HW8 Soln

1) $x = z,\ y = z'$

$$\frac{dx}{dt} = y \quad ; \quad \frac{dy}{dt} = z'' = -z - z'\ln\left(z^2 + 4z'^2\right)$$

$$= -x - y\ln\left(x^2 + 4y^2\right)$$

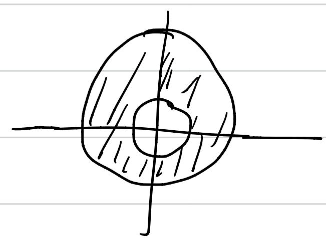
Equilibrium Points.

$\boxed{y=0} \Rightarrow -x = 0 \Rightarrow \boxed{x=0}$

doesn't work if set $x=0$ first (not fully continuous at origin due to ln)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let's look for an annulus that works:



$$\frac{d}{dt}(x^2+y^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$= 2xy + 2y\left(-x - y\ln(x^2+4y^2)\right)$$

$$= 2xy - 2xy - 2y^2\ln(x^2+4y^2)$$

$$= \underbrace{-2y^2}_{\leq 0}\ln(x^2+4y^2)$$

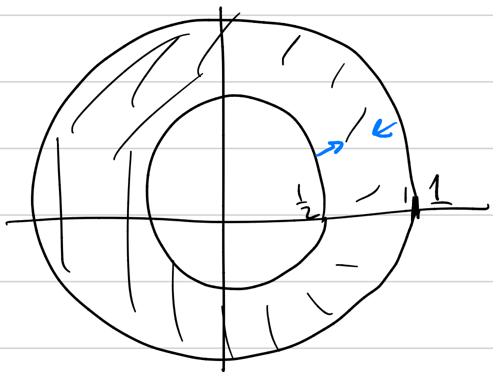
Analyze $\ln(x^2+4y^2)$, when is it positive? Negative?

$\ln(x^2+4y^2) < 0$ for $x^2+4y^2 < 1 \Rightarrow \underbrace{x^2+y^2+3y^2}_{} < 1$

let $\boxed{x^2+y^2 < \frac{1}{4}}$

$\ln(x^2+4y^2) > 0$ for $x^2+4y^2 > 1$

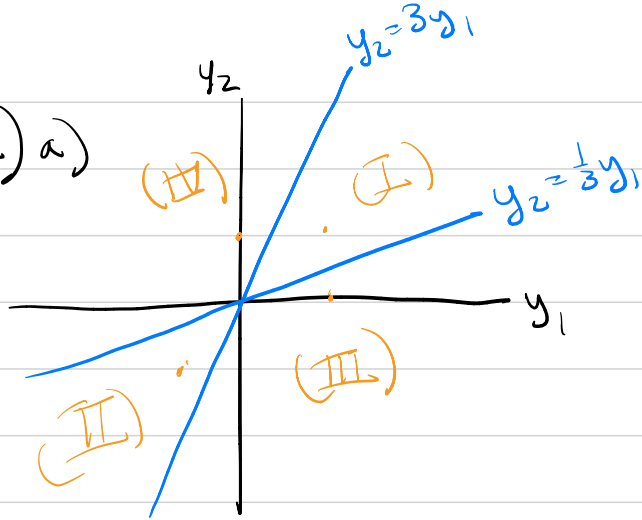
$x^2+y^2>1$ works for this.



So inside this region, no equil points, and solutions stay inside

By P-B Thm, converges to a periodic solution.

2) a)



Test point in each region!

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3y_1 + y_2 \\ y_1 - 3y_2 \end{bmatrix}$$

$(1,0) \to y_1' < 0$
$y_2' > 0$

$(0,1) \to y_1' > 0, \; y_2' < 0$

$(1,1) \to y_1' < 0, \; y_2' < 0$

$(-1,-1) \to y_1' > 0, \; y_2' > 0$

b) Suppose solution in I goes to III.

Then $y_2(t)$ has a local min at t^*, as y_2 dec. then inc.

