Module-2-Predicate Calculus

Dr. Nalliah M

Assistant Professor

Department of Mathematics

School of Advanced Sciences

Vellore Institute of Technology

Vellore,Tamil Nadu,India.

nalliah.moviri@vit.ac.in

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Propositional logic, studied in Module-1, cannot adequately express the meaning of all statements in mathematics and in natural language.

For example, suppose that we know that

"Every computer connected to the university network is functioning properly."

No rules of propositional logic allow us to conclude the truth of the statement.

"MATH3 is functioning properly,"

where MATH3 is one of the computers connected to the university network.

Likewise, we cannot use the rules of propositional logic to conclude from the statement

"CS2 is under attack by an intruder,"

where CS2 is a computer on the university network, to conclude the truth of

"There is a computer on the university network that is under attack by an intruder."

Predicate

Statements involving variables, such as "$x > 3$," "$x = y + 3$,"

"$x + y = z$,"

and "computer x is under attack by an intruder,"

and "computer x is functioning properly,"

are often found in mathematical assertions, in computer programs, and in system specifications.

These statements are neither true nor false when the values of the variables are not specified.

In this section, we will discuss the ways that propositions can be produced from such statements.

The statement "x is greater than 3" has two parts.

The first part, the variable x, is the subject of the statement.

The second partthe predicate, "is greater than 3"refers to a property that the subject of the statement can have.

We can denote the statement "x is greater than 3" by $P(x)$,

where P denotes the predicate "is greater than 3" and x is the variable.

The statement $P(x)$ is also said to be the value of the propositional function P at x.

Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

We can also have statements that involve more than one variable. For instance, consider the statement "$x = y + 3$." We can denote this statement by $Q(x, y)$, where x and y are variables and Q is the predicate. When values are assigned to the variables x and y, the statement $Q(x, y)$ has a truth value.

Similarly, we can let $R(x, y, z)$ denote the statement "$\mathrm{x} + \mathrm{y} = \mathrm{z}$." When values are assigned to the variables x, y, and z, this statement has a truth value.

Example

- Let P(x) denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

  We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$," is true. However, $P(2)$, which is the statement "$2 > 3$," is false.

Example

- Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?
  To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false. The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true.
- Let $R(x, y, z)$ denote the statement "$x + y = z$." What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?
  The proposition $R(1, 2, 3)$ is obtained by setting $x = 1, y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement "$1 + 2 = 3$," which is true. Also note that $R(0, 0, 1)$, which is the statement "$0 + 0 = 1$," is false.

In general, a statement involving the n variables $(x_1, x_2, \ldots, x_n)$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called an n-place predicate or a n-ary predicate. Propositional functions occur in computer programs, as the following Example demonstrates.

Example

Consider the statement if $x > 0$ then $x := x + 1$.

When this statement is encountered in a program, the value of the variable x at that point in the execution of the program is inserted into $P(x)$, which is "$x > 0$." If $P(x)$ is true for this value of x, the assignment statement $x := x + 1$ is executed, so the value of x is increased by 1. If $P(x)$ is false for this value of x, the assignment statement is not executed, so the value of x is not changed.

Quantifiers

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value.

However, there is another importantway, called quantification, to create a proposition from a propositional function.

Quantification expresses the extent to which a predicate is true over a range of elements.

In English, the words all, some, many, none, and few are used in quantifications.

Types of Quantification

We will focus on two types of quantification here:

- Univeal Quantification

  Universal quantification, which tells us that a predicate is true for every element under consideration, and

- Existential Quantification

  Existential quantification, which tells us that there is one or more element under consideration for which the predicate is true.

The area of logic that deals with predicates and quantifiers is called the predicate calculus.

THE UNIVERSAL QUANTIFIER

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the domain of discourse (or the universe of discourse), often just referred to as the domain. Such a

statement is expressed using universal quantification.

The universal quantification of $P(x)$ for a particular domain is the proposition that asserts that $P(x)$ is true for all values of x in this domain.

THE UNIVERSAL QUANTIFIER

Note that the domain specifies the possible values of the variable x.

The meaning of the universal quantification of $P(x)$ changes when we change the domain.

The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

Definition

The universal quantification of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x\ P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall x\ P(x)$ as "for all x $P(x)$" or "for every x $P(x)$." An element for which $P(x)$ is false is called a counterexample of $\forall x\ P(x)$.

Example

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x\ P(x)$, where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the quantification $\forall x\ P(x)$, is true.

