CS 754 - Assignment 2

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Question 1

The ISTA code is attached for all three parts. Here are the results:

(a) Denoising

Noisy image:

Denoised image:

(b) Note: Please increase the stride to run the code faster. (make stride=4)

Reconstructed image for stride=1:

Question 2

The proof of the RIP paper by Candes is shown in Fig. 1 to Fig. 5

Question 3

(a)

The oracular solution $\tilde{x}$ will be the solution to the following optimization problem:

$min||x_s||_1$ s.t. $||y - \phi_s x_s||_2^2 \leq \epsilon$ Here x_s consists of the subvector of original signal x with indices of non-zero values as provided by the oracle. The solution to this equation can be obtained as: $(y - \phi_s x_s)^T(y - \phi_s x_s) = 0$

Thus $y^T y - 2x_s^T \phi_s^T y + x_s^T \phi_s^T \phi_s x_s = 0$

Thus $-2\phi_s^T y + 2\phi_s^T \phi_s \tilde{x} = 0$

Thus $\tilde{x} = (\phi_s^T \phi_s)^{-1} \phi_s^T y$

(b)

$||\tilde{x} - x||_2 = ||(\phi_s^T \phi_s)^{-1} \phi_s^T y - x||_2 = ||(\phi_s^T \phi_s)^{-1} \phi_s^T (\phi x + \eta) - x||_2$

$= ||((\phi_s^T \phi_s)^{-1} \phi_s^T (\phi x)) + ((\phi_s^T \phi_s)^{-1} \phi_s^T \eta) - x||_2 = ||x + ((\phi_s^T \phi_s)^{-1} \phi_s^T \eta) - x||_2 = ||\phi_s^+ \eta||_2$

Now for any matrix A, we can prove that the largest singular value $||A||_2 = max\frac{||Ax||_2}{||x||_2}$

First performing SVD on A, we get $A = USV^T$

Thus $max\frac{||Ax||_2}{||x||_2} = max\frac{||USV^T x||_2}{||x||_2} = max\frac{||SV^T x||_2}{||x||_2}$ since U is unitary

Taking $V^T x = y$ and for $||x|| = 1$,we get $||y||_2 = ||x||_2 = 1$ and $max||Ay||_2 = \sigma$ for $y = (1, 0, 0...)$ Hence $||A||_2 = \sigma \geq \frac{||Ax||_2}{||x||_2}$ Thus we have

$||\tilde{x} - x||_2 = ||\phi_s^+ \eta||_2 \leq ||\phi^+||_2 ||\eta||_2$

(c)

