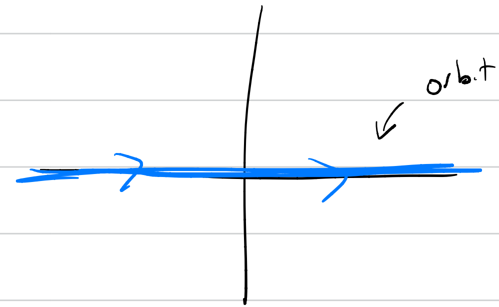
HW7 Soln

1) a) Will show x-axis is an orbit so nothing can cross it.

let $\boxed{y=0}$

$dx/dt = 1 + x^2$ → always > 0

$dy/dt = 0 + \tan(0) = 0$ → no y movement



b) Will show $y=x$ is made up of orbits that cannot be crossed.

Equil Points:

$$\left.\begin{array}{ll} y+x^2=0 & y=-x^2 \\ x+y^2=0 & x=-y^2 \end{array}\right\} (0,0),\ (-1,-1)$$

↳ $x=-x^2$

$\boxed{y=x}$

$$\frac{dx}{dt} = x + x^2 = x^3$$

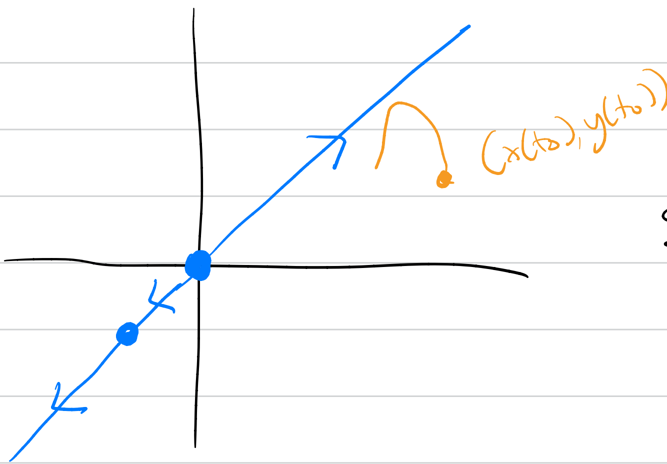
$$\frac{dy}{dt} = x + x^2 = x^3$$

So if on line $y=x$, orbit moves in direction $\begin{bmatrix} x^3 \\ x^3 \end{bmatrix}$

which is still on line!

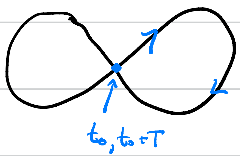
$x>0$, $x^3>0$

$x<0$, $x^3<0$



So if $x(t_0) \neq y(t_0)$, orbit can never intersect $y=x$.

c) No, this can't happen.



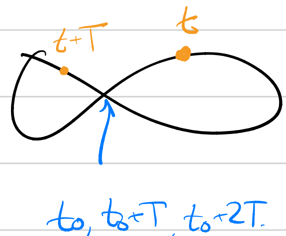
Let's say we start at intersection point at t_0.

We return to this point at $t = t_0 + T$. (traveling along right loop only)

So $\varphi(t_0) = \varphi(t_0 + T)$.

By Property 2, φ is periodic with $\varphi(t) \equiv \varphi(t+T)$ $\forall t$.

But this isn't true: Take $t_0 < t < t_0 + T$



$\varphi(t)$ is on RHS,

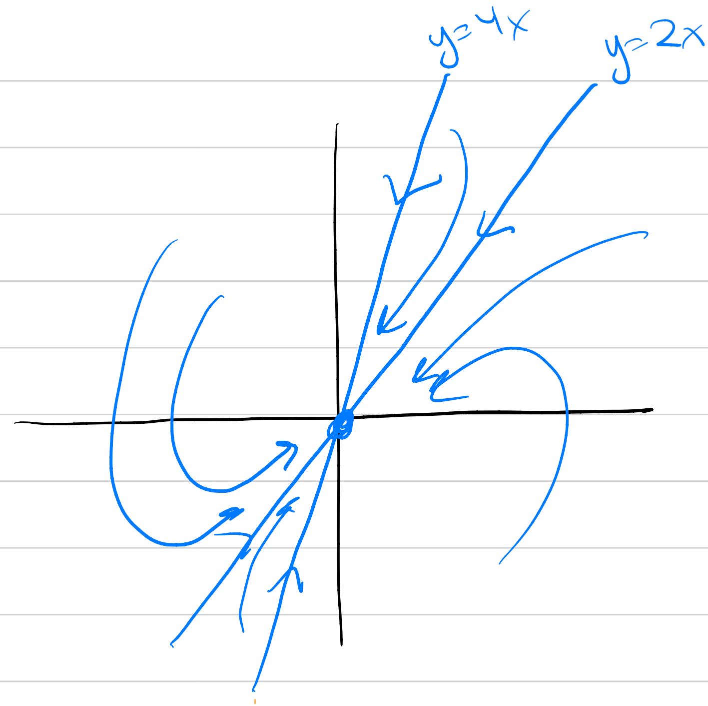
$\varphi(t+T)$ must be on LHS.

as $t_0 + T < t + T < t_0 + 2T$

$\therefore$ Contradiction.

