
Problem 7.28. A ***coloring*** of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

$$3COLOR = \{\langle G \rangle \mid G \text{ is colorable with 3 colors}\}.$$

Show that $3COLOR$ is NP-complete.

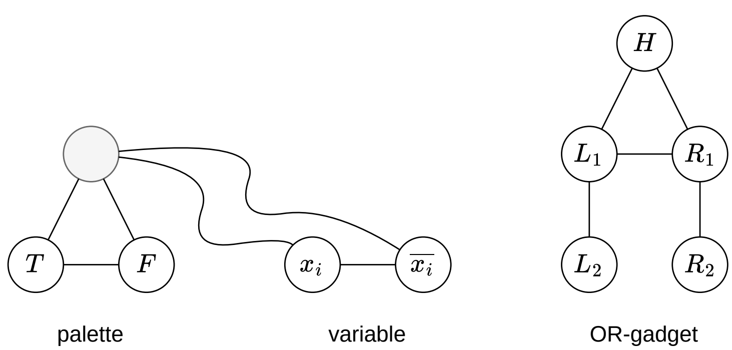
Proof. To show that $3COLOR$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the coloring of nodes. To prove the second part, we show that $\neq SAT$ is polynomial time reducible to $3COLOR$. The reduction converts a 3cnf-formula ϕ into a graph G, so that ϕ has a satisfying $\neq$*-assignment*, iff G is colorable with 3 colors.

Let ϕ be any 3cnf-formula containing m clauses $C_1, C_2, \cdots C_m$:

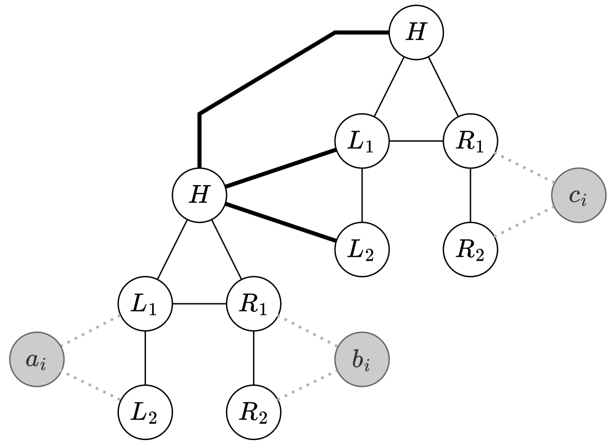
$$\phi = (a_1 \vee b_1 \vee c_1) \ \wedge (a_2 \vee b_2 \vee c_2) \ \wedge \cdots \wedge (a_m \vee b_m \vee c_m).$$

where each a, b and c is a literal x_i or $\overline{x_i}$, and $x_1, x_2 \cdots x_n$ are the n variables of ϕ. Now we show how to convert ϕ to graph G.

The graph contains gadgets that mimic the variables and clauses of the formula. The graph contains a palette gadget consisting of 3 mutually connected nodes. The two palette nodes are labeled T and F, and the third one is not labeled. The variable gadget for variable x is two adjacent nodes labeled x and $\overline{x}$. Both variable nodes are connected to the unlabeled palette node. The OR-gadget consists of 5 nodes labeled L_1, L_2, R_1, R_2 and H. Nodes L_1, R_1 and H are mutually connected to each other. Additionally, node L_1 is connected to L_2, and R_1 is connected to R_2.



