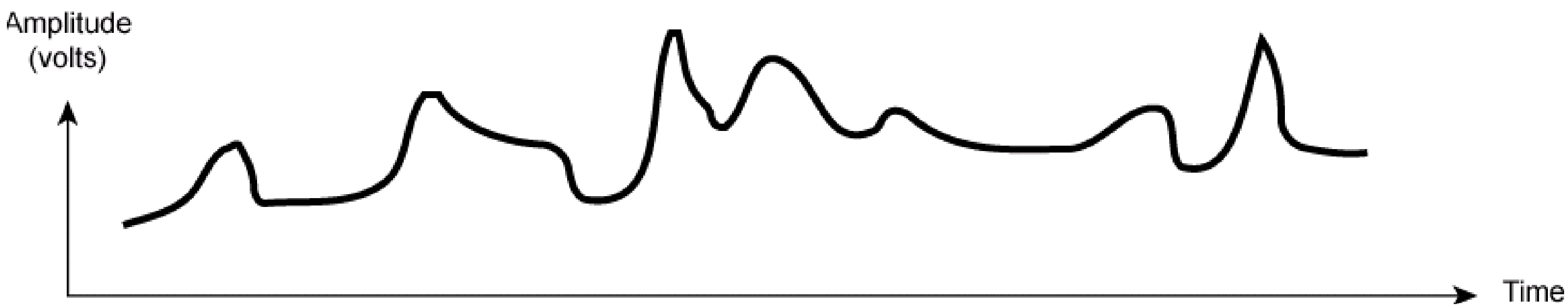
K. N. Toosi
University of Technology

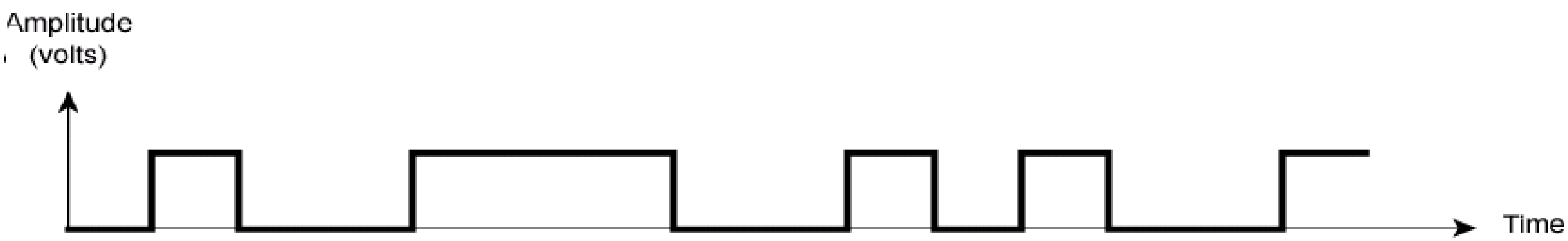
Computer Networks 1

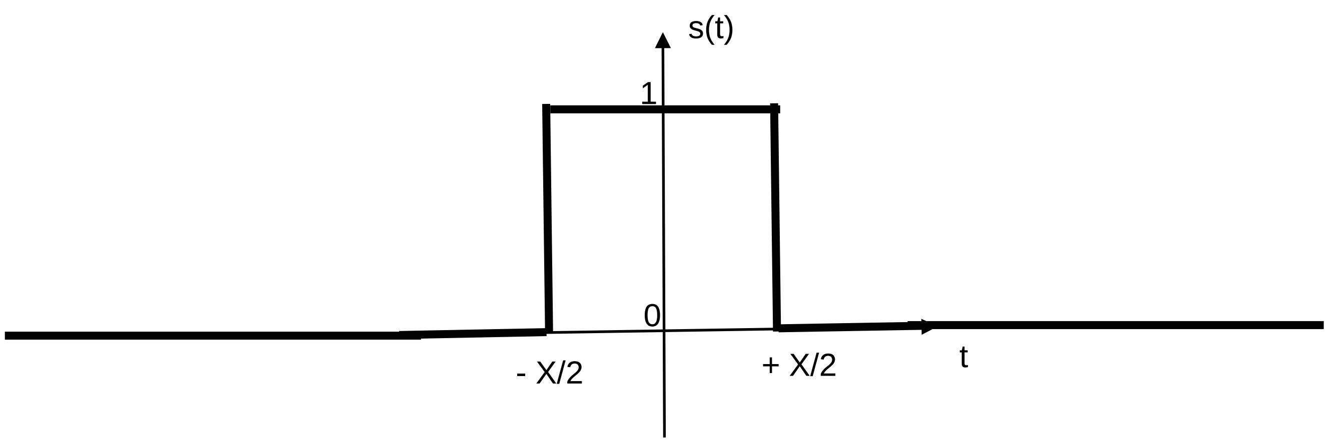
BANDWIDTH AND CAPACITY

Fatemeh Rezaei

APERIODIC SIGNALS IN TIME







PERIODIC SIGNALS

For any periodic wave:

$S(t+mT) = S(t); \quad 0 \leq t \leq T$

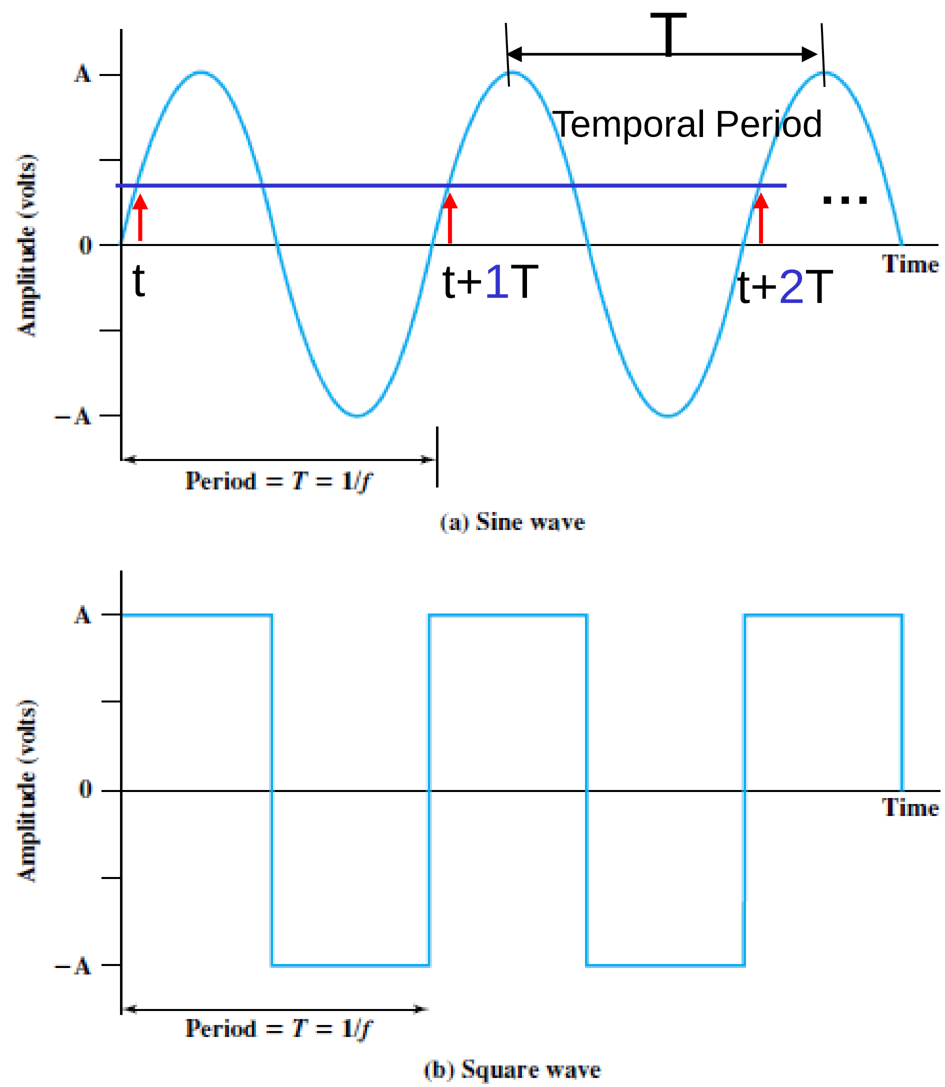
Where:

