Module-1-Inference Theory for Statement of Calculus

Dr. Nalliah M

Assistant Professor
Department of Mathematics
School of Advanced Sciences
Vellore Institute of Technology
Vellore,Tamil Nadu,India.

nalliah.moviri@vit.ac.in

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Tautological implication

A statement A is said to tautologically imply a statement B if and only if $A \rightarrow B$ is a tautology. It is denoted by $A \Rightarrow B$.

IMPLICATIONS FORMULA

- Simplication

  $P \wedge Q \Rightarrow P$

  $P \wedge Q \Rightarrow Q$

- Addition

  $P \Rightarrow P \vee Q$

  $Q \Rightarrow P \vee Q$

- $\neg P \Rightarrow P \rightarrow Q$
- $Q \Rightarrow P \rightarrow Q$
- $\neg(P \rightarrow Q) \Rightarrow P$

Tautological implication

IMPLICATIONS FORMULA

- $\neg(P \to Q) \Rightarrow \neg Q$
- $P \wedge (P \to Q) \Rightarrow Q$
- $\neg Q \wedge (P \to Q) \Rightarrow \neg P$
- $\neg P \wedge (P \vee Q) \Rightarrow Q$
- $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$
- $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$
- $\neg P \Rightarrow P \to Q$
- $Q \Rightarrow P \to Q$
- $\neg(P \to Q) \Rightarrow P$

Inference Theory for statement of calculus

An argument is a group of statements including one or more statements (premises) and one and only one conclusion.

A premise is a statement or statements that follow the proposition.
A premise is a statement in an argument that provides reason or support for the conclusion. There can be one or many premises in a single argument.

A conclusion is a statement in an argument that indicates of what the arguer is trying to convince the reader/listener. What is the argument trying to prove? There can be only one conclusion in a single argument.

Let $H_1, H_2, .., H_n$ be the n premises and C be the conclusion statement. Then the argument is denoted by

$$\begin{array}{l} H_1 \\ H_2 \\ H_3 \\ \quad \vdots \\ H_n \Rightarrow C \end{array}$$

OR

$H_1, H_2, H_3, \ldots, H_n \Rightarrow C$

OR

$H_1 \wedge H_2 \wedge H_3 \wedge \ldots, \wedge H_n \Rightarrow C$

Inference Theory for statement of calculus

An argument is valid if all the premises are true, then the conclusion CANNOT be false.

$$\begin{aligned} &H_1 \\ &H_2 \\ &H_3 \\ &\ \vdots \\ &H_n \Rightarrow C \end{aligned}$$

An argument is valid If C is true whenever $H_1, H_2, \ldots, H_n$ are true. Otherwise invalid.

Example

Consider the following argument involving propositions (premises):

"If you have a current password, then you can log onto the network."
"You have a current password."
Therefore,
"You can log onto the network."

Determine whether this is a valid argument.

That is, we would like to determine whether the conclusion "You can log onto the network" must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

Let P : You have a current password

Q : You can log onto the network.

Then the premises and conclusion are

$H_1 : P \to Q$

$H_2 : Q$

and $C : Q$

Therefore, the argument is $H_1, H_2 \Rightarrow C$.

To prove that the the argument $H_1, H_2 \Rightarrow C$ is valid.

That is to prove that $H_1 \wedge H_2 \to C$ is a tautology.

Use truth table, to prove the argument is valid.

We can always use a truth table to show that an argument form is valid.We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows. Fortunately, we do not have to resort to truth tables.

Instead, we can first establish the validity of some relatively simple argument forms, called rules of inference.

These rules of inference can be used as building blocks to construct more complicated valid argument forms.

Important rules of inference

We will now introduce the most important rules of inference in propositional logic.

