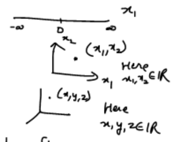
Geometry of LPP

Notation: $\mathbb{R}^n$: n-dimensional Vector space

$\mathbb{R}$

$\mathbb{R}^2$

$\mathbb{R}^3$



Vector Space.

$\mathbb{R}^n$

Elements of $\mathbb{R}^n$ If $X \in \mathbb{R}^n$ then

$$\boxed{X = (x_1, x_2, \ldots, x_n)^T}$$

↳ n-component vector such that $x_1, x_2, \ldots, x_n$ are all reals.

Definition

① **Line**: A line in $\mathbb{R}^n$ passing through the points X_1 and X_2 in $\mathbb{R}^n$ is the set

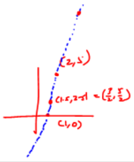
$$S = \{ X \in \mathbb{R}^n : X = \alpha_1 X_1 + \alpha_2 X_2, \ \alpha_1 + \alpha_2 = 1 \}$$

E.g. $X_1 = (2,5)$, $X_2 = (1,0)$

$\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$, $X = \frac{1}{2}(2,5) + \frac{1}{2}(1,0) = (1, \frac{5}{2}) + (\frac{1}{2}, 0) = (\frac{3}{2}, \frac{5}{2})$

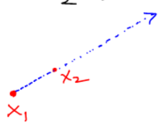
$\alpha_1 = 3, \alpha_2 = -2$

$X = 3(2,5) + (-2)(1,0)$
$= (6,15) + (-2,0) = (4,15)$



(2) **Half-line**: A half-line in $\mathbb{R}^n$ originating from $X_1 \in \mathbb{R}^n$ and proceeding towards the point $X_2 \in \mathbb{R}^n$ is the set

$$S = \{ X \in \mathbb{R}^n : X = \alpha_1 X_1 + \alpha_2 X_2, \ \alpha_1 + \alpha_2 = 1, \ \alpha_2 \geq 0 \}$$



$\alpha_2 = 0, \ \alpha_1 = 1 \Rightarrow X = X_1$

$\alpha_2 = 2, \ \alpha_1 = -1, \Rightarrow X = -X_1 + 2X_2$

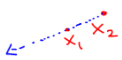
$\alpha_2 = 1, \ \alpha_1 = 0, \Rightarrow X = X_2$

$$S = \{ X \in \mathbb{R}^n : X = \alpha_1 X_1 + \alpha_2 X_2, \ \alpha_1 + \alpha_2 = 1, \ \alpha_1 \geq 0 \}$$

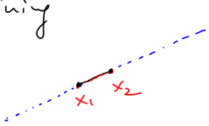
$\alpha_1 = 0, \alpha_2 = 1 \Rightarrow X = X_2$

$\alpha_1 = 1, \alpha_2 = 0 \Rightarrow X = X_1$

$\alpha_1 = 3, \alpha_2 = -2 \Rightarrow X$!



(3) Line Segment: A line segment in $\mathbb{R}^n$, joining two points x_1 and x_2 in $\mathbb{R}^n$ is the set



$$S=\left\{x\in\mathbb{R}^n:\ x=\alpha_1x_1+\alpha_2x_2,\ \underbrace{\alpha_1+\alpha_2=1,\ \alpha_1,\alpha_2\geq 0}\right\}$$

$$\downarrow$$

$$0\leq\alpha_1,\alpha_2\leq 1$$

(4) Convex linear Combination (clc): A point $x\in\mathbb{R}^n$ is called a clc of the points x_1 and x_2 in $\mathbb{R}^n$ if there exists $\alpha_1,\alpha_2\in\mathbb{R}$ such that

$$x=\alpha_1x_1+\alpha_2x_2,\ \underbrace{\alpha_1+\alpha_2=1,\ \alpha_1,\alpha_2\geq 0}_{0\leq\alpha_1,\alpha_2\leq 1}$$

