Single Qubit Gates

Identity Gate

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|\psi\rangle \; -\!\!\boxed{I}\!\!- \; |\psi'\rangle$$

$$I\,|0\rangle = |0\rangle\,, \quad I\,|1\rangle = |1\rangle$$

$$I(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}\,|1\rangle$$

$$I\,|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix}$$

Pauli X Gate

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|\psi\rangle \;—\boxed{X}—\; |\psi'\rangle$$

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle$$

$$X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}|1\rangle$$

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \sin(\pi/8)e^{i7\pi/4} \\ \cos(\pi/8) \end{pmatrix}$$

Pauli Y Gate

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|\psi\rangle \;-\!\!\boxed{Y}\!\!-\; |\psi'\rangle$$

$$Y|0\rangle = i|1\rangle, \quad Y|1\rangle = -i|0\rangle$$

$$Y(\alpha|0\rangle + \beta|1\rangle) = -i\beta|0\rangle + i\alpha|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}|1\rangle$$

$$Y|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} -i\sin(\pi/8)e^{i7\pi/4} \\ i\cos(\pi/8) \end{pmatrix}$$

Pauli Z Gate

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\psi\rangle \;—\boxed{Z}—\; |\psi'\rangle$$

$$Z\,|0\rangle = |0\rangle\,, \quad Z\,|1\rangle = -\,|1\rangle$$

$$Z(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle - \beta\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}\,|1\rangle$$

$$Z\,|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ -\sin(\pi/8)e^{i7\pi/4} \end{pmatrix}$$

Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|\psi\rangle \; - \boxed{H} - \; |\psi'\rangle$$

$$H\,|0\rangle = \tfrac{|0\rangle + |1\rangle}{\sqrt{2}}, \;\; H\,|1\rangle = \tfrac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right) e^{i7\pi/4}\,|1\rangle$$

$$H\,|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\pi/8) + \sin(\pi/8)e^{i7\pi/4} \\ \cos(\pi/8) - \sin(\pi/8)e^{i7\pi/4} \end{pmatrix}$$

Phase Gate

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

$$|\psi\rangle \;-\!\!\!-\; \boxed{P(\phi)} \;-\!\!\!-\; |\psi'\rangle$$

$$P(\phi)\,|0\rangle = |0\rangle\,, \qquad P(\phi)\,|1\rangle = e^{i\phi}\,|1\rangle$$

$$P(\phi)(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle + \beta e^{i\phi}\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right) e^{i7\pi/4}\,|1\rangle$$

$$P(\phi)\,|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i(7\pi/4+\phi)} \end{pmatrix}$$

- The phase gate rotates the qubit **around the Z-axis** by angle ϕ.
- $|0\rangle$ stays at the north pole, $|1\rangle$ stays at the south pole, but **any superposition on the equator rotates.**

Think of it as **spinning the state along the vertical axis without changing probabilities.**

S Gate

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$|\psi\rangle \;-\!\!\!-\!\boxed{S}\!-\!\!\!-\; |\psi'\rangle$$

$$S\,|0\rangle = |0\rangle\,, \qquad S\,|1\rangle = i\,|1\rangle$$

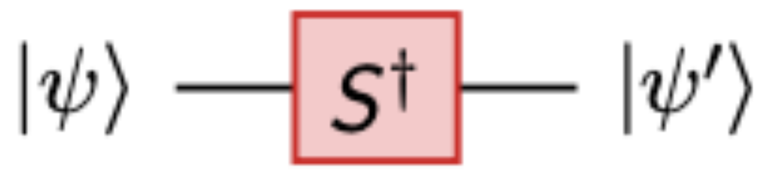
$$S(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle + i\beta\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right) e^{i7\pi/4}\,|1\rangle$$

$$S\,|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ i\sin(\pi/8)e^{i7\pi/4} \end{pmatrix}$$

S† Gate

$$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$



$$S^\dagger |0\rangle = |0\rangle, \qquad S^\dagger |1\rangle = -i\,|1\rangle$$

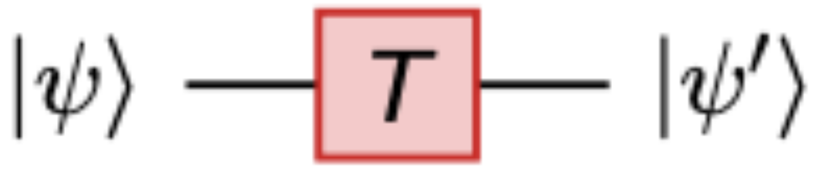
$$S^\dagger(\alpha |0\rangle + \beta |1\rangle) = \alpha |0\rangle - i\beta |1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right) |0\rangle + \sin\left(\frac{\pi}{8}\right) e^{i7\pi/4} |1\rangle$$

$$S^\dagger |\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ -i\sin(\pi/8)e^{i7\pi/4} \end{pmatrix}$$

T Gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$



$$T\,|0\rangle = |0\rangle\,, \qquad T\,|1\rangle = e^{i\pi/4}\,|1\rangle$$

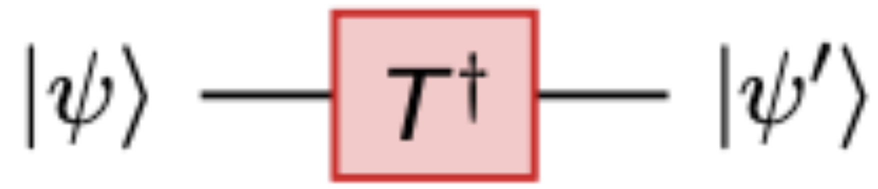
$$T(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle + \beta e^{i\pi/4}\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}\,|1\rangle$$

$$T\,|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{pmatrix}$$

T† Gate

$$T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$



$$T^\dagger |0\rangle = |0\rangle\,, \qquad T^\dagger |1\rangle = e^{-i\pi/4} |1\rangle$$

$$T^\dagger(\alpha |0\rangle + \beta |1\rangle) = \alpha |0\rangle + \beta e^{-i\pi/4} |1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right) |0\rangle + \sin\left(\frac{\pi}{8}\right) e^{i7\pi/4} |1\rangle$$

$$T^\dagger |\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8)e^{i7\pi/4} \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ -i\sin(\pi/8) \end{pmatrix}$$

Rotation-X Gate $R_X(\theta)$

$$|\psi\rangle \;—\; \boxed{R_X(\theta)} \;—\; |\psi'\rangle$$

$$R_X(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_X(\theta)\,|0\rangle = \cos\frac{\theta}{2}\,|0\rangle - i\sin\frac{\theta}{2}\,|1\rangle$$

$$R_X(\theta)\,|1\rangle = -i\sin\frac{\theta}{2}\,|0\rangle + \cos\frac{\theta}{2}\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}\,|1\rangle$$

$$R_X(\theta)\,|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\cos(\pi/8) - i\sin\frac{\theta}{2}\sin(\pi/8)e^{i7\pi/4} \\ -i\sin\frac{\theta}{2}\cos(\pi/8) + \cos\frac{\theta}{2}\sin(\pi/8)e^{i7\pi/4} \end{pmatrix}$$

Rotation-Y Gate $R_Y(\theta)$

$$R_Y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$|\psi\rangle \; - \boxed{R_Y(\theta)} - \; |\psi'\rangle$$

$$R_Y(\theta)\,|0\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,|1\rangle$$

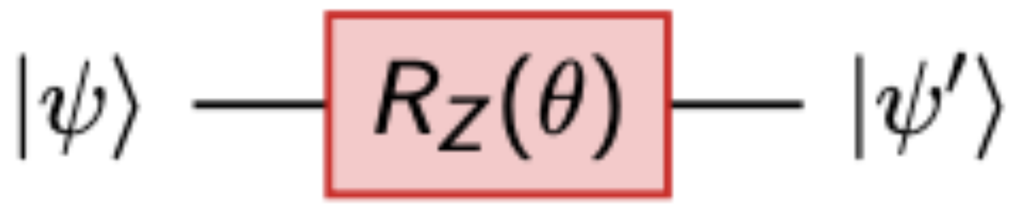
$$R_Y(\theta)\,|1\rangle = -\sin\frac{\theta}{2}\,|0\rangle + \cos\frac{\theta}{2}\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}\,|1\rangle$$

$$R_Y(\theta)\,|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\cos(\pi/8) - \sin\frac{\theta}{2}\sin(\pi/8)e^{i7\pi/4} \\ \sin\frac{\theta}{2}\cos(\pi/8) + \cos\frac{\theta}{2}\sin(\pi/8)e^{i7\pi/4} \end{pmatrix}$$

Rotation-Z Gate $R_Z(\theta)$

$$R_Z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$



$$R_Z(\theta)\,|0\rangle = e^{-i\theta/2}\,|0\rangle$$

$$R_Z(\theta)\,|1\rangle = e^{i\theta/2}\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)e^{i7\pi/4}\,|1\rangle$$

$$R_Z(\theta)\,|\psi\rangle = \begin{pmatrix} \cos(\pi/8)e^{-i\theta/2} \\ \sin(\pi/8)e^{i(7\pi/4+\theta/2)} \end{pmatrix}$$

S Gate is Reversible

S Gate:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Inverse ($S^\dagger$ Gate):

$$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

Check Reversibility:

$$S^\dagger S = S S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Conclusion:

✔ $S^\dagger$ exists → **S gate is reversible**