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Optimal Trading of Arbitrage Opportunities with Market Impact

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Abstract

This paper studies the optimal trading strategy of arbitrageurs in a dynamic economy where there are transaction costs and the arbitrageur's trades reduce (or eliminate) future arbitrage opportunities. In contrast to the standard textbook arbitrage trading strategy which has infinite present value, we show that an arbitrageur's expected discounted trading profits are finite. In addition, we show that it is rational for arbitrageurs not to trade the first time that arbitrage profits exceed their transaction costs. In fact, in our economy, arbitrage profits will often exceed the transaction cost band, diappear, then reappear again. The implications of this observation for the existing empirical literature testing for arbitrage opportunities is also discussed.

KEY WORDS: arbitrageurs, arbitrage opportunities, optimal trading strategies, liquidity risk, quantity impact on price.

1 Introduction

The standard textbook presentations of security valuation for options, derivatives, and even stock, always start with arbitrage-free pricing in a frictionless and competitive market, arguing that arbitrage opportunities cannot exist (see Duffie [8], Jarrow and Turnbull [9]). It is reasoned that arbitrage opportunities cannot exist because if they did, arbitrageurs could make infinite profits (in finite time) by trading on these mispricings. And, this is inconsistent with the existence of market equilibrium and stable market prices.

Of course, to apply these insights to actual markets we need to relax the frictionless market hypothesis. Given market frictions like transaction costs,

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collateral/margin requirements, and short sale restrictions, the argument is extended. In imperfect markets, arbitrage opportunities can exist but only to the extent that they cannot be exploited due to these market frictions (see Barles and Soner [1], Broadie, Cvitanic and Soner [3], Cuoco and Liu [6], Cvitanic, Pham and Touze [7], Jouini and Kallal [10], Naik and Uppal [12], and Soner, Shreve and Cvitanic [15]). In these economies, exploiting such limited arbitrage opportunities generates risky investment opportunities analogous to those obtained by investing in the securities themselves. In fact, the optimal trading strategies for exploiting such limited arbitrage opportunities have been studied by Tuckman and Vila [16] and Liu and Longstaff [11]. An implication of this line of research, however, is that arbitrage opportunities in excess of a "transaction cost" band still cannot exist in a well functioning economy.

Although insightful, this line of research has a limitation. In deriving its implications for market prices and practice, it maintains the competitive market assumption. The competitive market assumption imposes the condition that arbitrageurs act as price takers and their trades have no impact on the market price. This assumption is imposed for tractability, and not because it is true in practice. In fact, it is often argued that it is exactly the action of these arbitrageurs that force market prices to be nearly arbitrage free. And, for this argument to be true, rational arbitrageurs cannot be price takers.

The purpose of this paper is to investigate the effect of relaxing the price taking assumption on the optimal trading strategy for arbitrageurs. In this regard, our paper adds to the literature on optimal arbitrage trading strategies started by Tuckman and Vila [16] and Liu and Longstaff [11]. The standard textbook trade for a price taker exploiting an arbitrage opportunity is to trade as quickly and as often as possible. In a dynamic setting, when arbitrageurs know that their trades will reduce or eliminate future arbitrage opportunities, the standard trade may no longer be optimal. We study if this is indeed the case. Such a quantity impact on the market price is related to the asset pricing literature including liquidity risk, see Çetin, Jarrow, and Protter [4] and Çetin, Jarrow, Protter and Warachka [5].

To generate our insights, we construct a continuous trading economy with frictions - transaction costs - where arbitrage opportunities can exist. We take these arbitrage opportunities as exogenous with their magnitudes stochastic across time. In our economy, risk neutral arbitrageurs exist. However, these arbitrageurs' trades have a permanent impact on the market price. When they trade on an arbitrage opportunity, their trades reduce (or even eliminate) the arbitrage opportunity, limiting the magnitude of future mispricings. In this setting, we study the optimal trading strategy of arbitrageurs seeking to maximize their expected discounted revenues. This idealized set-up is a first approximation to the actual trading activities engaged in by hedge funds and proprietary trading operations at commercial and investment banks who search and trade on these limited arbitrage opportunities (an example might be a statistical arbitrage group trading a stock index against its futures contracts).

We obtain a new and important insight into market mispricings. When trades have market impact, arbitrage opportunities in excess of transaction costs can exist, even in rational settings. Arbitrage opportunities will be bounded in magnitude (the bound depends on the market's parameters), but at a higher level than the existing literature suggests. Furthermore, arbitrage opportunities will disappear at random times, to reappear again as time progresses.

The intuition is simple. Rational arbitrageurs will only trade if the arbitrage opportunities are sufficiently large. If they trade when the arbitrage opportunity is too small, their impact on the market price precludes them from trading on a potentially larger mispricing in the near future. It is the total revenues from all their trades across time that is important.

Our paper has an important implication for the empirical literature testing for the existence of arbitrage opportunities in excess of the "transaction cost" band. Violations of this condition is taken to be proof of a market failure and an inefficient market. Our paper shows that this conclusion is false, because with market impact, the non-exploitation of arbitrage opportunities in excess of this "transaction cost" band can be optimal. In this regard, the implications of this empirical literature investigating the efficiency of financial markets needs to be revisited and perhaps additional tests performed.

An outline for this paper is as follows. Section 2 presents the model set-up. Section 3 solves for the optimal trading strategy of an arbitrageur. This section contains the key results of the paper. Section 4 provides another example, while section 5 concludes. All proofs are contained in the appendix.

2 The Set-up

For pedagogic reasons, we choose a simple formulation to make the economics of the solution transparent. However, every aspect of our model can be generalized with correspondingly more abstract notation and less transparent economics.[1]

We consider a continuous trading economy with a time horizon $t \in [0, \infty)$. The randomness in the economy is characterized by a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions (see Protter [14]) where P is the statistical probability measure.

