
Problem 7.28. You are given a box and a collection of cards as indicated in the following figure. Because of the pegs in the box and the notches in the cards, each card will fit in the box in either of two ways. Each card contains two columns of holes, some of which may not be punched out. The puzzle is solved by placing all the cards in the box so as to completely cover the bottom of the box (i.e., every hole position is blocked by at least one card that has no hole there). Let $PUZZLE = \{\langle c_1, \cdots, c_k \rangle \mid \text{each } c_i \text{ represents a card and this collection of cards has a solution}\}$. Show that $PUZZLE$ is NP-complete.

Proof. To show that $PUZZLE$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the placement of cards. To prove the second part, we show that $\neq SAT$ is polynomial time reducible to $PUZZLE$. The reduction converts a 3cnf-formula ϕ into a collection of k cards, so that ϕ has a satisfying $\neq$-*assignment*, iff collection of cards $c_1, \cdots, c_k$ has a solution.

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The variable gadget for variable x is a collection of $3c$ nodes labeled with x and another $3c$ node labeled with $\overline{x}$, where c is the number of clauses. All nodes labeled x are connected with all nodes labeled $\overline{x}$. The clause gadget is a triangle of three edges connecting three nodes labeled with the literals appearing in the clause. Do not use the same node in more than one clause gadget. Finally, we choose k to be the maximum possible cut size of $l \times (3c)^2 + 2c$ in G, where l is the number of variables in ϕ.

Suppose that ϕ has satisfying $\neq$-*assignment*. Such an assignment to the variables of ϕ is one where each clause contains two literals with unequal truth values. In graph G, the vertices V can be paritioned into two disjoint subsets S and T as follows:

1. If some variable x is assigned true value, then put x nodes in S and $\overline{x}$ nodes in T.
2. If some variable x is assigned false value, then put x nodes in T and $\overline{x}$ nodes in S.

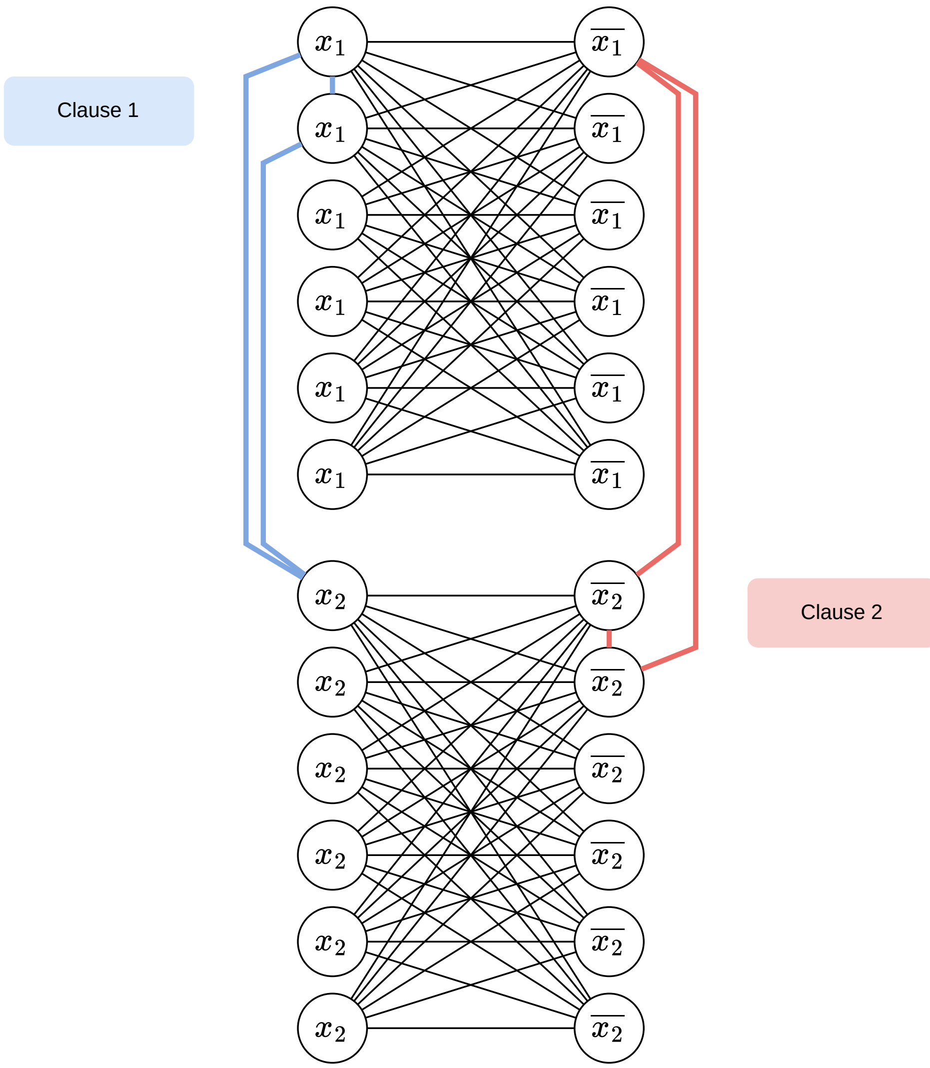
Clearly, in this parition all the $l \times (3c)^2$ edges in variable gadgets have one endpoint in S and other in T. Next, we show that the clause gadgets have $2c$ edges that have one endpoint in S and other in T. Let c_i be any clause $(y_1 \vee y_2 \vee y_3)$ in ϕ. Every clause c_i contains two literals with unequal truth values. These two literals can be:

1. Same variable x and $\overline{x}$.
2. Different variables, where both literals are non-negated x and y.
3. Different variables, where one literal is negated $\overline{x}$ and y.
4. Different variables, where both literals are negated $\overline{x}$ and $\overline{y}$.

In each case, a clause gadget has two edges that have one endpoint in S and other in T. Therefore, the graph G has a cut size of $l \times (3c)^2 + 2c$ or more.

Suppose the graph G has a cut size of $l \times (3c)^2 + 2c$ or more. Then, the separation of the vertices V into two disjoint subsets S and T gives the $\neq$-*assignment* for ϕ. If a node x_i is in S, then the variable x_i is assigned true, otherwise false.

$\square$



Graph G for $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2})$. Maximum cut size is obtained when S contains all nodes labeled x_1 and $\overline{x_2}$, and T contains nodes $\overline{x_1}$ and x_2.