HW5 Soln

1) a) $-1-y_2-e^{y_1}=0$

$y_2=-1-e^{y_1} \Rightarrow$

$$y_1^2+y_2(e^{y_1}-1)=0$$
$$y_1^2+(-1-e^{y_1})(e^{y_1}-1)=0$$
$$y_1^2+1-e^{2y_1}=0$$
$$\Rightarrow y_1=0$$

$\Rightarrow y_2=-1-e^0=-1-1=-2$

$\cancel{y_1}+\sin(y_3)=0 \qquad y_3=n\pi \quad n\in\mathbb{Z}.$

0

Equilibrium Values are $\begin{bmatrix}0\\-2\\n\pi\end{bmatrix}$ for $n\in\mathbb{Z}$.

b) $ay_1 + by_2 = 0$
$cy_1 + dy_2 = 0$ $\Rightarrow$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

as $ad - bc \neq 0$, A^{-1} exists.

So $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ← only equilibrium value.

c) $\left[\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} a & b & 0 \\ 0 & d - \frac{bc}{a} & 0 \end{array}\right]$

$=0$ as $ad - bc = 0$.

so $ay_1 + by_2 = 0$
$y_1 = \frac{-b}{a} y_2$

$\Rightarrow \begin{bmatrix} -\frac{bt}{a} \\ t \end{bmatrix}, \ t \in \mathbb{R}$

2) Matrix should be $\begin{pmatrix} -2 & 1 \\ 0 & 4 \end{pmatrix}$

$$y = c_1 e^{4t} \begin{bmatrix} 1 \\ 6 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a) $y = 2e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ b) $y = 2.01 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ c) $y = \frac{e^{4t}}{600} \begin{bmatrix} 1 \\ 6 \end{bmatrix} + \frac{1199}{2000} e^{-2t} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

d) Plug in $t = 10$

a) $y(10) \approx \begin{bmatrix} 4.12 \times 10^{-9} \\ 0 \end{bmatrix}$ b) $y(10) \approx \begin{bmatrix} 4.14 \times 10^{-9} \\ 0 \end{bmatrix}$ c) $y(10) \approx \begin{bmatrix} 3.92 \times 10^{14} \\ 2.35 \times 10^{15} \end{bmatrix}$

Huge difference.

Since a), c) start close but change drastically, this is unstable.

3) a) $x(t) \equiv 1$ is soln for I.C. $x(0) = 1$.

Change I.C. a little $\Rightarrow$ $x(0) = 1.01$ $\Rightarrow$ $x(t) = \frac{e^t}{-0.01 + e^t}$ which is always close to 1.

So this looks to be $\boxed{\text{stable}}$.

b) $x(0) = 0$.

$x(0) = 0.01$ $\Rightarrow$ $x(t) = \frac{e^t}{99 + e^t}$ which quickly moves away from 0 $\Rightarrow$ $\boxed{\text{Unstable}}$

c) $x(0) = 1$

$x(0) = 1.01$ $\Rightarrow$ $x(t) = \frac{1}{-0.099e^t + 1}$ which goes to 0 $\Rightarrow$ $\boxed{\text{Unstable}}$

$x(0) = 0$

$x(0) = 0.01$ $\Rightarrow$ $x(t) = \frac{1}{99e^t + 1}$ which is close to 0 $\Rightarrow$ $\boxed{\text{Stable}}$

4) a) "$\Rightarrow$" Let $\vec{y}(t)$ be a stable solution to $\vec{y}' = A\vec{y} + \vec{g}(t)$.

Let $\vec{\varphi}(t)$ be a solution to $\vec{y}' = A\vec{y}$ with $\vec{\varphi}(0)$ close to 0. $\Leftarrow$ we need to check if $\vec{\varphi}(t)$ is always close to 0.

Consider $\vec{y}(t) - \vec{\varphi}(t)$, which solves $\vec{y}' = A\vec{y} + \vec{g}(t)$

as $(\vec{y} - \vec{\varphi})' = A\vec{y} + \vec{g}(t) - A\vec{\varphi} = A(\vec{y} - \vec{\varphi}) + \vec{g}(t)$.

$(\vec{y}(0) - \vec{\varphi}(0))$ is close to $\vec{y}(0)$ as $\vec{\varphi}(0)$ small.

As $\vec{y}$ stable, $\vec{y}(t) - \vec{\varphi}(t)$ is always close to $\vec{y}(t)$.

$\Rightarrow \vec{\varphi}(t)$ is always close to 0 $\Rightarrow \vec{0}$ is a stable sol'n to $\vec{y}' = A\vec{y}$.

"$\Leftarrow$" Let $\vec{0}$ be a stable sol'n to $\vec{y}' = A\vec{y}$

Let $\vec{y}(t), \vec{\varphi}(t)$ be two soln to $\vec{y}' = A\vec{y} + \vec{g}(t)$ with $\vec{\varphi}(0) - \vec{y}(0)$ small.

$\vec{\varphi}(t) - \vec{y}(t)$ solves homogeneous, as $(y - \varphi)' = A\vec{y} + \vec{g}(t) - (A\vec{\varphi} + \vec{g}(t))$

$= A(\vec{y} - \vec{\varphi}) + 0$.

Show $\vec{y}(t) - \vec{\varphi}(t)$ always small.

$\Rightarrow$ As $\vec{0}$ is stable, and $\vec{\varphi}(0) - \vec{y}(0)$ close to 0,

$\vec{\varphi}(t) - \vec{y}(t)$ always small $\Rightarrow \vec{y}(t)$ stable soln of $\vec{y}' = A\vec{y} + \vec{g}(t)$

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b) $\vec{y}' = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \Rightarrow \lambda = -1 \pm i\sqrt{2}$

All eigenvalues have negative real part, so by (a), all solutions to

$$\vec{y}' = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \vec{y} + \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

are stable.

5) a) Equilibrium : $\begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \pm 1 \end{bmatrix}$.

$$A = \begin{bmatrix} 2y_1 & 2y_2 \\ 2y_2 & 2y_1 \end{bmatrix}$$

i) at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda = 2$ so unstable

ii) at $\begin{bmatrix} 0 \\ \pm 1 \end{bmatrix}$: $A = \begin{bmatrix} 0 & \pm 2 \\ \pm 2 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_1 = 2 \leftarrow \text{so unstable} \\ \lambda_2 = -2 \end{array}$

iii) at $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$: $A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \lambda = -2$ so stable

b) Equil: $y_1 + y_2 = n\pi \qquad (n \in \mathbb{Z})$

$y_2 = n\pi \qquad \Rightarrow \quad y_1 + y_1^3 = 0 \quad \Rightarrow \quad y_1 = 0$

$$\begin{bmatrix} 0 \\ n\pi \end{bmatrix} \qquad \text{Jacobian} \quad A = \begin{bmatrix} \sec^2(y_1+y_2) & \sec^2(y_1+y_2) \\ 1+3y_1^2 & 0 \end{bmatrix}$$

at $\begin{bmatrix} 0 \\ n\pi \end{bmatrix}$: $A = \begin{bmatrix} \sec^2(n\pi) & \sec^2(n\pi) \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$

As $\frac{1+\sqrt{5}}{2} > 0$, all equil. values are unstable.