Remark

Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. Note that if the domain is empty, then $\forall x\ P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.

Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "all of," "for each," "given any," "for arbitrary," "for each," and "for any."

Remark

It is best to avoid using "for any x" because it is often ambiguous as to whether "any" means "every" or "some." In some cases, "any" is unambiguous, such as when it is used in negatives, for example, "there is not any reason to avoid studying."

Note: When all the elements in the domain can be listed–say, $x_1, x_2, \ldots, x_n$–it follows that the universal quantification $\forall x\ P(x)$ is the same as the conjunction $P(x_1) \land P(x_2) \land, \ldots, \land P(x_n)$, because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

THE EXISTENTIAL QUANTIFIER

Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification.

With existential quantification, we form a proposition that is true if and only if $P(x)$ is true for at least one value of x in the domain.

Definition

The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$." We use the notation $\exists x\ P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

Without specifying the domain, the statement $\exists x\ P(x)$ has no meaning. Besides the phrase "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is."

The existential quantification $\exists x\ P(x)$ is read as "There is an x such that $P(x)$," "There is at least one x such that $P(x)$,"

or

"For some x $P(x)$."

Remark

Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x\ Q(x)$ is false whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element x in the domain for which $Q(x)$ is true.

Note: When all the elements in the domain can be listed–say, $x_1, x_2, \ldots, x_n$–it follows that the existential quantification $\exists x\ P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee, \ldots, \vee P(x_n)$, because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

THE UNIQUENESS QUANTIFIER

We have now introduced universal and existential quantifiers. These are the most important quantifiers in mathematics and computer science. However, there is no limitation on the number of different quantifiers we can define, such as "there are exactly two," "there are no more than three," "there are at least 100," and so on.

Of these other quantifiers, the one that is most often seen is the uniqueness quantifier, denoted by $\exists!$ or $\exists_1$.

The notation $\exists! x\ P(x)$ or $[\exists_1 x\ P(x)]$ states "There exists a unique x such that $P(x)$ is true."

(Other phrases for uniqueness quantification include "there is exactly one" and "there is one and only one.")

For instance, $\exists! x\ (x - 1 = 0)$, where the domain is the set of real numbers, states that there is a unique real number x such that $x - 1 = 0$. This is a true statement, as $x = 1$ is the unique real number such that $x - 1 = 0$.

Observe that we can use quantifiers and propositional logic to express uniqueness, so the uniqueness quantifier can be avoided.

Generally, it is best to stick with existential and universal quantifiers so that rules of inference for these quantifiers can be used.

Quantifiers with Restricted Domains

The restriction of a universal quantification is the same as the universal quantification of a conditional statement.

For instance, $\forall x < 0 \ (x^2 > 0)$ is another way of expressing $(\forall x)(x^2 > 0, x < 0)$.

On the other hand, the restriction of an existential quantification is the same as the existential quantification of a conjunction.

For instance, $\exists z > 0 \ (z^2 = 2)$ is another way of expressing $(\exists z)(z > 0 \wedge z^2 = 2)$.

Precedence of Quantifiers:

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

For example, $\forall x\ P(x) \vee Q(x)$ is the disjunction of $\forall x\ P(x)$ and $Q(x)$.

pause In other words, it means $(\forall x\ P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

Binding Variables

When a quantifier is used on the variable x, we say that this occurrence of the variable is bound.

An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

The part of a logical expression to which a quantifier is applied is called the scope of this quantifier.

Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.

Example

In the statement $\exists x(x + y = 1)$, *the variable* x *is bound by the existential quantification* $\exists$, *but*

the variable y *is free because it is not bound by a quantifier and*

no value is assigned to this variable. Therefore, in the statement $\exists x(x + y = 1)$, x *is bound, but* y *is free.*

In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x\ R(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not to the rest of the statement.

Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$. That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$.

Observe that we could have written our statement using two different variables x and y, as $\exists x(P(x) \wedge Q(x)) \vee \forall y \; R(x)$, because the scopes of the two quantifiers do not overlap.

In common usage, the same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap.

Translating from English into Logical Expressions

Example

Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

"For every student in this class, that student has studied calculus."

Next, we introduce a variable x so that our statement becomes

"For every student x in this class, x has studied calculus."

We introduce $C(x)$, which is the statement "x has studied calculus."

Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x\ C(x)$.

Now, we discuss the different approaches for expressing the same statement of the above example using predicates and quantifiers.

- We change the domain to consist of all people We will need to express our statement as

  "For every person x, if person x is a student in this class then x has studied calculus."

