INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v} + \boldsymbol{w} = (2, 3)$ and $\boldsymbol{v} - \boldsymbol{w} = (6, -1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0, 0)$.

3 This problem gives the diagonals $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\boldsymbol{v} = (3, 3)$ and $\boldsymbol{w} = (2, -2)$.

4 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

5 $\boldsymbol{u}+\boldsymbol{v} = (-2, 3, 1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w} = (0, 0, 0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w} =$ (add first answers) $=$ $(-2, 3, 1)$. The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same plane because a combination gives $(0, 0, 0)$. Stated another way: $\boldsymbol{u} = -\boldsymbol{v} - \boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 The components of every $c\boldsymbol{v} + d\boldsymbol{w}$ add to zero because the components of $\boldsymbol{v}$ and of $\boldsymbol{w}$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$. There is no solution to $c\boldsymbol{v}+d\boldsymbol{w} = (3, 3, 6)$ because $3 + 3 + 6$ is not zero.

7 The nine combinations $c(2, 1) + d(0, 1)$ with $c = 0, 1, 2$ and $d = (0, 1, 2)$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.

8 The other diagonal is $\boldsymbol{v} - \boldsymbol{w}$ (or else $\boldsymbol{w} - \boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$. Three possible parallelograms!

10 $\boldsymbol{i} - \boldsymbol{j} = (1, 1, 0)$ is in the base (x-y plane). $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} = (1, 1, 1)$ is the opposite corner from $(0, 0, 0)$. Points in the cube have $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 The combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1, 1, 0)$ fill the xy plane in xyz space.

13 Sum $=$ zero vector. Sum $= -2{:}00$ vector $= 8{:}00$ vector. $2{:}00$ is $30°$ from horizontal $= (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j} = (0, 1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\boldsymbol{j} = (0, 12)$.

15 The point $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v} + \boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c + d = 1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V} = -\boldsymbol{v} + 2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v} + c\boldsymbol{w}$ are on the line passing through $(0, 0)$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v} + \boldsymbol{w}$ and back beyond $(0, 0)$. With $c \geq 0$, half of this line is removed, leaving a *ray* that starts at $(0, 0)$.

18 The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $0 \leq c \leq 1$ and $0 \leq d \leq 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$ then $c\boldsymbol{v} + d\boldsymbol{w}$ fills the unit square. But when $\boldsymbol{v} = (a, 0)$ and $\boldsymbol{w} = (b, 0)$ these combinations only fill a segment of a line.

19 With $c \geq 0$ and $d \geq 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$, then the cone is the whole quadrant $x \geq 0$, $y \geq 0$. *Question*: What if $\boldsymbol{w} = -\boldsymbol{v}$? The cone opens to a half-space. But the combinations of $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (-1, 0)$ only fill a line.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

21 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

23 *All* vectors are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$.

24 The combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ fill one plane. The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill another plane. Those planes meet in a *line*: *only the vectors* $c\boldsymbol{v}$ are in both planes.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} =$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. *The solution is* $c = 2$ *and* $d = 4$. Then $2(1,2) + 4(3,1) = (14,8)$.

27 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

28 There are **6** unknown numbers $v_1, v_2, v_3, w_1, w_2, w_3$. The six equations come from the components of $\boldsymbol{v} + \boldsymbol{w} = (4,5,6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2,5,8)$. Add to find $2\boldsymbol{v} = (6,10,14)$ so $\boldsymbol{v} = (3,5,7)$ and $\boldsymbol{w} = (1,0,-1)$.

29 Fact: For any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane, some combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ is the zero vector (beyond the obvious $c = d = e = 0$). So if there is one combination $C\boldsymbol{u}+D\boldsymbol{v}+E\boldsymbol{w}$ that produces $\boldsymbol{b}$, there will be many more—just add c, d, e or $2c, 2d, 2e$ to the particular solution C, D, E.

The example has $3\boldsymbol{u} - 2\boldsymbol{v} + \boldsymbol{w} = 3(1,3) - 2(2,7) + 1(1,5) = (0,0)$. It also has $-2\boldsymbol{u} + 1\boldsymbol{v} + 0\boldsymbol{w} = \boldsymbol{b} = (0,1)$. Adding gives $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w} = (0,1)$. In this case c, d, e equal $3, -2, 1$ and $C, D, E = -2, 1, 0$.

Could another example have $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that could NOT combine to produce $\boldsymbol{b}$? Yes. The vectors $(1,1)$, $(2,2)$, $(3,3)$ are on a line and no combination produces $\boldsymbol{b}$. We can easily solve $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = 0$ but not $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w} = \boldsymbol{b}$.

30 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0,0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$.

31 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl@{\qquad}l@{\qquad}l} 2c & -d & & = 1 & \text{So } d = 2e & c = 3/4 \\ -c & +2d & -e & = 0 & \text{then } c = 3e & d = 2/4 \\ & -d & +2e & = 0 & \text{then } 4e = 1 & e = 1/4 \end{array}$$

Problem Set 1.2, page 18

1 $\boldsymbol{u}\cdot\boldsymbol{v} = -2.4 + 2.4 = 0$, $\boldsymbol{u}\cdot\boldsymbol{w} = -.6 + 1.6 = 1$, $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{w} = 0 + 1$, $\boldsymbol{w}\cdot\boldsymbol{v} = 4 + 6 = 10 = \boldsymbol{v}\cdot\boldsymbol{w}$.

2 $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = \sqrt{5}$. Then $|\boldsymbol{u}\cdot\boldsymbol{v}| = 0 < (1)(5)$ and $|\boldsymbol{v}\cdot\boldsymbol{w}| = 10 < 5\sqrt{5}$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{4}{5}, \frac{3}{5}) = (0.8, 0.6)$. The vectors $\boldsymbol{w}, (2, -1)$, and $-\boldsymbol{w}$ make $0^\circ, 90^\circ, 180^\circ$ angles with $\boldsymbol{w}$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (1/\sqrt{5}, 2/\sqrt{5})$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = 10/5\sqrt{5}$.

4 (a) $\boldsymbol{v}\cdot(-\boldsymbol{v}) = \mathbf{-1}$ (b) $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = \mathbf{0}$ so $\theta = 90^\circ$ (notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$) (c) $(\boldsymbol{v}-2\boldsymbol{w})\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{w}\cdot\boldsymbol{w} = 1 - 4 = \mathbf{-3}$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (1, 3)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2, 1, 2)/3$. $\boldsymbol{U}_1 = (3, -1)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-3, 1)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1, -2, 0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. They lie on a line. All vectors (x, y, z) with $x + y + z = 0$ lie on a *plane*. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line* in 3-dimensional space.

7 (a) $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60^\circ$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90^\circ$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60^\circ$ or $\pi/3$ (d) $\cos\theta = -1/\sqrt{2}$ so $\theta = 135^\circ$ or $3\pi/4$.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True: $\boldsymbol{u}\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + 2\boldsymbol{u}\cdot\boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u}-\boldsymbol{v}\|^2 = (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})$ splits into $\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = 0$.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1 + v_2w_2 = \boldsymbol{v}\cdot\boldsymbol{w} = 0$: perpendicular! The vectors $(1, 4)$ and $(1, -\frac{1}{4})$ are perpendicular.

10 Slopes $2/1$ and $-1/2$ multiply to give -1: then $\boldsymbol{v}\cdot\boldsymbol{w}=0$ and the vectors (the directions) are perpendicular.

11 $\boldsymbol{v}\cdot\boldsymbol{w}<0$ means angle $>90^\circ$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1,1)$ perpendicular to $(1,5)-c(1,1)$ if $(1,1)\cdot(1,5)-c(1,1)\cdot(1,1)=6-2c=0$ or $\boldsymbol{c=3}$; $\boldsymbol{v}\cdot(\boldsymbol{w}-c\boldsymbol{v})=0$ if $c=\boldsymbol{v}\cdot\boldsymbol{w}/\boldsymbol{v}\cdot\boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to constructing a perpendicular vector.

13 The plane perpendicular to $(1,0,1)$ contains all vectors $(c,d,-c)$. In that plane, $\boldsymbol{v}=(1,0,-1)$ and $\boldsymbol{w}=(0,1,0)$ are perpendicular.

14 One possibility among many: $\boldsymbol{u}=(1,-1,0,0)$, $\boldsymbol{v}=(0,0,1,-1)$, $\boldsymbol{w}=(1,1,-1,-1)$ and $(1,1,1,1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ in their 3D hyperplane and they will stay perpendicular."

