SWAP GATE → Two Qubit Gate

Action on computational basis

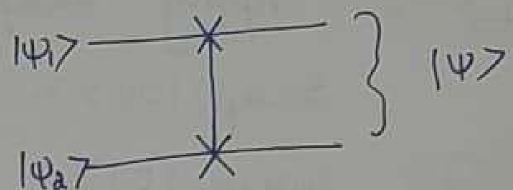
$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |10\rangle$

$|10\rangle \rightarrow |01\rangle$

$|11\rangle \rightarrow |11\rangle$

Symbol



Let $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} \otimes \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1\alpha_2\\ \alpha_1\beta_2\\ \beta_1\alpha_2\\ \beta_1\beta_2\end{pmatrix}$$

Action on arbitrary state

$$|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

$$\text{SWAP}|\psi\rangle = \alpha_0|00\rangle + \alpha_1|10\rangle + \alpha_2|01\rangle + \alpha_3|11\rangle$$

$$\text{Swap}|\psi\rangle = \begin{pmatrix}\alpha_0\\ \alpha_2\\ \alpha_1\\ \alpha_3\end{pmatrix}$$

Matrix Dirivation of SWAP GATE

<u>Step1</u>: fix the computational basis order:

$$|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle$$

<u>Step2</u> : Apply swap gate

Swap $|00\rangle = |00\rangle$ (1st col^n^)

swap $|01\rangle = |10\rangle$ (2nd col^n^)

swap $|10\rangle = |01\rangle$ (3rd col^n^)

swap $|11\rangle = |11\rangle$ (4th col^n^)

Now use results →

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Controlled - Z Gate   Two Qubit gate

Action on computational basis

$|00\rangle \to |00\rangle$

$|01\rangle \to |01\rangle$

$|10\rangle \Rightarrow |10\rangle$

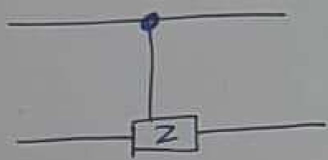
$|11\rangle \to -|11\rangle$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1|$

CZ gate is a two qubit gate that applies a Z (phase-flip) gate to the target qubit only when control qubit is $|1\rangle$

Gate Symbol



$|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$

$CZ|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle - \alpha_3|11\rangle$

Matrix derivation

Step1: fix computational basis order
$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Step2: Apply CZ to each basis vector

$CZ|00\rangle \to |00\rangle$

$CZ|01\rangle \to |01\rangle$

$CZ|10\rangle \to |10\rangle$

$CZ|11\rangle \to -|11\rangle$

Step3: write o/p vectors as columns

$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $-|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

Step4: construct the matrix column wise

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix}$$

Truth table

Input	Output
00	00
01	01
10	10
11	-11

ontrolled Phase gate CP(ϕ)

Action on computational basis

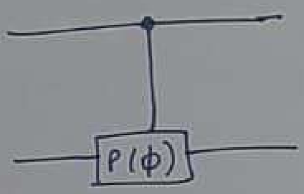
$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |10\rangle$

$|11\rangle \rightarrow e^{i\phi}|11\rangle$

Gate symbol



Truth table

Input	Output
00	00
01	01
10	10
11	$e^{i\phi}$ 11

Action on the Arbitrary state

$$|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

$$CP(\phi)|\psi\rangle = \alpha_0|00\rangle + \alpha_2|01\rangle + \alpha_3|10\rangle + \alpha_3 e^{i\phi}|11\rangle$$

Matrix derivation

Step1: Find the computational basis Order:

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Step2: Apply CP(ϕ) to each basis Vector

$CP(\phi)|00\rangle = |00\rangle$ $\quad CP(\phi)|01\rangle = |01\rangle$

$CP(\phi)|10\rangle = |10\rangle$ $\quad CP(\phi)|11\rangle = e^{i\phi}|11\rangle$

Step 3: write O/P vector as columns

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad e^{i\phi}|11\rangle = \begin{pmatrix}0\\0\\0\\e^{i\phi}\end{pmatrix}$$

Step 4: Construct the Matrix column-wise:

$$CP(\phi) = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & e^{i\phi}\end{pmatrix} = \begin{pmatrix}I & 0\\0 & e^{i\phi}\end{pmatrix}$$

Generic Controlled-U Gate

Action on Computational Basis

$|00\rangle \rightarrow |00\rangle$

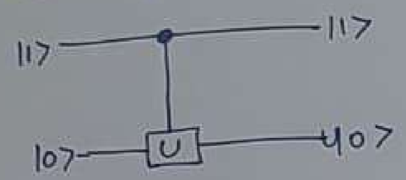
$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |1\rangle U|0\rangle$