Can compute $y_2''(t^*) = \frac{d}{dt} y_2' = \frac{d}{dt}[y_1 - 3y_2] = \left.\frac{dy_1}{dt} - 3\frac{dy_2}{dt}\right|_{t=t^*}$

$$= \underbrace{\frac{dy_1}{dt}}_{<0} - 3 \cdot 0$$

By 2nd deriv test, y_2 has a local max at t^*. Contradiction! I can't go to III?

Suppose solution in I goes to IV

Then $y_1(t)$ has a local min at t^*, as y_1 dec then inc.

$y_1''(t^*) = \left.\frac{dy_2}{dt} - 3\frac{dy_1}{dt}\right|_{t=t^*} = \frac{dy_2}{dt}(t^*) < 0$. Same contradiction!

Same reasoning for $\mathrm{II} \to \mathrm{III}$ or $\mathrm{II} \to \mathrm{IV}$.

Now, suppose we have solution that either stays in I or II.
That solution is monotonic, and bounded

in I: $x, y \geq 0$

in II: $x, y \leq 0$.

Thus, must converge to equil point $\to$ only option is $(0,0)$.

c) Suppose we have solution that either stays in III or IV.

This solution is monotonic and bounded

III	IV
$y_1 \geq 3y_1^{(0)}$	$y_1 \leq 3y_1^{(0)}$
$y_2 \leq \frac{1}{3}y_1^{(0)}$	$y_2 \geq \frac{1}{3}y_1^{(0)}$

Must go to ~~e~~ only equil point $(0,0)$.

3) $\dfrac{dx}{dt} = x - xy \qquad \dfrac{dy}{dt} = y - xy - 2y^2$

a) $x(\cancel{1}-y)$

$x = 0 \qquad y = 1$

$y - 2y^2 = 0 \qquad 1 - x - 2 = 0$

$y(1-2y) = 0 \qquad -x = 1$

$y = 0,\ y = \frac{1}{2} \qquad \boxed{x = -1}$

$\boxed{(0,0)} \quad \boxed{(0,1/2)} \quad \cancel{\boxed{(1,-1)}}$

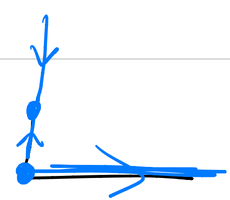
Can ignore as not in 1^{st} Q.

$\boxed{x\text{-axis}}$ $y = 0 \Rightarrow \dfrac{dy}{dt} = 0,\ \dfrac{dx}{dt} = x > 0$ in 1^{st} Q.

$\boxed{y\text{-axis}}$ $x = 0 \Rightarrow \dfrac{dx}{dt} = 0 \quad \dfrac{dy}{dt} = y - 2y^2$