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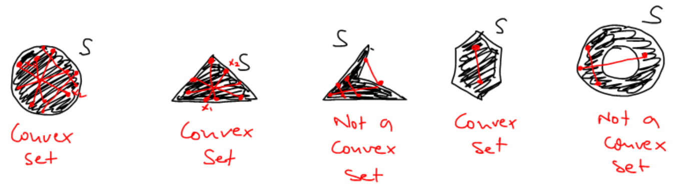
The line-segment joining x_1 & x_2 in $\mathbb{R}^n$ is the set of all clcs of x_1 & x_2

(5) Convex Set: Let $S\subseteq\mathbb{R}^n$. Then S is called a convex set iff for any two points $x_1,x_2\in S$, the point $x\in S$ where

$$x=\alpha_1x_1+\alpha_2x_2,\ \alpha_1+\alpha_2=1,\ \alpha_1,\alpha_2\geq 0$$

e.g.



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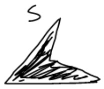
⑥ **Convex Hull**: Let $S \subseteq \mathbb{R}^n$. Then the smallest convex set containing the given set S is called the Convex Hull of S.

e.g.

S



not a convex set

$\hat{S}$

Convex set

If S is Convex set then convex hull of S is S itself.

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⑦ **Hyperplane**: A hyperplane in $\mathbb{R}^n$ is the set of points $X = (x_1, x_2, \dots, x_n)^T$ satisfying

$$S = \{X \in \mathbb{R}^n,\ C^T X = \alpha\}$$

where $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\alpha \in \mathbb{R}$

$$C^T X = \alpha \Rightarrow [c_1\ c_2\ \dots\ c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \alpha$$

$\Rightarrow \boxed{c_1 x_1 + c_2 x_2 + \dots + c_n x_n = \alpha} \to$ equality Constraint or hyperplane

LPP

$Ax = b$ — hyperplane

$Ax \le b$, $Ax \ge b$ — closed half spaces

$\boxed{2x_1 - x_2 = 9}$

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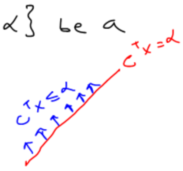
(8) Closed Half space : Let $H = \{X \in \mathbb{R}^n : c^T X = \alpha\}$ be a hyperplane in $\mathbb{R}^n$. Then the sets

$$H_1 = \{X \in \mathbb{R}^n : c^T X \le \alpha\} \text{ and}$$

$$H_2 = \{X \in \mathbb{R}^n : c^T X \ge \alpha\}$$

are called closed half-spaces generated by the hyperplane H.

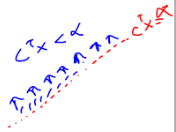


(9) open Half Space. Let $H = \{X \in \mathbb{R}^n : c^T X = \alpha\}$ be a hyperplane in $\mathbb{R}^n$. Then the sets

$$H_1 = \{X \in \mathbb{R}^n : c^T X < \alpha\} \text{ and}$$

$$H_2 = \{X \in \mathbb{R}^n : c^T X > \alpha\}$$

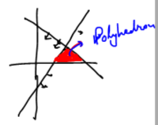
are called open half-spaces generated by hyperplane H.



(10) Polyhedral Set : A set formed by the intersection of finite number of closed half-spaces is termed as polyhedral or polyhedron.

$x_1, x_2 \ge 0$

$$\begin{cases} x_1 + x_2 \le 6 \\ 2x_1 - x_2 \ge 9 \end{cases}$$



Theorem : Line segment is a Convex set.

