Complementary Slackness theorem

Let $\bar{x}$ and $\bar{y}$ be feasible solutions to the primal-dual pair (*, **)

Primal

(*) Max $z = c^T x$

s.t. $Ax \le b$

$x \ge 0$

$\hookrightarrow (s^P)$

Dual

Min $w = b^T y$

(**) s.t. $A^T y \ge c$

$y \ge 0$ $\rightarrow (s^D)$

Then $\bar{x}$ and $\bar{y}$ are optimal to the respective problems (*) & (**) if and only if

$$\left.\begin{aligned} &\bar{y}^T (A\bar{x} - b) = 0 \\ \text{and} \quad &\bar{x}^T (c - A^T \bar{y}) = 0 \end{aligned}\right\} \text{Complementary slackness Conditions.}$$

① If in primal table, $x_k > 0$

then $s_k^D = 0$ $\rightarrow$ (k^{th} dual constraint is equality constraint)

② If in primal table, $s_k^P > 0$

then $y_k = 0$ in dual problem

$\downarrow$

(k^{th} dual variable is zero)

Q Consider the following LPP

Max $z = x_1 + 5x_2 + 3x_3$

s.t. $x_1 + 2x_2 + x_3 = 3$

$2x_1 - x_2 = 4$

$x_1, x_2, x_3 \ge 0$

If x_1 and x_3 are the basic variables in optimal solution table of the primal then find the optimal solution of dual.

Primal LPP

$$\text{Max } Z = x_1 + 5x_2 + 3x_3$$

s.t. $x_1 + 2x_2 + x_3 = 3 \longrightarrow y_1$ (unrestricted)

$2x_1 - x_2 = 4 \longrightarrow y_2$ (")

$\underbrace{x_1}_{\downarrow}, \underbrace{x_2}_{\downarrow}, \underbrace{x_3}_{\downarrow} \geq 0$

(" $\geq$ ")

Dual variables

Dual

Min $\phi = 3y_1 + 4y_2$

s.t. $y_1 + 2y_2 \geq 1$

$2y_1 - y_2 \geq 5$

$y_1 \geq 3$

y_1, y_2, y_3 are unrestricted in sign.

Standard form

$\Downarrow$

Min $\phi = 3y_1 + 4y_2$

s.t. $y_1 + 2y_2 - s_1 = 1$

$2y_1 - y_2 - s_2 = 5$

$y_1 - s_3 = 3$

y_1, y_2, y_3 are unrestricted

Since x_1 and x_3 are basic variables in primal optimal table

By Complementary Slackness theorem,

$s_1 = 0, s_3 = 0$

$y_1 + 2y_2 = 1 \implies \boxed{y_2 = -1}$

$\boxed{y_1 = 3}$

and Min $\phi = 3y_1 + 4y_2 = 3(3) + 4(-1)$
$= 5$

$\Rightarrow$ Optimal dual solution is $y_1 = 3$, $y_2 = -1$ and Min $\phi = 5$

Q Formulate the LPP from the following optimal table

C_B	B.V.	x_1	x_2	x_3	s_1	s_2	Sol.
	z	0	0	17/7	6/7	4/7	2 $\}$ $f(x_B)$
4	x_2	0	1	4/7	2/7	-1/7	0 $\}$ X_B
2	x_1	1	0	17/7	-1/7	4/7	1

C_B^T $\quad \alpha^1 \quad \alpha^2 \quad \alpha^3 \quad B^{-1}$

Max ✓

2 Constraints " $\leq$ "

s_1 & s_2 are starting basic variables

$$\begin{array}{ccc} & x_2 & x_1 \\ B = & \multicolumn{2}{c}{\begin{bmatrix} b & a \\ c & d \end{bmatrix}} \end{array}$$

Coefficients in Coefficient matrix

$b = BX_B$

$\alpha^1 = B^{-1}A_1$

$\alpha^2 = B^{-1}A_2$

$X_B = B^{-1}b$

$f(X_B) = C_B^T X_B$

$\alpha^j = B^{-1}A_j$

$z_j - c_j = C_B^T \alpha^j - c_j$

$z_3 - c_3 = C_B^T \alpha^3 - c_3$

$\Downarrow$

$\frac{17}{7} = C_B^T \alpha^3 - c_3$

Max $Z = 2x_1 + 4x_2 + c_3x_3$

s.t. $ax_1 + bx_2 + cx_3 \le b_1$

$dx_1 + ex_2 + fx_3 \le b_2$

$x_1, x_2, x_3 \ge 0$

$x_1, x_2 \ge 0$

find $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \\ \begin{array}{ccc} A_1 & A_2 & \end{array} \end{array}$

