Practice Question :-

Ques: Find the Eigenvalues & Eigen vectors of X operator.

Start with Matrix rep of X :-

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det |X - \lambda I| = 0$$

find λ

use $X|\psi\rangle = \lambda |\psi\rangle$

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

find $\det|X - \lambda I|$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0 \qquad \lambda^2 = 1$$

$$\boxed{\lambda = \pm 1}$$

Sin $(|0\rangle, |1\rangle)$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Use Equation

$$A|\phi\rangle = \lambda|\phi\rangle$$

where $|\phi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$\boxed{\text{for } \lambda = +1}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$b = a$$

$$a = b$$

$$|a|^2 + |a|^2 = 1$$

$$2|a|^2 = 1$$

$$|a|^2 = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{\sqrt{2}}$$

also $b = \frac{1}{\sqrt{2}}$

$\therefore$ eigen vector for $\lambda = 1$ is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Reminder

For normalizing $|u\rangle$
use! $\rightarrow \frac{|u\rangle}{||u||}$

classmate
Date
Page

for $\lambda = -1$
using $X|\phi\rangle = \lambda|\phi\rangle$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ a \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$b = -a$
$a = -b$

for $\lambda = -1$ $|\phi\rangle = \begin{pmatrix} a \\ -a \end{pmatrix}$
$= a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

~~We can fix $a=1, b=-1$~~
$\therefore (a,b) = (t,-t)\ \forall\ t \neq 0$

Norm = $\sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$$\boxed{\text{so } |\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$
normalized
Eigen vector

$|u_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\overset{|u_2\rangle}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ is not ortho normal.

$||u_1|| = \sqrt{(1)^2+(1)^2} = \sqrt{2}$
$||u_2|| = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$\therefore |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

another ortho normal basis set

Date ______
Page ______

Q: What is a Normalized Eigen vector

Ans: A vector is normalized if its length (norm) is 1

$$\|\vec{v}\| = 1$$

Here $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

norm is $\sqrt{|a|^2 + |b|^2} = \|\vec{v}\|$

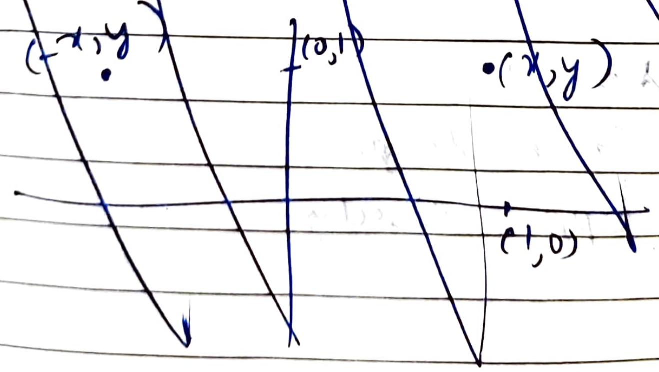
<u>Ortho normality</u>

✓ When the norm of a vector is unity vector is ortho normal $\sqrt{\langle \psi|\psi \rangle} = 1$

✓ Eigen value doesnot changes the direction. It only changes the magnitude (length) & possibly the sign.

Required

Consider a vector in 2-D



Not Required

Let $T(x,y) \to T(-x, y)$

$\cancel{(x,y) = x(1,0) + y(0,1)}$

~~[illegible]~~

$\boxed{\text{Eigen value means}}$

$$Av = \lambda v$$

where $v = |\phi\rangle$

$|\phi\rangle$ is sent to a multiple of itself
so the result is along same line.