> 0 if $y < 1/2$

< 0 if $y > 1/2$



b) $\frac{dx}{dt} = x - xy$

$x(1-y)$

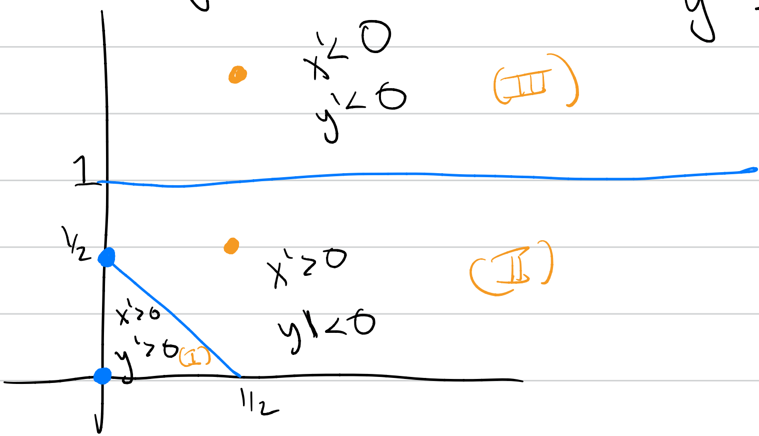
$y = 1$

$\frac{dy}{dt} = y - xy - 2y^2$

$y(1 - x - 2y)$

$2y = 1 - x$

$y = \frac{1}{2} - \frac{1}{2}x$



Test Points $(1/2, 1/2)$

$x' = \frac{1}{2} - \frac{1}{4} > 0$

$y' = \frac{1}{2} - \frac{1}{4} - 2 \cdot \frac{1}{4} < 0$

$(\frac{1}{2}, 2)$

$x' = \frac{1}{2} - \frac{1}{2} \cdot 2 < 0$

$y' = 2 - \frac{1}{2} \cdot 2 - 2 \cdot 4 < 0$

$(\frac{1}{16}, \frac{1}{18})$

$x' = \frac{1}{16} - \frac{1}{16^2} > 0$

$y' = \frac{1}{16} - \frac{1}{16^2} - 2 \cdot \frac{1}{16^2} > 0$

c) Suppose orbit stays in region (I).

As monotonic and bounded, must converge to equil point.

So must go to $(0,0)$ or $(0,\frac{1}{2})$.

But this is impossible as $x' > 0$ so x always increasing. $(x \not\to 0)$.

So orbit must leave (I).

Likewise, suppose orbit stays in (III).

As monotonic and bounded, must converge to equil point.

But there are no equil points!

So orbit must leave (III).

d) i) Suppose orbit moves from region II to region I.

At $t = t^*$, intersects line $y = \frac{1}{2} - \frac{1}{2}x$.

$y'(t^*) = 0$, and as y' going from negative to positive, y has a local min at t^*.

Look at $y''(t^*) = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}\left(y - xy - 2y^2\right)$

$$= \left.\frac{dy}{dt} - \frac{dx}{dt}y - x\frac{dy}{dt} - 4y\frac{dy}{dt}\right|_{t=t^*}$$

$$= 0 - \underbrace{\frac{dx}{dt}}_{>0} \cdot \underbrace{y}_{>0} - \underbrace{x}_{>0} \cdot 0 - 4y \cdot 0$$

$= -$ positive # < 0.

By 2^{nd} deriv test, y has a local max at t^*. Contradiction!

Orbit can't cross from II to I.

ii) Suppose orbit crosses II to III.

At t^*, $x'(t^*) = 0$

1^{st} deriv shows local max
2^{nd} deriv shows local min. $\Big\}$ as in part (i).

Thus orbits in II stay in II.

e) Every orbit by c), d) moves to II and stays in II.

As $x' > 0$ in II, $x(t)$ is always increasing.

If $x(t) \not\to \infty$, then $x(t)$ is bounded and would converge to equil point.

But no equil point, so $x(t) \to \infty$.

f) Same as e). $y' < 0$ in II, as no equil point $y(t) \to 0$.

4) a) $\det\left(\begin{bmatrix} 1-\lambda & \varepsilon \\ \varepsilon & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - \varepsilon^2 = 0$

$\lambda^2 - 2\lambda + (1-\varepsilon^2)$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1-\varepsilon^2)}}{2} = 1 \pm \sqrt{\cancel{1} - \cancel{1} + \varepsilon^2}$$

