L-10

Monday, September 18, 2023 2:41 PM

$$R^3 \longrightarrow \overset{P}{\text{2D plane g.t.t } \begin{bmatrix}0\\0\\0\end{bmatrix} \text{ in } R^3} \qquad \overset{L}{\text{a line g.t.t } \begin{bmatrix}0\\0\\0\end{bmatrix} \text{ in } R^3}$$

$$A = \begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}$$

(i) $P \cup L$
(ii) $P \cap L$ ✓ vector subspace

$C(A)$ = 2D plane in R^4

$$\underset{A}{\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix}} \underset{x}{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}} = \underset{b}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}$$

★★
(i) if b is not inside the columnspace → $b \notin C(A)$
(ii) if there is enough pivots equal to the components of b.

} that b is not solvable

NullSpace:

$$\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\\ 4 & 1 & 5\end{bmatrix} \begin{bmatrix}c\\ c\\ -c\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad c\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

a solution created by all $Ax=0$ is a vectorspace & it's called a nullspace.
the origin must be in the solution.

How To Find A Nullspace?

$$\underset{A}{\begin{bmatrix}1 & 2 & 2 & 2\\ 2 & 4 & 6 & 8\\ 3 & 6 & 8 & 10\end{bmatrix}} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix}1 & 2 & 2 & 2\\ 0 & 0 & 2 & 4\\ 3 & 6 & 8 & 10\end{bmatrix} \xrightarrow{R_3 = R_3 - 3R_1} \begin{bmatrix}1 & 2 & 2 & 2\\ 0 & 0 & 2 & 4\\ 0 & 0 & 2 & 4\end{bmatrix}$$

$$\Big\Downarrow R_3 = R_3 - R_2$$

$$\begin{bmatrix}1 & 2 & 2 & 2\\ 0 & 0 & 2 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Pivot column (1st), free column (2nd), Pivot column (3rd), free column (4th)

Rank of pivots = 2

$$A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 0$$

Apply 1 to each of the free variables on each pass.
the other free variables will be set to 0.

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the other free variables will be set to 0.

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0$

$\boxed{\begin{matrix} x_2 = 1 \\ x_4 = 0 \end{matrix}}$

$\rightarrow x_4 = 0$

$\therefore 2x_3 = 0 \Rightarrow x_3 = 0$

$\therefore x_1 + 2\times 1 + 0 + 0 = 0$

$\therefore x_1 = -2$

$x_2 = 1$

$x_3 = 0$

$x_4 = 0$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\begin{matrix} x_2 = 0 \\ x_4 = 1 \end{matrix}}$

$2x_3 + 4\cdot 1 = 0$

$\Rightarrow x_3 = -2$

$x_1 + 2\cdot 0 + 2\times -2 + 2\times 1 = 0$

$\therefore x_1 = 2$

$x_2 = 0$

$x_3 = -2$

$x_4 = 1$

$$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$** \quad \boxed{\therefore N(A) = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

* A nullspace is the solution set for $Ax = 0$.

no. of dependent column = no. of free column = no. of independent solution

= Dimension of the NullSpace.

L – 11

Thursday, September 21, 2023 2:41 PM

1. The set S of points $P(x,y,z)$ s.t. $\boxed{x-5y+2z=9}$ is a ___plane___ in $\mathbb{R}^3$. It is ___parallel___ to the ___plane___ 'S_0' of $P(x,y,z)$ s.t. $x-5y+2z=0$.

All points of S_0 have the form —

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}$$

$x-5y+2z=0$

$$\begin{bmatrix} 1 & \underset{\uparrow F}{-5} & \underset{\uparrow F}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y=0$, $z=1$

$\therefore x=-2$

$$c_2 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{x_{special}} \qquad c_1 \underbrace{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}}_{x_{special}}$$

$y=1$, $z=0$ $\therefore x=5$

$$A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \ldots\ldots$$ infinite solutions

The complete Sol^n of $Ax=b$:

A

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* spanning a space
* basis

basis: with a particular set of vectors, we must be able to span the whole R^n plane. The set of vectors can be called the basis of R^n plane.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Basis for columnspace?
(1^{st} & 3^{rd} columns)
(1^{st} & 2^{nd} columns)

$\dim(C(A)) = \text{rank}(r) = 2$

$\dim(N(A)) = 2$

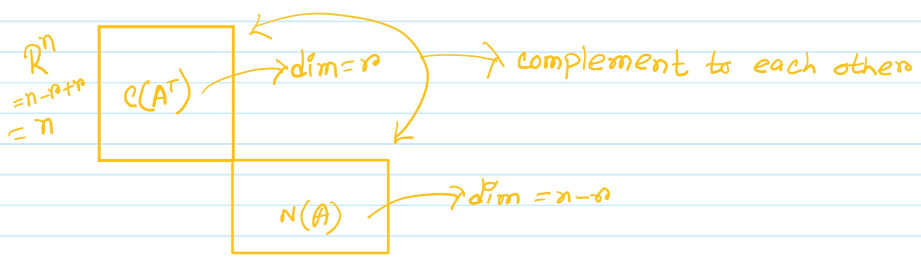
basis for $N(A) \longrightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

Four Fundamental Supspaces:

$$A = \begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -1/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Columnspace $\in R^m$	Nullspace $\in R^n$	Rowspace $\in R^n$	Left nullspace $\in R^m$
$\dim(C(A)) = r = 1$	$\dim(N(A)) = n - r = 3 - 1 = 2$	$\dim(C(A^T)) = r = 1$	$\dim(N(A^T)) = m - r = 2 - 1 = 1$
basis $= \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$	basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$	basis $= \left\{ \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \right\}$	basis $= \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Rowspace $\perp$ Nullspace.



