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Homework 1

1. Solve Kleinberg and Tardos, Chapter 1, Exercise 1. (5pts)

   Question: True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (*m, w*) such that *m* is ranked first on the preference list of *w* and *w* is ranked first on the preference list of *m*.

   Answer:

   False. We will prove this by assuming these are the following preferences of the two men – *m* and *m′*, and 2 women – *w* and *w′*:

   - *m* prefers *w* over *w′*
   - *m′* prefers *w′* over *w*
   - *w* prefers *m′* over *m*
   - *w′* prefers *m′* over *m*

   In the execution, we begin with man *m* proposing woman *w*. Since *w* prefers *m′*, she rejects *m*. Then, *m′* proposes *w′* and she accepts it because *m′* is the best valid partner for *w′*, and vice versa. But *w* prefers *m′*, hence leaving us with an instability. Thus, we proved that the result is not stable.

2. Solve Kleinberg and Tardos, Chapter 1, Exercise 2. (5pts)

   Question: True or false? Consider an instance of the Stable Matching Problem in which there exists a man *m* and a woman *w* such that *m* is ranked first on the preference list of *w* and *w* is ranked first on the preference list of *m*. Then in every stable matching S for this instance, the pair (*m, w*) belongs to *S*.

   Answer:

   True. We will prove this by assuming these are the following preferences of the two men – *m* and *m′*, and 2 women – *w* and *w′*:

   - *m* prefers *w* over *w′*
   - *m′* prefers *w′* over *w*
   - *w* prefers *m* over *m′*
   - *w′* prefers *m* over *m′*

   In the execution, *m* begins by proposing *w*. Since *m* is on the top of the list for *w*, she accepts *m* and (*m, w*) becomes a pair. Now, *m′* proposes his first preferred woman, *w′*. Since *w′* is free, (*m′*, *w′*) becomes a pair. Thus, we observe a Stable Matching in this case, and a pair (*m, w*) is obtained. The case is observed the same when woman proposes first instead of men.
   We can take another example to prove it again.

   - *m* prefers *w* over *w′*
   - *m′* prefers *w* over *w′*
   - *w* prefers *m* over *m′*
   - *w′* prefers *m* over *m′*

Here, we begin with m' proposing w. Since w is free, (m, w') becomes a pair. Now, m comes in the picture and proposes w. Since w prefers m over m', m' gets free and (m, w) becomes a pair. Then, m' proposes the next valid partner, w'. Since w' has no proposals yet, she accepts m' and a pair (m', w') gets created. Thus, again, we find that (m, w) is observed in the final stable matching S.

3. Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation. (5pts)

   Question: For some $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

   Answer: True.
   We can prove it by assuming women propose to men.

   Assume there are 2 men – m and m', and 2 women – w and w'. Here are the preferences of the group:

   - m prefers w over w'
   - m' prefers w over w'
   - w prefers m' over m
   - w' prefers m over m'

   We assume that woman w starts proposing to man m'. Since m' also prefers her back, a pair (m', w) is formed. Now, w' will propose her best valid partner m. Since m is free, m accepts the partner w', and a pair (m, w') is formed. Here, G-S Algorithm halts and we receive the pairs (m, w') and (m', w). Here, every woman is matched with their most preferred man ($w \rightarrow m'$ and $w' \rightarrow m$), but m' does not get the best valid partner.

   Now, we assume that men start proposing.

