Binding `MFront` with `FEniCS` for automated formulation and resolution of non-linear behaviours

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Outline

Nonlinear problems - Hyperelasticity

elastic potential + automatic differentiation with **UFL** operators

```
# Create mesh and define function space
mesh = UnitCubeMesh(24, 16, 16)
V = VectorFunctionSpace(mesh, "Lagrange", 1)

# Define functions
u  = Function(V)                # Displacement from previous iteration
v  = TestFunction(V)            # Test function
du = TrialFunction(V)           # Incremental displacement
dim = len(u)

F = Identity(dim) + grad(u)     # Deformation gradient
C = F.T*F
Ic, J = tr(C), det(F)           # Invariants of deformation tensors

# Elasticity parameters
E, nu = 10.0, 0.3
mu, lmbda = Constant(E/(2*(1 + nu))), Constant(E*nu/((1 + nu)*(1 - 2*nu)))
# potential (compressible neo-Hookean model)
psi = (mu/2)*(Ic - 3) - mu*ln(J) + (lmbda/2)*(ln(J))**2

# Total potential energy
f = Constant((0.0, -1.0, 0.0))
Pi = psi*dx - dot(f, u)*dx
```

Nonlinear problems - Hyperelasticity

```
# Compute first variation of Pi (directional derivative)
R = derivative(Pi, u, v)

# Compute Jacobian of R
J = derivative(R, u, du)

# Solve variational problem
solve(R == 0, u, bc, J=J)
```

we can specify more precisely which solver and parameters to use

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we can specify more precisely which solver and parameters to use

- Newton method
- PETSc SNES solver : line search, trust region
- PETSC TAO solver : bound-constrained minimization

we can choose which linear solver and preconditioner to use for the iterative process

also possible to **formulate yourself** a Newton method at the PDE level

Outline

Generic behaviours

Mechanics:

$$\Delta\varepsilon, \boldsymbol{\sigma}_n, \boldsymbol{Y}_n \to \boxed{\text{MFront}} \to \boldsymbol{\sigma}_{n+1}, \boldsymbol{Y}_{n+1}, \frac{\partial \boldsymbol{\sigma}}{\partial \Delta\varepsilon}$$

Generalized behaviours:

$$(\Delta\boldsymbol{g}^1, \dots \Delta\boldsymbol{g}^p)_n, (\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^p)_n, \boldsymbol{Y}_n \to \boxed{\text{MFront}} \to (\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^p)_{n+1}, \boldsymbol{Y}_{n+1}, \frac{\partial \boldsymbol{\sigma}^j}{\partial \Delta\boldsymbol{g}^k}$$

$\boldsymbol{g}^j$ are **gradients** (temp. gradient, strain, etc.) depending on the FE unknowns $\boldsymbol{u}$
$\boldsymbol{\sigma}^j$ are associated **fluxes** or **thermodynamic forces** (heat flux, stress, etc.)
$\dfrac{\partial \boldsymbol{\sigma}^j}{\partial \Delta\boldsymbol{g}^k}$ are **tangent blocks**
Work of internal forces density:

$$\delta w_{\text{int}} = \sum_{i=1}^{p} \boldsymbol{\sigma}^i \cdot \delta\boldsymbol{g}^i(\boldsymbol{v})$$

mgis.fenics Python package - Generalities

relies on the MGIS Python interface for loading a MFront library, calling the behaviour integration and giving access to fluxes, state variables and tangent operators

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General idea

Inside a custom `NonlinearProblem`, define $\boldsymbol{\sigma}^i$ on a `Quadrature` space and the generalized residual:

$$F(\boldsymbol{v}) = \sum_{i=1}^{p} \int_\Omega \boldsymbol{\sigma}^i(\boldsymbol{u}) \cdot \delta \boldsymbol{g}^i(\boldsymbol{v}) \,\mathrm{dx} - L(\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in V$$

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MGIS gives metadata to know on which blocks $\mathcal{B}(i)$ of gradients each flux $\boldsymbol{\sigma}_i$ depends:

$$a_{\text{tangent}}(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{p} \sum_{j \in \mathcal{B}(i)} \int_\Omega \delta \boldsymbol{g}^i(\boldsymbol{v}) \cdot \mathbb{T}_{\boldsymbol{g}^j}^{\boldsymbol{\sigma}^i} \cdot \delta \boldsymbol{g}^j(\boldsymbol{u}) \,\mathrm{dx}$$

with each tangent block $\mathbb{T}_{\boldsymbol{g}^j}^{\boldsymbol{\sigma}^i} = \dfrac{\partial \boldsymbol{\sigma}^i}{\partial \boldsymbol{g}^j}$ represented on a tensorial `Quadrature` space

mgis.fenics Python package - Registration concept

The **only** metadata not provided by **MGIS** is how the gradients (e.g. strain) are expressed as functions of the unknown fields $\boldsymbol{u}$ (e.g. displacement)