$|11\rangle \rightarrow |1\rangle U|1\rangle$

> CNOT and CZ are special controlled U-gates

Gate symbol



Action on the Arbitrary State

$|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$

$CU|\psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2 |1\rangle U|0\rangle + \alpha_3 |1\rangle U|1\rangle$

Matrix Derivation

Step1: Fix the computational basis order:

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

Step2: Apply CU to each basis vector:

$CU|00\rangle = |00\rangle$

$CU|01\rangle = |01\rangle$

$CU|10\rangle = |1\rangle U|0\rangle$

$CU|11\rangle = |1\rangle U|1\rangle$

Step3: write O/P vectors are columns

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{pmatrix}$$

$$U = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}$$

Step4: contruct the Matrix

$$\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}$$

3 Qubit Gates

1) Toffali Gate (CCNOT)

2) FREDKIN (CSWAP)

$\boxed{\text{TOFFOLI GATE (CCNOT)}}$ → 3 Qubit Gate

Action on computational basis

$|abc\rangle \rightarrow |ab, c \oplus ab\rangle \qquad a, b, c \in \{0, 1\}$

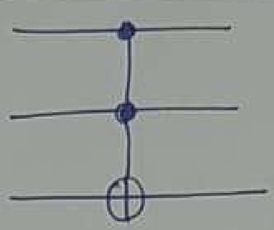
Truth Table

a	b	c	a	b	c
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

- 2 control qubits
- 1 target qubit
- Target flips if both control qubits are $|1\rangle$

$$|\psi\rangle = \sum_{i=0}^{7} \alpha_i |i\rangle$$

CIRCUIT SYMBOL



Matrix derivation

Step1: Fix computational basis set

$|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle$

Step2: apply CCNOT

$CCNOT\,|000\rangle \rightarrow |000\rangle$

$CCNOT\,|001\rangle \rightarrow |001\rangle$

$CCNOT\,|010\rangle \rightarrow |010\rangle$

$CCNOT\,|011\rangle \rightarrow |011\rangle$

$CCNOT\,|100\rangle \rightarrow |100\rangle$

$CCNOT\,|101\rangle \rightarrow |101\rangle$

$CCNOT\,|110\rangle \rightarrow |111\rangle$

$CCNOT\,|111\rangle \rightarrow |110\rangle$

Now $|000\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\|^{ly}$ find $|001\rangle, \ldots |110\rangle$

arrange as columns.

observe $CCNOT = \begin{pmatrix} I_6 & 0 \\ 0 & X \end{pmatrix}$

Controlled - Controlled - Z (CCZ gate)

3 Qubit gate

Action on computational basis

$$|abc\rangle \rightarrow (-1)^{abc}|abc\rangle \quad , a\,b\,c \in \{0,1\}$$

Truth table

a	b	c	a	b	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	0
1	1	1	1	1	-1

Gate symbol

Action on arbitrary state

$$|\psi\rangle = \sum_{i=0}^{7} \alpha_i |i\rangle$$

$$\sum_{i=0}^{6} \alpha_i |i\rangle - \alpha_7 |7\rangle$$

Matrix Derivation

Step 1. Computational basis

$|000\rangle, |001\rangle ---- |111\rangle$

Step 2: $|111\rangle \rightarrow -|111\rangle$

Others unchanged.

Step 3: Matrix

$\text{diag}\{1, 1, 1, 1, 1, 1, 1, -1\}$

Fredkin Gate (CSWAP)

$|a\ b\ c\rangle$

b, c, a can be 0 or 1

$|0\ bc\rangle \rightarrow |0\ bc\rangle$

$|1\ bc\rangle \rightarrow |1\ cb\rangle$

Swap b & c if a=1

Truth table

a	b	c	a	b	c
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1

Matrix Derivation

Step1: Basis order:

$$\{ |000\rangle, |001\rangle, \ldots |111\rangle \}$$

Step2:

$$|110\rangle \rightarrow |101\rangle$$

$$|101\rangle \rightarrow |110\rangle$$

Step3: Matrix

$$CSWAP = \begin{pmatrix} I_4 & 0 \\ 0 & SWAP \end{pmatrix}_{8\times 8}$$

Keyidea:- Target qubits are swapped only if control is 1.

Serial Quantum Gates

Quantum gates are said to be act in series when they are applied one after another on the same qubit (S).

Mathematical Description

If a quantum state $|\psi\rangle$ is acted upon by gates $U_1, U_2, \ldots U_n$ in sequence then

$$|\psi_{out}\rangle = U_n \ldots U_1 |\psi\rangle$$

Key properties

- Order of gates matter in general.
- Matrix multiplication is non- commutative.