Proof Let S be the line segment given by

$$S = \{X \in \mathbb{R}^n : X = \alpha_1 X_1 + \alpha_2 X_2,\ \alpha_1 + \alpha_2 = 1,\ \alpha_1, \alpha_2 \ge 0\} \quad \text{---(1)}$$

Let $Y_1, Y_2 \in S$. Then

$$Y_1 = \alpha_1^1 X_1 + \alpha_2^1 X_2,\quad \alpha_1^1 + \alpha_2^1 = 1,\quad \alpha_1^1, \alpha_2^1 \ge 0 \quad \text{---(2)}$$

$$Y_2 = \alpha_1^2 X_1 + \alpha_2^2 X_2,\quad \alpha_1^2 + \alpha_2^2 = 1,\quad \alpha_1^2, \alpha_2^2 \ge 0 \quad \text{---(3)}$$

Let Y be the clc of Y_1 & Y_2, so

$$Y = \beta_1 Y_1 + \beta_2 Y_2, \quad \beta_1+\beta_2 = 1, \quad \beta_1, \beta_2 \geq 0 \quad \text{— (4)}$$

From (2), (3) & (4), we have

$$Y = \beta_1(\alpha_1^1 X_1 + \alpha_2^1 X_2) + \beta_2(\alpha_1^2 X_1 + \alpha_2^2 X_2)$$

$$= (\beta_1\alpha_1^1 + \beta_2\alpha_1^2)X_1 + (\beta_1\alpha_2^1 + \beta_2\alpha_2^2)X_2$$

$$= \gamma_1 X_1 + \gamma_2 X_2 \quad \text{— (5)} \quad \text{where } \gamma_1 = \beta_1\alpha_1^1 + \beta_2\alpha_1^2$$

$$\gamma_2 = \beta_1\alpha_2^1 + \beta_2\alpha_2^2$$

Now $\gamma_1+\gamma_2 = \beta_1\alpha_1^1 + \beta_2\alpha_1^2 + \beta_1\alpha_2^1 + \beta_2\alpha_2^2 = \beta_1(\alpha_1^1+\alpha_2^1) + \beta_2(\alpha_1^2+\alpha_2^2)$

$= \beta_1 + \beta_2$ (Using (2) & (3))

$= 1$ (Using (4)) — (6)

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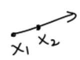
$$\left.\begin{array}{l}\gamma_1 = \beta_1\alpha_1^1 + \beta_2\alpha_1^2 \geq 0 \quad (\because \alpha_1^1, \alpha_1^2, \beta_1, \beta_2 \geq 0)\\ \text{Illy } \gamma_2 \geq 0\end{array}\right\} \text{— (7)}$$

From (5), (6) & (7), we have

$Y \in S$

$\Rightarrow$ S is a Convex set.

Theorem: Half-line is a Convex set.



Proof: Let S be a half-line,

$$S = \{X \in \mathbb{R}^n : X = \alpha_1 X_1 + \alpha_2 X_2, \ \alpha_1+\alpha_2 = 1, \ \alpha_2 \geq 0\} \text{ — (1)}$$

Let $Y_1, Y_2 \in S$. Then

$Y_1 = \alpha_1^1 X_1 + \alpha_2^1 X_2, \quad \alpha_1^1 + \alpha_2^1 = 1, \quad \alpha_2^1 \geq 0$ — (2)

$Y_2 = \alpha_1^2 X_1 + \alpha_2^2 X_2, \quad \alpha_1^2 + \alpha_2^2 = 1, \quad \alpha_2^2 \geq 0$ — (3)

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Let $Y \in S$ be clc of Y_1 & Y_2,

$Y = \beta_1 Y_1 + \beta_2 Y_2$, $\beta_1 + \beta_2 = 1$, $\beta_1, \beta_2 \geq 0$ — (4)

Now $Y = \beta_1 (\alpha_1^1 X_1 + \alpha_2^1 X_2) + \beta_2 (\alpha_1^2 X_1 + \alpha_2^2 X_2)$

$= (\beta_1 \alpha_1^1 + \beta_2 \alpha_1^2) X_1 + (\beta_1 \alpha_2^1 + \beta_2 \alpha_2^2) X_2$