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The user is required to provide this link with UFL expressions (**registration**):

```
mat_prop = {"YoungModulus": E, "PoissonRatio": nu,
            "HardeningSlope": H, "YieldStrength": sig0}
material = mf.MFrontNonlinearMaterial("src/libBehaviour.so",
               "IsotropicLinearHardeningPlasticity",
               hypothesis="plane_strain",
               material_properties=mat_prop)

u = Function(V)
problem = mf.MFrontNonlinearProblem(u, material,
                    quadrature_degree=2, bcs=bc)
problem.register_gradient("Strain", sym(grad(u)))
```

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```

can be done automatically for predefined keywords e.g. `"Strain"`, `"TemperatureGradient"`, etc.

mgis.fenics Python package - Overview

UFL `eps = sym(grad(u))`

registration `{"Strain": eps}`

array of values at integ. points `project(eps, Q)` → **MFront** → `sig_values` `Ct_values`

```
sig[:] = sig_values
Ct[:]  = Ct_values
```

Quadrature functions

```
F_int = inner(sig, eps_)*dx
```

UFL symbolic differentiation

```
a_tang = inner(inner(Ct, deps), eps_)*dx
```

```
assemble(F_int)
assemble(a_tang)
```
→ `solve`

nonlinear solver (Newton, SNES)

mgis.fenics Python package - Limitations

Pros

- Any mechanical behaviour can be solved easily.
- Can modify the residual form to account for time-dependent problems → automatic differentiation is preserved using MGIS metadata.
- Non-linear solvers can be chosen independently.
- Works in parallel.

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- Works in parallel.

Cons

- Constitutive update (call to MFront) is performed outside the assembler ($\Rightarrow$ one extra loop over quadrature points) and all tangent blocks $\mathbb{T}$ at all quadrature points must be stored in memory.
- Multi-material support is limited to spatially varying material properties.
- Memory transfers between FEniCS and MGIS objects have not been optimized.

$\Rightarrow$ to be improved with **FEniCS-X**

Outline

Stationary non-linear heat transfer

Non-linear Fourier law (UO_2): $\boldsymbol{j}(\nabla T, T) = -k(T)\nabla T$ with $k(T) = \dfrac{1}{A + BT}$ (✓)

$$a_{\mathrm{tangent}}(\widehat{T}, T^*) = \int_\Omega \nabla \widehat{T} \cdot \left(\frac{\partial \boldsymbol{j}}{\partial \boldsymbol{g}} \cdot \nabla T^* + \frac{\partial \boldsymbol{j}}{\partial T} \cdot T^* \right) \mathrm{dx}$$

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```
@DSL DefaultGenericBehaviour;
@Behaviour StationaryHeatTransfer;

@Gradient TemperatureGradient ∇T;
∇T.setGlossaryName("TemperatureGradient");

@Flux HeatFlux j;
j.setGlossaryName("HeatFlux");

@Integrator{
  // temperature at the end of the time step
  const auto T_ = T + ΔT;
  // thermal conductivity
  k = 1 / (A + B ⋅ T_);
  // heat flux
  j = -k ⋅ (∇T + Δ∇T);
} // end of @Integrator

@AdditionalTangentOperatorBlock ∂j/∂ΔT;
@TangentOperator {
  ∂j/∂Δ∇T = -k ⋅ tmatrix<N, N, real>::Id();
  ∂j/∂ΔT  =  B ⋅ k ⋅ k ⋅ (∇T + Δ∇T);
} // end of @TangentOperator
```

```
print(["d{}_d{}".format(*t) for t in material.
      get_tangent_block_names()])
```

```
['dHeatFlux_dTemperatureGradient',
 'dHeatFlux_dTemperature']
```

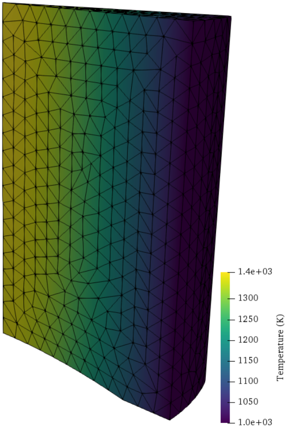
```
V = FunctionSpace(mesh, "CG", 1)
T = Function(V, name="Temperature")

problem = mf.MFrontNonlinearProblem(T, material,
      quadrature_degree=2, bcs=bc)
problem.set_loading(-r*T*dx)

