Quadratically Constrained Quadratic Program (QCQP)

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Abstract

In this project we study a certain class of nonconvex problems called QCQP.In the first section, we briefly introduce QCQP, motivations, and applications of this problem. Then we move on to different form of the problem. Next, after briefly introducing the background and literature in this matter, conditions that make the problem solvable in polynomial time are discussed. We will be also describing conditions in which the relaxation yields an approximate or exact answer. In section 4 different methods to relax the QCQP problem are discussed in detail. Finally nonconvex algorithms are studied and different methods of solving QCQP are compared to each other.

Keywords: QCQP, relaxation, convex optimization

1. Introduction

Quadratically constrained quadratic programming (QCQP) is an optimization problem in which both the objective function and the constraints are in quadratic forms. The general form of QCQP is given by

$$\begin{aligned} min_{x\in R^n} \quad & f_0(x) = x^T A_0 x + b_0^T x + c_0 \\ s.t. \quad & f_i(x) = x^T A_i x + b_i^T x + c_i \leq 0 \quad i = 1, 2, 3, \ldots, m \\ & (Gx = h) \end{aligned}$$

where $\{A_i\}_{i=0}^m$ are real symmetric matrices of dimension N and we have n equality constrains given by $h \in R^n$. QCQP is in general a non-convex problem which limits utilization of the conventional convex methods. This study focuses on the QCQP problem, and the methods that have been proposed in literature to tackle the problem.

There are several motivations behind studying QCQP. Because of its quadratic form, it can be counted as second-order approximation of nonlinear problems. In other words, it forms a connection between linear and nonlinear programming problems. It is also a subroutine used in nonlinear programming solution methods. Furthermore, QCQP is a generalization of commonly used models such as linear regression and generalized eigenvalue problems; it is a bridge to binary integer programming, and the first step toward non-convex optimization.

The problem of QCQP has wide applications in machine learning, portfolio optimization, knapsack, location-allocation, information network security, graph theory, signal processing (parametric model-based power spectrum sensing), economic equilibrium, combinatorial optimization, numerical partial differential equations, general nonlinear programming, communications and networking (multiuser detection, multicast beam-forming, the MAXCUT problem), radar (robust adaptive radar detection, optimum coded waveform design) to name but a few.[1][2][3][4]

Furthermore, note that two constraints $x_i(x_i-1) \leq 0$ and $x_i(x_i-1) \geq 0$ are equivalent to the constraint $x_i(x_i-1) = 0$, which is in turn equivalent to the constraint $x_i \in \{0,1\}$ (Boolean variable for $x_i^2 = 1$). Hence, any 0,1 integer program (in which all variables have to be either 0 or 1) can be formulated as a quadratically constrained quadratic program. Since 0,1 integer programming is NP-hard in general, QCQP is also NP-hard. Because any equation can be written as two inequalities, without loosing generality, we can merge $Gx = h$ into other $f_i(x)$ constraints. Therefore, for the rest of this study, we will not consider $Gx = h$ in the problem.

When all constraints are affine, the problem is a (nonconvex) quadratic program (QP). On the other hand, the optimization problem in which the constraints are in quadratic forms and the objective function is linear is equivalent to the original QCQP. This can be achieved by introducing a new variable w to the problem. In this formulation the objective is to minimize w such that $f_0(x) \leq w$.

$$\begin{aligned} min_{w\in R} \quad & w \\ s.t. \quad & f_0(x) \leq w \\ & f_i(x) \leq 0 \end{aligned} \tag{1}$$

Furthermore, by introducing an additional variable s to the inequalities, the problem can be demonstrated with only equality constraints ($f_i(x) + s_i^2 = 0 \quad i = 1, \ldots m \quad s \in R^m$).

The homogeneous form of the original QCQP problem is a QCQP where there are no linear terms in the variable. This can be attained by defining $\tilde{A}_i$ for $i = 1, \ldots m$:

$$\tilde{A}_i = \begin{bmatrix} \mathrm{A}_i & \mathrm{b}_i \\ \mathrm{b}_i^T & \mathrm{c}_i \end{bmatrix}$$

Therefore, the problem will have the following form:

$$\begin{aligned} min_{z\in R^n} \quad & f_0(z) = z^T \tilde{A}_0 z \\ s.t. \quad & f_i(z) = z^T \tilde{A}_i z \leq 0 \quad i = 1, 2, 3, \ldots, m \\ & z_{n+1}^2 = 1 \end{aligned}$$

Notice that in the homogeneous form there are n + 1 variables and m + 1 constraints. That is to say, the variable dimension and the number of constraints is now one larger than that

of the original QCQP form.
This problem is homogeneous in the sense that scaling z by a factor of $t \in R$ scales both the objective and lefthand sides of the constraints by a factor of t^2. It is easy to check that if z^* is a solution of the homogeneous problem, then the vector $(z_1^*/z_{n+1}^*, ..., z_n^*/z_{n+1}^*)$ is a solution of the original QCQP.[1]