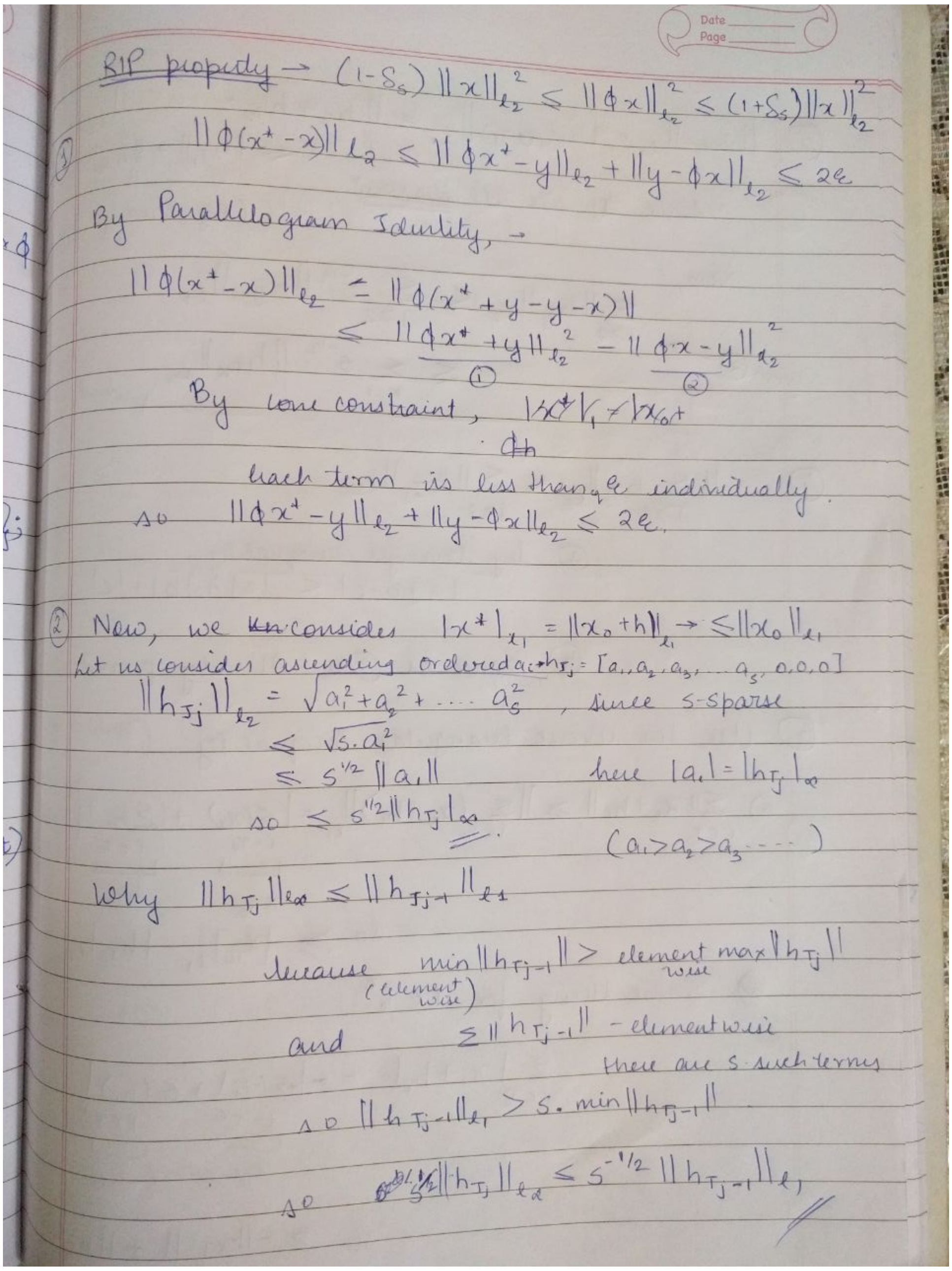
$||\phi_s^+||_2 = \sigma = max\frac{||\phi^+ x||}{||x||}$

$\geq \frac{||\phi^+ y||}{||y||} = \frac{||x_s||}{||\phi x||}$

Thus $\frac{||x_s||}{\sqrt{1-\delta_{2k}}||x||} \geq \sigma \geq \frac{||x_s||}{\sqrt{1+\delta_{2k}}||x||}$ (RIP Property)

Thus $\frac{1}{\sqrt{1-\delta_{2k}}} \geq \sigma \geq \frac{1}{\sqrt{1+\delta_{2k}}}$ $(||x_s|| = ||x||)$

(d)



RIP property → $(1-\delta_s)\|x\|_{\ell_2}^2 \le \|\phi x\|_{\ell_2}^2 \le (1+\delta_s)\|x\|_{\ell_2}^2$

(1) $\|\phi(x^* - x)\|_{\ell_2} \le \|\phi x^* - y\|_{\ell_2} + \|y - \phi x\|_{\ell_2} \le 2\epsilon$

By Parallelogram Identity, →

$\|\phi(x^* - x)\|_{\ell_2} = \|\phi(x^* + y - y - x)\|$

$\le \underbrace{\|\phi x^* + y\|_{\ell_2}^2}_{1} - \underbrace{\|\phi x - y\|_{\ell_2}^2}_{2}$

By cone constraint, $|x^*|_1 \neq |x_0 + \cancel{h}$

Each term is less than ϵ individually.

so $\|\phi x^* - y\|_{\ell_2} + \|y - \phi x\|_{\ell_2} \le 2\epsilon$.