$= \gamma_1 X_1 + \gamma_2 X_2$ — (5) where $\gamma_1 = \beta_1 \alpha_1^1 + \beta_2 \alpha_1^2$

$\gamma_2 = \beta_1 \alpha_2^1 + \beta_2 \alpha_2^2$

Now $\gamma_1 + \gamma_2 = \beta_1 (\alpha_1^1 + \alpha_2^1) + \beta_2 (\alpha_1^2 + \alpha_2^2)$

$= \beta_1 + \beta_2 = 1$ — (6) (Using (2), (3) & (4))

Now $\gamma_2 = \beta_1 \alpha_2^1 + \beta_2 \alpha_2^2 \geq 0$ — (7) ($\beta_1, \beta_2 \geq 0$, $\alpha_2^1, \alpha_2^2 \geq 0$)

From (5), (6) & (7), we have

$Y \in S$

$\Rightarrow$ S is convex set.

Theorem: Line is a convex set (Do yourself).

Theorem: Intersection of convex sets is a convex set.

Proof Let $S_1, S_2, \dots, S_m$ be convex sets in $\mathbb{R}^n$.

Let $S = \bigcap_{i=1}^{m} S_i$ — (1)

Let $X_1, X_2 \in S$. Then $X_1, X_2 \in S_i \ \forall \ i = 1, 2, \dots, m$. — (2)

Let X be the clc of X_1 and X_2. So

$$X = \alpha_1 X_1 + \alpha_2 X_2, \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_1, \alpha_2 \geq 0 \quad \text{--- (3)}$$

Since S_i is a convex set for all i, therefore
$X \in S_i$ for all $i = 1$ to m.

So $X \in \bigcap_{i=1}^{m} S_i \Rightarrow X \in S$

$\Rightarrow$ S is a convex set.

Theorem (a) Hyperplane is a convex set.
(b) Closed half-space is a convex set
(c) Intersection of two convex sets is a convex set.
(d) Finite intersection of convex sets is convex set.

(Do yourself)

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Theorem: Union of two convex sets may not be convex set.

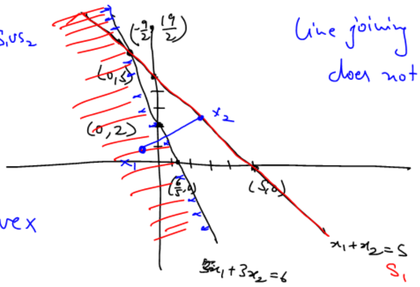
Eg. Let $S_1 = \{(x_1, x_2) : x_1 + x_2 = 5\}$ and
$S_2 = \{(x_1, x_2) : 5x_1 + 3x_2 \leq 6\}$

$S = S_1 \cup S_2$
$= \{(x_1, x_2) : x_1 + x_2 = 5 \text{ or } 5x_1 + 3x_2 \leq 6\}$

$x_1 + x_2 = 5 \begin{cases} x_1 = 0, x_2 = 5 \\ x_2 = 0, x_1 = 5 \end{cases} \begin{cases} (0,5) \\ (5,0) \end{cases}$

$5x_1 + 3x_2 = 6 \begin{cases} x_1 = 0, x_2 = 2 \\ x_2 = 0, x_1 = \frac{6}{5} \end{cases} \begin{cases} (0,2) \\ (\frac{6}{5}, 0) \end{cases}$

$S = S_1 \cup S_2$



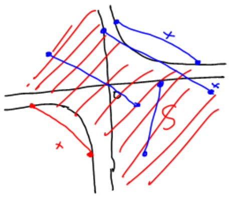
Line joining X_1 & X_2 does not belong to S

$\Rightarrow$ $S_1 \cup S_2$ is not a convex set.

Examples Q Show which of following are convex sets:

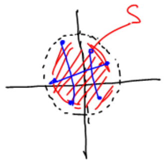
① $S = \{(x_1, x_2) : x_1 x_2 \le 1\}$.

