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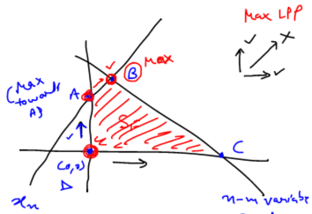
# Simplex Method (Standard form LPP)

* Iterative Method
* It always starts from one vertex (BFS) and goes to another adjacent vertex (BFS) so that the value of objective function is improved




LPP

$$\text{Max or Min } f(x) = C^T X$$
$$\text{s.t. } AX = b$$
$$X \geq 0,\ b \geq 0$$

$n-m$ variable equal to zero

$C^T = (C_1, C_2, \ldots, C_n)$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Basic variables $\leftarrow X_B$
Non-basic variables $\leftarrow X_N$

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let $x_1, x_2, \ldots, x_m$ are basic variables
$x_{m+1}, x_{m+2}, \ldots, x_n$ are non-basic variables

Basis ?

$$A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & \cdots & x_m & x_{m+1} & \cdots & x_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2m} & a_{2,m+1} & \cdots & a_{2n} \\ \vdots & & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \end{bmatrix} \\ \begin{array}{ccccccc} A_1 & A_2 & & A_m & A_{m+1} & & A_n \end{array} \end{array} = [B\ N]$$

$B$ ✓ $N$

$$X = \begin{bmatrix} X_B \\ X_N \end{bmatrix}$$

(Basis matrix)
(Invertible matrix)
$B^{-1}$ exist

$A_1, A_2, \ldots, A_m$ Constitute the basis matrix $B$.
$\Rightarrow A_1, A_2, \ldots, A_m$ are linearly independent

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Any $A_j \in \mathbb{R}^m$ can be expressed as a linear combination of basis elements, i.e., $A_1, A_2, \cdots, A_m$

$$\Rightarrow A_j = \alpha_1^j A_1 + \alpha_2^j A_2 + \cdots + \alpha_m^j A_m$$

$$= [A_1 \; A_2 \cdots A_m] \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \\ \vdots \\ \alpha_m^j \end{bmatrix} = B \underbrace{\alpha^j}_{\text{Co-ordinate vector.}}$$

$$\Rightarrow A_j = B\alpha^j \Rightarrow \boxed{\alpha^j = B^{-1} A_j} \quad , \quad j = m+1, m+2, \ldots, n$$

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$$f(X) = \overset{= c^T x}{c_1 x_1} + c_2 x_2 + \cdots + c_m x_m + c_{m+1} x_{m+1} + c_{m+2} x_{m+2} + \cdots + c_n x_n$$

$$= c_1 x_1 + c_2 x_2 + \cdots + c_m x_m + c_{m+1}(0) + c_{m+2}(0) + \cdots + c_n(0)$$

$$= c_1 x_1 + c_2 x_2 + \cdots + c_m x_m$$

$$= (c_1 \; c_2 \cdots c_m) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = C_B^T X_B$$

where $C_B^T = (c_1 \; c_2 \cdots c_m)$ (Cost vector corresponding to $X_B$ (basic variables))

$$\Rightarrow \boxed{f(X_B) = C_B^T X_B}$$

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Now $AX = b$

$$[B \; N]\begin{bmatrix} X_B \\ X_N \end{bmatrix} = b$$

$$\Rightarrow \underset{\substack{\downarrow \\ \text{non-basic variables}}}{BX_B + NX_N} = b \quad \Rightarrow \quad BX_B + N\underset{\substack{\downarrow \\ \text{zero vector}}}{(0)} = b$$

$$\Rightarrow BX_B = b \quad \Rightarrow \boxed{X_B = B^{-1}b}$$

Define, $z_j = C_B^T \alpha^j = C_B^T B^{-1} A_j$

$$\Rightarrow \boxed{z_j - c_j = C_B^T B^{-1} A_j - c_j} \quad \text{or} \quad \boxed{z_j - c_j = C_B^T \alpha^j - c_j}$$

