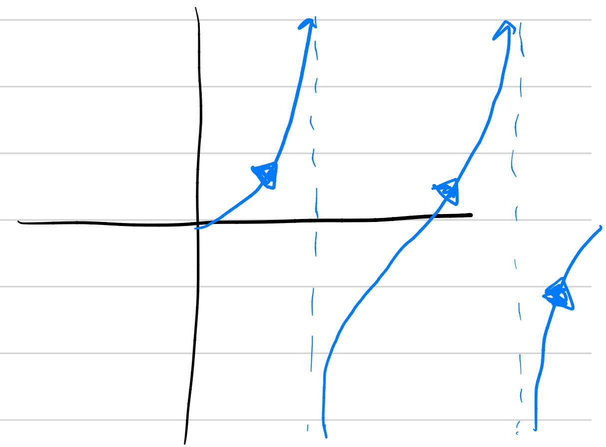
HW6 Soln

1) We will assume $t \geq 0$.

a) $x(t) = \tan(t)$, $y(t) = \tan(\tan(t))$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} u \\ \tan(u) \end{bmatrix}$$ where $u = \tan(t)$.



b) $x(t) = t + 4$, $y(t) = 3t^2 + 1$

let $u = t+4$. $\Rightarrow y = Au^2 + Bu + C$

$$= A(t+4)^2 + B(t+4) + C$$

$$= At^2 + (8A+B)t + (16A+4B+C) = 3t^2 + 1$$

$\Rightarrow A = 3$

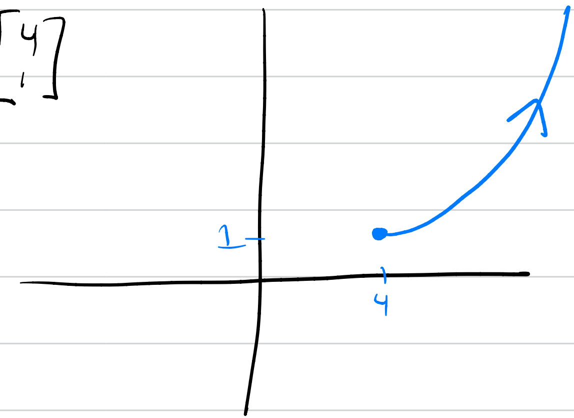
$B = -24$

$C = 49$

$y = 3u^2 - 24u + 49$

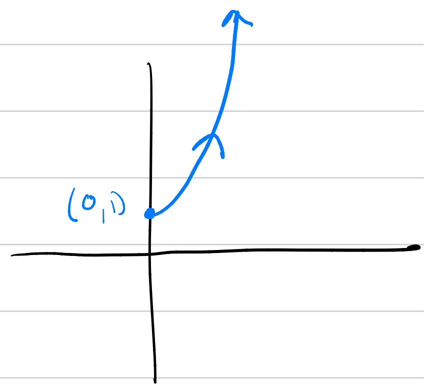
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ 3u^2 - 24u + 49 \end{bmatrix}$$

$t = 0 \Rightarrow \begin{bmatrix} 4 \\ 1 \end{bmatrix}$



c) $x(t) = \ln(1+t)$; $y(t) = e^t$

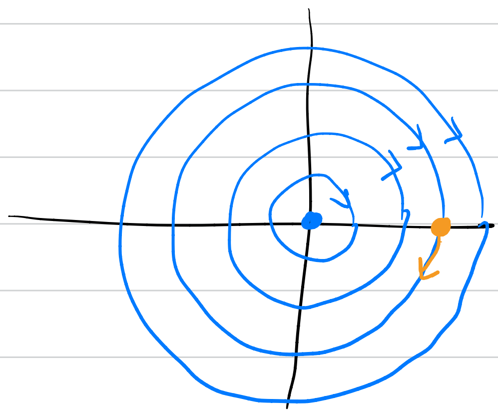
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ e^{e^u - 1} \end{bmatrix}$$



2) a) $x' = y(1+x^2+y^2)$
$y' = -x(1+x^2+y^2)$

Equil Points: $(0,0)$ only.

