Module-1-Mathematical Logic and Statement of Calculus

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Contents

Propositions and Logic

- **Introduction**

  Logic is to provide rules by which one can determine whether any particular argument or reasoning is valid(correct).

  Logic is concerend with all kinds of reasonings, whether they be legal arguments or mathematical proofs or conclusions in a scientific theory based upon a set of hypothesis.

  Because of the diversity of their application, these rules, called rules of inference, must be stated in general terms and must be independent of any particular argument or discipline involved.

  These rules should be independent of any particular language used in the arguments.

  The rules of logic specify the meaning of mathematical statements.

For instance, these rules help us understand and reason with statements such as "There exists an integer that is not the sum of two squares" and "For every positive integer n, the sum of the positive integers not exceeding n is $n(n+1)/2$.

Logic is the basis of all mathematical reasoning, and of all automated reasoning.

It has practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study.

- The area of logic that deals with propositions is called the propositional calculus or propositional logic.
- It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

Propositions

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Example

The following statements are propositions.

- Delhi is the capital of India
- $1+1=3$
- $20+10=30$

Propositions

Example

The following statements are not propositions.

- $x + 1 = 2$
- $x + y = z$
- Read this carefully
- I like apples

Simple and Compound propositions

The propositional variables are denoted by $p, q, r, s, \ldots,$ or $P, Q, R, \ldots,$

The truth value of a proposition is true, denoted by T, if it is a true proposition.

The truth value of a proposition is false, denoted by F, if it is a false proposition.

- Simple propositions, like
  "today is Tuesday"

  "it is raining"
- Compound propositions is two or more simple propositions can be combined to form compound propositions, like
  "today is Tuesday and it is raining" by using a connective ("and" in the example).

Propositions

The simplest connectives are "not," "and," "or," denoted by $\neg(\ \sim)$, $\wedge$, $\vee$ respectively.

- **Negation**

  Let P be a proposition. The negation of P, denoted by $\neg P$ ($\sim P$) is the statement "It is not the case that P".

  The proposition $\neg P$ is read as "not P".

  The truth value of the negation of P, $\neg P$, is the opposite of the truth value of P.

Truth Table for the Negation of a proposition

The truth table for the negation of a proposition P is defined as follows:

P	$\neg P$
T	F
F	T

The negation operator constructs a new proposition from a single existing proposition.

Truth Table for the Conjunction

Let P and Q be two propositions. Then the conjunction of P and Q, denoted by $P \wedge Q$, is the proposition "P and Q". The conjunction $P \wedge Q$ is true when both P and Q are true and is false otherwise.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table for the Disjunction

Let P and Q be two propositions. Then the disjunction of P and Q, denoted by $P \vee Q$, is the proposition "P or Q". The conjunction $P \vee Q$ is false when both P and Q are false and is true otherwise.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Problem

Let P denote the proposition "the sun is shining" and Q the proposition "the wind is blowing". Write expressions for the following

1. *"the sun is not shining,"*
2. *"the sun is shining and the wind is blowing," and*
3. *"the sun is shining but the wind is not blowing".*

Solution

1. $\neg P$ *denotes "the sun is not shining,"*
2. $P \vee Q$ *denotes "the sun is shining or the wind is blowing," and*
3. $P \wedge Q$ *denotes "the sun is shining and the wind is blowing."*

Problem

Write expressions for the following

1. *"the sun is not shining,"*
2. *"the sun is shining or the wind is blowing (may be both)," and*
3. *"the sun is not shining but the wind is blowing."*

Problem

Suppose P, Q and R mean "Joseph is here," "Nancy is here" and "Donna is here." Interpret $P \wedge \neg Q$ and $(P \wedge Q) \wedge R$.

Solution

1. $P \wedge \neg Q$ *means "Joseph is here but Nancy is not,"*
2. $(P \wedge Q) \wedge R$ *means "Joseph, Nancy and Donna are here."*

Conditional Statement

- Let P and Q be two propositions. Then the conditional statement $P \to Q$ is the proposition "if P, then Q," or "P implies Q"
- The conditional statement $P \to Q$ is false when P is true and Q is false, and true otherwise.
  In the conditional statement $P \to Q$, P is called the hypothesis (or premise) and Q is called the conclusion.
- The Truth Table for the conditional statement $P \to Q$ is given as follows:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Problem

Let P be the statement "Maria learns discrete mathematics" and Q the statement "Maria will find a good job." Express the statement $P \to Q$.

