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**Problem 7.30.** Let $SET - SPLITTING = \{\langle S, C\rangle \mid S$ is a finite set and $C = \{C_1, \ldots, C_k\}$ is a collection of subsets of $S,$ for some $k > 0,$ such that elements of $S$ can be colored red or blue so that no $C_i$ has all its elements colored with the same color$\}$. Show that $SET - SPLITTING$ is NP-complete.

*Proof.* To show that $SET - SPLITTING$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the assignment of red or blue colors to elements of set $S$. To prove the second part, we show that $\neq SAT$ is polynomial time reducible to $SET - SPLITTING$. The reduction converts a 3cnf-formula $\phi$ into a finite set $S$ and a collection $C = \{C_1, \ldots, C_k\}$ of subsets of S, for some $k > 0$, so that $\phi$ has a satisfying $\neq$-*assignment*, iff elements of set $S$ can be colored red or blue so that no $C_i$ has all its elements colored with the same color.

Let $\phi$ be any 3cnf-formula containing $m$ clauses $C_1, C_2, \cdots C_m$:

$$\phi = (a_1 \vee b_1 \vee c_1) \ \wedge (a_2 \vee b_2 \vee c_2) \ \wedge \cdots \wedge (a_m \vee b_m \vee c_m).$$

where each $a$, $b$ and $c$ is a literal $x_i$ or $\overline{x_i}$, and $x_1, x_2 \cdots x_n$ are the $n$ variables of $\phi$. Now we show how to convert $\phi$ to set $S$ and collection $C$. The set $S$ contains two literals $x_i$ and $\overline{x_i}$ for each variable $x_i$ in $\phi$.

$$S = \bigcup_{1 \le i \le n} \{x_i, \ \overline{x_i}\}$$

The collection $C$ contains a set $\{x_i, \overline{x_i}\}$ for each variable. Moreover, for each clause $(a_i \vee b_i \vee c_i)$ in $\phi$, $C$ contains a set $\{a_i, b_i, c_i\}$.

$$C = \Big( \bigcup_{1 \le i \le n} \Big\{\{x_i, \ \overline{x_i}\}\Big\}\Big) \bigcup \Big( \bigcup_{1 \le i \le m} \Big\{\{a_i, b_i, c_i\}\Big\}\Big)$$

Suppose that $\phi$ has satisfying $\neq$-*assignment*. Such an assignment to the variables of $\phi$ is one where each clause contains two literals with unequal truth values. Assign color to elements of $S$ for as follows:

1. If variable $x_i$ is true, then element $x_i$ gets red and $\overline{x_i}$ blue.
2. If variable $x_i$ is false, then element $x_i$ gets blue and $\overline{x_i}$ red.

The colors correspond to truth values, red for true and blue for false. An $\neq$-*assignment* to the variables of $\phi$ is one where each clause contains two literals with unequal truth values. Therefore, no $C_i$ contains elements of the same color.

Suppose that elements of the set $S$ can be colored red or blue so that no $C_i$ has all its elements colored with the same color. Then, the assignment of red or blue color to the elements of the set $S$ gives an $\neq$-*assignment* to the variables of $\phi$. □