t is time over first period

T is the waveform period

m is an integer

Signal behavior over **one period** describes behavior at all times

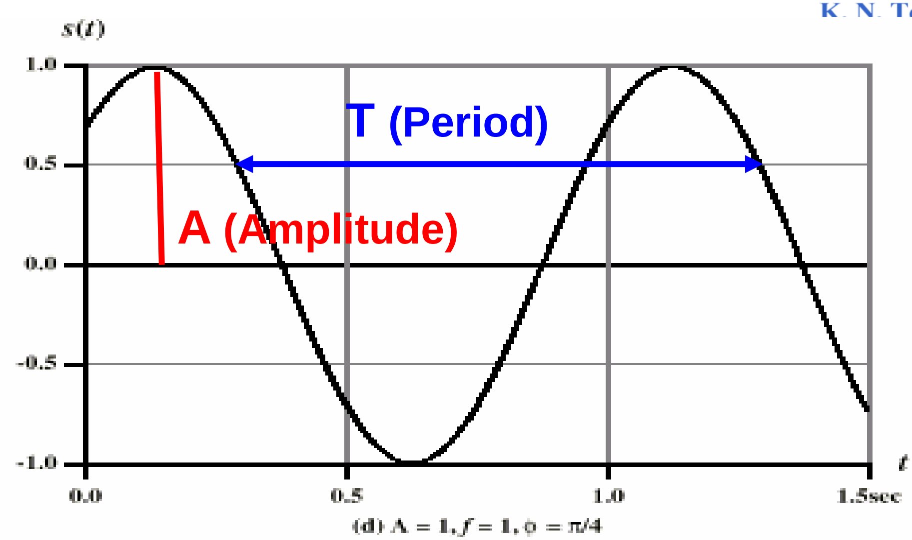


(a) Sine wave

(b) Square wave

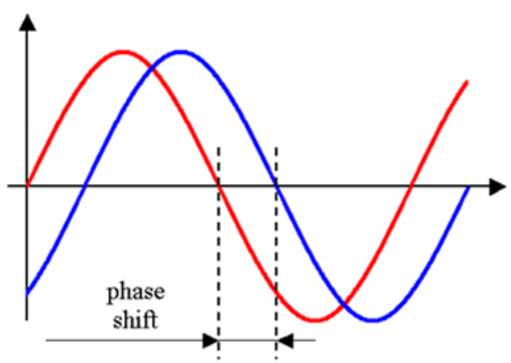
SINE WAVE

- $s(t) = A\sin(2\pi f t + \phi)$
- A: Peak Amplitude
  - Peak strength of signal, volts



- T: Temporal (time) Period, (S)
- f: Repetition Frequency, (Cycles/s) (Hz)
  - Measures how fast the signal varies with time
  - $f = 1/T$
- ω: Angular Frequency, (Radians/s)

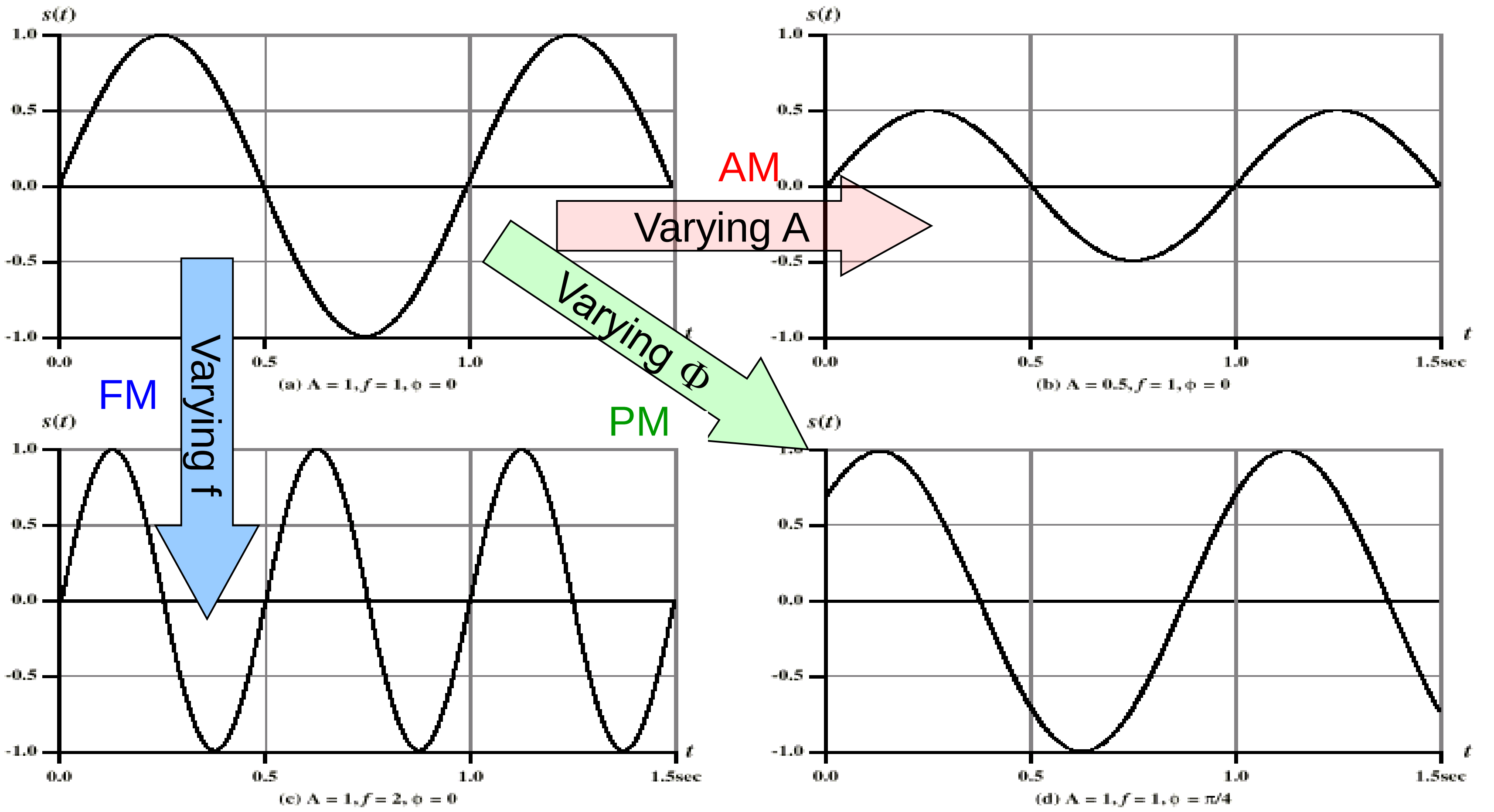
$\omega = 2\pi f = 2\pi / T$



- ϕ: Phase Angle, (Radians)

CONVEYING INFORMATION BY A SINE WAVE CARRIER

$$s(t) = A \sin(2\pi ft + \Phi)$$



WAVELENGTH

- λ (meters)
- Spatial period of the wave:
  - distance between two points in space on the wave propagation path where the wave has the same total phase
- Distance traveled by the wave during one temporal (time) cycle

$d_T = v\, T$

$v = \lambda\, f$

$d_T = (\lambda\, f)\, T = \lambda$

FREQUENCY DOMAIN CONCEPTS

- Response of systems to a sine waves is easy to analyze
- Fourier analysis shows that signals can be treated as
  - Sum of many sine wave components
    - having different frequencies, amplitudes, and phases
- Basis for frequency domain analysis
- Dealing with functions in the frequency domain is simpler