Solution

$P \to Q$ means "If Maria learns discrete mathematics, then she will find a good job."

Some new conditional statements

We can form some new conditional statements starting with a conditional statement $P \to Q$.

- The proposition $Q \to P$ is called the converse of $P \to Q$.
- The contrapositive of $P \to Q$ is the proposition $\neg Q \to \neg P$.
- The proposition $\neg P \to \neg Q$ is called the inverse of $P \to Q$.

Biconditional Statement

- Let P and Q be two propositions. Then the biconditional statement $P \leftrightarrow Q$ is the proposition "P if and only if Q," or "P implies Q"
- The biconditional statement $P \leftrightarrow Q$ is true when P and Q have the same truth values, and is false otherwise.
- Biconditional statements are also called bi-implications.
- The Truth Table for the biconditional statement $P \leftrightarrow Q$ is given as follows:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	F
F	F	T

Problem

Let P be the statement "You can take the flight," and Q the statement "You buy a ticket." Express the statement $P \leftrightarrow Q$.

Solution

$P \leftrightarrow Q$ *means "You can take the flight if and only if you buy a ticket."*

Truth Tables for some Compound Propositions

Problem

Find the truth table of $(P \wedge Q) \vee (Q \wedge \neg R)$.

Solution

P	Q	R	$\neg R$	$P \wedge Q$	$Q \wedge \neg R$	$(P \wedge Q) \vee (Q \wedge \neg R)$
T	T	T	F	T	F	T
T	T	F	T	T	T	T
T	F	T	F	F	F	F
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	T	F	T	T
F	F	T	F	F	F	F
F	F	F	T	F	F	F

Truth Tables for some Compound Propositions

Problem

Find the truth table of $(P \vee (Q \vee (\neg P \wedge \neg R)))$.

Truth Tables for some Compound Propositions

Problem

Find the truth table of $(P \wedge \neg Q) \rightarrow (Q \vee P)$.

Solution

P	Q	$\neg Q$	$P \wedge \neg Q$	$Q \vee P$	$(P \wedge \neg Q) \rightarrow (Q \vee P)$
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	F	T

Truth Tables for some Compound Propositions

Problem

. *Find the truth table of* $(P \vee \neg Q) \leftrightarrow (Q \rightarrow P)$.

Solution

P	Q	$\neg Q$	$P \vee \neg Q$	$Q \rightarrow P$	$(P \vee \neg Q) \leftrightarrow (Q \rightarrow P)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	F	F	T
F	F	T	T	T	T

Truth Tables for some Compound Propositions

Problem

. *Find the truth table for the following compound propositions*

1. $(P \vee \neg Q) \rightarrow (P \wedge Q)$
2. $P \wedge \neg P$
3. $P \vee \neg P$
4. $(P \vee \neg Q) \rightarrow Q$
5. $(P \vee Q) \rightarrow (P \wedge Q)$
6. $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$
7. $(P \rightarrow Q) \rightarrow (Q \rightarrow P)$
8. $(P \wedge Q) \vee R$
9. $(P \vee R) \wedge (Q \vee R)$
10. $P \vee (Q \vee R)$

Truth Tables for some Compound Propositions

Problem

. *Find the truth table for the following compound propositions*

1. $(P \vee Q) \vee R$
2. $(P \to Q) \to R$
3. $P \to (Q \to R)$
4. $(P \to R) \to (Q \to R)$
5. $(P \vee Q) \wedge \neg(P \wedge Q)$

Dual

The dual of a compound proposition that contains only the logical operators $\vee, \wedge$, and $\neg$ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each $\mathbf{T}$ by $\mathbf{F}$, and each $\mathbf{F}$ by $\mathbf{T}$. The dual of S is denoted by S^*.