For analysis, we fix a particular arbitrage opportunity to be considered by our arbitrageur. The arbitrage opportunity consists of a position in some traded security plus an offsetting position in the same cash flows, but constructed synthetically using an alternative set of traded assets.[2] For example, if the position in the traded security is being long a forward contract on a stock index, the offsetting position is being short a synthetic forward contract on the stock index constructed using a cash and carry strategy involving the underlying stocks themselves (see Jarrow and Turnbull [9]). Another example could be holding a long position in a European call option on a stock and creating an

[1] This assertion is documented in the appendix where it is easily seen that the verification theorems can be generalized to handle risk averse traders, more complex jump diffusion processes, and the inclusion of stochastic processes for the spot rate of interest, transaction costs, and market impact.

[2] This alternative set of traded assets could be a singleton.

identical but synthetic short call on the stock via delta hedging in the underlying stock.[3]

We now add a collection of five assumptions which completely characterize our economy.

Assumption 1. (Transaction Quantity) The arbitrage profits are measured for a single transaction unit where one buys and sells both the traded and synthetic asset, respectively.

For most transactions, to take advantage of an arbitrage opportunity, the trader would need to submit market orders at the quoted bid/ask prices and volumes. This is the transaction unit we are considering.

Our economy has market frictions, in particular we assume that every transaction involves a fixed transaction cost per trade, i.e.,

Assumption 2. (Fixed Cost per Trade)[4] The transaction cost per trade is $c > 0$.

These transaction costs include transaction fees plus any holding costs in the sense of Tuckman and Vila [16]. The existence of these transaction costs imply that arbitrage opportunities can rationally exist in our economy.

We let X_t denote the arbitrage profits per unit transaction at time t *before the consideration of transaction costs*. For example, if at time t the arbitrageur buys a security for q dollars and simultaneously sells the same security for p dollars where $p > q$, then the arbitrage profits at time t are $p - q > 0$. For our analysis, if the arbitrage opportunity involves a dynamic trading strategy over some fixed future time interval, one can view X_t as the present value of this dynamic arbitrage trading strategy. If the dynamic trading strategy cannot be executed with probability one, perhaps due to the violation of a short sale or collateral constraint, then X_t could be negative with positive probability. This is the case, for example, with the trading strategies contained in Liu and Longstaff [11].

We take as given a stochastic process for the evolution of these arbitrage opportunities.

Assumption 3. (Evolution of the Arbitrage Opportunity)

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \tag{1}$$

where $b : \mathbb{R} \to \mathbb{R}, \sigma : \mathbb{R} \to \mathbb{R}$, $\sigma(x) > 0$ for all x, and B_t is a standard Brownian motion on the filtered probability space. We assume that the measurability and

[3] Of course, this requires that the delta hedging trading strategy exactly replicates the cash flows to the traded call option. Sufficient conditions for such a trading strategy to exist are contained in any textbook on option pricing.

[4] We could also add a variable transaction cost equal to $\lambda X_{\tau -}$, then the profits become $(X_{\tau -} - \lambda X_{\tau -} - c)$, see Okesendal and Sulem [13], p. 95, exercise 6.3.

integrability conditions necessary for this stochastic differential equation to have a strong solution are satisfied (see Protter [14]).

As given, we do not restrict the arbitrage profits to be positive. There are two interesting cases in this regard. If trading in the arbitrage opportunity is completely unrestricted except for the transaction costs (a restriction might be short sale constraints or a collateral requirement), then one can impose the extra condition that $X_t \geq 0$ for all t. This would follow by just changing the sign of the initial position if the trading strategy first considered had non-positive profits. In some cases, however, the trading strategy could be restricted in such a manner that one direction of the trade could generate negative profits as in Liu and Longstaff [11]. The general form of expression (1) allows for both possibilities.

We note that the evolution of the arbitrage opportunity's profits as given in expression (1) are determined by the interaction of all market participants, *except our trader*. We take a partial equilibrium perspective in this regard. If there are other arbitrageurs in the economy, what the trader sees are the "remaining" arbitrage opportunities in the market available to him.

When our trader transacts, he will reduce (perhaps eliminate) these remaining arbitrage profits. The market impact of his trade is characterized in the next assumption.

Assumption 4. (Market Impact) When the trader transacts at time τ^-, due to his market impact, the change in the level of the arbitrage opportunity is

$$X_\tau = (1 - \delta)X_{\tau^-} \tag{2}$$

where $0 < \delta \leq 1$ is a constant and $X_{\tau^-} \equiv \lim_{t \to \tau,\ t < \tau} X_t$.

We see that after his transaction, the arbitrage profits will continue evolving according to expression (1), but starting from a different and lower level. Note that if $\delta = 1$ then the arbitrage opportunity disappears completely after his trade.

Assumption 5. (Default Free Spot Rate of Interest) The default free spot rate of interest $r > 0$ is constant.

We assume that our trader is risk neutral. The trader can transact when and as often as desired over the time horizon $[0, \infty)$. His transaction times are given by sequence of stopping times, denoted

$$w = (\tau_1, \tau_2, ...)$$

where $\tau_k \leq \tau_{k+1}$. We denote the set of all such transaction times as $w \in W$.

A state in our setting is a time s and arbitrage opportunity x. Given the state (s, x), the arbitrageur's objective function is given by the expected discounted profits from his trading activity $w \in W$, i.e.

$$J^w(s,x) \equiv E_{(s,x)} \left(\sum_{k=1}^{\infty} e^{-r\tau_k} (X_{\tau_k^-} - c) \right) \tag{3}$$

where the discounted profits from a transaction at time τ_k are:

$$e^{-r\tau_k} \left(X_{\tau_k^-} - c \right). \tag{4}$$

The decision problem is to find $w \in W$ to maximize $J^w(s,x)$. That is, we want to solve

$$\sup_{w \in W} J^w(s,x). \tag{5}$$

With market impact, the optimal trading strategy is not necessarily the textbook solution. The textbook solution is to trade at every time s that a positive arbitrage opportunity $x > c$ appears. For many diffusion process evolutions as in expression (1), this would imply an infinite number of trades over an open time interval starting at time s, yielding infinite expected discounted profits. However, with market impact, this need not be the case.

