
Problem 1.51. Let x and y be strings and let L be any language. We say that x and y are **distinguishable by** L if some string z exists whereby exactly one of the strings xz and yz is a member of L; otherwise, for every string z, we have $xz \in L$ whenever $yz \in L$ and we say that x and y are **indistinguishable by** L. If x and y are indistinguishable by L, we write $x \equiv_L y$. Show that $\equiv_L$ is an equivalence relation.

Proof. To show that $\equiv_L$ is an equivalence relation, first give formal definition of $\equiv_L$[1], and then show that $\equiv_L$ is:

1. Reflexive
2. Symmetric
3. Tranisitive

Definition $\equiv_L$. Let x and y be strings and let L be any language. Then $x \equiv_L y$ is defined as

$$(x \equiv_L y) \equiv (\forall z \mid : xz \in L \equiv yz \in L).$$

Part 1. Reflexivity: $x \equiv_L x$

$$\begin{array}{cl} & x \equiv_L x \\ = & \langle Definition\ of\ \equiv_L \rangle \\ & (\forall z \mid : xz \in L \equiv xz \in L) \\ = & \langle Reflexivity\ of \equiv (3.5) \rangle \\ & (\forall z \mid : true) \\ = & \langle (9.8) \rangle \\ & true \end{array}$$

[1]The formal method used in this proof is presented in the book **A Logical Approach to Discrete Math** by David Gries and Fred B. Schneider. To learn more about this method, checkout **Math 220: Formal Methods, Pepperdine University**. Course link is https://cslab.pepperdine.edu/warford/math220/.

Part 2. Symmetry: $x \equiv_L y \Rightarrow y \equiv_L x$

Proof by assuming conjuncts of the antecedent.

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\begin{aligned}
& y \equiv_L x \\
= \quad & \langle Definition\ of\ \ \equiv_L \rangle \\
& (\forall z \mid : yz \in L \equiv xz \in L) \\
= \quad & \langle Symmetry\ of \equiv (3.2) \rangle \\
& (\forall z \mid : xz \in L \equiv yz \in L) \\
= \quad & \langle Definition\ of\ \ \equiv_L \rangle \\
& x \equiv_L y \\
= \quad & \langle Assumption\ x \equiv_L y \rangle \\
& true
\end{aligned}
$$

Part 3. Transitivity: $x \equiv_L y \wedge y \equiv_L w \Rightarrow x \equiv_L w$

$$
\begin{aligned}
L \cdot H \cdot S: \quad & x \equiv_L y \wedge y \equiv_L w \\
= \quad & \langle Definition\ of\ \ \equiv_L \rangle \\
& (\forall z \mid : xz \in L \equiv yz \in L) \wedge (\forall z \mid : yz \in L \equiv wz \in L) \\
= \quad & \langle Distributivity\ (8.15) \rangle \\
& (\forall z \mid : xz \in L \equiv yz \in L \wedge yz \in L \equiv wz \in L) \\
= \quad & \langle Leibniz\ (3.84a) \rangle \\
& (\forall z \mid : xz \in L \equiv wz \in L \wedge yz \in L \equiv wz \in L) \\
\Rightarrow \quad & \langle Body\ weakening/strengthening\ (9.11) \rangle \\
& (\forall z \mid : xz \in L \equiv wz \in L) \\
= \quad & \langle Definition\ of\ \ \equiv_L \rangle \\
& x \equiv_L w
\end{aligned}
$$

□