
Problem 2.30. Use the pumping lemma to show that the following languages are not context free.

Part a. $\{0^n 1^n 0^n 1^n \mid n \geq 0\}$

Proof. The proof is by contradiction. Assume the language $A = \{0^n 1^n 0^n 1^n \mid n \geq 0\}$ is context free. Let p be the pumping length given by the pumping lemma. Choose the string $s = 0^p 1^p 0^p 1^p$. Clearly s is a member of A and of length at least p. The pumping lemma states that s can be pumped, but we show that it cannot be pumped.

1. Either v or y is non-empty. Without loss of generality, we only discuss the case when v is non-empty. If v is non-empty, then it may contain only one type of alphabet or both 0's and 1's. In former case, the string $uv^2xy^2z \notin A$, because it cannot contain equal number of 0's or 1's in each segment. In latter case, the string $uv^2xy^2z \notin A$ as it cannot contain exactly two segments of 0's and 1's.

2. Both v and y are non-empty. There are three possibilities. First, both v and y contain the same alphabet from the same segment of 0's or 1's. Second, v contains only one type of symbol from a segment and y only contains the other type of symbol from a neighboring segments. Third, v or y contain both 0's and 1's from two neighboring segments. In the first and second cases, the string $uv^2xy^2z \notin A$, because it cannot contain equal number of 0's and 1's in each segment. In the third case, the string $uv^2xy^2z \notin A$ as it cannot contain exactly two segments of 0's and 1's.

One of these cases must occur, and both cases result in a contradiction. $\square$

Part b. $\{t_1 \# t_2 \# \cdots \# t_k \mid k \geq 2,\ each\ t_i \in \{a, b\}^*,\ and\ t_i = t_j\ for\ some\ i \neq j\}$

Proof. $\square$