- $P, Q \Rightarrow P \wedge Q$
- $(P \rightarrow Q), (Q \rightarrow R) \Rightarrow P \rightarrow R$
- Modus ponens

  $P, P \rightarrow Q \Rightarrow Q$
- Disjunction Syllogism

  $\neg P, P \vee Q \Rightarrow Q$
- Modus Tollens

  $\neg Q, P \rightarrow Q \Rightarrow \neg P$

Problem

Determine whether the following argument is valid or invalid

1. $H_1 : P \rightarrow Q \quad H_2 : \neg(P \wedge Q) \quad C : \neg P$
2. $H_1 : P \quad H_2 : P \rightarrow Q \quad C : Q$
3. $H_1 : P \quad H_2 : P \rightarrow Q \quad C : P$
4. $H_{:} P \rightarrow Q \quad H_2 : \neg P \quad C : Q$
5. $H_1 : \neg P \quad H_2 : P \leftrightarrow Q \quad C : \neg(P \wedge Q)$
6. $H_1 : P \rightarrow Q \quad H_2 : Q \quad C : P$

Solution

1. $H_1 : P \to Q \quad H_2 : \neg(P \wedge Q) \quad C : \neg P$

P	Q	$P \wedge Q$	H_1	H_2	C	$H_1 \wedge H_2$	$H_1 \wedge H_2 \to C$
T	T	T	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	T	T	T	T
F	F	F	T	T	T	T	T

Since $H_1 \wedge H_2 \to C$ is a tautology, it follows that, this argument is valid.

Rules of Inference

We now describe the process of derivation by which one demonstrates that a particular formula is a valid consequence of a given set of premises. Before we do this, we give two rules of inference which are called rules P and T.

Rule P: Premise may be introduced at any point in the derivation.

Rule T: A formula S introduced in a derivation if S is a tautologically implied by any one or more of the proceeding formula on the derivation.

Problem

If there was a ball game, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no ball game. Show that these statements constitute a valid argument.

Proof.

Let P : There was a ball game

Q : Travelling was difficult

R : They arrived on time

Then $H_1 : P \to Q$, $H_2 : R \to \neg Q$, $H_3 : R$ and $C : \neg P$.

To prove that $H_1 \wedge H_2 \wedge H_3 \Rightarrow C$.

$\square$

Proof Cont...

Steps	Derivation	Rules	Reason
1	R	Rule P	
2	$R \to \neg Q$	Rule P	
3	$\neg Q$	Rule T,[1,2]	$a, a \to b \Rightarrow b$
4	$P \to Q$	Rule P	
5	$\neg P$	Rule T,[3,4]	$\neg b, a \to b \Rightarrow \neg a$

Hence $H_1, H_2, H_3 \Rightarrow C$ is valid.

Problem

Determine whether the conclusion C follows logically from the premises H_1 and H_2: $H_1 : P \to Q$ $H_2 : \neg(P \wedge Q)$ $C : \neg P$

Proof.

Steps	Derivation	Rules	Reason
1	$\neg(P \wedge Q)$	Rule P	
2	$\neg P \vee \neg Q$	Rule T,[1]	$\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b$
3	$\neg Q \vee \neg P$	Rule T,[2]	$a \vee b \Leftrightarrow b \vee a$
4	$Q \to \neg P$	Rule T,[3]	$a \to b \Leftrightarrow \neg a \vee b$
5	$P \to Q$	Rule P	
6	$P \to \neg P$	Rule T,[5,4]	$a \to b, b \to c \Rightarrow a \to c.$
7	$\neg P \vee \neg P$	Rule T,[6]	$a \to b \Leftrightarrow \neg a \vee b$
8	$\neg P$	Rule T,[7]	$a \vee a \Leftrightarrow a$

Problem

Problem

Determine whether the conclusion C follows logically from the premises H_1, H_2 and H_3

1. $H_1 : \neg P \qquad H_2 : P \leftrightarrow Q \qquad C : \neg(P \wedge Q)$
2. $H_1 : P \leftrightarrow Q \qquad H_2 : Q \rightarrow R \qquad H_3 : P \qquad C : R$

Problem

Show that $\neg R \to (S \to \neg T), \neg R \vee W, \neg P \to S, \neg W \Rightarrow T \to P$.