Formulas

$$\boxed{\begin{aligned} &X_B = B^{-1}b \\ &f(X_B) = C_B^T X_B \\ &\alpha^j = B^{-1}A_j \\ &z_j - c_j = C_B^T \alpha^j - c_j \end{aligned}}$$

(For this, all constraints are of type "$\le$")

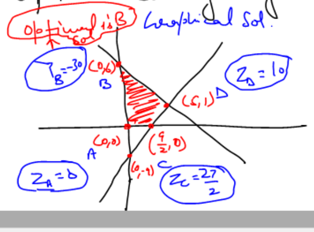
Q Use Simplex Method to find optimal sol. of the given LPP

$$\text{Min } Z = 3x_1 - 5x_2$$

$$\text{s.t.} \quad x_1 + x_2 \le 6$$

$$2x_1 - x_2 \le 9$$

$$x_1, x_2 \ge 0$$



## Standard form

$$\text{Min } Z = 3x_1 - 5x_2 + (0)s_1 + (0)s_2$$

s.t.

$$x_1 + x_2 + s_1 = 6$$
$$2x_1 - x_2 + s_2 = 9$$
$$x_1, x_2, s_1, s_2 \geq 0$$

$\rightarrow$ Min $Z = c^T X$

$AX = b$

$X \geq 0$

$c^T = (3 \quad -5 \quad 0 \quad 0)$

$X = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix}$,

$n$ = No. of unknowns = 4

$m$ = No. of equations = 2

$n - m = 2$ variables should be non-basic

$b = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

$$A = \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \\ \end{array}$$
$$A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \qquad \underbrace{\quad}_{N} \quad \underbrace{\quad}_{B}$$

$X_N = \{x_1, x_2\}$ $\qquad$ $X_B = \{s_1, s_2\}$

Here $X_B = \{s_1, s_2\}$ is a starting basic variable set.

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

## First Table of Simplex

$\rightarrow z_j - c_j$

| Basic variables | $x_1$ | $x_2$ | $s_1$ | $s_2$ | Solution |
|---|---|---|---|---|---|
| $f(x) = Z$ | $z_1 - c_1$ | $z_2 - c_2$ | $z_3 - c_3$ | $z_4 - c_4$ | $f(X_B)$ |
| Basic variables | $\alpha^1$ | $\alpha^2$ | $\alpha^3$ | $\alpha^4$ | $X_B$ |

For first table

$B, B^{-1}, b, c_B^T$

$X_B = B^{-1}b$

$f(X_B) = c_B^T X_B$

$\alpha^1 = B^{-1}A_1$

$\alpha^2 = B^{-1}A_2$

$\alpha^3 = B^{-1}A_3$

$\alpha^4 = B^{-1}A_4$

$z_1 - c_1 = c_B^T \alpha^1 - c_1$

$z_2 - c_2 = c_B^T \alpha^2 - c_2$

$z_3 - c_3 = c_B^T \alpha^3 - c_3$

$z_4 - c_4 = c_B^T \alpha^4 - c_4$

| Basic Variables | $x_1$ | $x_2$ | $s_1$ | $s_2$ | Sol. |
|---|---|---|---|---|---|
| $Z$ | $-3$ ($z_1 - c_1$) | $5$ ($z_2 - c_2$) | $0$ ($z_3 - c_3$) | $0$ ($z_4 - c_4$) | $0$ ($f(X_B)$) |
| $s_1$ | 1 | 1 | 1 | 0 | 6 |
| $s_2$ | 2 | $-1$ | 0 | 1 | 9 |
| | $\alpha^1$ | $\alpha^2$ | $\alpha^3$ | $\alpha^4$ | $X_B$ |