- If $\lambda > 0$
  Direction is same
  length of vector is scaled by $|\lambda|$

- $\lambda < 0$
  Direction is reversed (flipped)
  Length scaled by $|\lambda|$

- $\lambda = 1$
  Nothing changes

- $\lambda = -1$
  Same length
  opp. direction

Let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Apply pauli-X

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot v = v$$

Same length, same direction

Let $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Apply pauli X

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \cdot v$$

Same length
opposite direction

Date
Page

Spectral Decomposition Theorem

Any hermitian operator A acting on quantum states can be written as sum of its eigen values multiplied by projectors onto the corresponding eigen vectors.

i.e $A = \sum_i \lambda_i |u_i\rangle\langle u_i|$

λ_i = Eigen values of operator

u_i = basis of vector space.

$|u_i\rangle\langle u_i|$ = projector onto eigen vector

Pauli z - operator

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

eigen values = $\lambda_1 = 1$, $\lambda_2 = -1$

Eigen vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

∴ spectral decomposition

$Z = 1 \cdot |0\rangle\langle 0| - 1 |1\rangle\langle 1| =$

$$Z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties first

classmate

Date

Page

BASIS and Dimension

when a set of vectors is linearly independant and they span the space the set is called basis set.

The dimension of a vector space V is equal to the no. of elements in the basis set.

For C^2, basis set is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Again

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \alpha+\alpha \\ \alpha-\alpha \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \beta-\beta \\ \beta+\beta \end{pmatrix}$$

(0) If each element of a set of vectors is normalised and the elements are orthogonal w.r.t each other, we say the set is orthonormal.

self inner prod $= 1$

cross inner prod $= 0$

$$= \frac{1}{2}\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\beta\\ \beta\end{pmatrix}$$

norm $= \sqrt{(4/4)}$

$$- \frac{1}{2}\begin{pmatrix}\beta\\ -\beta\end{pmatrix}$$

$$\frac{\alpha}{2}\begin{pmatrix}1\\ 1\end{pmatrix} + \frac{\alpha}{2}\begin{pmatrix}1\\ -1\end{pmatrix} + \frac{\beta}{2}\begin{pmatrix}1\\ 1\end{pmatrix} - \frac{\beta}{2}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\frac{\alpha+\beta}{2}\begin{pmatrix}1\\ 1\end{pmatrix} + \frac{\alpha-\beta}{2}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$= \frac{\alpha+\beta}{2}|u_1\rangle + \frac{\alpha-\beta}{2}|u_2\rangle$$

where $|u_1\rangle = \begin{pmatrix}1\\ 1\end{pmatrix}$ $|u_2\rangle = \begin{pmatrix}1\\ -1\end{pmatrix}$

not orthonormal

$\therefore$ $|u_1\rangle$ & $|u_2\rangle$ is also another basis set that can be used for C^2.

$\Rightarrow$ A quantum state can be written as a linear combination of a basis set $|v_i\rangle$ with complex coefficients c_i

$$\boxed{|\psi\rangle = \sum_i c_i |v_i\rangle}$$

If norm of a vector is unity, we say vector is normalised.

classmate
Date
Page

1) Result:

(1) Eigen vectors of a Hermitian operator constitute an orthonormal basis set for given vector space. (vectors are orthogonal & are normalised)

(2) Eigen vectors of a Hermitian operator corresponding to diff eigen values are orthogonal.

(∵ If inner product b/w two vectors is 0 vectors are orthogonal
$\langle u|v\rangle = 0$

For X gate

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

E.V are $= \pm 1$

X is hermitian
Eigen values are real.

Norm of a vector $= \langle u|u\rangle$

Eigen vectors are

$$|\phi_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\phi_2\rangle \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$\langle \phi_1 | \phi_2 \rangle$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

Inner product = $\langle u|u\rangle$
if $\langle u|u\rangle = 1$ u is normal (~~ortho~~ normality)
$\langle u|v\rangle = 0$ u & v are orthogonal
norm = $\sqrt{\langle u|u\rangle} = \|u\|$

Hence the results.