(2) Now, we ~~kn~~ consider $|x^*|_{\ell_1} = \|x_0 + h\|_{\ell_1} \to \le \|x_0\|_{\ell_1}$

Let us consider ascending ordered $h_{T_j} = [a_1, a_2, a_3, \dots a_s, 0, 0, 0]$

$\|h_{T_j}\|_{\ell_2} = \sqrt{a_1^2 + a_2^2 + \dots a_s^2}$, since s-sparse

$\le \sqrt{s \cdot a_1^2}$

$\le s^{1/2}\|a_1\|$ here $|a_1| = |h_{T_j}|_\infty$

so $\le s^{1/2}\|h_{T_j}\|_\infty$

$(a_1 > a_2 > a_3 \dots)$

Why $\|h_{T_j}\|_{\ell_\infty} \le \|h_{T_{j-1}}\|_{\ell_1}$

because $\min\|h_{T_{j-1}}\|$ (element wise) $>$ element wise $\max\|h_{T_j}\|$

and $\sum\|h_{T_{j-1}}\|$ - elementwise

there are s such terms

so $\|h_{T_{j-1}}\|_{\ell_1} > s \cdot \min\|h_{T_{j-1}}\|$

so $s^{-1/2}\|h_{T_j}\|_{\ell_2} \le s^{-1/2}\|h_{T_{j-1}}\|_{\ell_1}$

Figure 1: Part 1 and 2.

Thus we get $\frac{\epsilon}{\sqrt{1-\delta_{2k}}} \geq ||x-\tilde{x}||_2 \geq \frac{\epsilon}{\sqrt{1+\delta_{2k}}}$ because $||\eta||_2 = \epsilon$

Theorem 3 solution:

$||x^* - x||_2 \leq \frac{C_0}{\sqrt{S}}||x - \tilde{(}x)|| + C_1\epsilon$

Thus we can see that even after the knowledge of the non-zero indices by an oracle, we are unable to improve the error bound by more than a constant value.

Question 4

Paper- Greedy Orthogonal Matching Pursuit Algorithm for Sparse Signal Recovery in Compressive Sensing