15 $\frac{1}{2}(x+y)=(2+8)/2=5$ and $5>4$; $\cos\theta=2\sqrt{16}/\sqrt{10}\sqrt{10}=\boldsymbol{8/10}$.

16 $\|\boldsymbol{v}\|^2=1+1+\cdots+1=9$ so $\|\boldsymbol{v}\|=\boldsymbol{3}$; $\boldsymbol{u}=\boldsymbol{v}/3=(\frac{1}{3},\ldots,\frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w}=(1,-1,0,\ldots,0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

17 $\cos\alpha=1/\sqrt{2}$, $\cos\beta=0$, $\cos\gamma=-1/\sqrt{2}$. For any vector $\boldsymbol{v}=(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$ the cosines with $(1,0,0)$ and $(0,0,1)$ are $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=(v_1^2+v_2^2+v_3^2)/\|\boldsymbol{v}\|^2=1$.

18 $\|\boldsymbol{v}\|^2=4^2+2^2=20$ and $\|\boldsymbol{w}\|^2=(-1)^2+2^2=5$. Pythagoras is $\|(3,4)\|^2=25=20+5$ for the length of the hypotenuse $\boldsymbol{v}+\boldsymbol{w}=(3,4)$.

19 Start from the rules $(1),(2),(3)$ for $\boldsymbol{v}\cdot\boldsymbol{w}=\boldsymbol{w}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})$ and $(c\boldsymbol{v})\cdot\boldsymbol{w}$. Use rule (2) for $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{v}+(\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{w}$. By rule (1) this is $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w})+\boldsymbol{w}\cdot(\boldsymbol{v}+\boldsymbol{w})$. Rule (2) again gives $\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}=\boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}$. Notice $\boldsymbol{v}\cdot\boldsymbol{w}=\boldsymbol{w}\cdot\boldsymbol{v}$! The main point is to feel free to open up parentheses.

20 We know that $(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w})=\boldsymbol{v}\cdot\boldsymbol{v}-2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$ for $\boldsymbol{v}\cdot\boldsymbol{w}$. Here θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. When $\theta<90^\circ$ this $\boldsymbol{v}\cdot\boldsymbol{w}$ is positive, so in this case $\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}$ is larger than $\|\boldsymbol{v}-\boldsymbol{w}\|^2$.

Pythagoras changes from equality $a^2+b^2=c^2$ to *inequality* when $\theta<90^\circ$ or $\theta>90^\circ$.

21 $2\boldsymbol{v}\cdot\boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2+2\|\boldsymbol{v}\|\|\boldsymbol{w}\|+\|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\|+\|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\|+\|\boldsymbol{w}\|$.

22 $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta-a) = \cos\beta\cos\alpha+\sin\beta\sin\alpha = v_1w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta-\alpha=\theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2+U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2+U_2^2)$. The whole line becomes $.96 \le (.6)(.8)+(.8)(.6) \le \frac{1}{2}(.6^2+.8^2)+\frac{1}{2}(.8^2+.6^2) = 1$. True: $.96 < 1$.

25 The cosine of θ is $x/\sqrt{x^2+y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than 1: $x^2/(x^2+y^2) \le 1$.

26–27 (with apologies for that typo !) These two lines add to $2||\boldsymbol{v}||^2 + 2||\boldsymbol{w}||^2$:

$$||\boldsymbol{v}+\boldsymbol{w}||^2 = (\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}$$
$$||\boldsymbol{v}-\boldsymbol{w}||^2 = (\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v}-\boldsymbol{v}\cdot\boldsymbol{w}-\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}$$

28 The vectors $\boldsymbol{w}=(x,y)$ with $(1,2)\cdot\boldsymbol{w} = x+2y = 5$ lie on a line in the xy plane. The shortest $\boldsymbol{w}$ on that line is $(1,2)$. (The Schwarz inequality $\|\boldsymbol{w}\| \ge \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\| = \sqrt{5}$ is an equality when $\cos\theta = 0$ and $\boldsymbol{w}=(1,2)$ and $\|\boldsymbol{w}\| = \sqrt{5}$.)

29 The length $\|\boldsymbol{v}-\boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

30 Three vectors in the plane could make angles greater than $90°$ with each other: for example $(1,0),(-1,4),(-1,-4)$. Four vectors could *not* do this ($360°$ total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n+1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n+1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n+1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

31 For a specific example, pick $\boldsymbol{v}=(1,2,-3)$ and then $\boldsymbol{w}=(-3,1,2)$. In this example $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120°$. This always happens when $x+y+z=0$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = xz + xy + yz = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$$

This is the same as $\boldsymbol{v} \cdot \boldsymbol{w} = 0 - \dfrac{1}{2}\,\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\cos\theta = \dfrac{1}{2}$.

32 Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \le$ arithmetic mean $A = (x+y+z)/3$. First there is equality in case $x = y = z$. Otherwise A is somewhere between the three positive numbers, say for example $z < A < y$.

Use the known inequality $g \le a$ for the *two* positive numbers x and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A)$ = same as A! So $a \ge g$ says that $A^3 \ge g^2 A = x(y + z - A)A$. But $(y + z - A)A = (y - A)(A - z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!

There are many proofs of $G = (x_1 x_2 \cdots x_n)^{1/n} \le A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing G on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all x's are equal.

33 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

34 The commands $V = \textbf{randn}\,(3,30)$; $D = \textbf{sqrt}\,(\textbf{diag}\,(V' * V))$; $U = V \backslash D$; will give 30 random unit vectors in the columns of U. Then $\boldsymbol{u}' * U$ is a row matrix of 30 dot products whose average absolute value should be close to $2/\pi$.

Problem Set 1.3, page 29

1 $3\boldsymbol{s}_1 + 4\boldsymbol{s}_2 + 5\boldsymbol{s}_3 = (3, 7, 12)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (3, 4, 5)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}.$$

2 The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side $=$ column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

3 $$\begin{aligned} y_1 \qquad\quad &= B_1 \\ y_1 + y_2 \quad &= B_2 \\ y_1 + y_2 + y_3 &= B_3 \end{aligned} \quad \text{gives} \quad \begin{aligned} y_1 &= B_1 \\ y_2 &= -B_1 + B_2 \\ y_3 &= - B_2 + B_3 \end{aligned} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: **independent** columns in A and S!

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\boldsymbol{w}_1 - 2\boldsymbol{w}_2 + \boldsymbol{w}_3 = \mathbf{0}$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual. Two solutions to $y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + y_3\boldsymbol{r}_3 = \mathbf{0}$ are $(Y_1, Y_2, Y_3) = (1, -2, 1)$ and $(2, -4, 2)$.

6 $c = \mathbf{3}$ $\quad \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & \mathbf{3} \end{bmatrix}$ has column $3 =$ column $1 -$ column 2

$c = \mathbf{-1}$ $\quad \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has column $3 = -$ column $1 +$ column 2

$c = \mathbf{0}$ $\quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ has column $3 = 3$ (column 1) $-$ column 2

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

8
$$\begin{array}{rcl} x_1 - 0 &=& b_1 \\ x_2 - x_1 &=& b_2 \\ x_3 - x_2 &=& b_3 \\ x_4 - x_3 &=& b_4 \end{array} \qquad \begin{array}{rcl} x_1 &=& b_1 \\ x_2 &=& b_1 + b_2 \\ x_3 &=& b_1 + b_2 + b_3 \\ x_4 &=& b_1 + b_2 + b_3 + b_4 \end{array} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\boldsymbol{b}$$

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{ any constant vector.}$$

10
$$\begin{array}{rcl} z_2 - z_1 &=& b_1 \\ z_3 - z_2 &=& b_2 \\ 0 - z_3 &=& b_3 \end{array} \qquad \begin{array}{rcr} z_1 &=& -b_1 - b_2 - b_3 \\ z_2 &=& - b_2 - b_3 \\ z_3 &=& - b_3 \end{array} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}\boldsymbol{b}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even size* seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{array}{l} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \qquad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ - x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$

The left side of the sum is zero

The right side is $b_1 + b_3 + b_5$

There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. We are given that the ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d must also be equal!

Problem Set 2.1, page 41

1 The row picture for $A = I$ has 3 perpendicular planes $x = 2$ and $y = 3$ and $z = 4$. Those are perpendicular to the x and y and z axes: $z = 4$ is a horizontal plane at height 4.