$\boxed{\text{ORTHONORMAL Set}}$

a Set of Vectors for which $\langle u|u\rangle = 1$ &
$\langle u|v\rangle = 0$
$\forall\, u, v \in$ Set

$\langle u|v\rangle = 0$
$\langle u|u\rangle = 1$

Eigen values of a unitary operator satisfy the following :-

1) The Eigen value of a unitary operator are complex numbers with modulus 1

SPECTRAL DECOMPOSITION

An operator A belonging to some vector space that is normal ($U^{+} = U^{-1}$) and has a diagonal matrix representation w.r.t some basis of that vector space. This result is known as spectral dec. theorem.

i.e we can write the operator in the form

$$A = \sum_{i=1}^{n} a_i |u_i\rangle\langle u_i|$$

where a_i = Eigen value

$|u_i\rangle$ = basis

For X Eigen values are ± 1

Eigen vectors are

$$|u_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad |u_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$|u_1\rangle\langle u_1| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}_{2\times 1} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}_{1\times 2}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$|u_2\rangle\langle u_2| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Using theorem,

$$X = 1 \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1/2 & 1/2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} \frac{1}{2} & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ Hence Proved.

Activity :- Do it for Z operator

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spectral Decomp

$\det |Z - \lambda I| = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$1 - \lambda = 0$

$-1 - \lambda = 0$

$\lambda = \pm 1$

$Z = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \;\; 0) \ldots$

$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ True

$Z|\phi\rangle = \lambda|\phi\rangle$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$

$a = \lambda a$	$-b = \lambda b$
$a(1-\lambda) = 0$	$b(1+\lambda) = 0$
$\lambda = 1$ or $a = 0$	$\lambda = -1$ or $b = 0$

For 1st case when $\lambda = 1$ we get $a \neq 0$

$a = \lambda a \Rightarrow a - \lambda a = 0 \therefore \lambda = 1$ or $a = 0$, $b = 0$

let $a = 1$, $b = 0$

E.vector $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Second case :- ($\lambda = -1$)

$a(1-\lambda) = 0$

$\lambda = 1$ or $a = 0$

$\Rightarrow \boxed{a = 0,\ b \neq 0}$

$b(1+\lambda) = 0$

$b = 0$ or $\lambda = -1$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Page

The Expectation Value of an operator

The expectation value of an operator is the mean or average value of that operator with respect to a given quantum state.

If a quantum state $|\psi\rangle$ is prepared many times, and we measure a given operator A each time, what is the average of the measurement results?

Expectation value is written as:

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

Question: A quantum system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

What is the average or expectation value of X in this state?

Ans: Recalling: $X|0\rangle = |1\rangle$

$X|1\rangle = |0\rangle$

$$X|\psi\rangle = X\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{3}} X|0\rangle + \sqrt{\frac{2}{3}}\, X|1\rangle$$

$$= \frac{1}{\sqrt{3}} |1\rangle + \sqrt{\frac{2}{3}}\, |0\rangle$$

Now $\langle \psi X | \psi \rangle$ — (1)

& $\langle \psi | = \frac{1}{\sqrt{3}} \langle 0 | + \sqrt{\frac{2}{3}} \langle 1 |$

$\therefore$ (1) becomes

$$\left(\frac{1}{\sqrt{3}} \langle 0| + \sqrt{\frac{2}{3}} \langle 1|\right)\left(\frac{1}{\sqrt{3}} |1\rangle + \sqrt{\frac{2}{3}} |0\rangle\right)$$

$$\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} \langle 0|1\rangle + \sqrt{\frac{2}{3}}\frac{1}{\sqrt{3}} \langle 1|1\rangle + \frac{1}{\sqrt{3}}\sqrt{\frac{2}{3}} \langle 0|0\rangle$$

$$+ \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}} \langle 1|0\rangle$$

Using $\langle 0|0\rangle = 1$ $\quad \langle 1|1\rangle = 1$ $\quad \langle 0|1\rangle = 0$ $\quad \langle 1|0\rangle = 0$

We get $\quad \frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}$

This value here told us that if we prepare a large number of systems in the state $|\psi\rangle$ and measure X on each of those systems, then average the results it will be $\frac{2\sqrt{2}}{3}$