With market impact, the trader has a nontrivial decision. Consider the trader standing at time s with an arbitrage opportunity $x > c$ currently available. If he trades now, he gets the immediate profit of $(x - c)$. And, after his trade, the arbitrage opportunity declines to $(1 - \delta)x$ due his market impact. If the arbitrage opportunity falls below c, then he cannot trade again at time s^+. Alternatively, if he doesn't trade at time s, the arbitrage opportunity may increase even further at time s^+, generating greater profits (even after considering the default free spot rate r). Of course, the arbitrage profits being stochastic may also decline. Hence, there is a decision to be made, based on the magnitude of the price impact versus the likelihood that the arbitrage opportunity will increase more than the discount rate over the time interval $[s, s + ds]$.

We illustrate that the textbook solution fails to hold, in general, by considering two simple examples in the subsequent sections. The failure of the textbook solution proves that in the presence of market impact, rational traders will not always trade when an arbitrage opportunity exists in excess of transaction costs. However, as shown below, such deviations will be bounded, and disappear at random times as arbitrageurs exploit these arbitrage opportunities.

3 Brownian Motion Arbitrage Profits

To show that it is rational for arbitrageurs to not trade the first time that the arbitrage opportunity exceeds the transaction cost, we consider the simplest example first, where the arbitrage profits evolve as a Brownian motion with drift,[5] i.e.

[5] This example is isomorphic to the problem studied by Willassen [17].

$$dX_t = bdt + \sigma dB_t \tag{6}$$

where b and $\sigma > 0$ are constants.

Because arbitrage profits can go negative for this evolution, this example is consistent with the arbitrage trading strategy being restricted in one direction. An example where only positive arbitrage profits are possible is contained in the next section.

Under this evolution we can prove the following theorem. The proof is contained in the appendix.

Theorem 1 *Let the continuation region $\mathcal{C} = \{x \leq x^*\}$ for some x^*.*

Let $(K > 0, x^* > c)$ solve the system of equations:

$$Ke^{\gamma^* x^*} = Ke^{\gamma^*(1-\delta)x^*} + (x^* - c) \quad \text{and} \tag{7}$$

$$\gamma^* Ke^{\gamma^* x^*} = \gamma^*(1-\delta)Ke^{\gamma^*(1-\delta)x^*} + 1 \quad \text{where} \tag{8}$$

$$\gamma^* = -\frac{b}{\sigma^2} + \sqrt{\left(\frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0. \tag{9}$$

Let (b, σ) be such that:

$$-r + b\gamma^*(1-\delta) + \frac{1}{2}\sigma^2\left[\gamma^*(1-\delta)\right]^2 \leq 0, \quad \text{and} \tag{10}$$

$$-r(x^* - c) + b \leq 0. \tag{11}$$

Then, the solution to the optimization problem is:

$$\sup_{w \in W} J^w(s,x) = \ e^{-rs}Ke^{\gamma^* x} \quad if \quad x < x^* \tag{12}$$

and the optimal strategy is to transact whenever x hits x^* from below.

This theorem characterizes the optimal trading strategy of an arbitrageur under the Brownian motion evolution. The solution exists if the parameters of the problem $(K, x^*, \delta, r, b, \sigma)$ satisfy expressions (7), (8), (10) and (11). In this case, the arbitrageurs expected discounted profits from trading are finite. This contrasts with the standard textbook characterization of an arbitrage trading strategy yielding infinite profits.

Theorem 1 is best understood by considering the graph contained in Figure 1. In Figure 1 the jagged line represents a realized evolution of the arbitrage opportunity's profits X_t. Also depicted is the transaction cost $c > 0$ and trigger level x^* represented as horizontal lines. The optimal trading strategy for the arbitrageur is to trade the first time that the trigger level x^* is hit from below.

For this sample realization, the first transaction takes place at time τ_1. Due to his market impact, the trade causes an immediate drop of δx^* in the arbitrage

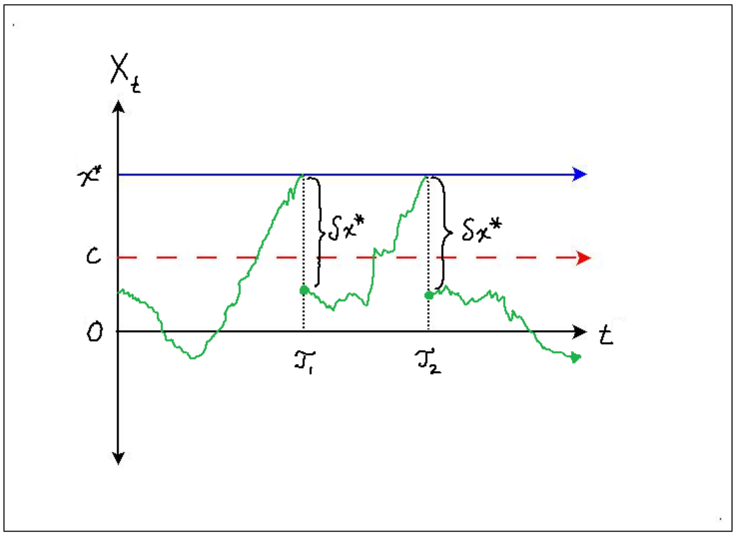

Figure 1: Optimal Transaction Times for a Sample Path of an Arbitrage Opportunity

profits. After the decline, the arbitrage profits evolution continues again, until time τ_2 when the trigger level is again hit and another transaction occurs.

Note that when the arbitrage profits lie within the interval $[c, x^*)$, no transaction occurs. This implies that it is optimal for the arbitrageur not to exploit any arbitrage opportunity that exceeds the transaction cost c but less than the trigger level x^*.

