Matrices & Operators

→ What is an operator?
An operator is a mathematical rule that can be applied to a function to transform it into another function

Ex:- Let $\hat{D} = \frac{d}{dx}$

$$\hat{D}(x\cos x) = \frac{d}{dx}x \cos x + x\frac{d}{dx}\cos x$$

$$= \cos x - x\sin x$$

Extending this idea to vector spaces.

- An operator $\hat{A}$ is a mathematical rule that transforms a Ket $|\psi\rangle$ into another Ket $|\phi\rangle$

i.e $\hat{A}|\psi\rangle = |\phi\rangle$

- Operators also act on duals. Result is another dual vector.

$\Rightarrow$ $\langle u|\hat{A} = \langle v|$

- Most of times, an operator transforms vectors into other vectors that belong to same space.

- An operator is linear if the following relationship holds :→ Given α and β $\in$ complex numbers and state vectors $|\psi_1\rangle$ and $|\psi_2\rangle$

$$\hat{A}\left(\alpha|\psi_1\rangle + \beta|\psi_2\rangle\right)$$

$$= \alpha\hat{A}|\psi_1\rangle + \beta\hat{A}|\psi_2\rangle$$

Generally,

$$\hat{A}\left(\sum_i \alpha_i |u_i\rangle\right) = \sum_i \alpha_i \left(\hat{A}|u_i\rangle\right)$$

- The simplest operator is the identity operator $\hat{I}$.
$\hat{I}$ is an operator that leaves the state of a system unchanged.

$$\hat{I}|\psi\rangle = |\psi\rangle$$

- Zero operator ($\hat{O}$) :- It transforms a vector into the zero vector.

$$\hat{O}|\psi\rangle = 0$$

- Operators are denoted by capital letters

OBSERVABLES

- In quantum theory, dynamical variables like position, momentum, angular momentum and energy are called observables.
- Observable are things we measure in order to characterize quantum state of a particle.
- An important postulate of Quantum theory is that there is an operator that corresponds to each physical observable.

The Pauli operators

A set of operators of fundamental importance are called Pauli operators.

The Pauli operators are denoted as:-

$\sigma_0, \sigma_1, \sigma_2$ and σ_3 OR

$\sigma_0, \sigma_x, \sigma_y$ and σ_z OR

I, X, Y, Z

1) Identity Operator

σ_0, I

when identity operator acts on computational basis states :-

$$\sigma_0 |0\rangle = |0\rangle$$

$$\sigma_0 |1\rangle = |1\rangle$$

2) σ_1 or σ_x or X operator.

It acts as follows:-

$$\sigma_1 |0\rangle = |1\rangle$$

$$\sigma_1 |1\rangle = |0\rangle$$

for this reason, X is sometimes known as NOT operator.

3) σ_2 or σ_y or Y pauli operators

$$\sigma_2 |0\rangle = -i |1\rangle$$

$$\sigma_2 |1\rangle = i |0\rangle$$

4) $\sigma_3 = \sigma_z = Z$

$$\sigma_3|0\rangle = |0\rangle$$

$$\sigma_3|1\rangle = -|1\rangle$$

OUTER PRODUCT

The product of a ket $|\psi\rangle$ with a dual $\langle\phi|$, written as $|\psi\rangle\langle\phi|$ is called outer product. This is an operator. ($|\psi\rangle\langle\phi|$ is a machine say.

You put a vector in you get a new vector out)

Apply $|\psi\rangle\langle\phi|$ to ket $|\chi\rangle$

$$|\psi\rangle\langle\phi||\chi\rangle$$

$$= |\psi\rangle \langle\phi|\chi\rangle$$

Now $\langle\phi|\chi\rangle$ denotes inner product of ϕ & χ vectors which is a scalar. Hence the operator transforms $|\chi\rangle$ into one proportional to $|\psi\rangle$.