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \quad C_B^T = (0 \quad 0)$$

($\because$ $s_1$ & $s_2$ are basic variables having cost zero)

$$X_B = B^{-1} b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$f(X_B) = C_B^T X_B = (0,0) \begin{bmatrix} 6 \\ 9 \end{bmatrix} = 0$$

$$\alpha^1 = B^{-1} A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\alpha^2 = B^{-1} A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\alpha^3 = B^{-1} A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\alpha^4 = B^{-1} A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$





($x_1$) $$z_1 - c_1 = C_B^T \alpha^1 - c_1 = (0 \quad 0) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 = -3$$

($x_2$) $$z_2 - c_2 = C_B^T \alpha^2 - c_2 = (0,0) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - (-5) = 5$$

($s_1$) $$z_3 - c_3 = C_B^T \alpha^3 - c_3 = (0,0) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = 0$$

($s_2$) $$z_4 - c_4 = C_B^T \alpha^4 - c_4 = (0,0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 0$$

Min $Z = 3x_1 - 5x_2 + 0s_1 + 0s_2$

$$Z - \underbrace{3x_1}_{z_1-c_1} + \underbrace{5x_2}_{z_2-c_2} + \underbrace{0s_1}_{z_3-c_3} + \underbrace{0s_2}_{z_4-c_4} = \underbrace{0}_{f(x_B)}$$

Row operations

$Z_{new} = Z_{old} - 5$ pivot row

| | | | | |
|---|---|---|---|---|
| -3 | 5 | 0 | 0 | 0 |
| -5 | -5 | -5 | 0 | -30 |
| -8 | 0 | -5 | 0 | -30 |

Min $Z = 3x_1 - 5x_2 + (0)s_1 + (0)s_2$

s.t. $x_1 + x_2 + s_1 = 6$

$2x_1 - x_2 + s_2 = 9$

$x_1, x_2, s_1, s_2 \geq 0$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

First table

$(x_1, x_2, s_1, s_2) = (0, 0, 6, 9)$

(Since $z_2 - c_2 \not\leq 0$ $\Rightarrow$ This table is not optimal)

| Basic Vari- | $x_1$ | $x_2$ | $s_1$ | $s_2$ | Sol. | Min Ratio |
|---|---|---|---|---|---|---|
| Z | -3 | 5 | 0 | 0 | 0 | |
| $s_1$ | 1 | $\boxed{1}$ $\alpha^2_1$ | 1 | 0 | 6 | $6/1 = 6$ |
| $s_2$ | 2 | -1 $\alpha^2_2$ | 0 | 1 | 9 | |

$x_1 = 0, x_2 = 0$

$\alpha^2$ $\quad$ $\geq 0$ $\quad$ $B^{-1}$

## Optimality Criteria

(i) For maximization problem, BFS $x_B$ is said to be optimal iff $z_j - c_j \geq 0 \quad \forall j$

(ii) For minimization problem, BFS $x_B$ is said to be optimal iff $z_j - c_j \leq 0 \quad \forall j$

## Entering variable ~~Criteria~~ (If table is not optimal)

A non-basic variable $x_k$ enters into the basis for which $z_k - c_k$ is

(a) most positive for minimization problem

(b) most negative for maximization problem





## Leaving Criteria

If one variable enters into the basis, then one variable must leave the basis.

Find $\min \left\{ \dfrac{b_i}{\alpha_i^k},\ \alpha_i^k > 0 \right\}$

Second

| Basic var. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | Sol. |
|---|---|---|---|---|---|
| $z$ | -8 | 0 | -5 | 0 | -30 |
| $x_2$ | 1 | 1 | 1 | 0 | 6 |
| $s_2$ | 3 | 0 | 1 | 1 | 15 |

$B^{-1}_{new}$

Here All $z_j - c_j \leq 0$ ⇒ optimal table

New $s_2$ row = old $s_2$ row + pivot row

| | | | | |
|---|---|---|---|---|
| 2 | -1 | 0 | 1 | 9 |
| 1 | 1 | 1 | 0 | 6 |
| 3 | 0 | 1 | 1 | 15 |

