

## SOLUTIONS TO EXAM 1

### Problem 1 (30 pts)

(a) Since the multipliers are all 3, the row operations we had goes:

- subtract three times row 1 to row 2;
- subtract three times row 1 to row 3;
- subtract three times row 2 to row 3,

where the end step gives us

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & g \end{bmatrix}.$$

So now we just need to reverse the row operations. Reversing the last step means adding three times row 2 to row 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & g+3 \end{bmatrix}.$$

Reversing the second step means adding three times row 1 to row 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 9 & g+6 \end{bmatrix},$$

And reversing the first step means adding three times row 1 to row 2:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 9 & g+6 \end{bmatrix}.$$

(b) To find the nullspace of  $A$ , we first note that there are three columns in  $A$ , and that the rank of  $A$  is 2 in the case when  $g = 2$ , so the dimension of  $N(A)$  is one. So we need to find just one vector that are in  $N(A)$ ; then this vector will be a basis for  $N(A)$ . By using Gaussian elimination to  $Ax = 0$ , we get  $Ux = 0$ , from which we

deduce that  $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$  is a solution.

Therefore,  $N(A) = c \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ .

(c) If  $g \neq 0$ , then we observe that  $C(U) = \mathbb{R}^3$ . Since row rank is equal to the column rank, the column rank (i.e. the dimension of the column space) of  $A$  is also 3. Since the column space of  $A$  is contained in  $\mathbb{R}^3$ , which is 3-dimensional, it must be the whole space as well. So  $C(A) = \mathbb{R}^3$ .

### Problem 2 (40 pts)

- (a) Any row of  $A$  is orthogonal to the two special solutions given in the problem. That is, any row

$$r = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

satisfies  $r \cdot s_1 = r \cdot s_2 = 0$ . This is just a system of two linear equations, so we need to solve the equation

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix} r = 0$$

whose complete solution is given by

$$c \begin{bmatrix} 1 \\ -3 \\ 0 \\ -6 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix},$$

from which we get the reduced row echelon form of  $A$  given by

$$\begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

- (b)  $R$  has two pivots, and therefore  $A$  has two pivots and  $r(A) = 2$ . Two independent columns in  $\mathbb{R}^2$  span  $\mathbb{R}^2$ , so  $C(A) = \mathbb{R}^2$ .

Partial credit was given if the student referred back to  $A$  for the column space and if they gave  $\mathbb{R}^2$  with incomplete reasoning. Most points were lost if they just gave  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  as the basis without indicating where they came from (or from reading them off of  $R$ , not  $A$ ). There were lots of right answers, with wrong (or no) reasons.

- (c) The free variables are  $x_2, x_4$  so the particular solution is

$$x_p = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix}.$$

The complete solution is

$$x_c = x_p + c_1 \cdot s_1 + c_2 \cdot s_2 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

- (d) We have

$$2 \cdot \begin{bmatrix} -3 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$

One can find these coefficients by inspection, or combining the special solutions:

$$2s_1 - s_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ -1 \end{bmatrix}.$$

**Problem 3** (30 pts)

(a) Let  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  be a column of  $X$ . Then  $x, y$  and  $z$  satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

from which we deduce that  $y = 0$ , and  $x = -z$ . So each column of  $X$  is a multiple of

the vector  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ . Since there are two columns of  $X$ ,  $X$  can be written as

$$\begin{bmatrix} a & b \\ 0 & 0 \\ -a & -b \end{bmatrix}.$$

The basis for this space of matrices is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) We first solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From which we see that we can take  $y = -1/2, x = 3/2, z = 0$ . This will be the first column of  $X$ .

We now solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And we can take  $y = 1/2, x = -1/2, z = 0$ . This is the second column of  $X$ .

So one possible solution for  $X$  is

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

(c) The set of complete solutions is given by

$$X_{\text{particular}} + X_{\text{special}} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix} + a \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$