It is important to note that a solution to the arbitrageurs optimization problem does not always exist. For example, if there is no liquidity impact ($\delta = 0$), then there is no solution to either expressions (7) and (8). When $\delta = 0$, the optimization is no longer an impulse control problem. Here, the optimal trading strategy is to trade continuously using a local time at every instant where $x > c$. This set of parameters gives back the standard textbook solution of infinite wealth from exploiting arbitrage opportunities.[6]

A range of parameters where it is easily seen that a solution exists is when $b \leq 0$ and $\delta = 1$. In this case arbitrage profits have a negative drift and after a trade by the arbitrageur, the arbitrage opportunity disappears. These are plausible conditions. Then, expression (10) simplifies to $-r \leq 0$ and expression (11) simplifies to $-r(x^* - c) + b \leq 0$, both of which are true. The $(K, x^* > c)$ that solve expressions (7) and (8) are $x^* = \frac{1}{\gamma^*} + c$ and $K = \frac{e^{-(1+\gamma^* c)}}{\gamma^*} > 0$, and the expected discounted profits are $Ke^{\frac{\sqrt{2r}}{\sigma}x}$ given an arbitrage opportunity of

[6] Another example where a solution doesn't exist is if $r = 0$. Here $\gamma^* = 0$. The reason is that the time value of money doesn't matter, and since the trader can trade up to time ∞, one can achieve unbounded profits using any $x^* > c$.

size x at time 0. More generally, theorem 1 characterizes the conditions under which solutions exist for the remaining range of the parameters, i.e. when $b > 0$ and $\delta < 1$.

In contrast to the standard textbook arbitrage trading strategy which has infinite present value, this example shows that an arbitrageurs' expected discounted trading profits are finite. In addition, it is rational for arbitrageurs not to trade the first time that arbitrage profits exceed their transaction costs. In fact, in this example, arbitrage mispricings will often exceed the transaction cost band, disappear, then reappear again.

These results are important for interpreting the empirical literature that tests for the existence of arbitrage opportunities which exceed the "transaction cost" band. In the existing literature, an arbitrage opportunity that exceeds this "transaction cost" band is taken to be evidence of an inefficient market. This is a false conclusion. For the validity of this conclusion the implicit assumption that there is no liquidity risk in the economy, i.e. investors trades do not have a quantity impact on the market price, must be true. There exists considerable evidence, however, that this implicit assumption is violated, even in very liquid markets. Consequently, the evidence against market efficiency needs to be reinterpreted with these new insights in mind.

4 Geometric Brownian Motion Arbitrage Profits

This section provides an example where arbitrage profits are restricted to be nonnegative, implying that the arbitrage opportunity trade is unrestricted in sign. For this illustration, we use geometric Brownian motion, i.e.

$$dX_t = bX_t dt + \sigma X_t dB_t \tag{13}$$

where b and $\sigma > 0$ are constants.

An analogous theorem can be proven that characterizes the arbitrageur's optimal trading strategy. The proof of this theorem is contained in the appendix.

Theorem 2 *Let the continuation region $\mathcal{C} = \{x \leq x^*\}$ for some x^*.*

Let $(K > 0, x^* > c)$ solve the system of equations:

$$K(x^*)^{\gamma^*} = K(1-\delta)^{\gamma^*}(x^*)^{\gamma^*} + (x^* - c) \quad \textit{and}$$

$$\gamma^* K(x^*)^{\gamma^*-1} = \gamma^* K(1-\delta)^{\gamma^*}(x^*)^{\gamma^*-1} + 1 \quad \textit{where}$$

$$\gamma^* = -\frac{\left(b - \frac{1}{2}\sigma^2\right)}{\sigma^2} + \sqrt{\left(\frac{\left(b - \frac{1}{2}\sigma^2\right)}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \quad \textit{and} \quad \gamma^* > 1.$$

Let (b, σ) be such that:

$$-r + b + \frac{1}{2}\sigma^2\gamma^*(\gamma^* - 1) \leq 0, \quad \textit{and}$$

$$-r(x^* - c) + bx^* \leq 0.$$

Then, the solution to the optimization problem is:

$$\sup_{w \in W} J^w(s, x) = \quad e^{-rs} K x^{\gamma^*} \quad if \quad x < x^*$$

and the optimal strategy is to transact whenever x hits x^* from below.

Since this theorem's representation is analogous to theorem 1, no further explanation is provided.

5 Conclusion

This paper studies the optimal trading strategy of arbitrageurs in a dynamic economy where there are transaction costs and the arbitrageur's trades reduce (or eliminate) future arbitrage opportunities. In contrast to the standard textbook arbitrage trading strategy which has infinite present value, we show that an arbitrageurs' expected discounted trading profits are finite. In addition, we show that it is rational for arbitrageurs not to trade the first time that arbitrage profits exceed their transaction costs. In fact, in our economy, arbitrage profits will often exceed the transaction cost band, disappear, then reappear again.

These results are important for interpreting the empirical literature that tests for the existence of arbitrage opportunities which exceed the "transaction cost" band. In the existing literature, an arbitrage opportunity that exceeds this "transaction cost" band is taken to be evidence of an inefficient market. This is a false conclusion. For the validity of this conclusion the implicit assumption that there is no liquidity risk in the economy, i.e. investors trades do not have a quantity impact on the market price, must be true. There exists considerable evidence, however, that this implicit assumption is violated, even in very liquid markets. Our understanding of market efficiency needs to be revisited and new tests designed with these insights in mind.

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6 Appendix

6.1 Verification Theorems

This section provides a method for determining the solution for the optimal trading strategy. We obtain this solution by transforming the problem into that of a standard impulse control problem, whose solutions are well studied (see Okesendal and Sulem [13]). All definitions for the regularity conditions used in this appendix can be obtained from Okesendal and Sulem [13].

An impulse control ν for such an optimization problem is a sequence of pairs $((\tau_1, \xi_1), (\tau_2, \xi_2), ...)$ where $\xi_k \in \mathbb{R}$ is an impulse and $w \in W$. We denote the set of all impulse controls starting from a given x as $I(x)$. Here, an impulse corresponds to a change in the arbitrage profits. In the general formulation of an impulse control problem, the impulses are part of the decision problem. For our problem, however, the impulse choice set $I(x)$ is a singleton because our trader has no choice on the impulse. The impulse's magnitude is determined exogenously by the market impact of his trade, i.e.

$$I(x) = \{(1-\delta)x\}.$$

In a dynamic programming context, we define the switch operator $\mathcal{M}(\cdot)$ by

$$\mathcal{M}\varphi(s,x) = \sup_{v \in I(x)} \left\{ \varphi(s,v) + e^{-rs}(x-c) \right\}.$$

Given that the impulse control set $I(x)$ is a singleton, this simplifies to:

$$\mathcal{M}\varphi(s,x) = \varphi(s,(1-\delta)x) + e^{-rs}(x-c). \tag{14}$$

The switch operator tells us how the value function $\varphi(s,x)$ changes after a transaction occurs at time s with arbitrage profits x.