Optimal sol $x_1 = 0$, $x_2 = 6$, $\text{Min } z = -30$

Q $\underset{\sim}{\text{Max}}$ $z = 5x_1 + 4x_2$

Simplex ? ( All "$\leq$" ) ✓

s.t. $3x_1 + 2x_2 \leq 12$

$x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

Standard form

Max $Z = 5x_1 + 4x_2$

s.t. $3x_1 + 2x_2 + s_1 = 12$

$x_1 + 2x_2 + s_2 = 6$

$-x_1 + x_2 + s_3 = 1$

$x_2 + s_4 = 2$

$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$





First Simplex table

✓ Simplex table ?

($z_j - c_j$ for basic variables are always zero)

($z_j - c_j \geq 0\ \forall j$) optimal ? × No

| Basic var. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | Sol | Min ratio |
|---|---|---|---|---|---|---|---|---|
| $Z$ | $-5$ | $-4$ | 0 | 0 | 0 | 0 | 0 | |
| $s_1$ | $\boxed{3}$ | 2 | 1 | 0 | 0 | 0 | 12 | $12/3 = 4$ Min |
| $s_2$ | 1 | 2 | 0 | 1 | 0 | 0 | 6 | $6/1 = 6$ |
| $s_3$ | $-1$ | 1 | 0 | 0 | 1 | 0 | 1 | |
| $s_4$ | 0 | 1 | 0 | 0 | 0 | 1 | 2 | |

optimal? ×

pivot row

| Basic var. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | Sol | Min ratio |
|---|---|---|---|---|---|---|---|---|
| $Z$ | 0 | $-2/3$ | $5/3$ | 0 | 0 | 0 | 20 | |
| $x_1$ | 1 | $2/3$ | $1/3$ | 0 | 0 | 0 | 4 | $4/(2/3) = 6$ |
| $s_2$ | 0 | $\boxed{4/3}$ | $-1/3$ | 1 | 0 | 0 | 2 | $2/(4/3) = 3/2 \to$ Min |
| $s_3$ | 0 | $5/3$ | $1/3$ | 0 | 1 | 0 | 5 | $5/(5/3) = 3$ |
| $s_4$ | 0 | 1 | 0 | 0 | 0 | 1 | 2 | $2/1 = 2$ |

New Z row = Old Z-row + 5 Pivot-row

$$\begin{array}{cccccc|c} -5 & -4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 10/3 & 5/3 & 0 & 0 & 0 & 20 \\ \hline 0 & -2/3 & 5/3 & 0 & 0 & 0 & 20 \end{array}$$

New $s_2$-row = old $s_2$-row − pivot row

$$\begin{array}{cccccc|c} 1 & 2 & 0 & 1 & 0 & 0 & 6 \\ -1 & -2/3 & -1/3 & 0 & 0 & 0 & -4 \\ \hline 0 & 4/3 & -1/3 & 1 & 0 & 0 & 2 \end{array}$$

New $s_3$-row = old $s_3$-row + pivot row

$$\begin{array}{cccccc|c} -1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2/3 & 1/3 & 0 & 0 & 0 & 4 \\ \hline 0 & 5/3 & 1/3 & 0 & 1 & 0 & 5 \end{array}$$





| Basic var. | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | Sol. |
|---|---|---|---|---|---|---|---|
| Z | 0 | 0 | 3/2 | 1/2 | 0 | 0 | 21 |
| $x_1$ | 1 | 0 | 1/6 | −1/2 | 0 | 0 | 3 |
| $x_2$ | 0 | 1 | −1/4 | 3/4 | 0 | 0 | 3/2 |
| $s_3$ | 0 | 0 | 3/4 | −5/4 | 1 | 0 | 5/2 |
| $s_4$ | 0 | 0 | 1/4 | −3/4 | 0 | 1 | 1/2 |

pivot → row

optimal Table ✓

Stop

New Z-row = old Z-row + $\frac{2}{3}$ pivot-row

$$\begin{array}{cccccc|c} 0 & -2/3 & 5/3 & 0 & 0 & 0 & 20 \\ 0 & 2/3 & -1/6 & 1/2 & 0 & 0 & 1 \\ \hline 0 & 0 & 3/2 & 1/2 & 0 & 0 & 21 \end{array}$$

