2. **Eulerian and Hamitonian Graphs**

Graph theory is typically dated as beginning with Leonhard Euler's solution to the problem of the Seven Bridges of Königsberg. Click below link for the description of the problem. Königsberg

2.1. **Eulerian circuit.** A closed walk in a graph an *Euler circuit* if it traverses every edge of the graph exactly once. A graph is *Eulerian* if it admits an Euler circuit.
Concatenation of two walks

$$v_1 \to v_2 \cdots \to v_i \to \cdots \to v_{k-1} \to v_k, \quad \text{and} \quad v_i \to v_l \to \cdots \to v_i$$

is defined to be the walk

$$v_1 \to v_2 \cdots \to v_i \to v_l \to \cdots \to v_i \to \cdots \to v_{k-1} \to v_k$$

Theorem 2.1. *A connected graph is Eulerian if and only if every vertex has even degree.*

Proof. The degree condition is clearly necessary: a vertex v appearing k times in an Euler circuit of the form

$$v_1 \to v_2 \cdots v_i \to v \to v_{i+1} \cdots v_k, v_{k+1} \cdots \to v \to \cdots v_1$$

must have degree $2k$ (not counting starting and ending vertex twice if v is as such).

Conversely, we show by induction on $||G||$ that every connected graph G with all degrees even has an Euler circuit. The induction starts trivially with $||G|| = 0$. Now let $||G|| \geq 1$. We can form a closed walk in G starting from an arbitrary vertex. This is possible because the degree of each vertex is even; for every edge that we traverse to reach/enter a vertex there must be an edge to exit. Let W be such a walk of maximal length. If the set of edges traversed in W is the same as $E(G)$, then W is an Euler circuit.

Suppose, therefore, that $G' := G - W$ (removing edges of W from G) has an edge. For every vertex $v \in G$, there are even number of edges incident to v in W, so the degree of each vertex of G' are again all even. Since G is connected, G' has an edge e incident with a vertex on W. By the induction hypothesis, the component C of G' containing e has an Euler tour. Concatenating this with W (suitably re-indexed), we obtain a closed walk in G that contradicts the maximal length of W. □

An *Eulerian trail* (or Eulerian path) is a walk that visits every edge exactly once (allowing for revisiting vertices).

2.2. **Hamiltonian circuit.** A *Hamiltonian cycle* in a graph is a closed cycle that visits each vertex of the graph exactly once. Hamiltonian cycles are named after Irish mathematician Sir William Rowan Hamilton, who studied such properties of graphs in the 19th century. William Hamilton, described a game on the graph of the dodecahedron in which one player specifies a 5-vertex path and the other must extend it to a spanning cycle. The game was marketed as the "Traveller's Dodecahedron", a wooden version in which the vertices were named for 20 important cities.

Until 1970s, interest in Hamiltonian cycles centered on their relationship to the four color problem, later study was stimulated by practical applications and by the issue of complexity.

- **Hamiltonian Path:** A path in a graph that visits every vertex exactly once.
- **Hamiltonian Cycle:** A cycle in a graph that visits every vertex exactly once and returns to the starting vertex.
- **Hamiltonian Graph:** A graph that contains a Hamiltonian cycle.

2.3. **Theorems and Results.** Not all graphs are Hamiltonian, and determining whether a given graph is Hamiltonian is a NP-complete problem. There are, however, a number of important results providing sufficient conditions for the existence of Hamiltonian cycle in a graph.

Theorem 2.2. Dirac's Theorem *Let G be a simple graph with $n \geq 3$ vertices. If every vertex of G has degree at least $\frac{n}{2}$, then G is Hamiltonian.*

Proof. Let G be a simple graph with n vertices where each vertex has degree at least $\frac{n}{2}$. Assume for contradiction that G is not Hamiltonian. Then G does not contain a Hamiltonian cycle.

Consider the longest cycle C in G. Since G is not Hamiltonian, there exists a vertex v in G not in C. Since G is connected and every vertex has degree at least $\frac{n}{2}$, v is adjacent to at least $\frac{n}{2}$ vertices of C.

Consider two consecutive vertices x and y on C. If v is adjacent to both x and y, then the cycle $C' = (v, x, C_{x \to y}, y, v)$ is longer than C, contradicting the assumption that C is the longest cycle. Thus, v cannot be adjacent to two consecutive vertices on C, and therefore, v must be adjacent to vertices separated by at least one vertex on C. But this implies degree of v can at most be $\frac{n-1}{2}$, contradicting the hypothesis of the theorem. Thus, the the largest cycle C cannot leave any vertex of G. Hence G must contain a Hamiltonian cycle. □

Remark 2.3. *A pentagon does not satisfy the hypothesis of Dirac's theorem stated above, but it is clearly a Hamilton graph. Thus the hypothesis is not necessary for the existence of Hamiltonian cycle in a graph.*

Theorem 2.4. Ore's Theorem *Let G be a simple graph with $n \geq 3$ vertices. If for every pair of non-adjacent vertices u and v, we have $\deg(u) + \deg(v) \geq n$, then G is Hamiltonian.*

The closure $[G]$ (also denoted by cl(G)) of a graph G with n vertices, obtained by repeatedly adding a new edge (u, v) connecting a nonadjacent pair of vertices u and v with $\deg(v) + \deg(u) \geq n$ until no more pairs with this property can be found.

