classmate
Date
Page

Qubit :-

The basic unit of information in quantum computing is called qubit, short form of quantum bit.

Qubit can be in one of two states. These states are refered as
$|0\rangle$ or $|1\rangle$

where $|\,\rangle$ is called a state, ket or a vector.

⇒ Difference between bit & qubit?

Bit (Classical computer):-
- can be 0 or 1
- Like a switch :- On or off.

Qubit (Quantum computer):-
- can be 0 or 1 or both at same time. (called Superposition state)
- Like on and off together.

A classical bit is like a coin: either heads or tails up.
A quantum bit can exist in a continuum of states between $|0\rangle$ and $|1\rangle$ until it's observed.

Date
Page

~~Possible~~ ~~of~~

The superposition state of a qubit is written as :- $|\psi\rangle$ where

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \text{——— (1)}$$

Here $\alpha, \beta \in$ complex numbers.

i.e α, β are of the form $x + iy$

where $i = \sqrt{-1}$

A qubit can exist in a superposition of states $|0\rangle$ and $|1\rangle$ but when you measure it, it is only going to be found in $|0\rangle$ or $|1\rangle$.

The Laws of quantum mechanics tell us that the modulus squared of α, β in (1) gives the probability of finding a qubit in state $|0\rangle$ or $|1\rangle$.

i.e

$|\alpha|^2$ = Tells us the probability of finding $|\psi\rangle$ in state $|0\rangle$.

$|\beta|^2$ = Tells us the probability of finding $|\psi\rangle$ in state $|1\rangle$.

Date ______
Page ______

Now, if an event has N possible outcomes and we label the probabilities of finding result i by p_i, then the condition that the probabilities sum to one is written as

$$\sum_{i=1}^{N} p_i = p_1 + p_2 + p_3 + \cdots p_N = 1$$

$\therefore$ The probability of finding a qubit in state $|0\rangle$ or $|1\rangle$ sums to 1

i.e $|\alpha|^2 + |\beta|^2 = 1$

for a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

if $|\alpha|^2 + |\beta|^2 = 1$

$|\psi\rangle$ is said to be normalised.

$|\alpha|^2 = \alpha\alpha^*$

where α^* is complex conjugate of α.

$|\beta|^2 = \beta\beta^*$

where β^* is complex conjugate of β.

Let $\alpha = x + iy$ then

$\alpha^* = x - iy$

$$|\alpha|^2 = \alpha \alpha^*$$
$$= (x+iy)(x-iy)$$
$$x^2 - ixy + ixy - i^2 y^2$$
$$= x^2 - i^2 y^2$$
$$= x^2 + y^2 \qquad (i^2 = -1)$$

$$\therefore \quad |\alpha|^2 = x^2 + y^2$$
$$|\alpha| = \sqrt{x^2 + y^2}$$

Example:- A qubit can be in following Superposition State:-

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\alpha = \frac{1}{\sqrt{2}} \qquad \beta = \frac{1}{\sqrt{2}}$$

Probability of finding $|\psi\rangle$ in State $|0\rangle$

$$|\alpha|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Probability of finding $|\psi\rangle$ in State $|1\rangle$

$$|\beta^2| = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Means, $|\psi\rangle$ when measured gives the result 0 fifty percent of the time and result 1 fifty percent of the time.

This state is sometimes denoted as $|+\rangle$.

⇒ Realization of a qubit

In an atom model, an electron can exist in either in 'ground' or 'excited' state, which we'll call $|0\rangle$ or $|1\rangle$ respectively.

By shining light on an atom with appropriate energy and for appropriate length of time, it is possible to move the electron from state $|0\rangle$ to $|1\rangle$ state and vice versa.

More Interestingly, by reducing the time we shine the light, electron initially in the state $|0\rangle$ can be moved 'halfway' between $|0\rangle$ and $|1\rangle$, into $|+\rangle$ state.