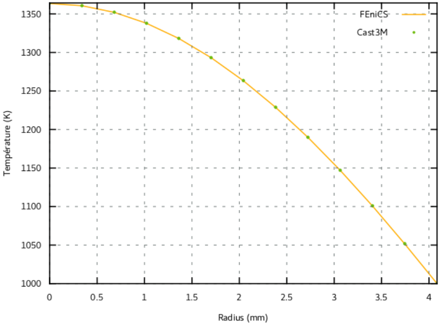
problem.register_gradient("TemperatureGradient",
      grad(T))

problem.solve(T.vector())
```

Stationary non-linear heat transfer



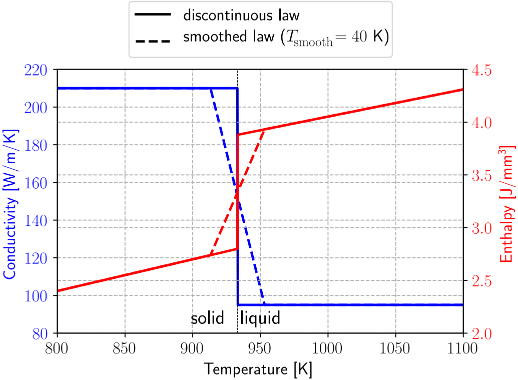
Mesh type	quad_deg	MFront	pure FEniCS
coarse	2	1.2 s	0.8 s
coarse	5	2.2 s	1.0 s
fine	2	62.8 s	58.4 s
fine	5	77.0 s	66.3 s



Transient heat equation with phase change

Transient heat equation with enthalpy formulation:

$$\frac{\partial h}{\partial t} = r - \operatorname{div} \boldsymbol{j}$$
$$\boldsymbol{j}(T, \nabla T) = -k(T)\nabla T$$



$$h(T) = \int_{T_0}^{T} \rho C_p dT = \begin{cases} c_s T & \text{if } T < T_m \\ c_l(T - T_m) + c_s T_m + \Delta h_{s/l} & \text{if } T > T_m \end{cases}$$

Smoothed version $T_{\text{smooth}} = 0.1$ K

Transient heat equation with phase change

Time integration with θ-scheme: $\star_{n+\theta} = \theta \star_{n+1} + (1-\theta)\star_n$

$$\left.\frac{\partial h}{\partial t}\right|_{t=t_{n+\theta}} \approx \frac{h_{n+1} - h_n}{\Delta t} = r_{n+\theta} - \operatorname{div} \boldsymbol{j}_{n+\theta}$$

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$$F(\widehat{T}) = \int_\Omega \left((h_{n+1}(T) - h_n)\,\widehat{T} - \Delta t(\theta \boldsymbol{j}_{n+1}(T, \nabla T) + (1-\theta)\boldsymbol{j}_n) \cdot \nabla \widehat{T} \right) \,\mathrm{dx} = 0$$

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@DSL DefaultGenericBehaviour;
@Behaviour HeatTransferPhaseChange;

@Gradient TemperatureGradient ∇T;
∇T.setGlossaryName("TemperatureGradient");

@Flux HeatFlux j;
j.setGlossaryName("HeatFlux");

@StateVariable real h;
h.setEntryName("Enthalpy"); //per unit of volume

@AdditionalTangentOperatorBlock ∂j/∂ΔT;
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```

```
h = problem.get_state_variable("Enthalpy")
j = problem.get_flux("HeatFlux")
problem.initialize()

j_old = j.copy(deepcopy=True)
h_old = h.copy(deepcopy=True)

