
Problem 5.21. Let $AMBIG_{CFG} = \{\langle G \rangle \mid G \text{ is an ambiguous CFG}\}$. Show that $AMBIG_{CFG}$ is undecidable.

Proof. To show that $AMBIG_{CFG}$ is undecidable, we give a reduction from PCP to $AMBIG_{CFG}$. Given an instance

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \dots, \left[\frac{t_k}{b_k}\right] \right\}$$

of the Post Correspondence Problem, construct a CFG G with the rules

$$\begin{aligned} &S \to T \mid B \\ &T \to t_1 T a_1 \mid \cdots \mid t_k T a_k \mid t_1 a_1 \mid \cdots \mid t_k a_k \\ &B \to b_1 B a_1 \mid \cdots \mid b_k B a_k \mid b_1 a_1 \mid \cdots \mid b_k a_k, \end{aligned}$$

where $a_1, a_2, \cdots, a_k$ are new terminal symbols[1]. Next we show that this reduction works.

If P is an instance of Post Correspondence Problem, then for every possible arrangement of the dominoes $i_1, i_2, \dots, i_l$, where $t_{i_1} t_{i_2} \dots t_{i_l}$ is the top string and $b_{i_1} t_{b_2} \dots b_{i_l}$ the bottom string, there exists exactly one derivation for the top and bottom strings in G. The terminal symbols $a_1, a_2, \cdots, a_k$, make sure that every string generated using the variable T has at most one left-most derivation even if some t_i and t_j are same. Same argument can be said for the variable B.

If $P \in PCP$, then there exists a match $i_1, i_2, \dots, i_l$, where $t_{i_1} t_{i_2} \dots t_{i_l} = b_{i_1} t_{b_2} \dots b_{i_l}$. In this case, the CFG G is guaranteed to have following two left-most derivations that produce the same string.

$$\begin{aligned} S &\to T \\ &\to t_{i_1} T a_{i_1} \\ &\to t_{i_1} t_{i_2} T a_{i_2} a_{i_1} \\ &\to t_{i_1} t_{i_2} \dots T \dots a_{i_2} a_{i_1} \\ &\to t_{i_1} t_{i_2} \dots t_{i_l} a_{i_l} \dots a_{i_2} a_{i_1} \end{aligned}$$

$$\begin{aligned} S &\to B \\ &\to b_{i_1} B a_{i_1} \\ &\to b_{i_1} b_{i_2} B a_{i_2} a_{i_1} \\ &\to b_{i_1} b_{i_2} \dots B \dots a_{i_2} a_{i_1} \\ &\to b_{i_1} b_{i_2} \dots b_{i_l} a_{i_l} \dots a_{i_2} a_{i_1} \end{aligned}$$

□

[1] This is a hint mentioned in the book.