Fits to Brody distribution, and calculations for density of states

In[960]:=

```
PBrody[x_, w_] := (1 + w) (Gamma[(2 + w)/(1 + w)])^(1+w) x^w Exp[- (Gamma[(2 + w)/(1 + w)] x)^(1+w)];
bins = Table[x, {x, 0, 5, 0.2}];
bFit = Table[0.5 * (bins[[i]] + bins[[i + 1]]), {i, 1, Length[bins] - 1}];
Ewindowsize = 30.0;
```

$$P_{Brody}(x, w) = (1 + w) \left(\Gamma\left[\frac{2 + w}{1 + w}\right]\right)^{1+w} x^w \exp\left[- \left(\Gamma\left[\frac{2 + w}{1 + w}\right] x\right)^{1+w}\right]$$

Run1

In[964]:=

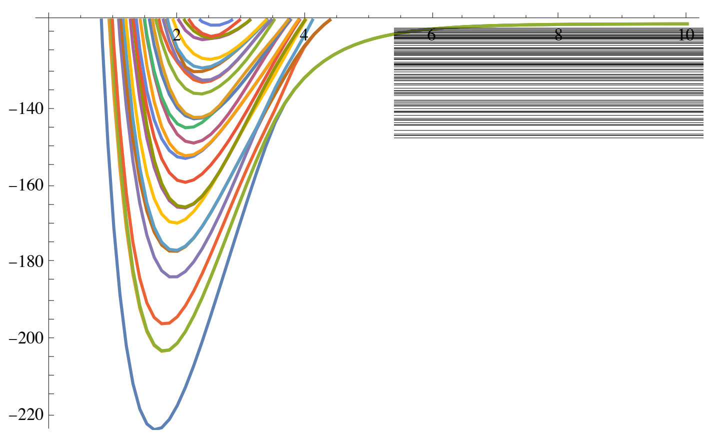
```
rundata1 = Import[
  "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run1/4BodySVD.par", "text"]
```

Out[964]=

```
xPoints (phi)        yPoints (theta)
60            100

LobattoPoints        NumChannels    Order
100             100        7

PotentialDepth        Rmin        Rmax        alpha (turns V on and off)
50.d0            0d0        10.d0        1.d0

m1        m2        m3        m4
1.d0        1.d0        1.0d0        1.0d0

xMin        xMax        yMin        yMax   (enter in units of pi)
0d0        0.25d0        0d0        0.5d0

Left        Right        Bot        Top
0        0        2        1



!  At top -- Phi(theta=pi/2,phi)=0 for odd parity so choose Top = 0
              Phi'(theta=pi/2,phi)=0 for even parity so choose Top = 1
```

```
In[965]:= SampleRange = Ecutoff1 - Ewindowsize < # < Ecutoff1 - 1. &;
curvestemp1 = Transpose[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run1/AdiabaticCurves.dat"]]
Curves1 = Table[Table[{curvestemp1[[1, i]], curvestemp1[[j, i]]},
    {i, 1, Length[curvestemp1[[1]]]}], {j, 2, Length[curvestemp1]}];
Evals1 = Flatten[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run1/Eigenvals.dat"]];
Ecutoff1 = Evals1[[4]]
Evals1 = Sort[Drop[Evals1, 1 ;; 4]];
eValsBound1  = Sort[Select[Evals1, SampleRange]]

Out[969]= -117.568

Out[971]= {-147.377, -146.741, -146.695, -146.36, -145.412, -144.061, -143.88, -143.376, -142.743,
 -142.602, -142.245, -141.472, -140.733, -140.529, -140.352, -139.715, -139.052, -138.87,
 -138.534, -138.169, -137.767, -137.487, -136.293, -135.917, -135.643, -135.538,
 -135.118, -134.951, -134.368, -133.556, -133.225, -133.194, -132.447, -132.201,
 -131.795, -131.662, -131.399, -130.951, -130.735, -130.115, -129.61, -129.386, -128.799,
 -128.435, -128.259, -128.127, -128.044, -127.892, -127.55, -126.891, -126.794, -126.518,
 -126.473, -125.808, -125.69, -125.444, -125.299, -125.059, -124.841, -124.513,
 -124.148, -123.941, -123.773, -123.169, -122.658, -122.487, -122.146, -121.74,
 -121.432, -121.354, -121.238, -121.163, -121.023, -120.757, -120.674, -120.461,
 -120.209, -119.883, -119.782, -119.474, -119.301, -118.969, -118.93, -118.646}

In[972]:= pcurves1 = ListPlot[Curves1, PlotMarkers → None,
   Joined → True, PlotRange → {Min[curvestemp1[[2]]], Ecutoff1 + 1}];
penergies1 = ListPlot[Table[{8, eValsBound1[[i]]}, {i, 1, Length[eValsBound1]}],
   PlotMarkers → Graphics[{Thickness[.001], Black, Line[{{15, 0}, {20, 0}}]}]];
Show[pcurves1, penergies1]
```

Out[974]=



In[975]:=
```
Es1 = Table[eValsBound1[[i + 1]] - eValsBound1[[i]], {i, 1, Length[eValsBound1] - 1}];
eTrim1 = Select[Es1, # > 0.0000001 &];
avg1 = Mean[eTrim1];
eTrim1 = eTrim1 / avg1;
(*Sort[eTrim1];*)
Length[Es1];
Length[eTrim1];
ρ1 = 1 / avg1;
EspaceBin1 = BinCounts[eTrim1, {bins}];
NormBrodyBinCs1 = Table[EspaceBin1[[i]] / Total[EspaceBin1] / (bins[[i + 1]] - bins[[i]]),
   {i, 1, Length[bins] - 1}];
brodyPdist1 = Table[{bFit[[i]], NormBrodyBinCs1[[i]]}, {i, 1, Length[bFit]}]
phist1 = ListPlot[brodyPdist1, InterpolationOrder → 0,
   Joined → True, PlotRange → All, PlotStyle → Black];
pars1 = FindFit[brodyPdist1, {PBrody[s, w], w > 0, w < 1}, {w}, s]
pfit1 = Plot[PBrody[s, w /. pars1], {s, 0, 5}, PlotRange → All, PlotStyle → Red];
```

Out[984]=
```
{{0.1, 0.240964}, {0.3, 0.662651}, {0.5, 0.843373}, {0.7, 0.722892},
 {0.9, 0.722892}, {1.1, 0.481928}, {1.3, 0.120482}, {1.5, 0.180723},
 {1.7, 0.240964}, {1.9, 0.361446}, {2.1, 0.120482}, {2.3, 0.120482}, {2.5, 0.},
 {2.7, 0.060241}, {2.9, 0.}, {3.1, 0.}, {3.3, 0.}, {3.5, 0.060241}, {3.7, 0.},
 {3.9, 0.060241}, {4.1, 0.}, {4.3, 0.}, {4.5, 0.}, {4.7, 0.}, {4.9, 0.}}
```

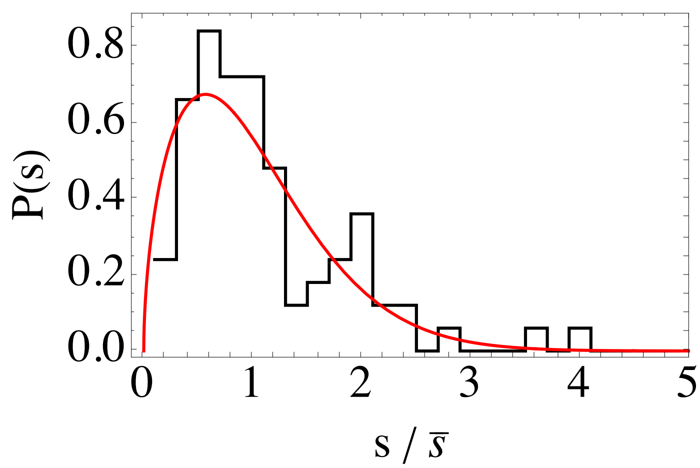
Out[986]=
```
{w → 0.544551}
```

In[988]:=
```
ρ1
```

Out[988]=
```
2.88884
```

In[989]:=
```
Show[phist1, pfit1, FrameLabel → {"s / s̅", "P(s)"}, Frame → True, LabelStyle → Large]
```

Out[989]=



Run2

In[990]:=
```
rundata2 = Import[
  "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run2/4BodySVD.par", "text"]
```

Out[990]=
```
xPoints (phi)        yPoints (theta)
60            100

LobattoPoints        NumChannels    Order
100             100        7

PotentialDepth        Rmin        Rmax        alpha (turns V on and off)
50.d0            0d0        10.d0        1.d0

m1        m2        m3        m4
1.d0        1.d0        1.0d0        1.0d0

xMin        xMax        yMin        yMax   (enter in units of pi)
0d0        0.25d0        0d0         0.5d0

Left        Right        Bot        Top
0        1        2        1



!  At top -- Phi(theta=pi/2,phi)=0 for odd parity so choose Top = 0
               Phi'(theta=pi/2,phi)=0 for even parity so choose Top = 1
```

In[991]:=
```
SampleRange = Ecutoff2 - Ewindowsize < # < Ecutoff2 - 1 &
curvestemp2 = Transpose[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run2/AdiabaticCurves.dat"]]
Curves2 = Table[Table[{curvestemp2[[1, i]], curvestemp2[[j, i]]},
    {i, 1, Length[curvestemp2[[1]]]}], {j, 2, Length[curvestemp2]}];
Evals2 = Flatten[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run2/Eigenvals.dat"]];
Ecutoff2 = Evals2[[4]]
Evals2 = Sort[Drop[Evals2, 1 ;; 4]];
eValsBound2  = Sort[Select[Evals2, SampleRange]]
```

