Sensitivity Analysis or Post-optimality Analysis

Cases

1. change in righthand side vector b
2. change in Cost vector
3. Addition of a constraint
4. Addition of a variable.

Possibilities

1. No change in Current solution (Remains optimal and feasible)
2. Current Solution becomes infeasible
3. Current Solution becomes non-optimal
4. Current solution becomes infeasible and non-optimal.

Case ① Change in right hand side vector b.

⇒ change will occur Only in X_B and $f(X_B)$

✓ $X_B = B^{-1}b$

✓ $f(X_B) = C_B^T X_B$

✗ $\alpha^j = B^{-1}A_j$

✗ $Z_j - C_j = C_B^T\alpha^j - C_j$

Q Min $f = -x_1 + x_2 + x_3$

s.t. $-2x_1 + x_2 + x_3 \geq 2$

$x_1 - 2x_2 + 2x_3 = 2$

$x_1, x_2, x_3 \geq 0$

Optimal Table

B.V	x_1	x_2	x_3	s_1	a_1	a_2	Sol.
f	-1	0	0	-1	$1-M$	-14	2
x_2	$-5/4$	1	0	$-1/2$	$1/2$	$-1/4$	$1/2$
x_3	$-3/4$	0	1	$-1/2$	$1/2$	$1/4$	$3/2$

B^{-1}

(a) change b to $(6,2)^T$

(b) Change b to $(6\ 16)^T$

Sol (a) change b to $(6, 2)^T$ $\qquad C_B^T = (1, 1)$

New $b_1 = 6$, New $b_2 = 2$, New $b = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = b_{new}$, $B^{-1} = \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix}$

$$\text{New } X_B = B^{-1} b_{new} = \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 7/2 \end{bmatrix} = X_{B_{new}}$$

$$\text{New } f(X_B) = C_B^T X_{B_{new}} = (1, 1) \begin{bmatrix} 5/2 \\ 7/2 \end{bmatrix} = \frac{12}{2} = 6$$

New Table

B.V	x_1	x_2	x_3	s_1	a_1	a_2	Sol.
f	-1	0	0	-1	$1-M$	-14	6
x_2	$-5/4$	1	0	$-1/2$	$1/2$	$-1/4$	$5/2$
x_3	$-3/4$	0	1	$-1/2$	$1/2$	$1/4$	$7/2$

optimal? ✓

feasible? ✓

$\Rightarrow$ optimal solution remains same
$x_1 = 0,\ x_2 = 5/2,\ x_3 = 7/2$ and Min $f = 6$

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(b) change b to $(6, 16)$

$b_{new} = \begin{pmatrix} 6 \\ 16 \end{pmatrix}$, $\quad C_B^T = (1, 1)$, $\quad B^{-1} = \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix}$

$$X_{B_{new}} = B^{-1} b_{new} = \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 6 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$f(X_{B_{new}}) = C_B^T X_{B_{new}} = (1\ 1) \begin{bmatrix} -1 \\ 7 \end{bmatrix} = 6$$

New table

	x_1	x_2	x_3	s_1	a_1	a_2	Sol.
f	-1	0	0	-1	$1-M$	-14	6
x_2	$\boxed{-5/4}$	1	0	$-1/2$	$1/2$	$-1/4$	-1
x_3	$-3/4$	0	1	$-1/2$	$1/2$	$1/4$	7
f	0	$-4/5$	0	$-3/5$	$\frac{3}{5}-M$	$-69/5$	$34/5$
x_1	1	$-4/5$	0	$2/5$	$-2/5$	$1/5$	$4/5$
x_3	0	$-3/5$	1	$-1/5$	$1/5$	$2/5$	$38/5$

optimal? ✓

feasible? ✗ (dual simplex Method)

most negative

$\text{Min}\left\{\left|\frac{-1}{-5/4}\right|, \left|\frac{-1}{-1/2}\right|\right\} = \frac{4}{5}$ for x_1

