Problem 7.27. A cut in an undirected graph is a separation of the vertices V into two disjoint subsets S and T. The size of a cut is the number of edges that have one endpoint in S and the other in T. Let

$$MAX\text{-}CUT = \{\langle G, k\rangle \mid G \text{ has a cut of size } k \text{ or more}\}.$$

Show that $MAX\text{-}CUT$ is NP-complete. You may assume the result of Problem 7.26.

Proof. To show that $MAX\text{-}CUT$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply two disjoint subsets of V having cut size of k or more. To prove the second part, we show that $\neq SAT$ is polynomial time reducible to $MAX\text{-}CUT$.

The reduction converts a 3cnf-formula ϕ into an undirected graph G and a number k, so that ϕ has a satisfying $\neq$*-assignment*, iff G has a cut of size k or more. The graph contains gadgets that mimic the variables and clauses of the formula.

The variable gadget for variable x is a collection of $3c$ nodes labeled with x and another $3c$ node labeled with $\overline{x}$, where c is the number of clauses. All nodes labeled x are connected with all nodes labeled $\overline{x}$. The clause gadget is a triangle of three edges connecting three nodes labeled with the literals appearing in the clause. Do not use the same node in more than one clause gadget. Finally, we choose k to be the maximum possible cut size of $l \times (3c)^2 + 2c$ in G, where l is the number of variables in ϕ.

Suppose that ϕ has satisfying $\neq$*-assignment.* Such an assignment to the variables of ϕ is one where each clause contains two literals with unequal truth values. In graph G, the vertices V can be paritioned into two disjoint subsets S and T as follows:

1. If some variable x is assigned true value, then put x nodes in S and $\overline{x}$ nodes in T.
2. If some variable x is assigned false value, then put x nodes in T and $\overline{x}$ nodes in S.

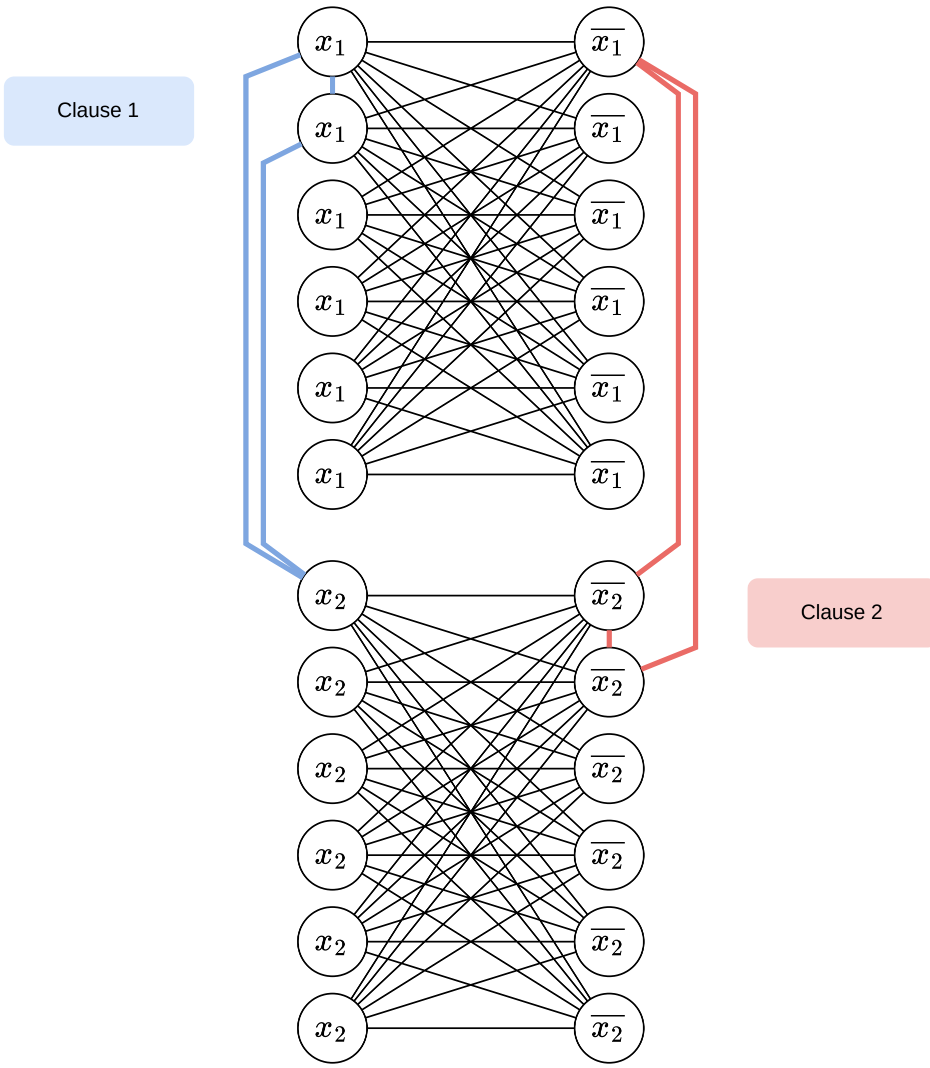
Clearly, in this parition all the $l \times (3c)^2$ edges in variable gadgets have one endpoint in S and other in T. Next, we show that the clause gadgets have $2c$ edges that have one endpoint in S and other in T. Let c_i be any clause $(y_1 \vee y_2 \vee y_3)$ in ϕ. Every clause c_i contains two literals with unequal truth values. These two literals can be:

1. Same variable x and $\overline{x}$.
2. Different variables, where both literals are non-negated x and y.
3. Different variables, where one literal is negated $\overline{x}$ and y.
4. Different variables, where both literals are negated $\overline{x}$ and $\overline{y}$.

In each case, a clause gadget has two edges that have one endpoint in S and other in T. Therefore, the graph G has a cut size of $l \times (3c)^2 + 2c$ or more.

Suppose the graph G has a cut size of $l \times (3c)^2 + 2c$ or more. Then, the separation of the vertices V into two disjoint subsets S and T gives the $\neq$-*assignment* for ϕ. If a node x_i is in S, then the variable x_i is assigned true, otherwise false.

$\square$



Graph G for $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2})$.
Maximum cut size is obtained when S contains all nodes labeled x_1 and $\overline{x_2}$, and T contains nodes $\overline{x_1}$ and x_2.