Expected Value Questions

1. A die is rolled until six appears.
What is the expected number of rolls.

Sol^n^ 1: Let x be the number of rolls until 6 appears.

$$E(x) = \sum P(x_i)\, x_i$$

$$= \frac{1}{6}\times 1 + \frac{1}{6}\left(\frac{5}{6}\right)\times 2 + \frac{1}{6}\left(\frac{5}{6}\right)^2\times 3 + \cdots$$

$$= \frac{1}{6}\left[\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1} k\right]$$

We know that (neat little formula):

$$1 + \sum_{K=1}^{\infty} x^K = \frac{1}{1-x}$$

Differentiating both sides

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$$\Rightarrow \left(1 + \sum_{k=1}^{\infty} x^k\right)' = \left(\frac{1}{1-x}\right)'$$

$$\Rightarrow \sum_{k=1}^{\infty} k\, x^{k-1} = \frac{1}{(1-x)^2}$$

Multiplying $(1-x)$ on both side

$$\Rightarrow (1-x) \sum_{k=1}^{\infty} k\, x^{k-1} = \frac{1}{1-x}$$

Replacing $x = \frac{5}{6}$

$$\Rightarrow \frac{1}{6} \sum_{k=1}^{\infty} k \left(\frac{5}{6}\right)^{k-1} = 6$$

$\therefore E(x) = 6$ Ans

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(*) In General [may be symmetric cases]

If $P(x) = p$ [Probability of happening is 'p']

then,

$E[\text{number of tries before } x] = \frac{1}{p}$

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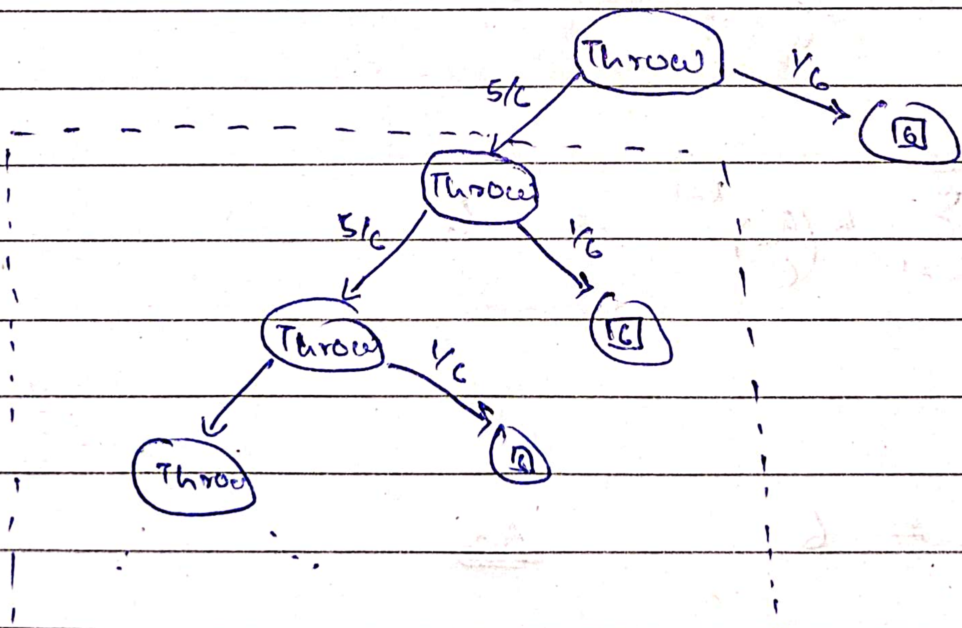
Expected 'x'

<u>Solution 2</u> : (clever Recursion)



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Left subtree == ~~tree~~ Parent Tree
↓
full

$$\therefore E[T] = \underbrace{1}_{\text{1 throw}} + \underbrace{\frac{1}{6} \times 0}_{} + \underbrace{\frac{5}{6}\left(E[T]\right)}_{}$$

After 1 throw going into right has $\frac{1}{6}$ probability and will take 0 throws to get 6

After 1 throw going into left has 5/6 prob. and will take E(x) to get 6.

$$\therefore \boxed{E[T] = 6} \quad \text{Ans.}$$

Solution 3: [law of large Numbers]

It states that as you repeat a random experiment a large number of times the outcome will converge to the theoritical expected value.

eg. If we throw a coin large no. of times the heads and tails will be $\approx$ 50%.

let we play 100 games ; then in 100 games
no. of 6's will be 100.

$$100 = \text{no. of } 6\text{'s}$$

$$\Rightarrow 100 \approx \frac{1}{6} \times \underset{\downarrow}{\text{no. of throws}}$$

$$\Rightarrow 100 \approx \frac{1}{6} \times \underbrace{100}_{\substack{\downarrow \\ 100 \\ \text{games}}} \times \underbrace{E[T]}_{\substack{\downarrow \\ \text{expected length} \\ \text{in each game}}}$$

$$\Rightarrow E[T] \approx 6$$

As no. of games increase $\approx \infty$

$$\Rightarrow E[T] = 6 \quad \text{Ans.}$$