2. Background

Research on quadratic programming (QP) began in the 1950s [5], and QP with a negative definite quadratic term is NP-hard. On the other hand, QP with convex objective function was shown to be polynomial-time solvable[6][7].
Then QCQP emerged with special conditions such as one constraint problem. Furthermore, semidefinite programming relaxation is closely related to QCQP. In order to solve QCQP global methods are widely used. Global methods for nonconvex problems always find an optimal point and certify it, but are often slow; the worst-case running time grows exponentially with problem size (unless P is NP). Many known algorithms for globally solving QCQP are based on the branch-and-bound framework. Branch-and-bound generally works by recursively splitting the feasible set into multiple parts and solving the problem restricted in each of the subdivision, typically via relaxation techniques. A popular variant of the branch-and-bound scheme is the branch-and-cut method, which incorporates cutting planes to tighten the subproblems generated from branching. Currently there are a number of solvers to solve QCQP including CLPEX provided by IBM which basically provides global methods for mixed-integer nonlinear programs, with limited support for nonconvex constraints.

3. Appropriate conditions on the original problem

Some special cases of QCQP provide conditions which enables solving the problem in polynomial time, i.e. the problem is no longer NP-hard. The following will describe each case in detail.

3.1. ***QCQP with Positive semi-definite A***

When all A_i matrices are positive semi-definite, it can be easily shown that QCQP is convex. Thus, it is easily solvable in polynomial time. In other words, in this case, second derivative of $f_i(x)$ is A_i which is positive semi-definite ($\nabla f_i(x) = A_i \succeq 0$). Therefore, the problem is convex and easily solvable using conventional convex methods (e.g. gradient descent, Newton's method, and so forth).

3.2. ***QCQP with one variable***

If the problem has only one variable, i.e., n = 1, then the feasible set is explicitly computable using elementary algebra. In other words, each constraints is equivalent to limiting x to one or two intervals, and the objective function is minimizing a quadratic function (parabola) on these intervals. The feasible set in this case is a collection of at most m + 1 disjoint, closed intervals on R.

$$min\ A_0x^2 + b_0x + c_0 \qquad s.t.\ \ A_ix^2 + b_ix + c_i \leq 0$$

Each constraint is, therefore, in the form of either of the two following:

$$\begin{cases} 1. \quad \alpha_i \leq x \leq \beta_i \\ 2. \quad x \leq \alpha_i \quad Or \quad x \geq \beta_i \qquad\qquad (\alpha_i \leq \beta_i) \end{cases}$$

It is possible to compute these intervals in $O(mlogm)$ time (using a binary search tree, for example). Then, minimizing a quadratic function in this feasible set can be done by evaluating the objective at the endpoints of the intervals, as well as checking the unconstrained minimizer (if there is one i.e. case 2). While one-variable problems are rarely interesting by themselves, their solvability are used in solving more complex problems.

3.3. ***QCQP with one/two constraints (nonconvex)***

This is the case with a single constraint (i.e., m = 1):

$$\begin{aligned} &min\ \ f_0(x) \\ &s.t.\ \ f_1(x) \leq 0 \end{aligned}$$

Even though this is not a convex problem in general, strong duality holds and the Lagrangian relaxation produces the optimal value.[1][4][8] A variant of this case with an equality constraint $f_1(x) = 0$ is also efficiently solvable.

In fact a more general case can be counted as solvable in polynomial time. The following cases are solvable QCQPs.

1. If QCQP has only two constraints, and one of the two constraints in the SDP relaxation is not binding
2. If the two constraint functions and the objective are all homogeneous quadratic functions
3. If one ellipsoidal and a linear complementarity constraint.

The first case can be easily solved in polynomial time. The trick here is to introduce a homogenized problem (HP) which is putting $x.t$ instead of x in the original problem on the condition that $t^2 = 1$.

$$\begin{aligned} min \quad & x^TA_0x - b_0^Txt \\ s.t. \quad & x^TA_ix - b_i^Txt + c_it^2 \leq 0 \quad i = 1, \dots, m \\ & t^2 = 1 \end{aligned}$$

Then the semidefinite programming relaxation of (HP) is derived (SP). The dual of (SP) is then called (SD), and the following proposition has been made.

The problem (SD) satisfies the slater regularity condition, either when at least one of the m constraints is ellipsoidal, or, when the objective function is strictly convex.