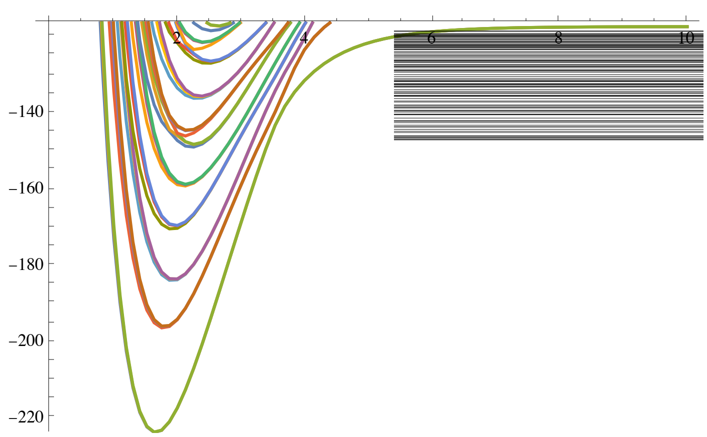
Out[991]= Ecutoff2 - Ewindowsize < #1 < Ecutoff2 - 1 &

Out[995]= -117.568

Out[997]= {-147.194, -147.049, -146.754, -146.507, -146.406, -146.117, -146.092, -145.273, -144.918, -144.43, -143.825, -143.729, -143.346, -142.673, -142.32, -142.117, -141.483, -140.631, -140.546, -140.437, -139.9, -139.613, -139.414, -138.869, -138.601, -138.582, -138.172, -138.063, -137.284, -137.014, -136.392, -135.819, -135.672, -135.518, -135.048, -134.816, -134.479, -134.038, -133.377, -133.222, -133.086, -132.752, -132.582, -132.526, -132.466, -131.954, -131.896, -131.688, -131.464, -131.092, -130.593, -130.09, -129.473, -129.162, -129., -128.855, -128.331, -128.014, -127.891, -127.843, -127.526, -127.471, -126.972, -126.758, -126.509, -126.462, -126.355, -126.03, -125.598, -125.264, -125.104, -124.74, -124.365, -124.055, -123.592, -123.424, -123.112, -123.058, -122.876, -122.721, -122.608, -122.405, -122.265, -122.06, -121.738, -121.552, -121.468, -121.277, -120.927, -120.908, -120.555, -120.527, -120.228, -120.048, -119.907, -119.85, -119.582, -119.562, -119.344, -118.811, -118.67}

In[998]:=
```
pcurves2 = ListPlot[Curves2, PlotMarkers → None,
   Joined → True, PlotRange → {Min[curvestemp2[[2]]], Ecutoff2 + 1}];
penergies2 = ListPlot[Table[{8, eValsBound2[[i]]}, {i, 1, Length[eValsBound2]}],
   PlotMarkers → Graphics[{Thickness[.001], Black, Line[{{15, 0}, {20, 0}}]}]];
Show[pcurves2, penergies2]
```

Out[1000]=



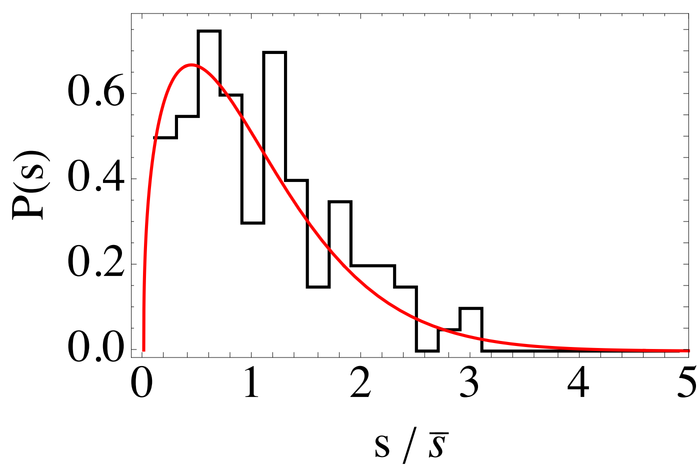
In[1001]:=
```
Es2 = Table[eValsBound2[[i + 1]] - eValsBound2[[i]], {i, 1, Length[eValsBound2] - 1}];
eTrim2 = Select[Es2, # > 0.0000001 &];
avg2 = Mean[eTrim2];
eTrim2 = eTrim2 / avg2;
(*Sort[eTrim2];*)
Length[Es2];
Length[eTrim2];
ρ2 = 1 / avg2;
EspaceBin2 = BinCounts[eTrim2, {bins}];
NormBrodyBinCs2 = Table[EspaceBin2[[i]] / Total[EspaceBin2] / (bins[[i + 1]] - bins[[i]]),
   {i, 1, Length[bins] - 1}];
brodyPdist2 = Table[{bFit[[i]], NormBrodyBinCs2[[i]]}, {i, 1, Length[bFit]}]
phist2 = ListPlot[brodyPdist2, InterpolationOrder → 0,
   Joined → True, PlotRange → All, PlotStyle → Black];
pars2 = FindFit[brodyPdist2, {PBrody[s, w], w > 0, w < 1}, {w}, s]
pfit2 = Plot[PBrody[s, w /. pars2], {s, 0, 5}, PlotRange → All, PlotStyle → Red];
```

Out[1010]=
```
{{0.1, 0.5}, {0.3, 0.55}, {0.5, 0.75}, {0.7, 0.6}, {0.9, 0.3}, {1.1, 0.7},
 {1.3, 0.4}, {1.5, 0.15}, {1.7, 0.35}, {1.9, 0.2}, {2.1, 0.2}, {2.3, 0.15},
 {2.5, 0.}, {2.7, 0.05}, {2.9, 0.1}, {3.1, 0.}, {3.3, 0.}, {3.5, 0.},
 {3.7, 0.}, {3.9, 0.}, {4.1, 0.}, {4.3, 0.}, {4.5, 0.}, {4.7, 0.}, {4.9, 0.}}
```

Out[1012]=
```
{w → 0.388477}
```

In[1014]:=
```
Show[phist2, pfit2, FrameLabel → {"s / s̄", "P(s)"}, Frame → True, LabelStyle → Large]
```

Out[1014]=



Run3

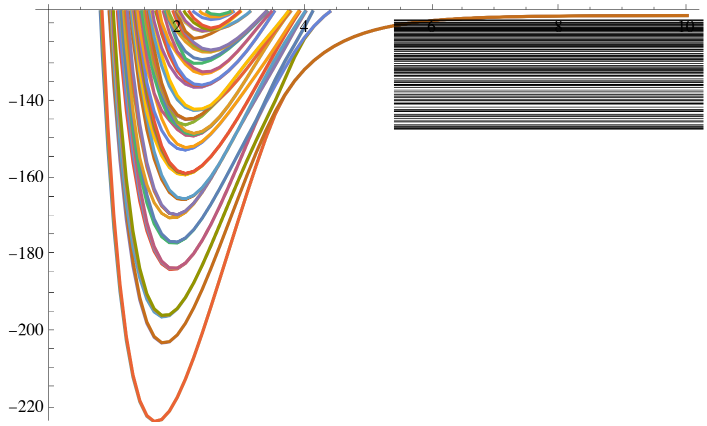
In[1015]:=
```
rundata3 = Import[
  "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run3/4BodySVD.par", "text"]
```

Out[1015]=
```
xPoints (phi)        yPoints (theta)
120            100

LobattoPoints        NumChannels    Order
100            160        7

PotentialDepth        Rmin        Rmax        alpha (turns V on and off)
50.d0            0d0        10.d0        1.d0

m1        m2        m3        m4
1.d0        1.d0        1.0d0        1.0d0

xMin        xMax        yMin        yMax   (enter in units of pi)
0d0        0.5d0        0d0        0.5d0

Left        Right        Bot        Top
0        0        2        1



!  At top -- Phi(theta=pi/2,phi)=0 for odd parity so choose Top = 0
               Phi'(theta=pi/2,phi)=0 for even parity so choose Top = 1
```

```
In[1016]:= SampleRange = Ecutoff3 - Ewindowsize < # < Ecutoff3 - 1 &
curvestemp3 = Transpose[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run3/AdiabaticCurves.dat"]]
Curves3 = Table[Table[{curvestemp3[[1, i]], curvestemp3[[j, i]]},
    {i, 1, Length[curvestemp3[[1]]]}], {j, 2, Length[curvestemp3]}];
Evals3 = Flatten[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50/run3/Eigenvals.dat"]];
Ecutoff3 = Evals3[[4]]
Evals3 = Sort[Drop[Evals3, 1 ;; 4]];
eValsBound3 = Sort[Select[Evals3, SampleRange]]

Out[1016]= Ecutoff3 - Ewindowsize < #1 < Ecutoff3 - 1 &

Out[1020]= -117.567

Out[1022]= {-147.377, -147.194, -147.049, -146.754, -146.741, -146.695, -146.507, -146.406,
 -146.36, -146.117, -146.092, -145.411, -145.273, -144.918, -144.43, -144.061,
 -143.88, -143.825, -143.729, -143.376, -143.346, -142.743, -142.673, -142.602,
 -142.32, -142.245, -142.117, -141.483, -141.472, -140.733, -140.631, -140.546,
 -140.529, -140.437, -140.352, -139.9, -139.715, -139.613, -139.414, -139.052,
 -138.87, -138.869, -138.601, -138.582, -138.534, -138.172, -138.169, -138.063,
 -137.767, -137.487, -137.284, -137.014, -136.392, -136.293, -135.917, -135.819,
 -135.672, -135.642, -135.538, -135.518, -135.118, -135.048, -134.951, -134.816,
 -134.479, -134.368, -134.038, -133.556, -133.377, -133.225, -133.222, -133.194,
 -133.086, -132.752, -132.582, -132.526, -132.466, -132.447, -132.201, -131.954,
 -131.896, -131.795, -131.688, -131.662, -131.464, -131.399, -131.092, -130.951,
 -130.735, -130.593, -130.115, -130.09, -129.61, -129.473, -129.386, -129.162, -129.,
 -128.855, -128.799, -128.435, -128.331, -128.259, -128.127, -128.044, -128.014,
 -127.892, -127.891, -127.843, -127.55, -127.526, -127.471, -126.972, -126.891,
 -126.794, -126.758, -126.518, -126.509, -126.473, -126.462, -126.355, -126.03,
 -125.808, -125.69, -125.598, -125.444, -125.299, -125.264, -125.104, -125.059,
 -124.841, -124.74, -124.513, -124.365, -124.148, -124.055, -123.941, -123.773,
 -123.592, -123.424, -123.169, -123.112, -123.058, -122.876, -122.721, -122.658,
 -122.608, -122.487, -122.405, -122.265, -122.146, -122.06, -121.74, -121.738,
 -121.552, -121.468, -121.432, -121.354, -121.277, -121.238, -121.163, -121.023,
 -120.927, -120.908, -120.757, -120.673, -120.554, -120.527, -120.461, -120.228,
 -120.209, -120.048, -119.907, -119.883, -119.85, -119.782, -119.582, -119.562,
 -119.473, -119.344, -119.301, -118.969, -118.93, -118.811, -118.67, -118.646}
```