   We assume the following table that has the preferences:

m_1	**m_2**	**m_3**	**w_1**	**w_2**	**w_3**
w_1	w_1	w_2	m_3	m_2	m_1
w_2	w_2	w_1	m_2	m_1	m_2
w_3	w_3	w_3	m_1	m_3	m_3

   When the execution begins with m_1, (m_1, w_1) make a pair. m_2 proposes w_1 and (m_2, w_1) make a pair, leaving m_1 free. Now, m_1 proposes w_2 and (m_1, w_2) make a pair. m_3 proposes w_2 but to no avail. m_3 then proposes w1 and (m_3, w_1) make a pair, rendering m_2 free. Now, m_2 proposes w_2 and (m_2, w_2) make a pair. Lastly, m_1 proposes his last preference, w_3 and make a pair (m_1, w_3). Thus, the final pairs are:

   $$(m_1, w_3), (m_2, w_2), (m_3, w_1)$$

   Hence, we observe that regardless of who proposes first, there exists a set of preferences where every woman is matched with their most preferred man.

4. Four students, a, b, c, and d are rooming in a dormitory. Each student ranks the others in strict order of preference. A *roommate matching* is defined as a partition of the students into two groups of two roommates each. A roommate matching is *stable* if no two students who are not roommates prefer each other over their roommate.

   Does a stable roommate matching always exist? If yes, give a proof. Otherwise, give an example of roommate preferences where no stable roommate matching exists. (8pts)

   Answer:

   There are no stable roommate matching that exists.

There can be 3 instances for roommates matching:

i. *ad* and *bc*
ii. *ac* and *bd*
iii. *ab* and *cd*

For this the following preferences would be:

a	*b*	*c*	*d*
b	*c*	*a*	*d*
c	*a*	*b*	*d*

In this case, roommate ***d***'s preference would not matter since for the remaining roommates, *d* is their last preference.

We look to the above-mentioned instances of roommate matching and observe the following instabilities:

i. *a* prefers *c* over *d* and *c* prefers *a* over *b*. Hence, this is an instability.
ii. *a* prefers *b* over *c* and *b* prefers *a* over *d*. Hence, this is an instability.
iii. *b* prefers *c* over *a* and *c* prefers *b* over *d*. Hence, this is an instability.

Hence, there are no stable roommate matching.

5. Solve Kleinberg and Tardos, Chapter 1, Exercise 4. (15pts)

Algorithm:

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Initially all students s ∈ S and h ∈ H are free
While there is a hospital h who has positions available and has not reached out to every student
 Choose such a hospital h
 Let s be the highest-ranked student in hospital h's preference list to whom h has not yet reached
 out
 If s has no offers, then
        (s, h) become a pair
 Else s has an offer from h'
        If s prefers offer from h' over h then
               h remains free
        Else s prefers h to h'
               (s, h) become engaged
               h' becomes free
        Endif
 Endif
End while
Return the set S of pairs
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Answer:

We take an example of 4 students – *S1, S2, S3, S4,* and 2 hospitals – *H1 and H2,* where each hospital has two slots open.

The preferences for the students will be:

- *S1 -> H1, H2*
- *S2 -> H2, H1*

- *S3 -> H2, H1*
- *S4 -> H1, H2*

The preference for the hospitals would be:

- *H1 -> S1, S3, S2, S4*
- *H2 -> S4, S1, S2, S3*

The algorithm starts running with H1 offering the position to its first preference, S1. Again, H1 has one slot left, and the position is offered to student S3. Now, H1's positions are filled and H2 begins by offering the position to S4. Student S4 accepts the offer. Now, S1 is being offered a position at H2, but since S1 prefers H1's position over H2, S1 rejects the offer. Hospital H2 then reaches out to Student S2, which accepts the offer. The result obtained is hospital H1 having the student S1 and S3 and hospital H2 having students S2 and S4.

The stable matching pairs do not contain any instability, and this can be proved. For the first type of instability, we require any student that has not been assigned to any hospital. Since that is not the case here, this instability cannot exist. For the second type of instability, one of the students, S4 is assigned a hospital, H2 for which it was not the preference. But since H1 prefers student S3 over S4, this instability also cannot exist.

Hence, there is always a stable assignment of students to hospitals with the given algorithm.