$x_1 x_2 = 1$



S is not convex set. ✓

② $S = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$



Convex set. ✓

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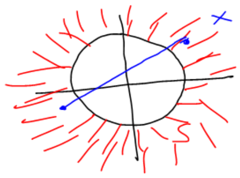
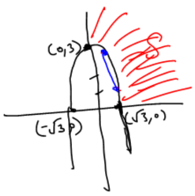
③ $S = \{(x_1, x_2) : x_1^2 + x_2^2 \ge 3\}$



Not Convex ✓

④ $S = \{(x_1, x_2) : x_2 - 3 \ge -x_1^2,\ x_1, x_2 \ge 0\}$



Not Convex set. ✓

$x_2 - 3 = -x_1^2$

$\boxed{x_1^2 = 3 - x_2}$

$x_2 = 0,\quad x_1 = \pm\sqrt{3}$

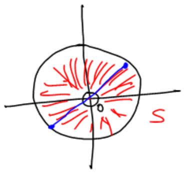
$(\sqrt{3}, 0)\ (-\sqrt{3}, 0)$

$x_1 = 0,\quad x_2 = 3$

$(0, 3)$

(5) $S=\{(x_1,x_2):\ 0<x_1^2+x_2^2\le 4\}$

S is not Convex ✓



Q(a) $S=\{(x_1,x_2):\ 2x_1+5x_2\le 20,\ x_1+2x_2\ge 6\}$

(b) $S=\{(x_1,x_2):\ -x_1-2x_2\le 2,\ -x_1+x_2\le 4,\ x_1\le 4\}$

(c) $S=\{(x_1,x_2):\ 4x_1\ge x_2^2\}$

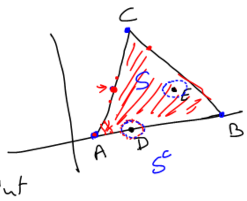
Show yourself.

Definitions

(1) Boundary points: Let $S\subset\mathbb{R}^n$. A point x_0 is called a boundary point of S if each deleted neighbourhood of x_0 intersects S and its compliment S^c.

(2) Interior point:

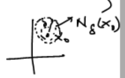
Let $S\subset\mathbb{R}^n$. A point $x_0\in S$ is said to be the interior point of S if there exist a neighbourhood of x_0 which is contained in S



Neighbourhood

If $x_0\in\mathbb{R}^n$. Then δ-neighbourhood of x_0, ($N_\delta(x_0)$) is the set of points satisfying

$N_\delta(x_0)=\{x\in\mathbb{R}^n / |x-x_0|<\delta\}$



$N_\delta(x_0)=\{x\in\mathbb{R}^n / 0<|x-x_0|<\delta\}$

deleted neighbourhood)

③ Vertex or Extreme point : Let $S \subseteq \mathbb{R}^n$ be a convex set. Then a point $X \in S$ is called a vertex of S if

$X \neq \alpha_1 X_1 + \alpha_2 X_2, \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_1, \alpha_2 \geq 0$. Thus, it is not possible to express X as a c.c of two distinct points of S.

Algebraic Method : (Use Standard form of LPP)

* All constraints are of equality type
* All b_i's are non-negative
* x_j's " " "

Min $f = 3x_1 - 5x_2$

s.t. $x_1 + x_2 \leq 6$

$2x_1 - x_2 \geq 9$

$x_1, x_2 \geq 0$

General LPP

⇓ Standard form

Min $f = 3x_1 - 5x_2$

s.t. $x_1 + x_2 + s_1 = 6$

$2x_1 - x_2 - s_2 = 9$

→ Linear system of equations with 2 equations and 4 unknowns

$x_1, x_2, s_1, s_2 \geq 0$ → $\boxed{\text{feasibility Conditions}}$

Let m: No. of equations
n: No. of unknowns

$\to$ $(n-m)$ variables should be taken as zero value

To get solution of the system, equate any two variables (when $m=2, n=4$) (In general $n-m$) equal to zero.