The column vectors are $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{j} = (0,1,0)$ and $\boldsymbol{k} = (0,0,1)$. Then $\boldsymbol{b} = (2,3,4)$ is the linear combination $2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes in a row picture are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The three column vectors are changed; but the same combination (coefficients z, produces $\boldsymbol{b} = 34$), $(4,9,16)$.

3 The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

4 If $z = 2$ then $x + y = 0$ and $x - y = 2$ give the point $(x,y,z) = (1,-1,2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5,1,0)$. Halfway between those is $(3,0,1)$.

5 If x, y, z satisfy the first two equations they also satisfy the third equation $=$ sum of the first two. The line $\mathbf{L}$ of solutions contains $\boldsymbol{v} = (1,1,0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$. (Notice that requirement $c + d = 1$. If you allow all c and d, you get a plane.)

6 Equation $1 +$ equation $2 -$ equation 3 is now $0 = -4$. The intersection line L of planes 1 and 2 misses plane 3: *no solution*.

7 Column $3 =$ Column 1 makes the matrix singular. For $\boldsymbol{b} = (2,3,5)$ the solutions are $(x,y,z) = (1,1,0)$ or $(0,1,1)$ and you can add any multiple of $(-1,0,1)$. $\boldsymbol{b} = (4,6,c)$ needs $c = 10$ for solvability (then $\boldsymbol{b}$ lies in the plane of the columns and the three equations add to $0 = 0$).

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3,3,3,2)$ is $\boldsymbol{x} = (0,0,1,2)$ if A has columns $(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)$. The equations are $x+y+z+t = 3$, $y+z+t = 3$, $z+t = 3$, $t = 2$. Solve them in reverse order!

9 (a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

10 Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x} = (18, 5, 0)$ and $(3, 4, 5, 5)$. By rows or by columns: **9** separate multiplications when A is 3 by 3.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

12 $A\boldsymbol{x}$ equals (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.

13 (a) $\boldsymbol{x}$ has n components and $A\boldsymbol{x}$ has m components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in n-dimensional space. The columns of A are in m-dimensional space.

14 $2x+3y+z+5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = \begin{bmatrix} 2 & 3 & 1 & 5 \end{bmatrix}$: *one row*. The solutions (x, y, z, t) fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

15 (a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = "identity" (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ = "permutation"

16 90° rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 180° rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

17 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces $\begin{bmatrix} y \\ z \\ x \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Q is the *inverse* of P. Later we write $QP = I$ and $Q = P^{-1}$.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

19 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$ and $E^{-1}E\boldsymbol{v}$ recovers $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

20 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis. The vector $\boldsymbol{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ projects to $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $P_2P_1\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

21 $R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by $45°$. The columns of R are the results from rotating $(1,0)$ and $(0,1)$!

22 The dot product $A\boldsymbol{x} = [\,1 \;\; 4 \;\; 5\,]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$ is zero for points (x, y, z) on a plane in three dimensions. The 3 columns of A are one-dimensional vectors.

23 $A = [\,1 \;\; 2 \;\; ; \;\; 3 \;\; 4\,]$ and $\boldsymbol{x} = [\,5 \;\; -2\,]'$ or $[\,5 \;\; ; \;\; -2\,]$ and $\boldsymbol{b} = [\,1 \;\; 7\,]'$ or $[\,1 \;\; ; \;\; 7\,]$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as two zeros.

24 $A * \boldsymbol{v} = [\,3 \;\; 4 \;\; 5\,]'$ and $\boldsymbol{v}' * \boldsymbol{v} = 50$. But $\boldsymbol{v} * A$ gives an error message from 3 by 1 times 3 by 3.

25 **ones**$(4,4)$ $*$ **ones**$(4,1)$ $=$ column vector $[\,4 \;\; 4 \;\; 4 \;\; 4\,]'$; $B * \boldsymbol{w} = [\,10 \;\; 10 \;\; 10 \;\; 10\,]'$.

26 The row picture has two lines meeting at the solution $(4, 2)$. The column picture will have $4(1, 1) + 2(-2, 1) = \mathbf{4}(\text{column } 1) + \mathbf{2}(\text{column } 2) = \text{right side } (0, 6)$.

27 The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally fill a ***line in* 3*-dimensional space***.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components add to 1. They are always positive. Their components still add to 1.

30 $\boldsymbol{u}_7$ and $\boldsymbol{v}_7$ have components adding to 1; they are close to $\boldsymbol{s} = (.6, .4)$. $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} =$ *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15, 15, 15)$; $M_4(1,1,1,1) = (34, 34, 34, 34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5,7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$. **Linearity** means: When w is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 53

1 Multiply equation 1 by $\ell_{21} = \frac{10}{2} = \mathbf{5}$ and subtract from equation 2 to find $2x + 3y = 1$ (unchanged) and $-6y = 6$. The pivots to circle are 2 and -6.

2 $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$. Multiplying the right side $(1, 11)$ by 4 will multiply the solution by 4 to give the new solution $(x, y) = (8, -4)$.

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $\mathbf{3y = 3}$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1 from equation 2. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$. Notice the "determinant of A" $= ad - bc$. It must be nonzero for this division.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x+2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. The two lines in the row picture are the same line, containing all solutions.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines $2x + 4y = 16$ and $4x + 8y = 32$ become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line in the row picture). The column picture has both columns along the same line.

10 The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = 20 - 4 = c = 16$.

11 (a) Another solution is $\frac{1}{2}(x+X, y+Y, z+Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to this upper triangular system; then comes back substitution.

$$\begin{array}{rcl} 2x + 3y + z = 8 & & x = 2 \\ y + 3z = 4 & \text{gives} & y = 1 \quad \text{If a zero is at the start of row 2 or row 3,} \\ 8z = 8 & & z = 1 \quad \text{that avoids a row operation.} \end{array}$$

13
$$\begin{array}{rcrcrcr} 2x - 3y \phantom{{}+z} = 3 & & 2x - 3y = 3 & & 2x - 3y = 3 & & x = 3 \\ 4x - 5y + z = 7 & \text{gives} & y + z = 1 & \text{and} & y + z = 1 & \text{and} & y = 1 \\ 2x - y - 3z = 5 & & 2y + 3z = 2 & & -5z = 0 & & z = 0 \end{array}$$

Here are steps $1, 2, 3$: Subtract $2 \times$ row 1 from row 2, subtract $1 \times$ row 1 from row 3, subtract $2 \times$ row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y-z=2$. Equation (3) is $y-z=3$. If $d=10$ exchange rows 2 and 3. If $d=11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. But equation (3) is the same so there is a *line of solutions* $(x,y,z)=(1,1,-1)$.

16 (a) **Example of 2 exchanges**
$$\begin{aligned} 0x+0y+2z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$
(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then breakdown**
$$\begin{aligned} 0x+3y+4z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$
(rows 1 and 3 are not consistent)

17 If row 1 $=$ row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and row 3 has no pivot. If column 2 $=$ column 1, then column 2 has no pivot.

18 *Example* $x+2y+3z=0$, $4x+8y+12z=0$, $5x+10y+15z=0$ has 9 different coefficients but rows 2 and 3 become $0=0$: infinitely many solutions to $A\boldsymbol{x}=\boldsymbol{0}$ but almost surely no solution to $A\boldsymbol{x}=\boldsymbol{b}$ for a random $\boldsymbol{b}$.

19 Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z=t-5$. If $q=-4$ the system is singular—no third pivot. Then if $t=5$ the third equation is $0=0$ which allows infinitely many solutions. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows 1+2 =row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution. The three planes in the row picture form a triangular tunnel.

21 (a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x+y=0$, $\frac{3}{2}y+z=0$, $\frac{4}{3}z+t=0$, $\frac{5}{4}t=5$ after elimination. Back substitution gives $t=4, z=-3, y=2, x=-1$. (b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1,2,3,4)$ instead of $(-1,2,-3,4)$.

22 The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.

23 If ordinary elimination leads to $x+y=1$ and $2y=3$, the original second equation could be $2y+\ell(x+y)=3+\ell$ for any ℓ. Then ℓ will be the multiplier to reach $2y=3$, by subtracting ℓ times equation 1 from equation 2.

24 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a=2$ or $a=0$. (You could notice that the determinant a^2-2a is zero for $a=2$ and $a=0$.)

25 $a=2$ (equal columns), $a=4$ (equal rows), $a=0$ (zero column).

26 Solvable for $s=10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2+s$ on the right sides). So 12 must agree with $2+s$, which makes $s=10$. The four equations for a,b,c,d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$,

$$A=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } U=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

27 Elimination leaves the diagonal matrix diag$(3,2,1)$ in $3x=3, 2y=2, z=2$. Then $x=1, y=1, z=2$.

28 $A(2,:)=A(2,:)-3*A(1,:)$ subtracts 3 times row 1 from row 2.

