
Problem 7.28. You are given a box and a collection of cards as indicated in the following figure. Because of the pegs in the box and the notches in the cards, each card will fit in the box in either of two ways. Each card contains two columns of holes, some of which may not be punched out. The puzzle is solved by placing all the cards in the box so as to completely cover the bottom of the box (i.e., every hole position is blocked by at least one card that has no hole there). Let $PUZZLE = \{\langle c_1, \cdots, c_k\rangle \mid \text{each } c_i \text{ represents a card and this collection of cards has a solution}\}$. Show that $PUZZLE$ is NP-complete.

Proof. To show that $PUZZLE$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the placement of cards. To prove the second part, we show that $\neq SAT$ is polynomial time reducible to $PUZZLE$. The reduction converts a 3cnf-formula ϕ into a collection of k cards, so that ϕ has a satisfying $\neq$-*assignment*, iff collection of cards $c_1, \cdots, c_k$ has a solution.

Let ϕ be any 3cnf-formula containing m clauses:

$$\phi = (a_1 \vee b_1 \vee c_1) \ \wedge (a_2 \vee b_2 \vee c_2) \ \wedge \cdots \wedge (a_m \vee b_m \vee c_m).$$

where each a, b and c is a literal x_i or $\overline{x_i}$, and $x_1, x_2 \cdots x_n$ are the n variables of ϕ. Now we show how to convert ϕ to collection of cards. The collection contains cards for variables, and each card has a row R_j of holes for every clause C_j. The top row R_1 is for the first clause C_1, and the bottom row R_m is for the last clause C_m.

1. Repeat for each variable $x_i = x_1,\ x_2, \cdots\ x_n$ in ϕ.
2. Add a card c_i.
3. Repeat for each clause $C_j = C_1, C_2, \cdots, C_m$:
4. Add row R_j of holes in card c_i.
5. If both literals x_i and $\overline{x_i}$ do not appear in C_j, then punch out both holes in R_j.
6. If only literal x_i appears in C_j, then punch out right hole in R_j.
7. If only literal $\overline{x_i}$ appears in C_j, then punch out left hole in R_j.
8. If both literals x_i and $\overline{x_i}$ appear in C_j, then leave both holes unpunched[1].

Suppose ϕ has a satisfying $\neq$-*assignment*. Such an assignment to the variables of ϕ is one where each clause contains two literals with unequal truth values. The collection of cards contains a card for every variable in ϕ. A solution can be constructed by flipping the cards for variables that are assigned false value. Let C_j be any clause $(a_j \vee b_j \vee c_j)$ in ϕ. Every clause C_j contains two literals with unequal truth values. These two literals can be:

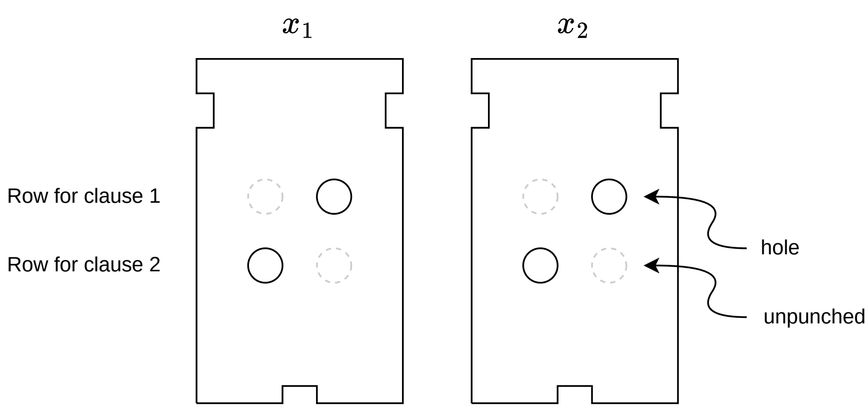
[1]A clause C_j that contains only one variable x_i, such as $(x_i, x_i, \overline{x_i})$ or $(x_i, \overline{x_i}, \overline{x_i})$, has only one card without any punched out holes in row R_j, and all the other cards have holes punched out in row R_j.

1. Same variable x and $\overline{x}$.
2. Different variables, where both literals are non-negated x and y.
3. Different variables, where one literal is negated $\overline{x}$ and y.
4. Different variables, where both literals are negated $\overline{x}$ and $\overline{y}$.

In every case, a row of holes for a clause has two opposite holes that are not punched out across all cards.

Suppose the collection of cards $c_1, \cdots, c_k$ has a solution. Then, the way each card is placed in the box gives the $\neq$-*assignment* for ϕ. If a card c_i is flipped before it is placed in the box, then the variable x_i is assigned false, otherwise true.

□



Collection of cards for $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2})$.