MATH 155 – Midterm Exam

April 30, 2021, 10:30am-11:35am, 65 minutes

- You may use your textbook, your notes, and a calculator, but you may **not** consult with any other internet resources or other people.
- Show all of your work for full credit. Partial credit will be given.
- Upload you exam to Gradescope within the allotted time (by 11:40am PDT). Please indicate to Gradescope which answers are on which of your uploaded pages.
- The exam is worth 40 points in total. Questions are on the next two pages.

Problem	Possible Points	Score
1	10	
2	10	
3	10	
4	10	
Total	40	

1. Consider the following matrix A,

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

You are given additional information about A:

- $\det(A - \lambda I) = (4-\lambda)^2(\lambda+2)$
- A has two eigenvectors:

  (a) $\lambda = 4$ has eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

  (b) $\lambda = -2$ has eigenvector $\begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}$.

Use this information to help you find the general solution to $\vec{y}\,' = A\vec{y}$.

2. (True or False). Answer the following questions true or false. Provide justification to receive full credit.

   (a) Consider a 3×3 constant coefficient system of ODEs $\vec{y}\,' = A\vec{y}$. True or false, it is possible for the following vector $\vec{y}$ to be a solution of this system:

   $$\vec{y} = e^t \begin{bmatrix} 2t \\ 0 \\ t^3 - t^2 + 1 \end{bmatrix}$$

   (b) Let $z_1(t)$ be a solution to $y'' + \sin(t)y = t^2e^t$. Let $z_2(t)$ be a solution to $y'' + \sin(t)y = 2$. True or false, $z_1(t) + 3z_2(t)$ is a solution to $y'' + \sin(t)y = t^2e^t + 6$.

3. Consider the second order ODE
$$y'' - y' = 0\ .$$
Note that there is no y term in this ODE, so it may be helpful to think of the ODE as
$$y'' - y' + 0 \cdot y = 0\ .$$
   (a) Rewrite this second order ODE as a linear system of first order ODEs of the form $\vec{y}\,' = A\vec{y}$.
   (b) Using your matrix from part (a), compute e^{At}.
   (c) **Optional for up to 3 extra credit points**: Use your answer from part (b) to solve the initial value problem
$$\vec{y}\,' = A\vec{y} + \begin{bmatrix} e^t \\ 0 \end{bmatrix} \qquad ; \qquad \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

4. Let $\vec{\phi}(t)$ solve the initial value problem: $\vec{y}\,' = A\vec{y}$ with $\vec{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   Let $\vec{u}_1(t)$ solve the initial value problem: $\vec{y}\,' = A\vec{y} + \vec{g}(t)$ with $\vec{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   Let $\vec{u}_2(t)$ solve the initial value problem: $\vec{y}\,' = A\vec{y} + \vec{g}(t)$ with $\vec{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

   Define $\vec{u}(t) = \vec{u}_2(t) - \vec{u}_1(t)$. Prove that $X(t)$ given by
$$X(t) = \begin{bmatrix} | & | \\ \vec{\phi}(t) & \vec{u}(t) \\ | & | \end{bmatrix}$$
   is a fundamental matrix solution for $\vec{y}\,' = A\vec{y}$.