
Problem 7.21. Let G represent an undirected graph. Also let

$$SPATH = \{\langle G, a, b, k\rangle \mid G \text{ contains a simple path of length at most } k \text{ from } a \text{ to } b\},$$

and

$$LPATH = \{\langle G, a, b, k\rangle \mid G \text{ contains a simple path of length at least } k \text{ from } a \text{ to } b\}.$$

Part a. Show that $SPATH \in P$.

Proof. We show that $SPATH \in P$ by presenting a polynomial time algorithm that decides $SPATH$. A polynomial time algorithm M for $SPATH$ operates as follows.

$M =$ "On input $\langle G, a, b, k\rangle$:

1. Unmark all nodes.
2. Assign each node a value of ∞.
3. Mark a, and set it's value 0.
4. Let integer $d = 0$.
5. Repeat until no additional nodes are marked:
6. Let $d = d + 1$.
7. Scan all edges of G. If an edge (u, v) exists between a marked node u and an unmarked node v, then mark v and set value of v to d.
8. If b is marked and value of b is at most k, then *Accept*. Otherwise *reject*."

Now we analyze this algorithm to show that it runs in polynomial time. Obviously, stages 1 to 4 and stage 8 are executed only once. Stage 7 runs at most m times because each time except the last it marks an additional node in G, where m is the number of nodes in G. Stage 6 also runs at most m times. Thus, the total number of stages used is at most $1 + 1 + m + m$, giving a polynomial in the size of G.

Stages 1 to 4 along with stages 6 and 8 are easily implemented in polynomial time on any reasonable deterministic model. Stage 7 involves a scan of the input and a test of whether certain nodes are marked and an update of node values, which also is easily implemented in polynomial time. Hence M is a polynomial time algorithm for $SPATH$. □

Part b. Show that $LPATH$ is NP-complete.

Proof. First, we need to show that $LPATH \in P$, which is easy as certificate is the path. Next, to show that all problems in NP are polynomial time reducible to $LPATH$, we show $3SAT \leq_p LPATH$. We show how to construct an integer k and an undirected graph G with two nodes, s and t, where a simple path of length at least k exists between s and t , iff ϕ is satisfiable.
Let ϕ be any Boolean formula in 3CNF containing m clauses:

$$\phi = (a_1 \vee b_1 \vee c_1) \ \wedge (a_2 \vee b_2 \vee c_2) \ \wedge \cdots \wedge (a_m \vee b_m \vee c_m).$$

where each a, b and c is a literal x_i or $\overline{x_i}$, and $x_1, x_2 \cdots x_n$ are the n variables of ϕ. Now we show how to convert ϕ to G. The graph contains nodes for literals, and each node's label is the same as the literal that it represents.

1. Repeat for each literal $l = a_1, \ b_1, \ c_1$ of the first clause C_1 in ϕ.
2. Add a node for literal l.
3. Repeat for each subsequent clause $C_2, C_3, \cdots, C_m$:
4. Add nodes for the three literal a_i, b_i and c_i in clause C_i.
5. If $i = 2$, then add an edge between node l and any non-conflicting nodes added in step 3.
6. If $i > 2$, then add an edge between a_{i-1} and any node added in step 3 that does not conlict with a_{i-1} and all the nodes reachable from a_{i-1}. Repeat this step for nodes b_{i-1} and c_{i-1}.
7. Add node s. Add edges between node s and nodes a_1, b_1 and c_1.
8. Add node t. Add edges between node t and all nodes for literals a_m, b_m and c_m.
9. $k = m + 1$.

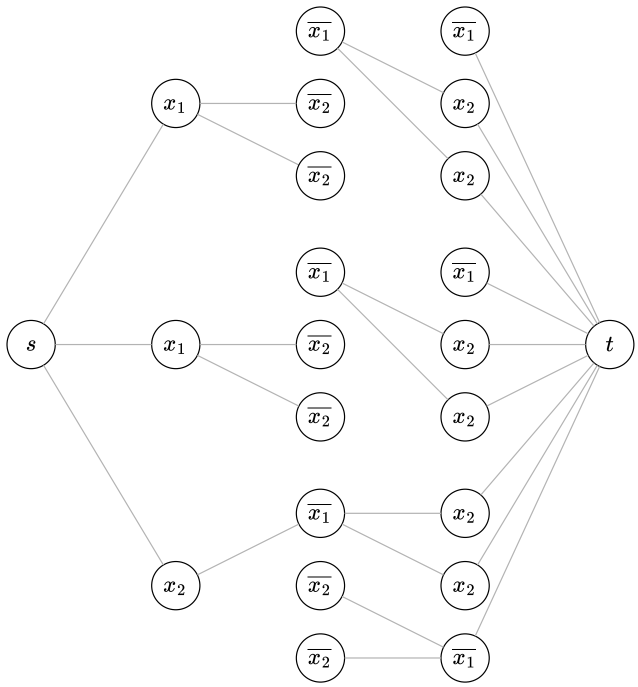
Now we analyze this algorithm to show that it runs in polynomial time. Obviously, stages 7, 8 and 9 are executed only once. Stage 2 runs 3 times. Stages 4, 5 and 6 execute at most m times. Thus, the total number of stages used is at most $1+1+1+3+m+m+m$, giving a polynomial in the size of ϕ. All stages are easily implemented in polynomial time on any reasonable deterministic model.

Now we demonstrate why this construction works. We show that graph G has a simple path of length at least k from s to t, iff ϕ is satisfiable.

Suppose that ϕ has a satisfying assignment. In that satisfying assignment, at least one literal is true in every clause. In graph G, there is an edge between node s and nodes of literals a_1, b_1 and c_1 of first cluase. One of these literal is true, say c_1 is true. Node c_1 would have an edge with at least one of the nodes a_2, b_2 and c_2 of second clause as one of these literals must be true, which means that they cannot all be $\overline{c_1}$. Therefore, there is at least one literal among a_2, b_2 and c_2 that does not conlict with c_1, say b_2, and node c_1 has an edge to node b_2. Similarly, node b_2

would have an edge to at least one of the nodes for literals of next clause and so on. Nodes for literals a_m, b_m and c_m have an edge to node t, so there is a path of length at least $m+1$ in G from s to t.

Suppose that G has a path of length at least k from s to t. No two nodes on this path conflict each other. Therefore, intermediate nodes $n_1, n_2, \cdots, n_{k-1}$ on the path $(s, n_1, n_2, \cdots, n_{k-1}, t)$ give a satisfying assignment for ϕ.



Graph G with $k = 4$ for
$\phi = (x_1 \vee x_1 \vee x_2) \ \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \ \wedge (\overline{x_1} \vee x_2 \vee x_2)$.

$\square$