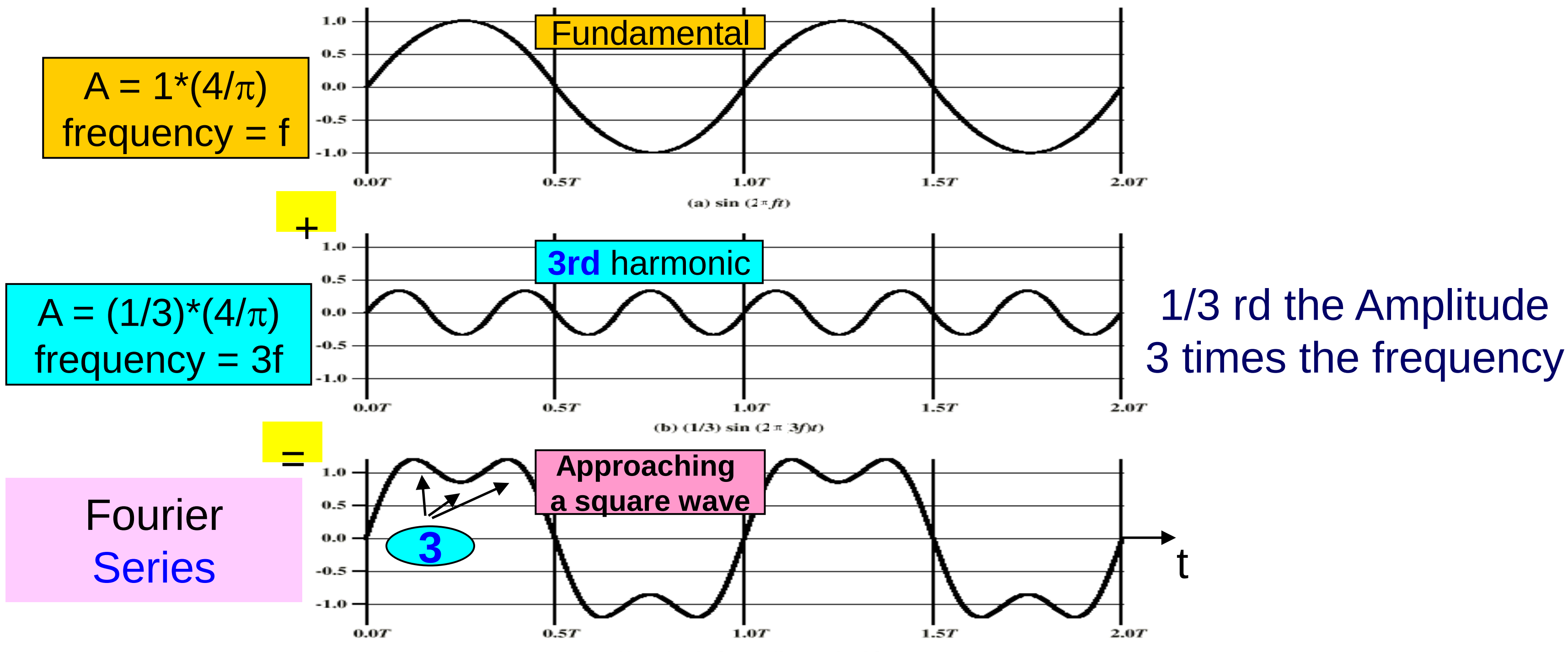
FREQUENCY DOMAIN REPRESENTATION

$$S(t) = (4/\pi f)\,[(\sin(2\pi ft) + (1/3)(\sin(2\pi(3f)t)]$$

3f :integer multiple of the first frequency ,f, known as **fundamental frequency**

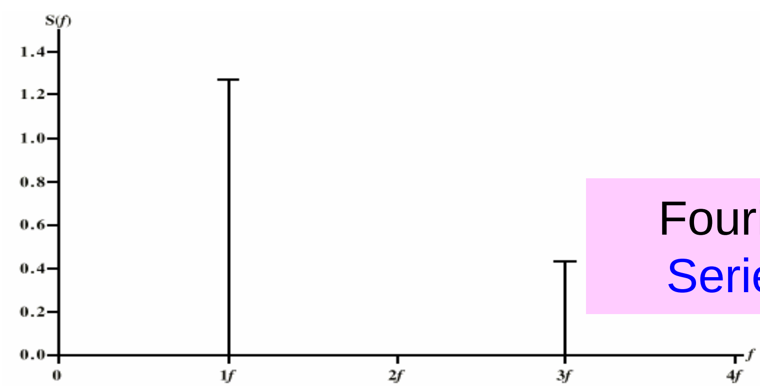
The period of the total signal is equal to the period of the **fundamental frequency**

The period of the component $\sin(2\pi ft)$ is T and the period of $s(t)$ is also T.