In[1023]:=
```
pcurves3 =
  ListPlot[Curves3, Joined → True, PlotRange → {Min[curvestemp3[[2]]], Ecutoff3 + 1}];
penergies3 = ListPlot[Table[{8, eValsBound3[[i]]}, {i, 1, Length[eValsBound3]}],
   PlotMarkers → Graphics[{Thickness[.001], Black, Line[{{15, 0}, {20, 0}}]}]];
Show[pcurves3, penergies3]
```

Out[1025]=



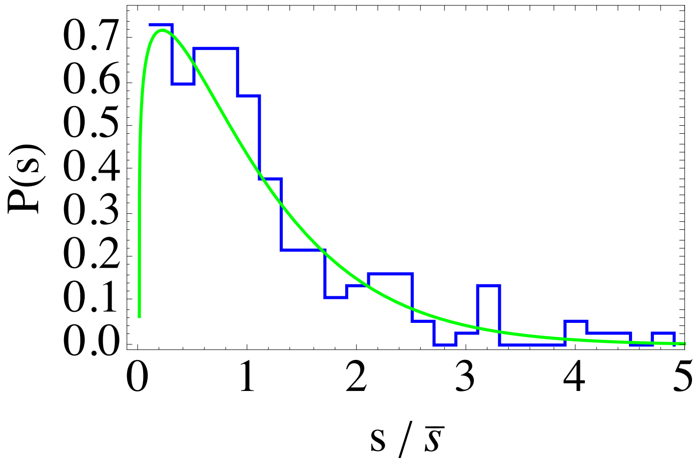
In[1026]:=
```
Es3 = Table[eValsBound3[[i + 1]] - eValsBound3[[i]], {i, 1, Length[eValsBound3] - 1}];
eTrim3  =  Select[Es3, # > 0.0000001 &];
avg3  =  Mean[eTrim3];
eTrim3  =  eTrim3 / avg3;
Sort[eTrim3];
Length[Es3];
Length[eTrim3];
ρ3 = 1 / avg3;
EspaceBin3 = BinCounts[eTrim3, {bins}];
NormBrodyBinCs3 = Table[EspaceBin3[[i]] / Total[EspaceBin3] / (bins[[i + 1]] - bins[[i]]),
   {i, 1, Length[bins] - 1}];
brodyPdist3  =  Table[{bFit[[i]], NormBrodyBinCs3[[i]]}, {i, 1, Length[bFit]}]
phist3 = ListPlot[brodyPdist3, InterpolationOrder → 0,
   Joined → True, PlotRange → All, PlotStyle → {Blue}];
pars3  =  FindFit[brodyPdist3, {PBrody[s, w], w > 0, w < 1}, {w}, s]
pfit3 = Plot[PBrody[s, w /. pars3], {s, 0, 5}, PlotRange → All, PlotStyle → {Green}];
```

Out[1036]=
```
{{0.1, 0.733696}, {0.3, 0.597826}, {0.5, 0.679348}, {0.7, 0.679348},
 {0.9, 0.570652}, {1.1, 0.380435}, {1.3, 0.217391}, {1.5, 0.217391}, {1.7, 0.108696},
 {1.9, 0.13587}, {2.1, 0.163043}, {2.3, 0.163043}, {2.5, 0.0543478}, {2.7, 0.},
 {2.9, 0.0271739}, {3.1, 0.13587}, {3.3, 0.}, {3.5, 0.}, {3.7, 0.}, {3.9, 0.0543478},
 {4.1, 0.0271739}, {4.3, 0.0271739}, {4.5, 0.}, {4.7, 0.0271739}, {4.9, 0.}}
```

Out[1038]=
```
{w → 0.17542}
```

In[1040]:=
```
pall3 = Show[phist3, pfit3, FrameLabel → {"s / s̄", "P(s)"}, Frame → True, LabelStyle → Large]
```

Out[1040]=



Analysis so far

Runs 1 and 2 have taken advantage of a possible symmetry by further reducing the coordinate space. If my guess is right, then the density of states for run3 should be roughly twice that of run 1 and 2 since I expect both "even and odd" states about phi = $\pi/4$ to be present in run 3, but only even states in run1 and odd states in run2

In[1041]:=
```
ρ1
ρ2
ρ3
```

Out[1041]= 2.88884

Out[1042]= 3.50581

Out[1043]= 6.40417

In[1044]:=
```
ρ1 + ρ2
```

Out[1044]= 6.39466

Further, I expect that if these states (even and odd) do not couple since the Hamiltonian is symmetric about $\phi=\pi/4$, then overlaying these two distributions will result in a distribution similar to that of run3

In[1045]:=
```
eValsBoundCombined = Sort[ Flatten[{eValsBound1, eValsBound2}]]
EsCombined = Table[eValsBoundCombined[[i + 1]] - eValsBoundCombined[[i]],
   {i, 1, Length[eValsBoundCombined] - 1}];
eTrimCombined = Select[EsCombined, # > 0.0000001 &];
avgCombined = Mean[eTrimCombined];
eTrimCombined = eTrimCombined / avgCombined;
(*Sort[eTrimCombined];*)
Length[eTrimCombined];
ρCombined = 1 / avgCombined;
EspaceBinCombined = BinCounts[eTrimCombined, {bins}];
NormBrodyBinCsCombined =
  Table[EspaceBinCombined[[i]] / Total[EspaceBinCombined] / (bins[[i + 1]] - bins[[i]]),
   {i, 1, Length[bins] - 1}];
brodyPdistCombined = Table[{bFit[[i]], NormBrodyBinCsCombined[[i]]}, {i, 1, Length[bFit]}]
phistCombined = ListPlot[brodyPdistCombined, InterpolationOrder → 0,
   Joined → True, PlotRange → All, PlotStyle → {Black, Dashed}];
parsCombined = FindFit[brodyPdistCombined, {PBrody[s, w], w > 0, w < 1}, {w}, s]
pfitCombined =
  Plot[PBrody[s, w /. parsCombined], {s, 0, 5}, PlotRange → All, PlotStyle → {Red, Dashed}];
```

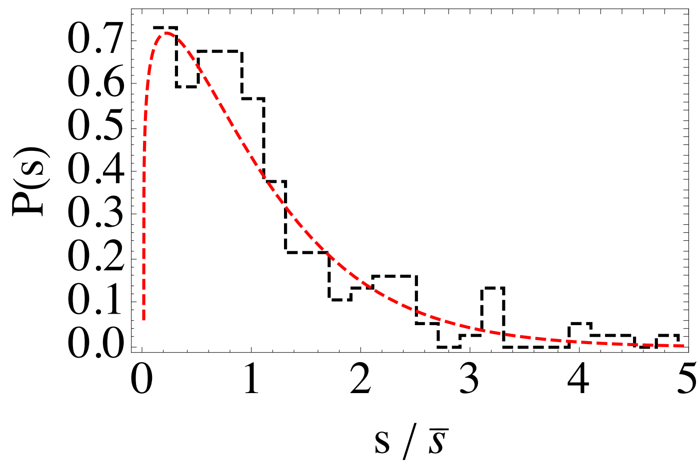
Out[1045]=
```
{-147.377, -147.194, -147.049, -146.754, -146.741, -146.695, -146.507, -146.406,
 -146.36, -146.117, -146.092, -145.412, -145.273, -144.918, -144.43, -144.061,
 -143.88, -143.825, -143.729, -143.376, -143.346, -142.743, -142.673, -142.602,
 -142.32, -142.245, -142.117, -141.483, -141.472, -140.733, -140.631, -140.546,
 -140.529, -140.437, -140.352, -139.9, -139.715, -139.613, -139.414, -139.052,
 -138.87, -138.869, -138.601, -138.582, -138.534, -138.172, -138.169, -138.063,
 -137.767, -137.487, -137.284, -137.014, -136.392, -136.293, -135.917, -135.819,
 -135.672, -135.643, -135.538, -135.518, -135.118, -135.048, -134.951, -134.816,
 -134.479, -134.368, -134.038, -133.556, -133.377, -133.225, -133.222, -133.194,
 -133.086, -132.752, -132.582, -132.526, -132.466, -132.447, -132.201, -131.954,
 -131.896, -131.795, -131.688, -131.662, -131.464, -131.399, -131.092, -130.951,
 -130.735, -130.593, -130.115, -130.09, -129.61, -129.473, -129.386, -129.162, -129.,
 -128.855, -128.799, -128.435, -128.331, -128.259, -128.127, -128.044, -128.014,
 -127.892, -127.891, -127.843, -127.55, -127.526, -127.471, -126.972, -126.891,
 -126.794, -126.758, -126.518, -126.509, -126.473, -126.462, -126.355, -126.03,
 -125.808, -125.69, -125.598, -125.444, -125.299, -125.264, -125.104, -125.059,
 -124.841, -124.74, -124.513, -124.365, -124.148, -124.055, -123.941, -123.773,
 -123.592, -123.424, -123.169, -123.112, -123.058, -122.876, -122.721, -122.658,
 -122.608, -122.487, -122.405, -122.265, -122.146, -122.06, -121.74, -121.738,
 -121.552, -121.468, -121.432, -121.354, -121.277, -121.238, -121.163, -121.023,
 -120.927, -120.908, -120.757, -120.674, -120.555, -120.527, -120.461, -120.228,
 -120.209, -120.048, -119.907, -119.883, -119.85, -119.782, -119.582, -119.562,
 -119.474, -119.344, -119.301, -118.969, -118.93, -118.811, -118.67, -118.646}
```