$A^T y = 0$

$(A^T y)^T = 0^T$

$\Rightarrow y^T A = 0$

$\Rightarrow \underbrace{[\ldots y \ldots]}_{\text{left nullspace}} A = [0 \ldots 0]$

left nullspace

$\searrow$ nullspace of the rowspace?

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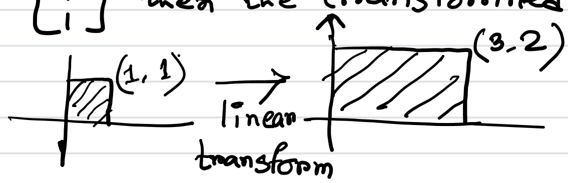
Determinant : to measure how much things are being squished or stretched out

$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ → scales $\hat{j}$ by a factor of 2.

$\begin{bmatrix} 3 \\ 0 \end{bmatrix}$: scales the $\hat{i}$ by a factor of 3

if this is a matrix and it's been multiplied with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then the transformed area:



(area has been stretched out)

area : 1 area : 6

∴ area has scaled by a factor of 6.

∴ determinant 6.

Determinant is 0 :

$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ has a dependent column. Here M represents a line in R^2 space. So whatever matrix we multiply with this, will be squished into a line.

M

∴ if we multiply a plane with this matrix, it will become line.

Dots & lines have no area. So, the determinant becomes zero.

Determinant is (−) : Determinant is negative because of orientation, namely inversion. When inverted, the determinant becomes negative.

⊛ In 3D, Determinants work with volume.

⊞ Properties of Determinants:

Properties of Determinants:

(i) $\det(I) = 1$

(ii) Exchanging any two rows of a matrix reverses the sign of the determinant.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(I) = 1 \qquad \det(P) = -1.$$

(iii) linear combination:

→ multiplying any row by a scalar factor:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times t = t(ad - bc)$$

→ Addition:
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

But, $\det(A) + \det(B) \neq \det(A+B)$.

(iv) If any two rows are equal, then the Determinant is 0.

(v) While elimination, if $l \times$ row i is substracted from a row k, the determinant remains unchanged.

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(vi) A complete row of zeros means $\det = 0$.

(vii) For any upper triangular matrix, the determinant is the product of the diagonal elements.

$$\det(U) = \begin{bmatrix} d_1 & a & a \\ 0 & d_2 & a \\ 0 & 0 & d_3 \end{bmatrix} = d_1 d_2 d_3$$

(viii)
$$\boxed{\begin{array}{lll} \det(A) = 0 & \text{when} & A \text{ is singular.} \\ \det(A) \neq 0 & \text{"} & A \text{ is invertible.} \end{array}}$$

Easy way to find Determinant:

Gauss-Jordan Elimination

Then multiply the diagonals.

Then multiply the diagonals.

(ix) $\boxed{\det(AB) = \det(A) \cdot \det(B)}$

If A is invertible, $AA^{-1} = I$

$$\therefore \det(AA^{-1}) = \det(I)$$

$$\Rightarrow \det(I) = 1 = \det(AA^{-1})$$

$$\therefore \boxed{\det(A^{-1}) = \frac{1}{\det(A)}} .$$

$$\underbrace{\det(LU) = \det(L)}_{\text{diagonal having 1s}} \cdot \det(U) = 1 \cdot x = x$$

$$\therefore \det(A) = x.$$

$$\det(A^T) = \det(U^T) \cdot \det(L^T) = \det(U^T) \cdot 1 = x .$$

$$\therefore \boxed{\det(A) = \det(A^T)}$$

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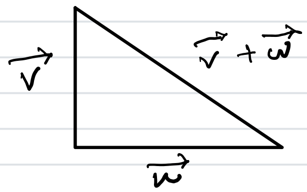
Orthogonality:

perpendicularity. If two vectors have a dot product = 0

or $\vec{v} \cdot \vec{w} = 0$ then $v^T w = 0$ or $v w^T = 0$



$$|v|^2 + |w|^2 = |v+w|^2$$

(applying Pythagorus)

$$\Rightarrow v \cdot v + w \cdot w = (v+w)(v+w)$$

$$\Rightarrow v^T v + w^T w = v^T v + v^T w + w^T v + w^T w$$

$$\Rightarrow v^T w + w^T v = 0$$

$$\Rightarrow 2 v^T w = 0 \Rightarrow v^T w = 0$$

$$\therefore \vec{v} \cdot \vec{w} = 0.$$

Example: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $x + y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$|x|^2 = 14$ $|y|^2 = 5$ $|x+y|^2 = 19$.

Orthogonality of subspaces:

Two subspaces are orthogonal if every vector in one subspace is orthogonal to every vector in another subspace.

if v & w are two subspaces,
$v^T w = 0$ for all vectors $\vec{v}$ in v and $\vec{w}$ in W.