Intuitively, this should be equal to the optimal value at time s^+ with new arbitrage profits $(1-\delta)x$ plus the profits from trading at time s, $e^{-rs}(x-c)$. By continuity of the value function $\varphi(s^+,(1-\delta)x) = \varphi(s,(1-\delta)x)$. This intuition describes expression (14). We note that it is the switch operator which explicitly recognizes the price impact of a transaction. It is implicitly recognized in the objection function (3) however because $X_{\tau_k^-}$ changes due to the market impact from the transaction at time τ_{k-1}.

Define the continuation set

$$\mathcal{C} = \{(t,x) : \varphi(t,x) > \mathcal{M}\varphi(t,x)\}. \tag{15}$$

The continuation set is where we do not transact at time t given arbitrage profits of size x.

The generator $\mathcal{A}$ of the diffusion process X_t is

$$\mathcal{A}\varphi(t,x) = \left(\frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\right)\varphi(t,x).$$

We can now state the two theorems used to characterize the solution.

Theorem 3 *Sufficient Conditions for Optimal Value Function*

Assume that the continuation region satisfies: (i) $\partial\mathcal{C}$ is a Lipschitz surface, and (ii) $E_{(s,x)}\left(\int_0^\infty 1_{\partial\mathcal{C}}(X_t)dt\right) = 0$ for all (s,x) and all admissible impulse controls.

Suppose there exists $\varphi(s,x) \geq 0$ which is uniformly integrable, C^1 everywhere, C^2 outside boundary of $\mathcal{C}$, with $\underset{s\to\infty}{lim}\,\varphi(s,X_s) = 0$ a.s. such that:

1. *$\varphi(s,x) \geq \mathcal{M}\varphi(s,x) = \varphi(s,(1-\delta)x) + e^{-rs}(x-c)$,*
2. *$\mathcal{A}\varphi(s,x) = 0$ on $\mathcal{C}$, and*
3. *$\mathcal{A}\varphi(s,x) \leq 0$ outside of $\mathcal{C}$.*

Then, $\varphi(s,x) \geq J^w(s,x)$ for all w.

Theorem 4 *Optimal Transaction Strategy*

Let $\varphi(s,x)$ satisfy the conditions of theorem 3, where
$\mathcal{M}\varphi(s,x) = \varphi(s,(1-\delta)x) + e^{-rs}(x-c)$.

Define $\widetilde{\tau}_k$ for $k = 0,1,2,....$ as follows.

Start at time $\widetilde{\tau}_0 = s$ with X_t evolving via expression (1) for $t \geq s$ with $X_s = x$.

Define $\widetilde{\tau}_1 = \inf\{t \geq \widetilde{\tau}_0 : \varphi(t,x) = \varphi(t,(1-\delta)x) + e^{-rt}(x-c)\}$. Transact at $\widetilde{\tau}_1$.

After the liquidity impact, $X_{\widetilde{\tau}_1} = (1-\delta)X_{\widetilde{\tau}_1^-}$. Starting at $X_{\widetilde{\tau}_1}$, X_t evolves via expression (1) for $t \geq \widetilde{\tau}_1$.

continuing....

Define $\widetilde{\tau}_k = \inf\{t \geq \widetilde{\tau}_{k-1} : \varphi(t,x) = \varphi(t,(1-\delta)x) + e^{-rt}(x-c)\}$. Transact at $\widetilde{\tau}_k$.

After the liquidity impact, $X_{\widetilde{\tau}_k} = (1-\delta)X_{\widetilde{\tau}_k^-}$. Starting at $X_{\widetilde{\tau}_k}$, X_t evolves via expression (1) for $t \geq \widetilde{\tau}_k$.

Then $\varphi(s,x) = J^{\widetilde{w}}(s,x) = \sup_{w\in W} J^w(s,x)$.

These are verification theorems because they requires knowledge of the value function $\varphi(s,x)$ and the continuation region $\mathcal{C}$. For many problems, however, these can be guessed, and then the verification theorems invoked.

6.2 Proof of theorems 3 and 4.

Proof. The proof of both theorems follow directly from Theorem 6.2, p. 83 of Okesendal and Sulem [13] with the following identifications.

1. $y = (x, s)$, we add time s to the state of the system,
2. $\mathcal{S} = \mathbb{R}^2$, no insolvency region,
3. $f = g = 0$, only a benefit at transaction times,
4. $\Gamma(y, \varsigma) = ((1 - \delta)y, 0)$, the second argument is time,
5. $K(y, \varsigma) = -c + x$, and
6. no jump process.

■

6.3 Proof of theorem 1.

Proof. The generator is: $\mathcal{A}\varphi(t, x) = \left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\right)\varphi(t, x)$.

The switch operator is: $\mathcal{M}\varphi(s, x) = \varphi(s, (1 - \delta)x) + e^{-rs}(x - c)$.

We guess a solution, then verify that it works.

Let us try a separable solution: $\varphi(s, x) = e^{-rs}\psi(x)$.

Then, $\mathcal{A}\left(e^{-rt}\psi(x)\right) = \left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\right)\left(e^{-rt}\psi(x)\right)$
$= -re^{-rt}\psi(x) + be^{-rt}\psi'(x) + \frac{1}{2}\sigma^2 e^{-rt}\psi''(x)$
$= e^{-rt}\left(-r\psi(x) + b\psi'(x) + \frac{1}{2}\sigma^2\psi''(x)\right)$
$= e^{-rt}\mathcal{A}_0\left(\psi(x)\right)$ where $\mathcal{A}_0\psi(x) = -r\psi(x) + b\psi'(x) + \frac{1}{2}\sigma^2\psi''(x)$. And,
$\mathcal{M}\left(e^{-rt}\psi(x)\right) = e^{-rt}\psi((1 - \delta)x) + e^{-rt}(x - c)$
$= e^{-rt}\mathcal{M}_0\psi(x)$ where $\mathcal{M}_0\psi(x) = \psi((1 - \delta)x) + (x - c)$.

