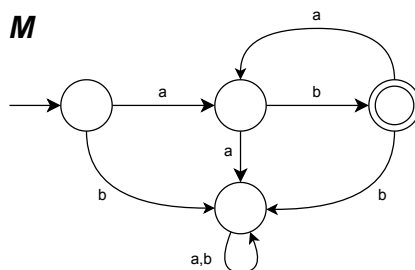


**Problem 1.66.** A homomorphism is a function  $f : \Sigma \rightarrow \Gamma^*$  from one alphabet to strings over another alphabet. We can extend  $f$  to operate on strings by defining  $f(w) = f(w_1)f(w_2)\cdots f(w_n)$ , where  $w = w_1w_2\cdots w_n$  and each  $w_i \in \Sigma$ . We further extend  $f$  to operate on languages by defining  $f(A) = \{f(w) \mid w \in A\}$ , for any language  $A$ .

**Part a.** Show, by giving a formal construction, that the class of regular languages is closed under homomorphism. In other words, given a DFA  $M$  that recognizes  $B$  and a homomorphism  $f$ , construct a finite automaton  $M'$  that recognizes  $f(B)$ .

*Proof Idea.* Let  $\Sigma = \{a, b\}$ ,  $\Gamma = \{0, 1\}$ , and  $B = (ab)^+$ . Define the homomorphism function  $f$  on  $\Sigma$  to be  $f(a) = 11$ , and  $f(b) = 00$ . Therefore,  $f(ab) = 1100$  and  $f(abab) = 11001100$ .

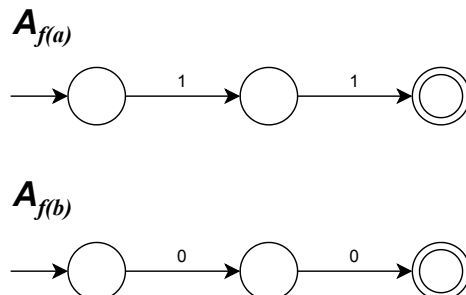
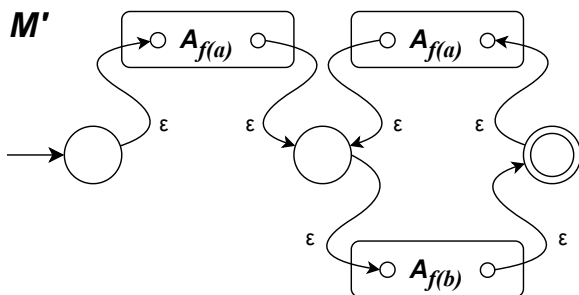
$$B = \{ab, abab, ababab, \dots\}, \text{ and } f(B) = \{1100, 11001100, 110011001100, \dots\}$$



State diagram of the DFA  $M$  that recognizes  $B$ .

Next, construct a finite automaton  $A_{f(a)}$  to recognize the string  $f(a)$  for each  $a \in \Sigma$ . The DFA  $M'$  to recognize  $f(B)$  can be constructed by taking the DFA  $M$  and by carrying out following steps for every transition between some initial state  $q_i$  and subsequent state  $q_j$  over some symbol  $a$ :

1. Remove the transition.
2. Add an  $\epsilon$ -transition that connects state  $q_i$  to the start state of the DFA  $A_{f(a)}$
3. Connect accept state of the DFA  $A_{f(a)}$  with  $q_j$  with an  $\epsilon$ -transition.
4. Make the accept state of  $A_{f(a)}$  a non-accept state.



Construction of the NFA  $M'$ . Unnecessary transitions are omitted for simplicity.

*Proof.* Todo: provide formal construction using above idea. □

Consider the machine  $M'$  that you constructed. Is it a DFA in every case?

If a given homomorphism function  $f$  maps two different symbols in  $\Sigma$  to a string over  $\Gamma$  that starts with the same symbol then  $M'$  can have non-deterministic transitions.

**Part b.** Show, by giving an example, that the class of non-regular languages is not closed under homomorphism.

*Proof.* Solution Replace this text with the details of your proof or solution. □