j_theta = theta*j + (1-theta)*j_old
problem.residual = (T_*(h - h_old)-dt*dot(
    grad(T_), j_theta))*problem.dx
problem.compute_tangent_form()
```

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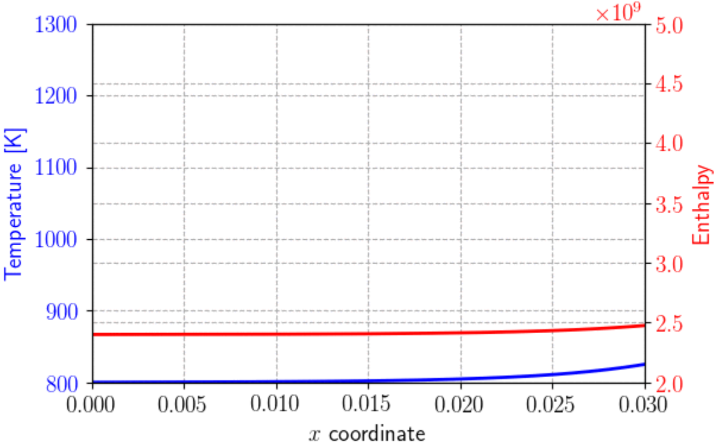

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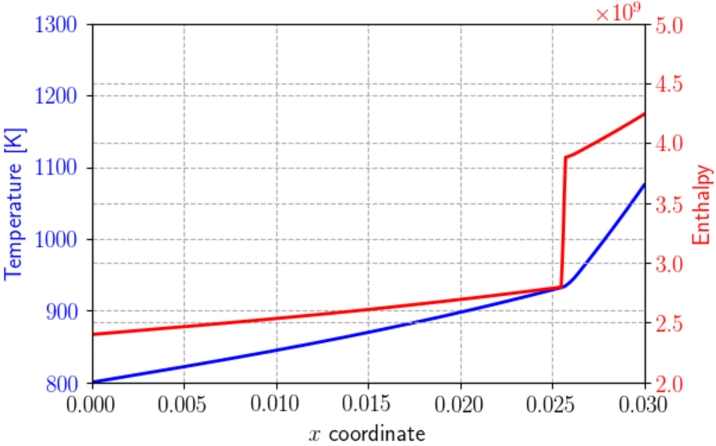

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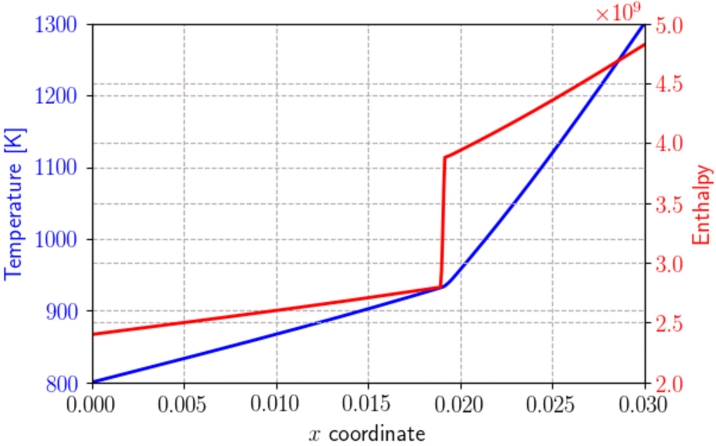

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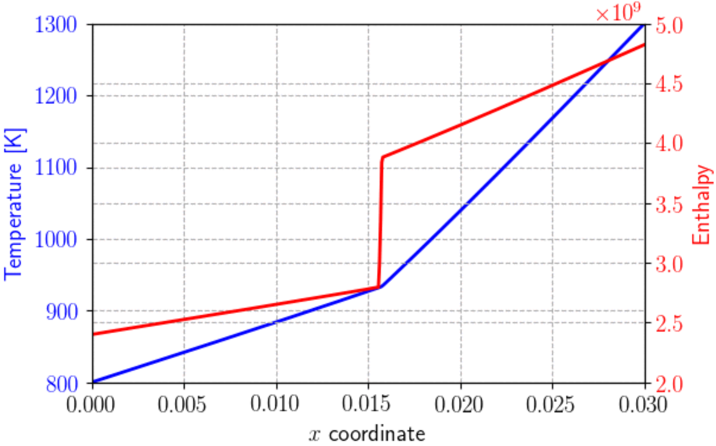

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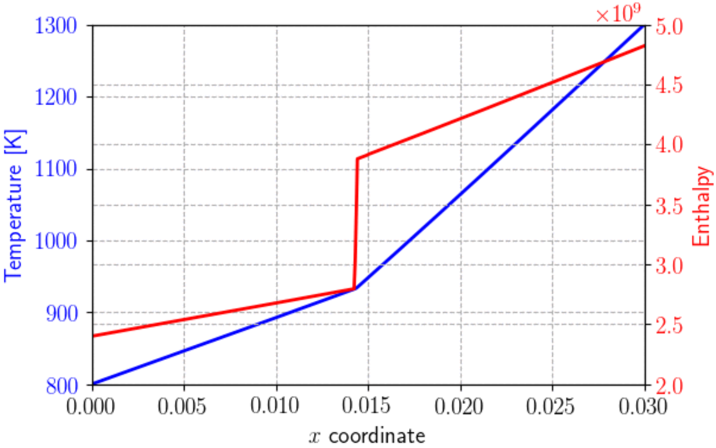

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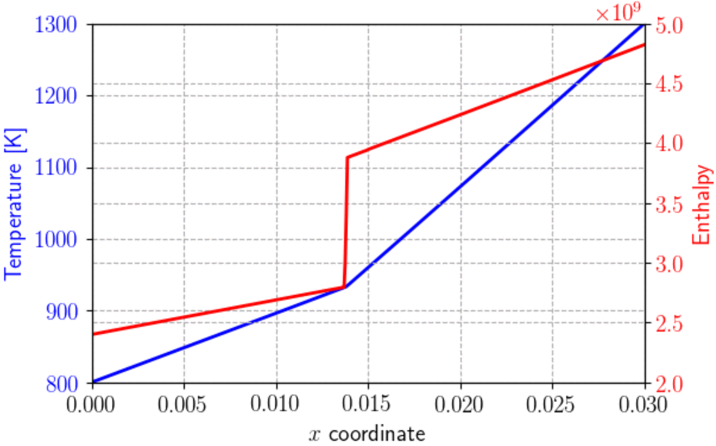

Extension to large strains

Quite simple using for instance $\boldsymbol{F}$ (*gradient*) and PK1 stress $\boldsymbol{P}$ (*flux*):
Residual is:

$$F(\boldsymbol{v}) = \int_\Omega \boldsymbol{P} : \delta\boldsymbol{F}(\boldsymbol{v}) \text{ dx} - W_{ext}(\boldsymbol{v}) = \int_\Omega \boldsymbol{P} : \nabla\boldsymbol{v} \text{ dx} - W_{ext}(\boldsymbol{v})$$

$\boldsymbol{F}$ and $\boldsymbol{P}$ are non-symmetric 2nd-order tensors

