1. **Basics of Graphs**

The graph are mathematical structures used to effectively model pairwise relations between objects. We will use the following standard notations throughout the text.

1.1. **Definition:** Formally, we define a *graph* to be an ordered pair $G = (V, E)$ where $E \subseteq [V]^2 := \{Y \subseteq V : |Y| = 2\}$ or
A *graph*, G is an ordered pair (V, E), where V is a finite set and $E \subseteq \binom{V}{2}$ is a set of pairs of elements in V.

- $G = (V, E)$ is an arbitrary (undirected, simple) graph
- $|V|$ is its number of vertices
- $|E|$ is its number of edges

Example 1.1. *In this example* $V = \{1, 2, 3, 4, 5, 6, 7\}$*, with edge set*
$E = \{\{1, 2\}, \{1, 5\}, \{2, 5\}, \{3, 4\}, \{5, 7\}\}$.

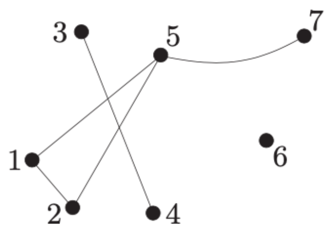
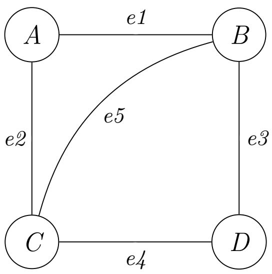


Fig. 1.1.1. The graph on $V = \{1, \ldots, 7\}$ with edge set
$E = \{\{1, 2\}, \{1, 5\}, \{2, 5\}, \{3, 4\}, \{5, 7\}\}$

Example 1.2. *Another example, where vertices are shown as* A, B, C *and* D *and edges labelled as* $e_1 := (A, B), e_2 := (A, C)e_3 := (B, D), e_4 := (C, D), e_5 := (C, D)$.



1.2. **Notations and Terminologies:** For a graph G, the set $V(G)$ is the *vertex set* of G and $E(G)$ is the *edge set.*

- The *order* of G is $|V(G)|$, often written $|G|$.
- The *size* of G is $|E(G)|$, often written $||G||$.
- *Self loop* An edge that is associated with same vertex as both its end vertices is called self loop. or, having an edge with both ends at the same vertex. Such edges are called *loop or self loop.*

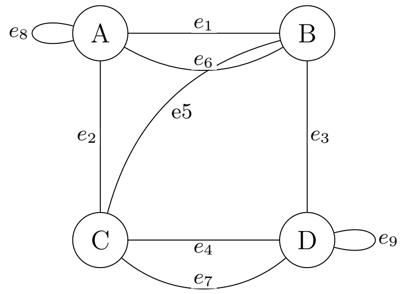
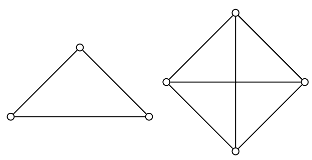


Fig. 1.1. graph with parallel edges and loops

- *Parallel edges* The edges associated with same pair of start & end vertices are called parallel edges, or having more than one edge between the same two vertices. Such edges are called *parallel edges*.

- *Simple graph* A graph that has neither self loop nor parallel edges is called a simple graph. The examples of graph above in Fig 1.1 and 1.2 are clearly simple graphs.
- We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ (rather than $v \in V(G)$), an edge $e \in G$, and so on.
- A vertex v is *incident with an edge* e if $v \in e$; then e is an edge at v. The set of all incidents edges to a vertex v is denoted by $|E(v)|$. The *degree of a vertex* v is defined to be $|E(v)|$.
- Two vertices x, y of G are *adjacent*, or neighbours, if x, y is an edge of G.
- A *pendant vertex* is a vertex whose degree is 1.
- An edge that has a pendant vertex as an end vertex is a pendant edge.
- An *isolated vertex* is a vertex whose degree is 0.
- A graph with no edges is called *Null Graph*.