  If $S(x)$ represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \to C(x))$.

  [Caution! Our statement cannot be expressed as $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus!]

- We are interested in the background of people in subjects

  We may prefer to use the two-variable quantifier $Q(x, y)$ for the statement

  "student x has studied subject y."

  Then we would replace $C(x)$ by $Q(x, calculus)$ in both approaches to obtain

  $\forall x Q(x, calculus)$ or $\forall x(S(x) \to Q(x, calculus))$.

Problem

Express the the following statements

1. *"Some student in this class has visited Mexico" and*
2. *"Every student in this class has visited either Canada or Mexico"*

using predicates and quantifiers. [Hint:1. $\exists x\ M(x)$ 2.$\forall x(C(x) \lor M(x))$.]

Example

Use predicates and quantifiers to express the system specifications

1. "Every mail message larger than one megabyte will be compressed"

   and

2. "If a user is active, at least one network link will be available."

For(1), let $S(m, y)$: "Mail message m is larger than y megabytes and let $C(m)$: "Mail message m will be compressed."

Therefore, $\forall x(S(m, 1) \rightarrow C(m))$.

For(2),[Hint:$\exists u\ A(u) \rightarrow \exists n S(n, available)$.]

Example

Use predicates and quantifiers to express the following statements

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

We can express these statements as:

1. $\forall x(P(x) \rightarrow Q(x))$
2. $\exists x(P(x) \wedge \neg R(x))$
3. $\exists x(Q(x) \wedge \neg R(x))$.

Notice that from the above example, the second statement cannot be written as $\exists x(P(x) \to \neg R(x))$. The reason is that $P(x) \to \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \wedge \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as $\exists x(Q(x) \to \neg R(x))$.

Problem

Use predicates and quantifiers to express the following statements

1. "All hummingbirds are richly colored."
2. "No large birds live on honey."
3. "Birds that do not live on honey are dull in color."
4. "Hummingbirds are small. "

Solution

Let $P(x), Q(x), R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x), Q(x), R(x)$, and $S(x)$.

1. $\forall x(P(x) \rightarrow S(x))$
2. $\neg\exists x(Q(x) \wedge R(x))$
3. $\forall x(\neg R(x) \rightarrow \neg S(x))$
4. $\forall x(P(x) \rightarrow \neg Q(x))$.

Problem

Symbolize the expression "All the world loves a lover."

Solution

Let $P(x)$: x is a person

$L(x)$: x is a lover

$R(x, y)$: x loves y

The required expression is $\forall x(P(x) \rightarrow \forall y(P(y) \wedge L(y) \rightarrow R(x, y)))$.

Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $A \Leftrightarrow B$ indicate that two statements A and B involving predicates and quantifiers are logically equivalent.

Equivalent formula:

- $\exists x(A(x) \vee B(x)) \Leftrightarrow \exists x A(x) \vee \exists x B(x)$
- $\forall x(A(x) \wedge B(x)) \Leftrightarrow \forall x A(x) \wedge \forall x B(x)$
- $\neg(\forall x(A(x)) \Leftrightarrow (\exists x)\neg A(x)$
- $\neg(\exists x(A(x)) \Leftrightarrow (\forall x)\neg A(x)$

Implication formula:

- $\forall x A(x) \vee \forall x B(x) \Rightarrow \forall x(A(x) \vee B(x))$
- $\exists x(A(x) \wedge B(x)) \Rightarrow \exists x A(x) \wedge \exists x B(x)$

Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions.We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

There are four rules such as

1. Universal Specification (US)
2. Existential Specification (ES)
3. Universal Generalization (UG)
4. Existential Generalization (EG)

Universal Specification (US):

From $\forall x A(x)$ one can conclude $A(y)$.

Example

Universal Specification is used when we conclude from the statement "All women are wise" that

"Lisa is wise," where Lisa is a member of the domain of all women.

Existential Specification (ES):

From $\exists x A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of derivation. This requirements can easily be met by choosing a new variable each time when ES is used. (The conditions of ES are more restictive than ordinarily required, but they do not affect the possibility of deriving any conclusion)

Existential Generalization (EG):

From $A(x)$ one can conclude $\exists y A(y)$.

Rules of Inference

Universal Generalization (UG):

From $A(x)$ one can conclude $\forall y A(y)$ provided that x is not free in any of given premises and provided that if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in $A(x)$.

Problem

Show that $\exists x M(x)$ follows logically from the premises $\forall x(H(x) \to M(x))$ and $\exists x H(x)$.

