Problem 4.31. Say that a variable A in CFG G is usable if it appears in some derivation of some string $w \in L(G)$. Given a CFG G and a variable A, consider the problem of testing whether A is usable. Formulate this problem as a language and show that it is decidable.

Proof. Let $U_{CFG} = \{\langle G, A \rangle \mid G$ is a CFG and A is a usable variable in $G\}$. To show that U_{CFG} is decidable, we present a **TM** I that decides it.

$I = $ "On input $\langle G, A \rangle$, where G is a CFG and A is a variable symbol:

1. Mark all terminal symbols in G.

2. Repeat until no new variables get marked:

3.   Mark any variable V where G has a rule $V \longrightarrow U_1 U_2 \cdots U_k$
       and each symbol $U_1, U_2, \cdots, U_k$ has already been marked.

4. If the start variable or A is not marked, *reject.*

5. Clear all marked symbols.

6. Perform *Breadth-First-Search* on start variable, and each time *Breadth-First-Search* procedure visits a variable or terminal symbol, mark it.

7. If A is not marked, *reject.* Otherwise *accept.*"

$\square$

Step 4 checks $L(G) \neq \phi$ and variable A can generate some string of terminals. In step 6 and 7, we check if A occurs in some derivation in G.