Clearly, dual(dual(S)) $= (S^*)^* = S$.

Problem

Find the dual of each of these compound propositions.

- $P \vee \neg Q$
- $P \wedge (Q \vee (R \wedge \mathbf{T}))$
- $(P \wedge \neg Q) \vee (Q \wedge \mathbf{F})$

Tautologies

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.
- A compound proposition that is always false is called a contradiction.
- A compound proposition that is neither a tautology nor a contradiction is called a contingency. .

Tautologies

Problem

Show that $(P \vee (\neg(P \wedge Q))$ *is a tautology.*

Solution

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$P \vee (\neg P \wedge Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Tautologies

Problem

Show that $(P \wedge Q) \wedge \neg(P \vee Q)$ *is contradiction.*

Solution

P	Q	$P \wedge Q$	$(P \vee Q)$	$\neg(P \vee Q)$	$(P \wedge Q) \wedge \neg(P \vee Q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called logically equivalent.

The compound propositions A and B are called logically equivalent if $A \leftrightarrow B$ is a tautology.

The notation $A \Leftrightarrow B$ denotes that A and B are logically equivalent.

Example

The compund propositions $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are logically equivalent.

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Problem

1. *Show that* $P \rightarrow Q$ *and* $\neg P \vee Q$ *are logically equivalent.(Use Truth Table)*
2. *Show that* $P \leftrightarrow Q$ *and* $(P \rightarrow Q) \wedge (Q \rightarrow P)$ *are logically equivalent.(Use Truth Table)*

Logical Equivalences

- Identity laws

  $P \wedge T \Leftrightarrow P$

  $P \vee F \Leftrightarrow P$

- Domination laws

  $P \vee T \Leftrightarrow T$

  $P \wedge F \Leftrightarrow F$

- Idempotent laws

  $P \vee P \Leftrightarrow P$

  $P \wedge P \Leftrightarrow P$

- Double negation law

  $\neg(\neg P) \Leftrightarrow P$

Logical Equivalences

- Commutative laws

  $P \vee Q \Leftrightarrow Q \vee P$

  $P \wedge Q \Leftrightarrow Q \wedge P$

- Associative laws

  $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

  $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

- Distributive laws

  $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

  $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

Logical Equivalences

- De Morgan's laws

  $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

  $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

- Absorption laws

  $P \vee (P \wedge Q) \Leftrightarrow P$

  $P \wedge (P \vee Q) \Leftrightarrow P$

- Negation laws

  $P \vee (\neg P) \Leftrightarrow T$

  $P \wedge (\neg P) \Leftrightarrow F$

Logical Equivalences

- $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
- $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
- $P \vee Q \Leftrightarrow \neg P \rightarrow Q$
- $P \wedge Q \Leftrightarrow \neg(P \rightarrow \neg Q)$
- $\neg(P \rightarrow \neg Q) \Leftrightarrow P \wedge Q$
- $(P \rightarrow Q) \wedge (P \rightarrow R) \Leftrightarrow P \rightarrow (Q \wedge R)$
- $(P \rightarrow Q) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$
- $(P \rightarrow Q) \vee (P \rightarrow R) \Leftrightarrow (P \rightarrow (Q \vee R)$
- $(P \rightarrow R) \vee (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

Logical Equivalences

- $P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$
- $P \leftrightarrow Q \Leftrightarrow \neg P \leftrightarrow \neg Q$
- $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$
- $\neg(P \leftrightarrow Q) \Leftrightarrow P \leftrightarrow \neg Q$

Logical Equivalences

Problem

Show that $\neg(P \to Q)$ *and* $P \wedge \neg Q$ *are logically equivalent.*
That is $\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$.