Solution

Step	Derivation	Rule	Reason
1	$\forall x(H(x) \to M(x))$	Rule P	
2	$H(y) \to M(y)$	Rule US,[1]	
3	$\exists x H(x)$	Rule P	
4	$H(y)$	Rule ES,[3]	
5	$M(y)$	Rule T,[4,2]	$a, a \to b \Rightarrow b$
6	$\exists x M(x)$	Rule EG,[5]	

Problem

Prove that $\exists x(P(x) \wedge Q(x)) \Rightarrow \exists x P(x) \wedge \exists x Q(x)$.

Solution

Step	Derivation	Rule	Reason
1	$\exists x(P(x) \wedge Q(x))$	Rule P	
2	$P(y) \wedge Q(y)$	Rule ES,[1]	
3	$P(y)$	Rule T,[2]	$a \wedge b \Rightarrow a$
4	$Q(y)$	Rule T,[2]	$a \wedge b \Rightarrow b$
5	$\exists x P(x)$	Rule EG,[3]	
6	$\exists x Q(x)$	Rule EG,[4]	
7	$\exists x P(x) \wedge \exists x Q(x)$	Rule T,[5,6]	$a, b \Rightarrow a \wedge b$.

Problem

Prove that $\exists x P(x) \wedge \exists x Q(x) \nRightarrow \exists x(P(x) \wedge Q(x))$.

Solution

Step	Derivation	Rule	Reason
1	$\exists x P(x) \wedge \exists x Q(x)$	Rule P	
2	$\exists x P(x)$	Rule T,[1]	$a \wedge b \Rightarrow a$
3	$\exists x Q(x)$	Rule T,[1]	$a \wedge b \Rightarrow b$
4	$P(y)$	Rule ES,[2]	
5	$Q(z)$	Rule ES,[3]	

Here we get the contradiction in the step 5.

Problem

Show that from

1. $\exists x(F(x) \wedge S(x)) \rightarrow \forall y(M(y) \rightarrow W(y))$
2. $\exists y(M(y) \wedge \neg W(y))$.

Conclusion $\forall x(F(x) \rightarrow \neg S(x))$.

Solution

Step	Derivation	Rule	Reason
1	$\exists y(M(y) \wedge \neg W(y))$	Rule P	
2	$M(z) \wedge \neg W(z)$	Rule ES,[1]	
3	$\neg(M(z) \rightarrow W(z))$	Rule T,[2]	$\neg(a \rightarrow b) \Leftrightarrow a \wedge \neg b$
4	$(\exists y)\neg(M(y) \rightarrow W(y))$	Rule EG,[3]	

Solution Cont...

Step	Derivation	Rule	Reason
5	$\neg(\forall y(M(y) \rightarrow W(y)))$	Rule T,[4]	$\neg(\forall x(A(x)) \Leftrightarrow (\exists x)\neg A(x)$
6	$\exists x(F(x) \wedge S(x))$		
	$\rightarrow \forall y(M(y) \rightarrow W(y))$	Rule P	
7	$\neg(\exists x(F(x) \wedge S(x)))$	Rule T,[5,6]	$\neg b, a \rightarrow b \Rightarrow \neg a$
8	$\forall x(\neg[F(x) \wedge S(x)])$	Rule T,[7]	$\neg(\exists x(A(x)) \Leftrightarrow (\forall x)\neg A(x)$
9	$\neg[F(y) \wedge S(y)]$	Rule US,[8]	
10	$\neg[F(y) \wedge \neg(\neg S(y))]$	Rule T,[9]	$\neg(\neg a) \Leftrightarrow a$
11	$\neg F(y) \vee \neg S(y)$	Rule T,[10]	$\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b$
12	$F(y) \rightarrow \neg S(y)$	Rule T,[11]	$a \rightarrow b \Leftrightarrow \neg a \vee b$
13	$(\forall x)[F(x) \rightarrow \neg S(x)]$	Rule UG,[12]	

Problem

Show that $\forall x(P(x) \vee Q(x)) \Rightarrow \forall x P(x) \vee \exists x Q(x)$ using indirect method.