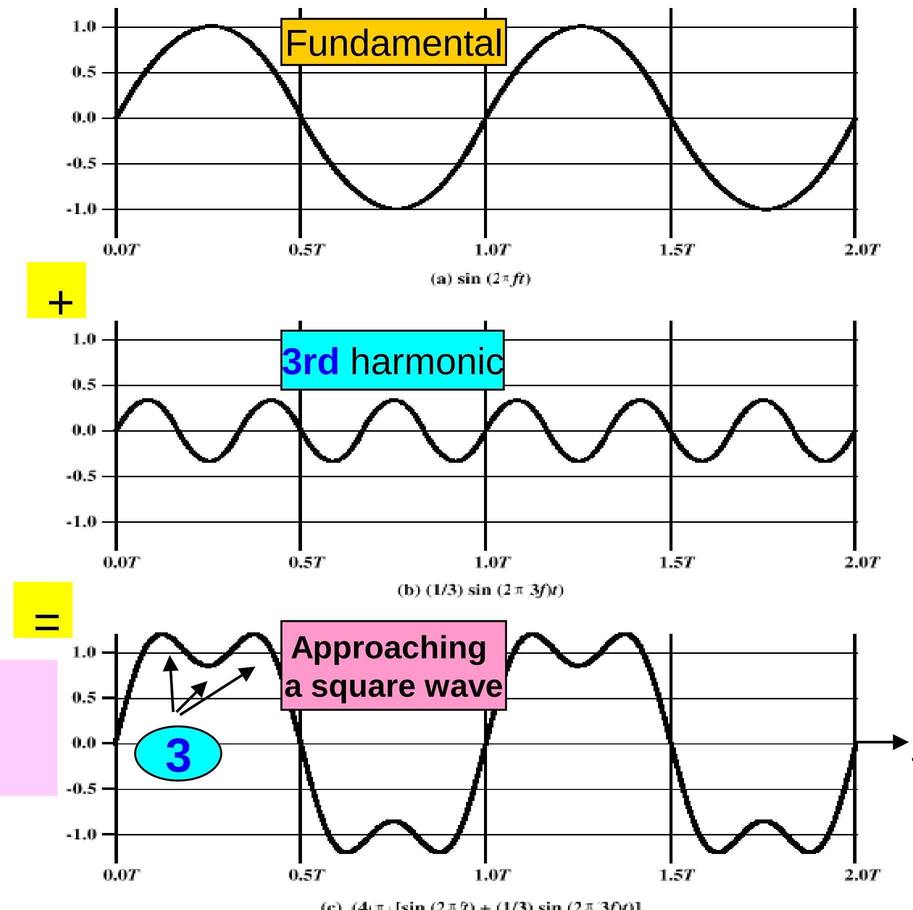
SIGNAL SPECTRUM

Frequency Spectrum



Fourier Series

f



Frequency Domain: S(f) vs f

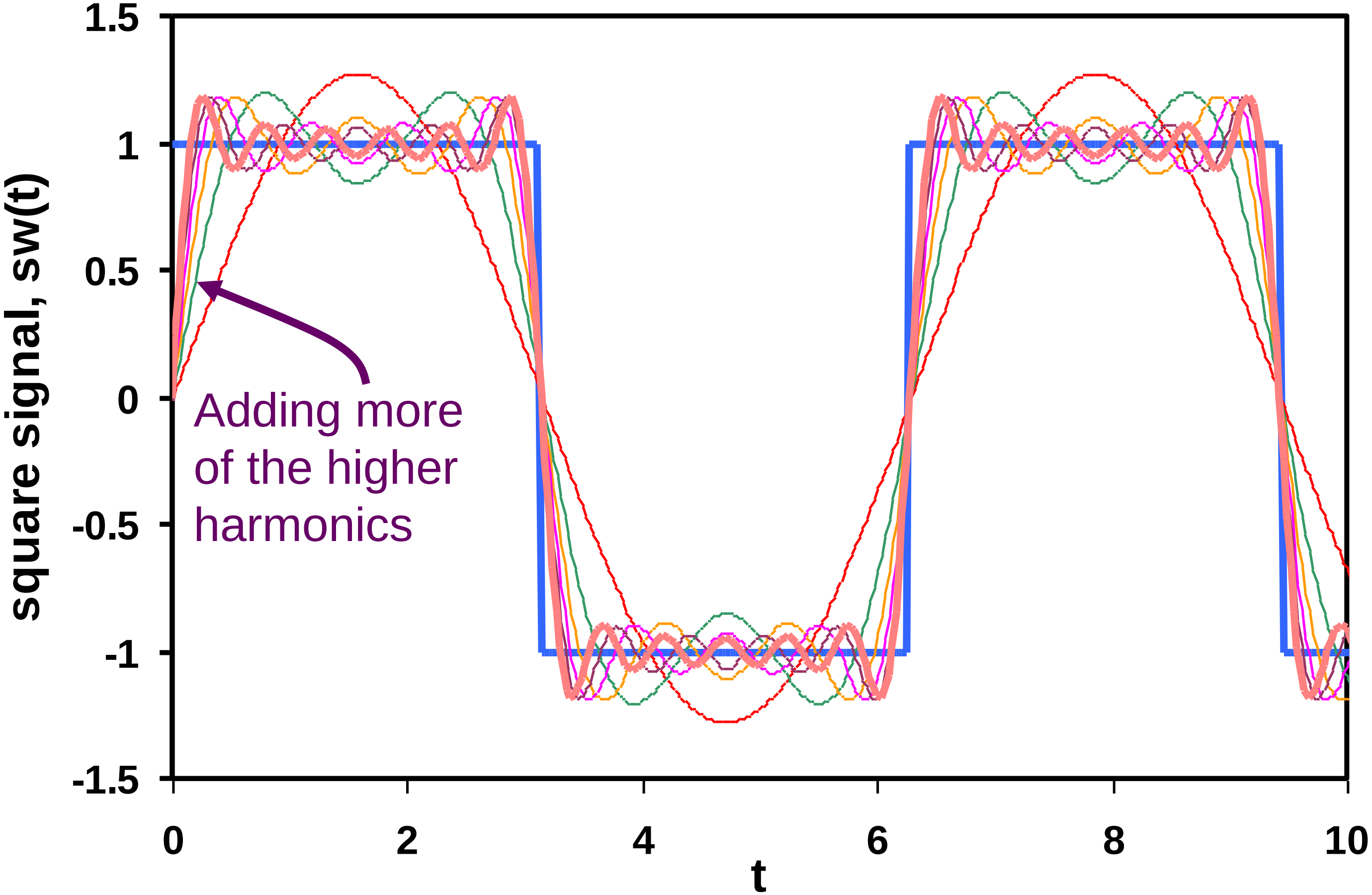
Discrete Function in f

Fourier Series

Time Domain: s(t) vs t

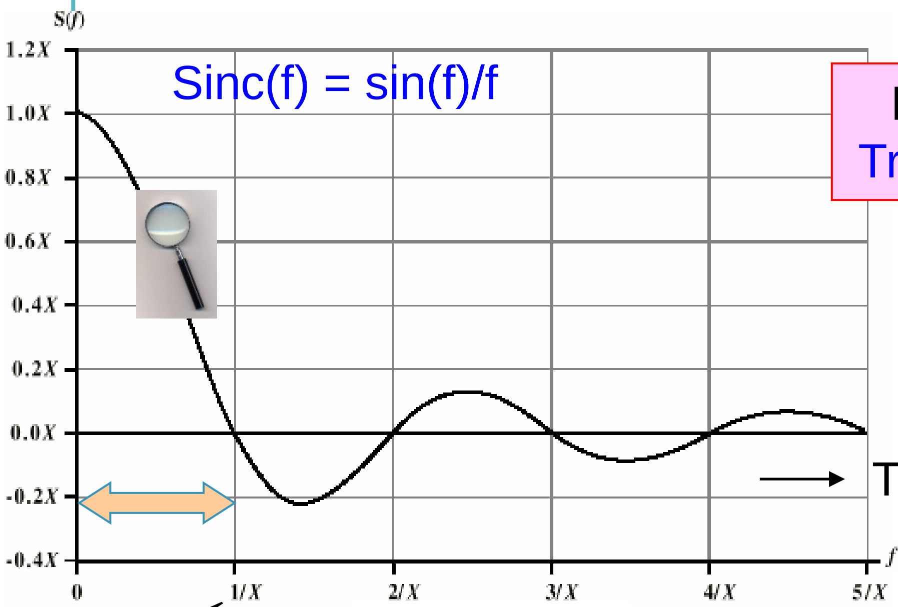
Periodic function in t

ASYMPTOTICALLY APPROACHING A SQUARE WAVE

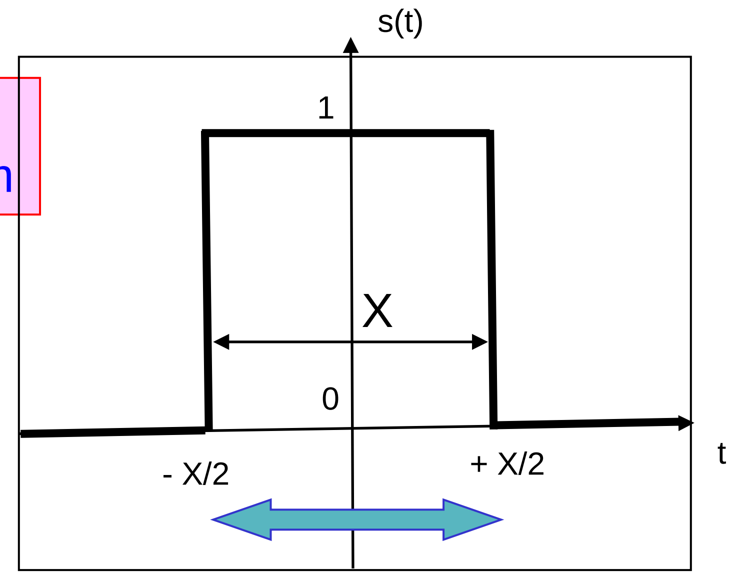


Combining the fundamental + an infinite number of odd harmonics at proper amplitudes

SPECTRUM OF A SINGLE SQUARE PULSE



Fourier Transform



frequency

time

1/X

Frequency Domain: S(f) vs f

Continuous Function in f

Fourier Transform

Time Domain: s(t) vs t

Aperiodic function in t

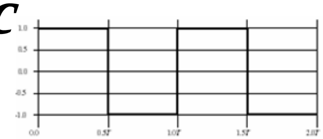
- What happens to the spectrum as the pulse gets broader …→ DC ?

- What happens to the spectrum as the pulse gets narrower …→ spike ?

FOURIER ANALYSIS

Signals in Time

Periodic



...

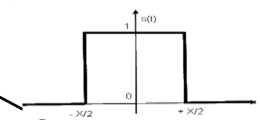
Discrete

DFS

Continuous

FS

Aperiodic



Discrete

DFT

Continuous

FT

Use Fourier Series

Use Fourier Transform

FS : Fourier Series

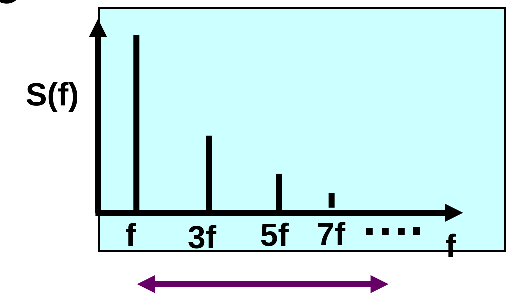
DFS : Discrete Fourier Series

FT : Fourier Transform

DFT : Discrete Fourier Transform

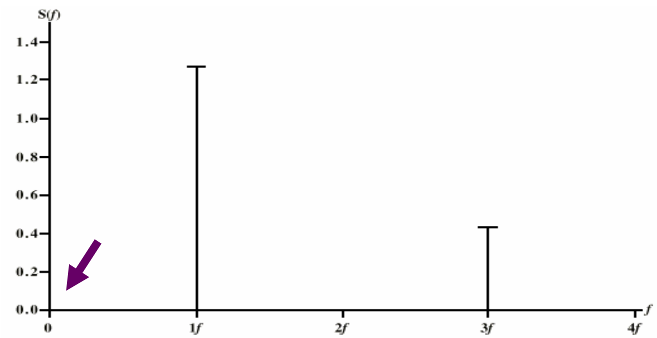
SPECTRUM & BANDWIDTH OF A SIGNAL

- Spectrum of a signal
  - Range of frequencies contained in a signal
- Absolute (theoretical) Bandwidth (BW):
  - Full width of spectrum = fmax- fmin
  - But in many situations, fmax = ∞!

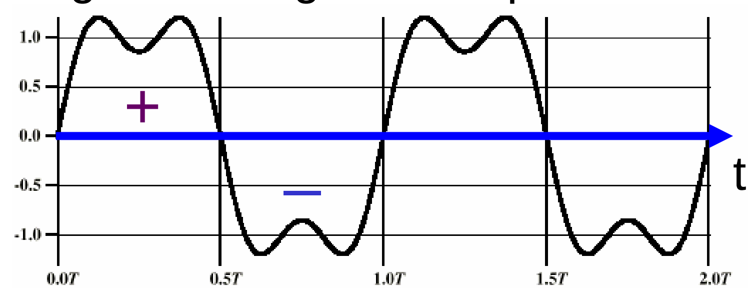


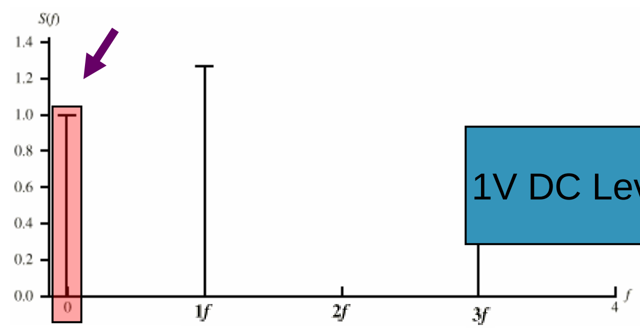
- Effective Bandwidth
  - Often called just *bandwidth*
  - Narrow band of frequencies containing *most* of the signal energy
  - e.g. that contains say 95% of the energy of the signal

