Exceptional Cases

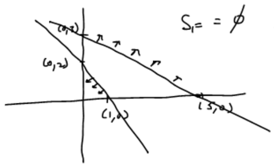
(1) Infeasible Solution (Non-existing Solution)

Max $f = 7x_1 - 8x_2$

s.t. $3x_1 + 5x_2 \geq 15$

$2x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$



$S_F = \phi$ No feasible solution.

Big-M Method:

Modified LPP

Max $f_a = 7x_1 - 8x_2 - M(a_1)$

s.t. $3x_1 + 5x_2 - s_1 + a_1 = 15$

$2x_1 + x_2 + s_2 = 2$

$x_1, x_2, s_1, s_2, a_1 \geq 0, \quad M > 0$ (large No.)

Basic var.	x_1	x_2 ↓	s_1	a_1	s_2	Sol	Min ratio
z	$\cancel{-7}$ $-7-3M$	$\cancel{8}$ $8-5M$	$\cancel{0}$ M	$\cancel{M}$ 0	0	$\cancel{0}$ $-15M$	
a_1	3	5	-1	1	0	15	$15/5 = 3$
← s_2	2	$\boxed{1}$	0	0	1	2	$2/1 = 2$ Min
z	$-23+7M$	0	M	0	$-8+5M$	$-16-5M$	
a_1	-7	0	-1	1	-5	5	
pivot row x_2	2	1	0	0	1	2	

New z-row = old z row − M a_1-row

New z row = old z row − $(8-5M)$ pivot row

New a_1-row = old a_1-row − 5 pivot row

→ optimal Table ($\because z_j - c_j \geq 0 \ \forall j$)

$a_1 = 5 \neq 0$, $x_2 = 2$, $x_1 = 0$

$\because$ Artificial variable takes non-zero value

$\therefore S_F = \phi$ for original problem

$\Rightarrow$ No feasible solution (Infeasible solution)

Two - Phase Method

Phase -I

$$\text{Min } Z_a = a_1$$

s.t.

$$3x_1 + 5x_2 - s_1 + a_1 = 15$$

$$2x_1 + x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

B.V.	x_1	x_2	s_1	a_1	s_2	Sol.	Min Ratio
Z_a	$\cancel{0}$ 3	$\cancel{0}$ 5	$\cancel{0}$ -1	$\cancel{-1}$ 0	0	$\cancel{0}$ 15	
a_1	3	5	-1	1	0	15	15/5=3
← s_2	2	$\boxed{1}$	0	0	1	2	2/1 = 2 Min
Z_a	-7	0	-1	0	-5	5	
a_1	-7	0	-1	1	-5	5	
pivot row x_2	2	1	0	0	1	2	

New Z_a-row = old Z_a-row + a_1-row

optimality Criteria of Phase -I: $\boxed{\text{All } z_j - c_j \leq 0}$

New Z_a-row = old Z_a-row − 5 pivot row

New a_1-row = old a_1-row − 5 pivot row.

→ optimal Table

Solution of Phase-I, $a_1 = 5$, $x_2 = 2$

⟹ $a_1 = 5 \neq 0$

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Original problem have $S_F = \phi$

⟹ No feasible Solution

Rule-1: If in the optimal table of Big-M method or Phase-I of Two-phase method, an artificial variables is not zero, then original LPP has no-feasible solution, i.e., $S_F = \phi$.

(2) Alternate optima (Multiple Solutions)

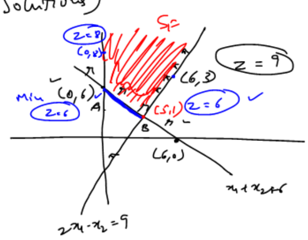
$$\text{Min } Z = x_1 + x_2$$

s.t.

$$x_1 + x_2 \geq 6$$

$$2x_1 - x_2 \leq 9$$

$$x_1, x_2 \geq 0$$



✓ Two-Phase Method

Phase-I

Min $Z_a = a_1$

s.t. $x_1 + x_2 - s_1 + a_1 = 6$

$2x_1 - x_2 + s_2 = 9$

$x_1, x_2, s_1, a_1, s_2 \geq 0$

① B.V	x_1	x_2 ↓	s_1	a_1	s_2	Sol.	Min ratio
Z_a	$\cancel{0}$ 1	$\cancel{0}$ 1	$\cancel{0}$ −1	$\cancel{-1}$ 0	0	$\cancel{0}$ 6	
← a_1	1	$\boxed{1}$	−1	1	0	6	$6/1 = 6$ (for x_1); $6/1 = 6$ (for x_2)
s_2	2	−1	0	0	1	9	$9/2 = 4.5$ Min (for x_1)

s_1 leave; x_1, x_2; a_1

② B.V	x_1	x_2	s_1	a_1	s_2	Sol.
Z_a	0	0	0	−1	0	0
x_2	1	1	−1	1	0	6
s_2	3	0	−1	1	1	15

