
Problem 7.33. In the following solitaire game, you are given an $m \times m$ board. On each of its m^2 positions lies either a blue stone, a red stone, or nothing at all. You play by removing stones from the board until each column contains only stones of a single color and each row contains at least one stone. You win if you achieve this objective. Winning may or may not be possible, depending upon the initial configuration. Let $SOLITAIRE = \{\langle G \rangle \mid G \text{ is a winnable game configuration}\}$. Prove that $SOLITAIRE$ is NP-complete.

Proof. To show that $SOLITAIRE$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the set of positions that needs to be cleared. To prove the second part, we show that $3SAT$ is polynomial time reducible to $SOLITAIRE$. The reduction converts a Boolean formula ϕ in 3CNF into an $m \times m$ board G having either a blue stone, a red stone, or nothing at all placed on each of its m^2 positions, so that G is a winnable game configuration, iff ϕ is satisfiable.

Let ϕ be any Boolean formula in 3CNF containing p clauses:

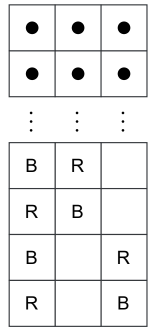
$$\phi = (a_1 \vee b_1 \vee c_1)\ \wedge (a_2 \vee b_2 \vee c_2)\ \wedge \cdots \wedge (a_p \vee b_p \vee c_p).$$

where each a, b and c is a literal x_i or $\overline{x_i}$, and $x_1, x_2 \cdots x_n$ are the n variables of ϕ. Now we show how to convert ϕ to G.

The board G contains a row for each clause in ϕ and 3 consecutive columns $3i-2$, $3i-1$ and $3i$ for each variable x_i in ϕ. For each clause C_i in ϕ, place stones in G for each of the three literals in C_i as follows:

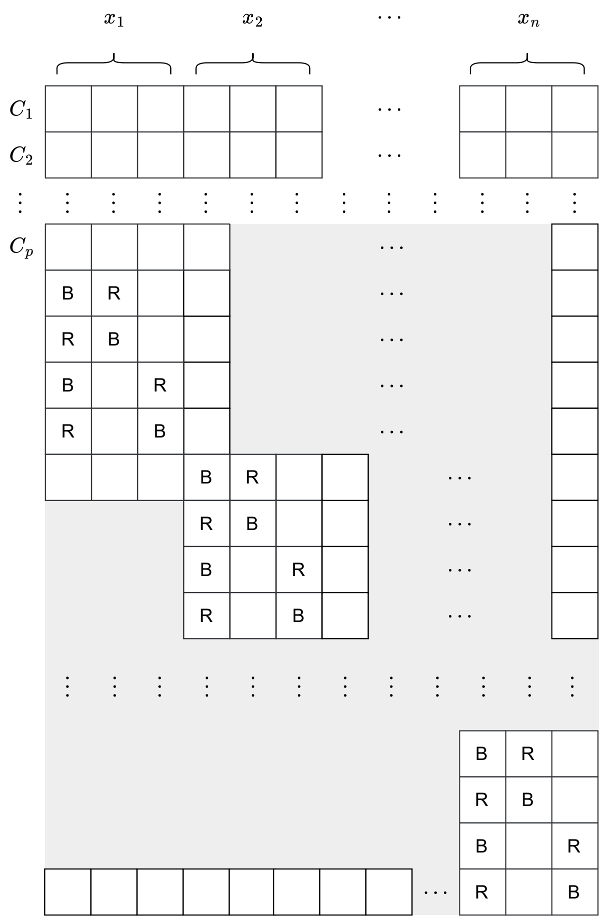
1. If a_i is literal x_j, then place a red stone at row i and column $3j-2$. Otherwise, place a blue stone.
2. If b_i is literal x_j, then place a red stone at row i and column $3j-1$. Otherwise, place a blue stone.
3. If c_i is literal x_j, then place a red stone at row i and column $3j$. Otherwise, place a blue stone.

Next, we make sure that all three columns of a variable can only contain stones of a single color. This constraint is imposed by using the following configuration.



The placement of stones in the bottom four rows ensure that all three columns can only contain stones of a single color.

The above configuration is repeatedly used in columns of every variable as shown in the following figure.



Cells in the shaded area of the board are empty.

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