$c_3 = ?$

$\alpha^1 = B^{-1}A_1$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{bmatrix}\begin{bmatrix} a \\ d \end{bmatrix} \Rightarrow \begin{array}{l} \frac{2}{7}a - \frac{1}{7}d = 0 \Rightarrow 2a - d = 0 \\ -\frac{1}{7}a + \frac{4}{7}d = 1 \Rightarrow -a + 4d = 7 \end{array} \left\{ \begin{array}{l} a = 1 \\ d = 2 \end{array} \right.$

$\Rightarrow A_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\alpha^2 = B^{-1}A_2 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{bmatrix}\begin{bmatrix} b \\ e \end{bmatrix} \Rightarrow \begin{array}{l} \frac{2}{7}b - \frac{1}{7}e = 1 \\ -\frac{1}{7}b + \frac{4}{7}e = 0 \end{array}$

$\Rightarrow \begin{array}{l} 2b - e = 7 \\ -b + 4e = 0 \end{array} \Rightarrow \begin{array}{l} b = 4 \\ e = 1 \end{array}$

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$\Rightarrow A_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\therefore B = \begin{bmatrix} b & a \\ e & d \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$

$\therefore X_B = B^{-1}b \Rightarrow b = BX_B = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow \boxed{b_1 = 1, b_2 = 2}$

$\alpha^3 = B^{-1}A_3 \Rightarrow \begin{bmatrix} 1/7 \\ 17/7 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{bmatrix}\begin{bmatrix} c \\ f \end{bmatrix} \Rightarrow \begin{array}{l} \frac{1}{7} = \frac{2}{7}c - \frac{1}{7}f \\ \frac{17}{7} = -\frac{1}{7}c + \frac{4}{7}f \end{array}$

$\Rightarrow \begin{array}{l} 2c - f = 1 \\ -c + 4f = 17 \Rightarrow \end{array} \begin{array}{l} 8c - 4f = 4 \\ \cancel{-2c + 8f} \\ -c + 4f = 17 \end{array} \Bigg\} \begin{array}{l} 7c = 21 \\ c = \frac{21}{7} = 3 \end{array}$

$$2c - f = 1$$

$$\Rightarrow 6 - f = 1 \Rightarrow f = 5$$

$$c = 3,\ f = 5 \qquad \Rightarrow A_3 = \begin{bmatrix} c \\ f \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$z_3 - c_3 = c_B^T \alpha^3 - c_3 \qquad \Rightarrow \frac{17}{7} = (4 \quad 2)\begin{pmatrix} 1/7 \\ 17/7 \end{pmatrix} - c_3$$

$$\Rightarrow \frac{17}{7} = \left(\frac{4}{7} + \frac{34}{7}\right) - c_3 \Rightarrow$$

$$\frac{17}{7} = \frac{38}{7} - c_3 \qquad \Rightarrow c_3 = 3$$

Final LPP

$$\text{Max } Z = 2x_1 + 4x_2 + 3x_3$$

$$\text{s.t. } x_1 + 4x_2 + 3x_3 \le 1$$

$$2x_1 + x_2 + 5x_3 \le 2$$

$$x_1, x_2, x_3 \ge 0$$

Q Use the Simplex Method to show that the following problem has unbounded solution:

$$\text{Max } Z = x_1 + x_2$$

$$\text{s.t. } 3x_1 - 4x_2 \ge -3 \longrightarrow -3x_1 + 4x_2 \le 3$$

$$x_1 - x_2 - x_3 = 0 \longrightarrow -x_1 + x_2 + x_3 = 0$$

$$x_1, x_2, x_3 \ge 0$$

Standard form

$$\text{Max } Z = x_1 + x_2$$

$$\text{s.t. } -3x_1 + 4x_2 + s_1 = 3$$

$$-x_1 + x_2 + x_3 = 0$$

$$x_1, x_2, x_3, s_1 \ge 0$$

Apply Simplex Method

Q Two Consecutive simplex tables of a LPP are

B.V	x_1	x_2	x_3	x_4	x_5	Sol.
Z	$A=\frac{9}{2}$	-1	3	0	0	
x_4	$\boxed{B=3}$	$C=2$	$D=2$	$I=1$	0	6
x_5	-1	2	$E=-1$	0	1	1