~~Ket with Ket~~

~~$|\psi\rangle$~~

bra/Dual with ~~Dual~~ Ket

$\langle\psi| \times |\phi\rangle$ = Inner product

Ket with bra/dual

$|\psi\rangle \times \langle\phi|$ = Outer product.

Question Describe the action of operator $\hat{A} = |0\rangle\langle 0| - |1\rangle\langle 1|$ on qubits

$$\hat{A}|\psi\rangle = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \alpha\left\{|0\rangle\langle 0||0\rangle - |1\rangle\langle 1||0\rangle\right\}$$

$$- \beta\left(|1\rangle\langle 1||1\rangle - |0\rangle\langle 0||1\rangle\right)$$

Using $\langle 0|0\rangle =$

$$= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \qquad 1\times1+0\times0 = 1$$

$\langle 0|1\rangle = 0$ $\qquad$ $\langle 1|1\rangle = 1$

$\langle 1|0\rangle = 0$

we get

$$\hat{A} = \alpha|0\rangle - \beta|1\rangle\langle 1|$$

$\therefore$ $\hat{A}$ has changed $|0\rangle$ to $\langle 0|$
and $|1\rangle$ to $-|1\rangle$

or $\hat{A}$ is the outer product representation of the Z operator.

<u>Ques:</u> Consider the action of $\hat{A} = |0\rangle\langle 0| + |1\rangle\langle 1|$ on an arbitrary qubit and deduce that this is the outer product representation of the identity operator.

~~Answer~~

Question:- Verify that the outer product representations of X and Y are given by $|0\rangle\langle 1| + |1\rangle\langle 0|$ and $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ by letting them act on state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

classmate

Date
Page

The Closure Relation

Recall $\hat{I}\{|\psi\rangle\} = |\psi\rangle$ $\quad\boxed{\hat{I} = |0\rangle\langle 0| + |1\rangle\langle 1|}$

The outer product representation of Identity operator is written as

$$\sum_{i=1}^{n} |u_i\rangle\langle u_i| = \hat{I} \qquad \text{—— (1)}$$

This equation states that given a basis set $\{|u_i\rangle\}$ in n dimensions, the identity operator can be written as eqn (1)

$\therefore$ for qubit, which is a 2-dimensional Hilbert space, we can write the identity operator using (1)

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \hat{I}$$

for a qutrit,

$$\hat{I} = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$$

Representation of Operators using matrices

→ Operators can be represented by matrices.
→ Action of an operator on a vector is reduced to simple matrix multiplication.

For an n-dimensional vector space, a operator can be represented as a ~~n-dimensional~~ $n \times n$ matrix.

$\boxed{\text{WHY Matrix}}$

- Quantum states are represented as Vectors
- Actions on vectors are done using matrix so

Operator = Matrix.

Operator Matrix

$$O = \begin{pmatrix} O_{00} & O_{01} \\ O_{10} & O_{11} \end{pmatrix}$$

where $O_{ij} = \langle i | \hat{O} | j \rangle$ or $\langle u_i | \hat{O} | u_j \rangle$

O_{ij} tells how much of i appears when operator acts on j.

$$\langle u_i | \hat{O} | u_j \rangle \neq \langle u_j | \hat{O} | u_i \rangle$$

Date
Page

Outer product & Matrix Representations

Let $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$

Outer product = Ket x dual

$$|\psi\rangle\langle\phi| = \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c^* & d^* \end{pmatrix}$$

$$\begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix}$$

$$|\phi\rangle\langle\psi| = \begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} a^* & b^* \end{pmatrix}$$

$$\begin{pmatrix} ca^* & cb^* \\ da^* & db^* \end{pmatrix}$$

Ques) - Let $|\psi\rangle = a|1\rangle + b|2\rangle + c|3\rangle$

$|\phi\rangle = e|1\rangle + f|2\rangle + g|3\rangle$

Show that the outer product $|\psi\langle\phi|$ is given by:-

$$|\psi\rangle\langle\phi| = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} e^* & f^* & g^* \end{pmatrix}$$

$$= \begin{pmatrix} ae^* & af^* & ag^* \\ be^* & bf^* & bg^* \\ ce^* & cf^* & cg^* \end{pmatrix}$$

Matrix Representation of Operators in 2-D Spaces

for qubits $\in$ 2-D Hilbert Space, we represent operators by 2x2 matrices

$$A = \begin{pmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{pmatrix}$$

Ques? Write down the matrix representation of operator Z.