Example: 2 A qubit can be in following superposition state:-

$$|\psi_1\rangle = \cos\theta\, |0\rangle + \sin\theta\, |1\rangle$$

is a valid qubit

as $\cos^2\theta + \sin^2\theta = 1$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$|\psi_2\rangle$ is a valid qubit

as $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

Representation of ~~a~~ qubit

1) Ket notation $|\ \rangle$

$|0\rangle$ $\qquad$ $|1\rangle$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

2) Bra notation :-

Bra is a hermitian conjugate of Ket notation.

Hermitian Conjugate is obtained by taking transpose of a matrix and complex conjugate of each element of a matrix.

Bra is represented by $\langle 0|$

Questions: For each of the following qubits, if a measurement is made, what is the probability that we find the qubit in state $|0\rangle$? What is the probability that we find the qubit in state $|1\rangle$?

(i) $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$

(ii) $|\psi\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

(iii) $|\chi\rangle = \frac{(1+i)}{\sqrt{3}}|0\rangle - \frac{i}{\sqrt{3}}|1\rangle$

Ans:- (i) $P(|\psi\rangle = |0\rangle) = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$

$P(|\psi\rangle = |1\rangle) = \frac{\sqrt{2}}{\sqrt{3}} = \frac{2}{3}$

verify. $\frac{1}{3} + \frac{2}{3} = 1$ hence valid.

(ii) $|\psi\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

in case of complex coeff (α, β), use complex conjugate while calculating modulus squared.

i.e $|\alpha|^2 = \left(\frac{i}{2}\right)\left(\frac{-i}{2}\right) = \frac{-i^2}{4}$

$= \frac{1}{4}$

$$|\beta|^2 = \frac{\sqrt{3}}{2} = \frac{3}{4}$$

$$\sum |\alpha|^2 + |\beta|^2 = \frac{1}{4} + \frac{3}{4} = 1.$$

(iii) $|\psi\rangle = \frac{(1+i)}{\sqrt{3}}|0\rangle - \frac{i}{\sqrt{3}}|1\rangle$

$$\alpha = \frac{1+i}{\sqrt{3}}, \quad \beta = \frac{-i}{\sqrt{3}}$$

$$|\alpha|^2 = \alpha\alpha^* = \frac{(1+i)(1-i)}{(\sqrt{3})(\sqrt{3})}$$

$$= \frac{1-i^2-i+i}{3} = \frac{2}{3}$$

$$|\beta|^2 = \beta\beta^* = \frac{-i}{\sqrt{3}} \frac{i}{\sqrt{3}} = \frac{-i^2}{3} = \frac{1}{3}$$

$$\sum |\alpha|^2 + |\beta|^2 = \frac{2}{3} + \frac{1}{3} = 1$$

Representation of a Qubit

A qubit is represented as a n dimensional column vector as follows

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In order to understand this representation of a qubit, lets revise vector space :-

Quantum computation takes place in an mathematical abstraction called a vector space.

Defn. → A vector space 'V' is a non empty set of vectors v, u called vectors for which following operations are defined:-

i) Vector Addition:- If $u, v \in V$

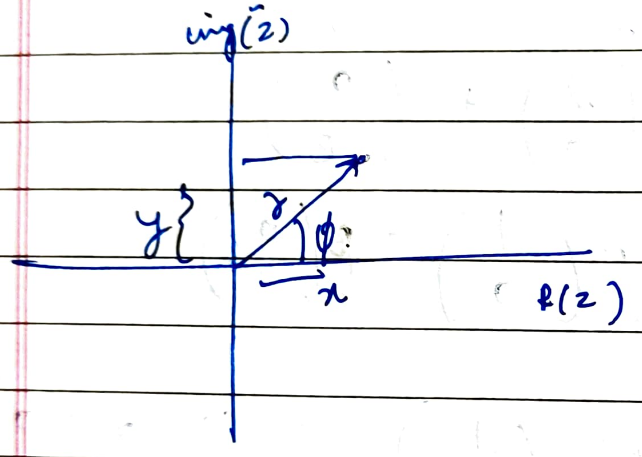
$w = u + v \in V$

BLOCK SPHERE REPRESENTATION OF A QUBIT

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \alpha, \beta \in C$

Basics of complex numbers

$Z = Re(Z) + i\, img(z)$



$\sin\phi = \frac{y}{r}$

$y = r\sin\phi$

$\cos\phi = \frac{x}{r}$

$x = r\cos\phi$

$$Z = r(\underbrace{\cos\phi + i\sin\phi}) \quad \text{Euler's identity}$$

$$z = re^{i\phi}$$

Polar rep of a complex number.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\Rightarrow \quad r_\alpha e^{i\phi_\alpha}|0\rangle + re^{i\phi_\beta}|1\rangle$

S M M B

Vector space Continued....