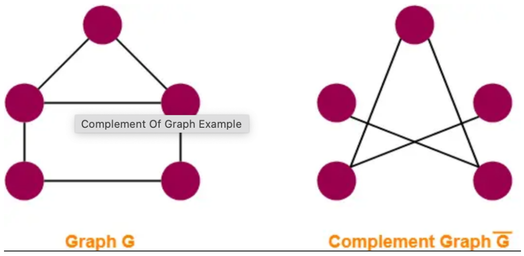
- The degree of a vertex is sometimes also known as valency.
- Two graphs G, H are said to be *isomorphic* if there exists a bijection $f : V(G) \to V(H)$ such that $(u, v) \in E(G) \iff (f(u), f(v)) \in E(H)$.
- A *subgraph* of G is another graph H with $V(H) \subseteq V(G), E(H) \subseteq E(G)$.
- A simple graph that contains every possible edge between all the vertices is called a *complete graph*. The complete graph with n vertices is denoted by K_n.
- A complete graph with n vertices is $([n], \binom{n}{2})$. Every graph of order $\leq n$ is a subgraph of K_n. The following graphs are complete graphs:



K_3 K_4

- The *minimum degree* of the vertices in a graph G is denoted $\delta(G)$ (which equals 0 if there is an isolated vertex in G). Similarly, we write $\Delta(G)$ as the maximum degree of vertices in G.

- The *complement or inverse* of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G. That is, to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there.



- A graph with infinite number of vertices is an example of *Infinite graph.* For example:

$$v_1 \to v_2 \to v_3 \to v_4 \to \cdots$$

Remark 1.3. *In this course, we only consider finite graphs, i.e. V and E are finite sets. We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ (rather than $v \in V(G)$), an edge $e \in G$, and so on.*

1.3. **Subgraphs.** A *subgraph* G' of a graph G is a graph G' whose vertex set and edge set are subsets of those of G. If G' is a subgraph of G, then G is said to be a *supergraph* of G'. An *induced subgraph* of a graph is another graph, formed from a subset of the vertices of the graph and all of the edges, from the original graph, connecting pairs of vertices in that subset.

1.4. **Graph operations.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the following graphs. Then

- $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$
- $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$
- $G_1 - G_2 = (V_1 \setminus V_2, E_1 \setminus E_2)$

Given two graph G and G' shown below, the union, difference and intersection $G \cup G'$, $G \cap G'$, respectively of these graphs as shown below:

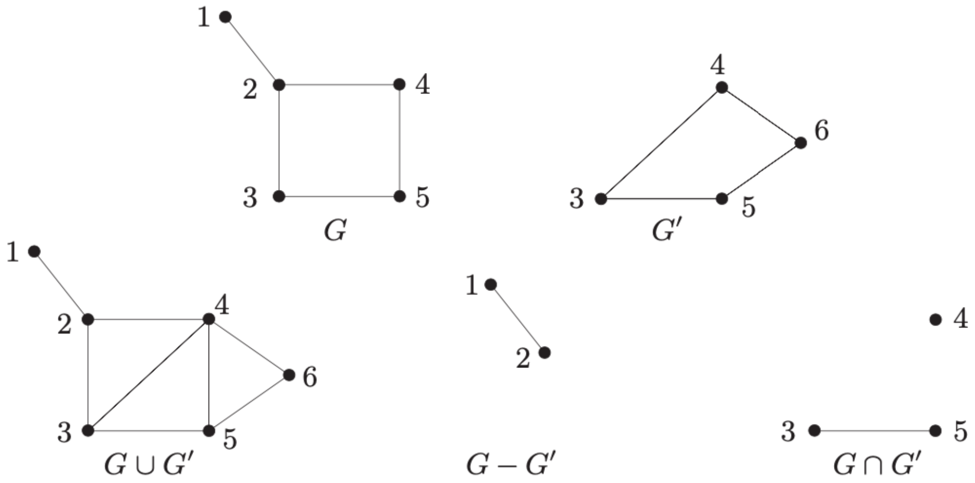


Fig. 1.1.2. Union, difference and intersection; the vertices 2,3,4 induce (or span) a triangle in $G \cup G'$ but not in G