Now Z operator behaves as follows:

$|0\rangle = |0\rangle \qquad |1\rangle = -|1\rangle$

$$Z = \begin{pmatrix} \langle 0|Z|0\rangle & \langle 0|Z|1\rangle \\ \langle 1|Z|0\rangle & \langle 1|Z|1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle 0|0\rangle & -\langle 0|1\rangle \\ \langle 1|0\rangle & -\langle 1|1\rangle \end{pmatrix}$$

Using $\langle 0|0\rangle = 1 \qquad \langle 1|0\rangle = 0$

$\langle 0|1\rangle = 0 \qquad \langle 1|1\rangle = 1$

we get

$$\boxed{Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

So outer product rep. of Z is

$$|0\rangle\langle 0| - |1\rangle\langle 1|$$

and

matrix representation is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Ques 1 Find the matrix representations of X, Y

Ques-2 Verify that the matrix representation of identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Answer 1 For operator X,

$$X|0\rangle = |1\rangle \qquad X|1\rangle = |0\rangle$$

$$X = \begin{pmatrix} \langle 0|X|0\rangle & \langle 0|X|1\rangle \\ \langle 1|X|0\rangle & \langle 1|X|1\rangle \end{pmatrix}$$

$$X = \begin{pmatrix} \langle 0|1\rangle & \langle 0|0\rangle \\ \langle 1|1\rangle & \langle 1|0\rangle \end{pmatrix}$$

$$\boxed{X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Date ______
Page ______

For operator Y

$$Y|0\rangle = +i|1\rangle \qquad Y|1\rangle = -i|0\rangle$$

$$Y = \begin{pmatrix} \langle 0|Y|0\rangle & \langle 0|Y|1\rangle \\ \langle 1|Y|0\rangle & \langle 1|Y|1\rangle \end{pmatrix}$$

$$\begin{matrix} i\langle 0|1\rangle & -i\langle 0|0\rangle \\ i\langle 1|1\rangle & -i\langle 1|0\rangle \end{matrix}$$

$$\boxed{Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

Now we are given $X|0\rangle = 0 \quad X|1\rangle = 1$

$Y|0\rangle = i|1\rangle \quad Y|1\rangle = -i|0\rangle$

$Z|0\rangle = 1 \quad Z|1\rangle = -|1\rangle$

$\longrightarrow$

Date
Page

Ques! The Pauli operators are given by:-

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ in the $\{|0\rangle, |1\rangle\}$

basis find the action of these operators on the basis states by considering the column vector representation of $|0\rangle$ and $|1\rangle$

Answer!- Given $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$X(|0\rangle) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0\times1 + 1\times0 \\ 1\times1 + 0\times0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$X(|1\rangle) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\times0 + 1\times1 \\ 1\times0 + 0\times1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

X is also called NOT operator or bit flip operator

HILBERT SPACE

A hilbert space is a space where quantum State live.

Defn:- A hilbert Space is a vector Space that is complete & equipped with an inner product.

A hilbert space has 4 Key features:-

1) Vector space:- $|\psi\rangle + |\phi\rangle$ (Add vectors)
$\alpha|\psi\rangle$ (Multiply by complex num)

(*) Quantum States are complex Vectors

2) Inner Product

Inner product allows us to compute
(i) Norm of a vector

$$||\psi|| = \sqrt{\langle\psi|\psi\rangle}$$

for a valid Qubit State
$$\langle\psi|\psi\rangle = 1$$

A single qubit lives in a 2-D hilbert Space

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

where $|\alpha|^2 + |\beta|^2 = 1$

HERMITIAN OPERATORS

Hermitian Adjoint :- The hermitian adjoint of an operator is obtained by taking transpose and complex conjugate of its matrices.