(ii) Scalar multiplication :- Defines multiplication of a vector by a number α such that

$$\alpha u \in V.$$

(iii) Associativity of addition :- Given $u, v, w \in V$

$$(u+v)+w = u+(v+w)$$

(iv) Zero vector :- $u+0 = 0+u = u$

For every vector $u \in V$, $\exists$ additive inverse of u s.t

$$u+(-u) = (-u)+u = 0$$

(v) Addition of vectors is commutative

So, a qubit is defined as element of n touple column vector $\mathbb{C}^n$ with elements $|a\rangle, |b\rangle, |c\rangle$ ---

where $|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$|b\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. so on so $|\psi\rangle$ is an element of

C^2 vector space.

Recall

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$\therefore$ $|\psi\rangle$ can be represented as a 2 touple column vector

as $$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Using properties of Vector space:-

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

here $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ forms as

<u>Basis Set</u>

Date
Page

Basis Set:- when a set of vectors is linearly independent (means they cannot be made from others) and they span the space (i.e these can be used to construct every vector in the space), the set is known as a basis set.

BLOCH SPHERE NOTATION.

The Block sphere is a useful tool for visualising the state of a qubit.
It is a sphere where any point on or inside represents a possible state of a qubit.

It is a geometrical representation of a qubit.
Now,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Rewrite is as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

- θ & ϕ defines a point on the sphere.
- A unit sphere in 3D
- A qubit is represented by an arrow from the centre to the surface.
- The arrow is called bloch vector.

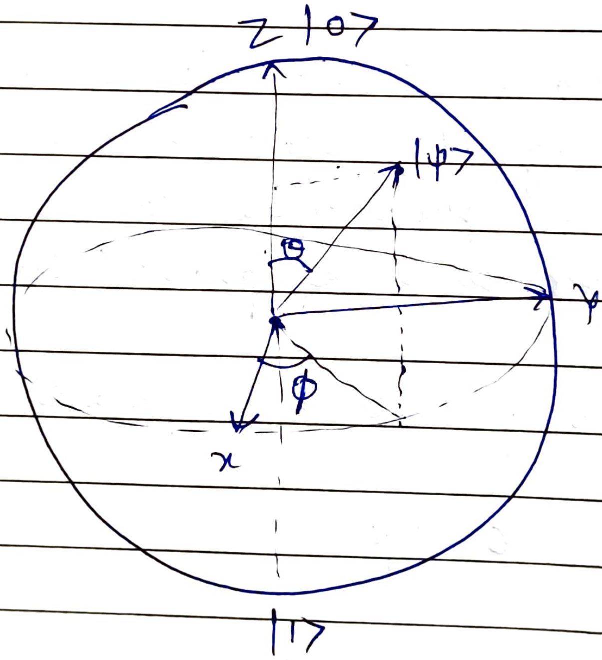
→

- The north pole represents $|0\rangle$.
- The south pole represents $|1\rangle$

Special superposition states:-

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$ (Equator pt)

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



Q why is Bloch sphere useful?

→ Single-qubit gates becomes easy to understand.

ex:- X-gate:- flip flops top ↔ bottom

classmate

Date
Page

Q: How much information is represented by a qubit.

Bloch sphere has infinitely many points, measurement behaves differently.

- Measuring a qubit gives 0 or 1.

from measurement we get only 1 bit information.

You cannot read full state information from one qubit.

You need infinitely many identical qubits.

Information is hidden, Information is there.

with multiple qubits, this hidden information grows exponentially.