Now Suppose that (SP) and (SD) both satisfy the slater condition and m = 2. Furthermore, suppose the primal problem (SP) has at least one non-binding constraint at optimality. Then, the original (QCQP) can be solved in polynomial time. In other words, suppose that (SP) and (SD) both satisfy the slater condition and m = 2. Furthermore, suppose that $f1(x) \leq 0$ for all x with $f_2(x) \geq 0$ and $f_1(x)$ and $f_2(x)$ do not share any common root. Then, (QCQP) can be solved in polynomial time.
There are other nontrivial domains that are claimed by the aforementioned results, such as the whole space with two non-intersecting ellipsoids taken away. Minimizing an indefinite quadratic function over such a domain, according to above, is easy.
The only assumption that was made in here is that slater regularity condition is satisfied, i.e., there exists x_0 such that $f_i(x_0) < 0$ for all $i = 1, \ldots, m$.
In the second case, Let X^* be the low-rank minimizer of (SDP) with rank $r \leq minm - 1, n$. Then we can quickly (in polynomial time) compute a feasible solution of the homogeneous QCQP such that $x^T A_0 x \leq (1/r) z^{SDP} \leq (1/r) z^*$ if $A_i \not\succeq 0$ for i=1,...,m.
Now for the third part, can be formulated as the following.

$$
\begin{aligned}
&min f_0(x) \\
&s.t. ||x||^2 \leq 1 \\
&\quad \bar{a}^T x \leq a_0 \\
&\quad \bar{b}^T x \leq b_0 \\
&\quad (a_0 - \bar{a}^T x)(b_0 - \bar{b}^T x) = 0
\end{aligned}
$$

Suppose that the SDP relaxation and it dual problem have complementary optimal solutions. Then, the optimal value of the SDP relaxation, which equals that of the dual by strong duality, is identical to the optimal value of Mathematical Program with Equilibrium Constraint (MPEC). In other words, the relaxation admits no gap. Moreover, an optimal solution for MPEC can be obtained in polynomial time, provided that the solution for its SDP relaxation is available.[4]

3.4. ***QCQP with one interval constraint***

Consider a variation of the previous part, with an interval constraint:

$$
\begin{aligned}
min \quad & f_0(x) \\
s.t. \quad & l \leq f_1(x) \leq u
\end{aligned}
$$

Solving this variant reduces to solving QCQP with one constraint twice, once with the upper bound constraint $f_1(x) \leq u$ only, and once with the lower bound constraint $f_1(x) \geq l$ only [3]. One of the solutions is guaranteed to be an optimal point of this problem.

3.5. ***QCQP with homogeneous constraints with one negative eigenvalue***

Consider a homogeneous constraint of the form $x^T A x \leq 0$, where A has exactly one negative eigenvalue. This constraint can be rewritten as a disjunction of two second-order cone (SOC) constraints. Let $A = Q\lambda Q^T$ be the eigenvalue decomposition of A, with $\lambda_1 < 0$. Then, $x^T Ax \leq 0$ if and only if

$$\sum_{i=2}^{n} \lambda_i (b_i^T x)^2 \leq -\lambda_1 (b_1^T x)^2$$

Then if $x^T Ax \leq 0$ one of the following is true.

$$||(\sqrt{|\lambda_2|} b_2^T x, \ldots, \sqrt{|\lambda_n|} b_n^T x)||_2 \leq \sqrt{|\lambda_1|} b_1^T x$$
$$||(\sqrt{|\lambda_2|} b_2^T x, \ldots, \sqrt{|\lambda_n|} b_n^T x)||_2 \leq -\sqrt{|\lambda_1|} b_1^T x$$

Suppose that the constraint $x^T Ax \leq 0$ was the only nonconvex constraint of the original QCQP. Then, we can solve two convex problems, one where the nonconvex constraint is replaced with the first inequality, and the other where the same constraint is replaced with the second one. In conclusion, the Hessian matrix of the Lagrangian function for (A), when evaluated with optimal Lagrangian multipliers at the optimal solution, can have at most one negative eigenvalue.

3.6. ***QCQP with Toeplitz-Hermitian quadratics***

If the quadratic terms are all homogeneous and $\{A_i\}_{i=0}^m$ are all Toeplitz, strong duality holds and there is a hidden convexity in the problem. In other words, For Toeplitz , (in)feasibility of the SDP relaxed problem is equivalent to (in)feasibility of the original. Furthermore, if the original QCQP is feasible, then it can be solved to global optimality in polynomial time.

4. Relaxations

A relaxation of an optimization problem is obtained when each nonconvex constraint is replaced with a looser constraint such that the resulting feasible set contains the feasible set of the original problem. This provides a lower bound on the optimal value of the nonconvex problem. Thus, two situations arise: either the optimal solution of the relaxed problem is in the original problem's feasible set or not. In the first case, this solution is also a solution of the original problem. If the solution does not belong to the feasible set of the original problem, it is still a good starting point for the nonconvex problem. Moreover, a convex relaxation is a relaxation for which replaces the original constraints with convex ones.

4.1. ***Spectral relaxation***

Let $\lambda \in R_+^m$ be an arbitrary vector in the nonnegative orthant, and consider the following optimization problem:

$$\begin{aligned} min \quad & f_0(x) \\ s.t. \quad & \sum_{i=1}^{m} \lambda_i f_i(x) \leq 0 \end{aligned}$$

Since λ is element-wise non-negative, every feasible point x of the original QCQP is also feasible in this problem. Thus, this is a relaxation of the original QCQP, and its optimal value f^{rl} is a lower bound on f^*. Since this is a QCQP with one constraint (as described before), it is a tractable problem that can be solved using methods via matrix pencil[9]. It is easy to see that the same idea extends to problems with equality constraints.

