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REMOVAL OF SINGULARITIES IN TRESCA AND MOHR–COULOMB YIELD FUNCTIONS

S. W. SLOAN

Department of Civil Engineering and Surveying, University of Newcastle, N.S.W., Australia

J. R. BOOKER

Department of Civil and Mining Engineering, University of Sydney, N.S.W., Australia

SUMMARY

The Tresca and Mohr–Coulomb yield functions are used widely in metal and soil plasticity computations. Both of these criteria, however, possess angular vertices at which the gradient with respect to the stresses, and hence the elastoplastic constitutive law, is undefined. This paper describes a modified yield function which may be used to 'round-off' these vertices. When used in conjunction with the parent yield function, the modified yield function results in a yield surface which is continuous and differentiable for all values of the stresses. The modified yield function is used in the vicinity of the vertices and is given in a form suitable for finite element programming.

INTRODUCTION

As noted by Nayak and Zienkiewicz,[1] it is convenient to represent any isotropic yield function by the three stress invariants σ_m, $\bar{\sigma}$ and θ. In terms of the stress tensor, σ_{ij}, these invariants are given by

$$\sigma_m = \frac{1}{3} \sigma_{ii}$$

$$\bar{\sigma} = \sqrt{J_2}$$

$$\theta = \frac{1}{3} \sin^{-1} (-3\sqrt{(3)}J_3/2\bar{\sigma}^3) \quad (-30° \leq \theta \leq 30°)$$

where

$$J_2 = \frac{1}{2} s_{ij}\, s_{ij}$$

$$J_3 = \frac{1}{3} s_{ij}\, s_{jk}\, s_{ki}$$

and

$$s_{ij} = \sigma_{ij} - \delta_{ij}\, \sigma_m$$

Following the usual convention, repeated indices imply summation and δ_{ij} denotes the Kronecker delta. Assuming that tensile stresses are defined to be positive the three principal stresses, in descending order of magnitude, are given by the equations

$$\sigma_1 = \frac{2}{\sqrt{3}} \bar{\sigma} \sin(\theta + 120°) + \sigma_m$$

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$$\sigma_2 = \frac{2}{\sqrt{3}}\,\bar{\sigma}\sin(\theta) + \sigma_m$$

$$\sigma_3 = \frac{2}{\sqrt{3}}\,\bar{\sigma}\sin(\theta - 120^\circ) + \sigma_m$$

In terms of the above invariants, the Tresca and Mohr–Coulomb yield functions may be expressed, respectively, as

$$F = \bar{\sigma}\cos\theta - c = 0$$

and

$$F = \sigma_m\sin\phi + \bar{\sigma}\left(\cos\theta - \frac{1}{\sqrt{3}}\sin\phi\sin\theta\right) - c\cos\phi = 0$$

where c denotes cohesion and ϕ denotes friction angle.

In the π-plane of principal stress space, the Mohr–Coulomb and Tresca yield surfaces appear as shown in Figure 1 (both surfaces are plotted for the same cohesion with a friction angle of 30° for the Mohr–Columb surface). If the principal stresses are ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then only one sector of this diagram needs to be considered, as illustrated in Figure 2. The singularities in the yield surfaces, where the gradient with respect to the stresses is undefined, occur at $\theta = \pm 30^\circ$. A satisfactory method for dealing with these singularities is needed as they are often encountered in finite element computations, particularly under conditions of axial symmetry. One technique for dealing with this problem has been discussed by Owen and Hinton.[2] In the vicinity of the vertices, their procedure uses the Von Mises criterion to round-off the Tresca criterion and the Drucker–Prager criterion to round-off the Mohr–Coulomb criterion. Although this approach permits the gradient to be computed for all values of θ, a jump in the gradient still occurs at the point where the changeover is made from one yield surface to the other. This may cause the computed stresses to be inaccurate if yielding takes place in the vicinity of a vertex.