Question 2 An operator acts on the qutrit basis states in the following way:

$$A|0\rangle = |0\rangle$$

$$A|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$A|2\rangle = 0$$

find $\langle A\rangle$ for the state

$$|\psi\rangle = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle.$$

Solⁿ: Now $\langle A\rangle = \langle\psi|A|\psi\rangle$

$$A|\psi\rangle = \frac{1}{2}A|0\rangle - \frac{i}{2}A|1\rangle + \frac{1}{\sqrt{2}}A|2\rangle$$

$$= \frac{1}{2}|0\rangle - \frac{i}{2\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{-i}{2\sqrt{2}}|0\rangle + |1\rangle\left(\frac{1}{2} - \frac{i}{2\sqrt{2}}\right)$$

$$\frac{-i}{2\sqrt{2}}|0\rangle + \frac{(\sqrt{2} - i)}{2\sqrt{2}}|1\rangle$$

So $\langle\psi| = \frac{1}{2}\langle 0| + \frac{i}{2}\langle 1| + \frac{1}{\sqrt{2}}\langle 2|$

$\therefore \langle\psi|A|\psi\rangle =$

$$\left(\frac{1}{2}\langle 0| + \frac{i}{2}\langle 1| + \frac{1}{\sqrt{2}}\langle 2|\right)\left(\frac{-i}{2\sqrt{2}}|0\rangle + \frac{(\sqrt{2}-i)}{2\sqrt{2}}|1\rangle\right)$$

$$\frac{-i}{4\sqrt{2}}\langle 0|0\rangle + \frac{\sqrt{2}-i}{4\sqrt{2}}\langle 0|1\rangle +$$

$$\frac{(-i^2)}{4\sqrt{2}}\langle 1|0\rangle + \frac{i(\sqrt{2}-i)}{4\sqrt{2}}\langle 1|1\rangle +$$

$$\frac{(-i)}{4}\langle 2|0\rangle + \frac{\sqrt{2}-i}{4}\langle 2|1\rangle$$

$$= \frac{-i}{4\sqrt{2}} + \frac{i(\sqrt{2} - i)}{4\sqrt{2}}$$

$$= \frac{-i + i\sqrt{2} + 1}{4\sqrt{2}} = \frac{(\sqrt{2}-1)i + 1}{4\sqrt{2}}$$

$\langle 2|0\rangle = 0$

for a qutrit $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}$$

Hence the result.

Obviously Prob of finding the qubit in state $|2\rangle$ when system is in state $|0\rangle$ is 0

UNITARY TRANSFORMATIONS

Learn $|+\rangle$ & $|-\rangle$ States

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

Page

$$|\psi\rangle = \frac{1}{2}\begin{pmatrix} \alpha+\alpha \\ \alpha-\alpha \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \beta-\beta \\ \beta+\beta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \beta \\ \beta \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\beta \\ \beta \end{pmatrix}$$

$$\cancel{\frac{1}{2}\left(\alpha+\beta\right.}$$

$$= \frac{\alpha}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\alpha}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\frac{\alpha+\beta}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\alpha-\beta}{2}\begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

$$|u_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |u_2\rangle = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

Normalizing $|u_1\rangle$ & $|u_2\rangle$

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

$$|u_1\rangle = |+\rangle$$

$$|u_2\rangle = |-\rangle$$

Unitary Transformation

can be used to Transform the matrix representation of an operator in one basis to a representation of that operator in another basis.

Consider a 2-D vector space C^2

1. Change of basis matrix ~~from~~ $|u_i\rangle$ to a basis matrix $|v_j\rangle$

is given by

$$U = \begin{pmatrix} \langle v_1|u_1\rangle & \langle v_1|u_2\rangle \\ \langle v_2|u_1\rangle & \langle v_2|u_2\rangle \end{pmatrix}$$

2. write a vector state $|\psi\rangle$ given in $|u_i\rangle$ basis to a state in $|v_i\rangle$ basis as follows

$$|\psi'\rangle = U|\psi\rangle$$

Note:- $|\psi'\rangle$ is same as $|\psi\rangle$

but it is expressed in $|v_i\rangle$ now.

