Related Distributions

- Probability Distributions are defined by the Probability Density Functions
- Probability Density Function
    - It is the probability that the variate has the value x
    - For a continuous distribution, the probability at a single point is zero
        - The probability is expressed in terms of the integral of the Probability Distribution Function between two points, a and b

          ⇒ $\int_{a}^{b} f(x)dx = Pr[a \leq x \leq b]$
    - For a discrete distribution, the probability is given by the Probability Distribution Function

      ⇒ $f(x) = Pr[a \leq x \leq b]$
- Cumulative Distribution Function
    - It is the probability that a variate takes a value less than or equal to x

      ⇒ $F(x) = Pr[X \leq x] = \alpha$
    - For a continuous distribution, it is given by

      ⇒ $F(x) = \int_{-\infty}^{x} f(t)dt$
    - For a discrete distribution, it is given by

      ⇒ $F(x) = \sum_{i=0}^{x} f(i)$
- Percent Point Function
    - It is the inverse of the Cumulative Distribution Function and it is also referred to as the Inverse Distribution Function
    - For this distribution function, calculate the probability that the variable is less than or equal to x for a given x

- The inverse of the function can be given by

  ⇒ $F^{-1}(x) = G(x)$

  ⇒ $F^{-1}(F(x)) = G(F(x)) = x = G(\alpha)$ as $F(x)$ which is equal to α

  ⇒ $Pr[X \leq G(\alpha)] = \alpha$

- Survival Function
  - It is the probability that the variate takes a value greater than x
  - It is given by

    ⇒ $S(x) = Pr[X > x] = 1 - F(x)$
  - The Survival Function should be compared to the Cumulative Distribution Function
- Inverse Survival Function
  - The Inverse Survival Function can be defined in terms of the percent point function

    ⇒ $S^{-1}(x) = Z(x)$

    ⇒ $S^{-1}(S(x)) = Z(S(x))$ by replacing x with $S(x)$

    ⇒ $x = Z(1 - F(x))$ as $S(x) = 1 - F(x)$

    ⇒ $x = Z(1 - \alpha) = G(\alpha)$ as $F(x) = \alpha$ and $x = G(\alpha)$

    ⇒ $Z(\alpha) = G(1 - \alpha)$ by replacing α with $1 - \alpha$
- Hazard Function
  - It is the ratio of the Probability Distribution Function to the Survival Function
  - It is given by

    ⇒ $h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}$
  - These are used in reliability measures and are sometimes referred to as the Conditional Failure Density Function
- Cumulative Hazard Function
  - It is the integral of the Hazard Function

- It is given by

  $\Rightarrow \mathrm{H(x)} = \int_{-\infty}^{x} \mathrm{h}(\mu)\mathrm{d}\mu$ where $\mathrm{h}(\mu) = \frac{\mathrm{f}(\mu)}{1 - \mathrm{F}(\mu)}$

  $\Rightarrow \mathrm{F}(\mu) = \int_{-\infty}^{\mu} \mathrm{f(t)dt}$ as $\mathrm{F(t)} = \int \mathrm{f(t)dt}$ so $\mathrm{F}'(\mathrm{t}) = \mathrm{f(t)}$

  $\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\mu} \int_{-\infty}^{\mu} \mathrm{f(t)dt} = \frac{\mathrm{d}(\mu)}{\mathrm{d}\mu}\mathrm{F}'(\mu) - \frac{\mathrm{d}(-\infty)}{\mathrm{d}\mu}\mathrm{F}'(-\infty) = \mathrm{F}'(\mu) = \mathrm{f}(\mu)$ as $\mathrm{F}'(-\infty) = 0$

  $\Rightarrow \mathrm{H(x)} = \int_{-\infty}^{x} \mathrm{h}(\mu)\mathrm{d}\mu = \int_{-\infty}^{x} \frac{\mathrm{f}(\mu)}{1 - \mathrm{F}(\mu)}\mathrm{d}\mu$ and $\mathrm{F}(\mu) = \mathrm{y}$

  $\Rightarrow \mathrm{F}'(\mu)\mathrm{d}\mu = \mathrm{dy} = \mathrm{f}(\mu)\mathrm{d}\mu$ so replacing this in $\mathrm{H(x)} = \int_{-\infty}^{x} \frac{\mathrm{f}(\mu)}{1 - \mathrm{F}(\mu)}\mathrm{d}\mu$

  $\Rightarrow \mathrm{H(x)} = \int_{-\infty}^{x} \frac{\mathrm{dy}}{1 - \mathrm{y}} = -\ln(1 - \mathrm{y})\Big|_{\mu=-\infty}^{\mu=x} = -\ln(1 - \mathrm{F}(\mu))\Big|_{\mu=-\infty}^{\mu=x}$

  $\Rightarrow -\ln(1 - \mathrm{F(x)}) - (-\ln(1 - \mathrm{F}(\mu))) = -\ln(1 - \mathrm{F(x)}) = \mathrm{H(x)}$

- These are used in reliability measures