This paper describes a new yield function which is suitable for rounding-off the vertices in Tresca and Mohr–Coulomb yield criteria. Due to the fitting procedure adopted, the resulting yield surface is continuous and differentiable with respect to the stress for all values of θ. Moreover, it is simple to implement and the parent functions may be approximated as closely as desired by adjusting one parameter.

THEORY

Since the Tresca yield function may be recovered from the Mohr–Coulomb yield function as a special case (with $\phi = 0^\circ$), it is necessary to concentrate only on the latter. If we define

$$\hat{\sigma}_m = \sigma_m - c\cot\phi$$

and the angles

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\sin\phi\right) \\ \beta &= \theta + \alpha \end{aligned} \tag{1}$$

then the Mohr Coulomb criterion may be expressed in the form

$$\hat{\sigma}_m\sin\phi + \bar{\sigma}\,\frac{\cos\beta}{\cos\alpha} = 0$$

This equation gives $\bar{\sigma}$ and $\partial\bar{\sigma}/\partial\theta$ as

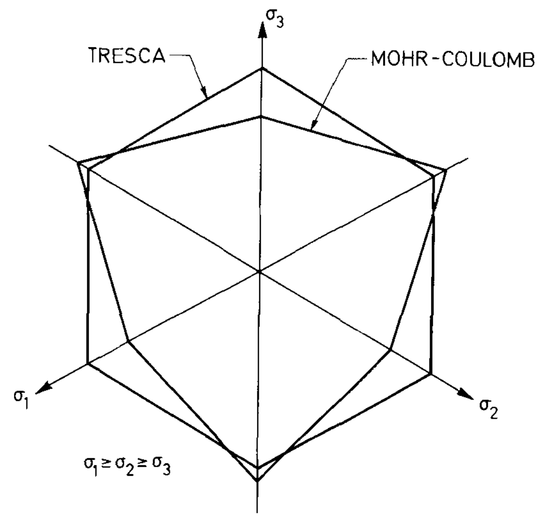

Figure 1. Mohr–Coulomb and Tresca yield surfaces in π-plane

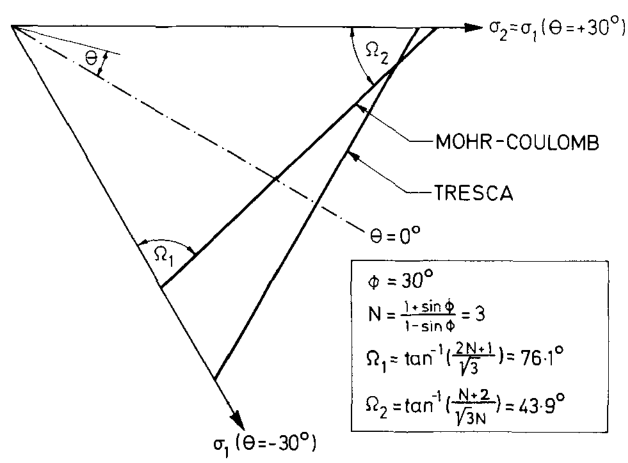

Figure 2. Mohr–Coulomb and Tresca yield surfaces in sector of π-plane ($\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$)

$$\bar{\sigma} = -\,\hat{\sigma}_m \sin\phi \,\frac{\cos\alpha}{\cos\beta} \tag{2}$$

$$\frac{\partial\bar{\sigma}}{\partial\theta} = \bar{\sigma}\tan\beta \tag{3}$$

To avoid the singularities at the vertices of the Mohr–Coulomb surface, we assume a different type of yield surface whenever θ approaches $\pm$ 30°. In practice, the modified yield criterion is used whenever

$$|\theta| \geq \theta_T$$

where θ_T is specified and represents the absolute value of the angle at which the transition occurs. To ensure that the Mohr–Coulomb yield surface is modelled with acceptable accuracy, θ_T should typically be in the range 25° to 29°. For a given $\hat{\sigma}_m$, the overall yield function is continuous and differentiable with respect to the stresses if the following conditions hold:

1. At a transition point, the $\bar{\sigma}$ invariant for the modified surface is identical to the $\bar{\sigma}$ invariant for the Mohr–Coulomb surface.