2) a) $\lambda_1 = -2$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

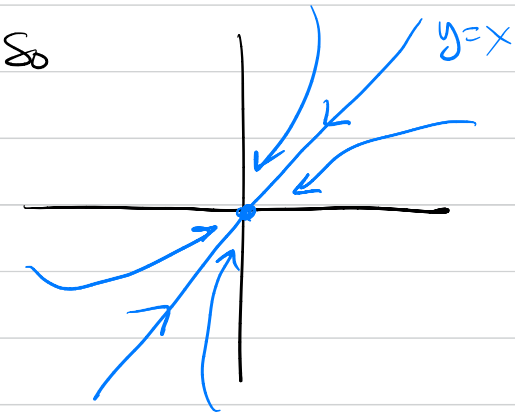
$\lambda_2 = -4$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$



$\lambda_2 < \lambda_1 < 0$

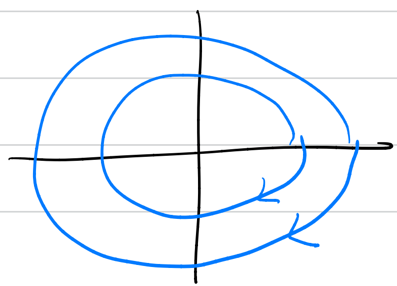
b) $\lambda_1 = -5$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\leftarrow$ only one eigenvector.

So



c) $\lambda = \pm i$

$(\alpha = 0)$



At $(x,0)$ with $x > 0$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x + 1(0) \\ -5x - 2(0) \end{bmatrix} = \begin{bmatrix} 2x \\ -5x \end{bmatrix}$$

$\uparrow$ y goes $\downarrow$

3) a) $x = z$ $\Rightarrow$ $\frac{dx}{dt} = z' = y$

$y = z'$ $\frac{dy}{dt} = z'' = -\frac{c}{m} z' - \frac{k}{m} z = -\frac{c}{m} y - \frac{k}{m} x$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

b) $\det(A - \lambda I) = -\lambda\left(-\lambda - \frac{c}{m}\right) - \left(-k/m\right) = 0$

$\lambda^2 + \frac{c}{m}\lambda + k/m = 0$

$$\lambda = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4k/m}}{2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$$\frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Case I: $c^2 - 4km > 0$ $\Rightarrow$ Then as $c > \sqrt{c^2 - 4km}$

both eigenvalues < 0.

Case II: $c^2 - 4km = 0$

$\lambda = -\frac{c}{2m}$: 1 or 2 eigenvectors?

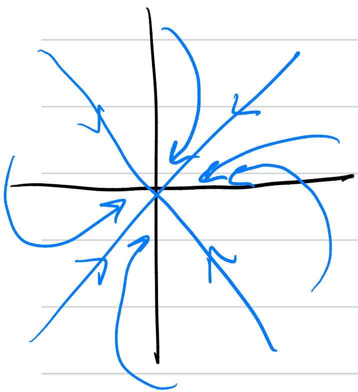
$$\begin{bmatrix} c/2m & 1 \\ -k/m & -c/2m \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -c/2m \end{bmatrix}$$

Only 1.

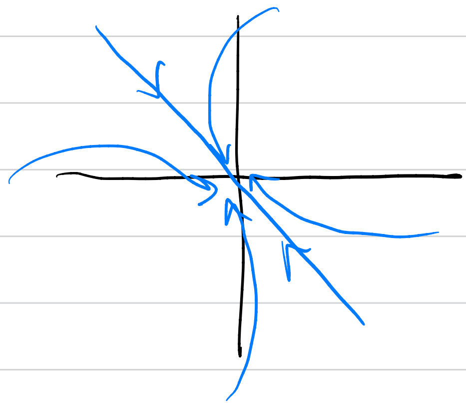
Case III: $c^2 - 4km < 0$

$$\lambda = \underbrace{-\frac{c}{2m}}_{\text{negative real part.}} \pm i\frac{\sqrt{4km - c^2}}{2m}$$

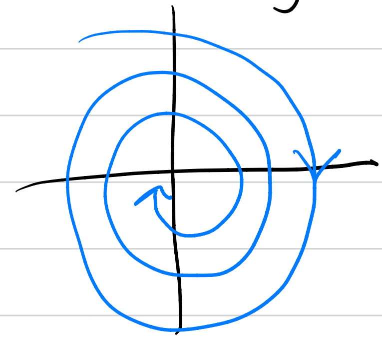
Case I: (Overdamped)