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Consistent tangent bilinear form is:

$$a_{\text{tangent}}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}} : \nabla \boldsymbol{v} dx$$

3 possible tangent operators $\dfrac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{F}}, \dfrac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}}, \dfrac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}_{GL}}$, obtained from the behaviour tangent operator using MFront conversion methods (extremely useful!)

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Logarithmic strain plasticity

- Hencky strain measure $\boldsymbol{H} = \dfrac{1}{2}\log(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F})$ ✗
- use a small strain constitutive relation on $\boldsymbol{H} \Rightarrow \boldsymbol{H} = \boldsymbol{H}^e + \boldsymbol{H}^p$

$$\boldsymbol{F} \longrightarrow \boldsymbol{H} \longrightarrow \boxed{\text{small strain law}} \longrightarrow \boldsymbol{T} \longrightarrow \boldsymbol{\sigma}$$

Logarithmic strain plasticity

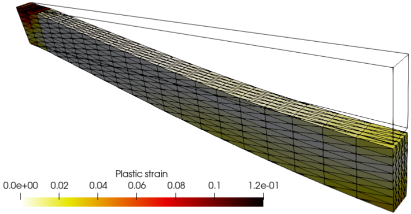
Use of the StandardElastoViscoPlasticity brick

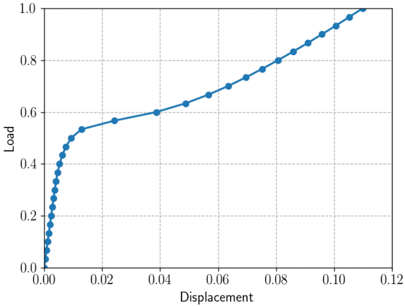
```
@DSL Implicit;