4.2. ***Lagrangian relaxation***

Lagrangian relaxation method can be considered as a generalization of the spectral relaxation. Consider the Lagrangian dual of the QCQP problem. Then for $\lambda \in R_+^m$:

$$\mathcal{L}(x,\lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) = x^T \tilde{A}(\lambda) x + \tilde{b}(\lambda)^T x + \tilde{c}(\lambda)$$

$$\tilde{A}(\lambda) = A_0 + \sum_{i=1}^{m} \lambda_i A_i \ , \quad \tilde{b}(\lambda) = b_0 + \sum_{i=1}^{m} \lambda_i b_i \ , \quad \tilde{c}(\lambda) = c_0 + \sum_{i=1}^{m} \lambda_i c_i,$$

The Lagrangian dual function is then

$$g(\lambda) = inf_x \mathcal{L}(x,\lambda) = \begin{cases} \tilde{c}(\lambda) - (1/4)\tilde{b}(\lambda)^T \tilde{A}(\lambda)^{\dagger} \tilde{b}(\lambda) & if \quad \tilde{A}(\lambda) \succeq 0 \ , \quad \tilde{b}(\lambda) \in Range(\tilde{A}(\lambda)) \\ -\infty \qquad otherwise & \end{cases}$$

Using the Schur complement, we can write the Lagrangian dual problem as a semidefinite program (SDP):

$$\begin{array}{rl} max & \alpha \\ s.t. & \lambda_i \geq 0 \quad i = 1, \ldots, n \end{array}$$

$$\left[\begin{array}{cc} \mathrm{A}(\lambda) & 1/2(\lambda) \\ 1/2(\lambda)^T & (\lambda) - \alpha \end{array} \right] \succeq 0$$

We can show that solving the Lagrangian relaxation is equivalent to finding the value of λ that achieves the best spectral bound. This property also implies a natural way of obtaining a candidate point $x^r l$; we first solve the lagrangian to obtain a solution λ^*. Then, using the value of λ^*, we solve the spectral relaxation. Its solution can be taken as a candidate point x^{rl}.

Let d_λ be the optimal value of the original QCQP for a given $\lambda \in R_+^m$, and d^* be the optimal value of the Lagrangian dual problem. Then,

$$d^* = sup_{\lambda \in R_+^m} \, d_\lambda$$

4.3. ***Semi-definite relaxation (SDR)***

In this section, we derive a semidefinite relaxation (SDR) using a technique called lifting. This SDR and the Lagrangian dual problem are Lagrangian duals of each other. To derive the SDR, we start by introducing a new variable $X = xx^T$ and rewriting the problem as:

$$\begin{array}{ll} min & Tr(A_0X) + b_0^T x \\ s.t. & Tr(A_iX) + b_i^T x + c_i \le 0 \quad i = 1, ..., m \end{array}$$

$$Z(X, x) = \begin{bmatrix} \mathrm{X} & \mathrm{x} \\ \mathrm{x}^T & 1 \end{bmatrix} \succeq 0$$

The optimal value f^{sdr} of this problem is then a lower bound on the optimal value f^* of the original QCQP. Under mild assumptions (e.g., feasibility of the original problem), this has the same optimal value as the original problem. If an optimal point (X^*, x^*) of the SDR relaxed problem satisfies $X^* = x^*x^{*T}$ (or equivalently, the rank of $Z(X^*, x^*)$ is one), then x^* is a solution of the original problem. In general, we can show that the SDR is tight when the original problem is convex. Indeed, for a homogeneous QCQP, one can transform the problem into a convex-ish problem. This is done by using $X = xx^t$ and the trace of the yielded matrix instead. This gives us a convex problem except for the fact that rank of the original x should not exceed 1.
In fact, a fundamental issue that one must address when using SDR is how to convert a globally optimal solution X^* to the new problem into a feasible solution $\tilde{x}$ to QCQP. Now, if X^* is of rank one, then there is nothing to do, for we can write $X^* = x^*x^*T$, and x^* will be a feasibleand in fact optimalsolution to problem. On the other hand, if the rank of X^* is larger than 1, then we must somehow extract from it, in an efficient manner, a vector $\tilde{x}$ that is feasible for the original problem.
In conclusion, for a real-valued (resp. complex-valued) homogeneous QCQP with 2 (resp. 3) constraints or less, SDR is not just a relaxation. It is tight, i.e., solving the SDR is equivalent to solving the original QCQP.
For a homogeneous separable QCQP, suppose that none of the solution $\{X_i^*\}_{i=1}^k$ to the SDR satisfies $X_i^* = 0$ for some i. Then, the SDR is tight if $m \le k + 2$ in the complex case; and if $m \le k + 1$ in the real case.