↳ optimal table of Phase-I

$a_1 = 0$ ⇒ Move on to Phase-II

SF unbounded

Phase-II

Simple format ✗. $\boxed{Z \to Z + x_2}$

Non-basic with $z_j - c_j = 0$

③ B.V.	x_1 ↓	x_2	s_1	s_2	Sol	Min ratio
Z	$\cancel{-1}$ 0	$\cancel{-1}$ 0	$\cancel{0}$ −1	0	$\cancel{0}$ 6	
x_2	1	1	−1	0	6	$6/1$
← s_2	$\boxed{3}$	0	−1	1	15	$15/3 = 5$

④ B.V.	x_1	x_2	s_1	s_2	Sol
Z	0	0	−1	0	6
x_2	0	1	$2/3$	$-1/3$	1
x_1	1	0	$-1/3$	$1/3$	5

↳ optimal table

optimal table ✓

~~One optimal~~ Sol. ~~from table~~ ③, $x_1 = 0$, $x_2 = 6$, Min $Z = 6$

Second optimal Sol. from table ④, $x_1 = 5$, $x_2 = 1$ with Min $Z = 6$

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All Alternate Sols

Let $X_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Then $X = \alpha_1 X_1 + \alpha_2 X_2$, $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 \geq 0$ (clc of X_1 & X_2)

$= \alpha_1 \begin{bmatrix} 0 \\ 6 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 5\alpha_2 \\ 6\alpha_1 + \alpha_2 \end{bmatrix}$, $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 \geq 0$

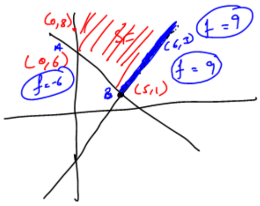
Here X is the clc of X_1 & X_2 and is also an optimal Solution

⇒ Line Segment joining X_1 and X_2 are all optimal solutions.

Rule 2 : If in the optimal table, $z_j - c_j = 0$ for a non-basic variable (say x_j) then LPP has alternate optima. One such solution can be obtained by bringing x_j into the basis, provided it is allowed.

e.g.

$$\text{Max } f = 2x_1 - x_2$$
$$\text{s.t. } x_1 + x_2 \geq 6$$
$$2x_1 - x_2 \leq 9$$
$$x_1, x_2 \geq 0$$



Two phase Method

Phase-I

$$\text{Min } f_a = a_1$$
$$\text{s.t. } x_1 + x_2 - s_1 + a_1 = 6$$
$$2x_1 - x_2 + s_2 = 9$$
$$x_1, x_2, s_1, a_1, s_2 \geq 0$$

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B.V	x_1	x_2	s_1	a_1	s_2	Sol.
f_a	$\cancel{0}$ 1	$\cancel{0}$ 1	$\cancel{0}$ −1	$\cancel{-1}$ 0	0	$\cancel{0}$ 6
← a_1	1	$\boxed{1}$	−1	1	0	6
s_2	2	−1	0	0	1	9
f_a	0	0	0	−1	0	0
x_2	1	1	−1	1	0	6
s_2	3	0	−1	1	1	15

Optimal Table of Phase I

Phase-II

B.V.	x_1	x_2	s_1	s_2	Sol.	Min
f	$\cancel{-2}$ −3	$\cancel{1}$ 0	$\cancel{0}$ 1	0	$\cancel{0}$ −6	
x_2	1	1	−1	0	6	$6/1 = 6$
← s_2	$\boxed{3}$	0	−1	1	15	$15/3 = 5$
f	0	0	0	1	9	
x_2	0	1	$-2/3$	$-1/3$	1	
x_1	1	0	$-1/3$	$1/3$	5	

~~last~~ table of

Optimal table

Sol. $x_1 = 5, x_2 = 1$

$\text{Max } f = 9$

Here s_1 is non-basic variable with $z_j - c_j = 0$ (there exists alternate optima)
$\Rightarrow$ s_1 can enter the basis

However, α^{s_1} having negative entries
$\Rightarrow$ No variable can leave the basis.

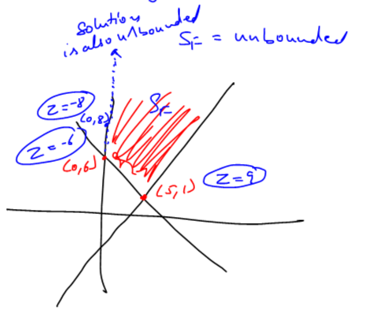
$\Rightarrow$ Multiple solutions exists and feasible region is unbounded.

