Assignment 2: CS 754, Advanced Image Processing

Due: 21st Feb before 11:55 pm

Remember the honor code while submitting this (and every other) assignment. All members of the group should work on and understand all parts of the assignment. We will adopt a zero-tolerance policy against any violation.

Submission instructions: You should ideally type out all the answers in Word (with the equation editor) or using Latex. In either case, prepare a pdf file. Create a single zip or rar file containing the report, code and sample outputs and name it as follows: A2-IdNumberOfFirstStudent-IdNumberOfSecondStudent.zip. (If you are doing the assignment alone, the name of the zip file is A2-IdNumber.zip). Upload the file on moodle BEFORE 11:55 pm on 21st Feb. Late assignments will be assessed a penalty of 50% per day late. Note that only one student per group should upload their work on moodle. Please preserve a copy of all your work until the end of the semester. If you have difficulties, please do not hesitate to seek help from me.

1. Your task here is to implement the ISTA algorithm for the following three cases:

   (a) Consider the barbara image from the homework folder. Add iid Gaussian noise of mean 0 and variance 10 (on a [0,255] scale) to it, using the 'randn' function in MATLAB. Thus $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{\eta}$ where $\boldsymbol{\eta} \sim \mathcal{N}(0, 100)$. You should obtain $\boldsymbol{x}$ from $\boldsymbol{y}$ using the fact that patches from $\boldsymbol{x}$ have a sparse or near-sparse representation in the 2D-DCT basis.

   (b) Divide the barbara image shared in the homework folder into patches of size 8×8. Let $\boldsymbol{x_i}$ be the vectorized version of the i^{th} patch. Consider the measurement $\boldsymbol{y_i} = \boldsymbol{\Phi x_i}$ where $\boldsymbol{\Phi}$ is a 32×64 matrix with entries drawn iid from $\mathcal{N}(0, 1)$. Note that $\boldsymbol{x_i}$ has a near-sparse representation in the 2D-DCT basis $\boldsymbol{U}$ which is computed in MATLAB as 'kron(dctmtx(8)',dctmtx(8)')'. In other words, $\boldsymbol{x_i} = \boldsymbol{U\theta_i}$ where $\boldsymbol{\theta_i}$ is a near-sparse vector. Your job is to reconstruct each $\mathbf{x_i}$ given $\boldsymbol{y_i}$ and $\boldsymbol{\Phi}$ using ISTA. Then you should reconstruct the image by averaging the overlapping patches. You should choose the α parameter in the ISTA algorithm judiciously. Choose $\lambda = 1$ (for a [0,255] image).

   (c) Consider a 100-dimensional sparse signal $\boldsymbol{x}$ containing 10 non-zero elements. Let this signal be convolved with a kernel $\boldsymbol{h} = [1, 2, 3, 4, 3, 2, 1]/16$ followed by addition of Gaussian noise of standard deviation equal to 5% of the magnitude of $\boldsymbol{x}$ to yield signal $\boldsymbol{y}$, i.e. $\boldsymbol{y} = \boldsymbol{h} * \boldsymbol{x} + \boldsymbol{\eta}$. Your job is to reconstruct $\boldsymbol{x}$ from $\boldsymbol{y}$ given $\boldsymbol{h}$. Be careful of how you create the matrix $\boldsymbol{A}$ in the ISTA algorithm. [40 = 10 + 20 + 10 points]

2. Refer to a copy of the paper 'The restricted isometry property and its implications for compressed sensing' in the homework folder. Your task is to open the paper and answer the question posed in each and every green-colored highlight. The task is essentially the complete proof of Theorem 3 done in class. [30 points = 2 points for the first 13 questions each and 4 points for the last one]

3. Consider compressive measurements $\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{\eta}$ of a purely sparse signal $\boldsymbol{x}$, where $\|\boldsymbol{\eta}\|_2 \leq \epsilon$. When we studied Theorem 3 in class, I had made a statement that the solution provided by the basis pursuit problem for a purely sparse signal comes very close (i.e. has an error that is only a constant factor worse than) an oracular solution. An oracular solution is defined as the solution that we could obtain if we knew in advance the indices (set S) the non-zero elements of the signal $\boldsymbol{x}$. There were many questions about this statement. This homework problem is the correct answer to those questions. For this, do as follows. In the following, we will assume that the inverse of $\boldsymbol{\Phi_S^T \Phi_S}$ exists.

   (a) Express the oracular solution $\tilde{\boldsymbol{x}}$ using a pseudo-inverse of the sub-matrix $\boldsymbol{\Phi_S}$. [5 points]

(b) Now, show that $\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2 = \|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\boldsymbol{\eta}\|_2 \leq \|\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}\|_2\|\boldsymbol{\eta}\|_2$. Here $\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger} \triangleq (\boldsymbol{\Phi}_{\boldsymbol{S}}^{\boldsymbol{T}}\boldsymbol{\Phi}_{\boldsymbol{S}})^{-1}\boldsymbol{\Phi}_{\boldsymbol{S}}^{\boldsymbol{T}}$ is standard notation for the pseudo-inverse of $\boldsymbol{\Phi}_{\boldsymbol{S}}$. The largest singular value of matrix $\boldsymbol{X}$ is denoted as $\|\boldsymbol{X}\|_2$. [3 points]

(c) Argue that the largest singular value of $\boldsymbol{\Phi}_{\boldsymbol{S}}^{\dagger}$ lies between $\dfrac{1}{\sqrt{1+\delta_{2k}}}$ and $\dfrac{1}{\sqrt{1-\delta_{2k}}}$ where $k = |S|$ and δ_{2k} is the RIC of $\boldsymbol{\Phi}$ of order $2k$. [4 points]

(d) This yields $\dfrac{\epsilon}{\sqrt{1+\delta_{2k}}} \leq \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 \leq \dfrac{\epsilon}{\sqrt{1-\delta_{2k}}}$. Argue that the solution given by Theorem 3 is only a constant factor worse than this solution. [3 points]

4. Here is our obligatory Google search question :-). Your task is to search for any one algorithm for compressive signal recovery which we have <u>not</u> covered in class, and do the following: (a) Write down an easy-to-follow pseudo-code with properly defined symbols, (b) Mention the paper(s) which first described this algorithm, (c) State the documented advantages of this method over OMP or ISTA, and (d) Include a link to software packages (if available) that implement this algorithm. [15 points]