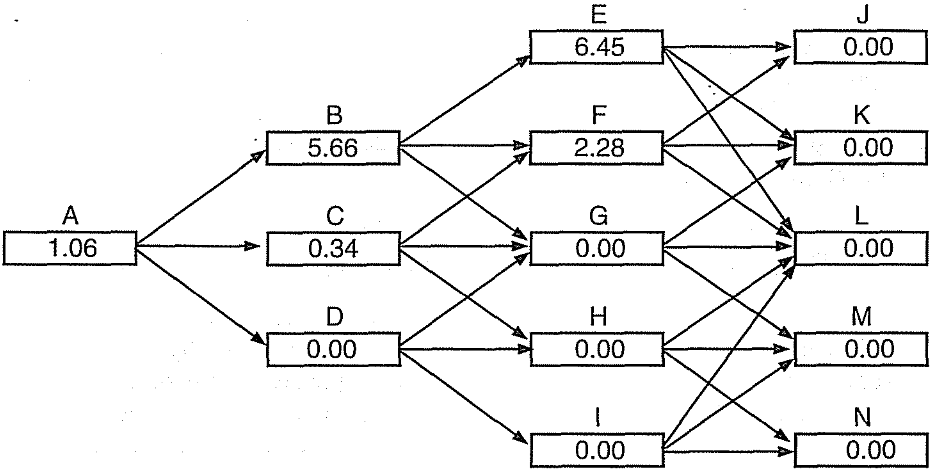
$$[0.1667 \times 0 + 0.6666 \times 0 + 0.1667 \times (5.31)]e^{-0.1\times 1} = 0.80$$

Figure 31.4 Valuation of option to abandon the project: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Figure 31.5 Valuation of option to expand the project: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

and the value at node A is

$$(0.1667 \times 0 + 0.6666 \times 0.80 + 0.1667 \times 9.65)e^{-0.1\times 1} = 1.94$$

The abandonment option is therefore worth \$1.94 million. It increases the value of the project from −\$0.54 million to +\$1.40 million. A project that was previously unattractive now has a positive value to shareholders.

Suppose next that the company has no abandonment option. Instead it has the option at any time to increase the scale of the project by 20%. The cost of doing this is \$2 million. Oil production increases from 2.0 to 2.4 million barrels. Variable costs remain \$17 per barrel and fixed costs increase by 20% from \$6.0 million to \$7.2 million. This is an American call option to buy 20% of the base project in Figure 31.3 for \$2 million. The option is valued in Figure 31.5. At node E, the option should be exercised. The payoff is $0.2 \times 42.24 - 2 = 6.45$. At node F, it should also be exercised for a payoff of $0.2 \times 21.42 - 2 = 2.28$. At nodes G, H, and I, the option should not be exercised. At node B, exercising is worth more than waiting and the option is worth $0.2 \times 38.32 - 2 = 5.66$. At node C, if the option is not exercised, it is worth

$$(0.1667 \times 2.28 + 0.6666 \times 0.00 + 0.1667 \times 0.00)e^{-0.1\times 1} = 0.34$$

If the option is exercised, it is worth $0.2 \times 10.80 - 2 = 0.16$. The option should therefore not be exercised at node C. At node A, if not exercised, the option is worth

$$(0.1667 \times 5.66 + 0.6666 \times 0.34 + 0.1667 \times 0.00)e^{-0.1\times 1} = 1.06$$

If the option is exercised it is worth $0.2 \times 14.46 - 2 = 0.89$. Early exercise is therefore not optimal at node A. In this case, the option increases the value of the project from -0.54 to $+0.52$. Again we find that a project that previously had a negative value now has a positive value.

The expansion option in Figure 31.5 is relatively easy to value because, once the option has been exercised, all subsequent cash inflows and outflows increase by 20%. In the case where fixed costs remain the same or increase by less than 20%, it is necessary to keep track of more information at the nodes of Figure 31.3. Specifically we need to record the following:

1. The present value of subsequent fixed costs
2. The present value of subsequent revenues net of variable costs

The payoff from exercising the option can then be calculated.

When a project has two or more options, they are typically not independent. The value of having both option A and option B is typically not the sum of the values of the two options. To illustrate this, suppose that the company we have been considering has both abandonment and expansion options. The project cannot be expanded if it has already been abandoned. Moreover, the value of the put option to abandon depends on whether the project has been expanded.[2]

These interactions between the options in our example can be handled by defining four states at each node:

1. Not already abandoned; not already expanded
2. Not already abandoned; already expanded
3. Already abandoned; not already expanded
4. Already abandoned; already expanded

When we roll back through the tree we calculate the combined value of the options at each node for all four alternatives. This approach to valuing path-dependent options is discussed in more detail in Section 24.4.

When there are several stochastic variables, the value of the base project is usually determined by Monte Carlo simulation. The valuation of the project's embedded options is then more difficult because a Monte Carlo simulation works from the beginning to the end of a project. When we reach a certain point, we do not have information on the present value of the project's future cash flows. However, the techniques mentioned in Section 24.7 for valuing American options using Monte Carlo simulation can sometimes be used.