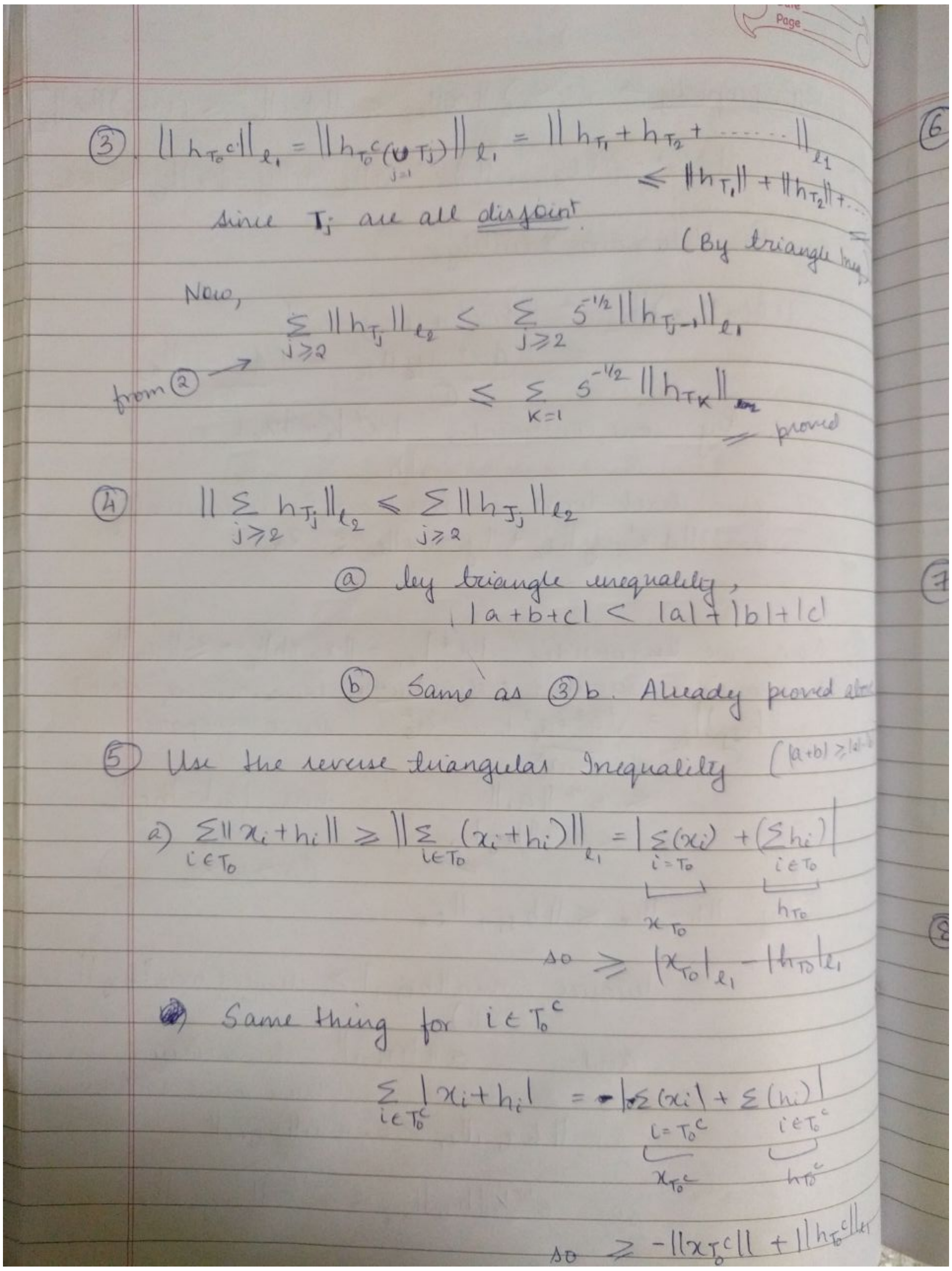