Definition 1.4. *A graph G is said to have been decomposed into two subgraphs H_1 and H_2 if*

$$H_1 \cup H_2 = G \quad \text{and} \quad H_1 \cap H_2 = \text{a null graph.}$$

i.e.,

- *Every edge of G occurs in either H_1 or H_2 but not in both.*
- *Some vertices, however, may occur in both H_1 and H_2.*

Definition 1.5. Deletion of Vertex and Edge, and Fusion in Graph
Vertex Deletion

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a vertex $v \in V$, the deletion of v from G results in a subgraph G' defined as follows:

$$G' = G - v = (V', E')$$

where $V' = V \setminus \{v\}$ and $E' = \{e \in E \mid v \notin e\}$. This means that G' is obtained by removing v and all edges incident to v.

Edge Deletion

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For an edge $e = \{u, v\} \in E$, the deletion of e from G results in a subgraph G' defined as follows:

$$G' = G - e = (V, E')$$

where $E' = E \setminus \{e\}$. This means that G' is obtained by removing e but keeping all vertices of G.

Example 1.6. *Illustration of the above process:*

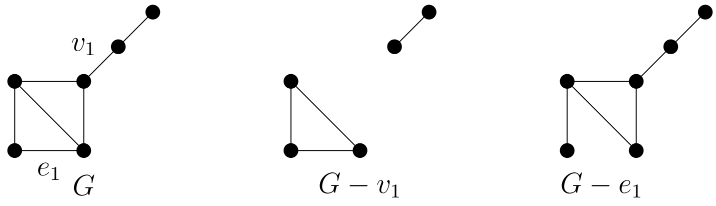


Fig. 1.2. Examples of vertex and edge deletion in a graph.

Fusion in Graph

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Fusion of two vertices $u, v \in V$ (also known as vertex contraction) results in a graph G' defined as follows:

- *Replace vertices u and v with a single new vertex w.*
- *Replace all edges $\{u, x\}$ and $\{v, x\}$ with edges $\{w, x\}$, removing any duplicate edges (if G is a simple graph).*
- *Remove any self-loops formed during this process.*
- *The number of vertices will always be reduced by one in fusion operation, one cannot say the same about edges.*

Formally, the new graph $G' = (V', E')$ is defined as:

$$V' = (V \setminus \{u, v\}) \cup \{w\}$$

$$E' = \{\{w, x\} \mid \{u, x\} \in E \text{ or } \{v, x\} \in E, x \neq u, x \neq v\}$$

Example 1.7. *Example of Fusion:*

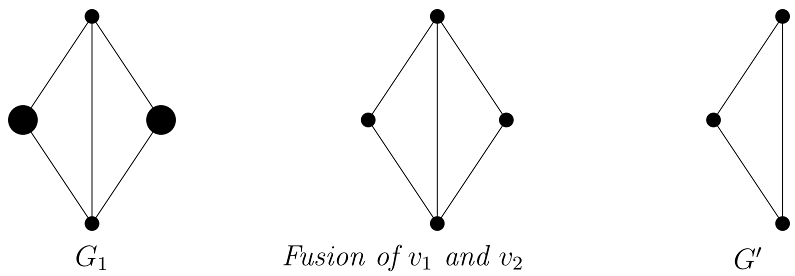


Fig. 1.3. Fusion of vertices v_1 and v_2 in a graph.

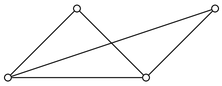
1.5. **Walks, Paths, and Cycles.** A *walk* in a graph $G = (V, E)$ is a finite sequence of vertices and edges, $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, such that $e_i = (v_{i-1}, v_i)$ for $i = 1, 2, \ldots, k$. The number k is the *length* of the walk. A walk is *closed* if $v_0 = v_k$ and *open* otherwise.

A *trail* is a walk in which all edges are distinct.

A *path* is a walk in which all vertices (and thus all edges) are distinct.

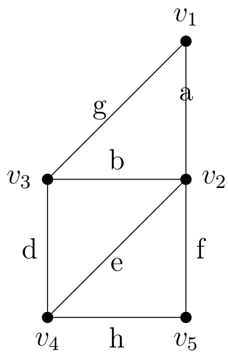
A *cycle* is a trail in which only the first and last vertices are equal.