We guess the continuation region takes the form $\mathcal{C} = \{x \le x^*\}$ for some x^*, to be determined.

Step 1

First, on the continuation region $\mathcal{C}$, $\mathcal{A}_0\psi(x) = 0$ subject to the continuity and differentiability conditions.

To find a solution on the continuation region, we guess $\psi(x) = Ke^{\gamma x}$. Then,
$\mathcal{A}_0\psi(x) = 0$
$-r\psi(x) + b\psi'(x) + \frac{1}{2}\sigma^2\psi''(x) = 0$
$-rKe^{\gamma x} + b\gamma Ke^{\gamma x} + \frac{1}{2}\sigma^2\gamma^2 Ke^{\gamma x} = 0$
$Ke^{\gamma x}\left(-r + b\gamma + \frac{1}{2}\sigma^2\gamma^2\right) = 0$
$\frac{2}{\sigma^2}Ke^{\gamma x}\left(-r + b\gamma + \frac{1}{2}\sigma^2\gamma^2\right) = 0$
$Ke^{\gamma x}h(\gamma) = 0$ where $h(\gamma) = -\frac{2r}{\sigma^2} + \frac{2b}{\sigma^2}\gamma + \gamma^2$.

We need to find the zeros of $h(\gamma)$.

By the quadratic formula, $\gamma^* = \frac{-\frac{2b}{\sigma^2} \pm \sqrt{\left(\frac{2b}{\sigma^2}\right)^2 + 4\frac{2r}{\sigma^2}}}{2} = -\frac{b}{\sigma^2} \pm \sqrt{\left(\frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$.

Choose $\gamma^* = -\frac{b}{\sigma^2} + \sqrt{\left(\frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0$.

This gives us our solution on the continuation region.

Define the value function as this solution, extended to the non-continuation region:

$$\psi(x) = \begin{array}{cc} Ke^{\gamma^* x} & if \;\; x < x^* \\ Ke^{\gamma^*(1-\delta)x} + (x-c) & if \;\; x \geq x^* \end{array}.$$

Now, continuity and differentiability at the boundary x^* require:

$Ke^{\gamma^* x^*} = Ke^{\gamma^*(1-\delta)x^*} + (x^* - c)Ke^{\gamma^* x^*} = Ke^{\gamma^*(1-\delta)x^*} + (x^* - c)$ and

$\gamma^* Ke^{\gamma^* x^*} = \gamma^*(1-\delta)Ke^{\gamma^*(1-\delta)x^*} + 1$.

This gives two equations in two unknowns (K, x^*). The solution determines the value function and continuation region explicitly.

We want to show that a solution exists with $x^* > c$ and $K > 0$.

Note that $K > 0$ is needed to get the value function positive.

The first equation yields:

$Ke^{\gamma^* x^*} = Ke^{\gamma^*(1-\delta)x^*} + (x^* - c)$

$Ke^{\gamma^* x^*}\left(1 - e^{-\gamma^* \delta x^*}\right) = (x^* - c)$

$K = \frac{(x^*-c)}{e^{\gamma^* x^*}\left(1-e^{-\gamma^* \delta x^*}\right)}$. This is positive if $x^* > c$.

The second equation is

$\gamma^* Ke^{\gamma^* x^*} = \gamma^*(1-\delta)Ke^{\gamma^*(1-\delta)x^*} + 1$.

$\gamma^* Ke^{\gamma^* x^*} - \gamma^*(1-\delta)Ke^{\gamma^*(1-\delta)x^*} - 1 = 0$.

Substitution for K from the first equation gives

$\frac{(x^*-c)}{e^{\gamma^* x^*}\left(1-e^{-\gamma^* \delta x^*}\right)}\left[\gamma^* e^{\gamma^* x^*} - \gamma^*(1-\delta)e^{\gamma^*(1-\delta)x^*}\right] - 1$.

Studying $k(x) = \frac{(x-c)}{e^{\gamma^* x}\left(1-e^{-\gamma^* \delta x}\right)}\left[\gamma^* e^{\gamma^* x} - \gamma^*(1-\delta)e^{\gamma^*(1-\delta)x}\right] - 1$

we see that $k(c) = -1 < 0$ and $\underset{x\to\infty}{lim}\, k(x) = +\infty$.

This implies that there exists an $x^* > c$ such that $k(x^*) = 0$.

This completes the proof of step 1.

Step 2

We need to verify that:

(i) $\psi(x) \geq \mathcal{M}_0\psi(x)$ for all x, and

(ii) $\mathcal{A}_0\psi(x) \leq 0$ for $x \geq x^*$.

[Proof of $\psi(x) \geq M_0\psi(x)$ for all x.]

Given the solution, we have

$$\mathcal{M}_0\psi(x) = \psi((1-\delta)x) + (x-c) = \begin{array}{cc} Ke^{\gamma^*(1-\delta)x} + (x-c) & if \;\; x < x^* \\ Ke^{\gamma^*(1-\delta)^2 x} + (x(1-\delta) - c) & if \;\; x \geq x^* \end{array}.$$

For $x \geq x^*$, we have $\psi(x) = Ke^{\gamma^*(1-\delta)x} + (x-c) \geq Ke^{\gamma^*(1-\delta)^2 x} + (x(1-\delta) - c)$ $= \mathcal{M}_0\psi(x)$ since $0 < (1-\delta) \leq 1$.

For $x < x^*$, we have $\psi(x) = Ke^{\gamma^* x}$ and $\mathcal{M}_0\psi(x) = Ke^{\gamma^*(1-\delta)x} + (x-c)$.

Consider the function

$F(x) = Ke^{\gamma^* x} - Ke^{\gamma^*(1-\delta)x} - (x-c)$

$= Ke^{\gamma^* x}(1 - e^{-\gamma^* \delta x}) - (x-c)$.