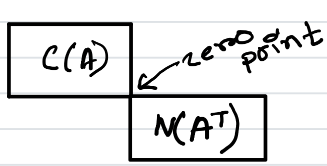
if v & w are two vectors in a single subspace,
orthogonality is possible if

$$\boxed{\dim(v) + \dim(w) = \dim(\text{Actual vectorspace})}$$

If, $\dim(v) + \dim(w) > \dim$ Actual vectorspace)

then orthogonality is impossible.

Zero is the only point where the null space meets the row space.



whole nullspace is perpendicular to whole row space.

$$\therefore \boxed{\begin{array}{l} C(A^T) \perp N(A) \text{ in } R^n \\ C(A) \perp N(A^T) \text{ in } R^m \end{array}}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dim(C(A^T)) = 1 \quad \dim(N(A)) = 2$

$\therefore$ total dim : $d(m(u)) + \dim(v) = \dim(R^3)$

$\therefore$ total Rank would be 3 in that case.

$\therefore$ These are complementary numbers.

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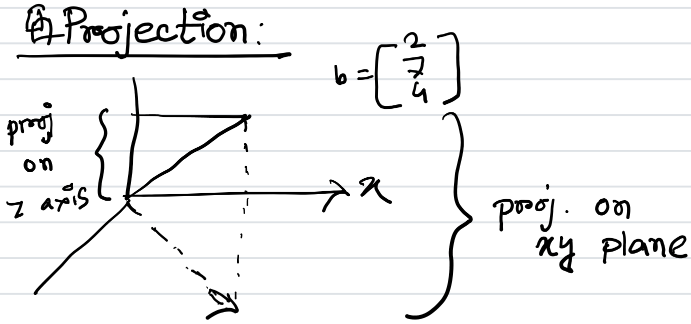
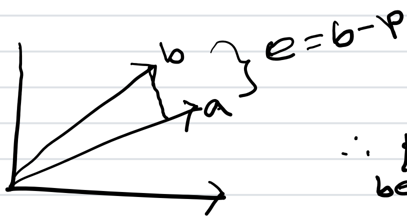
④ Projection:



In order to solve $Ax = b$, b needs to be in the $C(A)$

if $b \notin C(A)$ then we solve for $A\hat{x} = p$ (pseudo b)

[closest solve for $Ax = b$]



∴ p is some part of a and it can be described by $p = \hat{x}\, a$.

∴ e is perpendicular to p and a.

projection matrix P:

a matrix that if we multiply with b, gives us p.

∴ $\boxed{Pb = p}$

$e = b - p$

$\Rightarrow e = b - \hat{x}.a$

again, $a \cdot e = 0$

$\Rightarrow a(b - \hat{x}a) = 0$

$\Rightarrow a^T b - \hat{x}\, a^T a = 0$

∴ $\boxed{\hat{x} = \dfrac{a^T b}{a^T a}}$

$p = \hat{x} \cdot a$

$= \dfrac{a^T b}{a^T a} \cdot a$

∴ $p = \dfrac{a \cdot a^T}{a^T a}\, b = Pb$

∴ $\boxed{P = \dfrac{a a^T}{a^T a}}$

2 special cases:

① $b = a$ [projection onto itself]

in that case $\hat{x} = \dfrac{a^T a}{a^T a} = 1$

② $b \perp a$ [$a^T b = 0$]

(ii) $b \perp a \ [a^Tb = 0]$

$$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{0}{a^Ta} = 0$$

projection matrix goes through the origin.

$\therefore C(P) = a$ [A st. line going through origin]

rank = 1.

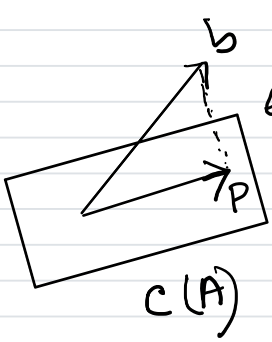
If a is doubled, P matrix unchanged
bcz b "

if p is doubled,

$$P \cdot P = \frac{aa^T}{a^Ta} \cdot \frac{aa^T}{a^Ta} = P \qquad \therefore P^2 = P$$

again, $P^T = \left(\frac{aa^T}{a^Ta}\right)^T = \frac{aa^T}{a^Ta} = P. \quad \therefore P^T = P$

(4) Projection onto subspace:



$e = b - p = b - A\hat{x}$

$A\hat{x} = p$

$\therefore$ entire subspace is orthogonal to e.

