3. **Tree**

Definition 3.1. *A tree is a connected graph that has no circuit. A forest is a graph that has no circuit, in other words, it is a graph whose each component is a tree.*

Example 3.2. *As the naming suggest, any real tree*

Theorem 3.3. *For a connected graph G with n vertices the following statements are equivalent.*
(a) G is a tree
(b) G has $n-1$ edges
(c) There is a unique path between any two given vertices in G.

Proof. $(a) \implies (b)$.
We will use induction on the number of vertices.
For one edge connected graph, we have two vertices. therefore (b) is true.
By induction, let us assume that (b) is true for all graphs G with $|V(G) \leq n$. To prove (b) for $|V(G) = n$
First claim, removing an edge e from G (not any vertex) leaves G disconnected with two component. Let an edge uv is removed from G then $G \setminus uv$ is disconnected with two component. Suppose not, then there is a path $\gamma := u \to$ from u to v, say of the form $\gamma := u \to v_1 \to v_2, \cdots, ..$. Compositing/ concatenating the edge/path $u \to v$ with γ we obtain a cycle in G, but G is a tree, hence our supposition cannot be true. Hence the claim.
Now suppose $G \setminus e = G_1 \cup G_2$, and the components G_1 and G_2 have n_1 and n_2 vertices respe
By induction $|E(G_1)| = V(G_1) - 1$ and $|E(G_2)| = V(G_2) - 1$. Adding these, we obtain

$$|E(G_1)| + |E(G_2)| = V(G_1) + V(G_2) - 2$$
$$\implies E(G) - 1 = V(G) - 2$$
$$\implies E(G) = V(G) - 1$$

$(b) \implies (a)$
Suppose. $|E(G)| = n-1$ but G is not a tree. We remove (one or more) edges from cycles to obtain a tree T_G (also called spanning tree), so $|E(G)| > |E(T_G)|$. On the other hand, $|V(T_G)| = n$, so, by the above implication $|E(T_G)| = n-1$. It follows that $|E(G)| > n-1$, a contradiction, hence G is a tree.
$(a) \implies (c)$
Suppose we have two distinct paths γ_1 and γ_2 from u to t v in a tree, then the concatenation $\gamma_2^{-1} \circ \gamma_1$ (walk from u to v along γ_1 first return to u along γ_2) produces a cycle, which we cannot have in a tree. Hence the implication.
$(c) \implies (a)$,
Suppose we have a cycle $(v_1, v_2, \cdots v_n, v_1)$ in G with property (3) then we can form two paths from v_1 to v_2 by traversing in two opposite directions, one is $v_1 \to v_2$ and the other is $v_1 \to v_n \to \cdots, v_2$

□

A **minimally connected graph** is a connected graph where the removal of any single edge results in the graph becoming disconnected. It is clear from the proof above that a tree is minimally connected, and vice-versa.

Theorem 3.4. *Any tree with at least two vertices has at least two pendant vertices (also called leaves).*

Proof. Consider a maximal path γ in T. If u and v are the end vertices of γ , then the only neighbour of u must be the vertex adjacent to u on γ. If u was adjacent to any other vertex on γ, then T would contain a cycle; if u was adjacent to any vertex w not on γ, extending γ to w would yield a path longer than γ, thereby contradicting the maximality of γ. Hence $d(u) = 1$, and by the same argument $d(v) = 1$.

Alternative proof: Let T be a tree with n vertices. By the previous theorem it has $n-1$ edges. By the Handshake Lemma

$$2(n-1) = \sum_{v \in V(G)} d(v)$$

Let's assume T has exactly one leaf, which means every other vertex has degree at least 2. Consequently,

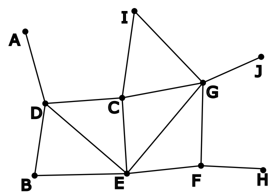
$$2(n-1) = \sum_{v \in V(G)} d(v) \geq 2(n-1) + 1$$

A contradiction. The same contradiction arises if there is no leaf in T. Hence proved. □

Definition 3.5. *For a graph G, length of a path in G is defined as the number of edges contained in the path. The distance between two vertices u and v in G is the minimum of the length of all possible paths between u and v. Eccentricity of any particular vertex v in G is the maximum distance from vertex v to any other vertex w,*

$$\mathcal{E}(v) = \max_{u \in G} d(u, v)$$

Center of G are the vertices having minimum ecentricity. Radius of G is the minimum of the ecentricities of all its vertices.



The ecentricity of some of the vertices in the above graph is given by: $\mathcal{E}(G) = 3$ (as $d(E, B) = 3$) $\mathcal{E}(G) = 3$ (as $d(C, H) = 3$), $\mathcal{E}(G) = 2$.
It is not difficult to verify that the vertex E has the least ecentricity. Hence, the center of this graph is the vertex E.

Theorem 3.6. *Every tree has either one or two centers.*

Proof. Let T be a tree. By the previous lemma T has at least two leaves of a tree. Let T' be the tree obtained by removing the leaves of T. Then we note that, for any vertex v in T', the ecentricity of v in T' is one less than the ecentricity of v in T. Hence T' has the same set of vertices as T. The theorem now follows by induction. □

Definition 3.7. *A spanning tree T of an undirected graph G is a subgraph that is a tree which includes all of the vertices of G.*

Theorem 3.8. *Every connected graph has a spanning tree.*

Proof. By induction on the number of edges. If G is connected and has zero edges, it is a single vertex, so G is already a tree.

Now suppose G has $m \geq 1$ edges. If G is a tree, it is its own spanning tree. Otherwise, G contains a cycle; remove one edge of this cycle. The resulting graph G' is still connected and has fewer edges, so it has a spanning tree; this is also a spanning tree for G. $\square$

In general, spanning trees are not unique, that is, a graph may have many spanning trees. It is possible for some edges to be in every spanning tree even if there are multiple spanning trees. For example, any pendant edge must be in every spanning tree, as must any edge whose removal disconnects the graph.