Introduction to the Theory of Computation **Mehshan Mustafa**
Chapter 1: Reqular Languages **December 9, 2024**

Problem 1.46. Prove that the following languages are not regular. You may use the pumping lemma and the closure of the class of regular languages under union, intersection, and complement.

Part c. $\{w \mid w \in \{0,1\}^* \textit{ is not a palindrome}\}$

Proof. The proof is by contadiction. Assume A is a regular language. Let p be the pumping lenght given by the pumping lemma. Let s be the string:

$$s = 0^p 1 0^{p+p!}$$

The string s is a member of A, and $|s| \geq p$, therefore the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$ the string $xy^iz \in A$. According to the condition 3 ($|xy| \leq p$) of the pumping lemma, y consists of 0s. Let $m + n = p$, where $0 \leq m < p$ and $1 \leq n \leq p$. Then

$$0^m \Big[0^n\Big]^i 1 0^{p+p!} \in A.$$

But for $i = 1 + \frac{p!}{n}$, the number of 0s on both sides of the 1 become the same and the resulting string is a palindrome, which is a contradiction. □

Part d. $\{wtw \mid w,\ t \in \{0,1\}^+\}$

Proof. The proof is by contradiction. Assume A is a regular language. Let p be the pumping length give by the pumping lemma. Choose $w = 1^p0$ and $t = 0$, so the string s is:

$$s = 1^p 0 0 1^p 0$$

The string s is a member of A, $|s| \geq p$, therefore the pumping lemma guarantees that the string s can be split into three pieces, $s = xyz$, where for any $i \geq 0$ the string $xy^iz \in A$. According to the condition 3 of the pumping lemma, y can only contain 1s. More specfically, y can contain any number of 1s from 1 to p. Also, it must be the case that w starts with 1 and ends with a 0. When $i > p$, there are more 1s at the start of s then the number of 1s before the last 0, so there is no way to pick a w. Therefore, $xy^{p+1}z \notin A$, which is a contradiction.

$$\underbrace{1......1}_{\text{more than } p \text{ 1s}}00\underbrace{1......1}_{p \text{ 1s}}0$$

When $i > p$, there are more 1s at the start of s.

□