$\therefore A \cdot e = 0$
$\Rightarrow A^Te = 0$
$\Rightarrow A^T(b - A\hat{x}) = 0$
$\Rightarrow A^Tb - A^TA\hat{x} = 0$

$$\therefore \hat{x} = \frac{A^Tb}{A^TA} = (A^TA)^{-1}A^Tb.$$

$$\therefore P = A\hat{x} = A(A^TA)^{-1}A^Tb$$

$$\therefore P = \frac{p}{b} = A(A^TA)^{-1}A^T$$

Steps for solving $A\hat{x} = p$:

Steps for solving $Ax = p$.

$$Ax = b.$$

$$\Rightarrow A^T A \hat{x} = A^T b$$

$$\Rightarrow \hat{x} = \frac{A^T b}{A^T A} = (A^T A)^{-1} A^T b$$

$$\therefore A\hat{x} = p$$

$$\therefore P = \frac{p}{b}.$$

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Methods for finding out Determinant:

3 methods $\longrightarrow$ (i) pivot formula
(ii) big " ✗
(iii) co-factor "

(i) The pivot formula: uses elimination method.

$$A = LU$$
$$\Rightarrow \det(A) = \det(L) + \det(U)$$
$$= 1 \cdot d_1 d_2 d_3 \ldots\ldots$$
$$= d_1 d_2 d_3 \ldots\ldots$$
$$\therefore \det(A) = \pm(d_1 d_2 d_3 \ldots\ldots d_n)$$

Ex:

$$A \begin{bmatrix} x & y & y \\ y & x & y \\ y & y & x \end{bmatrix} = \begin{bmatrix} x+2y & x+2y & x+2y \\ y & y & y \\ y & y & x \end{bmatrix}$$

$$\vdots$$

$$= (x+2y)(x-y)^2$$

$$\therefore \det(A) = (x+2y)(x-y)^2$$

The big formula:

for a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} \Rightarrow \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} - \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix}$$
$$\Rightarrow ad - bc$$

$\therefore$ for $n \times n \longrightarrow n!$ terms.

Determinant by co-factors:

co factor along any row, $C_{ij} = (-1)^{i+j} \det(A_{ij})$

$\therefore \det(A) = +a_{11}C_{11} + a_{12}(-C_{12}) + a_{13}(+C_{13})$

co factor along any row, $c_{ij} = (-1)^{i+j} \det(A_{ij})$

$\therefore \det(A) = +a_{11}c_{11} + a_{12}(-c_{12}) + a_{13}(+c_{13})$

False expansion Th^m:

if A is $n \times n$ matrix,

$$\text{if } A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\therefore a_{21}c_{11} + a_{22}c_{12} + a_{23}c_{13} = 0$$

$$\boxed{\begin{aligned} &\therefore \left(A \cdot (\text{cof } A)^T\right)_{ii} = a_i \cdot c_i = \det(A) \\ &\left(A \cdot (\text{cof } A)^T\right)_{ik} = 0 \end{aligned}}$$

$$\therefore A \cdot (\text{cof } A)^T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} \det(A) & 0 \\ 0 & \det(A) \end{bmatrix} = \det(A) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \det(A)$$

$$\therefore A (\text{cof } A)^T = \det(A) \cdot I$$

$$\Rightarrow A = \frac{(\text{cof } A)^T}{\det(A)} = I = AA^{-1}$$

$$\boxed{\Rightarrow A^{-1} = \frac{1}{\det(A)} \cdot (\text{cof } A)^T}$$

7) Cramer's rule:

$$x_1 = \frac{|D_1|}{A} \quad x_2 = \frac{|D_2|}{A} \quad x_3 = \frac{|D_3|}{A}$$

for $Ax = b$,
by replacing b in A.

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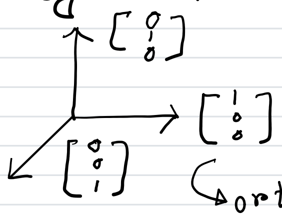
Orthogonal matrix: (Gram Schimdt Process)



all perpendicular unit vectors are independent.

$\hookrightarrow$ orthogonal vectors $[q_1 q_2 \dots q_n]$ are unit vectors

$$\therefore q_i^T q_j = \left\{ \begin{matrix} 0 \; ; \; i \neq j \\ 1 \; ; \; i = j \end{matrix} \right\}$$

$$Q = \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ q_1 & q_2 & \dots & q_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$\downarrow$

orthogonal metrix.

if square $\rightarrow$ orthogonal matrix

$$\therefore Q^TQ = \begin{bmatrix} \leftarrow q_1 \rightarrow \\ \leftarrow q_2 \rightarrow \\ \leftarrow q_3 \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ q_1 & q_2 & \dots & q_n \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = I \qquad \therefore \boxed{Q^TQ = I}$$

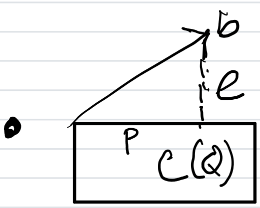
$\longrightarrow$ If Q is square, then $Q^{-1}Q = 1$.

$$\therefore Q^T = Q^{-1}$$

— all permutation matrices are orthogonal.

Normalization: divide by their value.

Ex: $Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$ $\checkmark$ orthonormal? not orthonormal?



since, proj. matrix,

$$P = A(A^TA)^{-1}A^T$$

$$\therefore P = Q(\underbrace{Q^TQ}_{I})^{-1}Q^T = Q\,Q^T = I \begin{bmatrix} \text{if } Q \\ \text{is square} \end{bmatrix}$$

$\therefore P = Pb = Ib = b \rightarrow$ can span the entire space because it's

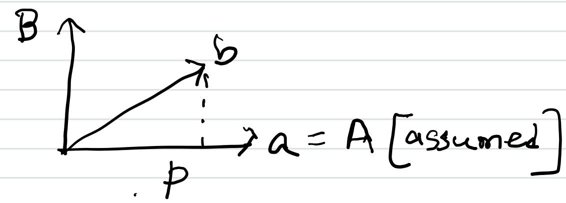
entire space
because it's
dependent

- $A\hat{x} = b$

$$A^T A \hat{x} = A^T b$$
$$\Rightarrow Q^T Q \hat{x} = Q^T b$$
$$\Rightarrow \hat{x} = Q^T b$$