Solution

Step	Derivation	Rules	Reason
1	$\neg[\forall x P(x) \vee \exists x Q(x)]$	Rule P(AP)	
2	$\neg[\forall x P(x)] \wedge \neg[\exists x Q(x)]$	Rule [1]	$\neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b$
3	$\neg[\forall x P(x)]$	Rule T,[2]	$a \wedge b \Rightarrow a$
4	$\neg[\exists x Q(x)]$	Rule T,[2]	$a \wedge b \Rightarrow b$
5	$\exists x(\neg P(x))$	Rule T,[3]	$\neg(\forall x(A(x)) \Leftrightarrow (\exists x)\neg A(x)$
6	$\forall x(\neg Q(x))$	Rule T,[4]	$\neg(\exists x(A(x)) \Leftrightarrow (\forall x)\neg A(x)$

Solution Cont...

Step	Derivation	Rules	Reason
7	$\neg P(y)$	Rule ES,[5]	
8	$\neg Q(y)$	Rule US,[6]	
9	$\neg P(y) \wedge \neg Q(y)$	Rule T,[7,8]	$a, b \Rightarrow a \wedge b$
10	$\neg[P(y) \vee Q(y)]$	Rule T,[9]	$\neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b$
11	$\forall x(P(x) \vee Q(x))$	Rule P	
12	$P(y) \vee Q(y)$	Rule US,[11]	
13	$[P(y) \vee Q(y)] \wedge$		
	$\neg[P(y) \vee Q(y)]$	RuleT,[12,10]	$a, b \rightarrow a \wedge b.$

Problem

Show that the premises "A student in this class has not read the book, and Everyone in this class passed the first exam imply the conclusion Someone who passed the first exam has not read the book."

Nested Quantifiers

In the above section, we avoided nested quantifiers, where one quantifier is within the scope of another, such as $\forall x \exists y(x + y = 0)$.

Note that everything within the scope of a quantifier can be thought of as a propositional function.

For Example,

$\forall x \exists y(x + y = 0)$.

is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.

Example

Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution

The quantification $\exists y \forall x Q(x, y)$ denotes the proposition

"There is a real number y such that for every real number x, $Q(x, y)$."

No matter what value of y is chosen, there is only one value of x for which $x + y = 0$.

Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false.

The quantification $\forall x \exists y Q(x, y)$ denotes the proposition

"For every real number x there is a real number y such that $Q(x, y)$."

Given a real number x, there is a real number y such that $x + y = 0$ namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true.

Example

Let $P(x, y)$ be the statement "$x + y = y + x$." What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

Answer: $\forall x \forall y P(x, y)$ is true and $\forall y \forall x P(x, y)$ is true.

Example

Translate the statement The sum of two positive integers is always positive into a logical expression.

Solution

Rewrite the given statement as follows

"For every two positive integers, the sum of these integers is positive."

We can express this as $\forall x \forall y(x + y > 0)$, where the domain for both variables consists of all positive integers.

Or

Rewrite the given statement as follows

"For every two integers, if these integers are both positive, then the sum of these integers is positive."

Next, we introduce the variables x and y to obtain "For all positive integers x and y, $x + y$ is positive."

$\forall x \forall y(x + y > 0)$,where $(x > 0) \wedge (y > 0)$.

Example

Translate the statement $\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$ into English, where $C(x)$ is "x has a computer," $F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution

The statement says that for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.

In other words, every student in your school has a computer or has a friend who has a computer.

Negating Nested Quantifiers

Example

Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.

Solution

By successively applying De Morgan's laws for quantifiers,

$$\begin{aligned} \neg[\forall x(\exists y(xy = 1))] &\Leftrightarrow \exists x[\neg(\exists y(xy = 1))] \\ &\Leftrightarrow \exists x \forall y[\neg(xy = 1)]. \end{aligned}$$

Because $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y(xy \neq 1)$.

Problem

Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world." Also, Express the negation of the statement.

Answer: The statement is $\exists w \forall a \exists f R(w, f, a)$, where $R(w, f, a)$ is "w has taken f on a."

Problem

Show that the conclusion $C : \neg P(a, b)$ follows logically from $\forall x \forall y[P(x, y) \rightarrow W(x, y)]$ and $\neg W(a, b)$.

Solution

Step	Derivation	Rules	Reason
1	$\forall x \forall y[P(x, y) \rightarrow W(x, y)]$	Rule P	
2	$\forall y[P(a, y) \rightarrow W(a, y)]$	Rule US,[1]	
3	$P(a, b) \rightarrow W(a, b)]$	Rule US,[2]	
4	$\neg W(a, b)$.	Rule P	
5	$\neg P(a, b)$	Rule T,[4,3]	$\neg q, p \rightarrow q \Rightarrow \neg p$

Thank you