
Problem 5.12. Consider the problem of determining whether a single-tape Turing machine ever writes a blank symbol over a non-blank symbol during the course of its computation on any input string. Formulate this problem as a language and show that it is undecidable.

Proof. Let $T = \{\langle M \rangle \mid M \text{ is a single-tape TM that writes a blank symbol over a non-blank symbol during the course of its computation on any input string}\}$. Show that A_{TM} reduces to T . Assume for the sake of contradiction that **TM** R decides T . Then construct a **TM** S that uses R to decide A_{TM} .

$S =$ “On input $\langle M, w \rangle$, where M is a **TM** and w is a string:

1. Use M and w to construct the following **TM** M_w .
 $M_w =$ “On any input:
 1. Simulate M on w .
 2. If the simulation shows that M accepts, write a blank symbol over a non-blank symbol.”
2. Run R on $\langle M_w \rangle$.
3. If R accepts, M accepts w , so *accept*. Otherwise, *reject*.”

Thus, if **TM** R exists, we can decide A_{TM} , but we know that A_{TM} is undecidable¹. By virtue of this contradiction, we can conclude that R does not exist. Therefore, T is undecidable. \square

¹Theorem 4.11 A_{TM} is undecidable.