Total ways of putting 2 variable zero $= {}^4C_2 = 6$ (cases)

$x_1 + x_2 + s_1 = 6$
$2x_1 - x_2 - s_2 = 9$
Sol. (x_1, x_2, s_1, s_2)
$f = 3x_1 - 5x_2$

Case 1: $x_1 = x_2 = 0 \Rightarrow s_1 = 6,\ s_2 = -9$
Sol. $\boxed{(0,0,6,-9),\ f = 0}$ ✗

Case 2: $x_1 = 0,\ s_1 = 0 \Rightarrow x_2 = 6,\ s_2 = -15$
$\boxed{(0,6,0,-15),\ f = -30}$ ✗

Case 3: $x_1 = 0,\ s_2 = 0 \Rightarrow x_2 = -9,\ s_1 = 15$
$\boxed{(0,-9,15,0),\ f = 45}$ ✗

Not feasible (i.e. not basic feasible solution)

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Case 4: $x_2 = 0,\ s_1 = 0 \Rightarrow x_1 = 6,\ s_2 = 3$

x_2, s_1 → Variables are known as non-basic variables
x_1, s_2 → Variables are known as basic variables

Non-degenerate BFS: $\boxed{(6,0,0,3),\ f = 18}$ ✓

Case 5: $x_2 = 0,\ s_2 = 0 \Rightarrow x_1 = \frac{9}{2},\ s_1 = \frac{3}{2}$

Non-degenerate BFS: $\boxed{\left(\frac{9}{2}, 0, \frac{3}{2}, 0\right),\ f = \frac{27}{2}}$ ✓

Case 6: $s_1 = s_2 = 0 \Rightarrow x_1 = 5,\ x_2 = 1$

Non-degenerate BFS: $\boxed{(5,1,0,0),\ f = 10}$ ✓ Min f

All these are basic feasible solutions (BFS)

Optimal solution is $(5,1)$ and Min $f = 10$

* All solutions are known as basic solutions.
How many of these basic solutions are feasible?

General

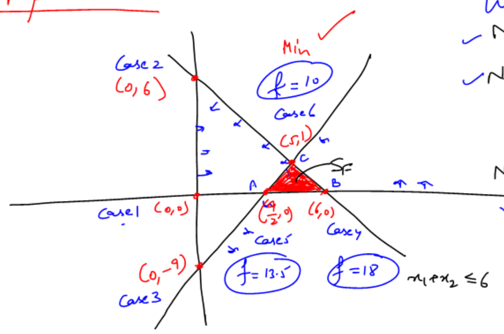
m: No. of equations

n: No. of unknowns

Put $(n-m)$ variables equal to zero and these variables are known as non-basic variables

The remaining variables (m) are known as basic variables

Graphical Representation



Graphically

- No. of vertices S_F = 3
- No. of BFS = 3 (Using Algebraic Method)

No of degenerate BFS = 0

Implies there is no degeneracy in the system

The no. of vertices of S_F and No. of BFS having one to one correspondance provided there is no degeneracy that is present in the system (LPP).

✓ Basic Solution → All solutions of linear system (All cases)

✓ Basic feasible solution (BFS) → Basic solutions having all variables ≥ 0. or Basic solutions that satisfy non-negative restrictions

Degenerate BFS: If some basic variables assume zero value then BFS is called degenerate BFS

Non-degenerate BFS: If all m basic variables in BFS are positive then it is called non-degenerate BFS

A point $X \in S_F$ (feasible region) is a vertex iff X is a BFS corresponding to some basis B.

$$AX = b \Rightarrow \underset{A}{[B, N]} \underset{x}{\begin{bmatrix} x_B \\ x_N \end{bmatrix}} = \underset{b}{[\ ]}$$

Basis (invertible matrix)

(Here all columns are linearly independent)

