
Problem 5.20. Prove that there exists an undecidable subset of $\{1\}^*$.

Proof Idea. Let $\mathcal{L} = \mathcal{P}(\{1\}^*)$ be the set of all subsets of $\{1\}^*$. To show that there exists an undecidable member of $\mathcal{L}$, we first observe that the set $\mathcal{B}$ of all *infinite binary sequences* is uncountable. We show that $\mathcal{L}$ is also uncountable by giving a correspondence with $\mathcal{B}$. If $\mathcal{L}$ is uncountable, then it would imply that there exists an undecidable member of $\mathcal{L}$, because $\mathcal{L}$ is a bigger set than the set of all Turing machines, which we know is countable according to the Corollary 4.18.

Proof. The proof is based on the proof of Corollary 4.18. An infinite binary sequence is an unending sequence of 0s and 1s. Let $\mathcal{B}$ be the set of all infinite binary sequences. We show that $\mathcal{B}$ is uncountable by using diagonalization. Suppose that the correspondence f exists between $\mathcal{N}$ and $\mathcal{B}$ as shown in the table below. We construct the desired x by giving its infinite binary sequence. Our objective is to ensure that $x \neq f(n)$ for any n. To ensure that $x \neq f(1)$, we let the first character of x to be different from the first character of $f(1) = \underline{1}000101\cdots$, which is 0. To ensure that $x \neq f(2)$, we let the second character of x to be different from the second character of $f(2) = 1\underline{0}01101\cdots$, so we choose 1. Continuing in this way down the diagonal of the table for f, we obtain all the characters of x, as shown in the following table. We know that x is not $f(n)$ for any n because it differs from $f(n)$ in the nth character.

n	$f(n)$
1	$\underline{1}000101\cdots$
2	$1\underline{0}01101\cdots$
3	$00\underline{1}1100\cdots$
$\vdots$	$\vdots$

$$x = 010\cdots$$

We show that $\mathcal{L}$ is uncountable by giving a correspondence with $\mathcal{B}$, thus showing that the two sets are the same size. Let $\{1^*\} = \{s_1, s_2, s_3, \cdots\}$. Each language $A \in \mathcal{L}$ has a unique sequence in $\mathcal{B}$. The ith bit of that sequence is a 1 if $s_i \in A$ and is a 0 if $s_i \notin A$, which is called the characteristic sequence $\mathcal{X}_A$ of A.

$$\begin{array}{rclcccccccc}
\{1\}^* & = & \{ & \varepsilon, & 1, & 11, & 111, & \cdots, & \} \\
A & = & \{ & & 1, & & 111, & \cdots, & \} \\
\mathcal{X}_A & = & & 0 & 1 & 0 & 1 & \cdots, &
\end{array}$$

The function $f : \mathcal{L} \longrightarrow \mathcal{B}$, where $f(A)$ equals the characteristic sequence of A, is one-to-one and onto, and hence is a correspondence. Therefore, as $\mathcal{B}$ is uncountable, $\mathcal{L}$ is uncountable as well. $\square$