New $x_1$-row = old $x_1$-row − $\frac{2}{3}$ pivot row

$$\begin{array}{cccccc|c} 1 & 2/3 & 1/3 & 0 & 0 & 0 & 4 \\ 0 & -2/3 & -1/6 & -1/2 & 0 & 0 & -1 \\ \hline 1 & 0 & 1/6 & -1/2 & 0 & 0 & 3 \end{array}$$

New $s_3$-row = old $s_3$-row − $\frac{5}{3}$ pivot-row

$$\begin{array}{cccccc|c} 0 & 5/3 & 1/3 & 0 & 1 & 0 & 5 \\ 0 & -5/3 & 5/12 & -5/4 & 0 & 0 & -5/2 \\ \hline 0 & 0 & 3/4 & -5/4 & 1 & 0 & 5/2 \end{array}$$

New $s_1$-row = old $s_1$-row − pivot row

$$\left|\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1/4 & -3/4 & 0 & 0 & -3/2 \\ \hline 0 & 0 & 1/4 & -3/4 & 0 & 1 & 1/2 \end{array}\right.$$

Optimal Sol. $x_1 = 3,\ x_2 = 3/2$

Optimal obj. func. value, Max $z = 21$



Q Max $z = 3x_1 + 2x_2 + 5x_3$

s.t. $x_1 + 2x_2 + x_3 \le 430$

$3x_1 + 2x_3 \le 460$

$x_1 + 4x_2 \le 420$

$x_1, x_2, x_3 \ge 0$

$B,\ B^{-1},\ b,\ C_B^T$

$X_B = B^{-1}b$

$f(X_B) = C_B^T X_B$

$\alpha^j = B^{-1}A_j$

$z_j - c_j = C_B^T\alpha^j - c_j$

One simplex iteration table is given by

| $C_B^T$ | Basic var. | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | Sol. |
|---|---|---|---|---|---|---|---|---|
| | $z$ | | | | | | | |
| 2 | $x_2$ | | | | | $-1/4$ | | |
| 5 | $x_3$ | | | | | $1/2$ | | |
| 0 | $s_3$ | | | | | 1 | | |

Without performing simplex iteration, find the missing entries in above table.

Sol (Without row-operations)

Current basic variables: $x_2, x_3, s_3$

$$A = \begin{array}{c} \\ \end{array}\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \end{array}$$

$$A = \left[\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$A_1$ (first column); $s_1, s_2, s_3$ columns: Identity

Standard form, Max $z = 3x_1 + 2x_2 + 5x_3 + 0\cdot s_1 + 0\cdot s_2 + 0\cdot s_3$

s.t. $x_1 + 2x_2 + x_3 + s_1 = 430$

$3x_1 + 2x_3 + s_2 = 460$

$x_1 + 4x_2 + s_3 = 420$

$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$

$$B = \begin{array}{c} \begin{matrix} x_2 & x_3 & s_3 \end{matrix} \\ \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix} \end{array}$$

<u>find $B^{-1}$</u>

$$B^{-1} = \begin{bmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 430 \\ 460 \\ 420 \end{bmatrix}, \quad C_B^T = (2 \;\; 5 \;\; 0)$$

$$X_B = B^{-1} b = \begin{bmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 430 \\ 460 \\ 420 \end{bmatrix} = \begin{bmatrix} 100 \\ 230 \\ 20 \end{bmatrix}$$

<u>Starting basic variables</u>: $s_1, s_2, s_3$ ($B^{-1}$ in every table is under $s_1, s_2$ & $s_3$)

$$f(X_B) = C_B^T X_B = (2 \;\; 5 \;\; 0) \begin{bmatrix} 100 \\ 230 \\ 20 \end{bmatrix} = 1350$$