$\therefore$ projection : $P = A\hat{x} = Q\hat{x} = QQ^T b = b$
[if Q is square]

Orthogonal matrix from 3 random vectors:



$$a \xrightarrow{\text{orthogonal}} A \xrightarrow{\text{orthogonal}} \frac{A}{||A||}$$
$$b \xrightarrow{\text{"}} B \xrightarrow{\text{"}} \frac{B}{||B||}$$

$\therefore$ conceptually, $\boxed{e = b}$ since $e \perp A$

$$\therefore b = e + p$$
$$\Rightarrow e = b - p = b - A\hat{x}$$
$$\therefore \boxed{B = b - A \cdot \frac{A^T b}{A^T A}}$$

Since, they are orthogonal,

$$A \cdot B = 0 = A^T B$$
$$= A^T \left(b - A \cdot \frac{A^T b}{A^T A} \right)$$
$$= \frac{(A^{-1} b)(A^T A) - (A^T b)(A^{-1} A)}{A^{-1} A}$$
$$= 0$$

If we have 3 vectors,

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} \cdot B$$

$$\therefore A^T C = 0 \quad ; \quad B^T C = 0$$

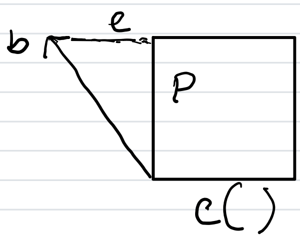
$\therefore$ last step : $q_1 = \frac{A}{||A||}$, $q_2 = \frac{B}{||B||}$

$$q_3 = \frac{C}{||C||}$$

$$q_3 = \frac{c}{\|c\|}$$

$$\therefore Q = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ q_1 & q_2 & q_3 & \cdots & q_n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}$$ is our orthogonal matrix

Least Square regression:



(i) if b doesn't fit in the column, we try to find the projection of it.

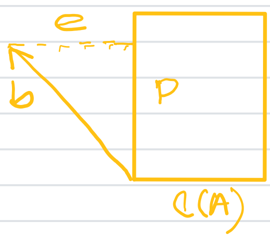
(ii) If b is in columnspace

(iii) If $b \perp C()$. Pb = columnspace.

L-19

Tuesday, November 14, 2023 11:03 PM

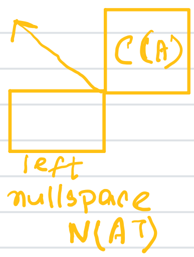
Least Square Regression:



- if b doesn't fit in $C(A)$ we try to find projection of it.
- If $b \in C(A)$, $Pb = b$
- if $b \perp C(A)$ $Pb = 0$

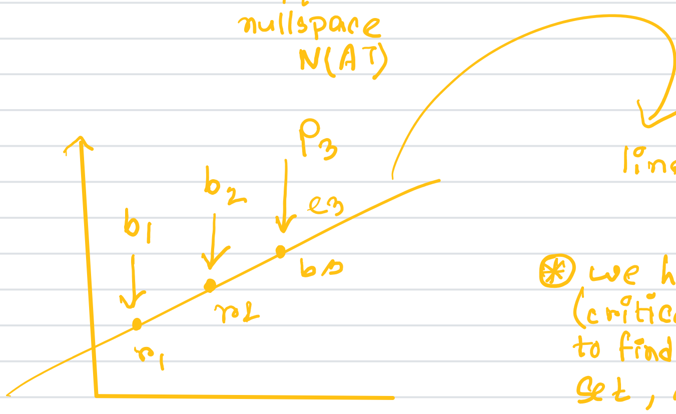
as left nullspace is orthogonal to $C(A)$,

$e \in$ left nullspace



when $Ax = b$ has no solution,

$A^T A \hat{x} = A^T b$



linear regression line

⊛ we have to minimize the errors (critical errors as much as possible to find the line that best fits the data set, as it minimizes the variance.

$Ax = b$

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2$
} no sol^n, cz more eq^ns than no of variable.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \rightarrow$ determine $\hat{x}$, P, p

$\|e\|^2 = \|Ax - b\|$ → eliminate negative values
→ focus more on weighted errors to minimize that

squaring in order

$Ax = b \longrightarrow$ no sol^n

$Ax = b \longrightarrow$ no solⁿ

$A^T A \hat{x} = A^T b$ $\qquad A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$\therefore A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

if $\hat{x}$ is unknown, $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$

$\therefore \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$$\left.\begin{aligned} 3\hat{C} + 6\hat{D} &= 5 \\ 6\hat{C} + 14\hat{D} &= 11 \end{aligned}\right\} \quad C = \frac{2}{3}, \; D = \frac{1}{2} \qquad \boxed{y = \frac{2}{3} + \frac{1}{2}t}$$

Total squared error: $e^2 = e_1^2 + e_2^2 + e_3^2$

$= (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

$= E$

$\therefore \frac{dE}{dC} = 2(C + D \cdot 0)^2 = 0$

$\therefore \frac{dE}{dC} = 2(0 + D)^2 = 0$

we have to solve C, D according to these.