2. At a transition point, $\partial\bar{\sigma}/\partial\theta$ for the modified surface is identical to $\partial\bar{\sigma}/\partial\theta$ for the Mohr–Coulomb surface.

In addition to these conditions, the modified surface must be convex and defined so that $\partial\bar{\sigma}/\partial\theta = 0$ when $\theta = \pm 30°$. One function which is capable of satisfying these requirements is of the form

$$F = \sigma_m \sin\phi + \bar{\sigma}\,(A - B\sin3\theta) - c\cos\phi = 0 \tag{4}$$

where A and B are functions of ϕ and θ. Rearranging this equation gives

$$\bar{\sigma} = -\hat{\sigma}_m \left(\frac{\sin\phi}{A - B\sin3\theta}\right) \tag{5}$$

$$\frac{\partial\bar{\sigma}}{\partial\theta} = \bar{\sigma}\left(\frac{3B\cos3\theta}{A - B\sin3\theta}\right) \tag{6}$$

In order to satisfy the first of the continuity conditions we use equations (2) and (5). At a transition point, the $\bar{\sigma}$ invariant for the modified surface is given by

$$\bar{\sigma} = -\hat{\sigma}_m \left(\frac{\sin\phi}{A - B\sin3\theta_t}\right) \tag{7}$$

where

$$\theta_t = \theta_T \sin(\theta)$$

and

$$\operatorname{sign}(\theta) = \begin{cases} +1 \text{ for } \theta > 0° \\ -1 \text{ for } \theta < 0° \end{cases}$$

From equation (2), the $\bar{\sigma}$ invariant for the Mohr–Coulomb surface at a transition point is

$$\bar{\sigma} = -\hat{\sigma}_m \sin\phi \frac{\cos\alpha}{\cos\beta_t} \tag{8}$$

where β_t is defined by equation (1) with $\theta = \theta_t$. Equation (7) and (8) and rearranging gives

$$A - B\sin3\theta_t = \frac{\cos\beta_t}{\cos\alpha} \tag{9}$$

In order to satisfy the second of the continuity conditions, we use equations (3) and (6). Matching $\partial\bar{\sigma}/\partial\theta$ for the two surfaces at $\theta = \theta_t$ leads to

$$A\tan\beta_t - B(3\cos3\theta_t + \tan\beta_t \sin3\theta_t) = 0 \tag{10}$$

Equations (9) and (10) define two equations in two unknowns and may be solved to give

$$A = \frac{1}{\cos\alpha}\left(\cos\beta_t + \frac{1}{3}\sin\beta_t \tan3\theta_t\right) \tag{11}$$

$$B = \frac{\sin\beta_t}{3\cos\alpha\cos3\theta_t} \tag{12}$$

Equations (11) and (12), together with equation (4), define the modified yield function. When this is used in conjunction with the Mohr–Coulomb function, the resultant yield function is differentiable for all values of θ between $-30°$ and $+30°$. Figure 3 illustrates the effect of rounding-off the vertices in the Mohr–Coulomb yield criterion using equation (4). This π-plane plot is for $\phi = 30°$ and $\theta_T =$

