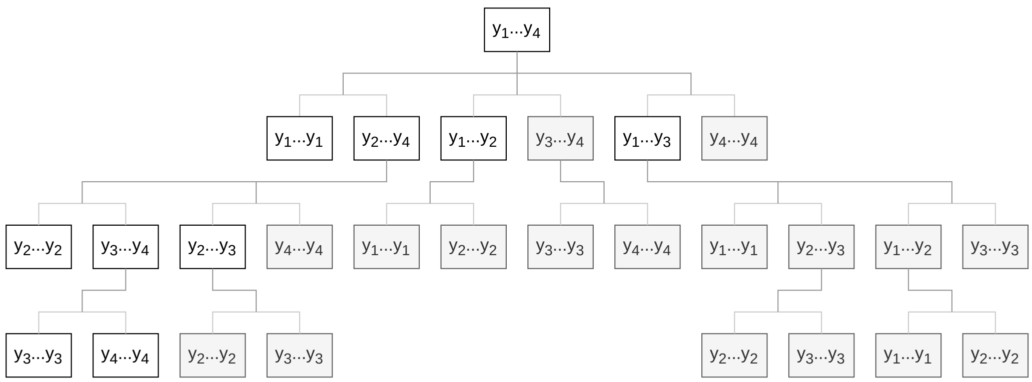
**Problem 7.15.** Show that P is closed under the star operation.

*Proof Idea.* For any language $A \in P$, a string $y = y_1 \cdots y_n \in A^*$, when one of following is true:

1. $y = \varepsilon$.
2. $y \in A$.
3. Both sub-strings of one of possible splits of $y$ are in $A^*$.




Tree of sub-problems for a string of length 4.
Non-shaded sub-problems are unique, whereas shaded are duplicate.

First, we give a recurive algorithm $C$ that tests if $A^*$ contains a string $y$. Secondly, we use *Dynamic Programming* (recursion + memoinzation) to obtain a polynomial time algorithm $D$.

*Proof.* Let $A$ be any language in $P$ and $T$ be the **TM** that decides $A$ in polynomial time.

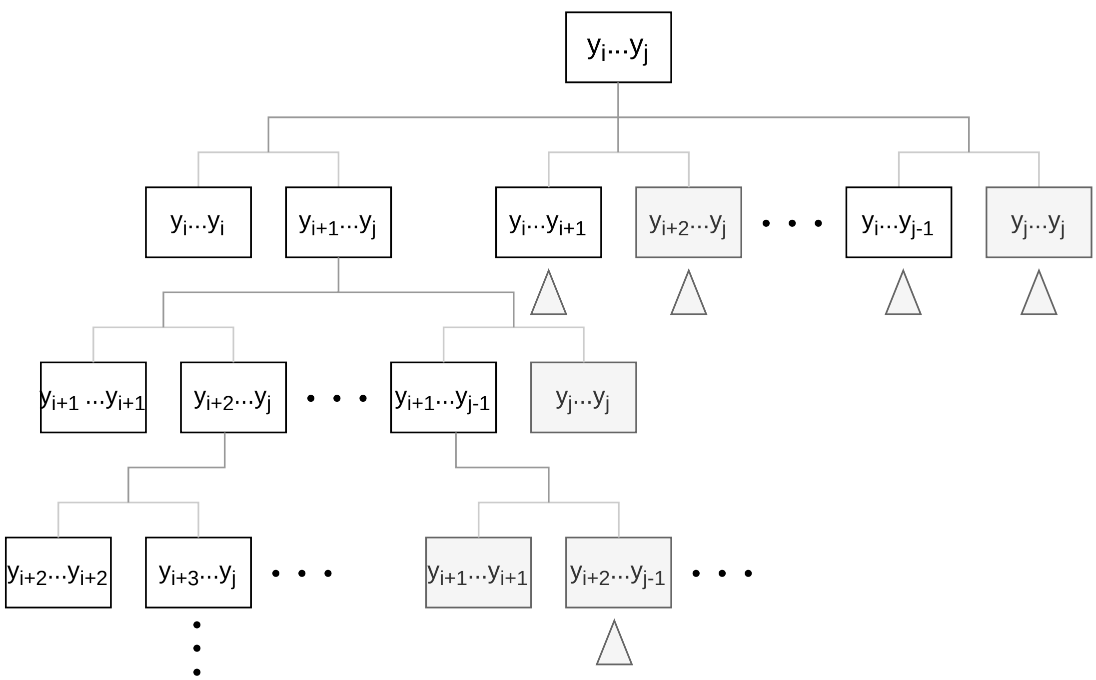
$C$ = "On input $\langle y, i, j \rangle$, where $y$ is a string and $i, j$ are integers:

1. If $y = \varepsilon$, then *accept*.
2. Use $T$ to check if $y_i \cdots y_j \in A$. If $T$ accepts, then *accept*.
3. Repeat for each $k$ between $i + 1$ and $j$.
4. Run $C$ on $\langle y, i, k - 1 \rangle$.
5. Run $C$ on $\langle y, k, j \rangle$.
6. *Accept*, if $C$ accepts in both cases.
7. *Reject*."

Then decide $A^*$ by starting with $i = 1$ and $j = |y|$.
$D =$ "On input $\langle y, i, j\rangle$, where $y$ is a string and $i, j$ are integers:

1. If $y = \varepsilon$, then *accept*.
2. If previously solved then answer same, else continue.
3. Use $T$ to check if $y_i \cdots y_j \in A$. If $T$ accepts, then *accept*.
4. Repeat for each $k$ between $i + 1$ and $j$.
5. Run $C$ on $\langle y, i, k - 1\rangle$.
6. Run $C$ on $\langle y, k, j\rangle$.
7. *Accept*, if $C$ accepts in both cases.
8. *Reject*."

Then decide $A^*$ by starting with $i = 1$ and $j = |y|$.




Tree of sub-problems for a string $y = y_i \cdots y_j$.
Shaded sub-problems are duplicate.

$\square$