For an operator $\hat{A}$, the hermitian adjoint is defined as $A^{\dagger}$.

$$\langle a|\hat{A}^{\dagger}|b\rangle = \langle b|\hat{A}|a\rangle^{*}$$

Take complex conjugate of all constants in the expression, replace all kets by bras and bras by kets

- $(\alpha\hat{A})^{\dagger} = \alpha^{*}A^{\dagger}$
- $(|\psi\rangle)^{\dagger} = \langle\psi|$
- $(\langle\psi|)^{\dagger} = (|\psi\rangle)$
- $(\hat{A}\hat{B})^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger}$
- $(\hat{A}|\psi\rangle)^{\dagger} = \langle\psi|\hat{A}^{\dagger}$
- $(\hat{A}\hat{B}|\psi\rangle)^{\dagger} = \langle\psi|\hat{B}^{\dagger}\hat{A}^{\dagger}$

~~If~~ $\cancel{\hat{A} = |a\rangle\langle b|}$

$\cancel{\hat{A}^{\dagger} = |b\rangle\langle a|}$

Page

Ques:- find the adjoint of operator

$$\hat{A} = 2|0\rangle\langle 1| - i|1\rangle\langle 0|$$

Take complex conjugate, replace kets by bras and bra by kets

$$\hat{A}^{+} = 2|1\rangle\langle 0| + i|0\rangle\langle 1|$$

Ques:- ~~Show~~ find adjoint of $\hat{B} = 3i|0\rangle\langle 0| + 2i|0\rangle\langle 1|$

$$-3i|0\rangle\langle 0| - 2i|1\rangle\langle 0|$$

$\boxed{\text{Hermitian Operator}}$

An operator $\hat{A}$ is said to be hermitian operator if $\hat{A} = \hat{A}^{+}$

eg:- $\hat{Y} = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)$

$$\hat{Y}^{+} = (i|1\rangle\langle 0| - i|0\rangle\langle 1|)$$

$$\hat{Y} = \hat{Y}^{+}$$

Page

ACTIVITY

Ques Outer product rep of

(i) $X = |0\rangle\langle 1| + |1\rangle\langle 0|$

(ii) $Y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$

(iii) $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Check if Pauli operators are hermitian operators.

UNITARY OPERATOR

An operator is said to be unitary if its adjoint is equal to its inverse.

Mathematically, $UU^\dagger = I = U^\dagger U$

$Y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$

$Y^\dagger = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$

Matrix $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$Y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Date
Page

$$YY^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\begin{matrix} 0\times 0 + (-i)\times i & 0\times(-i) + (-i)\times 0 \\ i\times 0 + 0\times i & i\times -i + 0\times 0 \end{matrix}$$

$$\begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Pauly Y is Unitary

Activity
Similarly show for Pauly X & Pauli Z

NORMAL OPERATORS

An operator is said to be Normal if $AA^{+} = A^{+}A$

for Pauli Y.

$$YY^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

find $Y^{+}Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$= \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ Pauli Y is normal.

check for Pauli X & Pauli Z.

<u>Note:-</u> Hermitian operators are normal
Unitary operators are normal.

<u>Ques:</u> Consider the basis states given by

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Show that matrix representation of X operator w.r.t this basis is

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

0

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Eigen Values & Eigen Vectors

A vector is said to be an eigen vector of an operator A if

$$A|\phi\rangle = \lambda|\psi\rangle$$

where λ is eigen value of an operator

Problem:- Given an operator, find eigen vector & eigen values.