6. Solve Kleinberg and Tardos, Chapter 1, Exercise 8. (10pts)

Answer:

We can resolve the question by giving an example of a set of preference lists for which if we switch the partners' preferences, the woman can improve the partner. We consider the following set of preferences with 3 men and 3 women, and the 3rd woman's preferences switched:

m_1	m_2	m_3	w_1	w_2	w_3	w_3'
w_3	w_1	w_3	m_1	m_1	m_2	m_2
w_1	w_3	w_1	m_2	m_2	m_1	m_3
w_2	w_2	w_2	m_3	m_3	m_3	m_1

We begin the algorithm's execution by m_1 first proposing to w_3 and (m_1, w_3) make a pair. m_2 proposes w_1 and they (m_2, w_1) make a pair. m_3 proposes w_3, but w_3 is engaged to m_1, hence m_3 moves on to propose w_2 and thus makes a pair (m_3, w_2). Thus, the execution ends, and we got the following pairs:

$$(m_1, w_3), (m_2, w_1), (m_3, w_2)$$

Now, when we switch w_3's preferences, the first step remains the same, and we get the same pair - (m_1, w_3'). m_2 proposes w_1 and we get the pair (m_2, w_1). Now, m_3 proposes to w_3' and since m_3 has a higher preference than m_1, (m_3, w_3') make a pair and m_1 is free. Now, m_1 proposes his next best preference i.e., w_1. Since w_1 meets her best valid partner, she leaves m_2 and thus (m_1, w_1) make a pair. The next step is m_2 proposing w_3' and w_3' finds her best valid partner. So, they create a pair (m_2, w_3'). After this, m_3 is free and goes on to propose w_1. But w_1 rejects him since she found her best preferred partner – m_1. Finally, m_3 proposes w_2 and they become a pair. The execution halts, and the final pairs are as follows:

$$(m_1, w_1), (m_2, w_3'), (m_3, w_2)$$

Thus, as we could observe here, in the above preference list, w_3 got the second-best preferred partner – m_1.

But in the false preference, she got the best partner for herself i.e., w_3'.

7. Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. (5pts)

   For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm, every man is matched with their most preferred woman.

   Answer: The given statement is True.

   Assume that there are 2 men – m and m', and 2 women – w and w', along with their preferences listed below:

   - m prefers w over w'
   - m' prefers w' over w
   - w prefers m over m'
   - w' prefers m over m'

   In the execution, we assume that when m starts by proposing to w, she accepts m, and a pair (m, w) has been created. Now, m' will move on to propose w'. Since w' is free, they both create a pair (m', w'). As we can see, the execution ends, and both the men are matched with their most preferred woman. Hence, there can exist a set of preferences where every man is matched with their most preferred woman.

8. Consider a stable marriage problem where the set of men is given by $M = m_1, m_2, ..., m_N$ and the set of women is $W = w_1, w_2, ..., w_N$. Consider their preference lists to have the following properties:

   $$\forall w_i \in W: w_i \text{ prefers } m_i \text{ over } m_j \quad \forall j > i$$

   $$\forall m_i \in M: m_i \text{ prefers } w_i \text{ over } w_j \quad \forall j > i$$

   Prove that a unique stable matching exists for this problem. Note: the $\forall$ symbol means "for all". (12pts)

   Answer:

   According to the properties stated above, and assuming three men m and three women w, here are the following preferences we assume for men:

   - m_1 -> w_1, w_2, w_3
   - m_2 -> w_2, w_3, w_1
   - m_3 -> w_3, w_1, w_2

   For women:

   - w_1 -> m_1, m_3, m_2
   - w_2 -> m_2, m_1, m_3
   - w_3 -> m_3, m_2, m_1

   As the execution begins, m_1 gets their first best valid partner – w_1. m_2 and m_3 also are in a similar situation – finding the best valid partner themselves i.e., w_2 and w_3. Thus, we find the following resulting pairs:

   $$(m_1, w_1), (m_2, w_2), (m_3, w_3)$$

   Thus, a stable matching exists for this problem.