4.4. ***Tightening relaxations***

Lower bounds obtained from relaxations can be improved by adding additional quadratic inequalities to QCQP that are satisfied by any solution of the original problem. In particular, redundant inequalities that hold for all feasible points of the original QCQP can still tighten the relaxation.
Consider the set of affine inequalities $Bx \le C$ which can be considered as several $f_i(x)$ displayed as a single matrix representation in the case of $A_i = 0$.
In this example, $(Bx-C)(Bx-C)^T \ge 0$ for any x satisfying the affine inequalities (element-wise). Each entry of the left-hand side has the form $(\hat{b}_i^T x - c_i)(\hat{b}_j^T x - c_j)$ where $\hat{b}_i$ are rows of B. These indefinite quadratic inequalities can be added to the original QCQP without changing the set of solutions.
In general, any exclusive disjunction of two affine inequalities can be encoded as a quadratic inequality. Let $a^T x \le b$ and $c^T x \le d$ be two affine inequalities such that for every feasible point of the original QCQP, exactly one of them holds. Then, $(a^T x - b)(c^T x - d) \le 0$ is a redundant quadratic inequality that holds for every feasible point x.

4.5. ***Relaxation of relaxations***

The Lagrangian and semidefinite relaxations are polynomial-time solvable, but in practice, can be expensive to solve as the dimension of the problem gets larger. In this section, we explore ways to further relax the relaxation methods discussed above to obtain lower bounds on f^* more efficiently.
Relaxing the Lagrangian relaxation can be yielded by the fact that weak duality implies that it is not necessary to solve the Lagrangian dual optimally in order to obtain a lower bound on f^*; any feasible point (λ, α) of the Lagrangian dual induces a lower bound on f^*. We note that α is easy to optimize given λ. In fact, when λ is fixed, optimizing over α is equivalent to solving the spectral relaxation with the same value of λ.
Now, we discuss relaxation methods for the SDR. Note that the semidefinite constraint $Z(X,x) \succeq 0$ can be written as an infinite collection of affine constraint $a^T Z(X,x)a \geq 0$ for all $a \in R^{n+1}$ of unit length, i.e., $\|a\|^2 = 1$. For example, if a is the ith unit vector, the resulting inequality states that $X_i i$ must be nonnegative. To approximate the optimal value f^{sdr}, one can generate affine inequalities to replace the semidefinite constraint and solve the resulting linear program (LP).

5. Nonconvex algorithms

5.1. ***Consensus-ADMM***

This is basically an improved version of the original ADMM. In this method, the problem is first reformulated in consensus optimization form, to which the alternating direction method of multipliers can be applied. To do so, we introduce m auxiliary variables $z_1, \ldots, z_m$,subject to $z_i = x, \quad i = 1, \ldots, m.$

$$\begin{aligned} min_{x,z_i} \quad & \sum_{i=1}^{m} f_i(z_i) + r_x \\ subjectto \quad & z_i = x, \quad i = 1, \ldots, m. \end{aligned}$$

Here r is just some additional regularization on x. Then the corresponding consensus-ADMM algorithm takes the form of the following iterations:

$$\begin{aligned} & (A_0 + m\rho I)^{-1}(b_0 + \rho \sum_{i=1}^{m}(z_i + u_i) \rightarrow x \\ & argmin_{z_i} ||z_i - x + u_i||^2 \rightarrow z_i \\ & s.t. \quad z_i^H A_i z_i - 2\mathcal{R}\{b_i^H z_i\} \leq c_i \\ & u_i + z_i - x \rightarrow u_i \end{aligned}$$

The reformulation is done in such a way that each of the subproblems is a QCQP with only one constraint (QCQP-1), which is efficiently solvable irrespective of (non)convexity. The proposed algorithm for minimizing a norm function is in the following.

Algorithm 3: Consensus-ADMM for (10).

initialize $\boldsymbol{x}$, $\boldsymbol{z}_i$ and $\boldsymbol{u}_i$;
for $i = 1, ..., m$ **do**
  Take the eigen-decomposition of $\boldsymbol{A}_i = \boldsymbol{Q}_i \boldsymbol{\Lambda}_i \boldsymbol{Q}_i^H$;
end
repeat
  $\boldsymbol{x} \leftarrow \frac{1}{m} \sum_{i=1}^{m} (\boldsymbol{z}_i + \boldsymbol{u}_i)$;
  for $i = 1, ...m$ **do**
    $\mu_i \leftarrow \arg_\mu \phi_i(\mu) = 0$ using Alg. 1 or 2;
    $\mu_i \leftarrow \max\{0, \mu_i\}$; `// for` $\leq$ `constraints`
    $\boldsymbol{z}_i \leftarrow \boldsymbol{Q}_i (\boldsymbol{I} + \mu_i \boldsymbol{\Lambda}_i)^{-1} \boldsymbol{Q}_i^H (\boldsymbol{x} - \boldsymbol{u}_i)$;
    $\boldsymbol{u}_i \leftarrow \boldsymbol{u}_i + \boldsymbol{z}_i - \boldsymbol{x}$;
  end
until $\boldsymbol{x}$ *feasible*;
$\rho = 1$;
repeat
  $\boldsymbol{x} \leftarrow \frac{1}{m+\rho^{-1}} \sum_{i=1}^{m} (\boldsymbol{z}_i + \boldsymbol{u}_i)$;
  for $i = 1, ...m$ **do**
    $\mu_i \leftarrow \arg_\mu \phi_i(\mu) = 0$ using Alg. 1 or 2;
    $\mu_i \leftarrow \max\{0, \mu_i\}$; `// for` $\leq$ `constraints`
    $\boldsymbol{z}_i \leftarrow \boldsymbol{Q}_i (\boldsymbol{I} + \mu_i \boldsymbol{\Lambda}_i)^{-1} \boldsymbol{Q}_i^H (\boldsymbol{x} - \boldsymbol{u}_i)$;
    $\boldsymbol{u}_i \leftarrow \boldsymbol{u}_i + \boldsymbol{z}_i - \boldsymbol{x}$;
  end
until *The successive difference of* $\boldsymbol{x}$ *is smaller than* ε;