29 The average pivots for rand(3) *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment— but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite ! *With row exchanges* in MATLAB's **lu** code, the averages $.75$ and $.50$ and $.365$ are much more stable (and should be predictable, also for **randn** with normal instead of uniform probability distribution for the numbers in A).

30 If $A(5,5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1,\ldots,j$ of A (when there are no row exchanges). If $A\boldsymbol{x}=\mathbf{0}$ then $U\boldsymbol{x}=\mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U just keeps the diagonal of A when A is *lower triangular.*

32 The question deals with 100 equations $A\boldsymbol{x}=\mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A =$ **ones** (100). A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through** $\mathbf{0}$. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 2.3, page 66

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}\boldsymbol{b} = (1, -5, -35)$ but $E_{21}E_{32}\boldsymbol{b} = (1, -5, 0)$. When E_{32} comes first, row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

Those E's are in the right order to give $MA = U$.

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b}$ has become $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$. Then back substitution gives $z = -5, y = \frac{1}{2}, x = \frac{1}{2}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 Example: $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31}, **add** 7 times row **1** to row **3**. The inverse of the elimination matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{-7} & 0 & 1 \end{bmatrix}$ is $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{7} & 0 & 1 \end{bmatrix}$. Multiplication confirms $EE^{-1} = I$.

8 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$. $\det M* = a(d - \ell b) - b(c - \ell a)$ reduces to $ad - bc$! Subtracting row 1 from row 2 doesn't change $\det M$.

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

11 An example with two negative pivots is $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. The diagonal entries can change sign during elimination.

12 The first product is $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

9 (a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

10 Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x} = (18, 5, 0)$ and $(3, 4, 5, 5)$. By rows or by columns: **9** separate multiplications when A is 3 by 3.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

12 $A\boldsymbol{x}$ equals (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.

13 (a) $\boldsymbol{x}$ has n components and $A\boldsymbol{x}$ has m components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in n-dimensional space. The columns of A are in m-dimensional space.

14 $2x+3y+z+5t=8$ is $A\boldsymbol{x}=\boldsymbol{b}$ with the 1 by 4 matrix $A=\begin{bmatrix}2 & 3 & 1 & 5\end{bmatrix}$: *one row*. The solutions (x,y,z,t) fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

15 (a) $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=$ "identity" (b) $P=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=$ "permutation"

16 $90°$ rotation from $R=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $180°$ rotation from $R^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=-I$.

17 $P=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$ produces $\begin{bmatrix}y\\ z\\ x\end{bmatrix}$ and $Q=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$ recovers $\begin{bmatrix}x\\ y\\ z\end{bmatrix}$. Q is the *inverse* of P. Later we write $QP=I$ and $Q=P^{-1}$.

18 $E=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$ and $E=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ subtract the first component from the second.

19 $E=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$ and $E^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$, $E\boldsymbol{v}=\begin{bmatrix}3\\ 4\\ 8\end{bmatrix}$ and $E^{-1}E\boldsymbol{v}$ recovers $\begin{bmatrix}3\\ 4\\ 5\end{bmatrix}$.

20 $P_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ projects onto the x-axis and $P_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ projects onto the y-axis.

The vector $\boldsymbol{v}=\begin{bmatrix}5\\ 7\end{bmatrix}$ projects to $P_1\boldsymbol{v}=\begin{bmatrix}5\\ 0\end{bmatrix}$ and $P_2P_1\boldsymbol{v}=\begin{bmatrix}0\\ 0\end{bmatrix}$.

21 $R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by $45°$. The columns of R are the results from rotating $(1, 0)$ and $(0, 1)$!

22 The dot product $A\boldsymbol{x} = [\,1 \;\; 4 \;\; 5\,]\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (1 by 3)(3 by 1) is zero for points (x, y, z) on a plane in three dimensions. The 3 columns of A are one-dimensional vectors.

23 $A = [\,1 \;\; 2 \;\; ; \;\; 3 \;\; 4\,]$ and $\boldsymbol{x} = [\,5 \;\; -2\,]'$ or $[\,5 \;\; ; \;\; -2\,]$ and $\boldsymbol{b} = [\,1 \;\; 7\,]'$ or $[\,1 \;\; ; \;\; 7\,]$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as two zeros.

24 $A * \boldsymbol{v} = [\,3 \;\; 4 \;\; 5\,]'$ and $\boldsymbol{v}' * \boldsymbol{v} = 50$. But $\boldsymbol{v} * A$ gives an error message from 3 by 1 times 3 by 3.

25 **ones**$(4, 4) *$ **ones**$(4, 1) =$ column vector $[\,4 \;\; 4 \;\; 4 \;\; 4\,]'$; $B * \boldsymbol{w} = [\,10 \;\; 10 \;\; 10 \;\; 10\,]'$.

26 The row picture has two lines meeting at the solution $(4, 2)$. The column picture will have $4(1, 1) + 2(-2, 1) =$ **4**(column 1) + **2**(column 2) = right side $(0, 6)$.

27 The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally fill a ***line in*** 3-***dimensional space***.

28 The row picture shov e 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components add to 1. They are always positive. Their components still add to 1.

30 $\boldsymbol{u}_7$ and $\boldsymbol{v}_7$ have components adding to 1; they are close to $\boldsymbol{s} = (.6, .4)$. $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ = *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15, 15, 15)$; $M_4(1,1,1,1) = (34, 34, 34, 34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$. **Linearity** means : When w is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 53

1 Multiply equation 1 by $\ell_{21} = \frac{10}{2} = \mathbf{5}$ and subtract from equation 2 to find $2x + 3y = 1$ (unchanged) and $-6y = 6$. The pivots to circle are 2 and -6.

2 $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$. Multiplying the right side $(1, 11)$ by 4 will multiply the solution by 4 to give the new solution $(x, y) = (8, -4)$.

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $\mathbf{3y = 3}$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1 from equation 2. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$. Notice the "determinant of A" $= ad - bc$. It m this division.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$.

Then all the points on the line $3x+2y=10$ are solutions, including $(0,5)$ and $(4,-1)$. The two lines in the row picture are the same line, containing all solutions.

6 Singular system if $b=4$, because $4x+8y$ is 2 times $2x+4y$. Then $g=32$ makes the lines $2x+4y=16$ and $4x+8y=32$ become the *same*: infinitely many solutions like $(8,0)$ and $(0,4)$.

7 If $a=2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a=0$, elimination will stop for a row exchange. Then $3y=-3$ gives $y=-1$ and $4x+6y=6$ gives $x=3$.

8 If $k=3$ elimination must fail: no solution. If $k=-3$, elimination gives $0=0$ in equation 2: infinitely many solutions. If $k=0$ a row exchange is needed: one solution.

9 On the left side, $6x-4y$ is 2 times $(3x-2y)$. Therefore we need $b_2=2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line in the row picture). The column picture has both columns along the same line.

10 The equation $y=1$ comes from elimination (subtract $x+y=5$ from $x+2y=6$). Then $x=4$ and $5x-4y=20-4=c=16$.

11 (a) Another solution is $\frac{1}{2}(x+X, y+Y, z+Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to this upper triangular system; then comes back substitution.

$$\begin{aligned} 2x+3y+z&=8 \\ y+3z&=4 \\ 8z&=8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x&=2 \\ y&=1 \\ z&=1 \end{aligned} \quad \begin{array}{l} \\ \text{If a zero is at the start of row 2 or row 3,} \\ \text{that avoids a row operation.} \end{array}$$

13
$$\begin{aligned} 2x-3y&=3 \\ 4x-5y+z&=7 \\ 2x-y-3z&=5 \end{aligned} \quad \text{gives} \quad \begin{aligned} 2x-3y&=3 \\ y+z&=1 \\ 2y+3z&=2 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x-3y&=3 \\ y+z&=1 \\ -5z&=0 \end{aligned} \quad \text{and} \quad \begin{aligned} x&=3 \\ y&=1 \\ z&=0 \end{aligned}$$

Here are steps $1,2,3$: Subtract $2\times$ row 1 from row 2, subtract $1\times$ row 1 from row 3, subtract $2\times$ row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y-z=2$. Equation (3) is $y-z=3$. If $d=10$ exchange rows 2 and 3. If $d=11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. But equation (3) is the same so there is a *line of solutions* $(x,y,z)=(1,1,-1)$.