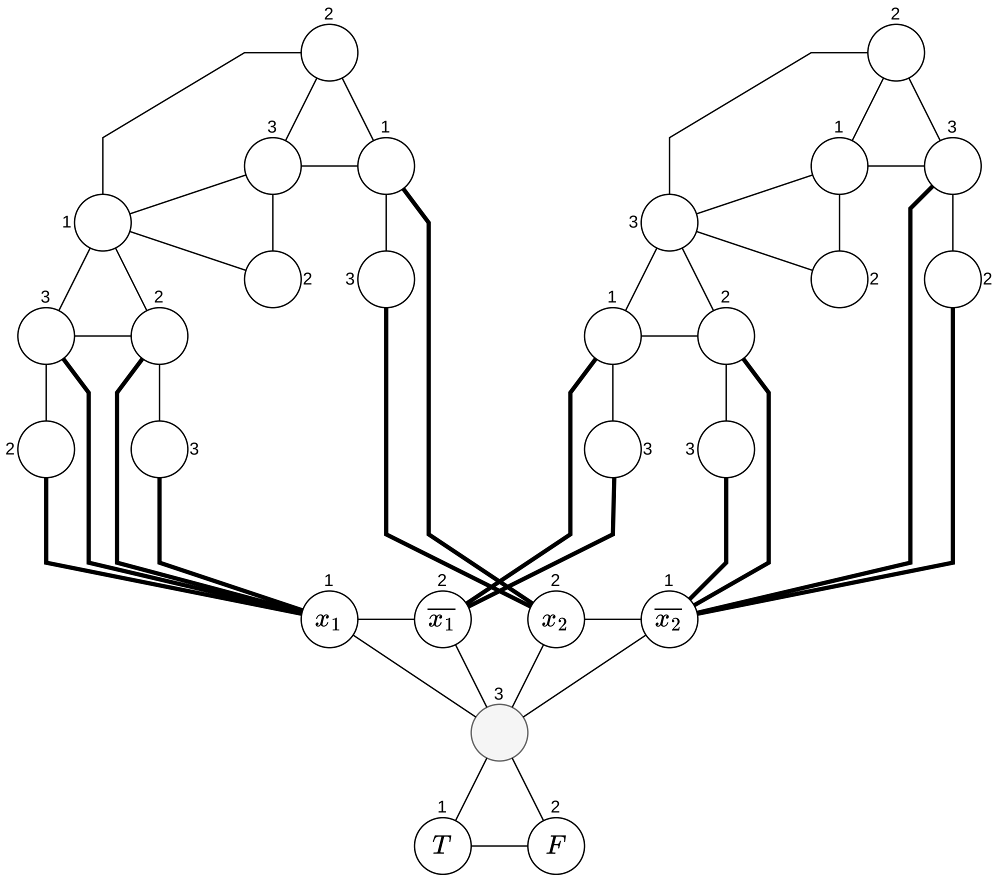
The clause gadget for clause C_i consists of two OR-gadgets. Nodes H, L_1 and L_2 of the second OR-gadget are connected to node H of the first OR-gadget. Nodes L_1 and L_2 of the first OR-gadget are connected to the node for literal a_i. Nodes R_1 and R_2 of the first OR-gadget are connected to the node for literal b_i. Nodes R_1 and R_2 of the second OR-gadget are connected to the node for literal c_i.



Clause gadget

Suppose that ϕ has satisfying $\neq$-*assignment*. Such an assignment to the variables of ϕ is one where each clause contains two literals with unequal truth values. Color the graph G for ϕ as follows:

1. Assign color 1 to T, 2 to F and 3 to unlabeled node.
2. If variable x is true, then assign color 1 to node x and color 2 to node $\overline{x}$. Otherwise, assign color 1 to $\overline{x}$ color 2 to node x.
3. Nodes L_1, L_2, R_1 and R_2 are easily colorable in 3 colors in each OR-gadget.
4. If both L and R nodes in an OR-gadget are connected to nodes of same color k, then the H node also gets the color k.
5. Two colors are available for the two H nodes in every clause gadget because the 3 pairs of L and R nodes in a clause gadget are not connected to literal nodes with same coloring.



Graph G for $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2})$.

Suppose the graph G is colorable with 3 colors. Then, the assignment of colors to nodes x_i and $\overline{x_i}$ gives the $\neq$-*assignment* to the variables of ϕ. If node x_i has same color as node T, then the variable x_i is assigned true value, other false.

$\square$