6. Results

In this section we ran different convex/nonconvex algorithms for a simple QCQP problem. You can see the plots for each method in the following.

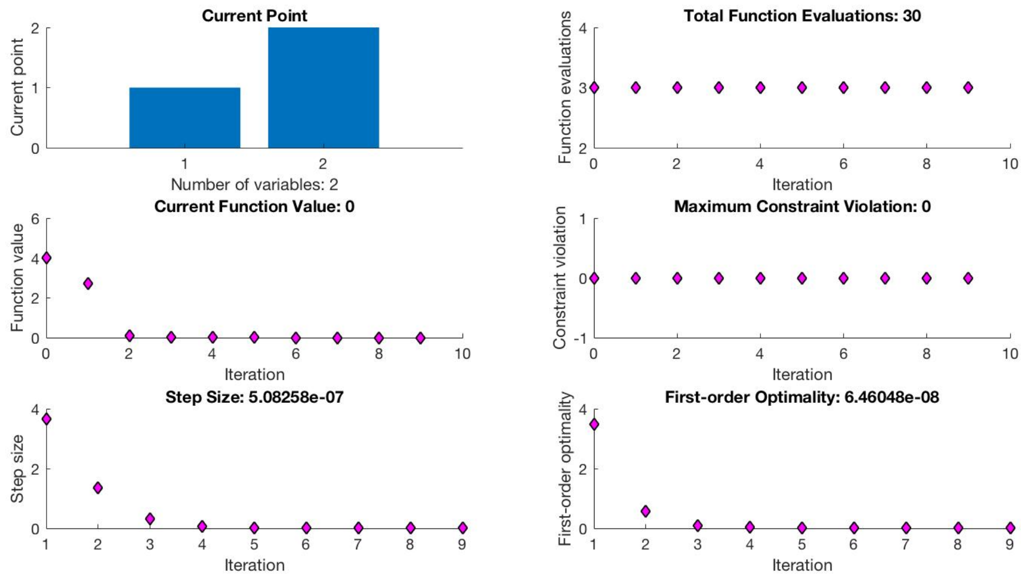

Figure 1: Finding minimum of a constrained nonlinear multivariable function using the interior-point algorithm (fmincon).

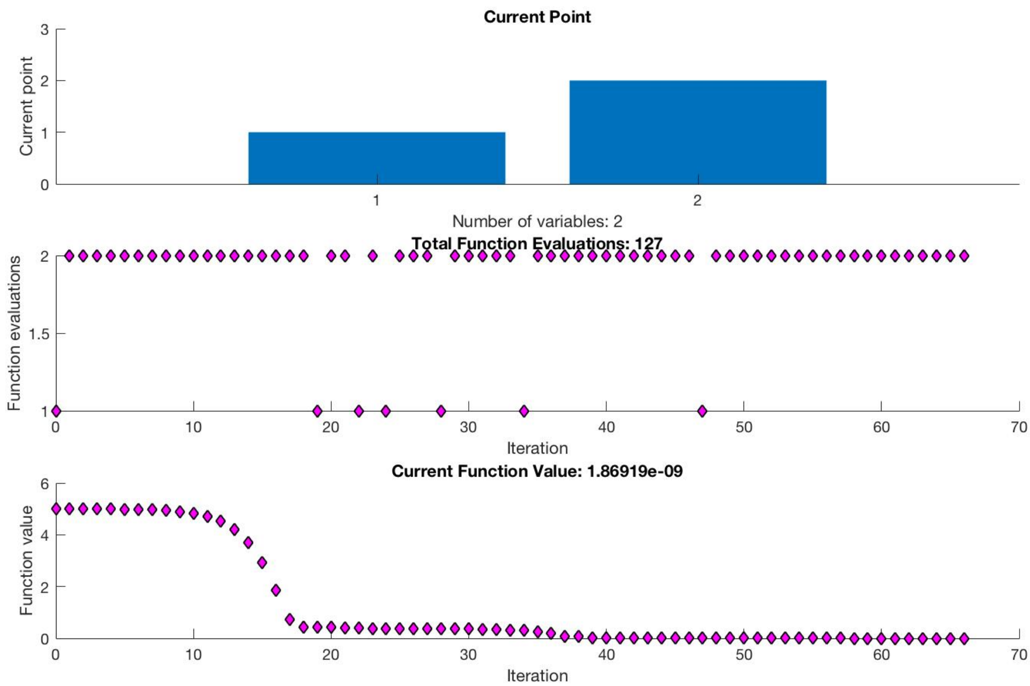

Figure 3: Finding a minimum of an unconstrained nonlinear multivariable function using a derivative-free method (fminsearch).