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| Basic var. | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | Sol |
|---|---|---|---|---|---|---|---|
| $z$ | 4 | 0 | 0 | 1 | 2 | 0 | 1350 |
| $x_2$ | $-1/4$ | 1 | 0 | $1/2$ | $-1/4$ | 0 | 100 |
| $x_3$ | $3/2$ | 0 | 1 | 0 | $1/2$ | 0 | 230 |
| $s_3$ | 2 | 0 | 0 | $-2$ | 1 | 1 | 20 |

$B^{-1}$

Simple Table?

<u>find</u> $\alpha^1 = B^{-1} A_1 = \begin{bmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 3/2 \\ 2 \end{bmatrix}$

<u>find $z_j - c_j$</u>

$$z_1 - c_1 = C_B^T \alpha^1 - c_1 = (2 \;\; 5 \;\; 0) \begin{bmatrix} -1/4 \\ 3/2 \\ 2 \end{bmatrix} - 3 = 4$$

$$z_4 - c_4 = C_B^T \alpha^4 - c_4 = (2 \;\; 5 \;\; 0) \begin{bmatrix} 1/2 \\ 0 \\ -2 \end{bmatrix} - 0 = 1$$

$$z_5 - c_5 = C_B^T \alpha^5 - c_5 = (2 \;\; 5 \;\; 0) \begin{bmatrix} -1/4 \\ 1/2 \\ 1 \end{bmatrix} - 0 = 2$$

# Big-M Method and Two-Phase Method

(Constraints contain "≤", "≥" and "=" type)

$$\text{Min } f = 3x_1 - 5x_2$$
$$\text{s.t. } x_1 + x_2 \le 6$$
$$2x_1 - x_2 \ge 9$$
$$2x_1 + 3x_2 = 12$$
$$x_1, x_2 \ge 0$$

Standard form

$$\text{Min } f = 3x_1 - 5x_2$$
$$\text{s.t. } x_1 + x_2 + s_1 = 6$$
$$2x_1 - x_2 - s_2 = 9$$
$$2x_1 + 3x_2 = 12$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Coefficient Matrix

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & -1 \\ 2 & 3 & 0 & 0 \end{bmatrix} \end{array} \rightarrow \text{Any identity matrix of order 3}$$

(Make identity matrix)

Add artificial variables in the constraints

Add $a_2$ in second constraint and $a_3$ in third constraint

Constraints

$$x_1 + x_2 + s_1 = 6$$
$$2x_1 - x_2 - s_2 + a_2 = 9$$
$$2x_1 + 3x_2 + a_3 = 12$$
$$x_1, x_2, s_1, s_2, a_2, a_3 \ge 0$$

Here $a_2$ and $a_3$ are artificial variables

Now Coefficient Matrix(new)

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_2 & s_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Identity

$\Rightarrow s_1, a_2, a_3$ Constitute starting basic variables

## (I) Big-M Method

The objective function is modified as follows:

Max problem: $f_a = f - M(\text{Sum of all artificial variables})$

Min problem: $f_a = f + M(\text{Sum of all artificial variables})$

$$\text{Min } f_a = 3x_1 - 5x_2 + M(a_2 + a_3)$$

Now problem becomes (Modified problem)

$$\text{Min } f_a = 3x_1 - 5x_2 + M(a_2 + a_3)$$

s.t.

$$x_1 + x_2 + s_1 = 6$$

$$2x_1 - x_2 - s_2 + a_2 = 9$$

$$2x_1 + 3x_2 + a_3 = 12$$

$$x_1, x_2, s_1, s_2, a_2, a_3 \geq 0, \quad M > 0 \text{ (very large Number)}$$

* If all the artificial variables in the optimal table are equal to zero, then the optimal solution is same for our original problem.