(3) Unbounded Solution

$$\text{Min } Z = 2x_1 - x_2$$

s.t.

$$x_1 + x_2 \geq 6$$
$$2x_1 - x_2 \leq 9$$
$$x_1, x_2 \geq 0$$



Two-Phase Method

Phase I: Min $Z_a = a_1$

✓ $z_j - c_j \leq 0$

s.t.

$$x_1 + x_2 - s_1 + a_1 = 6$$
$$2x_1 - x_2 + s_2 = 9$$
$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

B.V	x_1	x_2	s_1	a_1	s_2	Sol.
Z_a	$\cancel{0}$ 1	$\cancel{0}$ 1	$\cancel{0}$ −1	$\cancel{-1}$ 0	0	$\cancel{0}$ 6
← a_1	1	$\boxed{1}$	−1	1	0	6
s_2	2	−1	0	0	1	9
Z_a	0	0	0	−1	0	0
pivot row x_2	1	1	−1	1	0	6
s_2	3	0	−1	1	1	15

optimal table of phase-I

Phase-II Min $z_j - c_j \leq 0$

Not Simplex form

B.V.	x_1	x_2	s_1	s_2	Sol.
Z	$\cancel{-2}$ −3	$\cancel{1}$ 0	$\cancel{0}$ 1	0	$\cancel{0}$ −6
x_2	1	1	−1	0	6
s_2	3	0	−1	1	15

Here s_1 is entering var. but no. leaving variable

S_F unbounded $\Rightarrow$ Unbounded Solution.

Last table nonbasic $z_j - c_j \neq 0$ $\forall j \leq 0$

S_F unbounded

S_F unbounded

Solution exist (Min ratio rule fails)

Unbounded Solution (Entering variable No leaving variable)

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Rule 3 : In last table (may not optimal), $z_j - c_j \neq 0$ for non-basic variable and $\alpha^j \leq 0$, then S_F is unbounded. In such a case, the optimal solution is finite or unbounded. If S_F is unbounded, then the optimal solution is unbounded if

(i) LPP is max and $z_j - c_j < 0$ ↓

(ii) LPP is Min and $z_j - c_j > 0$ ↓

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(4) Degeneracy : terminate / Cyclic

In case a tie for minimum ratio (tie in the leaving variable) which may be broken arbitrarily, at least one basic variable will be zero in the next iteration and the new solution is said to be degenerate.

Q Max $z = 3x_1 - 5x_2$

s.t.

$$x_1 + x_2 \leq 6$$
$$2x_1 - x_2 \geq 9$$
$$x_1 + 2x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Phase I Min $z_a = a_2$

$z_j - c_j \geq 0$

s.t.

$$x_1 + x_2 + s_1 = 6$$
$$2x_1 - x_2 - s_2 + a_2 = 9$$
$$x_1 + 2x_2 + s_3 = 6$$
$$x_1, x_2, s_1, s_2, s_3, a_2 \geq 0$$

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B.V	x_1	x_2	s_2	s_1	a_2	s_3	Sol.	Min ratio
	~~0~~	~~0~~	~~0~~	0	~~-1~~	0	~~0~~	
r	2	-1	-1		0		9	
s_1	1	1	0	1	0	0	6	$6/1 = 6$
a_2	2	-1	-1	0	1	0	9	$9/2$ min
s_3	1	2	0	0	0	1	6	$6/1 = 6$
r	0	0	0	0	–	0	0	
s_1	0	$3/2$	$1/2$	1	–	0	$3/2$	
x_1	1	$-1/2$	$-1/2$	0	–	0	$9/2$	
s_3	0	$5/2$	$1/2$	0		1	$3/2$	

pivot row

Last of table of Phase I

Phase II (Max) $z_j - c_j \geq 0$ ✓

	x_1	x_2	s_2	s_1	s_3	Sol.	Min ratio
Z	~~-3~~	~~5~~	~~0~~	0	0	~~0~~	
	0	$7/2$	$-3/2$			$27/2$	
s_1	0	$3/2$	$\boxed{1/2}$	1	0	$3/2$	3
x_1	1	$-1/2$	$-1/2$	0	0	$9/2$	-
s_3	0	$5/2$	$1/2$	0	1	$3/2$	3
Z	0	8	0	3	0	18	
s_2	0	3	1	2	0	3	
x_1	1	1	0	1	0	6	
s_3	0	1	0	-1	1	0	

pivot row

optimal Table

s_3 is the basic variable with value zero
$\Rightarrow$ Solution is degenerate.

Optimal Sol is $x_1 = 6, x_2 = 0$, Max $Z = 18$