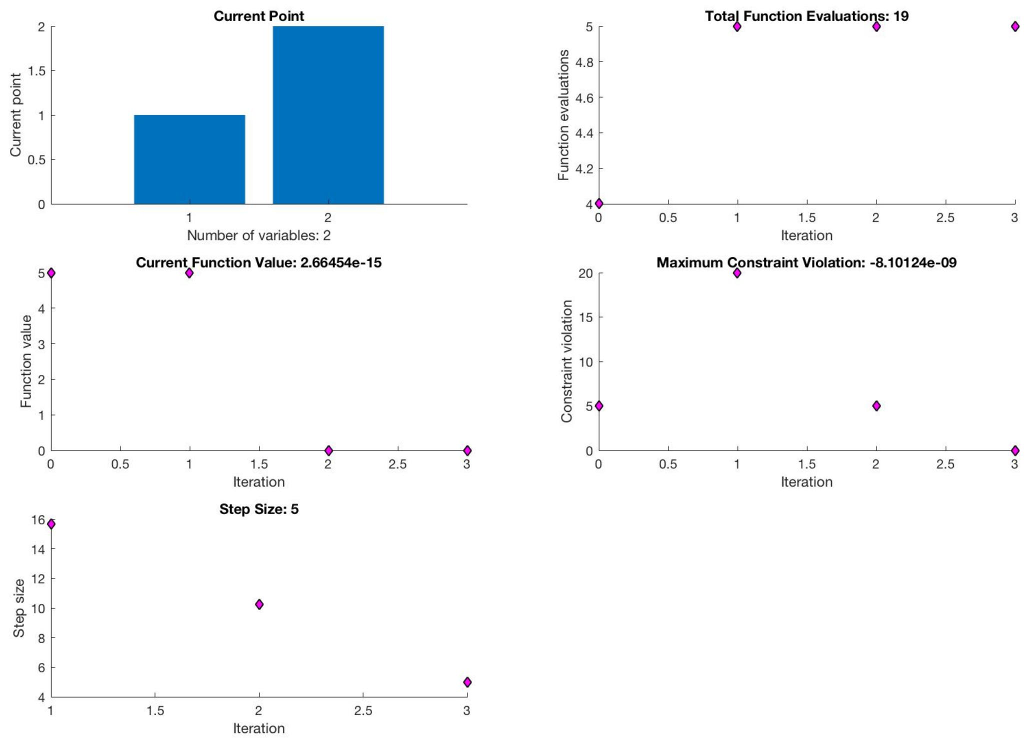

Figure 2: Finding a minimum of the worst-case value of a set of multivariable functions (minimax).

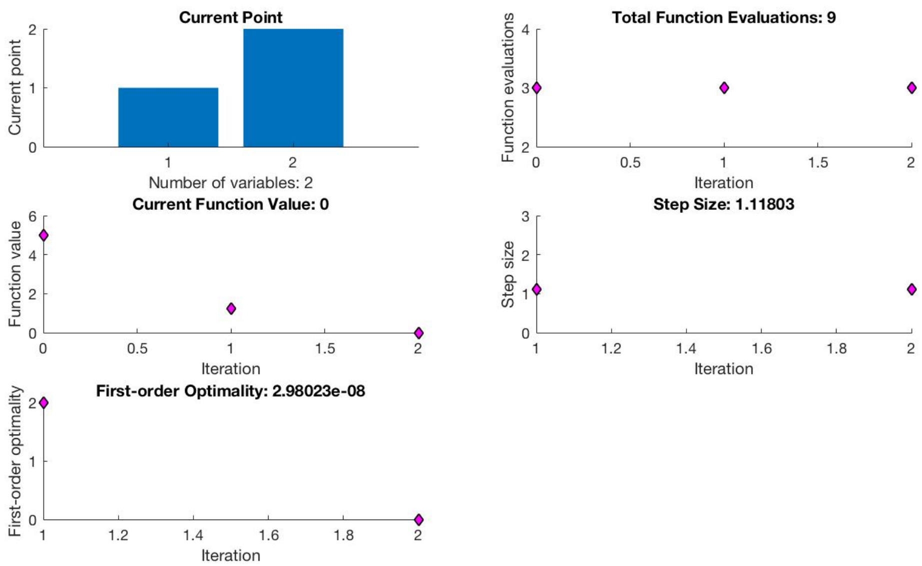

Figure 4: Finding a minimum of an unconstrained nonlinear multivariable function (fminunc).