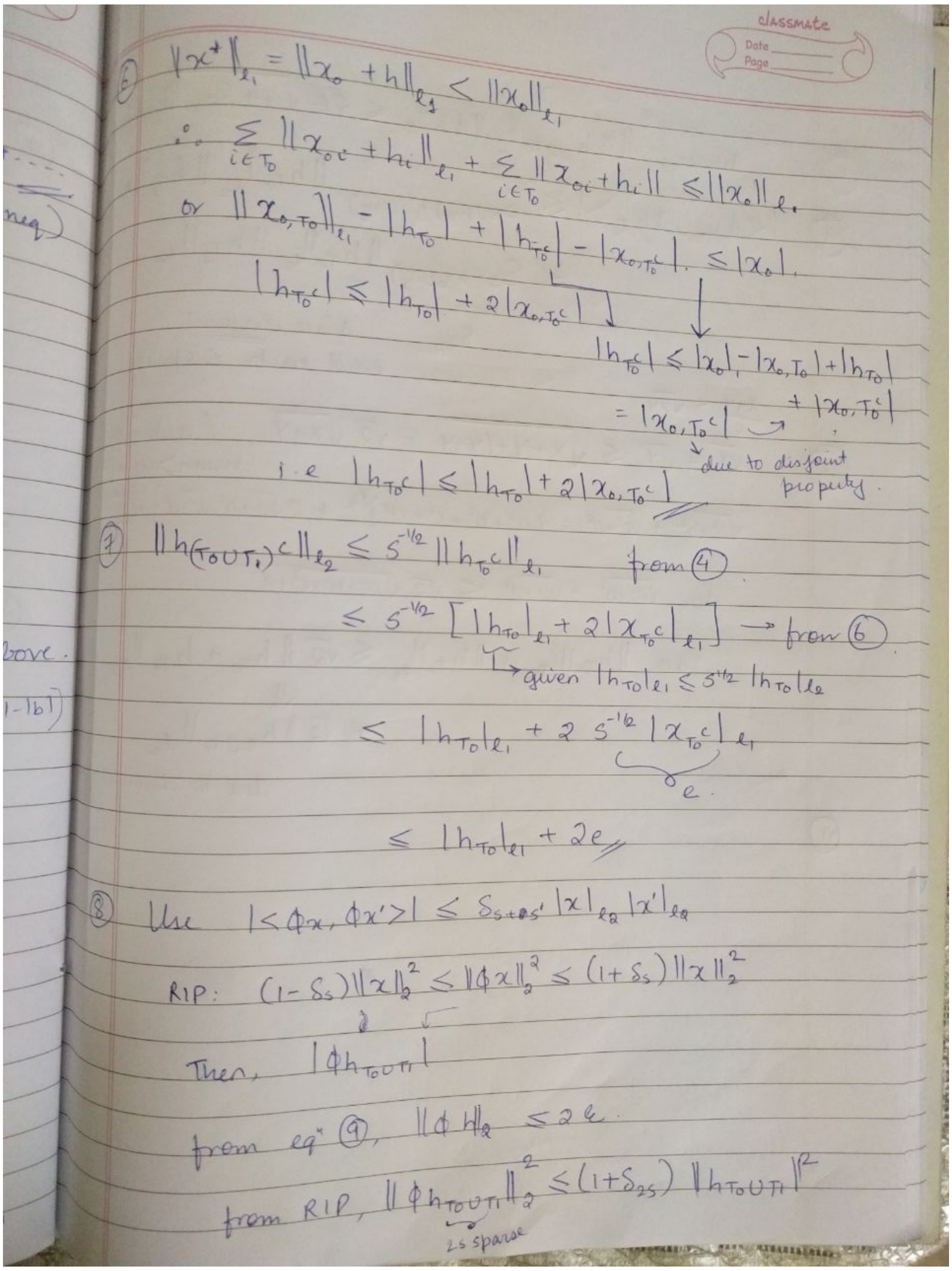