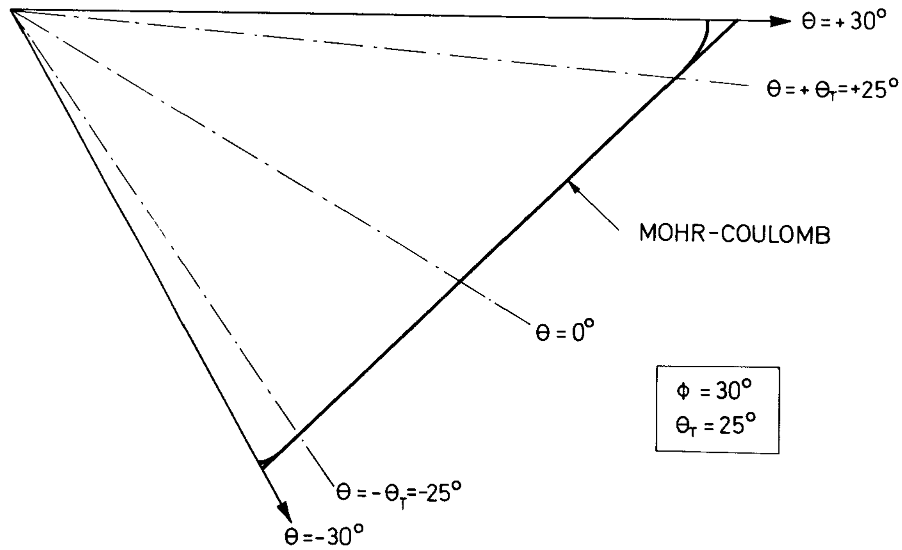

Figure 3. Mohr–Coulomb yield surface in π-plane with rounded vertices

25° (i.e. the modified surface is used whenever $|\theta| \geq 25°$). At $\theta = +30°$, the modified surface underestimates the $\bar{\sigma}$ invariant predicted by the Mohr–Coulomb criterion by approximately 4·3 per cent. At $\theta = -30°$, this difference is approximately 1·1 per cent.

When implementing the modified Mohr–Coulomb surface, the constants A and B may be computed efficiently once the transition angle θ_T is specified. Noting the trigonometric identities

$$\cos\beta_t = \cos(\theta_t + \alpha) = \cos\theta_t\cos\alpha - \sin\theta_t\sin\alpha$$

$$\sin\beta_t = \sin(\theta_t + \alpha) = \sin\theta_t\cos\alpha + \sin\alpha\cos\theta_t$$

it may be shown that

$$A = \frac{1}{3}\cos\theta_T\left(3 + \tan\theta_T\tan3\theta_T + \frac{1}{\sqrt{3}}\,\mathrm{sign}(\theta)(\tan3\theta_T - 3\tan\theta_T)\sin\phi\right)$$

$$B = \frac{1}{3}\,\frac{1}{\cos3\theta_T}\left(\mathrm{sign}(\theta)\sin\theta_T + \frac{1}{\sqrt{3}}\,\sin\phi\cos\theta_T\right)$$

Choosing, for example, $\theta_T = 25°$ this leads to

$$A = 1{\cdot}432052 + 0{\cdot}406942\ \mathrm{sign}(\theta)\sin\phi$$

$$B = 0{\cdot}544291\ \mathrm{sign}(\theta) + 0{\cdot}673903\ \sin\phi$$

where A and B have been computed to an accuracy of six decimal places.

The modified Tresca function is easily obtained from the modified Mohr–Coulomb function as a special case. Substituting $\phi = 0°$ and $\alpha = 0°$ into equations (4), (11) and (12) leads to

$$F = \bar{\sigma}\,(A - B\sin3\theta) - c = 0 \tag{13}$$

where

$$A = \cos\theta_t + \frac{1}{3}\sin3\theta_t\tan3\theta_t \tag{14}$$

$$B = \frac{\sin\theta_t}{3\cos3\theta_t} \tag{15}$$

Figure 4 illustrates the effect of rounding-off the vertices in the Tresca yield surface using equation (13). As for the Mohr–Coulomb example, this plot is in the π-plane and the modified surface is employed whenever $|\theta| \geq 25°$. At $\theta = \pm 30°$, the modified surface underestimates the $\bar{\sigma}$ invariant predicted by the Tresca criterion by approximately 2·5 per cent. In practice, θ_T should be chosen to lie between 25° and 29°. This ensures that the Tresca and Mohr–Coulomb yield criteria are modelled with acceptable accuracy. Note that θ_T should not be made too close to 30° in order to avoid ill-conditioning of the coefficients A and B.