Step 1- Find eigen values using characteristic equation

$$\det|A - \lambda I| = 0.$$

$\det|A| = ad - bc$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Step 2:- Once eigen values are known, eigen vectors can be found using equation

$$A|\phi| = \lambda|\phi\rangle$$

Ques: find eigen values of an operator with matrix representation

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\det|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-1-\lambda) - (-1)$$

$$-2 - 2\lambda + \lambda + \lambda^2 + 1$$

$$\boxed{\lambda^2 - \lambda - 1 = 0}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\boxed{\lambda = \frac{1 \pm \sqrt{5}}{2}}$$ Eigen values.

Ques: find eigen values and eigen vectors for the "$\pi/8$" gate, which has matrix representation

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

About T gate :- Single Qubit unitary Phase gate

The $\pi/4$ gate called T gate, is a unitary phase gate that adds a phase of $\pi/4$ to $|1\rangle$ and doesnot change $|0\rangle$.

i.e

$$T|0\rangle = |0\rangle$$

$$T|1\rangle = e^{i\pi/4}|1\rangle$$

Why $\pi/8$ (As phase is added by $\pi/4$)

$\pi/4$ rotation corresponds to $\pi/8$ on Bloch sphere hence $\pi/8$ gate.

T is unitary $\because TT^{+} = I$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \quad T^{+} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

$$TT^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{+i\pi/4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} e^{-i\pi/4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigen values & Eigen vector of T gate

step 1 $\quad \det |T - \lambda I| = 0$

$$\left| \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-\lambda & 0 \\ 0 & e^{i\pi/4} - \lambda \end{pmatrix} \right| = 0$$

$b e^{i\pi/4} - b = 0$

$b(e^{i\pi/4} - 1) = 0$

$(1-\lambda)(e^{i\pi/4} - \lambda) = 0$

$\boxed{\lambda = 1}$ $\boxed{\lambda = e^{i\pi/4}}$

we find now eigen vectors corresponding to $\boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = e^{i\pi/4}}$

let call $|\phi_1\rangle$ & $|\phi_2\rangle$

$T|\phi_1\rangle = \lambda|\phi_1\rangle \longrightarrow$ eigen vector eqn.

let $|\phi_1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$T|\phi_1\rangle = \lambda|\phi_1\rangle$ for $\boxed{\lambda = 1}$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ b e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$a = a$

$b e^{i\pi/4} = b$

$\cos \pi/4 + i \sin \pi/4 = 1$

$e^{i\theta}$ Euler form

$\cos\theta + i\sin\theta$

polar form

$\boxed{L \cdot H \cdot S}$

$\cos 45^\circ + i \sin 45^\circ$

Since $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \neq 1$

$\therefore \boxed{b = 0}$

$\therefore$ Eigen vector is $\begin{pmatrix} a \\ 0 \end{pmatrix}$

To find a, apply normalisation

$|a|^2 + |b^2| = 1$

as $b = 0$

$\therefore |a|^2 = 1$

$\Rightarrow \boxed{a = 1}$

$\therefore T|\phi_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = e^{i\pi/4}$ the eigen vector eqn is

$T|\phi_2\rangle = e^{i\pi/4}|\phi_2\rangle$

$|\phi_2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

Eigen Value Equation

$$\boxed{T|\psi\rangle = \lambda|\psi\rangle}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = e^{i\pi/4}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ be^{i\pi/4} \end{pmatrix} = e^{i\pi/4}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow a = e^{i\pi/4} a$$

$$be^{i\pi/4} = e^{i\pi/4} b \Rightarrow b = b$$

Same like eigen vector 1

we get $a=0$ $b=1$

$$|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

First eigen vector = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Second E.V = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Verify

Now $T(|\phi_1\rangle) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\phi_1\rangle$$

$$T|\phi_2\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\pi/4} \end{pmatrix} = e^{i\pi/4}|\phi_2\rangle$$

Practice Question:-

Ques: Find the Eigenvalues & Eigen vectors of X operator.

Start with Matrix rep of X

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det |X - \lambda I| = 0$$

find λ

use $X|\psi\rangle = \lambda|\psi\rangle$

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$