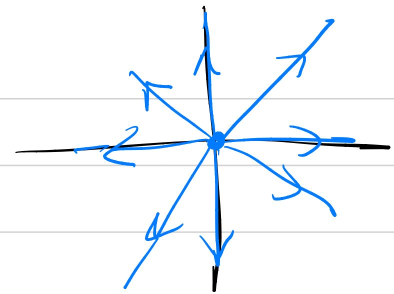
$$= 1 \pm |\varepsilon|$$

$\underbrace{\varepsilon = 1, -1}_{\text{gives } \lambda = 0.}$

$\underbrace{\varepsilon = 0}_{\text{only 1 } \lambda.}$

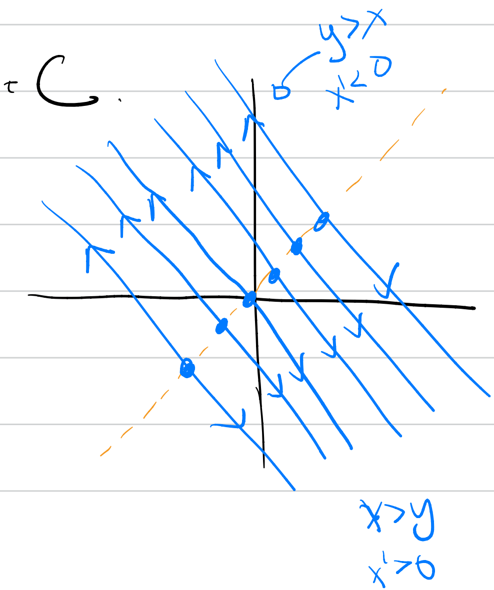
$\boxed{\varepsilon = 0}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda = 1$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$\boxed{\varepsilon = -1}$ $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - y \\ -x + y \end{bmatrix}$ Equil when $x = y$.

$$\frac{dy}{dx} = \frac{-y+x}{x-y} = -1 \Rightarrow dy = -dx \Rightarrow y = -x + C.$$



$\boxed{a=1}$

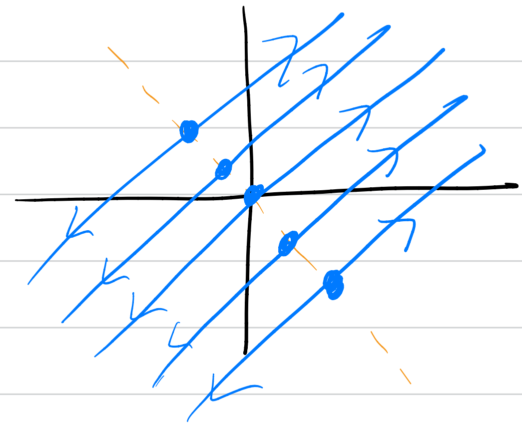
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$x' = x+y$

$y' = x+y$

$\frac{dy}{dx} = 1 \qquad y = x + C$

Equil when $y = -x$



b) $\begin{bmatrix} 0 & 2 \\ -2 & \varepsilon \end{bmatrix}$ $\begin{vmatrix} -\lambda & 2 \\ -2 & \varepsilon-\lambda \end{vmatrix} = -\lambda(\varepsilon-\lambda)+4$

$= \lambda^2 - \lambda\varepsilon + 4$

$$\lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 16}}{2}$$

$\boxed{\varepsilon = 4}$
$\boxed{\varepsilon = -4}$ } Only 1 λ

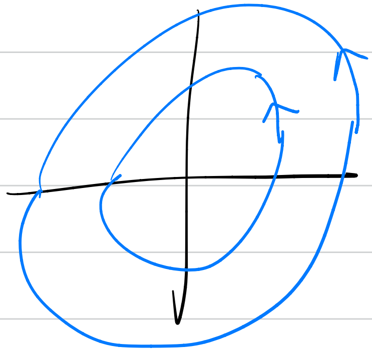
$\boxed{\varepsilon = 0}$ } λ has 0 real part.

$x>0$

Check $(x,0)$

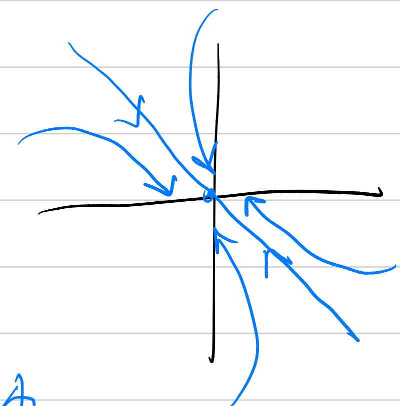
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2y \\ 2x \end{bmatrix} \qquad y'>0.$$

$\boxed{\varepsilon=0}$ $\lambda = \pm 2i$



$\boxed{\varepsilon=-4}$ $\lambda=-2$ $\begin{bmatrix} 0 & 2 \\ -2 & -4 \end{bmatrix}$

Only 1 eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$



$\boxed{\varepsilon=4}$ $\begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}$ $\lambda=2$

One eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$