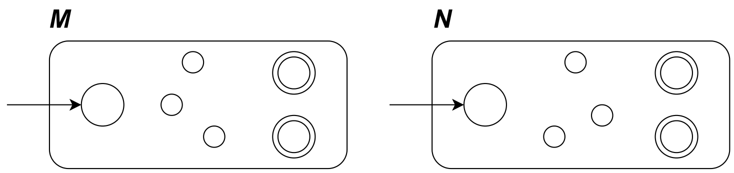
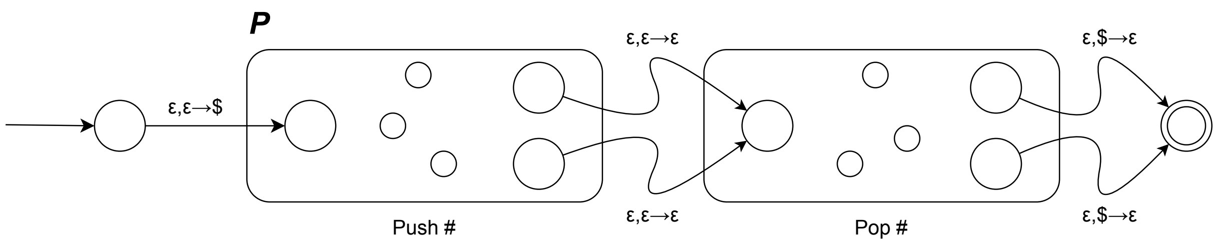
Problem 2.44. If A and B are languages, define $A \Diamond B = \{xy \mid x \in A \textit{ and } y \in B \textit{ and } |x| = |y|\}$. Show that if A and B are regular languages, then $A \Diamond B$ is a CFL.

Proof Idea.



Let M and N be the DFAs that recognize the languages A and B respectively.



PDA P that recognizes $A \Diamond B$.

Proof. The proof is by construction. Let $M = (Q_m, \Sigma_m, \delta_m, q_m, F_m)$ be the DFA that recognizes A, and $N = (Q_n, \Sigma_n, \delta_n, q_n, F_n)$ be the DFA that recognizes B. Construct the PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ as follows:

1. $Q = Q_m \cup Q_n \cup \{q_s, q_a\}$. The state q_s is the new start state, and q_a is the new accept state.
2. $\Sigma = \Sigma_m \cup \Sigma_n$
3. $\Gamma = \{\#, \$\}$
4. $q_0 = q_s$
5. $F = \{q_a\}$
6. Define $\delta(q, a, s)$ so that for any $q \in Q$, any $a \in \Sigma_\epsilon$, and any $s \in \Gamma_\epsilon$:

$$\delta(q,a,s) = \begin{cases} \{(q_m, \$)\} & q = q_s,\ a = \epsilon \textit{ and } s = \epsilon \\ \{(\delta_m(q,a), \#)\} & q \in Q_m,\ a \in \Sigma_m,\ \textit{and } s = \epsilon \\ \{(q_n, \epsilon)\} & q \in F_m,\ a = \epsilon,\ \textit{and } s = \epsilon \\ \{(\delta_n(q,a), \epsilon)\} & q \in Q_n,\ a \in \Sigma_n,\ \textit{and } s = \# \\ \{(q_s, \epsilon)\} & q \in F_n,\ a = \epsilon,\ \textit{and } s = \$ \end{cases}$$

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