Example:

Quantum circuit!-

$|0\rangle$ —[H]—[Z]—

$$\boxed{|\psi_{out}\rangle = Z H |0\rangle}$$

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{Z} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- swapping H & Z will give diff. result

check

Using $i = \cos\pi/2 + i\sin\pi/2 = e^{i\pi/2}$

$$STH|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i3\pi/4}|1\rangle\right)$$

PARALLEL QUANTUM GATES

Quantum gates act in parallel when they are applied simultaneously on different qubits in the same circuit layer.

Mathematical Description

Let $U_1, U_2, \ldots, U_n$ be single qubit gates acting on n distinct qubits. Their parallel action is given by

$$U_{parallel} = U_1 \otimes U_2 \otimes U_3 \cdots \otimes U_n$$

Action on n-qubit state

for an n qubit state $|\psi\rangle$

$$|\psi\rangle_{out} = (U_1 \otimes U_2 \cdots \otimes U_n)|\psi\rangle$$

Key Properties

- Parallel gates act in a single circuit depth
- Gates commute when acting on different qubits.
- System dimension scales as 2^n.

Example 2   H — T — H

$|0\rangle$ —[H]—[T]—[H]— $|\phi\rangle$

$|\phi\rangle = HTH|0\rangle$ Here $U = HTH$.

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$TH(|0\rangle) = \frac{1}{\sqrt{2}}(T|0\rangle + T|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$$

$$HTH|0\rangle = \frac{1}{\sqrt{2}}(H|0\rangle + e^{i\pi/4}H|1\rangle)$$

$$\frac{1}{\sqrt{2}}(|+\rangle + e^{i\pi/4}|-\rangle)$$

Example 3:   H — T — S

Quantum ckt: $|0\rangle$ —[H]—[T]—[S]—

$U = STH$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$TH|0\rangle = 1/\sqrt{2}\,(|0\rangle + e^{i\pi/4}|1\rangle)$$

$$STH|0\rangle = 1/\sqrt{2}\,(S|0\rangle + e^{i\pi/4}S|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + ie^{i\pi/4}|1\rangle)$$

(3) System dimension scales as 2^n.

If you have n qubits, the size of state space is 2^n.

Qubit	Possible states
1	2
2	2^2
3	2^3
$\vdots$	$\vdots$
n	2^n

As qubit grows, Information scales exponentially

Parallel Quantum gates : Example

Quantum circuit :-

$|0\rangle$ — $\boxed{H}$ —

$|0\rangle$ — $\boxed{X}$ —

$$|\psi\rangle = |0\rangle \otimes |0\rangle$$

$$H \otimes X|00\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$$

L.H.S $\quad H \otimes X|00\rangle = H|0\rangle \otimes X|0\rangle$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right) = R.H.S.$$

Property 1

Parallel gates act in a single circuit depth.

It Means:- If you apply gates to different qubits at the same time, it counts as one step in the circuit.

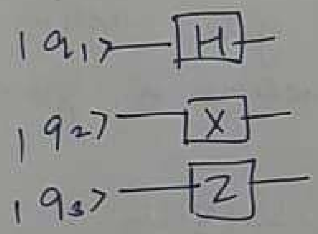
- Suppose you have 3 qubits

$|q_1 q_2 q_3\rangle$

- Apply H on $|q_1\rangle$

X on $|q_2\rangle$

Z on $|q_3\rangle$ at same time



- This is one depth (one layer)
- Many gates one depth.

Property 2

Gates commute when acting on different qubits.

It means:- Apply H on $|q_1\rangle$, X on $|q_2\rangle$

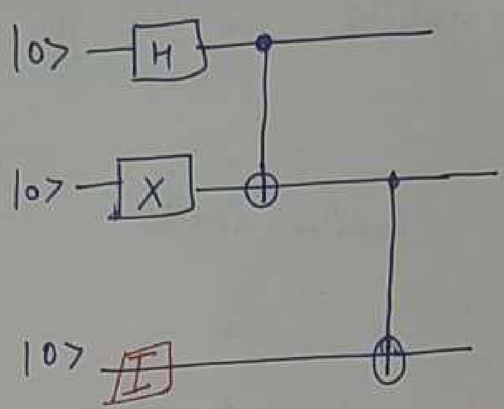
Case 1:- $(H \otimes X)$

Case 2:- $(X \otimes H)$

Both will give same result.

Each gate work on its qubit, they dont interfere.

Example:-



circuit:- 1st layer

$H \otimes X \otimes I$

Initial state $|\psi_0\rangle = |000\rangle$

$|\psi_1\rangle = H \otimes X \otimes I \, |000\rangle$

$= H|0\rangle \otimes X|0\rangle \circledast I|0\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle \otimes |0\rangle$

$\frac{1}{\sqrt{2}}\left(|010\rangle + |110\rangle\right)$

After 1st CNOT