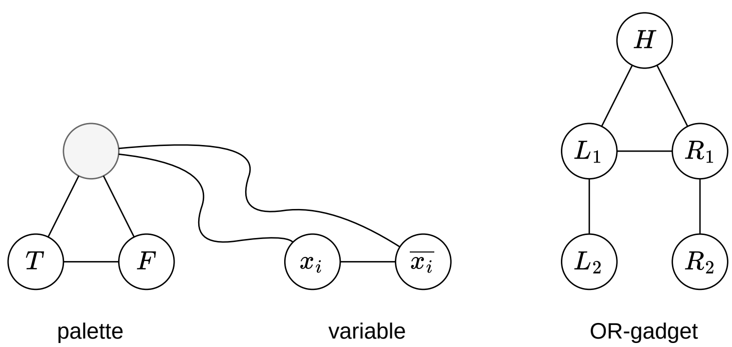
Problem 7.28. A ***coloring*** of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

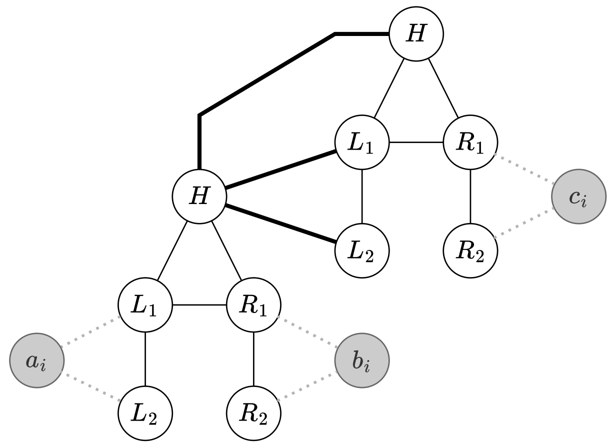
$$3COLOR = \{\langle G \rangle \mid G \text{ is colorable with 3 colors}\}.$$

Show that $3COLOR$ is NP-complete.

Proof. To show that $3COLOR$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the coloring of nodes. To prove the second part, we show that $\neq SAT$ is polynomial time reducible to $3COLOR$. The reduction converts a 3cnf-formula ϕ into a graph G, so that ϕ has a satisfying *$\neq$-assignment*, iff G is colorable with 3 colors. The graph contains gadgets that mimic the variables and clauses of the formula.

The graph contains a palette gadget consisting of 3 mutually connected nodes. The two palette nodes are labeled T and F, and the third one is not labeled. The variable gadget for variable x is two adjacent nodes labeled x and $\overline{x}$. Both variable nodes are connected to the unlabeled palette node. The OR-gadget consists of 5 nodes labeled L_1, L_2, R_1, R_2 and H. Nodes L_1, R_1 and H are mutually connected to each other. Additionally, node L_1 is connected to L_2, and R_1 is connected to R_2.





Clause gadget

Let ϕ be any 3cnf-formula containing m clauses:

$$\phi = (a_1 \vee b_1 \vee c_1) \ \wedge (a_2 \vee b_2 \vee c_2) \ \wedge \cdots \wedge (a_m \vee b_m \vee c_m).$$

where each a, b and c is a literal x_i or $\overline{x_i}$, and $x_1, x_2 \cdots x_n$ are the n variables of ϕ. Now we show how to convert ϕ to graph G.
In a satisfying $\neq$-*assignment*, the two disjunctions in every clause have different truth values.

$\square$