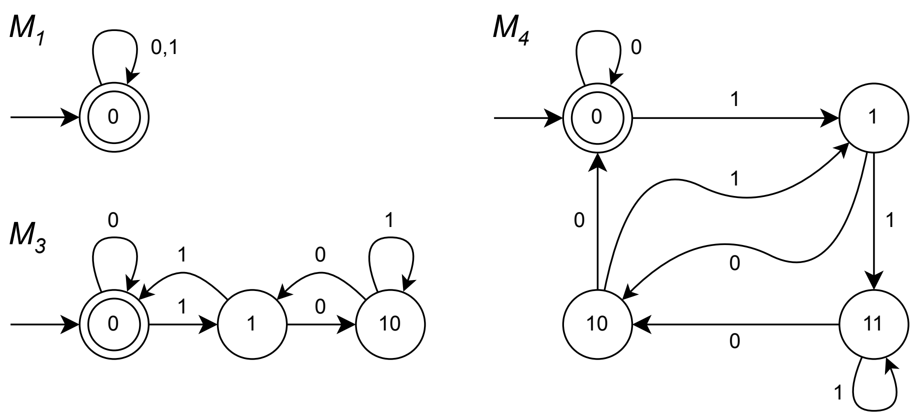
Problem 1.31. Let $C_n = \{x \mid x \textit{ is a binary number that is a multiple of } n\}$. Show that for each $n \geq 1$, the language C_n is regular.

Proof Idea. Design a finite automaton that simulates long division of x by n. Such a finite automaton will have a state for each possible remainder 0 to $n-1$. State 0 will be the only accept state.



State diagrams of DFAs that recognize B_1, B_3 and B_4. In each DFA, the states are binary numbers corresponding to each possible remainder.

Proof. The proof is by construction. Construct the DFA $M_n = (Q, \Sigma, \delta, q_0, F)$ to recognize C_n:

1. $Q = \{0, 1, 10, 11, 100, \cdots, (n-1)_2\}$, where $n \geq 1$. Set of binary numbers for all possible remainders of n.
2. $\Sigma = \{0, 1\}$
3. $q_0 = 0$
4. $F = \{0\}$
5. Define δ(q, a) so that for any $q \in Q$ and any $a \in \Sigma$:

$$\delta(q, a) = mod_2((q \circ a), n_2)$$

Here, n_2 is the binary representation of n, and $mod_2(i, j)$ is a function that takes two binary numbers i and j, and returns the remainder of the integer division $i \div j$. For example, $mod_2(101, 100) = 1$ and $mod_2(111, 100) = 11$. The first argument $(q \circ a)$ to the function mod_2, is the concatenation of the state q and input symbol a, which are both binary numbers.

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