@Behaviour LogarithmicStrainPlasticity;

@StrainMeasure Hencky;

@Brick StandardElastoViscoPlasticity{
  stress_potential : "Hooke" {
        young_modulus : 210e9,
        poisson_ratio : 0.3
        },
  inelastic_flow : "Plastic" {
    criterion : "Mises",
    isotropic_hardening : "Linear" {H : 500e6,
                                    R0 : 250e6}
  }
};
```

```
material = mf.MFrontNonlinearMaterial("./src/
     libBehaviour.so", "
     LogarithmicStrainPlasticity")
problem = mf.MFrontNonlinearProblem(u,
     material, bcs=bc)
problem.set_loading(dot(selfweight, u)*dx)

prm = problem.solver.parameters
prm["absolute_tolerance"] = 1e-6
prm["relative_tolerance"] = 1e-6
prm["linear_solver"] = "mumps"

for (i, t) in enumerate(load_steps[1:]):
    selfweight.t = t
    problem.solve(u.vector())
```

```
Automatic registration of 'DeformationGradient'
as I + (grad(Displacement))
```

Logarithmic strain plasticity





Phase-field approach to brittle fracture

Bourdin/Francfort/Marigo **variational phase-field approach**:

$$\boldsymbol{u}(t), d(t) = \underset{\boldsymbol{u},d}{\arg\min}\, \mathcal{E}_{pot}(\boldsymbol{u}, d) + \mathcal{E}_f(\boldsymbol{u}, d)$$

$$\mathcal{E}_{pot}(\boldsymbol{u}, d) = \int_\Omega (1-d)^2\psi^+(\boldsymbol{\varepsilon}) + \psi^-(\boldsymbol{\varepsilon}) \text{ dx} - W_{ext}(\boldsymbol{u})$$

$$\mathcal{E}_f(d) = \dfrac{G_c}{c_w}\int_\Omega \left(\dfrac{w(d)}{\ell_0} + \ell_0\|\nabla d\|^2\right) \text{ dx}$$

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tension/compression splittings:

$$\psi^+(\boldsymbol{\varepsilon}) = \frac{1}{2}\kappa\langle \mathrm{tr}(\boldsymbol{\varepsilon})\rangle_+^2 + \mu\boldsymbol{\varepsilon}^{dev} : \boldsymbol{\varepsilon}^{dev} \quad \text{(Amor et al.)} \ \checkmark$$

$$\psi^+(\boldsymbol{\varepsilon}) = \frac{1}{2}\lambda\langle \mathrm{tr}(\boldsymbol{\varepsilon})\rangle_+^2 + \mu\sum_I \langle \varepsilon_I\rangle_+^2 \quad \text{(Miehe et al.)} \ \text{✗}$$

Phase-field approach to brittle fracture

Alternate minimization

$$\boldsymbol{u}^k = \arg\min_{u} \mathcal{E}(\boldsymbol{u}, d^{k-1}) \qquad (u\text{-problem})$$

$$d^k = \arg\min_{d \text{ s.t. } d_{n-\mathbf{1}} \leq d} \mathcal{E}(\boldsymbol{u}^k, d) \qquad (d\text{-problem})$$

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(u-problem) is a **non-linear elasticity** problem with

$\boldsymbol{\sigma}(\boldsymbol{u}, d) = (1-d)^2 \dfrac{\partial \psi^+}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{u}) + \dfrac{\partial \psi^-}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{u})$ and d as an external state variable

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(d-problem) is **variational inequality** problem with ψ^+ as an external state variable: solved with `PETScTAOSolver`

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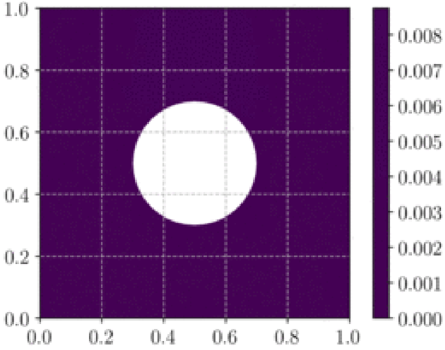
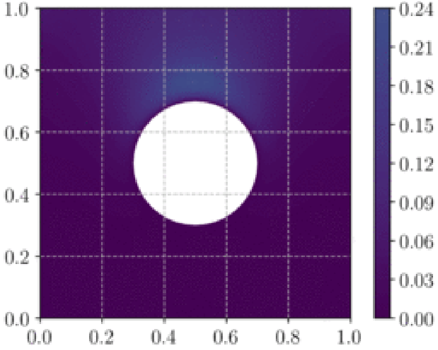
(d-problem) is **variational inequality** problem with ψ^+ as an external state variable: solved with `PETScTAOSolver`

```
problem_u = mf.MFrontNonlinearProblem(u, material_u, bcs=bcu)
problem_u.register_external_state_variable("Damage", d)
psi = problem_u.get_state_variable("PositiveEnergyDensity")

for (i, t) in enumerate(loading[1:]):
    Uimp.t = t
    while res > tol and j < Nitermax:
        problem_u.solve(u.vector()) # Solve displacement u-problem
        problem_d.solve(d.vector()) # Solve damage d-problem
```

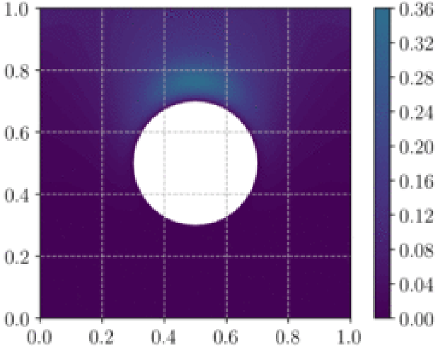
Phase-field approach to brittle fracture



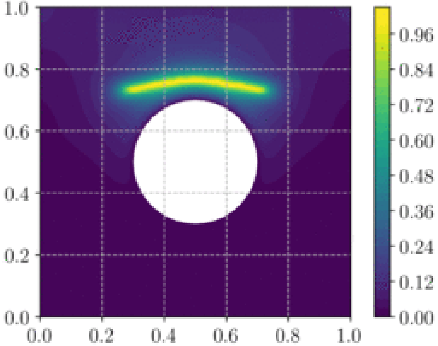
Phase-field approach to brittle fracture



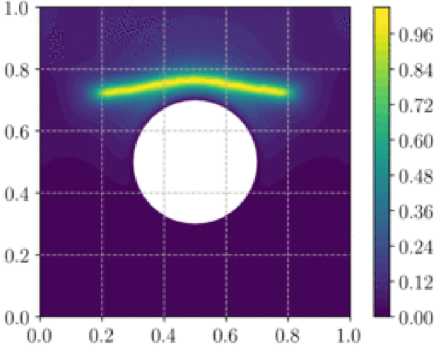
Phase-field approach to brittle fracture



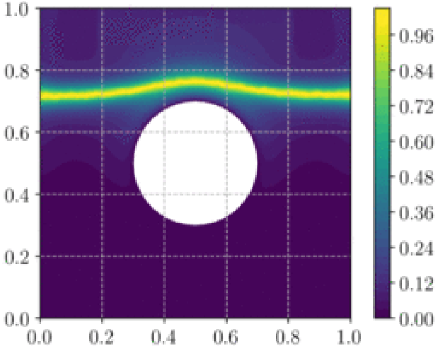
Phase-field approach to brittle fracture



Phase-field approach to brittle fracture



Phase-field approach to brittle fracture



Monolithic thermoelastoplasticity - MFront

Fully coupled thermomechanics ($\varepsilon^{\text{el}} = \varepsilon^{\text{to}} - \varepsilon^{\text{p}} - \varepsilon^{\text{th}}$):

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}(\varepsilon^{\text{el}})\boldsymbol{I} + 2\mu\varepsilon^{\text{el}}$$

$$s = \frac{C_\varepsilon}{T^{\text{ref}}}(T - T^{\text{ref}}) + \frac{\kappa}{\rho}\,\mathrm{tr}(\varepsilon^{\text{el}})$$

$$\boldsymbol{j} = -k\nabla T$$

