
Problem 5.13. A ***useless state*** in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Formulate this problem as a language and show that it is undecidable.

Proof. Let $T = \{\langle M \rangle \mid M \text{ is a } \mathbf{TM}, \text{ which has a useless state}\}$. Show that A_{TM} reduces to T, where $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a } \mathbf{TM} \text{ and } M \text{ accepts } w\}$. Assume for the sake of contradiction that **TM** R decides T. Then construct a **TM** S that uses R to decide A_{TM}. The idea is to modify the input **TM** M, so that the modified **TM** M_w does not have any useless states if M accepts w. We can do this by modifying M as follows:

1. Add special new state q_u.
2. Before processing the input, transition through all non-start states except q_u.
3. Simulate M on w.
4. If the simulation shows that M accepts w, transition to q_u.

$S =$ "On input $\langle M, w \rangle$, where M is a **TM** and w is a string:

1. Transform M to new **TM** M_w as discussed above.
2. Run R on $\langle M_w \rangle$.
3. If R rejects, M accepts w, so *accept*. Otherwise, *reject*."

Thus, if **TM** R exists, we can decide A_{TM}, but we know that A_{TM} is undecidable[1]. By virtue of this contradiction, we can conclude that R does not exist. Therefore, T is undecidable. □

[1] Theorem 4.11 A_{TM} is undecidable.