16 (a) **Example of 2 exchanges**

$$\begin{aligned} 0x+0y+2z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$

(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then breakdown**

$$\begin{aligned} 0x+3y+4z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$

(rows 1 and 3 are not consistent)

17 If row $1=$ row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and row 3 has no piv column 1, then column 2 has no pivot.

18 *Example* $x+2y+3z=0$, $4x+8y+12z=0$, $5x+10y+15z=0$ has 9 different coefficients but rows 2 and 3 become $0 = 0$: infinitely many solutions to $A\boldsymbol{x} = \mathbf{0}$ but almost surely no solution to $A\boldsymbol{x} = \boldsymbol{b}$ for a random $\boldsymbol{b}$.

19 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$ which allows infinitely many solutions. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2=$row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution. The three planes in the row picture form a triangular tunnel.

21 (a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x + y = 0$, $\frac{3}{2}y + z = 0$, $\frac{4}{3}z + t = 0$, $\frac{5}{4}t = 5$ after elimination. Back substitution gives $t = 4, z = -3, y = 2, x = -1$. (b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1, 2, 3, 4)$ instead of $(-1, 2, -3, 4)$.

22 The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.

23 If ordinary elimination leads to $x + y = 1$ and $2y = 3$, the original second equation could be $2y + \ell(x + y) = 3 + \ell$ for any ℓ. Then ℓ will be the multiplier to reach $2y = 3$, by subtracting ℓ times equation 1 from equation 2.

24 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$. (You could notice that the determinant $a^2 - 2a$ is zero for $a = 2$ and $a = 0$.)

25 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

26 Solvable for $s = 10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2 + s$ on the right sides). So 12 must agree with $2 + s$, which makes $s = 10$. The four equations for a, b, c, d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

27 Elimination leaves the diagonal matrix $\mathrm{diag}(3, 2, 1)$ in $3x = 3, 2y = 2, z = 2$. Then $x = 1, y = 1, z = 2$.

28 $A(2, :) = A(2, :) - 3 * A(1, :)$ subtracts 3 times row 1 from row 2.

29 The average pivots for rand(3) *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite ! *With row exchanges* in MATLAB's **lu** code, the averages .75 and .50 and .365 are much more stable (and shou[illegible]lso for **randn** with normal instead of uniform probability distribution for the numbers in A).

30 If $A(5,5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1, \ldots, j$ of A (when there are no row exchanges). If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U just keeps the diagonal of A when A is *lower triangular.*

32 The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A =$ **ones** (100). A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through 0**. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 2.3, page 66

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}\boldsymbol{b} = (1, -5, -35)$ but $E_{21}E_{32}\boldsymbol{b} = (1, -5, 0)$. When E_{32} comes first, row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.
Those E's are in the right order to give $MA = U$.

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b}$ has become $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$. Then back substitution gives $\boldsymbol{z} = -5, y = \frac{1}{2}, x = \frac{1}{2}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 Example: $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31}, **add** 7 times row **1** to row **3**. The inverse of the elimination matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\mathbf{7} & 0 & 1 \end{bmatrix} \text{ is } E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{7} & 0 & 1 \end{bmatrix}. \text{ Multiplication confirms } EE^{-1} = I.$$

8 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$. det $M* = a(d - \ell b) - b(c - \ell a)$ reduces to $ad - bc$! Subtracting row 1 from row 2 doesn't change det M.

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

11 An example with two negative pivots is $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. The diagonal entries can change sign during elimination.

12 The first product is $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

13 (a) E times the third column of B is the third column of EB. A column that starts at zero will stay at zero. (b) E could add row 2 to row 3 to change a zero row to a nonzero row.

14 E_{21} has $-\ell_{21}=\frac{1}{2}$, E_{32} has $-\ell_{32}=\frac{2}{3}$, E_{43} has $-\ell_{43}=\frac{3}{4}$. Otherwise the E's match I.

15 $a_{ij} = 2i - 3j$: $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & \mathbf{0} & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & \mathbf{-12} & -24 \end{bmatrix}$. The zero became -12, an example of *fill-in*. To remove that -12, choose $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

Every 3 by 3 matrix with entries $a_{ij} = ci + dj$ is **singular** !

16 (a) The ages of X and Y are x and y: $x - 2y = 0$ and $x + y = 33$; $x = 22$ and $y = 11$ (b) The line $y = mx + c$ contains $x = 2$, $y = 5$ and $x = 3$, $y = 7$ when $2m + c = 5$ and $3m + c = 7$. Then $m = 2$ is the slope.

17 The parabola $y = a + bx + cx^2$ goes through the 3 given points when $\begin{aligned} a + \ b + \ c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$.

Then $a = 2$, $b = 1$, and $c = 1$. This matrix with columns $(1, 1, 1)$, $(1, 2, 3)$, $(1, 4, 9)$ is a "Vandermonde matrix."

18 $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

19 $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. In the opposite order, two row exchanges give $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = I$. If M exchanges rows 2 and 3 then $M^2 = I$ (also $(-M)^2 = I$). There are many square roots of I: Any matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ has $M^2 = I$ if $a^2 + bc = 1$.

20 (a) Each column of EB is E times a column of B (b) $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{4}\\\mathbf{1} & \mathbf{2} & \mathbf{4}\end{bmatrix} =$ $\begin{bmatrix}1 & 2 & 4\\\mathbf{2} & \mathbf{4} & \mathbf{8}\end{bmatrix}$. All rows of EB are *multiples* of $\begin{bmatrix}1 & 2 & 4\end{bmatrix}$.

21 **No.** $E = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ and $F = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ give $EF = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ but $FE = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21} - a_{11}$ (c) $a_{21} - 2a_{11}$ (d) $(EA\boldsymbol{x})_1 = (A\boldsymbol{x})_1 = \sum a_{1j}x_j$.

23 $E(EA)$ subtracts 4 times row 1 from row 2 (EEA does the row operation twice). AE subtracts 2 times column 2 of A from column 1 (multiplication by E on the right side acts on **columns** instead of rows).

24 $\begin{bmatrix}A & \boldsymbol{b}\end{bmatrix} = \begin{bmatrix}2 & 3 & \mathbf{1}\\4 & 1 & \mathbf{17}\end{bmatrix} \to \begin{bmatrix}2 & 3 & \mathbf{1}\\0 & -5 & \mathbf{15}\end{bmatrix}$. The triangular system is $\begin{aligned}2x_1 + 3x_2 &= 1\\-5x_2 &= 15\end{aligned}$

Back substitution gives $x_1 = 5$ and $x_2 = -3$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3. Then the last equation is $0 = 0$ and the system has infinitely many solutions.

26 (a) Add two columns $\boldsymbol{b}$ and $\boldsymbol{b}^*$ to get $[A \;\; \boldsymbol{b} \;\; \boldsymbol{b}^*]$. The example has

$$\begin{bmatrix}1 & 4 & 1 & 0\\2 & 7 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 4 & 1 & 0\\0 & -1 & -2 & 1\end{bmatrix} \to \boldsymbol{x} = \begin{bmatrix}-7\\2\end{bmatrix} \text{ and } \boldsymbol{x}^* = \begin{bmatrix}4\\-1\end{bmatrix}.$$

27 (a) No solution if $d=0$ and $c\neq 0$ (b) Many solutions if $d=0=c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

29 $E = \begin{bmatrix}1 & 0 & 0 & 0\\-1 & 1 & 0 & 0\\0 & -1 & 1 & 0\\0 & 0 & -1 & 1\end{bmatrix}$ subtracts each row from the next row. The result $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 1 & 1 & 0\\0 & 1 & 2 & 1\end{bmatrix}$ still has multipliers $= 1$ in a 3 by 3 Pascal matrix. The product M of all elimination

matrices is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$. This "alternating sign Pascal matrix" is on page 91.

30 (a) $E = A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ will reduce row 2 of EM to $[2 \ \ 3]$.

(b) Then $F = B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ will reduce row 1 of FEM to $[1 \ \ 1]$.

(c) Then $E = A^{-1}$ twice will reduce row 2 of $EEFEM$ to $[0 \ \ 1]$

(d) Now $EEFEM = B$. Move E's and F's to get $\boldsymbol{M} = \boldsymbol{ABAAB}$. This question focuses on positive integer matrices M with $ad - bc = 1$. The same steps make the entries smaller and smaller until M is a product of A's and B's.