Now, since the line y is linear regression line that is orthogonal to every b point, y is the projection.

$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad p = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} \rightarrow$ by replacing t with (1, 2, 3) in y eqⁿ.

$b = p + e$

$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ 1/6 \end{bmatrix}$

$e \perp p$
$e \perp A$ can be proved from this.

this.

$A^T A \hat{x} = A^T b$

$\Rightarrow \hat{x} = \frac{A^T b}{A^T A} \Rightarrow A^T (A^T A)^{-1} b$

(if A has independent col^{m}s A is invertible.)

If $A^T A$ isn't invertible, $A^T A x = 0$

$\therefore x^T A^T A x = 0$

$\Rightarrow (Ax)^T \cdot Ax = 0$

$\Rightarrow y^T \cdot y = 0$

$\Rightarrow y^T \cdot y^T = 0 \longrightarrow$ only possible if $y = 0$

bcz a vector can't be perpendicular with itself.

$\therefore$ A has independent columns.

only $x = 0$ can lead to $Ax = 0$

$\therefore A^T A$ is invertible if it has independent columns.

L-20

Tuesday, November 14, 2023 8:10 AM

Least Square Approximation:

Aproximate solution is the closest point from b to $C(A)$. Basically the perpendiclar distance of b i.e. projection of b on $C(A)$.

$Ax = b$

$A\hat{x} = A^T b$

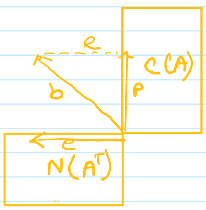
$p + e = b$ e=error

$e \perp C(A)$

if this is true,

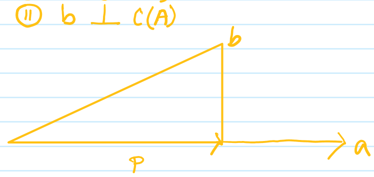
the e is definitely on left nullspace.

$Pb = p$



(i) $b \in C(A)$, $P = b$

(ii) $b \perp C(A)$



$P = a\hat{x}$

$b \perp a$, $P = 0$

$\therefore \hat{x} = 0$

the more b becomes perpendicular to a, p becomes smaller & when it's perpendicular, it lies on left nullspace & the solution is zero.

Least Square Approximation is a theory we use to solve relationships between independent variables.

we have to find the best st. line in which it has the shortest distance from the actual datasets.

$$\boxed{y = C + Dt}$$

$7 = C + 4D$

$10 = C + 5D$

$11 = C + 6D$

$$\underset{b}{\begin{bmatrix} 7 \\ 10 \\ 11 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}} \underset{\hat{x}}{\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}} = b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 15 \\ 15 & 77 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 28 \\ 144 \end{bmatrix}$$

$A^T A x = A^T b$

$$\Rightarrow \begin{bmatrix} 3 & 15 \\ 15 & 77 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 28 \\ 144 \end{bmatrix}$$

$\Rightarrow 3\hat{C} + 15\hat{D} = 28$

$15\hat{C} + 77\hat{D} = 144$

$\hat{C} = -2/3$

$\hat{D} = 2$

So, $t = 7$

$$y = -2/3 + 2 \times 7 = \frac{-2 + 42}{3} = \frac{40}{3} \approx 13$$

$$\boxed{y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \cdots\cdots + \beta_n x_n + \epsilon_0}$$

if A has all independent (hypothesis) columns, then $A^T A$ is invertible (we need to prove)

$A^T A \underbrace{x}_{\text{non-zero}} = 0$

$\Rightarrow x^T A^T A x = 0$

$\Rightarrow (Ax)^T Ax = 0$

$\Rightarrow y^T y = 0$ $[y = Ax]$

$\Rightarrow y \cdot y = 0$

$\Rightarrow y = 0$

So, $Ax = 0$ but $Ax \neq 0$

So, $A^T A x \neq 0$

$y = -2/3 + 2t$

$P_1 = -2/3 + 2 \times 4 = \frac{22}{3}$

$P_2 = -2/3 + 2 \times 5 = 28/3$

$P_3 = -2/3 + 2 \times 6 = 34/3$

$e = b - P$

$e_1 = 7 - \frac{22}{3} = 1/3$

$e_2 = 10 - \frac{28}{3} = 2/3$

$e_3 = 11 - 34/3 = -1/3$

$b = p + e$

$$\begin{bmatrix} 7 \\ 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 22/3 \\ 28/3 \\ 34/3 \end{bmatrix} + \underset{e}{\begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}}$$

errors

$P_2 = -\frac{2}{3} + 2 \times 5 = \frac{28}{3}$ $e_2 = 10 - \frac{28}{3} = \frac{2}{3}$

$P_3 = -\frac{2}{3} + 2 \times 6 = \frac{34}{3}$ $e_3 = 11 - \frac{34}{3} = -\frac{1}{3}$

$\begin{bmatrix} 11 \end{bmatrix}$ $\begin{bmatrix} \frac{34}{3} \end{bmatrix}$ $\begin{bmatrix} -\frac{1}{3} \end{bmatrix}$

e

error vector

if $p \cdot e = 0$

then $p \perp e$.