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \int y\,dy = \int -x\,dx \Rightarrow y^2 = x^2 + C \Rightarrow x^2+y^2 = C$$



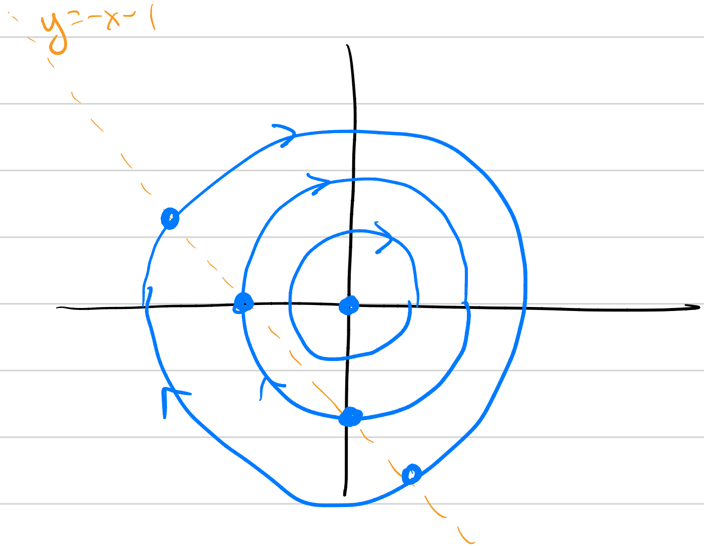
To get direction, look at a point such as $(x,0)$ where $x>0$ (•)

Then $y' = -x(1+x^2+y^2) < 0$

so y moves to

b) Same orbits, but new equilibrium points when $1+x+y=0$

$\Rightarrow y=-x-1$



c) $\frac{dy}{dx} = \frac{x+xy^2}{4y} = \frac{x(1+y^2)}{4y} \Rightarrow \int \frac{4y}{1+y^2}\,dy = \int x\,dx$

$2\ln(1+y^2) = \frac{1}{2}x^2 + C$

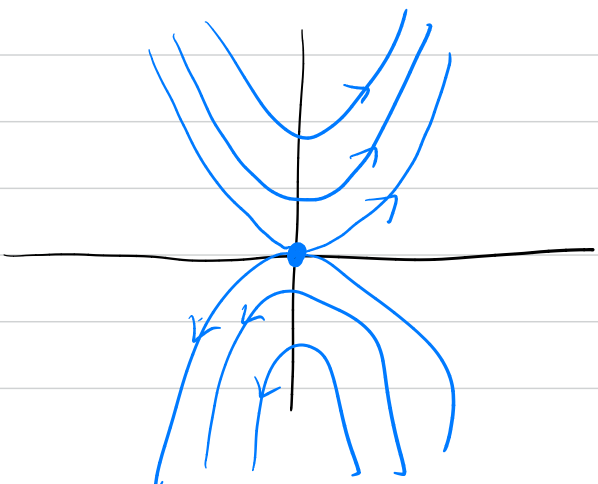
$$2\ln(1+y^2) = \frac{1}{2}x^2 + C$$

$$\ln(1+y^2) = \frac{1}{4}x^2 + C$$

$$1+y^2 = e^{\frac{1}{4}x^2 + C}$$

$$y = \pm\sqrt{Ae^{\frac{1}{4}x^2} - 1}$$

$A = e^C > 0.$



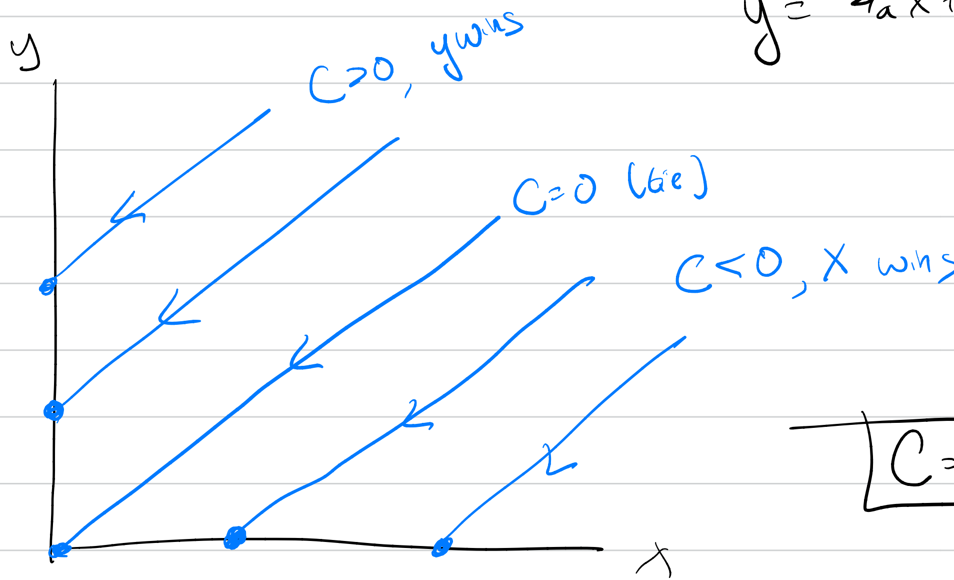
$x' = 4y$ so when $y>0$, x moves $\rightarrow$

