---

**Problem 1.41.** Let $B$ and $C$ be languages over $\Sigma = \{0, 1\}$. Define

$$B \overset{1}{\leftarrow} C = \{w \in B \mid for\ some\ y \in C,\ strings\ w\ and\ y\ contain\ equal\ numbers\ of\ 1s\}.$$

Show that the class of regular languages is closed under the $\overset{1}{\leftarrow}$ operation.

*Proof.* To show that the operation $B \overset{1}{\leftarrow} C$ is closed under the class of regular languages we use the closure properties of operations intersection, $Zero^-$ and $Zero^+$, and give a regular expression that uses these operations to describes operation $B \overset{1}{\leftarrow} C$. Let $A$ be any language over $\Sigma = \{0, 1\}$. Define

$$Zero^-(A) = \{w \mid w\ is\ constructed\ by\ removing\ all\ 0s\ from\ some\ string\ in\ A\}.$$

$$Zero^+(A) = \{w \mid w\ is\ constructed\ by\ inserting\ any\ number\ of\ 0s\ anywhere\ in\ some\ string\ of\ A\}.$$

The closure properties of $Zero^-$ and $Zero^+$ are proved in Corollaries 1 and 2. Let $B$ and $C$ be any regular languages over $\Sigma = \{0, 1\}$. The regular expression that describes the operation $B \overset{1}{\leftarrow} C$ is:

$$B \overset{1}{\leftarrow} C = Zero^+\Big(Zero^-(B) \cap Zero^-(C)\Big) \cap B$$

$\square$

**Corollary 1.** Let $A$ be any language over $\Sigma = \{0, 1\}$. Define

$$Zero^-(A) = \{w \mid w\ is\ constructed\ by\ removing\ all\ 0s\ from\ some\ string\ in\ A\}.$$

Show that the class of regular languages is closed under the $Zero^-$ operation.

*Proof.* The proof is by construction. Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that recognizes $A$. Construct the NFA $M' = (Q', \Sigma, \delta', q_0', F')$ to recognize $Zero^-(A)$:

1. $Q' = Q$
2. $q_0' = q_0$
3. $F' = \begin{cases} F & A \cap 0^+ = \phi \\ F \cup \{q_0\} & A \cap 0^+ \neq \phi \end{cases}$
4. Define $\delta'(q, a)$ so that for any $q \in Q'$ and any $a \in \Sigma$:

$$\delta'(q,\ a) = \begin{cases} \{\delta(q,\ a)\} & a \neq 0 \\ \phi & a = 0 \end{cases}$$

$\square$

**Corollary 2.** Let $A$ be any language over $\Sigma = \{0, 1\}$. Define

$$Zero^{+}(A) = \{w \mid w \textit{ is constructed by inserting any number of } 0s \textit{ anywhere in some string of } A\}.$$

Show that the class of regular languages is closed under the $Zero^{+}$ operation.

*Proof.* The proof is by construction. Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that recognizes $A$. Construct the NFA $M' = (Q', \Sigma, \delta', q_0', F')$ to recognize $Zero^{+}(A)$:

1. $Q' = Q$
2. $q_0' = q_0$
3. $F' = F$
4. Define $\delta'(q, a)$ so that for any $q \in Q'$ and any $a \in \Sigma$:

$$\delta'(q,\ a)\ = \begin{cases} \{\delta(q,\ a)\} & a \neq 0 \\ \{\delta(q,\ a),\ q\} & a = 0 \end{cases}$$

$\square$