
Problem 1.54. Consider the language $F = \{a^i b^j c^k \mid i,\ j,\ k \geq 0 \ and\ if\ i = 1\ then\ j = k\}$.

Part a. Show that F is not regular.

Proof. To show that F is not regular, we use **Myhill–Nerode theorem**, and show that the **index of $\boldsymbol{L}$**[1] is infinite. Define

$$X = \{w \mid w = ab^i,\ where\ i \geq 0\}.$$

The set X is infinite, and no two strings in X have the same number of b's. Take any two strings $w_a, w_b \in X$, let k be the number of b's in w_a, and let $z = c^k$, then $w_a z \in F$ and $w_b z \notin F$. Thus X is pairwise distinguishable by F, and X is the index of L. As X is infinite, therefore F is not a regular language. □

Part b. Show that F acts like a regular language in the pumping lemma. In other words, give a pumping length p and demonstrate that F satisfies the three conditions of the pumping lemma for this value of p.

Let the pumping length $p = 2$. For any string $s \in F$ with a length of at least 2, there can be four cases regarding the number of a's.

1. No a's.
   Split $s = xyz$, where $x = \epsilon$, y is the first symbol and z is the rest.
2. Eactly one a, and an equal number of b's and c's.
   Split $s = xyz$, where $x = \epsilon$, $y = a$ and z is the rest.
3. Exactly two a's.
   Split $s = xyz$, where $x = \epsilon$, $y = aa$ and z is the rest.
4. More than two a's.
   Split $s = xyz$, where $x = \epsilon$, $y = a$ and z is the rest.

In each case, the string s can be split into three pieces, $s = xyz$, satisfying the three conditions of the pumping lemma.

Part c. Explain why parts (a) and (b) do not contradict the pumping lemma.

The pumping lemma is an implication, where the antecedent is that a given language is regular, and the consequent is that all large enough strings in the given language can be split and pumped. In case of F, the antecedent is false, so the consequent may or may not be true.

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[1] Problem 1.52 **Myhill–Nerode theorem**.