Out[1054]= {{0.1, 0.733696}, {0.3, 0.597826}, {0.5, 0.679348}, {0.7, 0.679348},
{0.9, 0.570652}, {1.1, 0.380435}, {1.3, 0.217391}, {1.5, 0.217391}, {1.7, 0.108696},
{1.9, 0.13587}, {2.1, 0.163043}, {2.3, 0.163043}, {2.5, 0.0543478}, {2.7, 0.},
{2.9, 0.0271739}, {3.1, 0.13587}, {3.3, 0.}, {3.5, 0.}, {3.7, 0.}, {3.9, 0.0543478},
{4.1, 0.0271739}, {4.3, 0.0271739}, {4.5, 0.}, {4.7, 0.0271739}, {4.9, 0.}}

Out[1056]= {w → 0.17542}

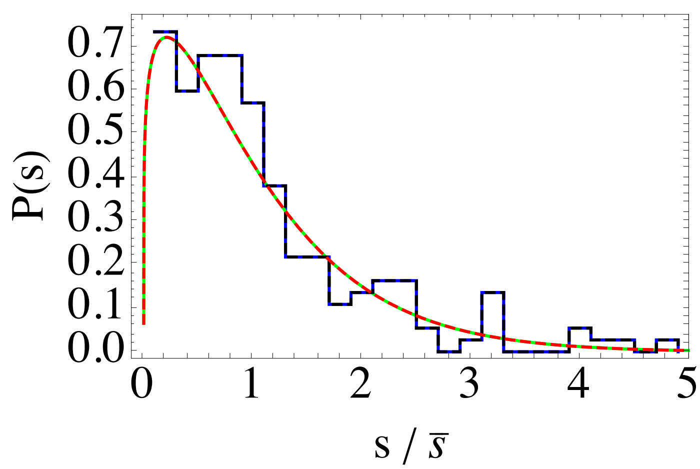
In[1058]:= **pcomb = Show[phistCombined, pfitCombined,
FrameLabel → {"s / s̄", "P(s)"}, Frame → True, LabelStyle → Large]**

Out[1058]=



In[1059]:= **Show[pall3, pcomb]**

Out[1059]=



Run4

In[1060]:=
```
rundata4 = Import[
  "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run4/4BodySVD.par", "text"]
```

Out[1060]=
```
xPoints (phi)        yPoints (theta)
120            100

LobattoPoints        NumChannels    Order
100            160        5

PotentialDepth        Rmin        Rmax        alpha (turns V on and off)
50.d0            0d0        10.d0        1.d0

m1        m2        m3        m4
1.d0        1.d0        1.5d0        1.5d0

xMin        xMax        yMin        yMax   (enter in units of pi)
0d0        0.5d0        0d0        0.5d0

Left        Right        Bot        Top
0        0        2        1



!  At top -- Phi(theta=pi/2,phi)=0 for odd parity so choose Top = 0
               Phi'(theta=pi/2,phi)=0 for even parity so choose Top = 1
```

In[1061]:=
```
SampleRange = Ecutoff4 - Ewindowsize < # < Ecutoff4 - 1 &
curvestemp4 = Transpose[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run4/AdiabaticCurves.dat"]]
Curves4 = Table[Table[{curvestemp4[[1, i]], curvestemp4[[j, i]]},
    {i, 1, Length[curvestemp4[[1]]]}], {j, 2, Length[curvestemp4]}];
Evals4 = Flatten[Import[
    "/Users/niravmehta/Documents/GitHub/4BodySVD/runs50-better/run4/Eigenvals.dat"]];
Ecutoff4 = Evals4[[4]]
Evals4 = Sort[Drop[Evals4, 1 ;; 4]];
eValsBound4 = Sort[Select[Evals4, SampleRange]]
```

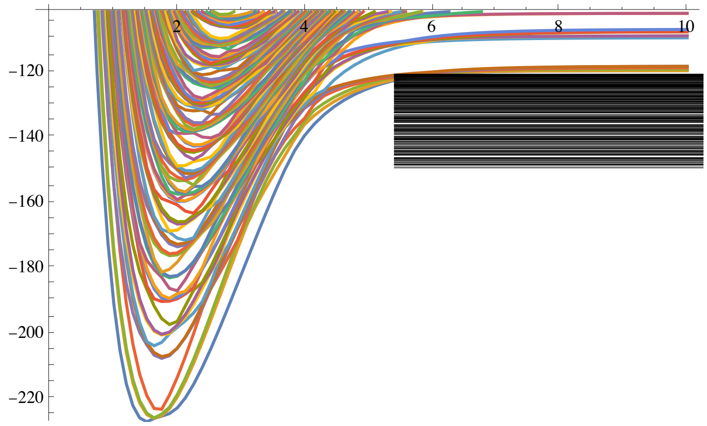
Out[1061]= Ecutoff4 - Ewindowsize < #1 < Ecutoff4 - 1 &

Out[1065]= -119.739

Out[1067]=
```
{-149.482, -149.163, -148.658, -148.509, -148.435, -148.392, -148.313, -148.179, -147.837,
 -147.669, -147.628, -147.309, -147.208, -146.937, -146.802, -146.773, -146.583,
 -146.397, -146.239, -145.623, -145.448, -145.355, -145.197, -144.946, -144.808,
 -144.565, -144.49, -144.347, -144.257, -144.198, -144.028, -143.674, -143.656, -143.477,
 -143.466, -143.194, -142.992, -142.89, -142.566, -142.262, -142.061, -142.008, -141.71,
 -141.573, -141.421, -141.227, -140.925, -140.768, -140.755, -140.607, -140.546,
 -140.445, -140.341, -140.112, -139.98, -139.896, -139.78, -139.323, -139.165,
 -138.943, -138.72, -138.468, -138.227, -138.195, -138.074, -138.025, -137.996,
 -137.81, -137.733, -137.627, -137.536, -137.503, -137.347, -137.104, -137.021,
 -136.821, -136.748, -136.661, -136.604, -136.427, -136.256, -136.204, -135.61,
 -135.553, -135.472, -135.314, -135.171, -135.131, -135.044, -134.967, -134.738,
 -134.575, -134.488, -134.414, -134.343, -134.236, -134.154, -134.044, -133.924,
 -133.746, -133.705, -133.48, -133.327, -133.034, -132.629, -132.589, -132.487,
 -132.449, -132.406, -132.319, -132.261, -131.981, -131.957, -131.698, -131.611,
 -131.517, -131.377, -131.251, -131.19, -131.083, -131.009, -130.984, -130.904,
 -130.696, -130.587, -130.526, -130.376, -130.332, -130.04, -129.944, -129.796, -129.7,
 -129.532, -129.383, -129.291, -129.231, -129.162, -129.039, -128.986, -128.902,
 -128.692, -128.587, -128.521, -128.412, -128.311, -128.249, -128.033, -127.839,
 -127.782, -127.746, -127.671, -127.64, -127.579, -127.539, -127.489, -127.389,
 -127.308, -127.092, -126.974, -126.756, -126.668, -126.537, -126.431, -126.386,
 -126.329, -126.159, -126.115, -126.056, -125.991, -125.891, -125.698, -125.445,
 -125.265, -125.114, -125.007, -124.973, -124.941, -124.78, -124.719, -124.692, -124.6,
 -124.556, -124.473, -124.423, -124.382, -124.365, -124.27, -124.237, -124.081,
 -124.006, -123.872, -123.824, -123.811, -123.716, -123.69, -123.472, -123.413,
 -123.381, -123.244, -123.199, -122.941, -122.931, -122.734, -122.661, -122.584,
 -122.509, -122.344, -122.319, -122.246, -122.197, -122.076, -121.998, -121.875,
 -121.827, -121.71, -121.659, -121.619, -121.51, -121.491, -121.417, -121.391,
 -121.28, -121.25, -121.16, -121.131, -121.071, -120.992, -120.92, -120.862, -120.774}
```

In[1068]:=
```
pcurves4 =
  ListPlot[Curves4, Joined → True, PlotRange → {Min[curvestemp4[[2]]], Ecutoff4 + 18}];
penergies4 = ListPlot[Table[{8, eValsBound4[[i]]}, {i, 1, Length[eValsBound4]}],
   PlotMarkers → Graphics[{Thickness[.001], Black, Line[{{15, 0}, {20, 0}}]}]];
Show[pcurves4, penergies4]
```