Recall that $F(x^*) = 0$ (this is continuity at the boundary) and $F'(x^*) = 0$ (this is continuity of the derivative at the boundary).

And $F''(x) = (\gamma^*)^2 Ke^{\gamma^* x} - (\gamma^*(1-\delta))^2 Ke^{\gamma^*(1-\delta)x} > 0$.

Hence, F increases as we move away from x^*, i.e. $F(x) \geq 0$ for $x < x^*$.

This implies $\psi(x) \geq \mathcal{M}_0\psi(x)$ for $x < x^*$, which completes the proof.

[Proof of $A_0\psi(x) \leq 0$ **for** $x \geq x^*$**]**

For $x > x^*$,

$\mathcal{A}_0\psi(x) = -r\psi(x) + b\psi'(x) + \frac{1}{2}\sigma^2\psi''(x)$

$= -r\left(Ke^{\gamma^*(1-\delta)x} + (x-c)\right) + b\left(\gamma^*(1-\delta)Ke^{\gamma^*(1-\delta)x} + 1\right) + \frac{1}{2}\sigma^2\left(\left[\gamma^*(1-\delta)\right]^2 Ke^{\gamma^*(1-\delta)x}\right).$

Then, $\mathcal{A}_0\psi(x) \leq 0$ if and only if

$-r\left(Ke^{\gamma^*(1-\delta)x} + (x-c)\right) + b\left(\gamma^*(1-\delta)Ke^{\gamma^*(1-\delta)x} + 1\right) + \frac{1}{2}\sigma^2\left(\left[\gamma^*(1-\delta)\right]^2 Ke^{\gamma^*(1-\delta)x}\right) \leq 0$ if and only if

$Ke^{\gamma^*(1-\delta)x}\left(-r + b\gamma^*(1-\delta) + \frac{1}{2}\sigma^2\left[\gamma^*(1-\delta)\right]^2\right) - r(x-c) + b \leq 0.$

This will be true if $-r+b\gamma^*(1-\delta)+\frac{1}{2}\sigma^2\left[\gamma^*(1-\delta)\right]^2 \leq 0$ and $-r(x^*-c)+b \leq 0$. ■

6.4 Proof of theorem 2.

Proof. The generator is: $\mathcal{A}\varphi(t,x) = \left(\frac{\partial}{\partial t} + bx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}\right)\varphi(t,x).$

The switch operator is: $\mathcal{M}\varphi(s,x) = \varphi(s,(1-\delta)x) + e^{-rs}(x-c).$

We guess a solution, then verify that it works.

Let us try a separable solution: $\varphi(s,x) = e^{-rs}\psi(x)$. Then,

$\mathcal{A}\left(e^{-rt}\psi(x)\right) = \left(\frac{\partial}{\partial t} + bx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}\right)\left(e^{-rt}\psi(x)\right)$

$= -re^{-rt}\psi(x) + be^{-rt}x\psi'(x) + \frac{1}{2}\sigma^2x^2e^{-rt}\psi''(x)$

$= e^{-rt}\left(-r\psi(x) + bx\psi'(x) + \frac{1}{2}\sigma^2x^2\psi''(x)\right)$

$= e^{-rt}\mathcal{A}_0\left(\psi(x)\right)$ where $\mathcal{A}_0\psi(x) = -r\psi(x) + bx\psi'(x) + \frac{1}{2}\sigma^2x^2\psi''(x)$. And,

$\mathcal{M}\left(e^{-rt}\psi(x)\right) = e^{-rt}\psi((1-\delta)x) + e^{-rs}(x-c)$

$= e^{-rt}\mathcal{M}_0\psi(x)$ where $\mathcal{M}_0\psi(x) = \psi((1-\delta)x) + (x-c)$.

We guess the continuation region takes the form $\mathcal{C} = \{x \leq x^*\}$ for some x^*, to be determined.

Step 1

First, on the continuation region $\mathcal{C}$, $\mathcal{A}_0\psi(x) = 0$ subject to the continuity and differentiability conditions.

To find a solution on the continuation region, we guess $\psi(x) = Kx^\gamma$.

Then, $\mathcal{A}_0\psi(x) = 0$

$-r\psi(x) + bx\psi'(x) + \frac{1}{2}\sigma^2x^2\psi''(x) = 0$

$-rKx^\gamma + bx\gamma Kx^{\gamma-1} + \frac{1}{2}\sigma^2x^2\gamma(\gamma-1)Kx^{\gamma-2} = 0$

$Kx^\gamma\left(-r + b\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1)\right) = 0$

$\frac{2}{\sigma^2}Kx^\gamma\left(-r + \left(b - \frac{1}{2}\sigma^2\right)\gamma + \frac{1}{2}\sigma^2\gamma^2\right) = 0$

$Kx^\gamma h(\gamma) = 0$ where $h(\gamma) = -\frac{2r}{\sigma^2} + \frac{2\left(b-\frac{1}{2}\sigma^2\right)}{\sigma^2}\gamma + \gamma^2.$

We need to find the zeros of $h(\gamma)$.

By the quadratic formula,

$$\gamma^* = \frac{-\frac{2\left(b-\frac{1}{2}\sigma^2\right)}{\sigma^2} \pm \sqrt{\left(\frac{2\left(b-\frac{1}{2}\sigma^2\right)}{\sigma^2}\right)^2 + 4\frac{2r}{\sigma^2}}}{2} = -\frac{\left(b-\frac{1}{2}\sigma^2\right)}{\sigma^2} \pm \sqrt{\left(\frac{\left(b-\frac{1}{2}\sigma^2\right)}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Choose $\gamma^* = -\frac{\left(b-\frac{1}{2}\sigma^2\right)}{\sigma^2} + \sqrt{\left(\frac{\left(b-\frac{1}{2}\sigma^2\right)}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0.$

This gives us our solution on the continuation region.

