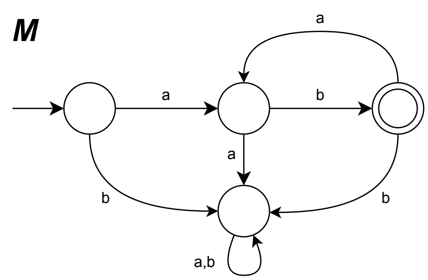
Introduction to the Theory of Computation — **Mehshan Mustafa**
Chapter 1: Regular Languages — **November 24, 2024**

Problem 1.66. A homomorphism is a function $f : \Sigma \longrightarrow \Gamma^*$ from one alphabet to strings over another alphabet. We can extend f to operate on strings by defining $f(w) = f(w_1)f(w_2)\cdots f(w_n)$, where $w = w_1w_2\cdots w_n$ and each $w_i \in \Sigma$. We further extend f to operate on languages by defining $f(A) = \{f(w) \mid w \in A\}$, for any language A.

Part a. Show, by giving a formal construction, that the class of regular languages is closed under homomorphism. In other words, given a DFA M that recognizes B and a homomorphism f, construct a finite automaton M' that recognizes $f(B)$.

Proof Idea. Let $\Sigma = \{a, b\}$, $\Gamma = \{0, 1\}$, and $B = (ab)^+$. Define the homomorphism function f on Σ to be $f(a) = 11$, and $f(b) = 00$. Therefore, $f(ab) = 1100$ and $f(abab) = 11001100$.

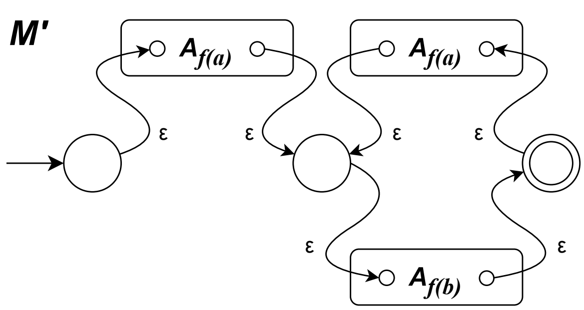
$$B = \{ab, abab, ababab, \cdots\}, and\ f(B) = \{1100, 11001100, 110011001100, \cdots\}$$

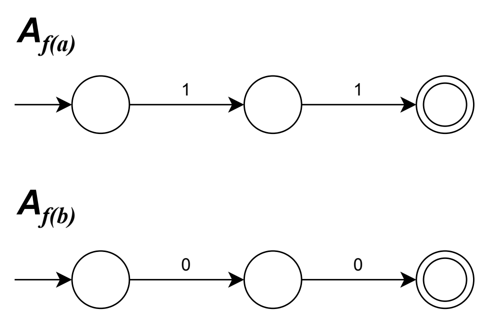


State diagram of the DFA M that recognizes B.

Next, construct a finite automaton $A_{f(a)}$ to recognize the string $f(a)$ for each $a \in \Sigma$. The DFA M' to recognize $f(B)$ can be constructed by taking the DFA M and by carrying out following steps for every transition between some initial state q_i and subsequent state q_j over some symbol a:

1. Remove the transition.
2. Add an ϵ-transition that connects state q_i to the start state of the DFA $A_{f(a)}$
3. Connect accept state of the DFA $A_{f(a)}$ with q_j with an ϵ-transition.
4. Make the accept state of $A_{f(a)}$ a non-accept state.





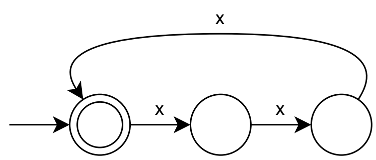
Construction of the NFA M'. Unnecessary transitions are omitted for simplicity.

Proof. To Do: proof by construction. □

Consider the machine M' that you constructed. Is it a DFA in every case?
If a given homomorphism function f maps two different symbols in Σ to a string over Γ that starts with the same symbol then M' can have non-deterministic transitions.

Part b. Show, by giving an example, that the class of non-regular languages is not closed under homomorphism.

Take $A = \{0^n 1^n 2^n \mid n \geq 0\}$ for example, which is a non-regular language[1]. Let $\Gamma = \{x, y\}$, and define homomorphism $f(a) = x$ for all $a \in \{0, 1, 2\}$. Then, $f(A) = \{\epsilon, x^3, x^6, x^9, x^{12}, \cdots, x^{3n}\}$ is a regular language that consists of only those strings, which have multiple of 3 $x's$.



A finite automaton that recognizes $f(A)$.

[1] Exercise 1.29 a.