Problem 2.30. Use the pumping lemma to show that the following languages are not context free.

Part a. $\{0^n1^n0^n1^n \mid n \geq 0\}$

Proof. The proof is by contradiction. Assume the language $A = \{0^n1^n0^n1^n \mid n \geq 0\}$ is context free. Let p be the pumping length given by the pumping lemma. Choose the string $s = 0^p1^p0^p1^p$. Clearly s is a member of A and of length at least p. The pumping lemma states that s can be pumped, but we show that it cannot be pumped.

1. Either v or y is non-empty. Without loss of generality, we only discuss the case when v is non-empty. If v is non-empty, then it may contain only one type of alphabet or both 0's and 1's. In former case, the string $uv^2xy^2z \notin A$, because it cannot contain equal number of 0's or 1's in each segment. In latter case, the string $uv^2xy^2z \notin A$ as it cannot contain exactly two segments of 0's and 1's.

2. Both v and y are non-empty. There are three possibilities. First, both v and y contain the same alphabet from the same segment of 0's or 1's. Second, v contains only one type of symbol from a segment and y only contains the other type of symbol from a neighboring segments. Third, v or y contain both 0's and 1's from two neighboring segments. In the first and second cases, the string $uv^2xy^2z \notin A$, because it cannot contain equal number of 0's and 1's in each segment. In the third case, the string $uv^2xy^2z \notin A$ as it cannot contain exactly two segments of 0's and 1's.

One of these cases must occur, and both cases result in a contradiction. □

Part d. $\{t_1\#t_2\#\cdots\#t_k \mid k \geq 2,\ each\ t_i \in \{a,b\}^*,\ and\ t_i = t_j\ for\ some\ i \neq j\}$

Proof. The proof is by contradiction. Assume the language $B = \{t_1\#t_2\#\cdots\#t_k \mid k \geq 2,\ each\ t_i \in \{a,b\}^*,\ and\ t_i = t_j\ for\ some\ i \neq j\}$ is context free. Let p be the pumping length given by the pumping lemma. Choose the string $s = a^pb^p\#a^pb^p$. Clearly s is a member of B and of length at least p. The pumping lemma states that s can be pumped, but we show that it cannot be pumped.

1. Either v or y contains the $\#$. The string $uv^0xy^0z \notin B$, because it does not contain any $\#'s$.

2. Neither v nor y contains the $\#$. There are two cases. First, both v and y are on the same side of the $\#$. Second, v contains any number of b's on the left of $\#$, while y contains any number of a's on the right. In both cases, the string $uv^2xy^2z \notin B$ as $t_1 \neq t_2$.

One of these cases must occur, and both cases result in a contradiction. □