Duality

Primal LPP $\longleftrightarrow$ Dual LPP

3-variable problem
50 constraints
→ standard form

3-constraints
50-variables

Obj function value is same

Rules for constructing the dual problem

Maximization Problem ("≤")	Minimization Problem ("≥")
Constraints	**Variables**
$\geq$	≤ 0
$\leq$	≥ 0
$=$	Unrestricted in sign
Variables	**Constraints**
≥ 0	$\geq$
≤ 0	$\leq$
Unrestricted	$=$

Q Primal Problem

Dual variables

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3$$

s.t.

$$x_1 + 2x_2 + x_3 \leq 10 \longrightarrow y_1 (\geq 0)$$

$$2x_1 - x_2 + 3x_3 = 8 \longrightarrow y_2 \text{ (unrestricted in sign)}$$

$$x_1, x_2, x_3 \geq 0$$

$x_1 \geq 0 \downarrow \geq$ constraint, $x_2 \geq 0 \downarrow \geq$, $x_3 \geq 0 \downarrow \geq$

Dual LPP

$$\text{Min } \omega = 10y_1 + 8y_2$$

s.t.

$$y_1 + y_2 \geq 5$$

$$2y_1 - y_2 \geq 12$$

$$y_1 + 3y_2 \geq 4$$

$y_1 \geq 0$, y_2 unrestricted in sign

Q Max $z = 5x_1 + 6x_2$

Dual variables

s.t. $x_1 + 2x_2 = 5 \longrightarrow y_1$ (unrestricted)

$-x_1 + 5x_2 \ge 3 \longrightarrow y_2 \ (\le 0)$

$4x_1 + 7x_2 \le 8 \longrightarrow y_3 \ (\ge 0)$

x_1 unrestricted, $x_2 \ge 0$

"=" "$\ge$"

Dual LPP

Min $\omega = 5y_1 + 3y_2 + 8y_3$

s.t. $y_1 - y_2 + 4y_3 = 5$

$2y_1 + 5y_2 + 7y_3 \ge 6$

y_1 unrestricted, $y_2 \le 0$, $y_3 \ge 0$

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* Dual of Dual is a primal

Q Min $z = 3x_1 + 5x_2 - x_3$

Dual variables

primal (1)

s.t. $x_1 + x_2 \ge 5 \longrightarrow y_1 \ (\ge 0)$

$2x_1 - x_2 + 3x_3 \le 6 \longrightarrow y_2 \ (\le 0)$

$x_1 \le 0$, $x_2 \ge 0$, x_3 unrestricted

"$\ge$" "$\le$" "="

Dual

Dual of (1)

(2)

Max $\omega = 5y_1 + 6y_2$

Dual variables

s.t. $y_1 + 2y_2 \ge 3 \longrightarrow x_1 \ (\le 0)$

$y_1 - y_2 \le 5 \longrightarrow x_2 \ (\ge 0)$

$3y_2 = -1 \longrightarrow x_3$ (unrestricted)

$y_1 \ge 0$, $y_2 \le 0$

"$\ge$" "$\le$"

Dual of Dual is primal

Dual

Dual of (2)

(3)

Min $z = 3x_1 + 5x_2 - x_3$

s.t. $x_1 + x_2 \ge 5$

$2x_1 - x_2 + 3x_3 \le 6$

$x_1 \le 0$, $x_2 \ge 0$, x_3 unrestr...

(1) $\equiv$ (3)

Primal Solution $\xrightarrow[\text{find}]{\text{Knows} \quad \text{Find}}$ Dual Solution

(Dual Solution $\xrightarrow{\text{Know}}$ find Primal Solution)

Two Methods

Method - I

$$\text{optimal Value of dual variable } y_i = \begin{pmatrix}\text{Optimal primal} \\ \text{Z-coefficient of} \\ \text{starting variable } (x_i) \text{ say}\end{pmatrix} + \begin{pmatrix}\text{Original objective} \\ \text{Coefficient of } x_i\end{pmatrix}$$

Exp. Primal Problem:

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3 \qquad \text{dual var}$$

s.t.

$$x_1 + 2x_2 + x_3 \le 10 \to y_1 (\ge)$$

$$2x_1 - x_2 + 3x_3 = 8 \to y_2 \text{ (unrestricted)}$$

$$\underbrace{x_1}_{\ge}, \underbrace{x_2}_{\ge}, \underbrace{x_3}_{\ge} \geq 0$$

Solve Primal Problem by Big-M Method and find the solution of Dual using the optimal table of primal

Big-M Method

$$\text{Max } Z_a = 5x_1 + 12x_2 + 4x_3 - Ma_2 + 0 \cdot s_1$$

s.t.