Figure 2: Part 3, 4 and 5.

(a)
In this paper, the authors analyze the iterative residual in the OMP algorithm. A greedy algorithm is introduced in this paper. This algorithm iteratively identifies more than one atoms using greedy atom identification, and then discards some atoms, which are of high similarity with the optimal atom. The authors claim that compared to an OMP algorithm, the proposed GOMP algorithm can provide better recovery performance. The algorithm gas been shown in Fig. 6
(b)
This method is taken from [1] by J. Li et. al., published in IEEE International Instrumentation and Measurement Technology Conference (I2MTC) Proceedings, and have been cited by numerous papers.

(c)
This algorithm (GOMP) inherits the atom identification from OMP algorithm. The GOMP



6) $\|x^*\|_{\ell_1} = \|x_0 + h\|_{\ell_1} \le \|x_0\|_{\ell_1}$

$\therefore \sum_{i \in T_0} \|x_{0i} + h_i\|_{\ell_1} + \sum_{i \in T_0} \|x_{0i} + h_i\| \le \|x_0\|_{\ell_1}$

or $\|x_{0,T_0}\|_{\ell_1} - |h_{T_0}| + |h_{T_0^c}| - |x_{0,T_0^c}|_1 \le |x_0|_1$

$|h_{T_0^c}| \le |h_{T_0}| + 2|x_{0,T_0^c}|$

$|h_{T_0^c}| \le |x_0|_1 - |x_{0,T_0}| + |h_{T_0}| + |x_{0,T_0^c}|$

$= |x_{0,T_0^c}|$ due to disjoint property.

i.e $|h_{T_0^c}| \le |h_{T_0}| + 2|x_{0,T_0^c}|$

7) $\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \le s^{-1/2} \|h_{T_0^c}\|_{\ell_1}$ from 4

$\le s^{-1/2} [|h_{T_0}|_{\ell_1} + 2|x_{T_0^c}|_{\ell_1}]$ → from 6

given $|h_{T_0}|_{\ell_1} \le s^{1/2} |h_{T_0}|_{\ell_2}$

$\le |h_{T_0}|_{\ell_1} + 2 \underbrace{s^{-1/2} |x_{T_0^c}|_{\ell_1}}_{e}$

$\le |h_{T_0}|_{\ell_1} + 2e$

8) Use $|\langle \Phi x, \Phi x' \rangle| \le \delta_{s+s'} |x|_{\ell_2} |x'|_{\ell_2}$

RIP: $(1-\delta_s)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1+\delta_s)\|x\|_2^2$

Then, $|\Phi h_{T_0 \cup T_1}|$

from eqn 9, $\|\Phi H\|_2 \le 2e$.

from RIP, $\|\Phi \underbrace{h_{T_0 \cup T_1}}_{2s\ \text{sparse}}\|_2^2 \le (1+\delta_{2s}) \|h_{T_0 \cup T_1}\|^2$

Figure 3: Part 6, 7 and 8.

algorithm adopts Greedy atom identification process to identify more atoms in each iteration, and then optimizes the selected atoms using similarity analysis. The authors claim that simulated results give excellent performances in recovering Gaussian and zero-one sparse signal.

