
Problem 3.20. Show that single-tape **TM**s that cannot write on the portion of the tape containing the input string recognize only regular languages.

The proof is in two parts. Call such **TM**s, single read-only input tape **TM**s, or SROI-TMs for short. First we show that SROI-TMs cannot recognize context-free and Turing-recognizable languages. Secondly, we show that such **TM**s can recognize regular languages.

Part a. SROI-TMs cannot recognize context free and Turing-recognizable languages.

Proof. Let $C = \{a^n b^n \mid n \geq 0\}$. Clearly, C is a context-free language. We show that no SROI-TM can recognize C. To recognize C, an SROI-TM would need to match each a with a corresponding b. A normal **TM** can mark matched a and b by writing a different symbol, but an SROI-TM cannot. Also, an SROI-TM cannot copy the input string to another part of the tape where it can write. To copy the input string, an SROI-TM would need to keep track of input symbols that are already copied, but it cannot. Similarly, it follows that no SROI-TM can recognize $\{a^n b^n c^n \mid n \geq 0\}$, which is Turing-recognizable. □

Part b. SROI-TMs recognize regular languages.

Proof. The proof is by construction. For any regular language A, let $M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$ be the DFA that recognizes it. Construct an SROI-TM $T = (Q^{'},\ \Sigma,\ \Gamma,\ \delta^{'},\ q_0^{'},\ q_{accept},\ q_{reject})$ to recognize A:

1. $Q^{'} = Q \cup \{q_{accept},\ q_{reject}\}$
2. $q_0^{'} = q_0$
3. $\Gamma = \Sigma \cup \{\sqcup\}$, where $\sqcup$ is the special blank symbol.
4. The transition function is $\delta^{'} : Q \times \Gamma \longrightarrow Q \times \{L,\ R\}$, so that the SROI-TM never writes any part of the tape, which satisfies the constraint that the SROI-TM cannot write on the portion of the tape containing the input string. Define $\delta^{'}(q, a)$ so that for any $q \,\varepsilon\, Q^{'}$ and any $a \,\varepsilon\, \Gamma$:

$$\delta'(q,\ a) \ = \begin{cases} (\delta(q,a),\ R) & a \in \Sigma \\ (q_{accept},\ R) & a = \sqcup \ and \ q \in F \\ (q_{reject},\ R) & a = \sqcup \ and \ q \notin F \end{cases}$$

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