18.06 MIDTERM 1 - SOLUTIONS

PROBLEM 1

(1) Use Gaussian elimination to put the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ in row echelon form.

Show all your steps! *(10 pts)*

Solution: Using Gaussian elimination we get:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_2+r_1} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_3+r_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_4-2r_3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so:

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) Use part (1) to write $A = LU$, where L is lower triangular and U is upper triangular.

Then express L as a product of elimination matrices $E_{ij}^{(\lambda)}$ for various $i > j$ and numbers λ. *(10 pts)*

Solution: We can rewrite the row operations in part (1) as multiplications by elimination matrices. The first step is given by $E_{21}^{(1)}$, the second by $E_{32}^{(1)}$ and the third by $E_{43}^{(-2)}$. Thus:

$$E_{43}^{(-2)} E_{32}^{(1)} E_{21}^{(1)} A = U$$

By using $(E_{ij}^{(\lambda)})^{-1} = E_{ij}^{(-\lambda)}$ we get:

$$A = E_{21}^{(-1)} E_{32}^{(-1)} E_{43}^{(2)} U$$

hence:

$$L = E_{21}^{(-1)} E_{32}^{(-1)} E_{43}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(3) Find a linear combination of the columns of A which is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *Hint: the answer to (1) may help.* *(5 pts)*

Solution: Since U is obtained from A by row operations, it suffices to solve the problem for U. By inspection, it's easy to see that the third column of U is equal to twice its first column. Hence the same is true for A:

$$\Big(\text{third column}\Big) - 2\Big(\text{first column}\Big) = 0$$

(4) Explain why for any 4×4 matrix X, the product AX cannot be invertible. *(5 pts)*

Solution: By part (3), the columns of A are linearly dependent (since the third column is a linear combination of the first column), so they span a vector space of dimension at most $3 < 4$. Since the columns of AX are linear combinations of the columns of A, we conclude that the columns of AX also span vector space of dimension at most $3 < 4$. So AX cannot be invertible, since invertible matrices have full dimensional column space.

PROBLEM 2

Consider the system of equations:

$$\begin{cases} a - 2b + 6c = 1 \\ -2a + 3b - 11c = -3 \end{cases} \qquad (*)$$

(1) Write the system as $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{b}$ for a suitably chosen 2×3 matrix A and 2×1 vector $\mathbf{b}$. *(5 pts)*

Solution:We can rewrite the system of equations as:

$$\underbrace{\begin{bmatrix} 1 & -2 & 6 \\ -2 & 3 & -11 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{\mathbf{b}}$$

(2) Use Gauss-Jordan elimination to put A from part (1) in $\boxed{\textbf{reduced}}$ row echelon form.

Show all your steps! *Hint: we recommend you actually do Gauss-Jordan elimination on the extended matrix* $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$*; it's a little bit more work, but it will pay off in part (4).*

(10 pts)

Solution: First add twice row 1 to row 2:

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & 1 \\ -2 & 3 & -11 & -3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 6 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right]$$

Then multiply row 2 by -1, to get all pivots equal to 1:

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & 6 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Finally, add twice row 2 to row 1:

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

(3) Write down the vector(s) in a basis for the nullspace of A. What is the dimension of this nullspace? **Explain how you know!** *(10 pts)*

Solution: The nullspace is unaffected by Gauss-Jordan elimination, so it is the set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \qquad \Leftrightarrow \qquad \begin{cases} a = -4c \\ b = c \end{cases}$$

The pivot variables are a and b, and the free variable is c. Recall that a basis vector is given by setting c equal to 1, and using the equations above to solve for a and b:

$$\text{a basis vector of } N(A) \text{ is } \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the dimension of $N(A)$ is 1.

(4) What is the general solution of the system $(*)$? *(10 pts)*

Solution: A particular solution can be obtained by setting the free variable c equal to 0, and solving for the pivot variables:

$$\begin{cases} a - 2b = 1 \\ -2a + 3b = -3 \end{cases} \qquad (*)$$

You can solve this 2×2 system in a number of ways (including back substitution) and we notice that $a = 3$, $b = 1$ is the solution. Equivalently, if you did Gauss-Jordan for the extended matrix in part (2), then the system is equivalent to:

$$\begin{cases} a \qquad + 4c = 3 \\ \qquad b - \quad c = 1 \end{cases} \qquad (**)$$

Setting the free variable $c = 0$ gives you, yet again ,$a = 3$, $b = 1$. Hence a particular solution of the equivalent systems (*) and (**) is:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

The general solution is given by adding the particular solution to an arbitrary element of the nullspace:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

for any number α.

PROBLEM 3

(1) Let V be the following vector subspace of $\mathbb{R}^2$:

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } 3x + 4y = 0 \right\}$$

Find a basis for $W = V^{\perp}$ (in other words, W is the orthogonal complement of V). *(5 pts)*

Solution: By the very definition of the vector space V, any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is orthogonal to the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, since:

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3x + 4y = 0$$

We conclude that a basis of W is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

In what follows, you may use the formula $P_{C(A)} = A(A^T A)^{-1} A^T$ for the projection matrix onto the column space $C(A)$ of any matrix A

(2) Compute the projection matrices P_V and P_W onto the subspaces from part (1). *(10 pts)*

Solution: We need matrices A and B whose column spaces are the vector spaces V and W, respectively. Since the vector space W is one-dimensional and spanned by the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, the natural candidate is:

$$B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Meanwhile, the vector space V is one dimensional (a line in the plane), so we must choose a single non-zero vector in V. One way to do so is to set one of the variables x, y equal to any number, and then solve the equation $3x + 4y = 0$ for the other variable. So we could say that V is spanned by the vector $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$. Hence we can take:

$$A = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Then we can calculate:

$$P_V = A(A^T A)^{-1} A^T = \frac{1}{25} \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix}$$

$$P_W = B(B^T B)^{-1} B^T = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

(3) Compute $P_V P_W$ and $P_W P_V$, where P_V and P_W are as in part (2). *(10 pts)*

Solution: It is straightforward to compute:

$$P_V P_W = \frac{1}{25}\begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \frac{1}{625}\begin{bmatrix} 16 \cdot 9 - 12 \cdot 12 & 16 \cdot 12 - 12 \cdot 16 \\ -12 \cdot 9 + 9 \cdot 12 & -12 \cdot 12 + 9 \cdot 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_W P_V = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \frac{1}{25}\begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} = \frac{1}{625}\begin{bmatrix} 9 \cdot 16 - 12 \cdot 12 & -12 \cdot 9 + 9 \cdot 12 \\ 16 \cdot 12 - 12 \cdot 16 & -12 \cdot 12 + 16 \cdot 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(4) Based on part (3), formulate a general principle by filling the blanks below:

For any vector spaces V and W, the projection matrices have the property that

$P_V P_W$ and $P_W P_V$ are ___0___ *if V and W are* ___orthogonal___

After formulating the principle above, justify it using a geometric argument (i.e. using the geometric interpretation of projections). *(10 pts)*

Solution: Taking the matrix $P_V P_W$ and multiplying it with any vector $\mathbf{b}$ means the same thing as projecting $\mathbf{b}$ onto the vector space W (this is the operation $\mathbf{b} \rightsquigarrow P_W \mathbf{b}$) and then projecting the result onto the vector space V (this is the operation $P_W \mathbf{b} \rightsquigarrow P_V P_W \mathbf{b}$). But if V and W are orthogonal to each other, then this sequence of two operations should give 0.