The original OMP algorithm selects only one atom to the approximation of support in each iteration, which makes number of iteration high when copping with large scale data and the application impractical. Hence, this GOMP algorithm is an attempt to identify more atoms in each iteration based on the following approach. The Greedy OMP (GOMP) algorithm applies the greedy atom identification and optimization of selected atoms to OMP algorithm.h is described as followed. Compared to the ISTA, this algorithm is quite different. Here the algorithm doesn't make use of iterative shrinkage step, which was the backbone of the ISTA algorithm.

(d)

`GNU libgomp` is the library present for the Greedy OMP algorithm. Its detailed documentation

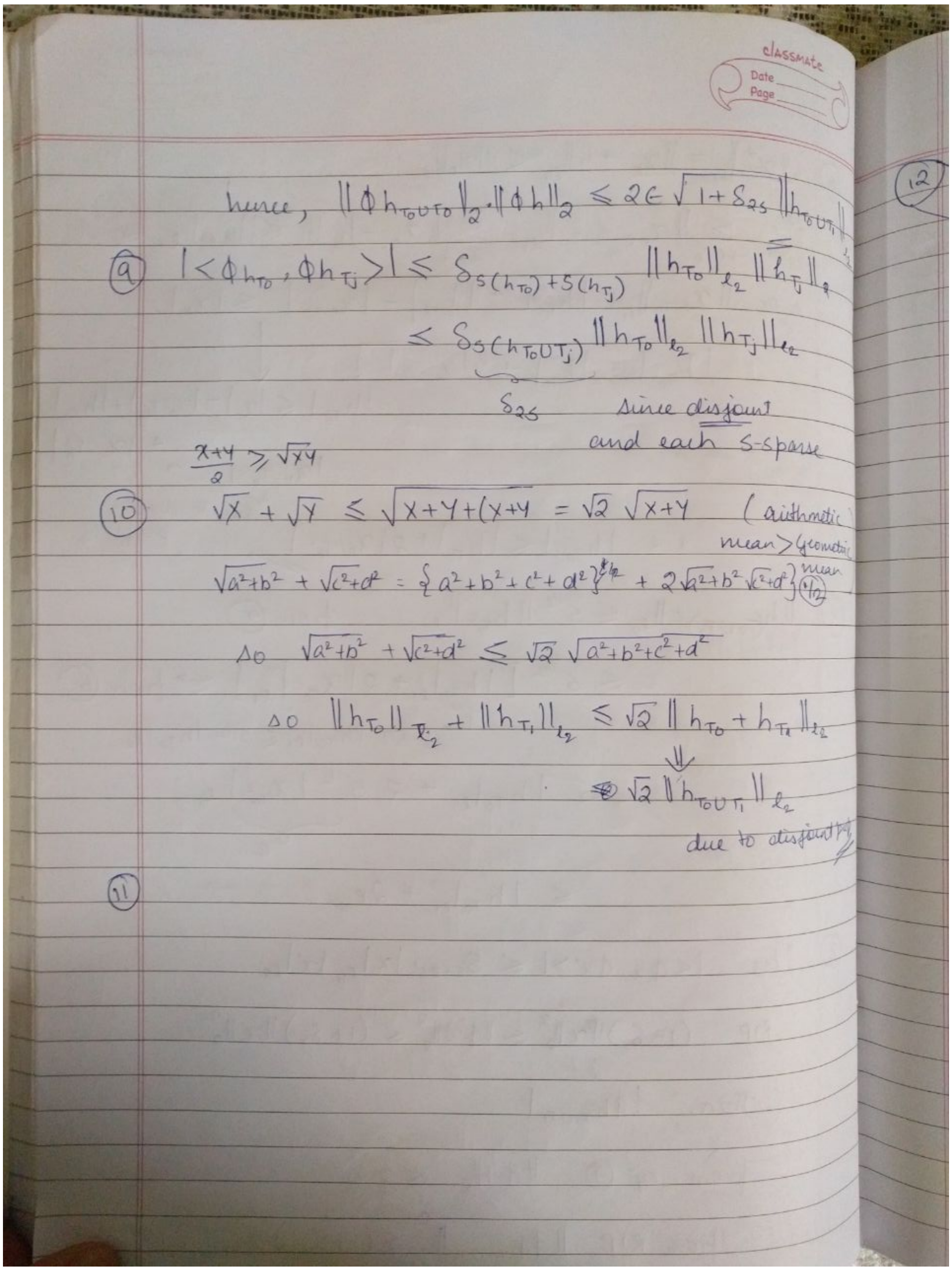
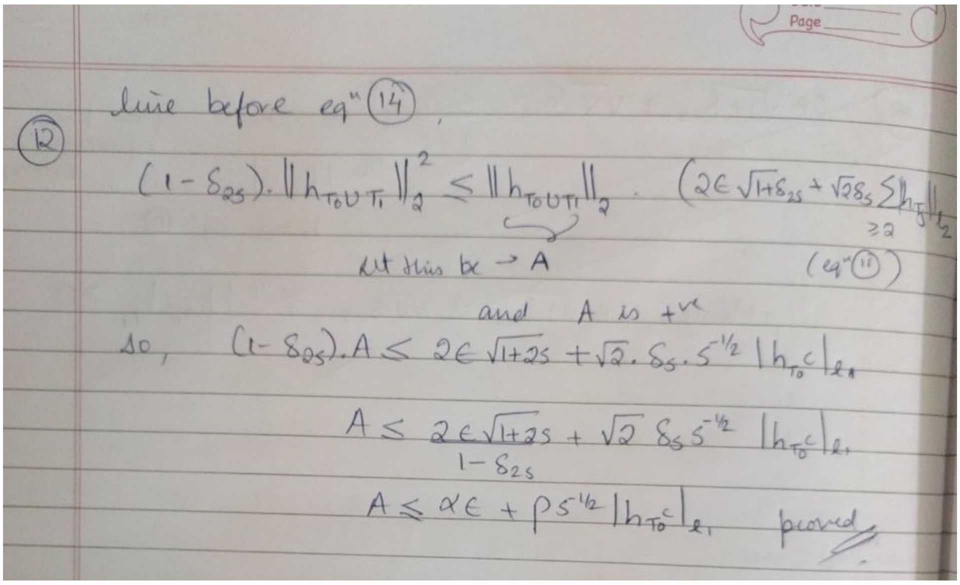


