Q.5 $X = \{x_1, x_2, \_\_\_ x_N\}$

e = first eigen vector (highest eigenvalue)

f is perpendicular to e
i.e $f^t e = 0$

Also $f^t f = 1$ [normalized]

C = Covariance matrix of f

So, the problem can be formulated as:

$$\max \; f^t C f \quad \text{[objective function]}$$
$$\left.\begin{array}{l} \text{s.t. } f^t f = 1 \\ \text{and } f^t e = 0 \end{array}\right\} \text{[constraints]}$$

$$\therefore E(f) = f^t C f - \lambda_1 (f^t f - 1) - \lambda_2 f^t e$$

Differentiating wrt. f,

$$\frac{\partial E}{\partial f} = Cf - \lambda_1 f - \lambda_2 e = 0$$

$$\therefore e^t C f - \lambda_1 e^t f - \lambda_2 e^t e = 0$$

Now, $e^t e = 1$
and $e^t C f = \lambda_1 e^t f = 0$
because $Ce = \lambda e \Rightarrow e^t C = \lambda e^t$
$\Rightarrow e^t C f = \lambda e^t f = 0$

Hence, $\lambda_2 = 0$

$\Rightarrow e^t C f = \lambda_1 e^t f$

$\Rightarrow C f = \lambda_1 e e^t f$

$\Rightarrow C f = \lambda_1 f$

$\Rightarrow$ f is also an eigenvector of C

Since $f^t e = 0$, f and e have distinct eigenvalues

$\Rightarrow$ it must correspond to second highest eigenvalue. $\lambda_1 = f^t C f$