Qunatum measurement theory

Step 1: Normalise the qubit

The qubit must satisfy:

$$\langle\psi|\psi\rangle = 1$$

That means:

$$|\alpha|^2 + |\beta|^2 = 1$$

If it is not normalized → probabilities will be wrong.

So first:

> ✓ Make sure the state is normalized.

Step 2: Find Eigenvalues and Eigenstates of the Operator

Suppose you are measuring operator A.

Solve:

$$A|\phi\rangle = \lambda|\phi\rangle$$

Then:

- λ → eigenvalues (possible results)
- $|\phi\rangle$ → eigenstates (final states)

Example:

For X, Y, Z → eigenvalues are ±1.

Step 3: Construct Projection Operators

For each eigenstate $|\phi_i\rangle$:

$$P_i = |\phi_i\rangle\langle\phi_i|$$

These are projectors.

They represent each possible outcome.

Step 4: Find Probabilities

If system is in $|\psi\rangle$, then:

$$P(i) = \langle\psi|P_i|\psi\rangle$$

This gives probability of outcome i.

Step 5: Construct Projection Operators

If outcome i occurs, new state becomes:

$$|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{P(i)}}$$

Which is just the eigenstate.

Step6: State After Measurement (Collapse)

If outcome i occurs, new state becomes:

$$|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{P(i)}}$$

Which is just the eigenstate.

Example

A qubit is in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$$

A measurement with respect to Y is made. Given that the eigenvalues of the Y matrix are ± 1, determine the probability that the measurement result is $+1$ and the probability that the measurement result is -1.

Step 1: Normalise the qubit

First we verify that the state is normalized

$$\langle\psi|\psi\rangle = \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{1}{2}\langle 1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$$

$$= \frac{3}{4}\langle 0|0\rangle - \frac{\sqrt{3}}{4}\langle 1|0\rangle - \frac{\sqrt{3}}{4}\langle 0|1\rangle + \frac{1}{4}\langle 1|1\rangle$$

$$= \frac{3}{4} + \frac{1}{4} = 1$$

Step 2: Find Eigenvalues and Eigenstates of the Operator

Since $\langle\psi|\psi\rangle = 1$ the state is normalized. Recall that $Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. You need to show that the eigenvectors of the Y matrix are

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

corresponding to the eigenvalues ± 1, respectively. The dual vectors in each case, found by computing the transpose of each vector can taking the complex conjugate of each element, are

$$\langle u_1| = (|u_1\rangle)^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}, \quad \langle u_2| = (|u_2\rangle)^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}$$

Step 3: Construct Projection Operators

The projection operators corresponding to each possible measurement result are

$$P_{+1} = |u_1\rangle\langle u_1| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$P_{-1} = |u_2\rangle\langle u_2| = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}\begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Writing the state $|\psi\rangle$ as a column vector, we have

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$$

Step 4: Find Probabilities

$$P_{+1}|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sqrt{3}+i \\ -1+i\sqrt{3} \end{pmatrix}$$

$$P_{-1}|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sqrt{3}-i \\ -1-i\sqrt{3} \end{pmatrix}$$

Now, if a measurement is made of the Y observable, the probability of finding $+1$ is

$$\Pr(+1) = \langle\psi|P_{+1}|\psi\rangle = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \end{pmatrix}\frac{1}{4}\begin{pmatrix} \sqrt{3}+i \\ -1+i\sqrt{3} \end{pmatrix}$$

$$= \frac{1}{8}(3+i\sqrt{3}+1-i\sqrt{3}) = \frac{1}{8}(3+1) = \frac{1}{2}$$

Similarly find

$$\Pr(-1) = \langle\psi|P_{-1}|\psi\rangle = \frac{1}{2}(\sqrt{3} \quad -1)\frac{1}{4}\begin{pmatrix} \sqrt{3}-i \\ -1-i\sqrt{3} \end{pmatrix}$$

$$= \frac{1}{8}(3-i\sqrt{3}+1+i\sqrt{3}) = \frac{1}{8}(3+1) = \frac{1}{2}$$

Example 2:

A system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{6}}|0\rangle + \sqrt{\frac{5}{6}}|1\rangle$$

A measurement is made with respect to the observable X. What is the expectation or average value?

Solution

The eigenvectors of $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are

$$|+_x\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-_x\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

The projection operator corresponding to a measurement of $+1$ is

$$\begin{aligned} P_+ &= |+_x\rangle\langle +_x| = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right) \\ &= \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right) \end{aligned}$$

The projection operator corresponding to a measurement of -1 is

$$\begin{aligned} P_- &= |-_x\rangle\langle -_x| = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0| - \langle 1|}{\sqrt{2}}\right) \\ &= \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \end{aligned}$$