L-21

Friday, November 17, 2023 10:41 AM

Eigenvalues & Eigenvectors:

prerequisites:

1. linear transformation by matrices
2. Change of basis
3. Determinants

$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \times v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ the basis & span of v would change

But $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\therefore Av_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ → no change in span

eigenvectors

eigenvalue

(values by which vectors are stretched/squeezed)

known as $\boxed{\text{eigenvectors}}$

$Ax = b$ → in case of eigenvectors, $\boxed{A\vec{v} = \lambda\vec{v}}$

$\Rightarrow A\vec{v} - \lambda\vec{v} = 0$

$\Rightarrow (A - \lambda I)\vec{v} = 0$

→ $\det(A - \lambda I) = 0$ (must be)

$\therefore (A - \lambda I)$ must be singular matrix

How to solve $Ax = \lambda x$:

$\det(A - \lambda I) = 0$

1. Find out λ
2. Then find the nullspace of $(A - \lambda I)$

Example: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\Rightarrow (3-\lambda)^2 - 1 = 0 \Rightarrow (3-\lambda) = \pm 1$

$\Rightarrow \lambda_1 = 2$

$\lambda_2 = 4$

$\lambda_1 + \lambda_2 = $ trace of A

$\lambda_1 \lambda_2 = \det(A)$

For, $\lambda_1 = 2$

$A - \lambda_1 I = \begin{bmatrix} 1 & 1 \end{bmatrix}$

For $\lambda_2 = 4$

$A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

For, $\lambda_1 = 2$

$A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$(A - \lambda_1 I) v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 4$

$A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$(A - \lambda_2 I) v_2 = 0$

$\Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

eigenvectors

⊛ If, $A + KI = B$

$\lambda_1, \lambda_2 \cdots \lambda_n \quad \lambda_1 + k, \lambda_2 + k, \cdots\cdots, \lambda_n + k$

& A, B will have the same eigenvectors.

$Ax = \lambda x$

$(A + 3I)x = Ax + 3Ix$

$= \lambda x + 3Ix$

$= (\lambda + 3)x$.

⊛ If A & B are different, $Ax = \lambda x$

$Bx = \alpha x$

we can't write, $(A+B)x = (\lambda + \alpha)x$

bcz for two diff matrix,
there should be different eigenvectors.

$[Bx_1 = \alpha x_1]$

(1) For a $n \times n$ matrix, there should be 'n' eigenvalues & 'n' eigenvectors.

(2) $\lambda_1 + \lambda_2 + \cdots\cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$ (trace)

(3) $\lambda_1 * \lambda_2 * \cdots\cdots * \lambda_n = \det(A)$

90° rotation $\longrightarrow Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ trace $= 0$, det $= 1$

$\lambda^2 - (\text{trace})\lambda + \det \Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda_1 = i, \lambda_2 = -i$

(4) For triangular matrices, we won't get 'n' eigenvalues & 'n' eigen matrices.

L-22

Tuesday, November 21, 2023 12:00 PM

Diagonalising a matrix:

Find the k-th power of A : A^k

$A \to n \times n \xrightarrow{\text{ideally}}$ n eigenvalues

n eigenvectors $\to x_1, x_2, \ldots, x_n$

$S = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ $AS = A \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Ax_1 & Ax_2 & \cdots & Ax_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \lambda_1 x_1 & \lambda_2 x_2 & \lambda_n x_n \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & \cdots & x_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & & \ddots & \\ \vdots & & & \lambda_n \end{bmatrix}$

$= S\Lambda$

$$\boxed{\begin{array}{l} AS = S\Lambda \\ \Rightarrow A = S\Lambda S^{-1} \end{array}}$$

$A^2 = S\Lambda S^{-1} \cdot S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

$$\boxed{A^k = S A^k S^{-1}}$$

* for the matrices having same eigen vectors, we cannot follow the process stated above. cz having same eigen vectors means dependent col^m & not invertible. So, there won't exist a S^{-1}.

In this case, we have to multiply manually.

$A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$ $\lambda_1 = 1$ $\lambda_2 = 6$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$S^{-1} = \frac{1}{det} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$

$A^{100} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}^{100} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 6^{100} \end{bmatrix}$

$$\boxed{\therefore A^{100} = \begin{bmatrix} 1 & 6^{100}-1 \\ 0 & 6^{100} \end{bmatrix}}$$

$u_0 \to$ vector

$u_{k+1} = Au_k$

$u_1 = Au_0 \to k=1$

$u_2 = A^2 u_0 \to k=2$

$\vdots$

$\boxed{\text{How to find } A^k \text{ ?}}$

1. $u_0 = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

$$\boxed{A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \cdots \cdots + c_n \lambda_n^k x_n}$$

$u_2 = A^2 u_0 \rightarrow k=2$ $\qquad \boxed{A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \dots + c_n \lambda_n^k x_n}$

$\vdots$

$u_k = A^k u_0$

Fibonacci Series:

$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$

$F_{k+2} = F_{k+1} + F_k \qquad u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

$u_{k+1} = A u_k$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = S \Lambda S^{-1}$

$u_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix} = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \Rightarrow \lambda^2 - \lambda - 1 = 0$

$\Rightarrow \lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lambda_1 = \dfrac{1+\sqrt{5}}{2} \qquad \lambda_2 = \dfrac{1-\sqrt{5}}{2}$