Out[1070]=



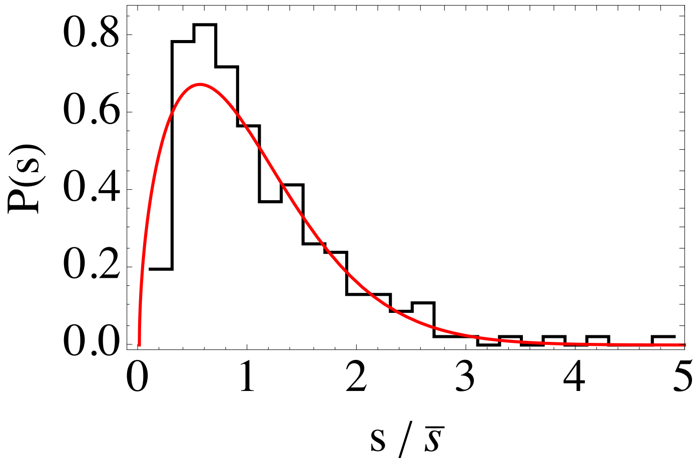
In[1071]:=
```
Es4 = Table[eValsBound4[[i + 1]] - eValsBound4[[i]], {i, 1, Length[eValsBound4] - 1}];
eTrim4 = Select[Es4, # > 0.0000001 &];
avg4 = Mean[eTrim4];
eTrim4 = eTrim4 / avg4;
Sort[eTrim4];
Length[Es4];
Length[eTrim4];
ρ4 = 1 / avg4;
EspaceBin4 = BinCounts[eTrim4, {bins}];
NormBrodyBinCs4 = Table[EspaceBin4[[i]] / Total[EspaceBin4] / (bins[[i + 1]] - bins[[i]]),
   {i, 1, Length[bins] - 1}];
brodyPdist4 = Table[{bFit[[i]], NormBrodyBinCs4[[i]]}, {i, 1, Length[bFit]}]
phist4 = ListPlot[brodyPdist4, InterpolationOrder → 0,
   Joined → True, PlotRange → All, PlotStyle → Black];
pars4 = FindFit[brodyPdist4, {PBrody[s, w], w > 0, w < 1}, {w}, s]
pfit4 = Plot[PBrody[s, w /. pars4], {s, 0, 5}, PlotRange → All, PlotStyle → Red];
```

Out[1081]=
```
{{0.1, 0.196507}, {0.3, 0.786026}, {0.5, 0.829694}, {0.7, 0.720524},
 {0.9, 0.567686}, {1.1, 0.371179}, {1.3, 0.414847}, {1.5, 0.262009}, {1.7, 0.240175},
 {1.9, 0.131004}, {2.1, 0.131004}, {2.3, 0.0873362}, {2.5, 0.10917}, {2.7, 0.0218341},
 {2.9, 0.0218341}, {3.1, 0.}, {3.3, 0.0218341}, {3.5, 0.}, {3.7, 0.0218341}, {3.9, 0.},
 {4.1, 0.0218341}, {4.3, 0.}, {4.5, 0.}, {4.7, 0.0218341}, {4.9, 0.0218341}}
```

Out[1083]=
```
{w → 0.530213}
```

In[1085]:=
```
pall4 = Show[phist4, pfit4, FrameLabel → {"s / s̄", "P(s)"}, Frame → True, LabelStyle → Large]
```

Out[1085]=



In[1086]:=
```
ρ4
```

Out[1086]=
```
7.9768
```

We need to more closely examine the identical particle symmetries and any other symmetries in the Hamiltonian in order to understand these spectra.

We defined the Jacobi coordinates in the H-tree above so that:

$$\vec{y}^{(12)} = S_{12}\,\vec{x} \tag{1}$$

where:

$$\frac{1}{\sqrt{\mu}}\begin{pmatrix} \sqrt{\mu_{12}} & -\sqrt{\mu_{12}} & 0 & 0 \\ 0 & 0 & \sqrt{\mu_{34}} & -\sqrt{\mu_{34}} \\ \frac{\sqrt{\mu_{12,34}}\ m_1}{m_1+m_2} & \frac{\sqrt{\mu_{12,34}}\ m_2}{m_1+m_2} & -\frac{\sqrt{\mu_{12,34}}\ m_3}{m_3+m_4} & -\frac{\sqrt{\mu_{12,34}}\ m_4}{m_3+m_4} \\ \frac{m_1}{\sqrt{M}} & \frac{m_2}{\sqrt{M}} & \frac{m_3}{\sqrt{M}} & \frac{m_4}{\sqrt{M}} \end{pmatrix} \tag{2}$$

To keep things as general as possible I'll assume for now that only particle 1 is of different mass and let $m_2 = m_3 = m_4 = m$, and $m_1 = \gamma\, m$ Then

$$\mu_{12} = \frac{\gamma\, m\, m}{\gamma\, m + m} = m\,\frac{\gamma}{1+\gamma} \longrightarrow \frac{m}{2} \tag{3}$$

$$\mu_{34} = \frac{m}{2} \tag{4}$$

$$\mu_{12,34} = \frac{(\gamma\, m + m)\,(2\, m)}{3\, m + \gamma\, m} = 2\, m\,\frac{(1+\gamma)}{(3+\gamma)} \longrightarrow m \tag{5}$$

$$\mu = \left(\frac{\gamma\, m\, m\, m\, m}{\gamma\, m + 3\, m}\right)^{1/3} = m\left(\frac{\gamma}{3+\gamma}\right)^{1/3} \longrightarrow \left(\frac{m}{2^{2/3}}\right) \tag{6}$$

So for the equal-mass case:

$$\frac{2^{1/3}}{\sqrt{m}}\begin{pmatrix} \sqrt{\frac{m}{2}} & -\sqrt{\frac{m}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{m}{2}} & -\sqrt{\frac{m}{2}} \\ \frac{\sqrt{m}}{2} & \frac{\sqrt{m}}{2} & -\frac{\sqrt{m}}{2} & -\frac{\sqrt{m}}{2} \\ \frac{m}{\sqrt{4m}} & \frac{m}{\sqrt{4m}} & \frac{m}{\sqrt{4m}} & \frac{m}{\sqrt{4m}} \end{pmatrix} = 2^{1/3}\begin{pmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$\vec{x} = S_{12}{}^{-1}\,\vec{y}^{(12)} \tag{8}$$

Then

$$\vec{y}^{(13)} = S_{13}\,\vec{x} = S_{13}\,S_{12}{}^{-1}\,\vec{y}^{(12)} \tag{9}$$

Now note that we could use the same matrix S_{12} to construct $\vec{y}^{(13)}$ if the matrix were to act on a column vector $\begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix}$ instead of the usual vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. We can then say:

$$\begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = P_{23}\,\vec{x} \tag{10}$$

Hence:

$$S_{13} = S_{12}\,P_{23} \tag{11}$$

and

$$\vec{y}^{(13)} = S_{13}\,\vec{x} = S_{13}\,S_{12}{}^{-1}\,\vec{y}^{(12)} = S_{12}\,P_{23}\,S_{12}{}^{-1}\,\vec{y}^{(12)} \tag{12}$$

Similarly,

$$\vec{y}^{(14)} = S_{14}\,\vec{x} = S_{14}\,S_{12}{}^{-1}\,\vec{y}^{(12)} = S_{12}\,P_{24}\,S_{12}{}^{-1}\,\vec{y}^{(12)} \tag{13}$$

where

$$P_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{14}$$

In[1]:= **S12 = $2^{1/3}\begin{pmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$;**

In[2]:=
$$P24 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

$$P23 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$P34 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$P14 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

$$P13 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$P12 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

In[3]:=
```
𝓅12 = Simplify[S12.P12.Inverse[S12]][[1 ;; 3, 1 ;; 3]]; TraditionalForm[𝓅12]
```

Out[∘]//TraditionalForm=
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[4]:=
```
𝓅34 = Simplify[S12.P34.Inverse[S12]][[1 ;; 3, 1 ;; 3]]; TraditionalForm[𝓅34]
```

Out[∘]//TraditionalForm=
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[5]:=
```
𝓅14 = Simplify[S12.P14.Inverse[S12]][[1 ;; 3, 1 ;; 3]]; TraditionalForm[𝓅14]
```

Out[∘]//TraditionalForm=
$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[6]:=
```
𝓅13 = Simplify[S12.P13.Inverse[S12]][[1 ;; 3, 1 ;; 3]]; TraditionalForm[𝓅13]
```

Out[∘]//TraditionalForm=
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[7]:=
```
𝓅23 = Simplify[S12.P23.Inverse[S12]][[1 ;; 3, 1 ;; 3]]; TraditionalForm[𝓅23]
```

Out[∘]//TraditionalForm=

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[8]:=
```
𝓅24 = Simplify[S12.P24.Inverse[S12]][[1 ;; 3, 1 ;; 3]]; TraditionalForm[𝓅24]
```

Out[∘]//TraditionalForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

In[9]:=
```
TraditionalForm[FullSimplify[𝓅23.𝓅14]]
```

Out[∘]//TraditionalForm=

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In[10]:=
```
TraditionalForm[FullSimplify[𝓅24.𝓅13]]
```

Out[∘]//TraditionalForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In[11]:=
```
TraditionalForm[FullSimplify[𝓅23.𝓅14.𝓅24.𝓅13]]
```