```
@Gradient StrainStensor εᵗᵒ;
εᵗᵒ.setGlossaryName("Strain");

@Flux StressStensor σ;
σ.setGlossaryName("Stress");

@Gradient TemperatureGradient ∇T;
∇T.setGlossaryName("TemperatureGradient");

@Flux HeatFlux j;
j.setGlossaryName("HeatFlux");

@StateVariable StrainStensor eel;
eel.setGlossaryName("ElasticStrain");
@StateVariable strain p;
p.setGlossaryName("EquivalentPlasticStrain");
@StateVariable real s;
s.setEntryName("EntropyPerUnitOfMass");

@TangentOperatorBlocks{∂σ/∂Δεᵗᵒ, ∂σ/∂ΔT, ∂s/∂ΔT, ∂s/∂Δεᵗᵒ, ∂j/∂Δ∇T};
```

```
@Integrator{
  const auto λ = computeLambda(E, ν);
  const auto μ = computeMu(E, ν);
  const auto κ = α ⋅ (2 ⋅ μ + 3 ⋅ λ);
  const auto ε = εᵗᵒ + Δεᵗᵒ;
  const auto εth = α ⋅ (T + ΔT - Tref) ⋅ I₂;

  eel += Δεᵗᵒ - εth;
  const auto se     = 2*μ*deviator(eel);
  const auto seq_e  = sigmaeq(se);
  const auto b      = seq_e-s0-H*p>stress{0};
  if(b){
```

[...]

```
  σ = λ ⋅ trace(eel) ⋅ I₂ + 2 ⋅ μ ⋅ eel ;
  s = Cₑ / Tref ⋅ (T + ΔT - Tref) + (κ / ρ) ⋅ trace(ε);
  j = -k ⋅ (∇T + Δ∇T);
  if (computeTangentOperator_) {
    ∂σ/∂ΔT = -κ ⋅ I₂;
    ∂s/∂ΔT = Cₑ / Tref;
    ∂s/∂Δεᵗᵒ = κ / ρ ⋅ I₂;
    ∂j/∂Δ∇T = -k ⋅ tmatrix<N, N, real>::Id();
  }
}
```

Monolithic thermoelastoplasticity - FEniCS

Quasi-static equilibrium + heat equation:

$$\operatorname{div} \boldsymbol{\sigma} = 0$$

$$\rho T^{\text{ref}} \frac{\partial s}{\partial t} + \operatorname{div} \boldsymbol{j} = 0$$

Monolithic thermoelastoplasticity - FEniCS

Variational formulation and implicit backward Euler:

$$\int_\Omega \boldsymbol{\sigma}_{n+1} : \nabla^s \widehat{\boldsymbol{u}} \,\mathrm{dx} = \int_{\partial\Omega} \boldsymbol{T} \cdot \widehat{\boldsymbol{u}} \,\mathrm{dS} \quad \forall \widehat{\boldsymbol{u}}$$

$$\int_\Omega \left(\rho T^{\mathrm{ref}} \frac{s_{n+1} - s_n}{\Delta t} - \boldsymbol{j}_{n+1}\right) \widehat{T} \,\mathrm{dx} = -\int_{\partial\Omega} q \widehat{T} \,\mathrm{dS} \quad \forall \widehat{T}$$

Monolithic thermoelastoplasticity - FEniCS

Variational formulation and implicit backward Euler:

$$\int_\Omega \boldsymbol{\sigma}_{n+1} : \nabla^s \widehat{\boldsymbol{u}} \,\mathrm{dx} = \int_{\partial\Omega} \boldsymbol{T} \cdot \widehat{\boldsymbol{u}} \,\mathrm{dS} \quad \forall \widehat{\boldsymbol{u}}$$

$$\int_\Omega (\rho T^{\text{ref}} \frac{s_{n+1} - s_n}{\Delta t} - \boldsymbol{j}_{n+1}) \widehat{T} \,\mathrm{dx} = -\int_{\partial\Omega} q \widehat{T} \,\mathrm{dS} \quad \forall \widehat{T}$$

