
Problem 1.72. Let M_1 and M_2 be DFAs that have k_1 and k_2 states, respectively, and then let $U = L(M_1) \cup L(M_2)$.

Part a. Show that if $U \neq \phi$, then U contains some string s, where $|s| < max(k_1,\ k_2)$.

Proof. The proof is by contradiction. Assume $U \neq \phi$ and U does not contain some string s, where $|s| < max(k_1,\ k_2)$. Without the loss of generality also assume that $k_1 > k_2$. As U is not empty, therefore all strings in U are of length at least k_1. Let w be some string in U of minimum length, say n:

$$w = w_1 w_2 w_3 \cdots w_n,\ 0 < k_2 < k_1 \leq n$$

Then, according to the definitions given in Problem 1.52 (Myhill–Nerode theorem), X is pairwise distinguishable by U, and $X \subseteq I$, where I is some index of U:

$$X = \{\epsilon,\ w_1,\ w_1w_2,\ w_1w_2w_3,\ w_1w_2w_3\cdots w_{n-1},\ w_1w_2w_3\cdots w_n\}$$

As $U = L(M_1) \cup L(M_2)$, so either $w \in L(M_1)$ or $w \in L(M_2)$. This means that X must also be the subset of some index of either $L(M_1)$ or $L(M_2)$. $|X| = n + 1$, therefore any DFA that recognizes a language containing w cannot have fewer than $n+1$ states. Hence, either $k_1 \geq n+1$ or $k_2 \geq n+1$, which is a contradiction. □

Part b. Show that if $U \neq \Sigma^*$, then U excludes some string s, where $|s| < k_1k_2$.

Proof. To show that, if $U \neq \Sigma^*$, then U excludes some string s, where $|s| < k_1k_2$, we show that the complement of U contains some string s, where $|s| < k_1k_2$. As $U \neq \Sigma^*$, so $\overline{U} \neq \phi$. Also, $\overline{U} = \overline{L(M_1)} \cup \overline{L(M_2)}$. If the DFAs M_1 and M_2 have k_1 and k_2 states, then the DFAs $\overline{M_1}$ and $\overline{M_2}$ that recognize $\overline{L(M_1)}$ and $\overline{L(M_2)}$ respectively, can be constructed with the same k_1 and k_2 states by swapping the accept and non-accept states. Therefore, according to the proof given in Part a, $\overline{U}$ contains some string s, where $|s| < max(k_1,\ k_2)$. Both k_1 and k_2 are positive integers, so $|s| < k_1k_2$. □