
Problem 5.35. Say that a variable A in CFG G is necessary if it appears in every derivation of some string $w \in G$. Let $NECESSARY_{CFG} = \{\langle G, A\rangle \mid A \text{ is a necessary variable in } G\}$.

Part a. Show that $NECESSARY_{CFG}$ is Turing-recognizable.

Proof. We construct a ***TM*** N which recognizes $NECESSARY_{CFG}$.

N = "On input $\langle G, A\rangle$, where G is a CFG and A is a symbol:

1. If A is not a variable in G, then *reject*.
2. Repeat for each $i = 0, 1, 2, \cdots$.
3. Generate all strings $s_1, s_2, s_3, \cdots$ of length i, such that each $s_i \in L(G)$.
4. If for some s_i and s_j, where $s_i = s_j$ and $i \neq j$ and A is included in the derivation of both s_i and s_j, then *accept*."

□

Part b. Show that $NECESSARY_{CFG}$ is undecidable.

Proof. □