The probability of finding each measurement result is

$$
\begin{aligned}
\Pr(+1) &= \langle\psi|P_+|\psi\rangle \\
&= \left(\frac{1}{\sqrt{6}}\langle 0| + \sqrt{\frac{5}{6}}\langle 1|\right)\left(\frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)\right)\left(\frac{1}{\sqrt{6}}|0\rangle + \sqrt{\frac{5}{6}}|1\rangle\right) \\
&= \left(\frac{1}{\sqrt{6}}\langle 0| + \sqrt{\frac{5}{6}}\langle 1|\right)\left(\frac{1+\sqrt{5}}{2\sqrt{6}}|0\rangle + \frac{1+\sqrt{5}}{2\sqrt{6}}|1\rangle\right) \\
&= \frac{6+2\sqrt{5}}{12} \\
\Pr(-1) &= \langle\psi|P_-|\psi\rangle \\
&= \left(\frac{1}{\sqrt{6}}\langle 0| + \sqrt{\frac{5}{6}}\langle 1|\right)\left(\frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)\right)\left(\frac{1}{\sqrt{6}}|0\rangle + \sqrt{\frac{5}{6}}|1\rangle\right) \\
&= \left(\frac{1}{\sqrt{6}}\langle 0| + \sqrt{\frac{5}{6}}\langle 1|\right)\left(\frac{1-\sqrt{5}}{2\sqrt{6}}|0\rangle + \frac{-1+\sqrt{5}}{2\sqrt{6}}|1\rangle\right) \\
&= \frac{6-2\sqrt{5}}{12}
\end{aligned}
$$

Notice that the probabilities sum to 1:

$$
\langle\psi|P_+|\psi\rangle + \langle\psi|P_-|\psi\rangle = \frac{6+2\sqrt{5}}{12} + \frac{6-2\sqrt{5}}{12} = 1
$$

The average value is

The average value is

$$\begin{aligned}\langle X\rangle &= (+1)\,\Pr(+1) + (-1)\,\Pr(-1)\\ &= \frac{6+2\sqrt{5}}{12} - \left(\frac{6-2\sqrt{5}}{12}\right) = \frac{\sqrt{5}}{3} \approx 0.75\end{aligned}$$

Suppose observable A has:

- Eigenvalues: a_i
- Eigenstates: $|\phi_i\rangle$

Then the projection operator is:

$$P_i = |\phi_i\rangle\langle\phi_i|$$

Now, the expectation value of A in state $|\psi\rangle$ is:

$$\boxed{\langle A\rangle = \sum_i a_i \,\langle\psi|P_i|\psi\rangle}$$

Measurements on composite systems

Example 1

Describe the action of the operators $P_0 \otimes I$ and $I \otimes P_1$ on the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

For two qubits, states look like:

$$|a\rangle \otimes |b\rangle = |ab\rangle$$

Operators act the same way:

$$(A \otimes B)(|a\rangle \otimes |b\rangle) = (A|a\rangle) \otimes (B|b\rangle)$$

This is the key rule.

The first operator, $P_0 \otimes I$, tells us to apply the projection operator, $P_0 = |0\rangle\langle 0|$, to the *first qubit* and to leave the second qubit alone. The result is

$$P_0 \otimes I|\psi\rangle = \frac{1}{\sqrt{2}}[(|0\rangle\langle 0|0\rangle) \otimes |1\rangle - (|0\rangle\langle 0|1\rangle \otimes |0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

The second operator, $I \otimes P_1$, tells us to leave the *first qubit alone* and to apply the projection operator $P_1 = |1\rangle\langle 1|$ to the second qubit. This gives

$$I \otimes P_1|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes (|1\rangle\langle 1|1\rangle) - |1\rangle \otimes (|1\rangle\langle 1|0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

Example 2

A system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

(a) What is the probability that measurement finds the system in the state $|\phi\rangle = |01\rangle$?

(b) What is the probability that measurement finds the first qubit in the state $|0\rangle$? What is the state of the system after measurement?

Recall Born Rule

If a system is in state $|\psi\rangle$, then the probability of getting outcome $|\phi\rangle$ is:

$$P = |\langle\phi|\psi\rangle|^2$$

Born rule can also be written as:

$$P = \langle\psi|P_\phi|\psi\rangle$$

where:

$$P_\phi = |\phi\rangle\langle\phi|$$

Because:

$$\langle\psi|P_\phi|\psi\rangle = |\langle\phi|\psi\rangle|^2$$

So same rule, different form.

Answer Example 2:

Given that the system is in the state $|\psi\rangle$, the probability of finding it in the state $|\phi\rangle = |01\rangle$ is calculated using the Born rule, which is $\Pr = |\langle\phi|\psi\rangle|^2$. Since $\langle 0|1\rangle = \langle 1|0\rangle = 0$, we have

$$\begin{aligned}
\langle\phi|\psi\rangle &= \langle 01|\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle\right) \\
&= \frac{1}{\sqrt{8}}\langle 0|0\rangle\langle 1|0\rangle + \sqrt{\frac{3}{8}}\langle 0|0\rangle\langle 1|1\rangle + \frac{1}{2}\langle 0|1\rangle\langle 1|0\rangle + \frac{1}{2}\langle 0|1\rangle\langle 1|1\rangle \\
&= \sqrt{\frac{3}{8}}
\end{aligned}$$