Problem:- Given an operator in $|u_i\rangle$ basis. write it in terms of $|v_i\rangle$ basis

$$A' = UAU^{+}$$

Ques: • find the change of basis matrix to go from the computational basis $\{|0\rangle, |1\rangle\}$ to $\{|+\rangle, |-\rangle\}$ basis.

• Write $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$ in $\{|+\rangle, |-\rangle\}$ basis.

• write Operator $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$ in $\{|\pm\rangle\}$ basis

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \langle 0| = (1 \quad 0)$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \langle 1| = (0 \quad 1)$

$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \langle +| = \frac{1}{\sqrt{2}}(1 \quad 1)$

$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad |-\rangle = \frac{1}{\sqrt{2}}(1 \quad -1)$

Single Qubit Gates

A gate is an abstraction that represents information processing.

A qubit state superposition is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α, β are complex & $|\alpha|^2 + |\beta|^2 = 1$

Single Qubit gates are the gates that change the state of a single qubit.
These are represented as 2×2 unitary matrices.

A quantum gate with n inputs and outputs can be represented by a matrix of degree 2^n.

For single qubits, $n=1$
so matrix required = deg 2.

A two qubit gate can be implemented using degree $= 2^2 = 4$ or 4×4

$$\hat{U}|\psi\rangle = |\phi\rangle$$

Let $\hat{U}^\dagger|\phi\rangle = |\psi\rangle$

$U^\dagger \hat{U}|\psi\rangle = I|\psi\rangle$

I

$U|\psi\rangle = |\psi\rangle \quad U|\psi\rangle = e^{i\theta}|\psi\rangle$

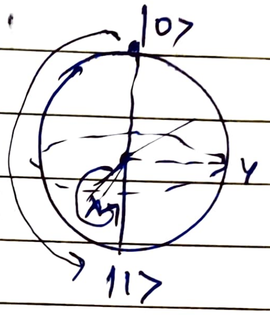
Single Qubit gates:

PAULI GATES

1) X gate (NOT gate)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

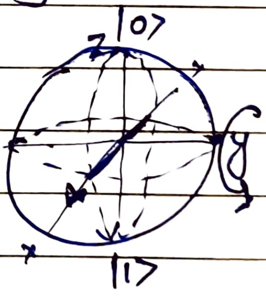
flips $|0\rangle \rightarrow |1\rangle$
$|1\rangle \rightarrow |0\rangle$



Symbol :- X

2) Pauli Y :- Pauli Y moves through imaginary space

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$



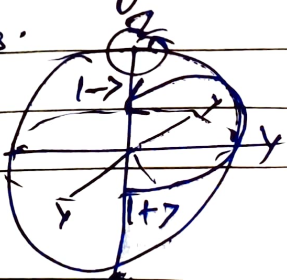
Symbol :- Y

$Y|0\rangle = i|1\rangle$

$Y|1\rangle = -i|0\rangle$

3) Pauli Z :- Pauli Z gate changes the state of qubit along the plane formed by vectors represented by two states.

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$|0\rangle \rightarrow |+\rangle$

$$|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Put a qubit on the equator of Bloch sphere.

$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$ $\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix}$

Projection Operators

A projection operator is an operator that can be formed by writing the outer product using single ket.

Given a state $|\psi\rangle$, the operator

$P = |\psi\rangle\langle\psi|$ is

projection operator.

Normal:-

$U^{\dagger}U = UU^{T}$

Hermitian-

$\hat{A} = \hat{A}^{\dagger}$

Unitary -

$U^{\dagger}U = U^{\dagger}U = I$

1) If state $|\psi\rangle$ is normalized, then projection operator is equal to its own square.

$$P^2 = P$$

2) If P_1 & P_2 are two projection operators s.t $P_1P_2 = P_2P_1$ then P_1P_2 is also a projection operator.

