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**Problem 5.9.** Let $T = \{\langle M \rangle \mid M \text{ is a } \textbf{TM} \text{ that accepts } w^R \text{ whenever it accepts } w\}$. Show that $T$ is undecidable.

*Proof.* Show that $A_{TM}$ reduces to $T$. Assume for the sake of contradiction that **TM** $R$ decides $T$. Then construct a **TM** $S$ that uses $R$ to decide $A_{TM}$.

$S =$ "On input $\langle M, w \rangle$, where $M$ is a **TM** and $w$ is a string:

1. Use $M$ and $w$ to construct the following TM $M_w$.
   $M_w =$ "On input $x$:
   1. If $x = w^R$, *accept.*
   2. If $x \neq w$, *reject.*
   3. Run $M$ on $x$ and output whatever $M$ outputs."
2. Run $R$ on $\langle M_w \rangle$.
3. If $R$ accepts, $M$ accepts $w$, so *accept.* Otherwise, *reject.*"

Thus, if **TM** $R$ exists, we can decide $A_{TM}$, but we know that $A_{TM}$ is undecidable[1]. By virtue of this contradiction, we can conclude that $R$ does not exist. Therefore, $T$ is undecidable. □

[1]Theorem 4.11 $A_{TM}$ is undecidable.