SIGNALS WITH A DC COMPONENT



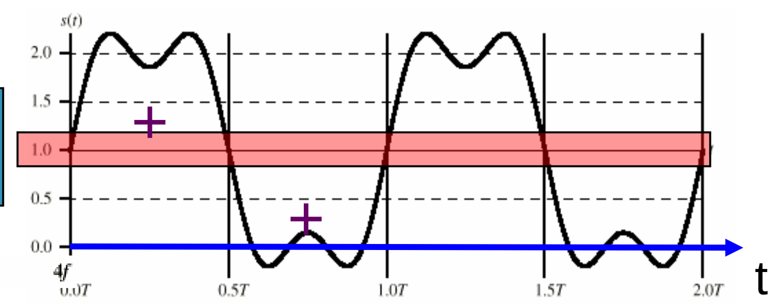
NO DC Component,
Signal average over a period = 0





1V DC Level

1V DC Component

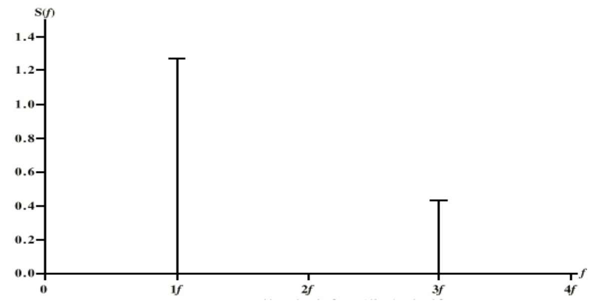


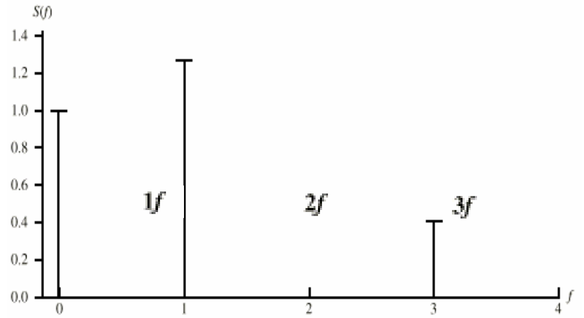
DC Component:

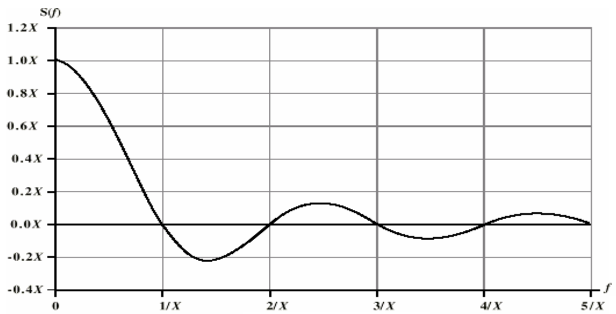
Component at zero frequency

Determines if fmin = 0 or not

BANDWIDTH FOR THESE SIGNALS



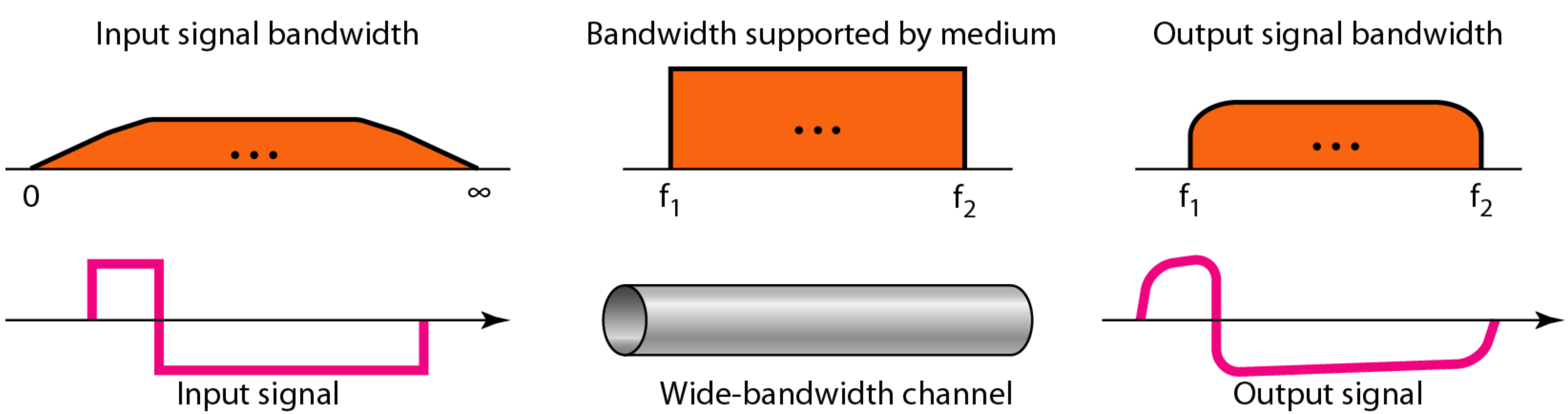




fmin	*fmax*	Absolute BW	Effective BW
1f	3f	2f	2f
0	3f	3f	3f
0	∞	∞	1/X ?

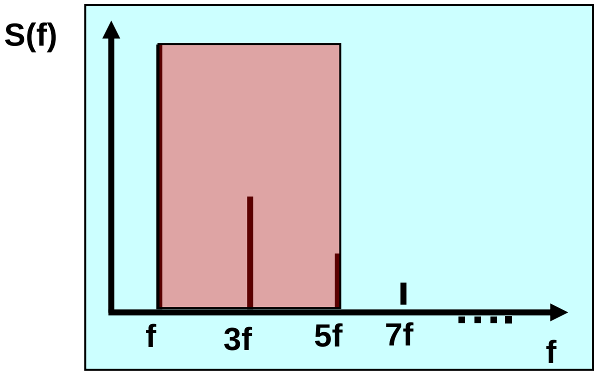
BANDWIDTH OF A TRANSMISSION SYSTEM

- Range of signal frequencies that are adequately passed by the system
- Transmission system (TX, medium, RX) acts as a filter
  - Poor transmission media, e.g. twisted pairs, have a narrow bandwidth

