
Problem 5.16. Let $\Gamma = \{0, 1, \sqcup\}$ be the tape alphabet for all **TM**s in this problem. Define the busy beaver function $BB : N \longrightarrow N$ as follows. For each value of k, consider all k-state **TM**s that halt when started with a blank tape. Let $BB(k)$ be the maximum number of 1s that remain on the tape among all of these machines. Show that BB is not a computable function.

Proof. The proof is by reduction from $HALT_{TM}$. Assume for the sake of contradiction that BB is computable and let R be the **TM** that computes it. It follow that R must have a procedure to decide if some **TM**s halts when started with a blank tape. Let T be a **TM** that implements such procedure. In other words $L(T) = \{\langle M\rangle \mid M$ is a **TM** that halts when started with a blank tape$\}$. We construct a **TM** S that uses T to decide $HALT_{TM}$.

$S =$ "On input $\langle M, w\rangle$, where M is a **TM** and w is a string:

1. Use M and w to construct the following **TM** M_w.
   $M_w =$ "On input x:
   1. If x is not empty string, then *reject*.
   2. Run M on w. If M halts, then output result is same as M."
2. Run T on $\langle M_w\rangle$.
3. If T accepts, M halts on input w, so *accept*. Otherwise, *reject*."

Thus, if **TM** R exists, then T exists. So we can decide $HALT_{TM}$, but we know that $HALT_{TM}$ is undecidable[1]. By virtue of this contradiction, we can conclude that R does not exist. Therefore, BB is uncomputable. □

[1] Theorem 5.1 $HALTM_{TM}$ is undecidable.