31 $E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & b & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & c & 1 \end{bmatrix}$,

$$E_{43}\,E_{32}\,E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ ab & b & 1 & \\ abc & bc & c & 1 \end{bmatrix}$$

Problem Set 2.4, page 77

1 If all entries of A, B, C, D are 1, then $BA = 3$ **ones**(5) is 5 by 5; $AB = 5$ **ones**(3) is 3 by 3; $ABD = 15$ **ones**$(3, 1)$ is 3 by 1. DC and $A(B + C)$ are not defined.

2 (a) A (column 2 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 5 of B)
(d) (Row 1 of C)D(column 1 of E). (Part (c) assumed 5 columns in B)

3 $AB + AC$ is the same as $A(B+C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$. (*Distributive law*).

4 $A(BC) = (AB)C$ by the *associative law*. In this example both answers are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
Column 1 of AB and row 2 of C are zero (then multiply columns times rows).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

6 $(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

14 The "reverse identity" P takes $(1,\ldots,n)$ into $(n,\ldots,1)$. When rows and also columns are reversed, the $1,1$ and n,n entries of A change places in PAP. So do the $1,n$ and $n,1$ entries. In general $(PAP)_{ij}$ is $(A)_{n-i+1,n-j+1}$.

15 (a) If P sends row 1 to row 4, then P^{T} sends row 4 to row 1 (b) $P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = P^{\mathrm{T}}$ with $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ moves all rows: 1 and 2 are exchanged, 3 and 4 are exchanged.

16 $A^2 - B^2$ and also ABA are symmetric if A and B are symmetric. But $(A+B)(A-B)$ and $ABAB$ are generally *not* symmetric.

17 (a) $S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = S^{\mathrm{T}}$ is not invertible (b) $S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ needs row exchange (c) $S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has pivots $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$: no real square root.

18 (a) $5+4+3+2+1 = 15$ independent entries if $S = S^{\mathrm{T}}$ (b) L has 10 and D has 5; total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

19 (a) The transpose of $A^{\mathrm{T}}SA$ is $A^{\mathrm{T}}S^{\mathrm{T}}A^{\mathrm{T}\,\mathrm{T}} = A^{\mathrm{T}}SA = n$ by n when $S^{\mathrm{T}} = S$ (any m by n matrix A) (b) $(A^{\mathrm{T}}A)_{jj} =$ (column j of A)$\cdot$ (column j of A) $=$ (length squared of column j) ≥ 0.

20 $$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = \boldsymbol{LDL}^{\mathrm{T}}.$$

21 Elimination on a symmetric 3 by 3 matrix leaves a symmetric lower right 2 by 2 matrix.
$$\begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \text{ lead to } \begin{bmatrix} -5 & -7 \\ -7 & -32 \end{bmatrix} \text{ and } \begin{bmatrix} d-b^2 & e-bc \\ e-bc & f-c^2 \end{bmatrix} : \text{symmetric!}$$

22 $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}; \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$

23 $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P$ and $L = U = I$. Elimination on this $A = P$ exchanges rows 1-2 then rows 2-3 then rows 3-4.

24 $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and a_{12} is the pivot, $A = L_1 P_1 U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

25 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

26 (a) $E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}$ puts 0 in the 2, 1 entry of $E_{21}A$. Then $E_{21}AE_{21}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ is still symmetric, with zero also in its 1, 2 entry. (b) Now use $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$ to make the 3, 2 entry zero and $E_{32}E_{21}AE_{21}^{\mathrm{T}}E_{32}^{\mathrm{T}} = D$ also has zero in its 2, 3 entry. Key point: Elimination from both sides (rows + columns) gives the symmetric LDL^{T}.

27 $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix} = A^{\mathrm{T}}$ has $0, 1, 2, 3$ in every row. I don't know any rules for a symmetric construction like this "Hankel matrix" with constant antidiagonals.

28 Reordering the rows and/or the columns of $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ will move the entry **a**. So the result cannot be the transpose (which doesn't move **a**).

29 (a) Total currents are $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$.

(b) Either way $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}$. Six terms.

30 $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x}$; $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix} \begin{array}{l} \text{1 truck} \\ \text{1 plane} \end{array}$

31 $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the *cost of inputs* while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the *value of outputs*.

32 $P^3 = I$ so three rotations for $360°$; P rotates every $\boldsymbol{v}$ around the $(1,1,1)$ line by $120°$.

33 $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \boldsymbol{EH}$ = (elementary matrix) times (symmetric matrix).

34 $L(U^{\mathrm{T}})^{-1}$ is lower triangular times lower triangular, so *lower triangular*. The transpose of $U^{\mathrm{T}}DU$ is $U^{\mathrm{T}}D^{\mathrm{T}}U^{\mathrm{T}\,\mathrm{T}} = U^{\mathrm{T}}DU$ again, so $U^{\mathrm{T}}DU$ is *symmetric*. The factorization multiplies lower triangular by symmetric to get LDU which is A.

35 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

36 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest $B = PL$ times southeast PU is $(PLP)U$ = upper triangular.

37 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same permutation*. And if $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^6 = I$.

38 To split the matrix M into (symmetric S) + (anti-symmetric A), the only choice is $S = \frac{1}{2}(M + M^{\mathrm{T}})$ and $A = \frac{1}{2}(M - M^{\mathrm{T}})$.

39 Start from $Q^{\mathrm{T}}Q = I$, as in $\begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 = 1$: *unit vectors*

(b) The off-diagonal entry is $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ (and in general $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$)

(c) The leading example for Q is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Problem Set 3.1, page 131

Note An interesting "max-plus" vector space comes from the real numbers $\mathbf{R}$ combined with $-\infty$. Change addition to give $x + y = \mathbf{max}(\boldsymbol{x}, \boldsymbol{y})$ and change multiplication to $xy = \textbf{usual } \boldsymbol{x} + \boldsymbol{y}$. Which y is the zero vector that gives $x + \mathbf{0} = \max(x, \mathbf{0}) = x$ for every x?

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$. Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

4 The zero vector in matrix space $\mathbf{M}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. The smallest subspace of $\mathbf{M}$ containing the matrix A consists of all matrices cA.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

6 When $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$, the combination $3\boldsymbol{f} - 4\boldsymbol{g}$ in function space is $\boldsymbol{h}(x) = 3\boldsymbol{f}(x) - 4\boldsymbol{g}(x) = 3x^2 - 20x$.

7 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $\boldsymbol{f}(cx)$ then $(c_1 + c_2)\boldsymbol{f} = \boldsymbol{f}((c_1 + c_2)x)$ is not generally the same as $c_1\boldsymbol{f} + c_2\boldsymbol{f} = \boldsymbol{f}(c_1 x) + \boldsymbol{f}(c_2 x)$.

8 If $(\boldsymbol{f} + \boldsymbol{g})(x)$ is the usual $\boldsymbol{f}(\boldsymbol{g}(x))$ then $(\boldsymbol{g} + \boldsymbol{f})x$ is $\boldsymbol{g}(\boldsymbol{f}(x))$ which is different. In Rule 2 both sides are $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{h}(x)))$. Rule 4 is broken because there might be no inverse function $\boldsymbol{f}^{-1}(x)$ such that $\boldsymbol{f}(\boldsymbol{f}^{-1}(x)) = x$. If the inverse function exists it will be the vector $-\boldsymbol{f}$.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions: $(1, 1) + (-1, 1) = (0, 2)$ is removed.

10 The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) the plane with $b_1 + b_2 + b_3 = 0$.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

12 For the plane $x + y - 2z = 4$, the sum of $(4, 0, 0)$ and $(0, 4, 0)$ is not on the plane. (The key is that this plane does not go through $(0, 0, 0)$.)

13 The parallel plane $\mathbf{P}_0$ has the equation $x + y - 2z = 0$. Pick two points, for example $(2, 0, 1)$ and $(0, 2, 1)$, and their sum $(2, 2, 2)$ is in $\mathbf{P}_0$.

14 (a) The subspaces of $\mathbf{R}^2$ are $\mathbf{R}^2$ itself, lines through $(0, 0)$, and $(0, 0)$ by itself (b) The subspaces of $\mathbf{D}$ are $\mathbf{D}$ itself, the zero matrix by itself, and all the "one-dimensional" subspaces that contain all multiples of one fixed matrix :

$$c \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \quad \text{for all} \quad c.$$

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$

(b) The plane and line probably intersect in the point $(0, 0, 0)$. *Could be a line* !

