
Problem 7.27. A cut in an undirected graph is a separation of the vertices V into two disjoint subsets S and T. The size of a cut is the number of edges that have one endpoint in S and the other in T. Let

$$MAX-CUT = \{\langle G, k\rangle \mid G \text{ has a cut of size } k \text{ or more}\}.$$

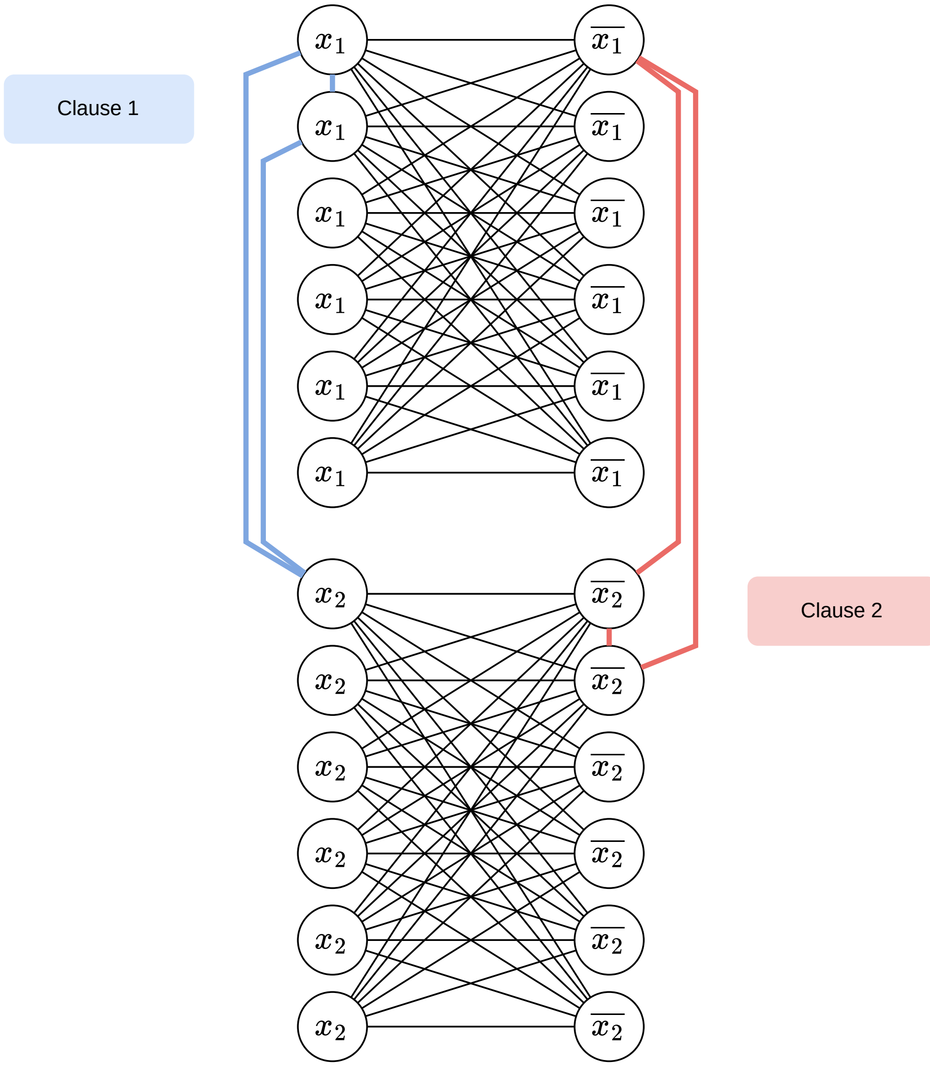
Show that $MAX-CUT$ is NP-complete. You may assume the result of Problem 7.26.

Proof. To show that $MAX-CUT$ is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply two disjoint subsets of V having cut size of k or more. To prove the second part, we show that $\neq SAT$ is polynomial time reducible to $MAX-CUT$.

The reduction converts a 3cnf-formula ϕ into an undirected graph G and a number k, so that ϕ has a satisfying $\neq$-*assignment*, iff G has a cut of size k. The graph contains gadgets that mimic the variables and clauses of the formula.

The variable gadget for variable x is a collection of $3c$ nodes labeled with x and another $3c$ node labeled with x, where c is the number of clauses. All nodes labeled x are connected with all nodes labeled x. The clause gadget is a triangle of three edges connecting three nodes labeled with the literals appearing in the clause. Do not use the same node in more than one clause gadget. Finally, we chose k to be the maximum possible cut size of $l \times (3c)^2 + 2c$ in G, where l is the number of variables in ϕ.
Phi cannot have a cluase with 3 same literals.

$\square$



Graph G for $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2})$.