```
Vue = VectorElement("CG", mesh.ufl_cell(), 2)  # displacement finite element
Vte = FiniteElement("CG", mesh.ufl_cell(), 1)  # temperature finite element
V = FunctionSpace(mesh, MixedElement([Vue, Vte]))

v = Function(V)
(u, Theta) = split(v)
problem = mf.MFrontNonlinearProblem(v, material, quadrature_degree=2, bcs=bcs)
problem.register_gradient("Strain", sym(grad(u)))
problem.register_gradient("TemperatureGradient", grad(Theta))
problem.register_external_state_variable("Temperature", Theta + Tref)
```

Monolithic thermoelastoplasticity - FEniCS

Variational formulation and implicit backward Euler:

$$\int_\Omega \boldsymbol{\sigma}_{n+1} : \nabla^s \widehat{\boldsymbol{u}} \, \mathrm{dx} = \int_{\partial\Omega} \boldsymbol{T} \cdot \widehat{\boldsymbol{u}} \, \mathrm{dS} \quad \forall \widehat{\boldsymbol{u}}$$

$$\int_\Omega \left(\rho T^{\text{ref}} \frac{s_{n+1} - s_n}{\Delta t} - \boldsymbol{j}_{n+1}\right) \widehat{T} \, \mathrm{dx} = -\int_{\partial\Omega} q \widehat{T} \, \mathrm{dS} \quad \forall \widehat{T}$$

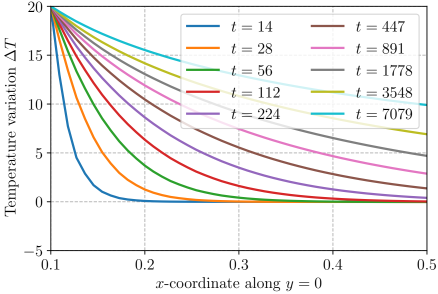
```
sig = problem.get_flux("Stress")
j = problem.get_flux("HeatFlux")
s = problem.get_state_variable("EntropyPerUnitOfMass")
problem.initialize()

s_old = s.copy(deepcopy=True)
v_ = TestFunction(V)
u_, T_ = split(v_)  # Displacement and temperature test functions
eps_ = problem.gradients["Strain"].variation(v_)
mech_residual = dot(sig, eps_)*problem.dx
thermal_residual = (rho*Tref*(s - s_old)/dt*T_ - dot(j, grad(T_)))*problem.dx
problem.residual = mech_residual + thermal_residual
problem.compute_tangent_form()
```

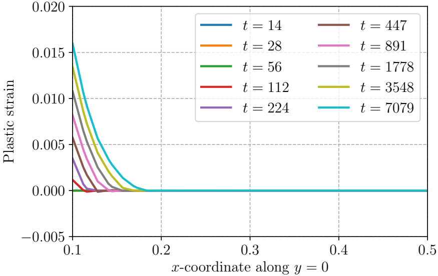
Monolithic thermoelastoplasticity - FEniCS

Variational formulation and implicit backward Euler:

$$\int_\Omega \boldsymbol{\sigma}_{n+1} : \nabla^s \widehat{\boldsymbol{u}} \, \mathrm{dx} = \int_{\partial\Omega} \boldsymbol{T} \cdot \widehat{\boldsymbol{u}} \, \mathrm{dS} \quad \forall \widehat{\boldsymbol{u}}$$

$$\int_\Omega (\rho T^{\mathrm{ref}} \frac{s_{n+1} - s_n}{\Delta t} - \boldsymbol{j}_{n+1}) \widehat{T} \, \mathrm{dx} = - \int_{\partial\Omega} q \widehat{T} \, \mathrm{dS} \quad \forall \widehat{T}$$





To conclude

Conclusions

- working `MGIS/FEniCS` binding although not fully optimized
- behaviour is **separated** from FEniCS implementation (function spaces, strain measures, equilibrium/evolution equations)
- wide range of (complex) applications with **simple** implementation

What's next ?

- tests!
- integration in **dolfin-x**
- multi-materials, plates/shells
- other examples:
  - Cosserat elastoplasticity (Raffaele Russo, Tamara Dancheva)
  - micromorphic crystal plasticity (Julien Sanahuja)
  - periodic polycrystals (orthotropic behaviour support)
  - **any ideas** ?