
Problem 1.39. The construction in Theorem 1.54 shows that every GNFA is equivalent to a GNFA with only two states. We can show that an opposite phenomenon occurs for DFAs. Prove that for every $k > 1$, a language $A_k \subseteq \{0,1\}^*$ exists that is recognized by a DFA with k states but not by one with only $k-1$ states.

Proof. Let A_k be the language of all strings of length at least $k-1$.

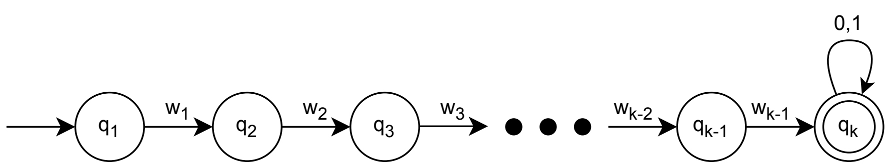
$$A_k = \{w \mid w \in \{0,1\}^* \text{ } and \text{ } |w| \geq k-1\}$$

$$w = w_1 w_2 w_3 \cdots w_{k-1} \cdots w_n, \text{ } where \text{ } n \geq k-1$$

The state diagram of a DFA D_k that recognizes A_k with k states is given below. Each symbol w_i, where $1 \leq i \leq k-1$, requires a transition between two states q_i and q_{i+1}. To construct a new DFA D_{k-1} that recognizes A_k with $k-1$ states by using D_k, there are three options:

1. Remove state q_1 and make q_2 the start state. First symbol w_1 is skipped.
2. Remove state q_k, make q_{k-1} the accept state and add a new transition from q_{k-1} to q_{k-1} for symbols 0 and 1. Symbol w_{k-1} is skipped.
3. Remove any intermediate state q_i, where $1 < i < k$, and add a new tranisition for symbols 0 and 1 from state q_{i-1} to q_{i+1}. Symbol w_i is skipped.

In all three cases the minimum length of strings, which are recognized by the DFA D_{k-1} is reduced by one. Therefore, there does not exist a DFA that recognizes A_k with $k-1$ states.



State diagrams of DFA D_k that recognizes A_k. Each $w_i \in \{0,1\}$.

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