Out[∘]//TraditionalForm=

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_{\mathrm{cm}} = \frac{m_1\, x_1 + m_2\, x_2 + m_3\, x_3 + m_4\, x_4}{m_1 + m_2 + m_3 + m_4} \tag{15}$$

$$\rho_1^{(12)} = x_1 - x_2 \tag{16}$$

$$\rho_2^{(12)} = x_3 - x_4 \tag{17}$$

$$\rho_3^{(12)} = \frac{m_1\, x_1 + m_2\, x_2}{m_1 + m_2} - \frac{m_3\, x_3 + m_4\, x_4}{m_3 + m_4} \tag{18}$$

and *mass-scaled* Jacobi coordinates as:

$$y_1^{(12)} = \sqrt{\frac{\mu_{12}}{\mu}}\ (x_1 - x_2) \tag{19}$$

$$y_2^{(12)} = \sqrt{\frac{\mu_{34}}{\mu}}\ (x_3 - x_4) \tag{20}$$

$$y_3^{(12)} = \sqrt{\frac{\mu_{12,34}}{\mu}}\ \left(\frac{m_1\, x_1 + m_2\, x_2}{m_1 + m_2} - \frac{m_3\, x_3 + m_4\, x_4}{m_3 + m_4}\right) \tag{21}$$

$$y_4^{(12)} = \sqrt{\frac{M}{\mu}}\; X_{\text{cm}} \tag{22}$$

The convention here is that the superscript (12) indicates this coordinate system is convenient for treating the interaction between particles 1 and 2. In matrix notation:

$$\begin{pmatrix} y_1^{(12)} \\ y_2^{(12)} \\ y_3^{(12)} \\ y_4^{(12)} \end{pmatrix} = \frac{1}{\sqrt{\mu}} \begin{pmatrix} \sqrt{\mu_{12}} & -\sqrt{\mu_{12}} & 0 & 0 \\ 0 & 0 & \sqrt{\mu_{34}} & -\sqrt{\mu_{34}} \\ \frac{\sqrt{\mu_{12,34}}\; m_1}{m_1+m_2} & \frac{\sqrt{\mu_{12,34}}\; m_2}{m_1+m_2} & -\frac{\sqrt{\mu_{12,34}}\; m_3}{m_3+m_4} & -\frac{\sqrt{\mu_{12,34}}\; m_4}{m_3+m_4} \\ \frac{m_1}{\sqrt{M}} & \frac{m_2}{\sqrt{M}} & \frac{m_3}{\sqrt{M}} & \frac{m_4}{\sqrt{M}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{23}$$

Now we define the hyperspherical coordinates as:

$$y_1^{(12)} = \cos\theta_{12} \tag{24}$$

$$y_2^{(12)} = \sin\theta_{12} \cos\phi_{12} \tag{25}$$

$$y_3^{(12)} = \sin\theta_{12} \sin\phi_{12} \tag{26}$$

In[12]:=
```
Clear[μ4]
```

In[13]:=
```
μ4 = (μ12 μ34 μ1234)^(1/3) /.
  {μ12 → (m1 m2/(m1 + m2)), μ34 → (m3 m4/(m3 + m4)), μ1234 -> ((m1 + m2) (m3 + m4)/(m1 + m2 + m3 + m4)), M → m1 + m2 + m3 + m4}
```

Out[∘]=

$$\left(\frac{\text{m1 m2 m3 m4}}{\text{m1} + \text{m2} + \text{m3} + \text{m4}}\right)^{1/3}$$

In[14]:=
$$A = \frac{1}{\sqrt{\mu 4}} \begin{pmatrix} \sqrt{\mu 12} & -\sqrt{\mu 12} & 0 & 0 \\ 0 & 0 & \sqrt{\mu 34} & -\sqrt{\mu 34} \\ \frac{\sqrt{\mu 1234}\, m1}{m1+m2} & \frac{\sqrt{\mu 1234}\, m2}{m1+m2} & -\frac{\sqrt{\mu 1234}\, m3}{m3+m4} & -\frac{\sqrt{\mu 1234}\, m4}{m3+m4} \\ \frac{m1}{\sqrt{M}} & \frac{m2}{\sqrt{M}} & \frac{m3}{\sqrt{M}} & \frac{m4}{\sqrt{M}} \end{pmatrix} \;/.\; \left\{\mu 12 \to \left(\frac{m1\, m2}{m1+m2}\right),\right.$$

$$\left.\mu 34 \to \left(\frac{m3\, m4}{m3+m4}\right),\ \mu 1234 \text{ -> } \left(\frac{(m1+m2)\ (m3+m4)}{m1+m2+m3+m4}\right),\ M \to m1+m2+m3+m4\right\} \text{ // FullSimplify}$$

Out[∘]=
$$\left\{\left\{\frac{\sqrt{\frac{m1\, m2}{m1+m2}}}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}},\ -\frac{\sqrt{\frac{m1\, m2}{m1+m2}}}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}},\ 0,\ 0\right\},\right.$$

$$\left\{0,\ 0,\ \frac{\sqrt{\frac{m3\, m4}{m3+m4}}}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}},\ -\frac{\sqrt{\frac{m3\, m4}{m3+m4}}}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}}\right\},\ \left\{\frac{m1\sqrt{\frac{(m1+m2)\ (m3+m4)}{m1+m2+m3+m4}}}{(m1+m2)\ \left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}},\right.$$

$$\left.\frac{m2\sqrt{\frac{(m1+m2)\ (m3+m4)}{m1+m2+m3+m4}}}{(m1+m2)\ \left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}},\ -\frac{m3\sqrt{\frac{(m1+m2)\ (m3+m4)}{m1+m2+m3+m4}}}{(m3+m4)\ \left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}},\ -\frac{m4\sqrt{\frac{(m1+m2)\ (m3+m4)}{m1+m2+m3+m4}}}{(m3+m4)\ \left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}}\right\},$$

$$\left\{\frac{m1}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}\sqrt{m1+m2+m3+m4}},\ \frac{m2}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}\sqrt{m1+m2+m3+m4}},\right.$$

$$\left.\left.\frac{m3}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}\sqrt{m1+m2+m3+m4}},\ \frac{m4}{\left(\frac{m1\, m2\, m3\, m4}{m1+m2+m3+m4}\right)^{1/6}\sqrt{m1+m2+m3+m4}}\right\}\right\}$$

Let’s just treat the equal mass case here...

In[15]:=
$$\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \text{FullSimplify}\left[\text{Inverse}[A].\begin{pmatrix} x \\ y \\ z \\ XCM \end{pmatrix} \;/.\; \{m1 \to m,\ m2 \to m,\ m3 \to \beta\, m,\ m4 \to \beta\, m\}\right];$$

In[16]:=
```
Clear[R]
```

In[17]:=
```
x12 = Simplify[x1 - x2]; FullSimplify[x12, {m > 0, β > 0}]
x12hyp = Simplify[x12, m > 0] /. {x → R Sin[θ] Cos[ϕ], y -> R Sin[θ] Sin[ϕ], z -> R Cos[θ]};
FullSimplify[x12hyp, m > 0]
```

Out[∘]=
$$\frac{2^{1/3}\, x\, \beta^{1/3}}{(1+\beta)^{1/6}}$$

Out[∘]=
$$2^{1/3}\, R\left(\frac{\beta^2}{1+\beta}\right)^{1/6} \text{Cos}[\phi]\ \text{Sin}[\theta]$$

In[19]:=
```
x13 = FullSimplify[x1 - x3, {m > 0, β > 0}]
x13hyp = Simplify[x13] /. {x → R Sin[θ] Cos[ϕ], y → R Sin[θ] Sin[ϕ], z → R Cos[θ]};
FullSimplify[x13hyp, {m > 0, β > 0}]
```

Out[•]=
$$\frac{\sqrt{m}\, x\, \beta - y\sqrt{m\,\beta} + z\sqrt{m\,\beta\,(1+\beta)}}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

Out[•]=
$$\frac{R\left(\sqrt{m\,\beta\,(1+\beta)}\,\text{Cos}[\theta] + \text{Sin}[\theta]\left(\sqrt{m}\,\beta\,\text{Cos}[\phi] - \sqrt{m\,\beta}\,\text{Sin}[\phi]\right)\right)}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

In[21]:=
```
x14 = FullSimplify[x1 - x4, {m > 0, β > 0}]
x14hyp = Simplify[x14] /. {x → R Sin[θ] Cos[ϕ], y -> R Sin[θ] Sin[ϕ], z -> R Cos[θ]};
FullSimplify[x14hyp, {m > 0, β > 0}]
```

Out[•]=
$$\frac{\sqrt{m}\, x\, \beta + y\sqrt{m\,\beta} + z\sqrt{m\,\beta\,(1+\beta)}}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

Out[•]=
$$\frac{R\left(\sqrt{m\,\beta\,(1+\beta)}\,\text{Cos}[\theta] + \text{Sin}[\theta]\left(\sqrt{m}\,\beta\,\text{Cos}[\phi] + \sqrt{m\,\beta}\,\text{Sin}[\phi]\right)\right)}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

In[23]:=
```
x23 = Simplify[x2 - x3, {m > 0, β > 0}]
x23hyp = Simplify[x23] /. {x → R Sin[θ] Cos[ϕ], y -> R Sin[θ] Sin[ϕ], z -> R Cos[θ]};
FullSimplify[x23hyp, {m > 0, β > 0}]
```

Out[•]=
$$-\frac{\sqrt{m}\, x\, \beta + y\sqrt{m\,\beta} - z\sqrt{m\,\beta\,(1+\beta)}}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

Out[•]=
$$\frac{R\left(\sqrt{m\,\beta\,(1+\beta)}\,\text{Cos}[\theta] - \text{Sin}[\theta]\left(\sqrt{m}\,\beta\,\text{Cos}[\phi] + \sqrt{m\,\beta}\,\text{Sin}[\phi]\right)\right)}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

In[25]:=
```
x24 = Simplify[x2 - x4, {m > 0, β > 0}]
x24hyp = Simplify[x24] /. {x → R Sin[θ] Cos[ϕ], y -> R Sin[θ] Sin[ϕ], z -> R Cos[θ]};
FullSimplify[x24hyp, {m > 0, β > 0}]
```

Out[•]=
$$\frac{-\sqrt{m}\, x\, \beta + y\sqrt{m\,\beta} + z\sqrt{m\,\beta\,(1+\beta)}}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

Out[•]=
$$\frac{R\left(\sqrt{m\,\beta\,(1+\beta)}\,\text{Cos}[\theta] + \text{Sin}[\theta]\left(-\sqrt{m}\,\beta\,\text{Cos}[\phi] + \sqrt{m\,\beta}\,\text{Sin}[\phi]\right)\right)}{2^{2/3}\sqrt{m}\,\beta^{2/3}\,(1+\beta)^{1/6}}$$