3) Given a vector space of n dimensions and basis set $|1\rangle, |2\rangle --- |n\rangle$

then $P = \sum_{i=1}^{n} |i\rangle\langle i|$

- Spectral decomposition eqn

$$A = \sum_{i=1}^{n} a_i |u_i\rangle\langle u_i|$$

$$= \sum_{i=1}^{n} a_i P_i$$

u_1 = eigen vector

a_i = eigen value

Activity $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$$P_+ = |+\rangle\langle+| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}$$

$$\left\{\begin{array}{l} P_+ = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \quad \cancel{\frac{1}{2}(1\times1 + 1\times1)} = 1 \\ P_- = |-\rangle\langle-| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\end{array}\right.$$

also $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Completeness say $P_+ + P_- = I$

$U(|\psi\rangle) = \phi$

$U^\dagger(|\phi\rangle) = \psi$

$\underline{U^\dagger U}|\psi\rangle = |\psi\rangle$

I

$P_+ = |+\rangle\langle+|$

$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$

$\frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|\Big)$

$P_- = |-\rangle\langle -|$

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)$

$\frac{1}{2}\Big(|0\rangle\langle 0| - |1\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 1|\Big)$

$P_+ - P_- = \frac{1}{\cancel{2}}\Big(\cancel{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\Big)$

$P_+ + P_- = |0\rangle\langle 0| + |1\rangle\langle 1| = I$

$$P_+ + P_- = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I$$

$$\rightarrow \boxed{\sum P_i = I}$$

(*) Any operator can be written in terms of diff projection operators.

$P_0 = \cancel{\quad} \; |0\rangle\langle 0|$

$P_1 = |1\rangle\langle 1|$

$$Z = (+1)P_0 + (-1)P_1$$

Eigen vectors $|0\rangle, |1\rangle$

eigen values $+1, -1$

$$P(r)(i) = |P_i|\psi\rangle|^2$$

$$= \langle\psi|P_i^{\dagger}P_i|\psi\rangle$$

$\because P$ is hermitian

$$\langle\psi|P_i^2|\psi\rangle$$

$$= \langle\psi|P_i|\psi\rangle$$

classmate
Date
Page

Probability of finding the i^{th} outcome if a measurement is made is $\langle\psi|P_i|\psi\rangle$

$$Pr(i) = |P_i|\psi\rangle|^2$$

$$=$$

Probability

For any vector $|v\rangle$ $\qquad |v|^2 = \langle v|v\rangle$

~~$P|\psi|^2 = \quad P_i\langle\psi||P_i\psi\rangle$~~

Here $P_i|\psi|^2 = \langle P_i\psi|P_i\psi\rangle$

$$\langle\psi|P_i^{\dagger}P_i\psi\rangle$$

For Projection operators are hermitian

$$\Rightarrow P_i|\psi\rangle^2 = \langle\psi|P_i^2|\psi\rangle$$

Now $P_i = |u_i\rangle\langle u_i|$

$$P_i^2 = |u_i\rangle\langle u_i||u_i\rangle\langle u_i|$$

$$= |u_i\rangle\langle u_i|$$

$$= P_i$$

$$\Rightarrow \boxed{P_i|\psi|^2 = \langle\psi P_i\psi\rangle}$$

Ques:- A qubit is in State

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

Using Projection operators P_0 and P_1 indicate the probability of finding $|0\rangle$ and the probability of finding $|1\rangle$ when a measurement is made.

