Problem 2.36. Give an example of a language that is not context free but that acts like a CFL in the pumping lemma. Prove that your example works.

Informal description. The language $A = \{a^n b^n c^n \mid n \geq 0\}$ is easily seen to be a non-CFL. So is $B = \{a^i b^j c^j d^k \mid i, j, k \geq 0,\ if\ i = 1\ then\ k = j\}$. We show that the language B acts like a CFL in the pumping lemma.

Proof. Assume the language $B = \{a^i b^j c^j d^k \mid i, j, k \geq 0,\ if\ i = 1\ then\ k = j\}$ is context free. Let p be the pumping length given by the pumping lemma. The pumping length p is at least 2, because the strings a and b cannot be divided and pumped. The strings ab, cc and dd can be divided and pumped according to the method given below.

Choose s to be any string, such that $s \in B$ and $|s| \geq p$. The pumping lemma states that s can be pumped. There are three cases:

1. $i = 0$. In other words, a's are missing. There are two sub-cases:
   (a) $j = 0$. This means that the string s contains only d's. Let $u = \varepsilon$, v be the first d, $x = \varepsilon$, $y = \varepsilon$, and z is the rest.
   (b) $j > 0$. Let $u = \varepsilon$, v be the last b, $x = \varepsilon$, y be the first c, and z is the rest.
2. $i = 1$. In this case the string s contains equal number of b's, c's and d's. Let $u = \varepsilon$, $v = a$, $x = \varepsilon$, $y = \varepsilon$, and z is the rest.
3. $i > 1$. There are three sub-cases:
   (a) $j = 0$, and $k = 0$. The string s only contains a's. Let $u = \varepsilon$, v be the first a, $x = \varepsilon$, $y = \varepsilon$, and z is the rest.
   (b) $j = 0$, and $k > 0$. Let $u = \varepsilon$, v be the first d, $x = \varepsilon$, $y = \varepsilon$, and z is the rest.
   (c) $j > 0$. Let $u = \varepsilon$, v be the last b, $x = \varepsilon$, y be the first c, and z is the rest.

This method of dividing the string s works no matter what s is, and all three conditions of the pumping lemma are satisfied:

1. $uv^i xy^i z \in B,\ for\ each\ i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

Therefore, even though the language B is not a CFL, but it acts like a CFL in the pumping lemma. □