
Problem 2.31. Let B be the language of all palindromes over $0, 1$ containing equal numbers of 0s and 1s. Show that B is not context free.

Proof. The proof is by contradiction. Assume the language B is context free. Let p be the pumping length given by the pumping lemma. Choose the string $s = 0^p1^p1^p0^p$. Clearly $s \in B$ and $|s| \geq p$. The pumping lemma states that s can be pumped, but we show that it cannot be pumped.

1. Either v or y is non-empty. Without the loss of generality, we only discuss the case when v is non-empty. If v is non-empty, then it may contain only one type of alphabet or both 0's and 1's. In former case, the string $uv^2xy^2z \notin B$, because it cannot contain equal number of 0's and 1's. In latter case, the string $uv^2xy^2z \notin B$, because it is not a palindrome.

2. Both v and y are non-empty. There are three possibilities. First, both v and y contain the same type of alphabet from the same segment of either 0's or 1's. Second, v and y contain different type of alphabets from two neighboring segments of 0's and 1's. Third, both v and y contain 1's, but from two different segments of 1's. In all three cases, the string $uv^2xy^2z \notin B$, because either it contains unequal number of 0's and 1's or the string is not a palindrome.

One of these cases must occur, and both cases result in a contradiction. Therefore, the language B is not context free. $\square$