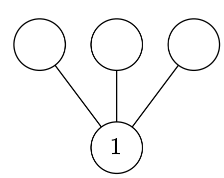
Problem 7.32. This problem is inspired by the single-player game *Minesweeper*, generalized to an arbitrary graph. Let G be an undirected graph, where each node either contains a single, hidden *mine* or is empty. The player chooses nodes, one by one. If the player chooses a node containing a mine, the player loses. If the player chooses an empty node, the player learns the number of neighboring nodes containing mines. (A neighboring node is one connected to the chosen node by an edge.) The player wins if and when all empty nodes have been so chosen.

In the *mine consistency problem*, you are given a graph G along with numbers labeling some of G's nodes. You must determine whether a placement of mines on the remaining nodes is possible, so that any node v that is labeled m has exactly m neighboring nodes containing mines. Formulate this problem as a language and show that it is NP-complete.

Proof. Let $MINE - CONSISTENCY = \{\langle G \rangle \mid G$ is an undirected graph along with numbers labeling some of G's nodes, and a placement of mines on the remaining nodes is possible, so that any node v that is labeled m has exactly m neighboring nodes containing mines.$\}$.

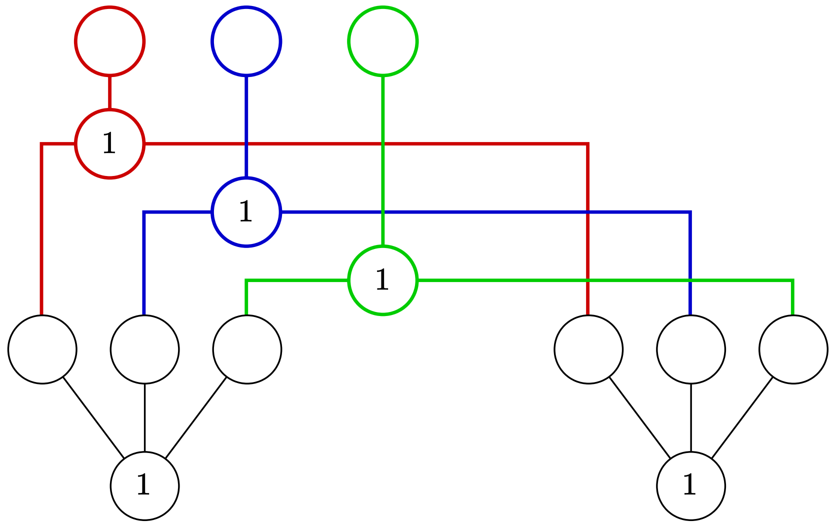
To show that $MINE - CONSISTENCY$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the placement of mines. To prove the second part, we show that $3COLOR$ is polynomial time reducible to $MINE - CONSISTENCY$. The reduction converts an undirected graph G_1 into undirected graph G_2 along with numbers labeling some of G_2's nodes, so that G_1 is colorable with 3 colors, iff a placement of mines on the remaining nodes of G_2 is possible, so that any node v that is labeled m has exactly m neighboring nodes containing mines.

Let G_1 be any undirected graph. Now we show how to convert G_1 to G_2 along with numbers labeling some of G_2's nodes. The graph G_2 contains gadgets for the nodes and edges in the graph G_1. The node gadget contains four nodes, where one of the four nodes has label 1 and it is connected to the other three nodes with an edge. The three unlabeled nodes represents the three colors of the palette.



Node gadget

The edge gadget consists of a structure that connects the unlabeled nodes in two node gadgets, so that it is not possible to place mines in both of the two corresponding unlabeled nodes.



Edge gadget

Suppose that the graph G_1 is colorable with 3 colors. Then, the assignment of color to each node v in G_1 gives the placement of mine in v's node gadget in G_2.

Suppose a placement of mines on the remaining nodes of G_2 is possible, so that any node v that is labeled m has exactly m neighboring nodes containing mines. Then, the placement of mines in G_2 gives the 3 coloring of the nodes in graph G_1. $\square$