Therefore the probability is

$$\Pr = |\langle\phi|\psi\rangle|^2 = \frac{3}{8}$$

To find the probability that measurement finds the first qubit in the state $|0\rangle$, we can apply $P_0 \otimes I = |0\rangle\langle 0| \otimes I$ to the state. So the projection operator P_0 is applied to the first qubit and the identity operator to the second qubit, leaving the second qubit unchanged. This obtains

$$\begin{aligned}
P_0 \otimes I|\psi\rangle &= (|0\rangle\langle 0| \otimes I)\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle\right) \\
&= \frac{1}{\sqrt{8}}|0\rangle\langle 0|0\rangle \otimes |0\rangle + \sqrt{\frac{3}{8}}|0\rangle\langle 0|0\rangle \otimes |1\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle \otimes |1\rangle \\
&= \frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle
\end{aligned}$$

The probability of obtaining this result is

$$\begin{aligned}
\mathrm{Pr} &= \langle\psi|P_0 \otimes I|\psi\rangle \\
&= \left(\frac{1}{\sqrt{8}}\langle 00| + \sqrt{\frac{3}{8}}\langle 01| + \frac{1}{2}\langle 10| + \frac{1}{2}\langle 11|\right)\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle\right) \\
&= \frac{1}{8} + \frac{3}{8} = \frac{1}{2}
\end{aligned}$$

State of the system after measurement

$$|\psi'\rangle = \frac{\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle}{\sqrt{\langle\psi|P_0 \otimes I|\psi\rangle}} = \sqrt{2}\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle\right)$$

$$= \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|01\rangle$$

Try

A three-qubit system is in the state

$$|\psi\rangle = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle$$

(a) Is the state normalized? What is the probability that the system is found in the state $|000\rangle$ if all 3 qubits are measured?

(b) What is the probability that a measurement on the first qubit only gives 0? What is the postmeasurement state of the system?

Solution

To determine if the state is normalized, we compute the sum of the squares of the coefficients:

$$\sum_i |c_i|^2 = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)\left(\frac{\sqrt{2}-i}{\sqrt{20}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right)$$
$$+ \left(\frac{i}{2}\right)\left(-\frac{i}{2}\right)$$
$$= \frac{3}{20} + \frac{1}{2} + \frac{1}{10} + \frac{1}{4} = \frac{20}{20} = 1$$

So the state is normalized. The probability the system is found in state $|000\rangle$ if all three qubits are measured is

$$\Pr(000) = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)\left(\frac{\sqrt{2}-i}{\sqrt{20}}\right) = \frac{3}{20} = 0.15$$

The probability that a measurement on the first qubit is zero can be found by acting on the state with the operator $P_0 \otimes I \otimes I$ and computing $\langle\psi|P_0 \otimes I \otimes I|\psi\rangle$. This will project onto the $|0\rangle$ state for the first qubit while leaving the second and third qubits

The probability that a measurement on the first qubit is zero can be found by acting on the state with the operator $P_0 \otimes I \otimes I$ and computing $\langle\psi|P_0 \otimes I \otimes I|\psi\rangle$. This will project onto the $|0\rangle$ state for the first qubit while leaving the second and third qubits

$$\begin{aligned}
P_0 \otimes I \otimes I|\psi\rangle &= \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)(|0\rangle\langle 0| \otimes I \otimes I)|000\rangle + \frac{1}{\sqrt{2}}(|0\rangle\langle 0| \otimes I \otimes I)|001\rangle \\
&+ \frac{1}{\sqrt{10}}(|0\rangle\langle 0| \otimes I \otimes I)|011\rangle + \frac{i}{2}(|0\rangle\langle 0| \otimes I \otimes I)|111\rangle \\
&= \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle
\end{aligned}$$

The last term vanishes, since $\langle 0|1\rangle = 0$ So

$$\frac{i}{2}(|0\rangle\langle 0| \otimes I \otimes I)|111\rangle = \frac{i}{2}(|0\rangle\langle 0|1\rangle) \otimes |1\rangle \otimes |1\rangle = 0$$

Hence the probability that measurement on the first qubit finds 0 is

$$\langle\psi|P_0 \otimes I \otimes I|\psi\rangle = \left|\frac{\sqrt{2}+i}{\sqrt{20}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{10}}\right|^2 = \frac{3}{20} + \frac{1}{2} + \frac{1}{10} = \frac{3}{4}$$

The postmeasurement state is

$$\begin{aligned}
|\psi'\rangle &= \frac{P_0 \otimes I \otimes I|\psi\rangle}{\sqrt{\langle\psi|P_0 \otimes I \otimes I|\psi\rangle}} = \sqrt{\frac{4}{3}}\left(\left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle\right) \\
&= \left(\frac{\sqrt{2}+i}{\sqrt{15}}\right)|000\rangle + \sqrt{\frac{2}{3}}|001\rangle + \sqrt{\frac{2}{15}}|011\rangle
\end{aligned}$$