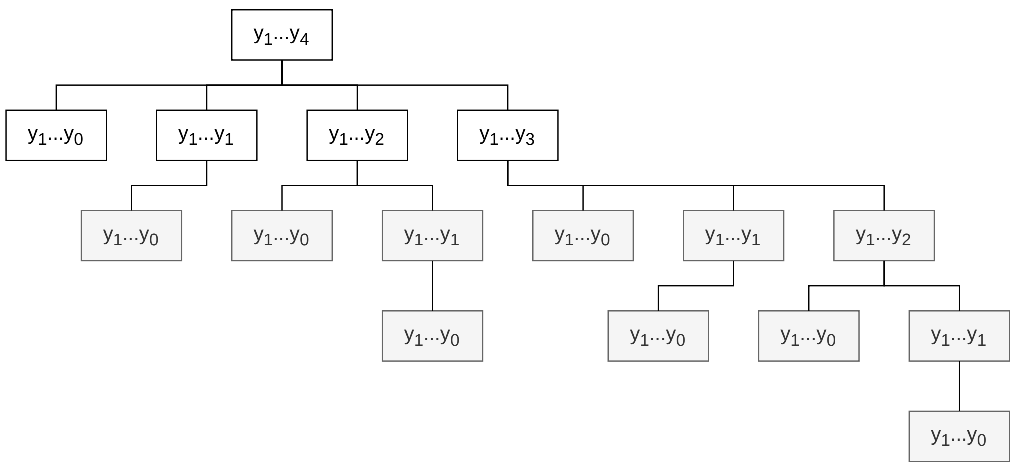
Introduction to the Theory of Computation **Mehshan Mustafa**
Chapter 7: Time Complexity **August 20, 2025**

Problem 7.15. Show that P is closed under the star operation.

Proof Idea. For $A \in P$, a string $y = y_i \cdots y_j \in A^*$, where $i \leq j$, when one of following is true:

1. $y_i \cdots y_j = \varepsilon$.
2. $y_i \cdots y_{k-1} \in A^*$ and $y_k \cdots y_j \in A$, for some $i \leq k \leq j$.



Tree of sub-problems for a string of length 4.
Non-shaded sub-problems are unique, whereas shaded are duplicate.

First, we give a recurive algorithm C that tests if A^* contains a string y. Secondly, we use *Dynamic Programming* (recursion + memoinzation) to obtain a polynomial time algorithm D.

Proof. Let A be any language in P, and S be the **TM** that decides A in polynomial time.

C = "On input $\langle y, i, j \rangle$, where y is a string and i, j are integers:

1. If $y_i \cdots y_j = \varepsilon$, then *accept*.
2. Repeat for each k between i and j.
3. Run C on $\langle y, i, k-1 \rangle$.
4. Use S to check if $y_k \cdots y_j \in A$.
5. *Accept*, if both C and S accept.
6. *Reject*."

Then decide A^* by starting with $i = 1$ and $j = |y|$.

C takes non-polynomial time as it makes 2^n calls for a string for length n. We add memoinzation in C to obtain algorithm D, which makes $O(n)$. Therefore, time used by D is polynomial.

$D =$ "On input $\langle y, i, j \rangle$, where y is a string and i, j are integers:

1. If previously solved then answer same, else continue.
2. If $y_i \cdots y_j = \varepsilon$, then *accept*.
3. Repeat for each k between i and j.
4. Run C on $\langle y, i, k-1 \rangle$.
5. Use S to check if $y_k \cdots y_j \in A$.
6. *Accept*, if both C and S accept.
7. *Reject*."

Then decide A^* by starting with $i = 1$ and $j = |y|$. $\square$