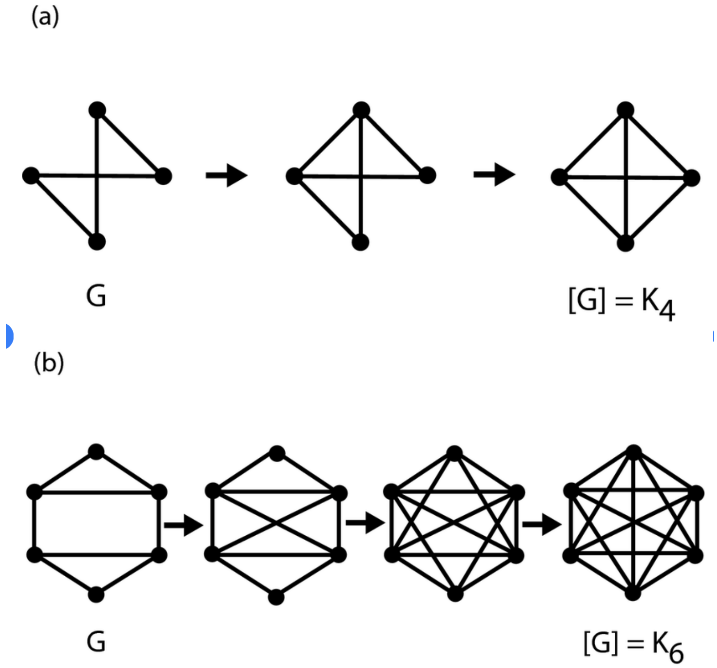


Fig. 2.1. Steps to form closure of a graph

Theorem 2.5. Bondy-Chvátal's Theorem *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Proof. TBD □

Corollary 2.6. *The Dirac theorem and Ore's theorem can be seen as straightforward consequences of the Bondy-Chvátal's Theorem.*

Definition 2.7. *Graph Connectivity: The connectivity of a graph, denoted $\kappa(G)$, is the minimum number of vertices whose removal disconnects the graph or reduces it to a single vertex.*

Higher Connectivity Increases Likelihood of a Hamiltonian Path: Higher connectivity (e.g., 2-connected or 3-connected) guarantees multiple vertex-disjoint paths between vertices, providing flexibility to construct a Hamiltonian cycle by choosing alternative routes to avoid revisiting vertices. For example, C_5 is 2-connected, and it admits a Hamiltonian cycle. In contrast, a graph with a cut vertex (1-connected), like two triangles joined at a single vertex, does not admit a Hamiltonian cycle.

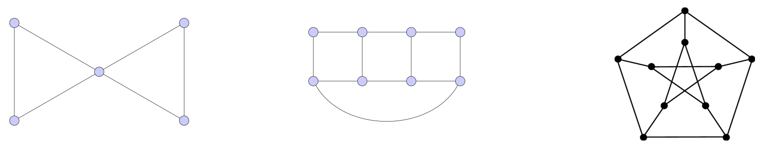
Application Hamiltonian graphs are a significant area of study in graph theory, with applications in various fields such as computer science (computational complexity theory), logistics (TSP), and biology. The results discussed above, though provides only sufficient conditions, but offers insight into the structure of such graphs.

2.4. **Practice Problems Set 2.** 9. For which integers m and n is $K_{m,n}$ Eulerian or Hamiltonian graph?

10. Is a Eulerian circuit necessarily a cycle? Prove or find a counterexample.

11. For which positive integers n does K_n have an Eulerian circuit?

12. Prove or disprove:
(i) Every Eulerian bipartite graph has an even number of edges.
(ii) Every Eulerian simple graph with an even number of vertices has an even number of edges.
13. Let v be a cut vertex of a simple graph G. prove that $\bar{G} - v$ is connected.
14. Which of the following graphs are Hamiltonian? If they are Hamiltonian identify a Hamiltonian cycle. If they are not, explain briefly why.



15. A group of n people are going out to dinner,where $n \equiv 0(mod\ 2)$ and $n \geq 3$. If every person going to dinner is friends with at least half the group, prove it is possible to seat the friends around a circular table so each person is seated next to two friends.
7. Determine if each statement is true or false. If true, provide a brief proof. If false, find an explicit counterexample.
(a) A graph of order $n \geq 4$ that contains a triangle cannot be Hamiltonian.
(c) If there exists a Hamiltonian path between any two vertices in a graph, then the graph is Hamiltonian.