optimal? ✓ feasible? ✓

Optimal solution is

$x_1 = \frac{4}{5}, \quad x_2 = 0, \quad x_3 = \frac{38}{5}$

and Min $f = \frac{34}{5}$

Case 2 change in Cost vector

- change in cost of non-basic variable
- change in cost of basic variable

Q Max $Z = 2x_1 + 3x_2 + 4x_3$

s.t. $x_1 + 2x_2 + 3x_3 \le 11$

$2x_1 + 3x_2 + 2x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

$C_B^T = (4 \ \ 2)$

Optimal table

B.V	x_1	x_2	x_3	s_1	s_2	Sol
Z	0	1/2	0	1	1/2	16
x_3	0	1/4	1	1/2	-1/4	3
x_1	1	5/4	0	-1/2	3/4	2
		α^2		α^4	α^5	

Subcase 1: change in the cost of Non-basic variable

(a) change the cost of x_2 to 4, i.e $C_2^{new} = 4$

~~$X_B = B^{-1}b$~~
~~$f(X_B) = C_B^T X_B$~~
~~$\alpha^j = B^{-1}A_j$~~
$Z_j - C_j = C_B^T \alpha^j - C_j$

$Z_j - C_j$ is evaluated only for that non-basic variable whose cost been changed

New $Z_2 - C_2 = C_B^T \alpha^2 - C_2^{new} = (4, 2)\begin{bmatrix} 1/4 \\ 5/4 \end{bmatrix} - 4 = -1/2$

New table

optimal? ✗
feasible? ✓

B.V	x_1	x_2	x_3	s_1	s_2	Sol.	Min ratio
Z	0	-1/2	0	1	1/2	16	
x_3	0	1/4	1	1/2	-1/4	3	$3/\frac{1}{4} = 12$
x_1	1	$\boxed{5/4}$	0	-1/2	3/4	2	$2/\frac{5}{4} = \frac{8}{5}$ min
Z	2/5	0	0	4/5	4/5	84/5	
x_3	-1/5	0	1	3/5	-2/5	13/5	
x_2	4/5	1	0	-2/5	3/5	8/5	

optimal? ✓
feasible? ✓

New optimal Solution $x_1 = 0, \ x_2 = 8/5, \ x_3 = 13/5$

and Max $Z = 84/5$

Subcase 2 : change in the cost of basic variable

(a) change the cost of x_3 to -2, i.e, $c_3 = -2$

(C_B^T will also change)

✓ $C_{B new}^T = (-2 \;\; 2)$

Here x_2, s_1, s_2 are three non-basic variables

$\Rightarrow$ Evaluate $z_j - c_j$ corresponding to x_2, s_1 & s_2,

For x_2 New $z_2 - c_2 = C_{B new}^T \alpha^2 - c_2 = (-2 \;\; 2)\begin{bmatrix} 1/4 \\ 5/4 \end{bmatrix} - 3 = -1$

For s_1 New $z_4 - c_4 = C_{B new}^T \alpha^4 - c_4 = (-2 \;\; 2)\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} - 0 = -2$

× $X_B = B^{-1}b$

✓ $f(X_B) = C_B^T X_B$

× $\alpha^j = B^{-1}A_j$

✓ $z_j - c_j = C_B^T \alpha^j - c_j$

Evaluate $z_j - c_j$ corresponding to all non-basic variables

For s_2 New $z_5 - c_5 = C_{B new}^T \alpha^5 - c_5 = (-2 \;\; 2)\begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix} - 0 = 2$

New $f(X_B) = C_{B new}^T X_B = (-2 \;\; 2)\begin{bmatrix} 3 \\ 2 \end{bmatrix} = -2$