Example. Consider the following graph $G = (V, E)$:



In this graph:

- $v_1, e_1, v_2, e_2, v_3, e_3, v_4$ is a walk of length 3.
- $v_1, e_1, v_2, e_2, v_3, e_5, v_1$ is a closed walk of length 3.
- $v_1, e_1, v_2, e_2, v_3, e_3, v_4$ is a path of length 3.
- $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$ is a cycle of length 4.



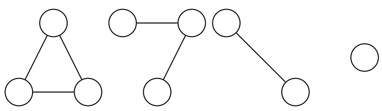
In above graph,

- Example of an open walk: $v_1 \to v_2 \to v_4 \to v_3 \to v_2$
- Example of a closed walk: $v_1 \to v_2 \to v_4 \to v_5 \to v_2 \to v_1$
- Example of a trail: $v_1 \to v_2 \to v_4 \to v_5 \to v_2$
- Example of a path: $v_1 \to v_2 \to v_4 \to v_5$
- Example of a cycle: $v_2 \to v_4 \to v_5 \to v_2$

1.6. **Connected Graphs.** A graph is *connected* if there is a $u - v$ path (a path from u to v in G for every pair $u, v \in V(G)$.

- A null graph of more than 1-vertex is a disconnected graph.
- The *components* of G are the maximal connected subgraphs of G (induced by the equivalence classes of the relation $u \leftrightarrow v \iff u = v$ or there is a $u - v$ path).

Example 1.8. *The following graph has 4 components.*



Theorem 1.9. *A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2.*

Proof:

(⇒) Suppose G is disconnected. Then there exist two nonempty disjoint subsets V_1 and V_2 of V such that there is no edge between any vertex in V_1 and any vertex in V_2.

This follows directly from the definition of a disconnected graph, where there exist at least two components with no edges connecting them.

($\Leftarrow$) Conversely, suppose V can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that there is no edge between V_1 and V_2. Then there are no paths connecting vertices in V_1 to vertices in V_2. Therefore, G is disconnected.

1.7. **The degree of a vertex.**

Lemma 1.10. *(Euler's handshaking lemma). The sum of the degrees of the vertices of a graph is equal to twice the number of edges. Or we say the graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, satisfies*

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

Proof. Each edge of G contributes 2 to the total degree count $\sum_{i=1}^{n} d(v_i)$, one for each incident vertices. Therefore, m edges of G contributes $2m$ to the sum $\sum_{i=1}^{n} d(v_i)$. Since each count of $\sum_{i=1}^{n} d(v_i)$ must be coming from one of the edges of G, hence $\sum_{i=1}^{n} d(v_i) = 2m$. $\square$

Theorem 1.11. *The number of vertices of odd degree in a graph is always even.*

Proof. Let G be a graph with vertices $v_1, v_2, \cdots, v_n$ and m edges. Then by the Handshake lemma,

$$\begin{aligned} &\sum_{i=1}^{n} d(v_i) = 2m \\ \implies &\sum_{d(v_i) \text{ is odd}} d(v_i) + \sum_{d(v_i) \text{ is even}} d(v_i) = 2m \\ \implies &\sum_{d(v_i) \text{ is odd}} d(v_i) = 2m - \sum_{d(v_i) \text{ is even}} d(v_i) \end{aligned}$$

RHS is an even number because it is the sum of even numbers. Therefore, the LHS is an even number. Hence proved.

Theorem 1.12. *If a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.*

Proof: Consider a graph G with exactly two vertices u and v of odd degree. If there is no path between u and v, then u and v lie in two different connected components of G.

In any connected component, the sum of the degrees of all vertices is even, since it is twice the number of edges (each edge contributes 2 to the sum of degrees).

However, if u and v are in different components, then each component has an odd sum of degrees (since each has one vertex of odd degree). This contradicts the fact that the sum of the degrees of vertices in any graph is always even. Thus, there must be a path joining u and v.