Proof.

$$\begin{aligned} \neg(P \to Q) &\Leftrightarrow \neg(\neg P \vee Q) \\ &\Leftrightarrow \neg(\neg P) \wedge \neg Q \\ &\Leftrightarrow P \wedge \neg Q \end{aligned}$$

$a \to b \Leftrightarrow \neg a \vee b$

$\neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b$

$\neg(\neg a) \Leftrightarrow a$

$\square$

Problem

Show that $\neg(P \vee (\neg P \wedge Q))$ *and* $\neg P \wedge \neg Q$ *are logically equivalent.*

Proof.

$$
\begin{aligned}
\neg(P \vee (\neg P \wedge Q)) &\Leftrightarrow \neg P \wedge \neg(\neg P \wedge Q) && \neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b \\
&\Leftrightarrow \neg P \wedge (\neg(\neg P) \vee \neg Q) && \neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b \\
&\Leftrightarrow \neg P \wedge (P \vee \neg Q) && \neg(\neg a) \Leftrightarrow a \\
&\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q) && a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (a \wedge c) \\
&\Leftrightarrow F \vee (\neg P \wedge \neg Q) && \neg a \wedge a \Leftrightarrow F \\
&\Leftrightarrow (\neg P \wedge \neg Q) \vee F && a \vee b \Leftrightarrow b \vee a \\
&\Leftrightarrow (\neg P \wedge \neg Q) && a \vee F \Leftrightarrow a
\end{aligned}
$$

□

$$
\begin{aligned}
\neg(P \vee (\neg P \wedge Q)) &\Leftrightarrow \neg P \wedge \neg(\neg P \wedge Q) && \neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b \\
&\Leftrightarrow \neg P \wedge (\neg(\neg P) \vee \neg Q) && \neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b \\
&\Leftrightarrow \neg P \wedge (P \vee \neg Q) && \neg(\neg a) \Leftrightarrow a \\
&\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q) && a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (a \wedge c \\
&\Leftrightarrow F \vee (\neg P \wedge \neg Q) && (\neg a \wedge a) \Leftrightarrow F \\
&\Leftrightarrow (\neg P \wedge \neg Q) \vee F && a \vee b \Leftrightarrow b \vee a \\
&\Leftrightarrow (\neg P \wedge \neg Q) && a \vee F \Leftrightarrow a
\end{aligned}
$$

Problem

Show that $(P \wedge Q) \rightarrow (P \vee Q)$ *is a tautology.*

Proof.

$$
\begin{aligned}
(P \wedge Q) \rightarrow (P \vee Q) &\Leftrightarrow \neg(P \wedge Q) \vee (P \vee Q) && a \rightarrow b \Leftrightarrow \neg a \vee b \\
&\Leftrightarrow (\neg P \vee \neg Q) \vee (P \vee Q) && \neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b \\
&\Leftrightarrow (\neg P \vee \neg Q) \vee (Q \vee P) && a \vee b \Leftrightarrow b \vee a \\
&\Leftrightarrow (\neg P \vee \neg Q) \vee (Q \vee P) && \\
&\Leftrightarrow \neg P \vee (\neg Q \vee (Q \vee P)) && (a \vee b) \vee c \Leftrightarrow a \vee (b \vee c) \\
&\Leftrightarrow \neg P \vee ((\neg Q \vee Q) \vee P) && a \vee (b \vee c) \Leftrightarrow (a \vee b) \vee c \\
&\Leftrightarrow \neg P \vee (P \vee (\neg Q \vee Q)) && a \vee b \Leftrightarrow b \vee a \\
&\Leftrightarrow (\neg P \vee P) \vee (\neg Q \vee Q) && a \vee (b \vee c) \Leftrightarrow (a \vee b) \vee c \\
&\Leftrightarrow T \vee T && \neg a \vee a \Leftrightarrow T \\
&\Leftrightarrow T && a \vee T \Leftrightarrow T
\end{aligned}
$$