(c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

16 The smallest subspace containing a plane $\mathbf{P}$ and a line $\mathbf{L}$ is *either* $\mathbf{P}$ (when the line $\mathbf{L}$ is in the plane $\mathbf{P}$) *or* $\mathbf{R}^3$ (when $\mathbf{L}$ is not in $\mathbf{P}$).

17 (a) The invertible matrices do not include the zero matrix, so they are not a subspace

(b) The sum of singular matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular: not a subspace.

18 (a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with $A^{\mathrm{T}} = -A$ do form a subspace (c) *False*: The sum of two unsymmetric matrices could be symmetric.

19 The column space of A is the x-axis $=$ all vectors $(x, 0, 0)$: a *line*. The column space of B is the xy plane $=$ all vectors $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

20 (a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + b_3$ in equation 3: Solution only if $b_3 = -b_1$.

21 A combination of the columns of C is also a combination of the columns of A. Then $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have the same column space. $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has a different column space. The key word is "space".

22 (a) Solution for every $\boldsymbol{b}$ (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$$[A \ \ \boldsymbol{b}] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{matrix} \text{(larger column space)} \\ \text{(no solution to } A\boldsymbol{x} = \boldsymbol{b}\text{)} \end{matrix} \quad \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix} \begin{matrix} (\boldsymbol{b} \text{ is in column space}) \\ (A\boldsymbol{x} = \boldsymbol{b} \text{ has a solution}) \end{matrix}$$

24 The column space of AB is *contained in* (possibly equal to) the column space of A. The example $B =$ zero matrix and $A \neq 0$ is a case when $AB =$ zero matrix has a smaller column space (it is just the zero space $\mathbf{Z}$) than A.

25 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

26 The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

27 (a) *False*: Vectors that are *not* in a column space don't form a subspace. (b) *True*: Only the zero matrix has $\boldsymbol{C}(A) = \{\mathbf{0}\}$. (c) *True*: $\boldsymbol{C}(A) = \boldsymbol{C}(2A)$. (d) *False*: $\boldsymbol{C}(A - I) \neq \boldsymbol{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

28 $A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\boldsymbol{C}(A) =$ line in $\mathbf{R}^3$.

29 When $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for all $\boldsymbol{b}$, every $\boldsymbol{b}$ is in the column space of A. So that space is $\boldsymbol{C}(A) = \mathbf{R}^9$.

30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S}+\boldsymbol{T}$, then $\boldsymbol{u}=\boldsymbol{s}_1+\boldsymbol{t}_1$ and $\boldsymbol{v}=\boldsymbol{s}_2+\boldsymbol{t}_2$. So $\boldsymbol{u}+\boldsymbol{v}=(\boldsymbol{s}_1+\boldsymbol{s}_2)+(\boldsymbol{t}_1+\boldsymbol{t}_2)$ is also in $\boldsymbol{S}+\boldsymbol{T}$. And so is $c\boldsymbol{u}=c\boldsymbol{s}_1+c\boldsymbol{t}_1$: $\boldsymbol{S}+\boldsymbol{T}=$ *subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S}\cup\boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S}+\boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S}=\boldsymbol{C}(A)$ and $\boldsymbol{T}=\boldsymbol{C}(B)$ then $\boldsymbol{S}+\boldsymbol{T}$ is the column space of $M=[\,A\quad B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A\quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ has a larger column space than $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ exactly when A is *invertible*.

Problem Set 3.2, page 142

1 (a) $U=\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$ Free variables x_2, x_4, x_5 Pivot variables x_1, x_3 (b) $U=\begin{bmatrix}2&4&2\\0&4&4\\0&0&0\end{bmatrix}$ Free x_3 Pivot x_1, x_2

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2,1,0,0,0)$, $(0,0,-2,1,0)$, $(0,0,-3,0,1)$

(b) Free variable x_3: solution $(1,-1,1)$. Special solution for each free variable.

3 $R=\begin{bmatrix}1&2&0&0&0\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$, $R=\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$, R has the same nullspace as U and A.

4 (a) Special solutions $(3,1,0)$ and $(5,0,1)$ (b) $(3,1,0)$. **Total of pivot and free is $\boldsymbol{n}$.**

5 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

6 $$\begin{bmatrix}0&\mathbf{1}&1&1&1&1&1\\0&0&0&\mathbf{1}&1&1&1\\0&0&0&0&\mathbf{1}&1&1\\0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\mathbf{1}&1&1&1&1&1&1\\0&0&\mathbf{1}&1&1&1&1\\0&0&0&0&0&\mathbf{1}&1\\0&0&0&0&0&0&\mathbf{1}\end{bmatrix}\begin{bmatrix}0&0&0&\mathbf{1}&1&1&1\\0&0&0&0&0&\mathbf{1}&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$$

7 $\begin{bmatrix} \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Notice the identity matrix in the pivot columns of these *reduced* row echelon forms R.

8 If column 4 of a 3 by 5 matrix is all zero then x_4 is a *free* variable. Its special solution is $\boldsymbol{x} = (0, 0, 0, 1, 0)$, because 1 will multiply that zero column to give $A\boldsymbol{x} = \mathbf{0}$.

9 If column $1 =$ column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

10 If a matrix has n columns and r pivots, there are $\boldsymbol{n - r}$ special solutions. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $\boldsymbol{r = n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All those statements are important!

11 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

12 $A = [1 \quad -3 \quad -1]$ gives the plane $x - 3y - z = 0$; y and z are free variables. The special solutions are $(3, 1, 0)$ and $(1, 0, 1)$.

13 Fill in **12** then **3** then **1** to get the complete solution in $\mathbf{R}^3$ to $x - 3y - z = 12$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix} = \text{one particular solution} + \text{all nullspace solutions.}$$

14 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (this nullspace is a line in $\mathbf{R}^5$).

15 To produce special solutions $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$ with free variables x_3, x_4: $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$ and A can be any invertible 2 by 2 matrix times this R.

16 The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is the line through the special solution $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

17 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in $\boldsymbol{N}(A)$. *Which other* A's?

18 This construction is impossible for 3 by 3 ! 2 pivot columns and 2 free variables.

19 $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ has $(1, 1, 1)$ in $\boldsymbol{C}(A)$ and only the line (c, c, c, c) in $\boldsymbol{N}(A)$.

20 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$. Notice that $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not A^{T}.

21 If nullspace $=$ column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

22 If A times every column of B is zero, the column space of B is contained in the *nullspace* of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\boldsymbol{C}(B)$ *equals* $\boldsymbol{N}(A)$. For $B = 0$, $\boldsymbol{C}(B)$ is smaller than $\boldsymbol{N}(A)$.

23 For $A =$ random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with a fourth row of zeros. What is R for a random 3 by 4 matrix?

24 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows that (a)(b)(c) are all false. Notice $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

25 If $\boldsymbol{N}(A) =$ line through $\boldsymbol{x} = (2, 1, 0, 1)$, A has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

26 $R = [\,1 \;\; -2 \;\; -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$. Any zero rows come after those rows.

27 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's !

28 One reason that R is the same for A and $-A$: They have the same nullspace. (They also have the same row space. They also have the same column space, but that is not required for two matrices to share the same R. R tells us the nullspace and row space.)

29 The nullspace of $B = [\,A \;\; A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

30 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection*.

31 (a) $R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank 1 (b) $R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank 2

(c) $R = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank 1

32 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops. The solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are combinations of $(-1, 0, 0, 1, -1, 0)$ and $(0, 0, -1, -1, 0, 1)$ and $(0, -1, 0, 0, 1, -1)$. Those are flows around the 3 small loops.

33 (a) and (c) are correct; (b) is completely false; (d) is false because R might have 1's in nonpivot columns.

34 $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_B = [\,R_A \;\; R_A\,]$ $R_C \longrightarrow \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix} \longrightarrow$ Zero rows go to the bottom

35 If all pivot variables come last then $R = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$. The nullspace matrix is $N = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

36 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

37 A and A^{T} have the same rank r = number of pivots. But *pivcol* (the column number) is 2 for this matrix A and 1 for A^{T}: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

38 Special solutions in $N = [-2 \ \ -4 \ \ 1 \ \ 0; -3 \ \ -5 \ \ 0 \ \ 1]$ and $[1 \ \ 0 \ \ 0; 0 \ \ -2 \ \ 1]$.

39 The new entries keep rank 1: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & \mathbf{4} & \mathbf{8} \\ 4 & \mathbf{8} & \mathbf{16} \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{2} & 6 & \mathbf{-3} \\ 1 & \mathbf{3} & \mathbf{-3/2} \\ 2 & 6 & -3 \end{bmatrix}$, $M = \begin{bmatrix} a & b \\ c & \boldsymbol{bc/a} \end{bmatrix}$.