Date
Page

Multiple qubits

Suppose we have two qubits, if these were two classical bits, then there would be four possible states. (00, 01, 10, 11)

So a 2 qubit system has 4 computational basis states

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

Vector describing the two qubit is

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

where $$\sum_{x \in \{0,1\}^2} |\alpha_x|^2 = 1$$

where $\{0,1\}^2$ = means set of strings of length 2. where each letter being 0 or 1.

Qutrit (Quantum trit)

A qutrit is a 3 level quantum state

- States $|0\rangle \quad |1\rangle \quad |2\rangle$

~~Here~~

- General State

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$$

- Normalization :-

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$$

Born's Rule

Born's rule connects quantum mechanics to real measurement, without it, quantum mechanics cannot make predictions.
It explains why quantum results are probabilistic.

Statement:- If a qubit is in the state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Then.

Probability of measuring $0 = |\alpha|^2$

" " " $1 = |\beta|^2$

with $|\alpha|^2 + |\beta|^2 = 1$

Qudit

- A qudit is a generalisation to d levels.
- Basis states:-

$$|0\rangle, |1\rangle, \text{-----}, |d-1\rangle$$

- General state :-

$$|\psi\rangle = \sum_{K=0}^{d-1} \alpha_K |K\rangle$$

- Normalization:-

$$\sum_{K=0}^{d-1} |\alpha_K|^2 = 1$$

$\Rightarrow$ Qubit = Qudit with $d=2$
Qudrit = qudit with $d=3$

Measurement using Born's Rule

$$|\psi\rangle = \frac{1}{\sqrt{6}}|0\rangle + \frac{2}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{6}}|2\rangle$$

with $P(0) = (1/\sqrt{6})^2 = 1/6$

$P(1) = (2/\sqrt{6})^2 = 4/6$

$P(2) = (1/\sqrt{6})^2 = 1/6$

Clearly, $\sum_{i=1}^{3} P_i = 1$

- The state collapses to measured basis state.

Geometry

1) No bloch sphere
$\because$ Bloch sphere represents a qubit. and for qudit the state space is higher. So NO 3D Visualisation exists.

Qudits & Qudrits are important $\because$ these can encode more information per particle.

Bra - Ket formalisation

In Quantum $\langle a|$ is called dual vector or bra corresponding to $|u\rangle$.

Ket is a column vector, dual vector is a row vector whose elements are complex conjugates of elements of column vector.

Let $|\phi\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

~~gis~~ here $|\phi\rangle$ is a 2-d vector as

$$|\phi\rangle = \begin{pmatrix} i/2 \\ \sqrt{3}/2 \end{pmatrix}$$

Dual vector is found by computing

$$\langle \phi| = \begin{pmatrix} \frac{-i}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Ques! - Two vectors in C^3 are given by

$$|a\rangle = \begin{pmatrix} -2 \\ 4i \\ 1 \end{pmatrix} \qquad |b\rangle = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

Find $\langle a|$ and $\langle b|$

Date
Page

$$|a\rangle = \begin{pmatrix} -2 \\ 4i \\ 1 \end{pmatrix}$$

$$\langle a| = \begin{pmatrix} -2 & -4i & 1 \end{pmatrix}$$

$$\langle b| = \begin{pmatrix} 1 & 0 & -i \end{pmatrix}$$

Inner Product of vectors

Inner Product between two vectors $\in C^n$ is a complex number.

Let $|u\rangle$ & $|v\rangle \in C^n$

then Inner product of $|u\rangle$ and $|v\rangle$ is denoted as

$$\langle u|v\rangle$$

Let $|u\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$$\langle u| = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_n^* \end{pmatrix}$$

$$|v\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

then Inner Product

$$\langle u|v\rangle = (a_1^* \quad a_2^* \; \cdots \; a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1^* b_1 + a_2^* b_2 + \cdots a_n^* b_n$$

$$= \sum_{i=1}^{n} a_i^* b_i$$

Ques!- $|a\rangle = \begin{pmatrix} -2 \\ 4i \\ 1 \end{pmatrix}$ $\quad |b\rangle = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

find $\langle a|b\rangle$.

$$(-2 \quad -4i \quad 1) \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$(-2 \times 1) + (-4i \times 0) + (1 \times i)$$

$$= -2 + 0 + i = -2 + i$$

If $\forall \; u, v \in C^n$

$\langle u|v\rangle = 0$

u & v are orthogonal to one another.