New table

B.V	x_1	x_2	x_3	s_1	s_2	Sol.
z	0	-1	0	-2	2	-2
x_3	0	$1/4$	1	$\boxed{1/2}$	$-1/4$	3
x_1	1	$5/4$	0	$-1/2$	$3/4$	2
z	0	0	4	0	1	10
s_1	0	$1/2$	2	1	$-1/2$	6
x_1	1	$3/2$	1	0	$1/2$	5

optimal? ×
feasible? ✓

optimal ✓
feasible ✓

New optimal solution is $x_1 = 5$, $x_2 = 0$, $x_3 = 0$ and Max $z = 10$

Case 3 Addition of a Constraint

Q Min $Z = x_1 - 2x_2 + x_3$

s.t. $x_1 + 2x_2 - 2x_3 \le 4$

$x_1 - x_3 \le 3$

$2x_1 - x_2 + 2x_3 \le 2$

$x_1, x_2, x_3 \ge 0$

optimal table

B.V	x_1	x_2	x_3	s_1	s_2	s_3	Sol.
z	$-9/2$	0	0	$-3/2$	0	-1	-8
x_2	3	1	0	1	0	1	6
s_2	$7/2$	0	0	$1/2$	1	1	7
x_3	$5/2$	0	1	$1/2$	0	1	4

Find solution if

(i) $x_2 + x_3 = 10$ is added

(ii) $x_3 - x_2 = -2$ " "

(iii) $x_2 + x_3 \le 10$ " "

(iv) $x_1 + x_2 = 4$ " "

(v) $x_1 + x_2 = 7$ " "

(vi) $x_1 + x_2 \le 4$ is added

(vii) $x_1 + x_2 \ge 7$ is added

Optimal Solution of Original LPP

$x_1 = 0, x_2 = 6, x_3 = 4$

Min $z = -8$

Sol. (i) New Constraint is $x_2 + x_3 = 10$

Since $x_2 = 6, x_3 = 4 \Rightarrow 10 = 10$ (Constraint is satisfied)

$\Rightarrow$ optimal solution remains same.

(ii) New Constraint is $x_3 - x_2 = -2$

Here $x_3 = 4, x_2 = 6 \Rightarrow x_3 - x_2 = 4 - 6 = -2 \Rightarrow$ Satisfied ✓

$\Rightarrow$ optimal Solution remains same

(iii) New Constraint $x_2 + x_3 \le 10$

Here $x_2 = 6, x_3 = 4 \Rightarrow 10 \le 10$ (Satisfied)

$\Rightarrow$ Optimal solution remains same.

(iv) New Constraint is $x_1 + x_2 = 4$

Here $x_1 = 0,\ x_2 = 6$

$\Rightarrow \quad 0 + 6 \neq 4$ (Not satisfied)

Add this constraint

$$x_1 + x_2 = 4 \begin{cases} x_1 + x_2 \leq 4 \;\rightarrow\; 6 \leq 4 \text{ (Not satisfied)} \\ x_1 + x_2 \geq 4 \;\rightarrow\; 0 + 6 \geq 4 \text{ (Satisfied)} \end{cases}$$

$\Rightarrow$ Add constraint $x_1 + x_2 \leq 4$ ✓

i.e. $x_1 + x_2 + s_4 = 4$

New table is given by

B.V	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.
Z	$-9/2$	0	0	$-3/2$	0	-1	0	-8
x_2	3	1	0	1	0	1	0	6
s_2	$7/2$	0	0	$1/2$	1	1	0	7
x_3	$5/2$	0	1	$1/2$	0	1	0	4
s_4 ←	$\cancel{1}$ -2	$\cancel{1}$ 0	0	$\cancel{0}$ -1	0	$\boxed{\cancel{0}\ -1}$	1	$\cancel{4}$ -2 ←

firstly Check it is in Simplex format?