Figure 4: Part 10 and 11.

is presented in the following two links:
https://gcc.gnu.org/projects/gomp/
https://gcc.gnu.org/onlinedocs/libgomp/

References

[1] J. Li, Z. Wu, H. Feng, Q. Wang, and Y. Liu. Greedy orthogonal matching pursuit algorithm for sparse signal recovery in compressive sensing. In *2014 IEEE International Instrumentation and Measurement Technology Conference (I2MTC) Proceedings*, pages 1355–1358, May 2014.



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(12) line before eqn (14),

$$(1-\delta_{2s}) \cdot \|h_{T_0 \cup T_1}\|_2^2 \le \|h_{T_0 \cup T_1}\|_2 \cdot \left(2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_s \sum_{j \ge 2} \|h_{T_j}\|_{\ell_2}\right)$$

let this be $\rightarrow$ A (eqn (11))

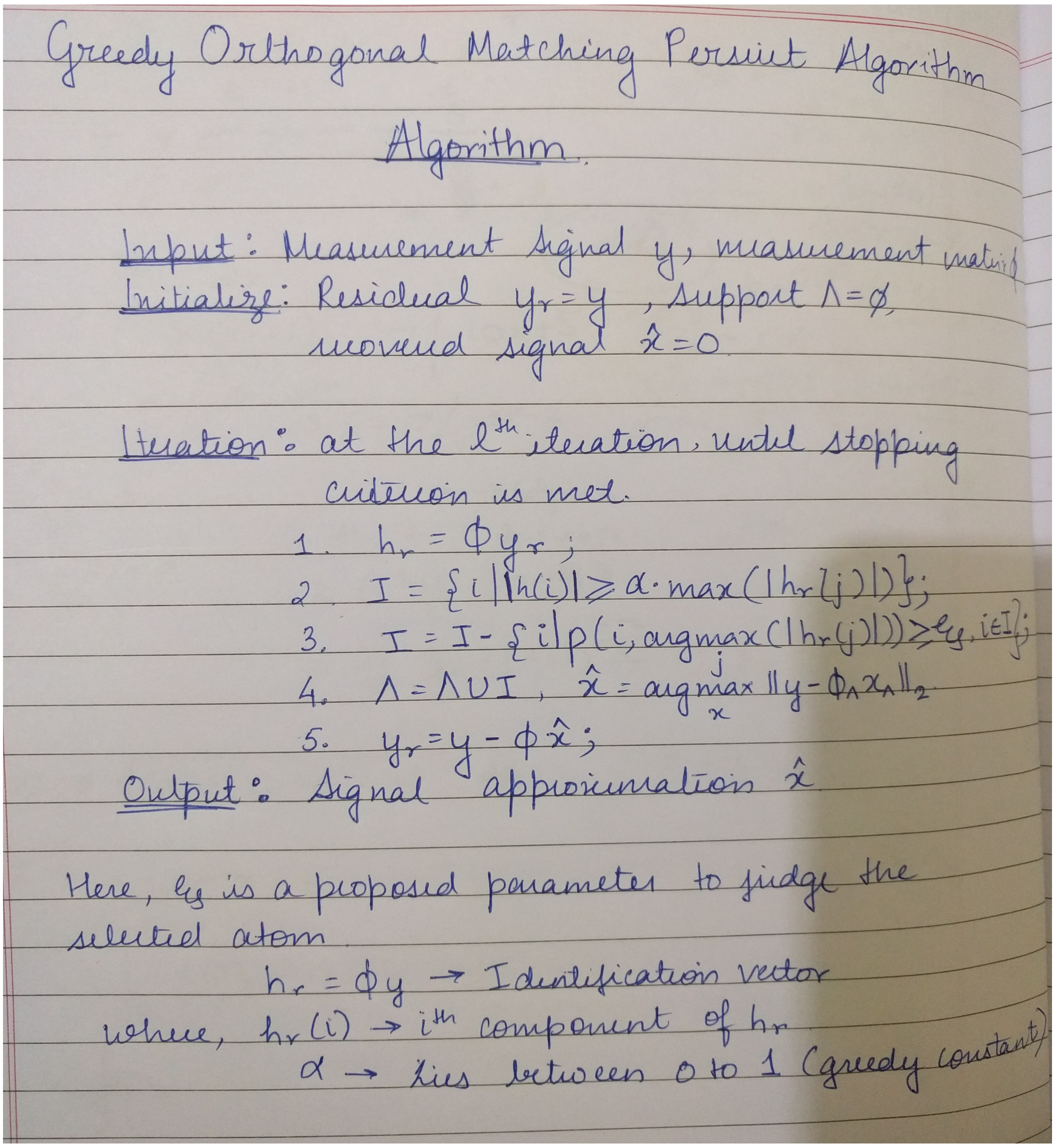
and A is +ve

so, $(1-\delta_{2s}).A \le 2\epsilon\sqrt{1+2s} + \sqrt{2}.\delta_s.s^{-1/2}\,|h_{T_0^c}|_{\ell_1}$

$$A \le \frac{2\epsilon\sqrt{1+2s} + \sqrt{2}\,\delta_s s^{-1/2}\,|h_{T_0^c}|_{\ell_1}}{1-\delta_{2s}}$$

$$A \le \alpha\epsilon + \rho s^{-1/2}\,|h_{T_0^c}|_{\ell_1}$$ proved

Figure 5: Part 12.



Greedy Orthogonal Matching Persuit Algorithm

Algorithm.

Input: Measurement Signal y, measurement matrix ϕ

Initialize: Residual $y_r = y$, support $\Lambda = \phi$, recovered signal $\hat{x} = 0$

Iteration: at the l^{th} iteration, until stopping criterion is met.

1. $h_r = \Phi y_r$;
2. $I = \{i \mid |h(i)| \geq \alpha \cdot \max(|h_r[j]|)\}$;
3. $I = I - \{i \mid p(i, \arg\max_j(|h_r(j)|)) \geq e_g, i \in I\}$;
4. $\Lambda = \Lambda \cup I$, $\hat{x} = \arg\max_x \|y - \Phi_\Lambda x_\Lambda\|_2$
5. $y_r = y - \phi \hat{x}$;

Output: Signal approximation $\hat{x}$

Here, e_g is a proposed parameter to judge the selected atom.

$h_r = \phi y \rightarrow$ Identification vector

where, $h_r(i) \rightarrow i^{th}$ component of h_r

$\alpha \rightarrow$ lies between 0 to 1 (greedy constant)

Figure 6: GOMP Algorithm.