□

$$
\begin{aligned}
(P \wedge Q) \to (P \vee Q) &\Leftrightarrow \neg(P \wedge Q) \vee (P \vee Q) && a \to b \Leftrightarrow \neg a \vee b \\
&\Leftrightarrow (\neg P \vee \neg Q) \vee (P \vee Q) && \neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b \\
&\Leftrightarrow (\neg P \vee \neg Q) \vee (Q \vee P) && a \vee b \Leftrightarrow b \vee a \\
&\Leftrightarrow (\neg P \vee \neg Q) \vee (Q \vee P) && \\
&\Leftrightarrow \neg P \vee (\neg Q \vee (Q \vee P)) && (a \vee b) \vee c \Leftrightarrow a \vee (b \vee c) \\
&\Leftrightarrow \neg P \vee ((\neg Q \vee Q) \vee P) && a \vee (b \vee c) \Leftrightarrow (a \vee b) \vee c \\
&\Leftrightarrow \neg P \vee (P \vee (\neg Q \vee Q)) && a \vee b \Leftrightarrow b \vee a \\
&\Leftrightarrow (\neg P \vee P) \vee (\neg Q \vee Q) && a \vee (b \vee c) \Leftrightarrow (a \vee b) \vee c \\
&\Leftrightarrow T \vee T && \neg a \vee a \Leftrightarrow T \\
&\Leftrightarrow T && a \vee T \Leftrightarrow T
\end{aligned}
$$

Problems

1. Prove the equivalences $(P \vee Q) \wedge \neg P \Leftrightarrow \neg P \wedge Q$
2. Prove the equivalence $\neg(P \vee (\neg P \wedge Q)) \Leftrightarrow \neg P \wedge \neg Q$
3. Prove the equivalence $\neg(\neg((P \vee Q) \wedge R) \vee \neg Q) \Leftrightarrow Q \wedge R$
4. Prove the equivalence $(\neg P \wedge (\neg Q \wedge R)) \vee ((Q \wedge R) \vee (P \wedge R)) \Leftrightarrow R$
5. Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.
6. Prove the equivalence $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$
7. Prove the equivalence $(P \vee Q) \vee ((Q \vee \neg R) \wedge (P \vee R)) \Leftrightarrow P \vee Q$
8. Prove the equivalence $(P \leftrightarrow Q) \Leftrightarrow (\neg P \wedge \neg Q) \vee (P \wedge Q)$

Normal Forms

- Normal form is a reduction of the given statement formula to the standard forms usually called normal forms.
- An elementary product

  A product of a variable and their negation in a formula is called an elementary product.
- An elementary sum

  The sum of variables and their negation in a formula is called an elementary sum.

Normal Forms

- Disjunctive Normal Form DNF

  A formula which is equivalent to given formula which consists of sum of elementary products is called a disjunctive normal form DNF.

- Conjunctive Normal Form CNF

  A formula which is equivalent to given formula which consists of a product of an elementary sums is called a conjunctive normal form CNF.

Example

- DNF for $(P \to Q) \land \neg Q$

$$
\begin{aligned}
(P \to Q) \land \neg Q &\Leftrightarrow (\neg P \lor Q) \land \neg Q && a \to b \Leftrightarrow \neg a \lor b \\
&\Leftrightarrow \neg Q \land (\neg P \lor Q) && a \lor b \Leftrightarrow b \lor a \\
&\Leftrightarrow (\neg Q \land \neg P) \lor (\neg Q \land Q) && a \land (b \lor c) \Leftrightarrow (a \land b) \lor (a \land c)
\end{aligned}
$$

This is the required DNF.

- CNF for $(P \to Q) \land \neg Q$

$$
\begin{aligned}
(P \to Q) \land \neg Q &\Leftrightarrow (\neg P \lor Q) \land \neg Q && a \to b \Leftrightarrow \neg a \lor b \\
&\Leftrightarrow (\neg P \lor Q) \land (\neg Q \lor \neg Q) && a \lor a \Leftrightarrow a
\end{aligned}
$$

This is the required CNF.

- **Minterms**

  Let $P_1, P_2, \ldots, P_n$ be n variables. Then the expression $x_1 \wedge x_2 \wedge, \ldots, \wedge x_n$ is is a called minterm if $x_i = P_i$ or $\neg P_i$, for all i.

- **Maxterms**

  Let $P_1, P_2, \ldots, P_n$ be n variables. Then the expression $x_1 \vee x_2 \vee, \ldots, \vee x_n$ is is a called maxterm if $x_i = P_i$ or $\neg P_i$, for all i.