New s_4 row = old s_4 row $-$ x_2 row

B.V	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.
Z	$-5/2$	0	0	$-1/2$	0	0	-1	-6
x_2	1	1	0	0	0	0	1	4
s_2	$3/2$	0	0	$-1/2$	1	0	0	5
x_3	$1/2$	0	1	$-1/2$	0	0	1	2
s_3	2	0	0	1	0	1	-1	2

optimality ✓
feasibility ✗

$\text{Min}\left\{ \left|\frac{-9/2}{-2}\right|, \left|\frac{-3/2}{-1}\right|, \left|\frac{-1}{-1}\right| \right\}$
$= \text{Min}\left\{ \frac{9}{4}, \frac{3}{2}, 1 \right\}$
$= 1$ for s_3

optimal table

New optimal Solution is $x_1 = 0,\ x_2 = 4,\ x_3 = 2$
and Min $Z = -6$

(v) Add Constraint $x_1 + x_2 = 7$

$x_1 = 0, x_2 = 6 \Rightarrow 0 + 6 \neq 7$ (Not satisfied)

$x_1 + x_2 = 7$ <
- $x_1 + x_2 \leq 7 \Rightarrow 0 + 6 \leq 7$ (Satisfied)
- $x_1 + x_2 \geq 7 \Rightarrow 0 + 6 \ngeq 7$ (Not satisfied)

$\Rightarrow$ Add constraint $\boxed{x_1 + x_2 \geq 7}$ (No Artificial variable will be added)

$\Rightarrow x_1 + x_2 - s_4 = 7 \Rightarrow \boxed{-x_1 - x_2 + s_4 = -7}$

New table is given by

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.
z	$-9/2$	0	0	$-3/2$	0	-1	0	-8
x_2	3	1	0	1	0	1	0	6
s_2	$7/2$	0	0	$1/2$	1	1	0	7
x_3	$5/2$	0	1	$1/2$	0	1	0	4
← s_4	$\cancel{-1}$ 2	$\cancel{-1}$ 0	0	$\cancel{0}$ 1	0	$\cancel{0}$ 1	1	$\cancel{-7}$ -1 ←

Simplex form? ✗

s_4 will leave the basis

But no entering variable

$\Rightarrow$ No feasible solution for new problem

(vi) Add a constraint $x_1 + x_2 \le 4$

(Look part (iv))

(vii) Add a constraint $x_1 + x_2 \ge 7$

(Look part (v))

Case 4 Addition of a variable

(Big-M Method)

$$\text{Min } Z = 2x_1 + x_2$$
$$\text{s.t. } 3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \ge 6$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

Optimal table

C_B^T	B.V	x_1	x_2	s_2	a_1	a_2	s_3	Sol.
	Z	0	0	$-4/5$	$\frac{2}{5}-M$	$\frac{1}{5}-M$	0	$12/5$
2	x_1	1	0	$4/5$	$3/5$	$-1/5$	0	$3/5$
1	x_2	0	1	$-3/5$	$-4/5$	$3/5$	0	$6/5$
0	s_3	0	0	1	1	-1	1	0

B^{-1}

Add x_3 here $c_3 = \frac{1}{2}$, $A_3 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

(i) Add a new variable x_3 having cost $\frac{1}{2}$ and column vector $(0, 5, 2)^T$

Find $z_3 - c_3$ and α^3 corresponding to x_3

$$\alpha^3 = B^{-1} A_3 = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$$

$$z_3 - c_3 = C_B^T \alpha^3 - c_3 = (2 \ 1 \ 0) \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} - \frac{1}{2} = -2 + 3 - \frac{1}{2} = 1/2$$

New table

B.V	x_1	x_2	x_3	s_2	a_1	a_2	s_3	Sol.
Z	0	0	$1/2$	$-4/5$	$\frac{2}{5}-M$	$\frac{1}{5}-M$	0	$12/5$
x_1	1	0	-1	$4/5$	$3/5$	$-1/5$	0	$3/5$
$\leftarrow x_2$	0	1	$\boxed{3}$	$-3/5$	$-4/5$	$3/5$	0	$6/5$
s_3	0	0	-3	1	1	-1	1	0
Z	0	$-1/6$	0	$-1/5$	$\frac{8}{5}-M$	$\frac{1}{10}-M$	0	$11/5$
x_1	1	$1/3$	0	0	$1/3$	0	0	1
x_3	0	$1/3$	1	$-1/5$	$-4/15$	$1/5$	0	$2/5$
s_3	0	1	0	$2/5$	$1/5$	$-2/5$	1	$6/5$

feasible? ✓

optimal? ✗ $z_j - c_j \le 0$ All

optimal table

optimal solution is $x_1 = 1, x_2 = 0, x_3 = 2/5$

Min $Z = 11/5$

(ii) Add new variable x_3 with cost 3 and column vector $(1,2,3)^T$

Here $C_3 = 3$ and $A_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

find α^3

find $z_3 - c_3$

Add in the table.