$$x_1 + 2x_2 + x_3 + s_1 = 10$$

$$2x_1 - x_2 + 3x_3 + a_2 = 8$$

$$x_1, x_2, x_3, s_1, a_2 \ge 0, \quad M > 0 \text{ (large No.)}$$

Solve this by yourself Using Big-M Method

Optimal table of primal

Basic Var.	x_1	x_2	x_3	s_1	a_2	Sol.
Z_a	0	0	3/5	29/5	$-\frac{2}{5}+M$	274/5
x_2	0	1	$-1/5$	2/5	$-1/5$	12/5
x_1	1	0	7/5	1/5	2/5	26/5

B^{-1}

Optimal Solution of primal is

$x_1 = \frac{26}{5}$, $x_2 = \frac{12}{5}$, $x_3 = 0$ and $\text{Max } Z = \frac{274}{5}$

Dual

$$\text{Min } W = 10y_1 + 8y_2$$

s.t.

$$y_1 + 2y_2 \ge 5$$

$$2y_1 - y_2 \ge 12$$

$$y_1 + 3y_2 \ge 4$$

$y_1 \ge 0$, y_2 unrestricted in sign.

Dual Solution from Primal table

Starting primal basic variables	s_1	a_2
Z-equation coefficients	$\frac{29}{5}$	$-\frac{2}{5}+M$
Original objective coefficients	0	$-M$
Dual variables (y_1, y_2)	$\frac{29}{5}$	$-\frac{2}{5}$
	y_1	y_2

$\therefore$ Optimal Dual Solution is

$$y_1 = \frac{29}{5}, \quad y_2 = -\frac{2}{5}$$

and $\text{Min } w = \frac{274}{5}$

Method 2

$$\text{Optimal Values of dual variables } (y_i) = \underbrace{\begin{pmatrix}\text{Row vector of original objective coefficients of optimal primal basic variables}\end{pmatrix}}_{C_B^T} \times \underbrace{\begin{pmatrix}\text{Optimal primal inverse}\end{pmatrix}}_{B^{-1}}$$

$$= C_B^T B^{-1}$$

Same exp

$$(y_1, y_2) = C_B^T B^{-1}$$

$$= (\text{original obj. coeff. of } x_2, x_1) * (\text{optimal inverse})$$

$$= (12 \quad 5)\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix} = \begin{bmatrix} \frac{29}{5} & -\frac{2}{5} \end{bmatrix}$$

Dual sol.

$\Rightarrow y_1 = \frac{29}{5}, \quad y_2 = -\frac{2}{5}$ and $\text{Min } w = \frac{274}{5}$

Weak Duality theorem

For any pair of feasible primal and dual solutions,

(objective func. value in maximization problem) $\le$ (objective function value in minimization problem)

Strong Duality theorem

If primal problem has an optimal solution, then its dual also has an optimal solution and objective function values of both are equal.

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Dual Simplex Method (Optimality Retained, feasibility achieved)

* The simplex method (algorithm) starts feasible and continue to be feasible until the optimum is reached.
* However, the dual simplex starts infeasible and remains infeasible until feasibility is restored.
* Dual simplex method is applicable if optimality condition is satisfied but feasibility condition is not satisfied.

Simplex Method (Big-M, Two phase)	Dual Simplex Method
$\le$, $\ge$, $=$, $b \ge 0$	"$\le$", $\boxed{b<0,\ b=0,\ b>0}$
s_i, a_i	
→ Feasibility Retains	→ Optimality Retains ($z_j - c_j \le 0$ Min, $z_j - c_j \ge 0$ Max)
→ Achieve optimality	→ Feasibility is achieved

Algorithm : (No artificial variable added)

① Write the constraints as $Ax = b$ where A contains identity matrix

"$\le$" , "$\ge$" $\xrightarrow[\text{by } (-1)]{\text{multiply}}$ "$\le$" , "$=$" $<$ "$\le$" / "$\ge$" $\xrightarrow[\text{by } -1]{\text{multiply}}$ "$\le$"

(only slack variables are added)

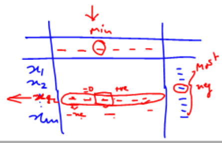
② Leaving Criteria : Let $x_1, x_2, \dots, x_m$ be basic variables in a simplex table. The variable x_r will leave the basis if

$$x_r = \min_{1 \le i \le m} \{ x_i : x_i < 0 \}$$

(Most negative will leave the basis)

③ Entering Criteria : The non-basic variable x_k will enter into the basis if

$$\left| \frac{z_k - c_k}{\alpha_r^k} \right| = \min_{1 \le j \le n} \left\{ \left| \frac{z_j - c_j}{\alpha_r^j} \right| , \ \alpha_r^j < 0 \right\}$$



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Remark : If $\alpha_r^j \ge 0 \ \forall j$, then $S_F = \phi$, i.e. no solution.