- Principle disjunctive Normal form(PDNF)

  For a given formula an equivalent formula consisting of disjunction of minterms only is known as principle disjunctive Normal form.

- Principle conjunctive Normal form(PCNF)

  For a given formula an equivalent formula consisting of conjunction of maxterms only is known as principle disjunctive Normal form.

Problem

Obtain a PDNF and PCNF of $\phi : (\neg P \to R) \wedge (Q \leftrightarrow P)$

Solution

P	Q	R	$\neg P$	$\neg P \to R$	$Q \leftrightarrow P$	ϕ
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

Solution Cont...

The minterms only for T in the formula ϕ is given below

P	Q	R	ϕ	$Minterms(T)$
T	T	T	T	$P \wedge Q \wedge R$
T	T	F	T	$P \wedge Q \wedge \neg R$
F	F	T	T	$\neg P \wedge \neg Q \wedge R$

Therefore, the required PDNF of ϕ is
$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$.

Solution Cont...

The minterms only for F in the formula ϕ is given below

P	Q	R	ϕ	$Minterms(F)$
T	F	T	F	$P \wedge \neg Q \wedge R$
T	F	F	F	$P \wedge \neg Q \wedge \neg R$
F	T	T	F	$\neg P \wedge Q \wedge R$
F	T	F	F	$\neg P \wedge Q \wedge \neg R$
F	F	F	F	$\neg P \wedge \neg Q \wedge \neg R$

The PCNF of ϕ: $\neg($
$(P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R))$.

Solution Cont...

The PCNF of ϕ: $\neg((P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R))$.

$\Leftrightarrow \neg(P \wedge \neg Q \wedge R) \wedge \neg(P \wedge \neg Q \wedge \neg R) \wedge \neg(\neg P \wedge Q \wedge R) \wedge \neg(\neg P \wedge Q \wedge \neg R) \wedge \neg(\neg P \wedge \neg Q \wedge \neg R))$ $\quad \neg(a \vee b \vee c) \Leftrightarrow \neg a \wedge \neg b \wedge \neg c$.

$\Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R))$ $\quad \neg(a \wedge b \wedge c) \Leftrightarrow \neg a \vee \neg b \vee \neg c$.

This is the required PCNF of ϕ.

Problem

Find the PDNF of $\Phi : P \to (P \wedge (Q \to P))$ without using truth table.

Solution

$P \to (P \wedge (Q \to P))$

$\Leftrightarrow \neg P \vee (P \wedge (Q \to P))$ $\quad a \to b \Leftrightarrow \neg a \vee b$

$\Leftrightarrow \neg P \vee (P \wedge (\neg Q \vee P))$ $\quad a \to b \Leftrightarrow \neg a \vee b$

$\Leftrightarrow \neg P \vee ((P \wedge \neg Q) \vee (P \wedge P))$ $\quad a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (a \wedge c)$

$\Leftrightarrow \neg P \vee ((P \wedge \neg Q) \vee P)$ $\quad a \wedge a \Leftrightarrow a$

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee ((P \wedge \neg Q) \vee (P \wedge (Q \vee \neg Q)))$ $\quad a \wedge T(a \vee \neg a) \Leftrightarrow a$

$\Leftrightarrow ((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee ((P \wedge \neg Q) \vee ((P \wedge Q) \vee (P \wedge \neg Q)))$ $\quad a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (a \wedge c)$

Solution Cont...

$\Leftrightarrow ((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee ((P \wedge \neg Q) \vee ((P \wedge \neg Q) \vee (P \wedge Q)))$ $a \vee b \Leftrightarrow b \vee a$

$\Leftrightarrow ((\neg P \wedge Q) \vee (\neg P \wedge \neg Q)) \vee ((P \wedge \neg Q) \vee (P \wedge \neg Q)) \vee (P \wedge Q)$ $a \vee (b \vee c) \Leftrightarrow (a \vee b) \vee c$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge \neg Q) \vee (P \wedge Q)$ $a \vee a \Leftrightarrow a$

This is the required PDNF.

Remark

The PDF of the above formula Φ is covered all the possible minterms for two variables. Hence the PCNF does not exist.

Thank you