find New X_B and $f(x_B)$

$$X_{B_{new}} = B^{-1} b_{new} = \begin{bmatrix} 1/3 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}b_1 \\ -2b_1+18 \end{bmatrix} \begin{pmatrix} \geq 0 \\ \geq 0 \end{pmatrix} \text{ for feasibility}$$

For feasibility, $\frac{1}{3}b_1 \geq 0 \Rightarrow b_1 \geq 0$

and $-2b_1 + 18 \geq 0 \Rightarrow b_1 \leq 9$

$\boxed{0 \leq b_1 \leq 9}$

Feasibility Range for b_1

For Second Constraint, Let the right hand side be b_2

$$b_{new} = \begin{bmatrix} 8 \\ b_2 \end{bmatrix}$$

For feasibility

$$X_{B_{new}} = B^{-1} b_{new} = \begin{bmatrix} 1/3 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ b_2 \end{bmatrix} = \begin{bmatrix} 8/3 \\ -16+b_2 \end{bmatrix} \begin{matrix} \dashrightarrow \geq 0 \checkmark \\ \dashrightarrow \geq 0 \end{matrix}$$

$$\Rightarrow -16 + b_2 \geq 0 \quad \Rightarrow b_2 \geq 16 \quad \Rightarrow \text{Feasibility Range for } b_2 \text{ is } \boxed{16 \leq b_2 < \infty}$$

OR

$$b_2 \in [16, \infty)$$

(ii) Optimality Ranges

For first variable x_1, Let the cost of x_1 be c_1 (Here x_1 is non-basic variable)

find $z_1 - c_1$

- ~~$X_B = B^{-1}b$~~
- ~~$f(X_B) = C_B^T X_B$~~
- ~~$\alpha^j = B^{-1}A_j$~~
- $z_j - c_j = C_B^T \alpha^j - c_j$

Only for x_1

Here

$$\text{New } z_1 - c_1 = C_B^T \alpha^1 - c_1 = (3 \ \ 0)\begin{bmatrix} 2/3 \\ -2 \end{bmatrix} - c_1$$

$$= 2 - c_1$$

For optimality, $2 - c_1 \geq 0 \Rightarrow c_1 \leq 2$

optimality Range for c_1 is $(-\infty, 2]$

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For Second variable x_2 let the cost of x_2 be c_2

Since x_2 is basic variable and change in its cost will change C_B^T

So find new C_B^T, $z_j - c_j$ for all non-basic variables and $f(X_B)$.

$$\text{New } C_B^T = (c_2 \ \ 0)$$

For x_1: $$z_1 - c_1 = C_{B\,new}^T \alpha^1 - c_1 = (c_2 \ \ 0)\begin{bmatrix} 2/3 \\ -2 \end{bmatrix} - 2 = \frac{2}{3}c_2 - 2$$

For x_3: $$z_3 - c_3 = C_{B\,new}^T \alpha^3 - c_3 = (c_2 \ \ 0)\begin{bmatrix} 1/3 \\ -2 \end{bmatrix} - 0 = \frac{1}{3}c_2$$

For optimality, $\frac{2}{3}c_2 - 2 \geq 0 \Rightarrow c_2 \geq 3$

and $\frac{1}{3}c_2 \geq 0 \Rightarrow c_2 \geq 0$

$$\boxed{c_2 \geq 3}$$

$\therefore$ optimality Range for c_2 is $[3, \infty)$