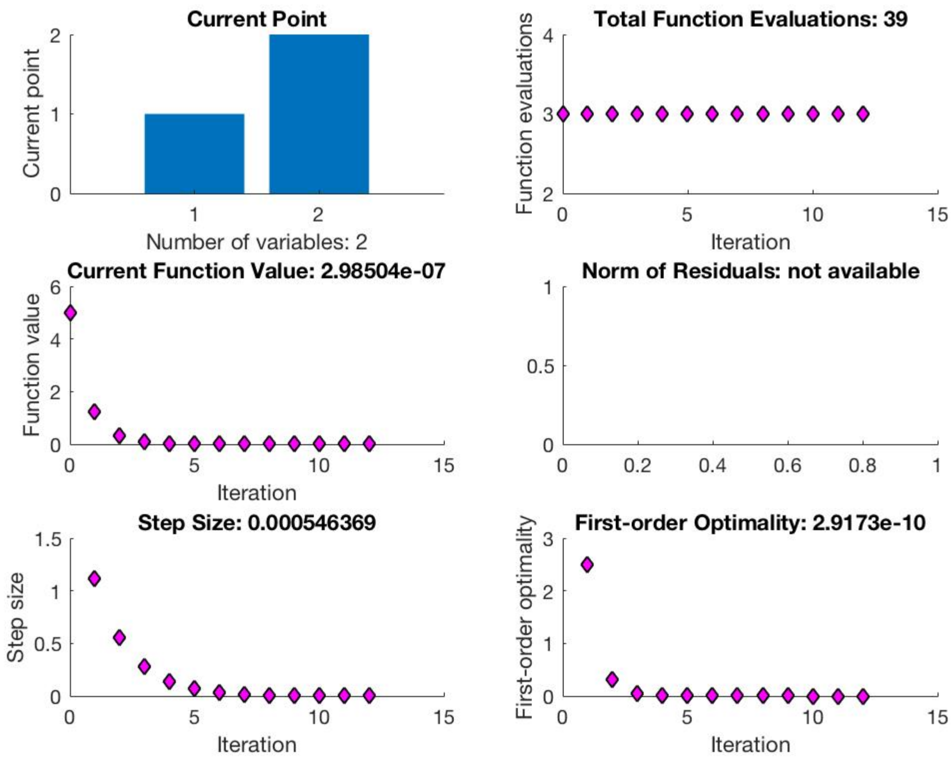

Figure 5: Solving a system of nonlinear equations(Levenberg-Marquardt).

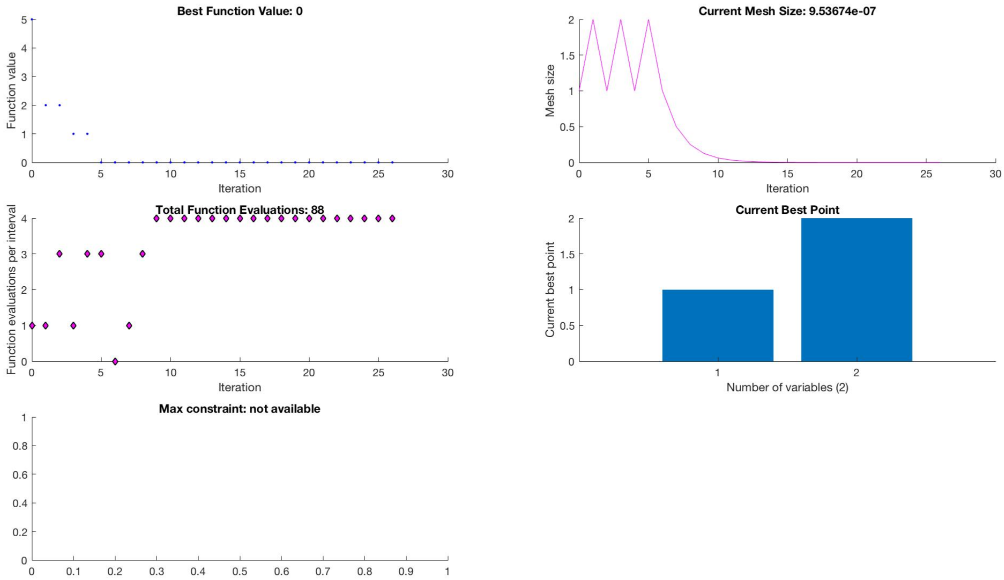

Figure 6: Pattern search looks for a local minimum based on an adaptive mesh that, in the absence of linear constraints, is aligned with the coordinate directions (patternsearch).

As it can be illustrated by the figures above, one can observe that methods such as 'minimax' and 'fminunc' generate the result in shorter iterations in comparison with other methods. First, we should note that 'fminunc' generates results when the problem is unconstrained which is not true in our case. Therefore, generating the true optimum point was by luck and managable by choosing an appropriate starting point. Furthermore, fminsearch seems to take a lot of iterations to get the answer. This is again because it is an unconstrained method and has to check many points to find the optimal value. All the methods yielded the right answer to the problem due to the simple nature of our problem. In general, one can say nonconvex algorithms are slower because they are searching globally while convex algorithms are less precise and in order to choose one algorithm, one must see the required features.

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