In[27]:=
```
x34 = Simplify[x3 - x4, {m > 0, β > 0}]
x34hyp = Simplify[x3 - x4, {m > 0, β > 0}] /.
   {x → R Sin[θ] Cos[ϕ], y -> R Sin[θ] Sin[ϕ], z -> R Cos[θ]};
FullSimplify[x34hyp, {m > 0, β > 0}]
```

Out[]= $$\frac{2^{1/3}\,y}{(\beta\,(1+\beta))^{1/6}}$$

Out[]= $$\frac{2^{1/3}\,R\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]}{(\beta\,(1+\beta))^{1/6}}$$

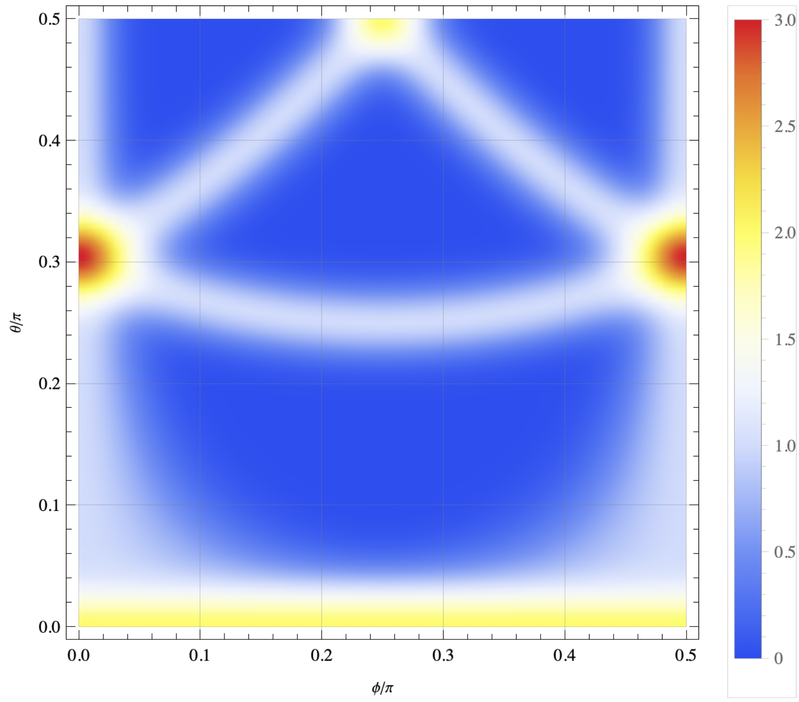
In[68]:=
```
Clear[unequalmassxij]
```

In[69]:=
```
unequalmassxij[R_, β_] =
  FullSimplify[{x12hyp, x13hyp, x14hyp, x23hyp, x24hyp, x34hyp} /. {m → 1, θ → t π, ϕ → f π}]
```

Out[]= $$\left\{2^{1/3}\,R\left(\frac{\beta^2}{1+\beta}\right)^{1/6}\mathrm{Cos}[f\,\pi]\,\mathrm{Sin}[\pi\,t],\right.$$
$$\frac{R\left(\sqrt{\beta\,(1+\beta)}\,\mathrm{Cos}[\pi\,t]+\beta\,\mathrm{Cos}[f\,\pi]\,\mathrm{Sin}[\pi\,t]-\sqrt{\beta}\,\mathrm{Sin}[f\,\pi]\,\mathrm{Sin}[\pi\,t]\right)}{2^{2/3}\,\beta^{2/3}\,(1+\beta)^{1/6}},$$
$$\frac{R\left(\sqrt{\beta\,(1+\beta)}\,\mathrm{Cos}[\pi\,t]+\beta\,\mathrm{Cos}[f\,\pi]\,\mathrm{Sin}[\pi\,t]+\sqrt{\beta}\,\mathrm{Sin}[f\,\pi]\,\mathrm{Sin}[\pi\,t]\right)}{2^{2/3}\,\beta^{2/3}\,(1+\beta)^{1/6}},$$
$$\frac{R\left(\sqrt{\beta\,(1+\beta)}\,\mathrm{Cos}[\pi\,t]-\sqrt{\beta}\left(\sqrt{\beta}\,\mathrm{Cos}[f\,\pi]+\mathrm{Sin}[f\,\pi]\right)\mathrm{Sin}[\pi\,t]\right)}{2^{2/3}\,\beta^{2/3}\,(1+\beta)^{1/6}},$$
$$\left.\frac{R\left(\sqrt{\beta\,(1+\beta)}\,\mathrm{Cos}[\pi\,t]-\beta\,\mathrm{Cos}[f\,\pi]\,\mathrm{Sin}[\pi\,t]+\sqrt{\beta}\,\mathrm{Sin}[f\,\pi]\,\mathrm{Sin}[\pi\,t]\right)}{2^{2/3}\,\beta^{2/3}\,(1+\beta)^{1/6}},\ \frac{2^{1/3}\,R\,\mathrm{Sin}[f\,\pi]\,\mathrm{Sin}[\pi\,t]}{\left(\beta+\beta^2\right)^{1/6}}\right\}$$

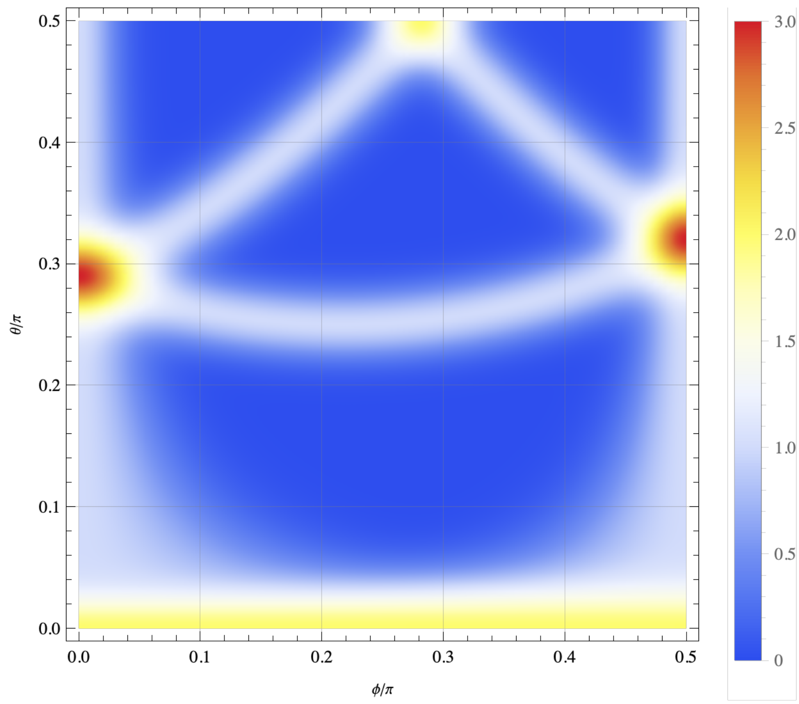
In[70]:=
```
beta = 1.0;
DensityPlot[Sum[Exp[-unequalmassxij[10, beta][[i]]^2], {i, 1, 6}] == 0,
 {f, 0.0, .5}, {t, 0, 0.5}, GridLines → Automatic, ColorFunction → "TemperatureMap",
 FrameLabel → {"ϕ/π", "θ/π"}, PlotPoints → 100, PlotLegends → Automatic]
```

Out[ ]=



In[71]:=
```
beta = 1.5;
DensityPlot[Sum[Exp[-unequalmassxij[10, beta][[i]]^2] /. β → beta, {i, 1, 6}] == 0,
 {f, 0.0, .5}, {t, 0, 0.5}, GridLines → Automatic, ColorFunction → "TemperatureMap",
 FrameLabel → {"ϕ/π", "θ/π"}, PlotPoints → 100, PlotLegends → Automatic]
```

Out[∘]=



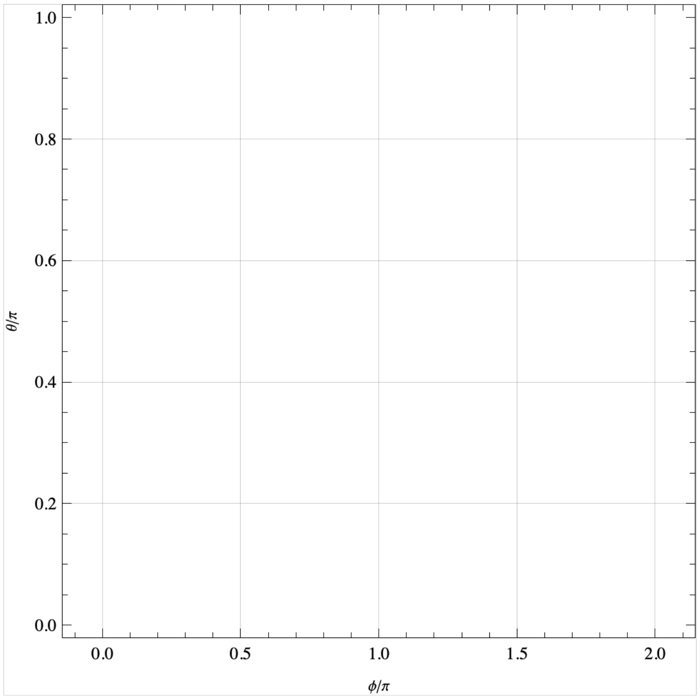
In[59]:=
```
unequalmassxij[[1]] == 0
```

Out[∘]= $5 \times 2^{1/3} \left(\frac{\beta^2}{1+\beta}\right)^{1/6} \text{Cos}[f\,\pi]\,\text{Sin}[\pi\,t] == 0$