Define the value function as this solution, extended to the non-continuation region:

$$\psi(x) = \begin{array}{cc} Kx^{\gamma^*} & if \;\; x < x^* \\ K(1-\delta)^{\gamma^*}x^{\gamma^*} + (x-c) & if \;\; x \geq x^* \end{array}.$$

Now, continuity and differentiability at the boundary x^* require:

$K(x^*)^{\gamma^*} = K(1-\delta)^{\gamma^*}(x^*)^{\gamma^*} + (x^*-c)$ and

$\gamma^* K(x^*)^{\gamma^*-1} = \gamma^* K(1-\delta)^{\gamma^*}(x^*)^{\gamma^*-1} + 1.$

This gives two equations in two unknowns (K, x^*). The solution determines the value function and continuation region explicitly.

We want to show that a solution exists with $x^* > c$ and $K > 0$.

Note that $K > 0$ is needed to get the value function positive.

The first equation yields:

$K(x^*)^{\gamma^*}(1-(1-\delta)^{\gamma^*}) = (x^*-c).$

$K = \frac{(x^*-c)}{(x^*)^{\gamma^*}(1-(1-\delta)^{\gamma^*})}$. This is positive if $x^* > c$.

The second equation is

$\gamma^* K(x^*)^{\gamma^*-1} = \gamma^* K(1-\delta)^{\gamma^*}(x^*)^{\gamma^*-1} + 1.$

$\gamma^* K(x^*)^{\gamma^*-1}(1-(1-\delta)^{\gamma^*}) - 1 = 0.$

Substitution for K from the first equation gives

$\gamma^* \frac{(x^*-c)}{(x^*)^{\gamma^*}(1-(1-\delta)^{\gamma^*})}(x^*)^{\gamma^*-1}(1-(1-\delta)^{\gamma^*}) - 1 = 0.$

$\gamma^*(1-\frac{c}{x^*}) - 1 = 0.$

Studying $k(x^*) = \gamma^*(1-\frac{c}{x^*}) - 1$ we see that $k(c) = -1 < 0$ and

$\underset{x^*\to\infty}{lim} k(x^*) > 0$ if $\gamma^* > 1$.

This implies that there exists an $x^* > c$ such that $k(x^*) = 0$ if $\gamma^* > 1$.

This completes the proof of step 1.

Step 2

We need to verify that:

(i) $\psi(x) \geq \mathcal{M}_0\psi(x)$ for all x, and

(ii) $\mathcal{A}_0\psi(x) \leq 0$ for $x \geq x^*$.

[Proof of $\psi(x) \geq M_0\psi(x)$ for all x]

Given the solution, we have

$$\mathcal{M}_0\psi(x) = \psi((1-\delta)x)+(x-c) = \begin{array}{cc} K(1-\delta)^{\gamma^*}x^{\gamma^*} + (x-c) & if \;\; x < x^* \\ K(1-\delta)^{2\gamma^*}x^{\gamma^*} + (x(1-\delta)-c) & if \;\; x \geq x^* \end{array}.$$

For $x \geq x^*$, we have $\psi(x) = K(1-\delta)^{\gamma^*}x^{\gamma^*} + (x-c) \geq K(1-\delta)^{2\gamma^*}x^{\gamma^*} + (x(1-\delta)-c)$

$= \mathcal{M}_0\psi(x)$ since $0 < (1-\delta) \leq 1$ and $\gamma^* > 2$.

For $x < x^*$, we have $\psi(x) = Kx^{\gamma^*}$ and $\mathcal{M}_0\psi(x) = K(1-\delta)^{\gamma^*}x^{\gamma^*} + (x-c)$.

Consider the function

$F(x) = Kx^{\gamma^*} - K(1-\delta)^{\gamma^*}x^{\gamma^*} - (x-c)$

$= Kx^{\gamma^*}(1-(1-\delta)^{\gamma^*}) - (x-c).$

Recall that $F(x^*) = 0$ (this is continuity at the boundary) and $F'(x^*) = 0$ (this is continuity of the derivative at the boundary).

And $F''(x) = \gamma^* K x^{\gamma^*-1}(1-(1-\delta)^{\gamma^*}) > 0$. Hence, F increases as we move away from x^*, i.e. $F(x) \geq 0$ for $x < x^*$.

This implies $\psi(x) \geq \mathcal{M}_0\psi(x)$ for $x < x^*$, which completes the proof.

[**Proof of** $A_0\psi(x) \leq 0$ **for** $x \geq x^*$]

For $x > x^*$,

$\mathcal{A}_0\psi(x) = -r\psi(x) + bx\psi'(x) + \frac{1}{2}\sigma^2 x^2 \psi''(x)$

$= -r\left(K(1-\delta)^{\gamma^*} x^{\gamma^*} + (x-c)\right) + bx\left(\gamma^* K(1-\delta)^{\gamma^*} x^{\gamma^*-1} + 1\right) + \frac{1}{2}\sigma^2 x^2\left(\gamma^*\left(\gamma^*-1\right)K(1-\delta)^{\gamma^*} x^{\gamma^*-2}\right)$

$= -r\left(K(1-\delta)^{\gamma^*} x^{\gamma^*} + (x-c)\right) + b\left(\gamma^* K(1-\delta)^{\gamma^*} x^{\gamma^*} + x\right) + \frac{1}{2}\sigma^2\left(\gamma^*\left(\gamma^*-1\right)K(1-\delta)^{\gamma^*} x^{\gamma^*}\right)$

Then, $\mathcal{A}_0\psi(x) \leq 0$ if and only if

$-r\left(K(1-\delta)^{\gamma^*} x^{\gamma^*} + (x-c)\right) + b\left(\gamma^* K(1-\delta)^{\gamma^*} x^{\gamma^*} + x\right) + \frac{1}{2}\sigma^2\left(\gamma^*\left(\gamma^*-1\right)K(1-\delta)^{\gamma^*} x^{\gamma^*}\right) \leq 0$ if and only if

$K(1-\delta)^{\gamma^*} x^{\gamma^*}\left(-r + b + \frac{1}{2}\sigma^2\gamma^*\left(\gamma^*-1\right)\right) - r(x-c) + bx \leq 0.$

This will be true if $\left(-r + b + \frac{1}{2}\sigma^2\gamma^*\left(\gamma^*-1\right)\right) \leq 0$ and $-r(x^*-c) + bx^* \leq 0$.

■