[Basic variables $= X_B$

Non-basic variables $= X_N$

Q Solve the following LPP using Algebraic Method. Also Compare your result using Graphical method and discuss the relationship between the set of BFSs and vertices.

$$\text{Min } f = 3x_1 - 5x_2$$

s.t.

$$x_1 + x_2 \leq 6$$
$$2x_1 - x_2 \geq 9$$
$$x_1 + 2x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Sol. Standard form:

$$\text{Min } f = 3x_1 - 5x_2$$

s.t.

$$x_1 + x_2 + s_1 = 6$$
$$2x_1 - x_2 - s_2 = 9$$
$$x_1 + 2x_2 + s_3 = 6$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

m = No. of equations $= 3$

n = No. of unknowns $= 5$

$n - m = 2$ variables equal to zero.

Non basic Variables	Basic Variables	Basic Solution $(x_1, x_2, s_1, s_2, s_3)$	Feasible (BFS)	vertex	Objective function value	Degenerate/Non-Degenerate BFS
x_1, x_2	s_1, s_2, s_3	$(0,0,6,-9,6)$	✗	D	—	
x_1, s_1	x_2, s_2, s_3	$(0,6,0,-15,-6)$	✗	F	—	
x_1, s_2	x_2, s_1, s_3	$(0,-9,15,0,24)$	✗	H	—	
x_1, s_3	x_2, s_1, s_2	$(0,3,3,-12,0)$	✗	E	—	
x_2, s_1	x_1, s_2, s_3	$(6,0,0,3,0)$	✓	C	18	Degenerate BFS
x_2, s_2	x_1, s_1, s_3	$(\frac{9}{2},0,\frac{3}{2},0,\frac{3}{2})$	✓	A	27/2	Non-degenerate BFS
x_2, s_3	x_1, s_1, s_2	$(6,0,0,3,0)$	✓	C	18	Degenerate BFS
s_1, s_2	x_1, x_2, s_3	$(5,1,0,0,-1)$	✗	G	—	
s_1, s_3	x_1, x_2, s_2	$(6,0,0,3,0)$	✓	C	18	Degenerate BFS
s_2, s_3	x_1, x_2, s_1	$(\frac{24}{5},\frac{3}{5},\frac{3}{5},0,0)$	✓	B	57/5	Non-degenerate.

$^nC_m = {}^5C_3$ cases $= 10$ cases.

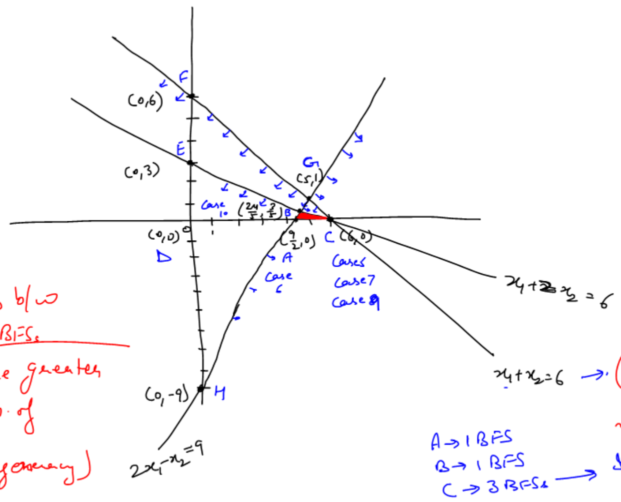
Total BFS = 5, Total vertices (S_F) = 3

$x_1 + 2x_2 = 6$

$x_1 = 0,\ x_2 = 3$

$x_2 = 0,\ x_1 = 6$

$(0,3),\ (6,0)$



Relationship b/w vertices & BFSs

No. of BFSs are greater than the no. of vertices. (Due to degeneracy)

$x_1 + x_2 = 6$ → (Redundant constraint. Removal of this constraint does not effect the S_F)

A → 1 BFS

B → 1 BFS

C → 3 BFSs → It happens due to degeneracy.