Norm of a vector

Inner product = $\langle u|u\rangle$

Norm $\|u\| = \sqrt{\langle u|u\rangle}$

If $\|u\| = 1$

i.e if norm of a vector is normalised we say that vector is normalised.

i.e $\langle a|a\rangle = 1$

then $|a\rangle$ is normalised.

If a vector is not normalised, then we can normalise it as follows:-

1) Compute the norm of vector
2) Divide the vector by its norm

i.e $\boxed{|\tilde{u}\rangle = \dfrac{|u\rangle}{\|u\|}}$

$\frac{i}{\sqrt{5}} \times \frac{-i}{\sqrt{5}} = \frac{-i^2}{5} = \frac{1}{5}$

CLASSMATE

Date
Page

Practice

Question:- A quantum state $|\phi_1\rangle$ is given as $|\phi_1\rangle = \frac{1}{\sqrt{5}}(i|0\rangle + 2|1\rangle)$

1) Explore that the state is normalised or not.

2) Indicate the probability of finding $|0\rangle$ and $|1\rangle$ when a measurement is made on $|\phi_1\rangle$

$$\left|\frac{i}{\sqrt{5}}\right|^2 + \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{+1}{5} + \frac{4}{5} = \frac{5}{5}$$

$$= 1$$

Normalised.

$$|i|^2 = (i)(i)^* = (i)(-i)$$
$$= -i^2$$
$$= -(-1)$$
$$= 1$$

Question Consider two - qubit quantum state :-

$$|\phi\rangle = \sqrt{\frac{1}{11}}|00\rangle + \sqrt{\frac{5}{11}}|01\rangle + \sqrt{\frac{3}{11}}|10\rangle + \frac{2}{\sqrt{11}}|11\rangle$$

(1) Explore that the state is normalised or not.

(2) Examine the probabilities of outcome 0 and 1. if second qubit of $|\phi\rangle$ is measured.

Ans:- (1) For normalization:
sum of sq. of all coeff = 1

$$\left(\frac{1}{\sqrt{11}}\right)^2 + \left(\sqrt{\frac{5}{11}}\right)^2 + \left(\frac{3}{\sqrt{11}}\right)^2 + \left(\frac{2}{\sqrt{11}}\right)^2$$

$$\frac{1+5+3+2}{11} = \frac{11}{11} = 1$$

The state of ϕ is normalised.

(2) Probability of second qubit = 0

$|00\rangle = |1/\sqrt{11}|^2 = 1/11$

$|10\rangle = \left(\sqrt{\frac{3}{11}}\right)^2 = 3/11$

Total Prob = $\frac{1+3}{11} = \frac{4}{11}$

Probability that the qubit is in state 1 second qubit is 1

$|01\rangle$, $\langle 11\rangle$

$$P(1) = \frac{5}{11} + \frac{2}{11} = \frac{7}{11}$$

Ques: A quantum is in state

$$\frac{(1-i)}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

If a measurement is made, what is the probability that the system is in state $|0\rangle$ or in state $|1\rangle$?

Ans: $$P(|0\rangle) = \left|\frac{1-i}{\sqrt{3}}\right|^2$$

$|\alpha|^2 = \alpha\alpha^*$

$$= \frac{(1-i)(1+i)}{\sqrt{3} * \sqrt{3}}$$

$$= \frac{1 - i + i - i^2}{3} = \frac{1+1}{3}$$

$$= \frac{2}{3}$$

$$P[|1\rangle) = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$P(|0\rangle) + P(|1\rangle) = 1$$

which is true

classmate

Date

Page

Ques: Two quantum states are

$$|a\rangle = \begin{pmatrix} -4i \\ 2 \end{pmatrix}, \quad |b\rangle = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

1) find $\langle a|$
2) find $\langle b|$
3) find $\langle a|b\rangle$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$