Q.6]

(a) $\underset{n\times n}{P} = \underset{n\times m}{A^T}\,\underset{m\times n}{A} \qquad \underset{m\times m}{Q} = \underset{m\times n}{A}\,\underset{n\times m}{A^T}$

$$y^t P y = (y^t A^T A y)$$

$$= (Ay)^t (Ay)$$

$$= r^t r = \|r\|^2 \geq 0$$

where $\underset{m\times 1}{r} = \underset{m\times n}{A}\,\underset{n\times 1}{y}$, a vector

Similarly, $z^t Q z = z^t A A^T z$

$$= (A^T z)^T (A^T z)$$

$$= \|A^T z\|^2 \geq 0$$

Eigen values of a positive semi-definite matrices are always positive
This is because
$\forall x, \quad x^T A x \geq 0$

If $\quad Ax = \lambda x$
$\lambda = x^T A x \geq 0$
Hence, all eigenvalues of P & Q are non-negative

(b) $\quad Pu = \lambda u$

$\therefore A^T A u = \lambda u$

Premultiply by A

$$(AA^T)(Au) = \lambda(Au)$$

Hence $Q\ (Au) = \lambda\ (Au)$
Hence Au is the eigenvector of Q with eigenvalue λ.

Similarly,
$$Qv = \mu v$$
$$AA^T v = \mu v$$
Premultiply by A^T

$$A^T A A^T v = \mu A^T v$$
$$P(A^T v) = \mu (A^T v)$$
Hence, $A^T v$ is eigenvector of P with eigenvalue μ

Since, u is an eigenvector of P, it will have same no. of elements as rows or columns of P

Hence $u \longrightarrow$ 'n' elements

Similarly $v \longrightarrow$ 'm' elements

(c) $Q v_i = \lambda v_i$

$\therefore A A^T v_i = \lambda v_i$

Divide on both sides by $\|A^T v_i\|^2$

$$\therefore A \frac{(A^T v_i)}{\|A^T v_i\|^2} = \frac{\lambda}{\|A^T v_i\|^2} v_i$$

$$\therefore A u_i = \frac{\lambda}{\|A^T v_i\|^2} v_i$$

$$\therefore \boxed{A u_i = \gamma_i v_i}$$

$$\gamma_i = \frac{\lambda}{\|A^T v_i\|^2} \geq 0$$

since $\lambda \geq 0$ (all eigenvalues of Q are non-negative)