Case II: (Crit. Damped)



Case III (Underdamped)



(Look at $(x,0)$, $x>0$)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ -k/m\, x \end{bmatrix} \leftarrow <0$$

c) Recall that motion is $z(t) = x(t)$ (ignore $y(t)$)

I: Can cross y-axis once, spring snaps back to place after 1st oscillation

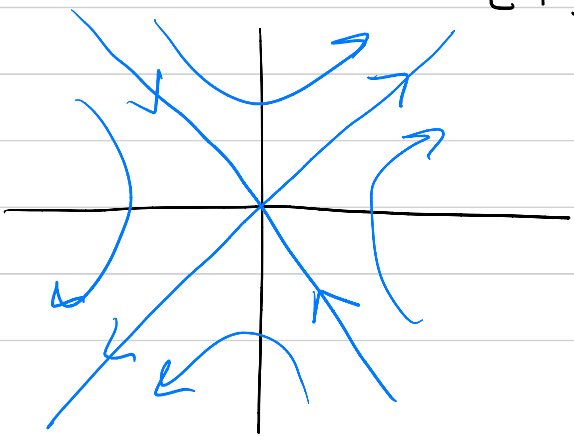
II: Same as I, but not as quickly.

III: Spring oscillates back and forth with oscillations getting smaller.

4) a) $A = \begin{bmatrix} 0 & 1 \\ 1+6x^2 & 0 \end{bmatrix} \Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\vec{z}\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{z}$

b) $\lambda = \pm 1$ $\quad \vec{v} = \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix}$

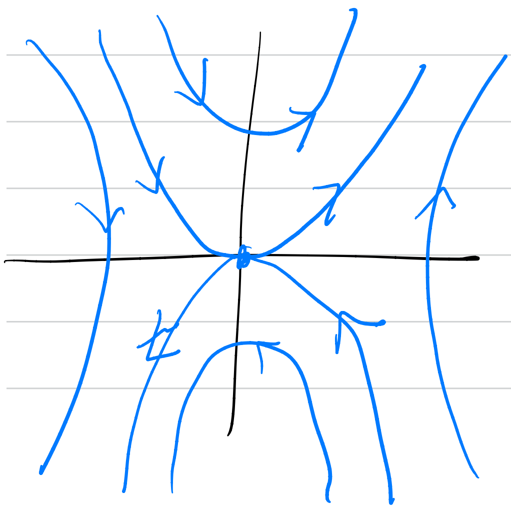


c) $\frac{dy}{dx} = \frac{x+2x^3}{y}$ $\Rightarrow \int y \, dy = \int x+2x^3 \, dx$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{2}x^4 + C$$

$$y = \pm\underbrace{\sqrt{x^4 + x^2 + C}}$$

looks like parabola with C as y intercept. when $\sqrt{}$ makes sense.



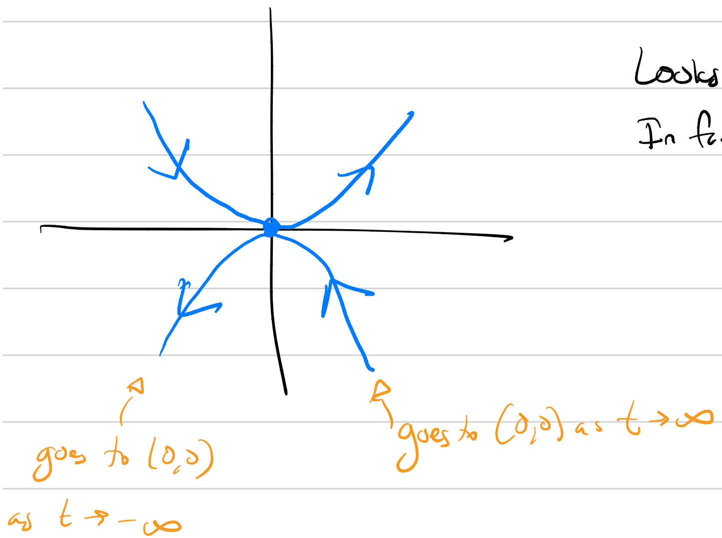
To get signs, take values and look at eq.

ex: $(0,y)$ $(y>0)$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} > 0$

d) Can see that there are two orbits towards origin $(C=0)$.

$$y = \pm\sqrt{x^4 + x^2}.$$

From phase portrait:



Looks like lines from linearized phase portrait.

In fact, for small x,

$$y = \pm\sqrt{x^4+x^2} \approx \pm\sqrt{x^2} = \pm x$$

which are precisely the lines from (b).