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**Problem 7.33.** In the following solitaire game, you are given an $m \times m$ board. On each of its $m^2$ positions lies either a blue stone, a red stone, or nothing at all. You play by removing stones from the board until each column contains only stones of a single color and each row contains at least one stone. You win if you achieve this objective. Winning may or may not be possible, depending upon the initial configuration. Let $SOLITAIRE = \{\langle G \rangle \mid G \text{ is a winnable game configuration}\}$. Prove that $SOLITAIRE$ is NP-complete.

*Proof.* To show that $SOLITAIRE$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the set of positions that needs to be cleared. To prove the second part, we show that $3SAT$ is polynomial time reducible to $SOLITAIRE$. The reduction converts a Boolean formula $\phi$ in 3CNF into an $m \times m$ board $G$ having either a blue stone, a red stone, or nothing at all placed on each of its $m^2$ positions, so that $G$ is a winnable game configuration, iff $\phi$ is satisfiable.

Let $\phi$ be any Boolean formula in 3CNF containing $p$ clauses:

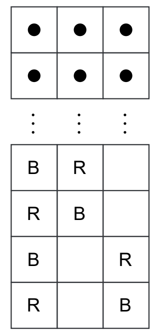
$$\phi = (a_1 \vee b_1 \vee c_1) \ \wedge (a_2 \vee b_2 \vee c_2) \ \wedge \cdots \wedge (a_p \vee b_p \vee c_p).$$

where each $a$, $b$ and $c$ is a literal $x_i$ or $\overline{x_i}$, and $x_1, x_2 \cdots x_n$ are the $n$ variables of $\phi$. Now we show how to convert $\phi$ to $G$.

The board $G$ contains a row for each clause in $\phi$ and 3 consecutive columns $3i-2$, $3i-1$ and $3i$ for each variable $x_i$ in $\phi$. For each clause $C_i$ in $\phi$, place stones in $G$ for each of the three literals in $C_i$ as follows:

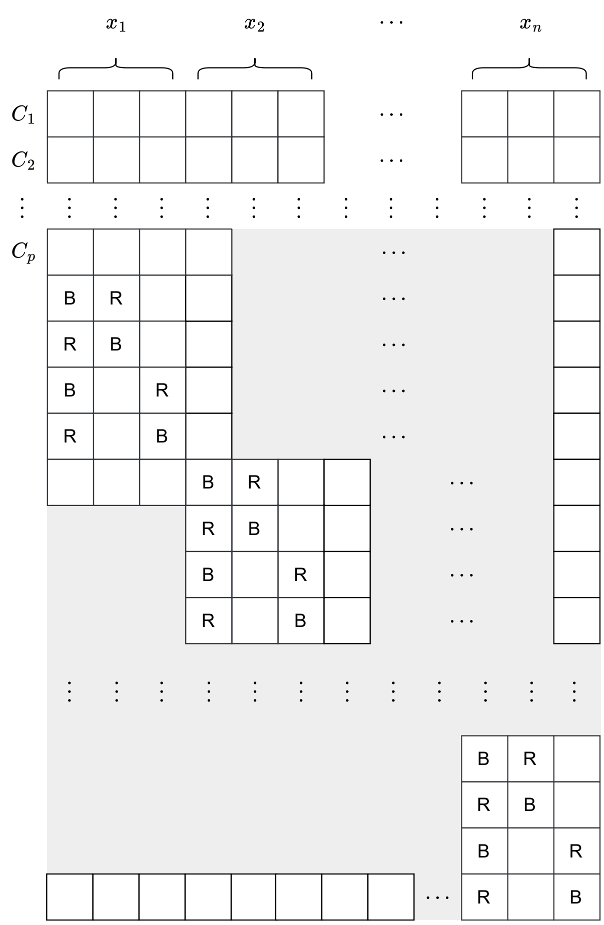
1. If $a_i$ is literal $x_j$, then place a red stone at row $i$ and column $3j-2$. Otherwise, place a blue stone.
2. If $b_i$ is literal $x_j$, then place a red stone at row $i$ and column $3j-1$. Otherwise, place a blue stone.
3. If $c_i$ is literal $x_j$, then place a red stone at row $i$ and column $3j$. Otherwise, place a blue stone.

Next, we make sure that all three columns of a variable can only contain stones of a single color. This constraint is imposed by using the following configuration.




The placement of stones in the bottom four rows ensure that all three columns can only contain stones of a single color.

The above configuration is repeatedly used in columns of every variable as shown in the following figure.




Cells in the shaded area of the board are empty.

Suppose $\phi$ has a satisfiable assignment. Then, at least one literal in every clause must be true. If variable $x_i$ is assigned true (false) value, then remove the blue (red) stones from the corresponding columns. So stones corresponding to true literals remain. Because every clause has a true literal, every row has a stone.

Suppose $G$ has a winnable game configuration. Take a game solution. If the red (blue) stones were removed from columns, set the corresponding variable false (true). Every row has a stone remaining, so every clause has a true literal. Therefore, $\phi$ is satisfied. $\square$