40 If A has rank 1, the column space is a *line* in $\mathbf{R}^m$. The nullspace is a *plane* in $\mathbf{R}^n$ (given by one equation). The nullspace matrix N is n by $n-1$ (with $n-1$ special solutions in its columns). The column space of A^{T} is a *line* in $\mathbf{R}^n$.

41 $\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

42 With rank 1, the second row of R is a zero row.

43 Invertible r by r submatrices
Use pivot rows and columns $S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $S = [\,1\,]$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

44 P has rank r (the same as A) because elimination produces the same pivot columns.

45 The rank of R^{T} is also r. The example matrix A has rank 2 with invertible S:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

46 The product of rank one matrices has rank one or zero. These particular matrices have rank$(AB) = 1$; rank$(AC) = 1$ except $AC = 0$ if $c = -1/2$.

47 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

48 (a) By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns, then column j of AB is the same combination of earlier columns of AB. Then rank $(AB) \le$ rank (B). No new pivot columns! (b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank of AB stays the same for $A_1 = I$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. It drops to zero for $A_2 = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$.

49 If we know that $\operatorname{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \le \operatorname{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.

50 We are given $AB = I$ which has rank n. Then rank$(AB) \le$ rank(A) forces rank$(A) = n$. This means that A is invertible. The right-inverse B is also a left-inverse: $BA = I$ and $B = A^{-1}$.

51 Certainly A and B have at most $\operatorname{rank} 2$. Then their product AB has at most $\operatorname{rank} 2$. Since BA is 3 by 3, it cannot be I even if $AB = I$.

52 (a) A and B will both have the same nullspace and row space as the R they share.

(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

53 $A = (\text{pivot columns})(\text{nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} \text{columns} \\ \text{times rows} \end{matrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

54 If $c = 1, R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_2, x_3, x_4 free. If $c \ne 1, R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_3, x_4 free. Special solutions in $N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for $c = 1$) and $N =$

$\begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (for $c \neq 1$). If $c = 1$, $R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and x_1 free; if $c = 2$, $R = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ and x_2 free; $R = I$ if $c \neq 1, 2$. Special solutions in $N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(c = 1)$ or $N = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $(c = 2)$ or $N = 2$ by 0 empty matrix.

55 $A = \begin{bmatrix} I & I \end{bmatrix}$ has $N = \begin{bmatrix} I \\ -I \end{bmatrix}$; $B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$ has the same N; $C = \begin{bmatrix} I & I & I \end{bmatrix}$ has $N = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}$.

56 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$ (pivot column) (first row) $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

57 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

58 $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } n-r \\ m-r \text{ by } r & m-r \text{ by } n-r \end{bmatrix}$; $\textbf{rref}(R^{\mathrm{T}}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\textbf{rref}(R^{\mathrm{T}}R) =$ same R

59 $R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has $R^{\mathrm{T}}R = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and this matrix row reduces to $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R \\ \text{zero row} \end{bmatrix}$. Always $R^{\mathrm{T}}R$ has the same nullspace as R, so its row reduced form must be R with $n - m$ extra zero rows. R is determined by its nullspace and shape !

60 The *row-column reduced echelon form* is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; I is r by r.

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1 $\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix}$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $\boldsymbol{s}_1 = (-1, -1, 1, 0)$ and $\boldsymbol{s}_2 = (2, -2, 0, 1)$; $\boldsymbol{x}_{complete} = \boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$;

$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ gives the particular solution $x_p = (4, -1, 0, 0)$.

2 $\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix}$ Then $[\,R \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A) =$ line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

3 $\boldsymbol{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. The matrix is singular but the equations are still solvable; $\boldsymbol{b}$ is in the column space. Our particular solution has free variable $y = 0$.

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1)$.

5 $\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix}$ solvable if $b_3 - 2b_1 - b_2 = 0$.

Back-substitution gives the particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{x} = \mathbf{0}$: $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$ One more step gives $[0\ 0\ 0\ 0] =$ row $3 - 2$ (row 2) $+ 4$(row 1) **provided $b_3 - 2b_2 + 4b_1 = 0$.**

8 (a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.

(b) We need $b_3 = 2b_2$, because $(\text{row } 3) - 2(\text{row } 2) = \mathbf{0}$.

9 $L\begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix}$

$= \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$; particular $\boldsymbol{x}_p = (-9, 0, 3, 0)$ means $-9(1, 2, 3) + 3(3, 8, 7) = (0, 6, -6)$. This is $A\boldsymbol{x}_p = \boldsymbol{b}$.

10 $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_{\text{null}} = (c, c, c)$. Many possible A !

11 A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and also $\boldsymbol{x} = \mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$

(b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}$, $A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$

(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)

(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \boldsymbol{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

15 If row 3 of U has no pivot, that is a *zero row*. $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided $c_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ *might not be solvable*, because U may have other zero rows needing more $c_i = 0$.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\, I \quad F \,]$ for any 3 by 2 matrix F.

17 The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every *column*. The solution is *unique* (if there is a solution). The nullspace contains only the *zero vector*. Then $\mathbf{R} = \mathbf{rref}(A) = \begin{bmatrix} I & (4 \text{ by } 4) \\ 0 & (2 \text{ by } 4) \end{bmatrix}$.

18 Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

19 Both matrices A have rank 2. Always $A^{\mathrm{T}}A$ and AA^{T} have **the same rank** as A.

20 $A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$; $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \mathbf{2} & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}$.

21 (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The second equation in part (b) removed one special solution from the nullspace.

22 If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and also $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $A(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{0}$ and we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$: the solution $\boldsymbol{x}$ is not unique. But there will be **no solution** to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

23 For $A, q = 3$ gives rank 1, every other q gives rank 2. For $B, q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

24 (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [\, x \,] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\, b \,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A

has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

25 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

26 $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to R = \begin{bmatrix} \mathbf{1} & 0 & -2 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \to R = I.$

27 $R = I$ when A is square and invertible—so for a triangular matrix, all diagonal entries must be nonzero.

28 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

29 $[R\ \boldsymbol{d}] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$ leads to $\boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $[R\ \boldsymbol{d}] = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$:

this has no solution because of the 3rd equation

30 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$; $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

31 For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$, the only solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

32 $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ factors into $LU = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rank is $r = 2$. The special solution to $A\boldsymbol{x} = \mathbf{0}$ and $U\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{s} = (-7, 2, 1)$. Since

$\boldsymbol{b} = (1, 3, 6, 5)$ is also the last column of A, a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $(0, 0, 1)$ and the complete solution is $\boldsymbol{x} = (0, 0, 1) + c\boldsymbol{s}$. (Or use the particular solution $\boldsymbol{x}_p = (7, -2, 0)$ with free variable $x_3 = 0$.)

For $\boldsymbol{b} = (1, 0, 0, 0)$ elimination leads to $U\boldsymbol{x} = (1, -1, 0, 1)$ and the fourth equation is $0 = 1$. No solution for this $\boldsymbol{b}$.

33 If the complete solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

34 (a) If $\boldsymbol{s} = (2, 3, 1, 0)$ is the only special solution to $A\boldsymbol{x} = \boldsymbol{0}$, the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ (a line of solutions). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is *not free* in $\boldsymbol{s}$, and R must be $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$, because A and R have *full row rank* $r = 3$.

35 For the $-1, 2, -1$ matrix K(9 by 9) and constant right side $\boldsymbol{b} = (10, \cdots, 10)$, the solution $\boldsymbol{x} = K^{-1}\boldsymbol{b} = (45, 80, 105, 120, 125, 120, 105, 80, 45)$ rises and falls along the parabola $x_i = 50i - 5i^2$. (A formula for K^{-1} is later in the text.)

36 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \boldsymbol{0}$). If $\boldsymbol{b} =$ column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

37 The column space of R (m by n with rank r) spanned by its r pivot columns (the first r columns of an m by m identity matrix).

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1 $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent. But $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{c} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is solved by $\boldsymbol{c} = (1, 1, -4, 1)$. Then $\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{v}_4 = \mathbf{0}$ (dependent).

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors in $\mathbf{R}^4$ are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 $U\boldsymbol{x} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$ (by back substitution). A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}; A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ columns add to $\mathbf{0}$.

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A. This is because $EA = U$ for the matrix E that subtracts 2 times row 1 from row 4. So A and U have the same nullspace (same dependencies of columns).