$P_0 = |0\rangle\langle 0|$ $\qquad$ $P_1 = |1\rangle\langle 1|$

Now ~~$Pr(0) =$~~

~~$P_0|\psi\rangle =$~~ ~~$\langle\psi|P_0|\psi\rangle$~~

~~$P_1|$~~

$Pr(0) = \underline{\langle\psi|P_0|\psi\rangle}$ $\qquad$ $Pr(1) = \langle\psi|P_1|\psi\rangle$

Q: A three State System is in the State

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle$$

Write down the necessary projection operators and calculate the probabilities $Pr(0)$, $Pr(1)$ and $Pr(2)$

Page

Tensor Products

Let H_1 and H_2 are two hilbert spaces of dimension N_1 and N_2. We can put H_1 & H_2 together to construct a large hilbert space H.

$H = H_1 \otimes H_2$ $\otimes$ = tensor product.

If $\dim(H_1) = N_1$

$\dim(H_2) = N_2$

then $\boxed{\dim(H) = N_1 N_2}$

A state vector $\in H$ is tensor product of state vectors $\in H_1$ and H_2.

Let $|\phi\rangle \in H_1$ and $|\chi\rangle \in H_2$

Let $|\psi\rangle \in H$

& $H = H_1 \otimes H_2$

then $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$

Properties

(1) Tensor product of two vectors is linear

$|\phi\rangle \otimes \left[|\chi_1\rangle + |\chi_2\rangle \right]$

$= |\phi\rangle \otimes |\chi_1\rangle + |\phi\rangle \otimes |\chi_2\rangle$

$$[|\phi_1\rangle + |\phi_2\rangle] \otimes |\chi\rangle$$

$$= |\phi_1\rangle \otimes |\chi\rangle + |\phi_2\rangle \otimes |\chi\rangle$$

(2) Tensor product is linear w.r.t to scalars :

$$|\phi\rangle \otimes \alpha|\chi\rangle = \alpha|\phi\rangle \otimes |\chi\rangle$$

Basis of Hilbert space H

Let $|u_i\rangle$ is basis set of H_1,

$|v_i\rangle$ is basis set of H_2

then basis set of H is constructed as

$$|\psi_i\rangle = |u_i\rangle \otimes |v_i\rangle$$

or $|\psi_i\rangle = |u_i\rangle|v_i\rangle = |u_i v_i\rangle$

Let $|u_i\rangle = \{|0\rangle, |1\rangle\}$

$|v_i\rangle = \{|0\rangle, |1\rangle\}$

then

$$\boxed{\begin{aligned} |\psi_i\rangle = \; & |0\rangle|0\rangle = |00\rangle \\ & |0\rangle|1\rangle = |01\rangle \\ & |1\rangle|0\rangle = |10\rangle \\ & |1\rangle|1\rangle = |11\rangle \end{aligned}}$$

Again let $|u_i\rangle$ = basis set of H_1
$|v_i\rangle$ = basis set of H_2

then

$$|\phi\rangle = \sum \alpha_i |u_i\rangle$$

$$|\psi\rangle = \sum \beta_i |v_i\rangle$$

let $|\chi\rangle = |\phi\rangle \otimes |\psi\rangle$

$$= \sum \alpha_i |u_i\rangle \otimes \sum \beta_i |v_i\rangle$$

$$= \alpha_i \beta_i \sum |u_i\rangle \otimes |v_i\rangle$$

<u>Ques</u>: let $|\phi\rangle \in H_1$ with basis vectors $|x\rangle, |y\rangle$

$$|\phi\rangle = a_x|x\rangle + a_y|y\rangle$$

let $|\psi\rangle \in H_2$ with basis vectors $|u\rangle, |v\rangle$

$$|\psi\rangle = b_x|u\rangle + b_y|v\rangle$$

find $|\chi\rangle = |\phi\rangle \otimes |\psi\rangle$

$$= (a_x|x\rangle + a_y|y\rangle) \otimes (b_x|u\rangle + b_y|v\rangle)$$

$$= a_x b_x |x\rangle \otimes |u\rangle + a_y b_y |y\rangle \otimes |v\rangle + a_x b_y |x\rangle \otimes |v\rangle + a_y b_x |y\rangle|u\rangle$$