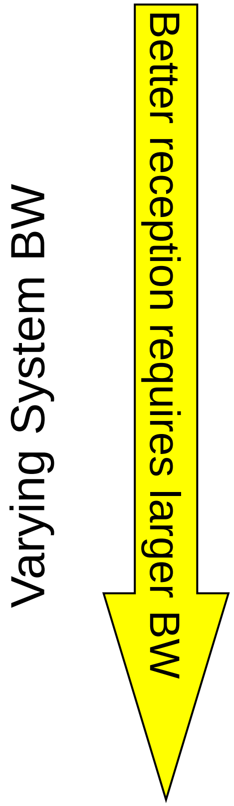


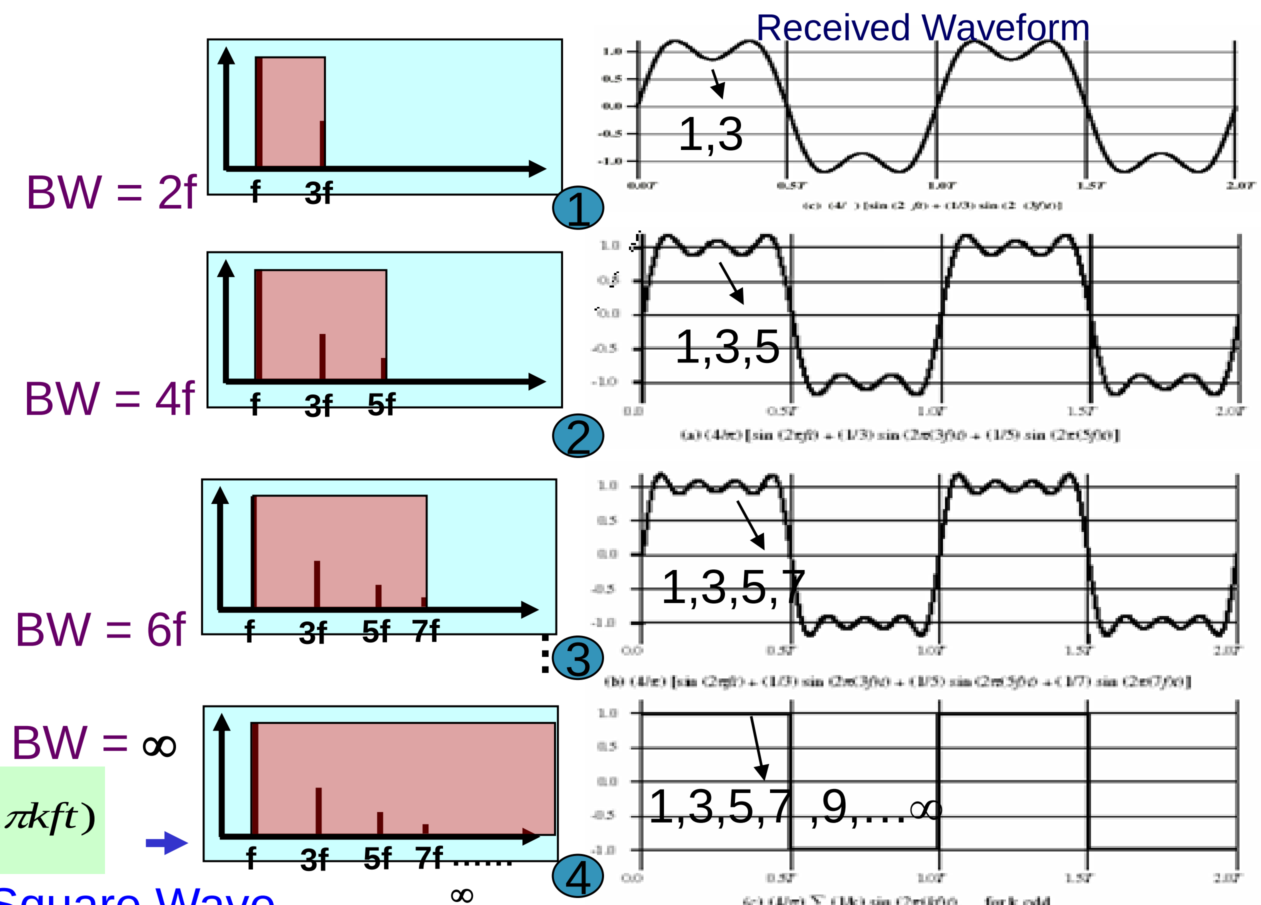
BANDWIDTH OF A TRANSMISSION SYSTEM

- Narrow media bandwidth effectively cuts off higher frequency signal components
  - poor signal quality at receiver
  - limiting the signal frequencies (Hz) that can be used for transmission
    - limiting the data rates that can be used (bps), examples:
    - Twisted pair: 4 KHz BW → 100 Kbps
    - Optical fiber: 4 THz BW → 10 Gbps



LIMITING EFFECT OF **SYSTEM** BANDWIDTH





$$s(t) = \frac{4}{\pi} \sum_{k \text{ odd, } k=1}^{\infty} \frac{1}{k} \sin(2\pi k f t)$$

Fourier Series for a Square Wave

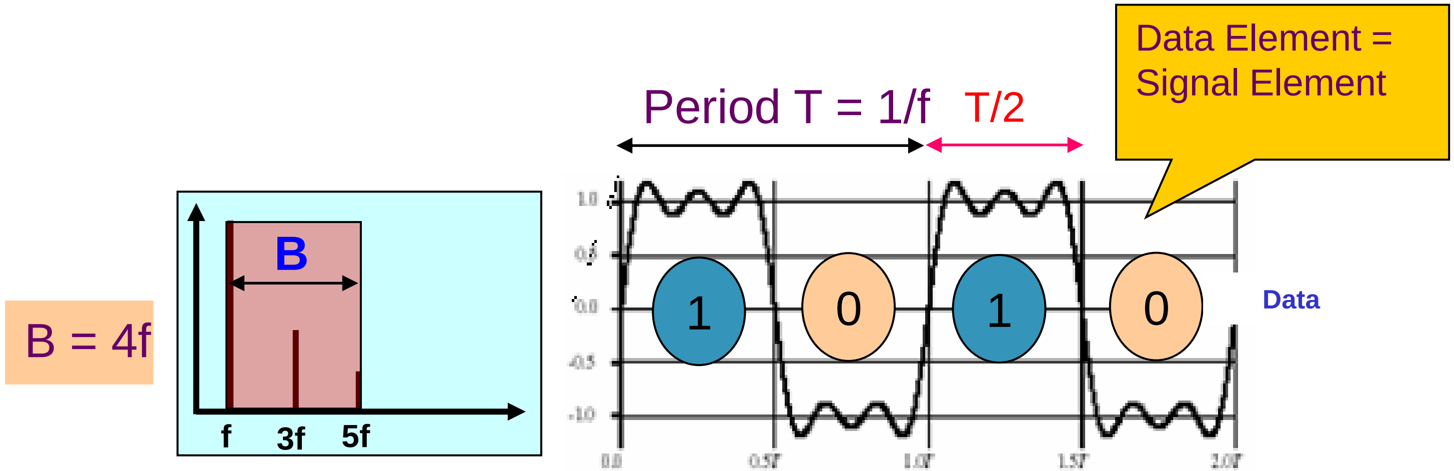
SYSTEM BANDWIDTH AND ACHIEVABLE DATA RATES

- Any transmission system supports only a limited range of frequencies (bandwidth) for satisfactory transmission
- For example, this bandwidth is largest for expensive optical fibers and smallest for cheap twisted pair wires
- Bandwidth is money → Economize in its use

SYSTEM BANDWIDTH AND ACHIEVABLE DATA RATES

- Limited system bandwidth degrades higher frequency components of the signal transmitted
  - $\Rightarrow$ poorer received waveforms
  - $\Rightarrow$ more difficult to interpret the signal at the receiver (especially with noise)
  - $\Rightarrow$ Data Errors
- More degradation occurs when higher data rates are used (signal will have higher frequency components)

BANDWIDTH AND DATA RATES



(a) (4/π) [sin (2πft) + (1/3) sin (2π(3f)t) + (1/5) sin (2π(5f)t)]

1- Data rate relates to f

Data rate = 1/(T/2) = (2/T) (bps)= 2f (bps)

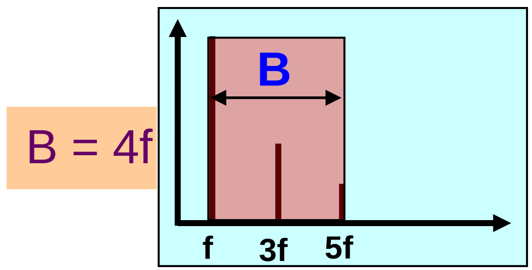
2- Bandwidth relates to quality of Fourier harmonics in the signal

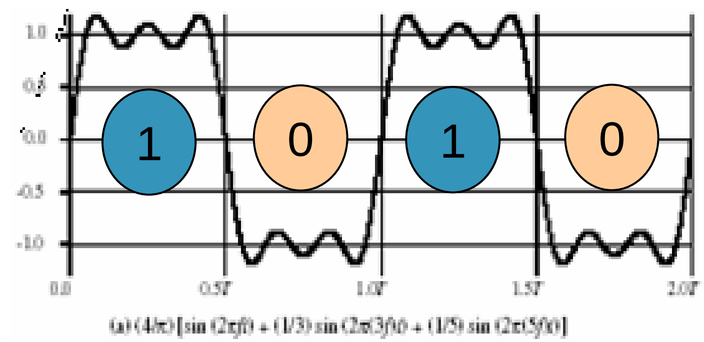
e.g. harmonics 1f, 3f,5f : B=4f

From 1 and 2:

Given a bandwidth B, Data rate = 2f = B/2

BANDWIDTH AND DATA RATES EXAMPLE

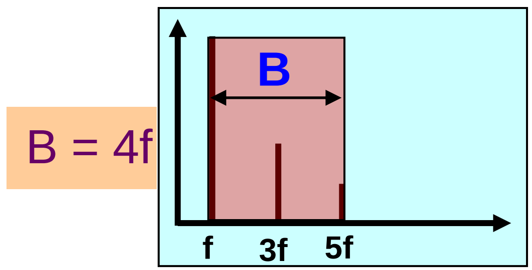


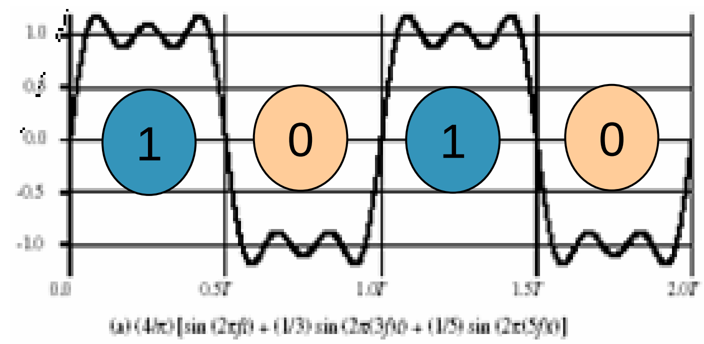


f=1Mhz , Data rate=?

f=2MHz, Data rate= ?

