
Problem 5.14. Consider the problem of determining whether a Turing machine M on an input w ever attempts to move its head left when its head is on the left-most tape cell. Formulate this problem as a language and show that it is undecidable.

Proof. Let $T = \{\langle M, w\rangle \mid M$ is a **TM**, which on input w moves its head left when its head is on the left-most tape cell$\}$.

Show that A_{TM} reduces to T, where $A_{TM} = \{\langle M, w\rangle \mid M$ is a **TM** and M accepts $w\}$. Assume for the sake of contradiction that **TM** R decides T. Then construct a **TM** S that uses R to decide A_{TM}. The idea is to construct a new **TM** M_w, which only moves its head left when its head is on the left-most tape cell if M accepts w. We construct **TM** M_w using 2 tapes as follows.

$M_w =$ "On input w:

1. Simulate M on w using the second tape.
2. If the simulation shows that M accepts, move the head of the first tape left."

Next, we construct **TM** S to decide A_{TM}.

$S =$ "On input $\langle M, w\rangle$, where M is a **TM** and w is a string:

1. Use M and w to construct **TM** M_w as discussed above.
2. Convert the 2 tape **TM** M_w to equivalent single tape **TM** M'.
3. Run R on $\langle M', w\rangle$.
4. If R accepts, M accepts w, so *accept*. Otherwise, *reject*."

Thus, if **TM** R exists, we can decide A_{TM}, but we know that A_{TM} is undecidable[1]. By virtue of this contradiction, we can conclude that R does not exist. Therefore, T is undecidable. $\square$

[1] Theorem 4.11 A_{TM} is undecidable.