Proof.

Steps	Derivation	Rules	Reason
1	$\neg R \vee W$	Rule P	
2	$R \to W$	Rule T,[1]	$a \to b \Leftrightarrow \neg a \vee b$
3	$\neg W$	Rule P	
4	$\neg R$	Rule T,[3,2]	$\neg b, a \to b \Rightarrow \neg a$
5	$\neg R \to (S \to \neg T)$	Rule P	
6	$S \to \neg T$	Rule T,[4,5]	$a, a \to b \Rightarrow b$

□

Proof Cont...

Steps	Derivation	Rules	Reason
7	$\neg P \to S$	Rule P	
8	$\neg P \to \neg T$	Rule T,[7,6]	$a \to b, b \to c \Rightarrow a \to c$
9	$\neg\neg P \vee \neg T$	Rule T,[8]	$a \to b \Leftrightarrow \neg a \vee b$
10	$P \vee \neg T$	Rule T,[9]	$\neg(\neg a) \Leftrightarrow a$
11	$\neg T \vee P$	Rule T,[10]	$a \vee b \Leftrightarrow b \vee a$
12	$T \to P$	Rule T,[11]	$a \to b \Leftrightarrow \neg a \vee b$

Problem

Show that

$C \vee D, C \vee D \to \neg H, \neg H \to (A \wedge \neg B), (A \wedge \neg B) \to (R \vee S) \Rightarrow R \vee .S$

Proof.

Steps	Derivation	Rules	Reason
1	$C \vee D$	Rule P	
2	$C \vee D \to \neg H$	Rule P	
3	$\neg H$	Rule T,[1,2]	$a, a \to b \Rightarrow b$
4	$\neg H \to (A \wedge \neg B)$	Rule P	
5	$A \wedge \neg B$	Rule T,[3,4]	$a, a \to b \Rightarrow b$
6	$(A \wedge \neg B) \to (R \vee S)$	Rule P	
7	$R \vee S$	Rule T,[5,6]	$a, a \to b \Rightarrow b$

□

Rule CP

Rule CP: If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

Remark

The conclusion is of the form $R \to S$. In such case R is taken as an additional premises and S is derived from the given premises and R.

Problem

Show that $R \to S$ can be derived from the premises $P \to (Q \to S), \neg R \vee P$ and Q.

Proof.

Steps	Derivation	Rules	Reason
1	$\neg R \vee P$	Rule P	
2	R	Rule CP (Additional Premise)	
3	$\neg(\neg R)$	Rule T,[2]	$\neg(\neg a) \Leftrightarrow a$
4	P	Rule T,[3,1]	$\neg a, (a \vee b) \Rightarrow b$
5	$P \to (Q \to S)$	Rule P	
6	$Q \to S$	Rule T,[4,5]	$a, a \to b \Rightarrow b$
7	Q	Rule P	
8	S	Rule T,[7,6]	$a, a \to b \Rightarrow b$

Consistency of premises and indirect Method of Proof

A set of formulas $H_1, H_2, H_3, \ldots, H_n$ is said to be consistent if $H_1 \wedge H_2 \wedge H_3 \wedge, \ldots, \wedge H_n$ has truth values T for some assignment of the truth values to the variables appearing in $H_1, H_2, H_3, \ldots, H_n$.

If for every assignment of the truth values to the variables at least one of the forumula $H_1, H_2, H_3, \ldots, H_n$ is false. Then the set of formulas $H_1, H_2, H_3, \ldots, H_n$ are called inconsistent.

A set of formulas $H_1, H_2, H_3, \ldots, H_n$ are called inconsistent if $H_1 \wedge H_2 \wedge H_3 \wedge, \ldots, \wedge H_n \Rightarrow R \wedge \neg R$ where R is any formula and note that it is a contradiction.

Problem

Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$ using indirect method.

Proof.