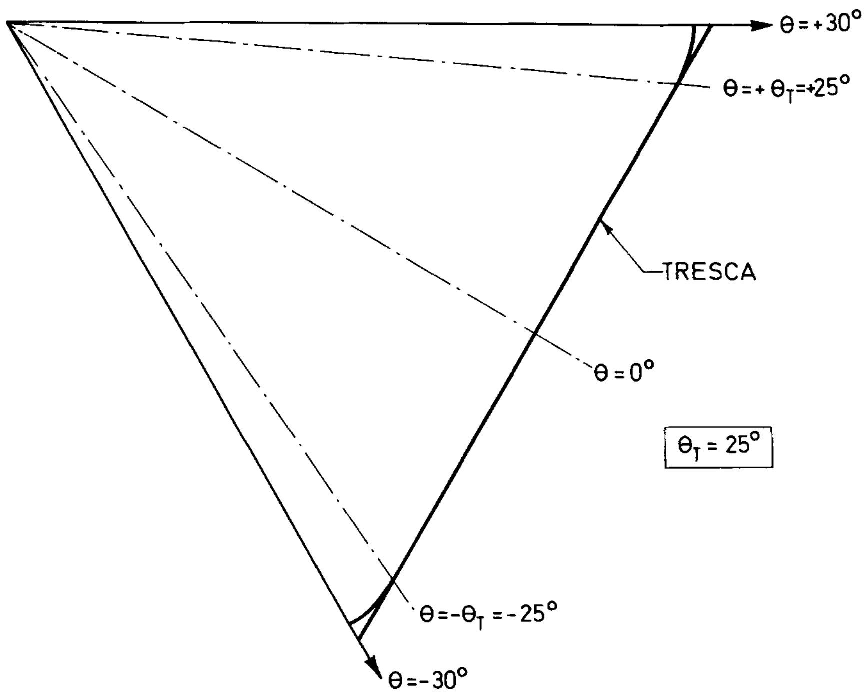

Figure 4. Tresca yield surface in π-plane with rounded vertices

EVALUATION OF GRADIENT TO YIELD SURFACE

In order to evaluate the elastoplastic stress–strain matrix in finite element analysis, it is necessary to compute the gradient of the yield function with respect to the stresses. As noted by Nayak and Zienkiewicz,[1] this operation may be implemented conveniently for various types of isotropic yield criteria by evaluating

$$\frac{\partial F}{\partial \boldsymbol{\sigma}} = C_1 \frac{\partial \sigma_m}{\partial \boldsymbol{\sigma}} + C_2 \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} + C_3 \frac{\partial J_3}{\partial \boldsymbol{\sigma}}$$

where

$$C_1 = \frac{\partial F}{\partial \sigma_m}$$

$$C_2 = \frac{\partial F}{\partial \bar{\sigma}} - \tan 3\theta \, \bar{\sigma}^{-1} \frac{\partial F}{\partial \theta}$$

$$C_3 = -\frac{\sqrt{3}}{2} \frac{1}{\cos 3\theta} \bar{\sigma}^{-3} \frac{\partial F}{\partial \theta}$$

and $\boldsymbol{\sigma}$ denotes the vector of stress components. This representation is convenient for finite element programming since different yield criteria may be incorporated merely by inserting the appropriate constants C_1, C_2 and C_3.

The appropriate constants for the modified Mohr–Coulomb criterion are given by

$$C_1 = \sin\phi$$
$$C_2 = A + 2B \sin 3\theta$$
$$C_3 = \frac{3}{2} \sqrt{3}\, \bar{\sigma}^{-2} B$$

where A and B are defined by equations (11) and (12). Similarly for the modified Tresca criterion

$$C_1 = 0$$
$$C_2 = A + 2B \sin 3\theta$$
$$C_3 = \frac{3}{2} \sqrt{3}\, \bar{\sigma}^{-2} B$$

where A and B are defined by equations (14) and (15).

CONCLUSIONS

A modified yield function, which may be used to 'round-off' the vertices in the Tresca and Mohr–Coulomb yield functions, is derived. The modified yield function is used in the vicinity of the vertices and results in a yield surface which is continuous and differentiable for all values of the stresses. The Tresca and Mohr–Coulomb yield functions may be approximated as closely as desired by adjusting one parameter.

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