$\approx 1.62 \qquad \approx -0.62$

$x_1 = ? \qquad x_2 = ?$

$\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda - \lambda^2 + 1 \\ \lambda - \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$A = S \Lambda S^{-1}$

$u_{k+1} = A^k u_0 = A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = S \Lambda^k S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix}$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_1^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k \end{bmatrix} \quad \begin{matrix} F_{k+2} \\ F_{k+1} \end{matrix}$$

$$\boxed{F_{k+1} = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}}$$

$$k = 99, \quad F_{100} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{99} - \left(\frac{1-\sqrt{5}}{2}\right)^{99}}{\sqrt{5}}$$

L-23

Friday, November 24, 2023 9:26 AM

Singular value Decomposition (SVD):

$A = S\Lambda S^{-1} = x_1\lambda_1 y_1 + x_2\lambda_2 y_2 + \cdots + x_n\lambda_n y_n$

→ most important part

* $Ax = \lambda x$ → this eq^n doesn't imply if A is a rectangular matrix, bcz rectangular matrices don't have eigenvalues & eigenvectors

so, for rectangular matrices we do singular value decomposition, rather than eigen value decomposition.

in case of symmetric matrices,

1. All eigen values are real
2. all eigen vectors are real & orthonormal

Proof: (by contradiction)

$\lambda = a + ib$, $\bar{\lambda} = a - ib$

if A is symmetric, $Ax = \lambda x$ —— (I)

$A\bar{x} = \bar{\lambda}\bar{x}$ —— (II)

$\bar{x}^T A^T = \bar{x}^T \bar{\lambda} = \bar{x}^T A$

(A symmetric)

$\times x$ $\bar{x}^T Ax = \bar{x}^T \bar{\lambda} x$ —— (III)

(I) $\times \bar{x}^T \Rightarrow \bar{x}^T Ax = \bar{x}^T \lambda x$ —— (IV)

$\therefore \bar{x}^T \lambda x = \bar{x}^T \bar{\lambda} x$

$\therefore \lambda = \bar{\lambda}$ [iff $b=0$]

but it contradicts our assumption.

Therefore, λ can't be imaginary.

$\therefore$ All eigen values are real (proved)

Spectral Th^m:

$A = S\Lambda S^{-1}$ → orthonormal

orthogonal

$A = Q\Lambda Q^{-1}$ $[Q^{-1} = Q^T]$

$= Q\Lambda Q^T = [q_1\ q_2 \cdots q_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} [q_1^T\ q_2^T \cdots q_n^T]$

$\boxed{\begin{matrix} qq^T = P \\ q^Tq = 1 \end{matrix}}$

$= q_1\lambda_1 q_1^T + q_2\lambda_2 q_2^T + \cdots + q_n\lambda_n q_n^T$

$= \lambda_1(q_1q_1^T) + \cdots + \lambda_n(q_nq_n^T)$

$$\boxed{\begin{array}{c} qq^T = I \\ q^Tq = 1 \end{array}}$$

$= q_1\lambda_1 q_1^T + q_2\lambda_2 q_2^T + \cdots\cdots + q_n\lambda_n q_n^T$

$= \lambda_1(q_1 q_1^T) + \cdots\cdots + \lambda_n(q_n q_n^T)$

SVD:

$A = U\Sigma V^T$ → right singular matrix

U → left singular matrix

Σ → singular values

$A v_1 = \sigma_1 u_1$ $\quad [v_1, v_2, \cdots, v_r \to$ orthogonal vectors$]$ → from row space of A

$A v_2 = \sigma_2 u_2$ $\quad [u_1, u_2, \cdots\cdots, u_r \to$ " " $]$

$\vdots$

$A v_r = \sigma_r u_r$

$$A[v_1 \; v_2 \cdots\cdots v_n] = [u_1 \; u_2 \cdots u_n]\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

$\Rightarrow AV = U\Sigma$

$\Rightarrow AVV^{-1} = U\Sigma V^{-1}$

$\Rightarrow \boxed{A = U\Sigma V^T}$

$A^TA = (U\Sigma V^T)^T U\Sigma V^T$

$= V\Sigma^T \underbrace{U^TU}_{1} \Sigma V^T = \underset{Q}{V}(\underset{\Lambda}{\Sigma^T\Sigma})\underset{Q^T}{V^T}$

7.2

$AA^T = U\Sigma V^T V\Sigma^T U^T$

$= \underset{Q}{U}(\underset{\Lambda}{\Sigma\Sigma^T})\underset{Q^T}{U^T}$

$$\Sigma^2 = \Sigma^T\Sigma = \Sigma\Sigma^T = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$$

$$\Sigma = \sqrt{\Lambda} = \begin{bmatrix} \sqrt{\sigma_1^2} & & & \\ & \sqrt{\sigma_2^2} & & \\ & & \ddots & \\ & & & \sqrt{\sigma_n^2} \end{bmatrix}$$

$A = U\Sigma V^T$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ u_1 & u_2 & \cdots & u_r \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}\begin{bmatrix} v_1^T & v_2^T & \cdots & v_n^T \end{bmatrix}$$

$= u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots\cdots + u_n\sigma_n v_n^T$

70% – 80%

PCA