$y<0$, x moves $\leftarrow$

3) a) Equil: Anywhere either $x=0$ or $y=0$.

$$\frac{dy}{dx} = \frac{-bxy}{-axy} = \frac{b}{a} \Rightarrow \int dy = \int \frac{b}{a}\,dx$$

$$y = b/a\, x + C$$



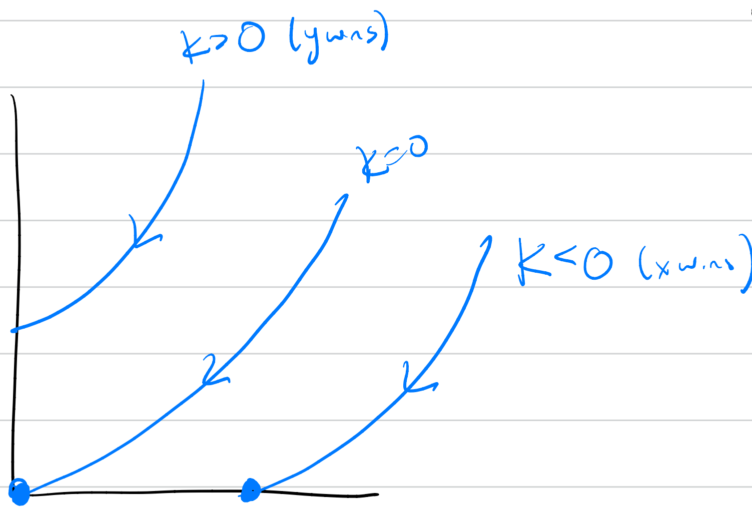
$$\boxed{C = ay_0 - bx_0}$$

b) Equil-Points: Whenever $y=0$.

$$\frac{dy}{dx} = \frac{-bxy - cy}{-ay} = \frac{-bx - c}{-a}$$

$$\int dy = \int \frac{b}{a}x + c \; dx$$

$$y = \frac{b}{2a}x^2 + cx + K$$



Upward parabola, y intercept of K.

$$K = y_0 - \frac{b}{2a}x_0^2 - cx_0$$

4) a) $\frac{dy}{dx} = \frac{cxy + dy}{axy + bx} = \frac{y(cx + d)}{x(ay + b)}$ $\quad \frac{ay + b}{y} \, dy = \frac{cx + d}{x} \, dx$

$$\int a + \frac{b}{y} \, dy = \int c + \frac{d}{x} \, dx$$

$$\boxed{ay + b\ln|y| = cx + d\ln|x| + K}$$

b) $\boxed{x = 0}$ (y-axis)

$x'(t) = 0$

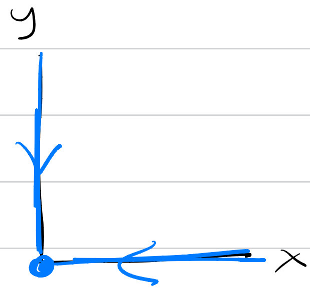
$y'(t) = -dy \Rightarrow y(t) = y_0 e^{-dt}$ So x doesn't change and y moves down to 0.

$\boxed{y = 0}$ (x-axis)

$x'(t) = -bx \Rightarrow x(t) = x_0 e^{-bt}$

$y'(t) = 0$

$\}$ Same but flipped

So 

This is different than #3 as in those, points along axes were either equil points or part of some other orbit (not axis).

c) Since orbits can't cross (Property 1), no orbit in 1^{st} quadrant can actually reach $x=0$ or $y=0$.

(Orbit could go to origin in theory, but that's a tie).

5) $\frac{d}{dt}(x^2+y^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$$= 2x(-1-y+x^2) + 2y(x+xy)$$

$$= -2x - \cancel{2xy} + 2x^3 + \cancel{2xy} + 2xy^2$$

$$= 2x(x^2+y^2-1)$$

$\frac{d}{dt}(x^2+y^2)$ measures how the radius (squared) changes in time.

If $x^2+y^2=1$, the radius doesn't change (so could move around unit circle).

If $x^2+y^2<1$, $\frac{d}{dt}(x^2+y^2)<0$ so moves closer to origin.

So unit circle is made up or orbits that cannot be intersected.

Anything inside unit circle must stay there!