Steps	Derivation	Rules	Reason
1	$\neg\neg(P \wedge Q)$	Rule P (Additional Premise)	
2	$P \wedge Q$	Rule T,[1]	$\neg(\neg a) \Leftrightarrow a$
3	P	Rule T,[2]	$a \wedge b \Rightarrow a$
4	$\neg P \wedge \neg Q$	Rule P	
5	$\neg P$	Rule T,[4]	$a \wedge b \Rightarrow a$
6	$P \wedge \neg P$	Rule T,[3,5]	$a, b \Rightarrow a \wedge b$

□

Problem

Show that the following premises are inconsistent

- If Jack misses many classes through illness, then he fails high school.
- If Jack fails high school then he is uneducated
- If Jack reads a lot of books,then he is not uneducated.
- Jack misses many classes trough illness and reads a lot of books.

Proof.

Let E : Jack misses many classes through illness

S : Jack fails high school

A : Jack reads a lot of books

H : Jack is uneducated.

□

Proof Cont...

Therefore,

- $E \rightarrow S$
- $S \rightarrow H$
- $A \rightarrow \neg H$
- $E \wedge A$

To prove that $E \rightarrow S, S \rightarrow H, A \rightarrow \neg H, E \wedge A \Rightarrow R \wedge \neg R$.

Steps	Derivation	Rules	Reason
1	$E \rightarrow S$	Rule P	
2	$S \rightarrow H$	Rule P	
3	$E \rightarrow H$	Rule T,[1,2]	$a \rightarrow b, b \rightarrow c \Rightarrow a \rightarrow c$
4	$A \rightarrow \neg H$	Rule T	
5	$\neg\neg H \rightarrow \neg A$	Rule T,[4]	$a \rightarrow b \Rightarrow \neg b \rightarrow \neg a$

Proof Cont...

Steps	Derivation	Rules	Reason
6	$H \rightarrow \neg A$	Rule T,[5]	$\neg\neg a \Rightarrow a$
7	$E \rightarrow \neg A$	Rule T,[3,6]	$a \rightarrow b, b \rightarrow c \Rightarrow a \rightarrow c$
8	$\neg E \vee \neg A$	Rule T,[7]	$a \rightarrow b \Rightarrow \neg a \vee b$
9	$\neg(E \wedge A)$	Rule T,[8]	$\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b$
10	$E \wedge A$	Rule P	
11	$(E \wedge A) \wedge (\neg(E \wedge A))$	Rule T,[9,10]	$a, b \Rightarrow a \wedge b$

Problem

Prove by an indirect method

- $\neg Q, P \to Q, P \vee R \Rightarrow R$
- $R \to \neg Q, R \vee S, S \to \neg Q, P \to Q \Rightarrow \neg P$

Proof.

For (2)

Steps	Derivation	Rules	Reason
1	$\neg\neg P$	Rule P(Additional Premise)	
2	P	Rule T,[1]	
3	$P \to Q$	Rule P	
4	Q	Rule T,[2,3]	$a, a \to b \Rightarrow b$
5	$S \to \neg Q$	Rule P	

Proof Cont...

Steps	Derivation	Rules	Reason
6	$Q \to \neg S$	Rule T,[5]	$a \to b \Rightarrow \neg b \to \neg a$
7	$\neg S$	Rule T,[4,6]	$a, a \to b \Rightarrow b$
8	$R \vee S$	Rule P	
9	$\neg R \to S$	Rule T,[8]	$a \to b \Leftrightarrow \neg a \vee b$
10	$\neg S \to R$	Rule T,[9]	$a \to b \Rightarrow \neg b \to \neg a$
11	R	Rule T,[7,10]	$a, a \to b \Rightarrow b$
12	$R \to \neg Q$	Rule P	
13	$\neg Q$	Rule T,[11,12]	$a, a \to b \Rightarrow b$
14	$Q \wedge \neg Q$	Rule T,[4,13]	$a, b \Rightarrow a \wedge b$

Thank you