As an illustration of this point, Schwartz and Moon (2000) explain how their Amazon.com analysis outlined in Busness Snapshot 31.1 could be extended to take account of the option to abandon (i.e. the option to declare bankruptcy) when the value of future cash flows is negative.[3] At each time step, a polynomial relationship between the value of not abandoning and variables such as the current revenue, revenue growth rate, volatilities, cash balances, and loss carry forwards is assumed. Each simulation trial

[2] As it happens, the two options do not interact in Figures 31.4 and 31.5. However, the interactions between the options would become an issue if a larger tree with smaller time steps were built.

[3] The analysis in Section 31.4 assumed that bankruptcy occurs when the cash balance falls below zero, but this is not necessarily optimal for Amazon.com.

provides an observation for obtaining a least-squares estimate of the relationship at each time. This is the Longstaff and Schwartz approach of Section 24.7.[4]

SUMMARY

In this chapter, we have investigated how the ideas developed earlier in the book can be applied to the valuation of real assets and options on real assets. We have shown how the risk-neutral valuation principle can be used to value an asset dependent on any set of variables. The expected growth rate of each variable is adjusted to reflect its market price of risk. The value of the asset is then the present value of its expected cash flows discounted at the risk-free rate.

Risk-neutral valuation provides an internally consistent approach to capital investment appraisal. It also makes it possible to value the options that are embedded in many of the projects that are encountered in practice. We have illustrated the approach by applying it to the valuation of Amazon.com at the end of 1999 and the valuation of an oil project.

FURTHER READING

Amran, M., and N. Kulatilaka, *Real Options*, Boston, MA: Harvard Business School Press, 1999.

Copeland, T., T. Koller, and J. Murrin, *Valuation: Measuring and Managing the Value of Companies*, 3rd edn. New York: Wiley, 2000.

Copeland, T., and V. Antikarov, *Real Options: A Practitioners Guide*, New York: Texere, 2001.

Schwartz, E. S., and M. Moon, "Rational Pricing of Internet Companies," *Financial Analysts Journal*, May/June (2000): 62–75.

Trigeorgis, L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation*, Cambridge, MA: MIT Press, 1996.

Questions and Problems (Answers in Solutions Manual)

31.1. Explain the difference between the net present value approach and the risk-neutral valuation approach for valuing a new capital investment opportunity. What are the advantages of the risk-neutral valuation approach for valuing real options?

31.2. The market price of risk for copper is 0.5, the volatility of copper prices is 20% per annum, the spot price is 80 cents per pound, and the 6-month futures price is 75 cents per pound. What is the expected percentage growth rate in copper prices over the next 6 months?

31.3. Consider a commodity with constant volatility σ and an expected growth rate that is a function solely of time. Show that, in the traditional risk-neutral world,

$$\ln S_T \sim \phi[(\ln F(T) - \tfrac{1}{2}\sigma^2 T,\ \sigma\sqrt{T}]$$

[4] F. A. Longstaff and E. S. Schwartz, "Valuing American Options by Simulation: A Simple Least-Squares Approach," *Review of Financial Studies*, 14, 1 (Spring 2001): 113–47.

where S_T is the value of the commodity at time T and $F(t)$ is the futures price at time 0 for a contract maturing at time t.

31.4. Derive a relationship between the convenience yield of a commodity and its market price of risk.

31.5. The correlation between a company's gross revenue and the market index is 0.2. The excess return of the market over the risk-free rate is 6% and the volatility of the market index is 18%. What is the market price of risk for the company's revenue?

31.6. A company can buy an option for the delivery of 1 million barrels of oil in 3 years at \$25 per barrel. The 3-year futures price of oil is \$24 per barrel. The risk-free interest rate is 5% per annum with continuous compounding and the volatility of the futures price is 20% per annum. How much is the option worth?

31.7. A driver entering into a car lease agreement can obtain the right to buy the car in 4 years for \$10,000. The current value of the car is \$30,000. The value of the car, S, is expected to follow the process

$$dS = \mu S\,dt + \sigma S\,dz$$

where $\mu = -0.25$, $\sigma = 0.15$, and dz is a Wiener process. The market price of risk for the car price is estimated to be -0.1. What is the value of the option? Assume that the risk-free rate for all maturities is 6%.

Assignment Questions

31.8. Suppose that the spot price, 6-month futures price, and 12-month futures price for wheat are 250, 260, and 270 cents per bushel, respectively. Suppose that the price of wheat follows the process in equation (31.4) with $a = 0.05$ and $\sigma = 0.15$. Construct a two-time-step tree for the price of wheat in a risk-neutral world.

A farmer has a project that involves an expenditure of \$10,000 and a further expenditure of \$90,000 in 6 months. It will increase wheat that is harvested and sold by 40,000 bushels in 1 year. What is the value of the project? Suppose that the farmer can abandon the project in 6 months and avoid paying the \$90,000 cost at that time. What is the value of the abandonment option? Assume a risk-free rate of 5% with continuous compounding.

31.9. In the oil example considered in Section 31.6:

(a) What is the value of the abandonment option if it costs \$3 million rather than zero?

(b) What is the value of the expansion option if it costs \$5 million rather than \$2 million?