

**Problem 3.15.** Show that the collection of decidable languages is closed under the operation of.

**Part b.** concatenation.

For any two decidable languages  $L_1$  and  $L_2$ , let  $M_1$  and  $M_2$  be the **TMs** that decide them. For a string  $w = w_1w_2w_3 \cdots w_n$ , define split  $W = (P, S)$ , where  $P, S \in \Sigma^*$ , and  $w = PS$ . In other words, the strings  $P$  and  $S$  are prefix and suffix of  $w$ , such that their concatenation is equivalent to  $w$ . We construct a **TM**  $M'$  that decides the concatenation of  $L_1$  and  $L_2$ :

“On input  $w$ :

1. Calculate the list of all possible splits  $W_1 = (P_1, S_1), W_2 = (P_2, S_2), \dots, W_n = (P_n, S_n)$  of  $w$ .
2. Repeat the following for each  $i = 1, 2, 3, \dots$ .
3. Run  $M_1$  on  $P_i$ , and  $M_2$  on  $S_i$ .
4. If in any computation, both  $M_1$  and  $M_2$  accept, *accept*. ”

**Part c.** star.

For any decidable language  $L$ , let  $M$  be the **TM** that decides it.

For any non-empty string  $w = w_1w_2w_3 \cdots w_n$ , define partition  $P = (p_1, p_2, \dots, p_k)$ , where  $1 \leq k \leq n$ , each  $p_i \in \Sigma^*$ , and  $w = p_1p_2 \cdots p_k$ . We construct a **TM**  $M'$  that decides  $L^*$ :

“On input  $w$ :

1. If  $w = \varepsilon$ , *accept*.
2. For non-empty  $w$ , generate list of all possible partitions  $P_1, P_2, P_3, \dots$ .
3. Repeat the following for each  $i = 1, 2, 3, \dots$ .
4. Run  $M$  on each  $p \in P_i$ .
5. If in any computation,  $M$  accepts all  $p \in P_i$ , *accept*.”

**Part d.** complementation.

For any decidable language  $L$ , let  $M$  be the **TM** that decides it. Construct a **TM**  $M'$  that decides the complement of  $L$ :

“On input  $w$ :

1. Run  $M$  on  $w$ . If it accepts, *reject*. Otherwise, *accept*.”

**Part e.** intersection.

For any two decidable languages  $L_1$  and  $L_2$ , let  $M_1$  and  $M_2$  be the **TMs** that decide them. We construct a **TM**  $M'$  that decides the intersection of  $L_1$  and  $L_2$ :

“On input  $w$ :

1. Run  $M_1$  on  $w$ . If it rejects, *reject*.
2. Run  $M_2$  on  $w$ . If it accepts, *accept*. Otherwise, *reject*.”