Exp Min $Z = 6x_1 + 9x_2$

s.t. $x_1 + 2x_2 \ge 3$

$-x_1 + x_2 \le 5$

$-2x_1 + x_2 \ge 1$

$x_1, x_2 \ge 0$

Can I apply Dual Simplex Method ?

$\longrightarrow$

Min $Z = 6x_1 + 9x_2$

s.t. $-x_1 - 2x_2 \le -3$

$-x_1 + x_2 \le 5$

$2x_1 - x_2 \le -1$

$x_1, x_2 \ge 0$

Make Identity

Min $Z = 6x_1 + 9x_2$

s.t. $-x_1 - 2x_2 + s_1 = -3$

$-x_1 + x_2 + s_2 = 5$

$2x_1 - x_2 + s_3 = -1$

$x_1, x_2, s_1, s_2, s_3 \ge 0$

(Here starting basic variables are s_1, s_2 & s_3 which forms identity matrix)

Basic var.	x_1	x_2	s_1	s_2	s_3	Sol.
z	-6	-9	0	0	0	0
s_1	-1	$\boxed{-2}$	1	0	0	-3
s_2	-1	1	0	1	0	5
s_3	2	-1	0	0	1	-1
z	$-3/2$	0	$-9/2$	0	0	$27/2$
x_2 (pivot row)	$1/2$	1	$-1/2$	0	0	$3/2$
s_2	$-3/2$	0	$1/2$	1	0	$7/2$
s_3	$5/2$	0	$-1/2$	0	1	$1/2$

Optimality is here All $z_j - c_j \le 0$

Most negative

Feasibility Disturbed

$$\text{Min}\left\{\left|\frac{-6}{-1}\right|, \left|\frac{-9}{-2}\right|\right\} = \text{Min}\left\{6, \frac{9}{2}\right\} = \frac{9}{2}$$

for x_2

New z row = old z row + 9 pivot row

-6	-9	0	0	0	0
$9/2$	9	$-9/2$	0	0	$27/2$
$-3/2$	0	$-9/2$	0	0	$27/2$

New s_2 row = old s_2 row − pivot row

New s_3 row = old s_3 row + pivot row

↳ optimal table ($\because$ All $z_j - c_j \le 0$ & $b_i \ge 0 \ \forall i$)

optimal solution, $x_1 = 0$, $x_2 = 3/2$

and Min $z = 27/2$

Q Min $Z = 4x_1 + 2x_2$

s.t. $x_1 + x_2 = 1$ < $x_1 + x_2 \le 1$, $x_1 + x_2 \ge 1$

$3x_1 - x_2 \ge 2$

$x_1, x_2 \ge 0$

Min $Z = 4x_1 + 2x_2$

s.t. $x_1 + x_2 \le 1$

$-x_1 - x_2 \le -1$

$-3x_1 + x_2 \le -2$

$x_1, x_2 \ge 0$

Make Identity

Min $Z = 4x_1 + 2x_2$

s.t $x_1 + x_2 + s_1 = 1$

$-x_1 - x_2 + s_2 = -1$

$-3x_1 + x_2 + s_3 = -2$

$x_1, x_2, s_1, s_2, s_3 \ge 0$

Basic Var.	x_1	x_2	s_1	s_2	s_3	Sol.
z	-4	-2	0	0	0	0
s_1	1	1	1	0	0	1
s_2	-1	-1	0	1	0	-1
s_3	$\boxed{-3}$	1	0	0	1	-2
z	0	$-10/3$	0	0	$-4/3$	$8/3$
s_1	0	$4/3$	1	0	$1/3$	$1/3$
s_2	0	$\boxed{-4/3}$	0	1	$-1/3$	$-1/3$
x_1	1	$-1/3$	0	0	$-1/3$	$2/3$
z	0	0	0	$-5/2$	$-1/2$	$7/2$
s_1	0	0	1	1	0	0
x_2	0	1	0	$-3/4$	$1/4$	$1/4$
x_1	1	0	0	$-1/4$	$-1/4$	$3/4$

$$\text{Min}\left\{ \underset{x_2}{\left|\frac{-10/3}{-4/3}\right|}, \underset{s_3}{\left|\frac{-4/3}{-1/3}\right|} \right\} = \text{Min}\{2.5, 4\} = 2.5$$

$\hookrightarrow$ for x_2

Optimal table

Sol. is $x_1 = 3/4$, $x_2 = 1/4$

Min $z = \frac{7}{2}$