BANDWIDTH AND DATA RATES EXAMPLE





f=1Mhz , bit time =.5 micro sec, Data rate=2Mbps

f=2MHz, , bit time =.25 micro sec, Data rate= 4Mbps

DOUBLING THE DATA RATE

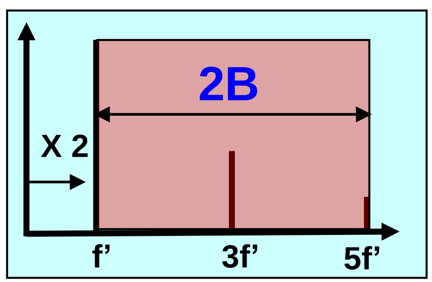
To double the data rate you need to double f: f'=2f
Two ways to do this…

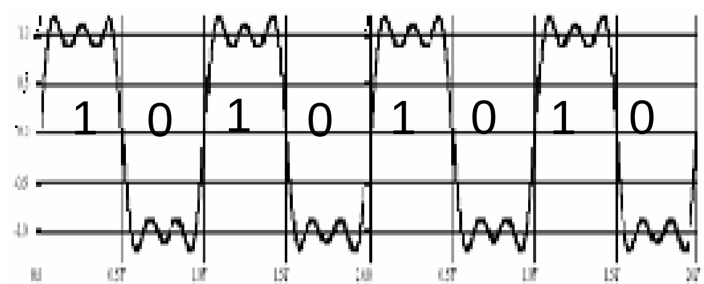
1. Double the bandwidth,
same received waveform quality (same RX conditions & error rate)

New BW=B'= 4f'= 8f = 2B

Data rate = 2f' = 2(2f)= 4f = B

e.g.
f'=2MHz
2B=8MHz
Data rate=4Mbps





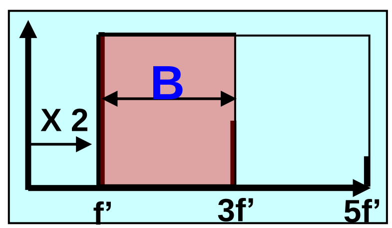
DOUBLING THE DATA RATE

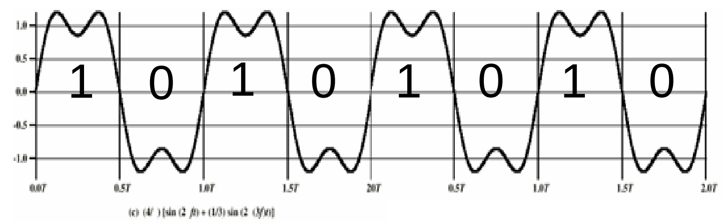
2. Same bandwidth, B, but tolerate poorer received waveform
(needs better receiver, higher S/N ratio, or tolerating more errors in data)

Bandwidth :B= 4f= 2f’

Data rate = 2f’ = 2(2f) = 4f = B

B = 2f’





e.g.
f’=2MHz ,
B=4MHz
Data rate=4Mbps

MAXIMUM DATA RATE (LINK CHANNEL CAPACITY)

- Bandwidth of transmission system
- Signal to noise ratio (SNR)
- Receiver type
- Acceptable error performance

BANDWIDTH & DATA RATES: TRADEOFFS…

- Increasing the data rate (bps) while keeping BW the same (to economize)
- Means working with inferior (poorer) waveforms at the receiver, which may require:
  - Ensuring higher signal to noise ratio (SNR) at RX
  - More sensitive (& costly!) receiver
  - Suffering from higher bit error rates
    - Tolerate them?
    - Add more efficient means for error detection and correction

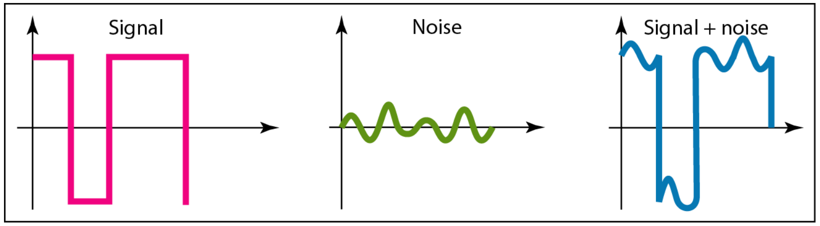
CHANNEL CAPACITY

- Channel capacity: Maximum data rate usable under a given set of communication conditions
  - Max rate at which data can be communicated on the channel, bits per second (bps)
- How channel BW (B), signal level, noise and impairments, and the amount of data error that can be tolerated limit the channel capacity?
- In general, Max possible data rate, C, on a given channel

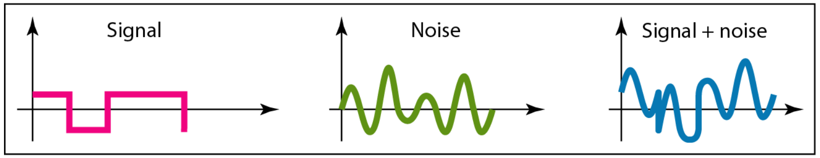
Function (B, Signal to noise, Bit error rate allowed)

CHANNEL CAPACITY

- Bandwidth: BW of the transmitted signal as constrained by the transmission system, (Hz)
- Signal relative to Noise,
  - SNR = signal power/noise power ratio
  - (<u>Higher</u> SNR → <u>better</u> communication conditions → higher C)



a. Large SNR



b. Small SNR

CHANNEL CAPACITY

- Bandwidth: BW of the transmitted signal as constrained by the transmission system, (Hz)
- Signal relative to Noise,
  - SNR = signal power/noise power ratio
  - (Higher SNR → better communication conditions → higher C)
- Bit error rate (BER) allowed:
  - in (bits received in error)/(total bits transmitted)
  - Equal to the bit error probability
  - Higher allowed → higher usable data rates → higher C

CHANNEL CAPACITY, C:

C (bps) = F(B, SNR, BER)

Three Formulations under different assumptions:

Idealistic

Assumptions	Formulation
Ideal: Noise-free, Error-free: C = F(B)	Nyquist
Noisy, Error-free: C = F(B, SNR)	Shannon
Practical: Noisy, Error: C = F(B, SNR, BER)	E_b/N_0 Vs Error Rate

Realistic

BANDWIDTH (OR SPECTRAL) EFFICIENCY (BE):

- Measures how well we are utilizing a given bandwidth to send data at a high rate….
- Can be greater than 1 (not like engineering efficiencies)
- The larger the better

$$BE = \frac{Channel\ Capacity\ C}{Bandwidth\ B}, \qquad bps / Hz$$

1. **NYQUIST** CHANNEL CAPACITY: (NOISE-FREE, ERROR-FREE)

- Idealized, theoretical
  - Assumes a noise-free → error-free channel
- If rate of signal transmission is 2B then a signal with frequency components up to B Hz is sufficient to carry that signalling rate
- Given bandwidth B, highest signalling rate possible is 2B signal elements/s
- How much data rate does this represent?

(depends on how many bits are represented by each signal element!)

  - Given a binary signal (1,0), data rate is same as signal rate → Data rate supported by a BW of B Hz is 2B bps → $C = 2B$
  - For the same B, data rate can be increased by sending one of M different signals (symbols): as each signal level now represents $\log_2 M$ bits
- Generalized Nyquist Channel Capacity, $C = 2B \log_2 M$ (bps)
- Bandwidth efficiency = $C/B = 2 \log_2 M$ (bits/s)/Hz : Dimensionless quantity

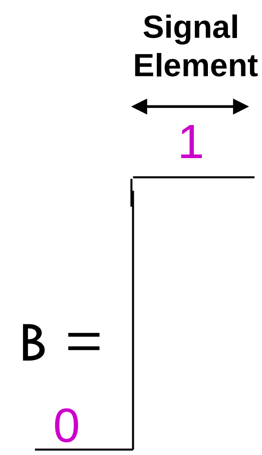
NYQUIST BANDWIDTH: EXAMPLE

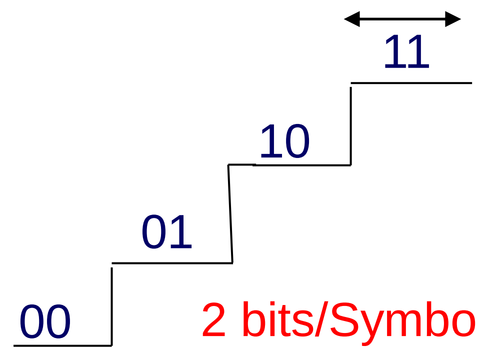
- $C = 2B \log_2 M$ bits/s
  - C = Nyquist Channel Capacity
  - B = Bandwidth
  - M = Number of discrete signal levels (symbols) used
- Data on telephone Channel:
  3400-300 = 3100 Hz
- With a binary signal (M = 2 symbols, e.g. 2 amplitudes):

  $C = 2B \log_2 2 = 2B \times 1 = 6200$ bps
- With a quadnary signal (M = 4 symbols):

  $C = 2B \log_2 4 = 2B \times 2 = 4B = 12{,}400$ bps
- Channel capacity increased, but

  disadvantage: Larger number of signal levels (M) makes it more difficult for the receiver to determine data correctly in the presence of noise





2 bits/Symbol
i.e. 2 bits /signal element

2. **SHANNON** CAPACITY FORMULA: (NOISY, ERROR-FREE)

- Highest error-free data rate in the presence of noise
- Signal to noise ratio SNR = signal / noise levels

$SNR_{dB} = 10 \log_{10} (\text{SNR ratio})$

- Errors are less likely with lower noise (larger SNR ratios). This allows higher error-free data rates i.e. larger Shannon channel capacities

- Shannon Capacity $C = B \log_2(1+SNR)$:

Highest data rate transmitted error-free with a given noise level

- For a given BW, the larger the SNR the higher the data rate I can use without introducing errors
- C/B: Spectral (bandwidth) efficiency, BE, (bps/Hz) (>1)
- Larger BEs mean better utilization of a given bandwidth B for transmitting data fast.

