
Problem 1.32. Let

$$\Sigma_3 = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \dots, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

Σ_3 contains all size 3 columns of 0s and 1s. A string of symbols in Σ_3 gives three rows of 0s and 1s. Consider each row to be a binary number and let

$$B = \{w \in \Sigma_3^* \,|\, the\, bottom\, row\, of\, w\, is\, the\, sum\, of\, the\, top\, two\, rows\}$$

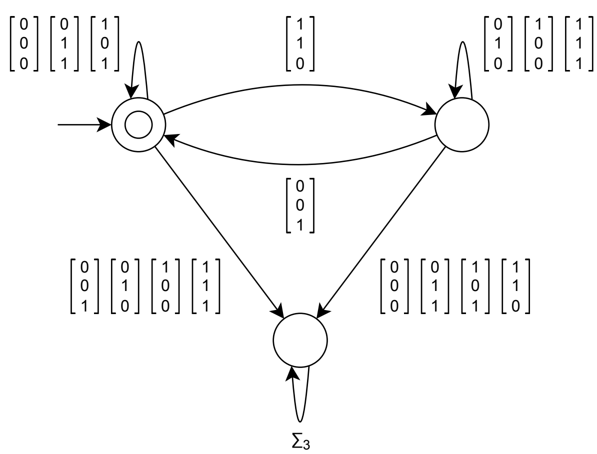
For example,

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \in B,\, but \begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\1\end{bmatrix} \notin B.$$

Show that B is regular.

Proof Idea. To prove B is regular, we show that B's reverse B^R is regular. According to the proof given in Problem 1.31, if B^R is regular, then its reverse B is also regular.

Proof. The proof is by construction. The following state diagram shows the construction of a DFA that recognizes B^R.



State diagram of a DFA that recognizes B^R.

□

Problem 1.33. Let

$$\Sigma_2 = \left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$$

Here, Σ_2 contains all columns of 0s and 1s of height two. A string of symbols in Σ_2 gives two rows of 0s and 1s. Consider each row to be a binary number and let

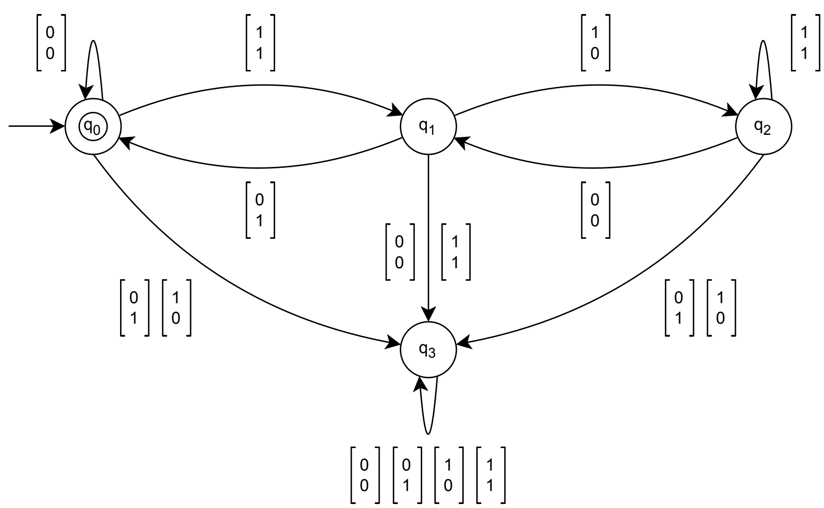
$$C = \{w \in \Sigma_2^* \,|\, the\, bottom\, row\, of\, w\, is\, three\, times\, the\, top\, row\}$$

For example,

$$\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} \in C, \, but \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \notin C.$$

Show that C is regular.

Proof Idea. To prove C is regular, we show that C's reverse C^R is regular. According to the proof given in Problem 1.31, if C^R is regular, then its reverse C is also regular.



State diagram of a DFA that recognizes C^R.

Proof. The proof is by construction. The above state diagram shows the construction of a DFA that recognizes C^R. $\square$

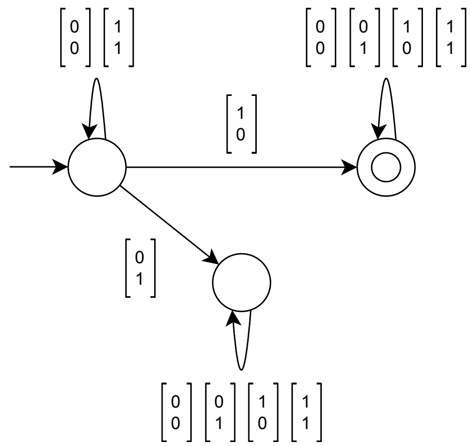
Problem 1.34. Let Σ_2 be the same as in Problem 1.33. Consider each row to be a binary number and let

$$D = \{w \in \Sigma_2^* \mid \textit{the top row of } w \textit{ is a larger number than is the bottom row}\}$$

For example,

$$\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} \in D, \, but \begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} \notin D$$

Show that D is regular.



State diagram of a DFA that recognizes D.

Proof. The proof is by construction. The given state diagram shows the construction of a DFA that recognizes D. $\square$

Problem 1.35. Let Σ_2 be the same as in Problem 1.33. Consider the top and bottom rows to be strings of 0s and 1s, and let

$$E = \{w \in \Sigma_2^* \mid \textit{the bottom row of } w \textit{ is the reverse of the top row}\}$$

Show that E is not regular.

Proof. The proof is by contradiction. Assume that E is regular. Let p be the pumping length given by the pumping lemma. Choose s to be the string:

$$s = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^p \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^p$$

Therefore, $s \in E$ and $|s| \geq p$, so the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$, $xy^iz \in E$. According to condition 3 (*i.e.* $|xy| \leq p$) of the pumping lemma, y can only contain $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$s. The string $xyyz$ has more $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$s than $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$s, so its bottom row is not the reverse of its top row, which is a contradiction. Therefore, E is not regular. $\square$

Problem 1.35. Let Σ_2 be the same as in Problem 1.33. Consider the top and bottom rows to be strings of 0s and 1s, and let

$$E = \{w \in \Sigma_2^* | \textit{ the bottom row of } w \textit{ is the reverse of the top row of } w\}.$$

Proof. $\square$