x_1 entering variable
x_4 leaving variable
$B=3$
$C=2$
$D=2$
$F=\frac{6}{3}=2$

B.V	x_1	x_2	x_3	x_4	x_5	Sol.
Z	0	-4	$J=0$	$K=-\frac{3}{2}$	0	
x_1	$G=1$	$2/3$	$2/3$	$1/3$	0	$F=2$
x_5	$H=0$	$8/3$	$-1/3$	$1/3$	1	3

pivot row

2 rows

Find the values of "A to K".

Znew row = old zrow − A times pivot row (x_1 − row)

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$$\begin{array}{ccccc} A & -1 & 3 & 0 & 0 \\ -A & -\frac{2}{3}A & -\frac{2}{3}A & -\frac{1}{3}A & 0 \\ \hline 0 & -4 & J & K & 0 \end{array}$$

$$\Rightarrow -1-\frac{2}{3}A=-4 \Rightarrow \frac{2}{3}A=3 \Rightarrow \boxed{A=\frac{9}{2}}$$

$$3-\frac{2}{3}A=J \Rightarrow 3-\frac{2}{3}\left(\frac{9}{2}\right)=J \Rightarrow \boxed{J=0}$$

$$-\frac{1}{3}A=K \Rightarrow \boxed{K=-\frac{3}{2}}$$

old x_5-row = New x_5-row + pivot row

$\boxed{E=-1}$

Q Solve the following systems of equations using Simplex Method. (or variants)

(a)
$$2x_1 + x_2 - x_3 = 1$$
$$-2x_1 + 2x_2 - x_3 = -2$$
$$x_1 + x_3 = 3$$
$$x_1, x_2, x_3 \geq 0$$

Min $Z = a_1 + a_2 + a_3$

s.t. $2x_1 + x_2 - x_3 + a_1 = 1$

$2x_1 - 2x_2 + x_3 + a_2 = 2$

$x_1 + x_3 + a_3 = 3$

$x_1, x_2, x_3, a_1, a_2, a_3 \geq 0$

(Apply Phase-I of Two-Phase Method)

(b)
$$x_1 - x_2 + x_3 = 1$$
$$x_1 + x_3 = 2$$
$$2x_1 + x_2 + 2x_3 = 3$$

Min $Z = a_1 + a_2 + a_3$

s.t. $x_1 - x_2 + x_3 + a_1 = 1$

$x_1 + x_3 + a_2 = 2$

$2x_1 + x_2 + 2x_3 + a_3 = 3$

x_1, x_2, x_3 are unrestricted in sign

$a_1, a_2, a_3 \geq 0$

(Firstly convert x_1, x_2, x_3 from unrestricted to restricted and Apply phase-I)

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Q Solve by the Simplex Method (without using artificial variables)

(a) Min $Z = -5x_1 - 3x_2$

s.t. $x_1 + x_2 + x_3 = 2$

$5x_1 + 2x_2 + x_4 = 10$

$3x_1 + 8x_2 + x_5 = 12$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Here x_3, x_4, x_5 will act as starting basic variables that makes the identity.

(Apply simplex)

(It is already in standard form)

(b) Max $Z = 3x_1 + x_2 + 2x_3$

s.t. $12x_1 + 3x_2 + 6x_3 + 3x_4 = 9$

$8x_1 + x_2 - 4x_3 + 2x_5 = 10$

$3x_1 - x_6 = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Max $Z = 3x_1 + x_2 + 2x_3$

s.t. $4x_1 + x_2 + 2x_3 + x_4 = 3$

$4x_1 + \frac{1}{2}x_2 - 2x_3 + x_5 = 5$

$-3x_1 + x_6 = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

(Here x_4, x_5 & x_6 are starting basic variables)