
Problem 7.22. Let $DOUBLE\text{-}SAT = \{\langle\phi\rangle \mid \phi \text{ has at least two satisfying assignments}\}$.
Show that $DOUBLE\text{-}SAT$ is NP-complete.

Proof. The proof is in two parts. First, we show that $DOUBLE\text{-}SAT \in NP$. Second, we show that every $A \in P$ is polynomial time reducable to $DOUBLE\text{-}SAT$.

Part a. Show that $DOUBLE\text{-}SAT \in NP$.

A certificate is simply two satisfying assignments.

Part b. Show that every $A \in P$ is polynomial time reducable to $DOUBLE\text{-}SAT$.

To show that every $A \in P$ is polynomial time reducable to $DOUBLE\text{-}SAT$, we give a polynomial time reduction F from SAT to $DOUBLE\text{-}SAT$.

$F =$ "On input $\langle\phi\rangle$, where ϕ is a satisfiable Boolean formula:

1. Let α be a new Boolean variable, which is not in ϕ.
2. Let $\phi_2 = \phi \wedge (\alpha \vee \overline{\alpha})$.
3. Output $\langle\phi_2\rangle$."

If ϕ is satisfiable, then ϕ_2 has at least two satisfiable assignments. First, where $\alpha = T$, and the second where $\alpha = F$. If $\phi_2 = \phi \wedge (\alpha \vee \overline{\alpha})$ has at least two satisfiable assignments, then ϕ is satisfiable. □