
Problem 7.37. Let $U = \{\langle M, x, \#^t\rangle \mid \text{NTM } M \text{ accepts } x \text{ within } t \text{ steps on at least one branch}\}$. Note that M isn't required to halt on all branches. Show that U is NP-complete.

Proof. To show that U is NP-complete, we must show that it is in NP and that all NP-problems are polynomial time reducible to it. The first part is easy; a certificate is simply the accepting computation history for M on x. To prove the second part, we show that $3SAT$ is polynomial time reducible to U. The reduction converts a Boolean formula ϕ in 3CNF into an NTM M, a string x and an integer t, so that ϕ is satisfiable, iff M accepts x within t steps on at least one branch.

Let ϕ be any Boolean formula in 3CNF containing m clauses:

$$\phi = (a_1 \vee b_1 \vee c_1) \ \wedge (a_2 \vee b_2 \vee c_2) \ \wedge \cdots \wedge (a_m \vee b_m \vee c_m).$$

where each a, b and c is a literal x_i or $\overline{x_i}$, and $x_1, x_2, \ldots, x_n$ are the n variables of ϕ. Now we show how to convert ϕ to M, x and t. Let x to be the string $\#_1\#_2\cdots\#_n$, where each # symbol represents a variable in ϕ. Construct M as follows:

$M =$ "On input $\langle x\rangle$, where x is a string of the form $\#_1\#_2\cdots\#_n$:

1. Assign all possible combination of truth values to the n variables by nondeterministically replacing each # symbol with T ($true$) and F ($false$).
2. Repeat for each clause C_i.
3. Evaluate the clause $(a_i \vee b_i \vee c_i)$ according to the assigned truth values.
4. *Reject*, if the clause evaluates to $false$.
5. *Accept*."

The value of t should be set to the maximum number of steps required to evaluate all m clauses in a formal definition of NTM M.

Suppose ϕ has a satisfiable assignment. Then, M must have that assignment on one of its branches of computation. Therefore, M must accept x within t steps on at least one branch.

Suppose M accepts x within t steps on at least one branch. Then, the symbols T ($true$) and F ($false$) written on tape on an accepting branch give a satisfiable assignment to the variables of ϕ. □