* If some or all artificial variables have non-zero value in last table, then the modified problem becomes different from original problem. Thus, the original problem have no solution.





| Basic var. | $x_1$ | $x_2$ | $s_2$ | $s_1$ | $a_2$ | $a_3$ | Sol. | Min ratio |
|---|---|---|---|---|---|---|---|---|
| Z | $\cancel{-3}$ $4M-3$ | $\cancel{5}$ $2M+5$ | $\cancel{0}$ $-M$ | 0 | $-\cancel{M}$ 0 | $-\cancel{M}$ 0 | $\cancel{0}$ $21M$ | |
| $s_1$ | 1 | 1 | 0 | 1 | 0 | 0 | 6 | 6/1 |
| ← $a_2$ | $\boxed{2}$ | −1 | −1 | 0 | 1 | 0 | 9 | 9/2 |
| $a_3$ | 2 | 3 | 0 | 0 | 0 | 1 | 12 | 12/2 |
| Z | 0 | $4M+\frac{7}{2}$ | $M-\frac{3}{2}$ | 0 | $-2M+\frac{3}{2}$ | 0 | $3M+\frac{27}{2}$ | |
| $s_1$ | 0 | 3/2 | 1/2 | 1 | −1/2 | 0 | 3/2 | $\frac{3/2}{3/2}=1$ |
| pivot row $x_1$ | 1 | −1/2 | −1/2 | 0 | 1/2 | 0 | 9/2 | |
| ← $a_3$ | 0 | $\boxed{4}$ | 1 | 0 | −1 | 1 | 3 | 3/4 Min |
| Z | 0 | 0 | −19/8 | 0 | $-M+\frac{19}{8}$ | $-M-\frac{7}{8}$ | 87/8 | |
| $s_1$ | 0 | 0 | 1/8 | 1 | −1/8 | −3/8 | 3/8 | |
| $x_1$ | 1 | 0 | −3/8 | 0 | 3/8 | 1/8 | 39/8 | |
| $x_2$ | 0 | 1 | 1/4 | 0 | −1/4 | 1/4 | 3/4 | |

Min → Simplex table ✗ ?

Min

New table optimal?

Optimal Sol. $x_1 = \frac{39}{8}$, $x_2 = \frac{3}{4}$ $(\because a_2 = a_3 = 0)$

$\text{Min } f = \frac{87}{8}$

↳ optimal table $\because z_j - c_j \leq 0 \ \forall j$

New Z-row = old Z-row + M $a_2$ row + M $a_3$ row

| | | | | | | |
|---|---|---|---|---|---|---|
| −3 | 5 | 0 | 0 | −M | −M | 0 |
| 2M | −M | −M | 0 | M | 0 | 9M |
| 2M | 3M | 0 | 0 | 0 | M | 12M |
| 4M−3 | 2M+5 | −M | 0 | 0 | 0 | 21M |

New Z-row = old Z-row + (−4M+3) pivot row

| | | | | | | |
|---|---|---|---|---|---|---|
| 4M−3 | 2M+5 | −M | 0 | 0 | 0 | 21M |
| −4M+3 | $2M-\frac{3}{2}$ | $2M-\frac{3}{2}$ | 0 | $-2M+\frac{3}{2}$ | 0 | $-18M+\frac{27}{2}$ |
| 0 | | | | | | |

New $s_1$-row = old $s_1$-row − pivot-row

New $a_3$-row = old $a_3$-row − 2 pivot-row

New Z-row = old Z row $-\left(4M+\frac{7}{2}\right)$ pivot-row

New $s_1$-row = old $s_1$-row $-\frac{3}{2}$ pivot row

New $x_1$-row = old $x_1$-row $+\frac{1}{2}$ pivot row

## (II) Two-Phase Method

**Phase-I:** It is applied to find the starting BFS

**Phase-II:** If starting BFS found in Phase-I, then phase-II is applied to find the optimal solution of the given problem.