Inner Product

Let $|\psi_1\rangle = |\phi_1\rangle \otimes |\chi_1\rangle$

$|\psi_2\rangle = |\phi_2\rangle \otimes |\chi_2\rangle$

where $|\psi_1\rangle, |\psi_2\rangle \in H$

$$H = H_1 \otimes H_2$$

and $|\phi_1\rangle, |\phi_2\rangle \in H_1$

$|\chi_1\rangle, |\chi_2\rangle \in H_2$

then $\langle\psi_1|\psi_2\rangle =$

$$\left(\langle\phi_1| \otimes \langle\chi_1|\right)\left(|\phi_2\rangle \otimes |\chi_2\rangle\right)$$

$$\langle\phi_1|\phi_2\rangle\langle\chi_1|\chi_2\rangle$$

$\Rightarrow$ Take inner products of vectors belonging to H_1 & H_2 & multiply them together

Ques Use $|+\rangle$ $|-\rangle$ states to construct a basis of C^4 & verify the basis is orthonormal.

$H_1 = \{|+\rangle, |-\rangle\}$

$H_2 = \{|+\rangle, |-\rangle\}$

$\therefore H = H_1 \otimes H_2$

$\therefore \{|w_1\rangle, |w_2\rangle, |w_3\rangle, |w_4\rangle\}$ be basis of H

where

$|w_1\rangle = |+\rangle|+\rangle$

$|w_2\rangle = |+\rangle|-\rangle$

$|w_3\rangle = |-\rangle|+\rangle$

$|w_4\rangle = |-\rangle|-\rangle$

for a basis to be orthonormal

$\langle u_i|u_i\rangle = 1$ & $\langle u_j|v_i\rangle = 0$

$\therefore$ Checking $\langle w_i|w_i\rangle = 1$ & $\langle w_i|w_j\rangle = 0$

$$\langle w_1|w_1\rangle = \langle +|\langle +|\,|+\rangle|+\rangle$$
$$= \langle +|(\langle +|+\rangle)|+\rangle$$
$$= \langle +|+\rangle$$
$$= 1$$

using $\langle + | + \rangle = 1$

$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $\langle + | = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}$

$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ $\langle - | = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}$

Similarly check for $\langle w_2 | w_2 \rangle$, $\langle w_3 | w_3 \rangle$, $\langle w_4 | w_4 \rangle$

and $\langle w_1 | w_2 \rangle$, $\langle w_2 | w_3 \rangle$, $\langle w_3 | w_4 \rangle$
$\langle w_4 | w_1 \rangle = 0$

Ques: Given $\langle a | b \rangle = 4$ $\langle c | d \rangle = 7$
Calculate $\langle \psi | \phi \rangle$ where

$|\psi\rangle = |a\rangle \otimes |c\rangle$ and
$|\phi\rangle = |b\rangle \otimes |d\rangle$

$$\langle \psi | \phi \rangle = (\langle a | \otimes \langle c |)(|b\rangle \otimes |d\rangle)$$
$$= \langle a | b \rangle \cdot \langle c | d \rangle$$
$$= 4 \cdot 7 = 28$$

Ques: Given $\langle a | b \rangle = 1$ and $\langle c | d \rangle = -2$
Calculate $\langle \psi | \phi \rangle$
where

$|\psi\rangle = |a\rangle \otimes |c\rangle$ and
$|\phi\rangle = |b\rangle \otimes |d\rangle$

$$\langle \psi | \phi \rangle = \langle a | c \rangle \cdot \langle b | d \rangle$$
$$= (1)(-2) = -2$$

Not all states can be written as
$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$

If you cannot do this → the state is entangled.

$$|\psi\rangle = \frac{1}{2}\{|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\}$$

let $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|\chi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$|\phi\rangle \otimes |\chi\rangle = \frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right)$$

$$= |\psi\rangle$$

$\therefore$ $|\psi\rangle$ is product state.

Try for $|\psi\rangle = \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)$

Tensor Products of Column Vectors