In[542]:=
```
c12[β_] := ContourPlot[Evaluate[unequalmassxij[[1]] == 0], {f, -0.1, 2.1}, {t, 0, 1},
  GridLines → Automatic, ContourStyle → Black, FrameLabel → {"ϕ/π", "θ/π"}]
c13[β_] := ContourPlot[Evaluate[unequalmassxij[[2]] == 0], {f, -0.1, 2.1}, {t, 0, 1},
   GridLines → Automatic, ContourStyle → Red, FrameLabel → {"ϕ/π", "θ/π"}];
c14[β_] := ContourPlot[Evaluate[unequalmassxij[[3]] == 0], {f, -0.1, 2.1}, {t, 0, 1},
   GridLines → Automatic, ContourStyle → Blue, FrameLabel → {"ϕ/π", "θ/π"}];
c23[β_] := ContourPlot[Evaluate[unequalmassxij[[4]] == 0], {f, -0.1, 2.1}, {t, 0, 1},
   GridLines → Automatic, ContourStyle → Green, FrameLabel → {"ϕ/π", "θ/π"}];
c24[β_] := ContourPlot[Evaluate[unequalmassxij[[5]] == 0], {f, -0.1, 2.1}, {t, 0, 1},
   GridLines → Automatic, ContourStyle → Orange, FrameLabel → {"ϕ/π", "θ/π"}];
c34[β_] := ContourPlot[Evaluate[unequalmassxij[[6]] == 0], {f, -0.1, 2.1}, {t, 0, 1},
   GridLines → Automatic, ContourStyle → Yellow, FrameLabel → {"ϕ/π", "θ/π"}];
```

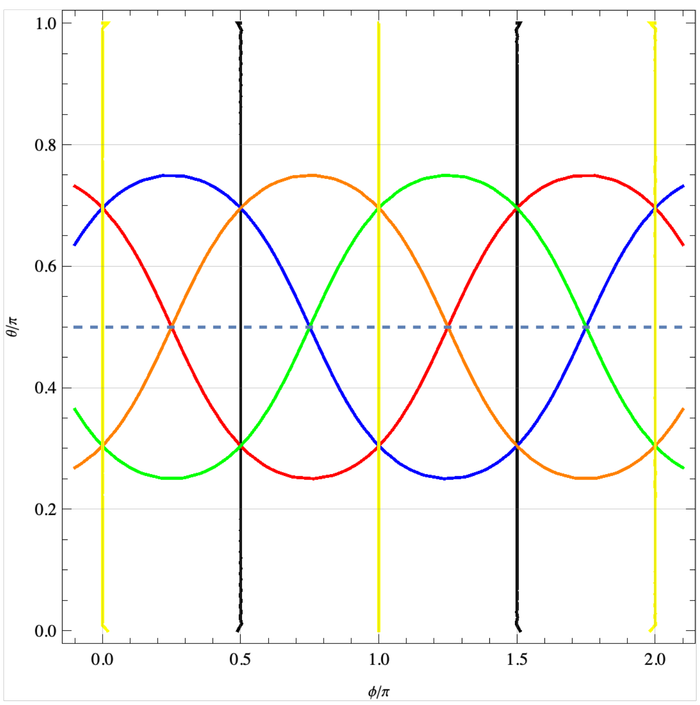
In[548]:=
```
c12[1.0]
```

Out[]=



In[1089]:=
```
pall = Show[{c12, c13, c14, c23, c24, c34}, Plot[0.5, {x, -.1, 2.1}, PlotStyle → Dashed]]
```

Out[]=

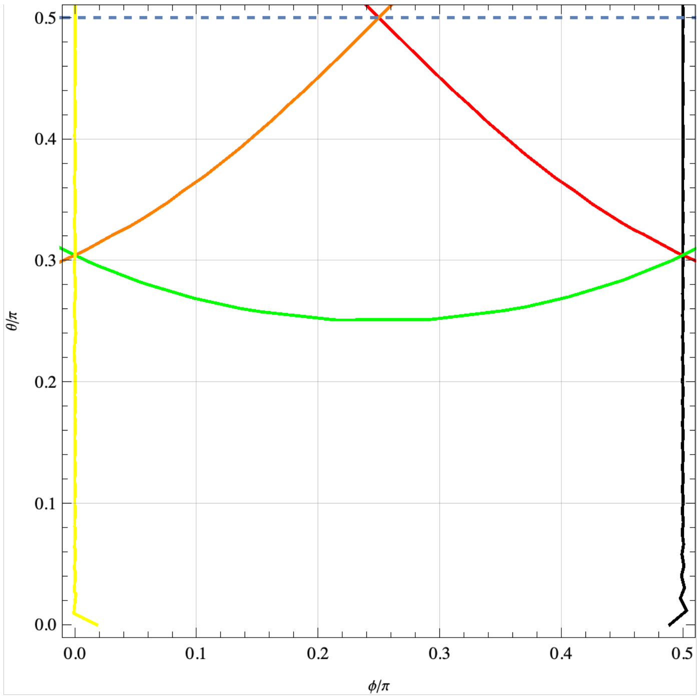


In[1087]:=
```
cptable = Table[ContourPlot[Evaluate[unequalmassxij[[i]] == 0 /. {R → 1, α → 1}],
    {f, 0, .5}, {t, 0, .5}, GridLines → Automatic, ContourStyle → Black], {i, 1, 6}];
```

In[1090]:=

```
Show[pall, PlotRange → {{0, 0.5}, {0, 0.5}}]
```

Out[∘]=



▪ Construct a simpler model with numerical solution

Consider a 3D Hamiltonian of the form

$$H = \frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu r^2 + k_{xy}\,x\,y + k_{xz}\,x\,z + k_{yz}\,y\,z \tag{27}$$

$$H = \frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu r^2 + k_{xy}\,x\,y + k_{xz}\,x\,z + k_{yz}\,y\,z \tag{28}$$

Sakurai, Modern QM page 217 (translated to my preferred notation)

$$\int d\Omega\, Y_l^{m^*}(\Omega)\, Y_{l_1}^{m_1}(\Omega)\, Y_{l_2}^{m_2}(\Omega) = \sqrt{\frac{(2\,l_1+1)\,(2\,l_2+1)}{4\,\pi\,(2\,l+1)}}\,\langle l_1\,0,\;\; l_2\,0|(l_1\,l_2)\,l\,0\rangle\,\langle l_1\,m_1\,l_2\,m_2\,|(l_1\,l_2)\,l\,m\rangle \tag{29}$$

In[284]:=

```
ME3[l_, m_, l1_, m1_, l2_, m2_] := Sqrt[((2 l1 + 1) (2 l2 + 1))/(4 π (2 l + 1))]
   ClebschGordan[{l1, 0}, {l2, 0}, {l, 0}] × ClebschGordan[{l1, m1}, {l2, m2}, {l, m}]
```

So if we need matrix elements of the form:

$$\int d\Omega\, Y_l^{m^*}(\Omega)\, Y_2^{m_1}(\Omega)\, Y_{l_2}^{m_2}(\Omega) = \sqrt{\frac{5\,(2\,l_2+1)}{4\,\pi\,(2\,l+1)}}\,\langle 2,\,0,\;\; l_2,\,0|(l_1\,l_2)\,l\,0\rangle\,\langle 2\,m_1\,l_2\,m_2\,|(2\,l_2)\,l\,m\rangle \tag{30}$$

In[286]:=
```
Simplify[ME3[l, m, 2, m1, l2, m2]]
```

Out[286]=

$$\begin{cases} \frac{1}{2\,(2+l-l2)\,(l-l2)!}\,(-1)^{-3\,l2+m}\,(1+2\,l)\left(l^3+l^2\,(1+l2)-l\,(2+2\,l2+l2^2)-l2\,(2+3\,l2+l2^2)\right)\sqrt{\frac{5+10\,l2}{\pi+2\,l\,\pi}}\,\sqrt{\frac{(2+l-l2)!}{(-1+l+l2)\,(l+l2)\,(1+l+l2)\,(2+l+l2)\,(3+l+l2)\,(2-l+l2)!}}\,\text{ThreeJSymbol}[\{2, m1\}, \{l2, m2\}, \{l, -m\}] & l\in\mathbb{Z}\,\&\&\,l2\in\mathbb{Z}\,\&\&\,l\geq 0\,\&\&\,l2\geq 0\,\&\&\,l+l2\geq 2\,\&\&\,l2\leq 2+l\,\&\&\,l\leq 2+l2 \\ 0 & \text{True} \end{cases}$$

Now, since we know

$$x\,y = r^2\sqrt{\frac{30}{\pi}}\;i\left(Y_2^{-2}-Y_2^2\right) \tag{31}$$

In[287]:=

$$\text{MExy}[R\_, l\_, m\_, l2\_, m2\_] := R^2\, i \sqrt{\frac{30}{\pi}}\;(\text{ME3}[l, m, 2, -2, l2, m2] - \text{ME3}[l, m, 2, 2, l2, m2])$$

Now the issue is if we want to apply this model to test the two-component 4-fermion problem in 1D, then we need to know how to construct systematically the symmetrized 4-fermion harmonics. We can specify the subset of harmonics by the boundary conditions implied by identical particle symmetry:

$$\psi(\phi=0,\,\theta)=\psi(\phi=\pi/2,\,\theta)=0 \tag{32}$$

requires a superposition $\psi \sim Y_l^m - (-1)^m\,Y_l^{-m}$

In[283]:=
```
Integrate[SphericalHarmonicY[lb, mb, θ, ϕ] SphericalHarmonicY[k, q, θ, ϕ]
  SphericalHarmonicY[lk, mk, θ, ϕ] Sin[θ], {θ, 0, π}, {ϕ, 0, 2 π}]
```

Out[283]=

$$\int_0^{\pi}\int_0^{2\pi}\text{Sin}[\theta]\,\text{SphericalHarmonicY}[k, q, \theta, \phi]\,\text{SphericalHarmonicY}[lb, mb, \theta, \phi]\,\text{SphericalHarmonicY}[lk, mk, \theta, \phi]\,d\phi\,d\theta$$