**Objective functions**

- **Phase I:** Min $Z_a$ = Sum of all artificial variables
- **Phase II:** Original obj. function



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**Q** Min $f = 3x_1 - 5x_2$

s.t.

$$x_1 + x_2 \le 6$$
$$2x_1 - x_2 \ge 9$$
$$2x_1 + 3x_2 = 12$$
$$x_1, x_2 \ge 0$$

**Standard form:** Min $f = 3x_1 - 5x_2$

s.t.

$$x_1 + x_2 + s_1 = 6$$
$$2x_1 - x_2 - s_2 = 9$$
$$2x_1 + 3x_2 = 12$$
$$x_1, x_2, s_1, s_2 \ge 0$$

**Phase-I:** Min $Z_a = a_2 + a_3$

s.t.

$$x_1 + x_2 + s_1 = 6$$
$$2x_1 - x_2 - s_2 + a_2 = 9$$
$$2x_1 + 3x_2 + a_3 = 12$$
$$x_1, x_2, s_1, s_2, a_2, a_3 \ge 0$$

| Basic var. | $x_1$ | $x_2$ | $s_2$ | $s_1$ | $a_2$ | $a_3$ | Sol. | Min ratio |
|---|---|---|---|---|---|---|---|---|
| $Z$ | $\cancel{0}$ 4 | $\cancel{0}$ 2 | $\cancel{0}$ −1 | 0 | $\cancel{-1}$ 0 | $\cancel{-1}$ 0 | $\cancel{0}$ 21 | |
| $s_1$ | 1 | 1 | 0 | 1 | 0 | 0 | 6 | 6/1 |
| ← $a_2$ | $\boxed{2}$ | −1 | −1 | 0 | 1 | 0 | 9 | 9/2 Min |
| $a_3$ | 2 | 3 | 0 | 0 | 0 | 1 | 12 | 12/2 |
| $Z$ | 0 | 4 | 1 | 0 | – | 0 | 3 | Min ratio |
| $s_1$ | 0 | 3/2 | 1/2 | 1 | – | 0 | 3/2 | 1 |
| pivot row $x_1$ | 1 | −1/2 | −1/2 | 0 | 1/2 | 0 | 9/2 | |
| ← $a_3$ | 0 | $\boxed{4}$ | 1 | 0 | – | 1 | 3 | 3/4 Min |
| $Z$ | 0 | 0 | 0 | 0 | – | – | 0 | |
| $s_1$ | 0 | 0 | 1/8 | 1 | – | – | 3/8 | |
| $x_1$ | 1 | 0 | −3/8 | 0 | – | – | 39/8 | |
| $x_2$ | 0 | 1 | 1/4 | 0 | – | – | 3/4 | |

→ Simplex table? ✗

New Z-row = old Z-row + $a_2$-row + $a_3$ row

New Z-row = old-Z row − 4 pivot row

New $s_1$-row = old $s_1$ row − pivot row

New $a_3$-row = old $a_3$ row − 2 pivot row

→ last table of Phase I

Optimal table of phase-I

## Phase-II

| Basic var. | $x_1$ | $x_2$ | $s_2$ | $s_1$ | Sol. |
|---|---|---|---|---|---|
| $Z$ | $\cancel{-3}$ 0 | $\cancel{5}$ 0 | $\cancel{0}$ −19/8 | 0 | $\cancel{0}$ 87/8 |
| $s_1$ | 0 | 0 | 1/8 | 1 | 3/8 |
| $x_1$ | 1 | 0 | −3/8 | 0 | 39/8 |
| $x_2$ | 0 | 1 | 1/4 | 0 | 3/4 |

→ Simplex table? ✗

New Z row = old-Z row + 3 $x_1$ row − 5 $x_2$ row

↳ SF bounded

optimal table

Optimal Sol. $x_1 = \frac{39}{8}$, $x_2 = \frac{3}{4}$

Min $f = \frac{87}{8}$