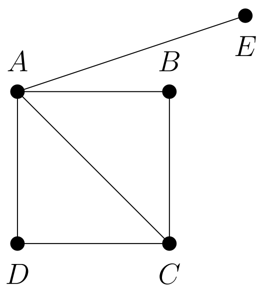
1.8. **Adjacency and incident matrix.** For a simple graph with vertex set $U = \{u_1, \cdots, u_n\}$, the adjacency matrix is a square $n \times n$ matrix A such that its element A_{ij} is one when there is an edge from vertex u_i to vertex u_j, and zero when there is no edge.



G

$$A_G = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

In general, for a graph (not necessarily simple) with vertex set $U = \{u_1, \cdots, u_n\}$, the adjacency matrix is a square $n \times n$ matrix A such that its element A_{ij} for $i \neq j$ is the number of edges from vertex u_i to vertex u_j, and zero when there is no edge. A_{ii} is two times the number of loops based at u_i. Clearly, adjacency matrix is symmetric.

The **incidence matrix** of a simple graph G with n vertices and m edges is a $n \times m$ matrix B_G, where n and m are the numbers of vertices and edges respectively, such that

$$(B_G)_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is incident with edge } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

For example, the incidence matrix of the graph shown on the right is a matrix consisting of 4 rows (corresponding to the four vertices, 1–4) and 4 columns (corresponding to the four edges, e_1, e_2, e_3, e_4):

The *degree matrix* D_G os $n \times n$ diagonal matrix with (i, i) entry given by degree of i-th vertex v_i .
Exercise: $B_G B_G^T = A_G + D_G$.

Definition 1.14. *For a simple graph G, the Laplacian matrix of G is defined to be $L_G = D_G - A_G$.*

- The sum the rows of the Laplacian matrix (or the columns, since it's symmetric), is always zero.
- Laplacian is a singular matrix
- Null space L_G is one dimensional

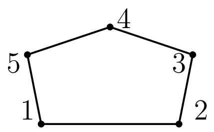
- L_G is positive semi-definite matrix, consequently, eigenvalues of L_G are non-negative.

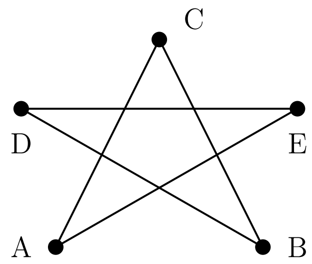
Theorem 1.15 (Matrix tree theorem/Kirchoff Theorem). *For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the non-zero eigenvalues of its Laplacian matrix. Then the number of spanning trees of G is*

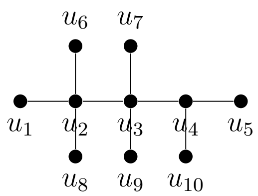
$$t(G) = \frac{1}{n}\lambda_1\lambda_2 \cdots \lambda_{n-1}\,.$$

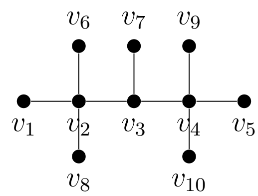
Proof. Beyond the scope of this course. $\square$

1.9. **Isomorphism of graphs.** Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomoprhic if there exist a bijective function (called isomorphism) $f : V \to V'$ such that $f(e) \in E'$ for every edge $e \in E$.
Example: The following two graphs are isomorphic. It is easy to verify that following bijection $f : V \to V$ given by $1 \to A, 2 \to C, 3 \to B, 4 \to D, 5 \to E$ defines an isomorphism.







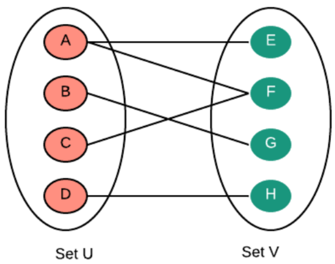


Consider the above two graph. An isomorphism must map a vertex of degree d of the first graph to a vertex of degree d in the second graph. Therefore, the only vertex with degree 3, u_4 must be mapped to the only vertex of degree 3, v_3 . An isomorphism must also map adjacent vertices to adjacent vertices. Subsequently, u_5 (adjacent to u_4) must map to a degree 2 vertex adjacent to v_3. But both the vertices adjacent to v_3 (v_2 and v_4) have degree 4. Hence, an isomorphism is impossible. Hence, there doesn't exist any isomorphism between the two graph.

Theorem 1.16. *Two graphs G and H are isomorphic if and only if there is a permutation matrix P such that $A_G = PA_HP^{-1}$*

Proof. Exercise. $\square$

1.10. **Bipartite Graph.** A bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint and independent sets U and V, that is, every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the parts of the graph.



Bipartite Graph OR Bi-Graph

A complete bipartite graph is a bipartite graph where every vertex of the first set is connected to every vertex of the second set. It is denoted by $K_{m,n}$ where m and n denote the number of vertices in the first and second set respectively.

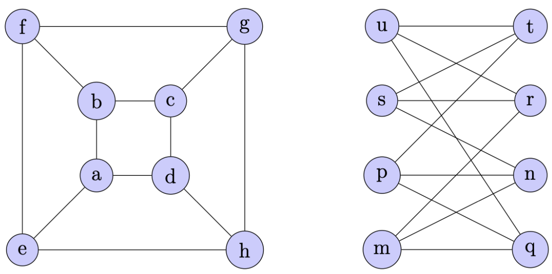
1.11. **Practice problems set 1.** 1. Can there exist a graph on 13 vertices and 31 edges, with three vertices of degree 1, and seven vertices of degree 4? Explain.

2. Given the following degree sequences either construct a graph with such a degree sequence, or explain why this would be impossible.
(a) 1,1,1,1,1,1
(b) 5,4,3,2,1
(c) 6,6,4,2,2,2,2,1

3. What is the maximum number of vertices on a graph that has 35 edges and every vertex has degree ≤ 3?

4. Suppose all vertices in a graph, G, have odd degree, k. Prove that k divides $|E(G)|$.

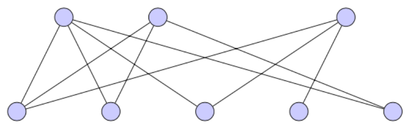
5. Given the following two graphs, write an explicit isomorphism between them.



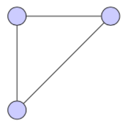
6. Draw all non-isomorphic graphs with n vertices for $n = 3$ and 4.

7. Can a graph have K_3 subgraph and be bipartite? Explain.

8. Let G be the following graph:



a) Is the following a subgraph of G?



(b) Draw an induced subgraph of G with exactly 3 edges.