SHANNON CAPACITY FORMULA: COMMENTS

- Formula says: for data rates $\leq$ calculated C, it is theoretically possible to find an encoding scheme that achieves error-free transmission at the given SNR… But it does not say how!

Also:

- It is a theoretical approach based on thermal (white) noise only. But in practice, we also have impulse noise, attenuation and delay distortions, etc…
  - So, maximum error-free data rates measured in practice are expected to be lower than the C predicted by the Shannon formula due to the greater noise
- However, maximum error-free data rates can be used to compare practical systems: The higher that rate the better the system…

SHANNON CAPACITY FORMULA: COMMENTS CONTD.

- Formula suggests that changes in B and SNR can be done arbitrarily and independently… but

→ In practice, this may not be the case!

- Higher SNR obtained through excessive amplification may also introduce nonlinearities → increased distortion and inter-modulation noise … which reduces SNR!
- High Bandwidth B opens the system up for more thermal noise (kTB), and therefore reduces SNR!

SHANNON CAPACITY FORMULA: EXAMPLE

- Spectrum of communication channel extends from 3 MHz to 4 MHz
- SNR = 24dB
- Then B = 4MHz – 3MHz = 1MHz

  $SNR_{dB} = 24dB = 10 \log_{10} (SNR)$

  SNR (ratio) = $\log^{-1}_{10} (24/10) = 10^{24/10} = 251$
- Using Shannon's formula: $C = B \log_2 (1+ SNR)$

  $C = 10^6 * \log_2(1+251) \sim 10^6 * 8 = 8$ Mbps
- Based on Nyquist's formula, determine M that gives the above channel capacity:

  $C = 2B \log_2 M$

  $8 * 10^6 = 2 * (10^6) * \log_2 M$

  $4 = \log_2 M$

  $M = 16$

3. E_b/N_0 VS ERROR RATE FORMULATION (NOISE AND ERROR ARE BOTH SPECIFIED TOGETHER)

- Handling both noise and a quantified error rate simultaneously
- We introduce E_b/N_0: A standard quality measure of three channel parameters (B, SNR, R) and can also be independently related to the error rate

3. E_B/N_0 VS ERROR RATE FORMULATION (NOISE AND ERROR ARE BOTH SPECIFIED TOGETHER)

- It expresses SNR in a manner related to the data rate, R
  - E_b = Signal energy in one bit interval (Joules)

    = Signal power (Watts) x bit interval T_b (second)

    = S x (1/R) = S/R
  - N_0 = Noise power (watts) in 1 Hz = kT. Two formulations:

$$\frac{E_b}{N_0} = \frac{ST_b}{N_0} = \frac{S/R}{kT} = \frac{S}{kTR}$$

$T_b = 1/R$

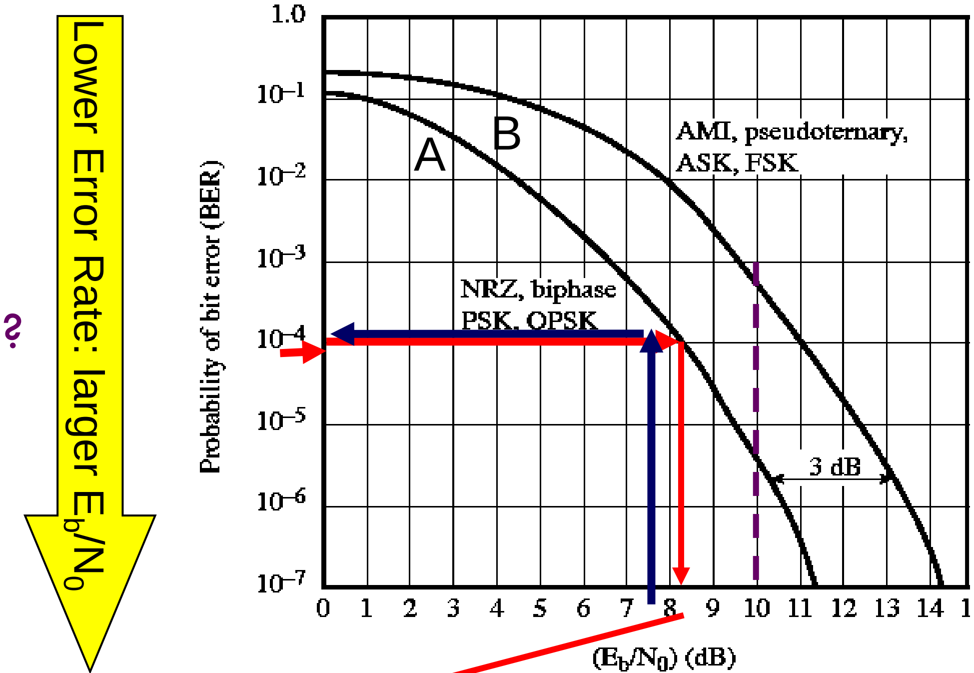
$$\frac{E_b}{N_0} = \frac{S/R}{N_0} = \frac{S}{N}\frac{B}{R} = SNR\left(\frac{B}{R}\right)$$

= SNR/BE

BER CURVE

- Which encoding scheme is better: A or B?

BER vs E_b/N_0 curve for a given encoding scheme



$$\left(\frac{E_b}{N_0}\right)_{dB} = S_{dBW} - 10\log R - 10\log k - 10\log T$$

$$= S_{dBW} - 10\log R + 228.6\ dBW - 10\log T$$

$$\frac{E_b}{N_0} = \frac{S/R}{N_0} = \frac{S}{N}\frac{B}{R} = SNR\left(\frac{B}{R}\right) = \frac{SNR}{BE}$$

Max R = C, BE = C/B

EXAMPLE:

$$\left(\frac{E_b}{N_0}\right)_{dB} = S_{dBW} - 10\log R - 10\log k - 10\log T$$

$$= S_{dBW} - 10\log R + 228.6\ dBW - 10\log T$$

- Given:
- The effective noise temperature, T, is 290°K
- The data rate, R, is 2400 bps
- Would like to operate with a bit error rate of 10^{-4} (e.g. 1 error in 10^4 bits)

What is the minimum signal level required for the received signal?

- From curve,

a minimum E_b/N_o needed to achieve a bit error rate of 10^{-4} = 8.4 dB

- 8.4 = S(dBW) – 10 log 2400 + 228.6 dBW – 10 log290
  = S(dBW) – (10)(3.38) + 228.6 – (10)(2.46)

S = -161.8 dBW

E_B/N_0 IN TERMS OF BE, ASSUMING SHANNON CHANNEL CAPACITY

From Shannon's formula:

$C = B \log_2(1+SNR)$

We have:

$$SNR = (2^{C/B} - 1) = (2^{BE} - 1)$$

From the E_b/N_0 formula:

$$\frac{E_b}{N_0} = \frac{SNR}{BE} = \frac{1}{BE}(2^{BE} - 1)$$

C/B (bps/Hz) is the spectral (bandwidth) efficiency BE based on Shannon channel capacity

EXAMPLE

Find the minimum E_b/N_0 required to achieve a Shannon bandwidth efficiency ($BE=C_{Shannon}/B$) of 6 bps/Hz:

$$\frac{E_b}{N_0} = \frac{1}{BE}(2^{BE} - 1)$$

Substituting in the equation above:

$$E_b/N_0 = (1/6)\,(2^6 - 1) = 10.5 = 10.21 \text{ dB}$$

SNR is mostly applicable to analog signal and E_b/N_0 digital signal , where we are dealing with bits