
Problem 5.31. Let

$$f(x) = \begin{cases} 3x+1 & \text{for odd } x \\ x/2 & \text{for even } x \end{cases}$$

for any natural number x. If you start with an integer x and iterate f, you obtain a sequence, $x, f(x), f(f(x)), \ldots$. Stop if you ever hit 1. For example, if $x = 17$, you get the sequence 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Extensive computer tests have shown that every starting point between 1 and a large positive integer gives a sequence that ends in 1. But the question of whether all positive starting points end up at 1 is unsolved; it is called the $3x+1$ problem. Suppose that A_{TM} were decidable by a **TM** H. Use H to describe a **TM** that is guaranteed to state the answer to the $3x+1$ problem.

Proof. We construct a **TM** S that is guaranteed to state the answer to the $3x+1$ problem. S rejects if there exists some x for which the $3x+1$ problem does not end up at 1. Otherwise, S accepts.

$S =$ "On any input:

1. Construct **TM** M to solve the $3x+1$ problem for some x.
2. $M =$ "On input x, where $x \geq 1$:
    1. Start with x and iterate f to obtain each term of the sequence $x, f(x), f(f(x)), \ldots$ step by step.
    2. If some term ever equals 1, then *accept*."
3. Construct the following **TM** F that runs the **TM** M on every positive integer. M enters its accept state only if there exists some x for which the $3x+1$ problem does not end up at 1.
4. $F =$ "On any input:
    1. Repeat for each $i = 1, 2, 3, \ldots$.
    2. Run H on $\langle M, i \rangle$.
    3. If H rejects, M rejects i by looping, so *accept*."
5. Run H on $\langle F, F \rangle$.
6. If H accepts, then there exists some x for which the $3x+1$ problem does not end up at 1, so *reject*. Otherwise, *accept*.

$\square$