Strange Attractors & Dynamical Systems
From Van der Pol Oscillators to Deterministic Chaos

Dynamical Systems Exploration

Mathematical Physics

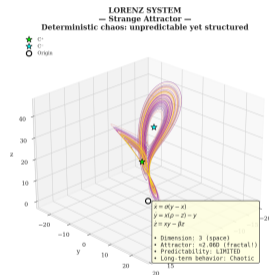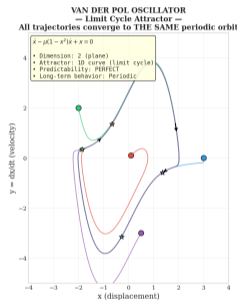December 8, 2025

Outline

The Nature of Dynamical Systems

Key Question:

What happens to systems over long time periods?

Attractors are the geometric structures toward which dynamical systems evolve.

- Fixed points (stable equilibria)
- Limit cycles (periodic orbits)
- Strange attractors (chaotic behavior)

From Order to Chaos

The Lyapunov Exponent

Measures the rate of separation of infinitesimally close trajectories:

$$|\delta\mathbf{x}(t)| \sim |\delta\mathbf{x}_0|e^{\lambda t}$$

- $\lambda < 0$: Stable (trajectories converge)
- $\lambda = 0$: Marginally stable
- $\lambda > 0$: Chaotic (trajectories diverge)

Deterministic Chaos

Simple equations can produce unpredictable behavior!

The Van der Pol Equation

Governing Equation (1927)

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

First-order system form:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1-x^2)y - x \end{cases}$$

- $\mu > 0$: Nonlinear damping parameter
- Discovered in vacuum tube circuits
- First example of a **limit cycle**

Key Feature

Amplitude-dependent damping creates self-sustained oscillations
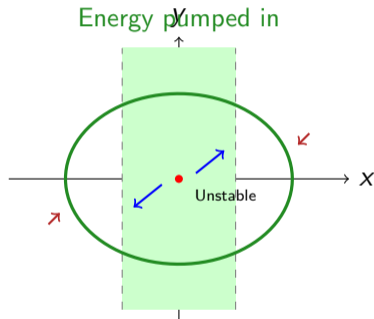
Energy Pumping Mechanism

Effective damping coefficient:

$$\gamma_{\text{eff}}(x) = -\mu(1 - x^2)$$

Condition	Damping	Effect
$|x| < 1$	Negative	Energy IN
$|x| = 1$	Zero	Balance
$|x| > 1$	Positive	Energy OUT

Result: Self-regulating limit cycle

Energy pumped in

Unstable

x

Phase Plane Analysis: Nullclines

Nullclines are curves where $\dot{x} = 0$ or $\dot{y} = 0$:

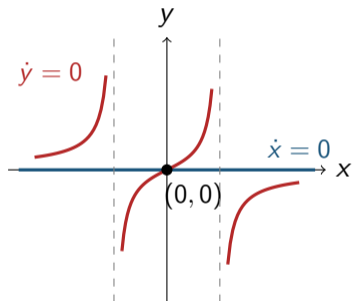x-**nullcline** ($\dot{x} = 0$):

$$y = 0 \quad \text{(the } x\text{-axis)}$$

y-**nullcline** ($\dot{y} = 0$):

$$y = \frac{x}{\mu(1 - x^2)}$$

Equilibria occur at nullcline intersections.

Only one equilibrium: $(0, 0)$

Equilibrium Analysis: The Jacobian Matrix

General Jacobian

For $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$:

$$J(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\mu xy - 1 & \mu(1 - x^2) \end{pmatrix}$$

At the origin $(x, y) = (0, 0)$:

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

- Trace: $\text{tr}(J) = \mu > 0$ (sum of eigenvalues)
- Determinant: $\det(J) = 1 > 0$ (product of eigenvalues)

Eigenvalue Analysis: Stability Classification

Characteristic equation: $\lambda^2 - \mu\lambda + 1 = 0$

Eigenvalues

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

Discriminant: $\Delta = \mu^2 - 4$

μ range	Eigenvalues	Type	Stability
$\mu = 0$	$\pm i$	Center	Neutral
$0 < \mu < 2$	Complex, Re > 0	Unstable spiral	Unstable
$\mu = 2$	$\lambda = 1$ (repeated)	Unstable node	Unstable
$\mu > 2$	Real, both > 0	Unstable node	Unstable

Conclusion

For **any** $\mu > 0$: origin is unstable — trajectories escape!

The Poincaré-Bendixson Theorem

Question: If the origin is unstable, where do trajectories go?

Poincaré-Bendixson Theorem

If a trajectory in $\mathbb{R}^2$ enters a closed, bounded region R containing **no equilibria** and never leaves, then it must approach a **periodic orbit**.

Application to Van der Pol:

1. Origin is unstable $\Rightarrow$ trajectories spiral *outward*
2. Energy dissipation when $|x| > 1 \Rightarrow$ trajectories are *bounded*
3. Trajectories enter an annular region with no equilibria
4. $\therefore$ A **limit cycle** must exist!

Key Insight

The limit cycle is the *unique* stable attractor for $\mu > 0$

Relaxation Oscillations: Large μ

When $\mu \gg 1$, the system exhibits **relaxation oscillations**:

1. **Slow phase:** Trajectory creeps along the cubic nullcline
2. **Fast phase:** Rapid jump between branches

Period scaling:

$$T \approx (3 - 2\ln 2)\mu \approx 1.614\mu$$

Biological examples:

- Heartbeat (SA node)
- Neural action potentials
- Predator-prey cycles
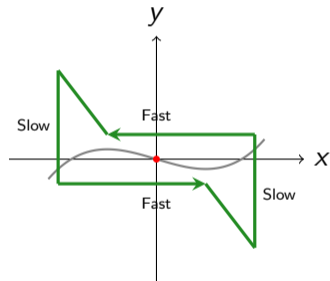
Relaxation oscillation for $\mu \gg 1$

Heartbeat Modeling with Van der Pol

The Van der Pol oscillator models **cardiac rhythms**:

- Sinoatrial node acts as a *relaxation oscillator*
- Self-sustaining periodic signal
- Robust against perturbations

FitzHugh-Nagumo Model

$$\begin{cases} \dot{v} = v - \frac{v^3}{3} - w + I_{\text{ext}} \\ \dot{w} = \epsilon(v + a - bw) \end{cases}$$

A simplified Hodgkin-Huxley model

Clinical connections:

Normal	Stable limit cycle (60-100 bpm)
Arrhythmia	Bifurcation from healthy cycle
Pacemaker	Artificial periodic forcing

Unstable Stable Limit Cycle
Large μ
Relaxation oscillation
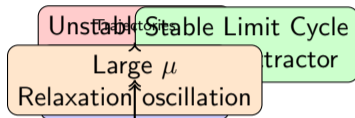
Key Results

- **Unique equilibrium** at origin — always unstable for $\mu > 0$
- **Limit cycle** guaranteed by Poincaré-Bendixson theorem
- **Applications:** Electronics, cardiology, neuroscience

Lorenz Equations

The System (1963)

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases}$$

Standard parameters:

- $\sigma = 10$ (Prandtl number)
- $\rho = 28$ (Rayleigh number)
- $\beta = \frac{8}{3}$ (geometric factor)

Originally derived from atmospheric convection—now the icon of chaos theory.

THE LORENZ STRANGE ATTRACTOR
σ = 10, ρ = 28, β = 8/3

Classic View
(The Butterfly Wings)

★ C⁺
★ C⁻
○ Origin
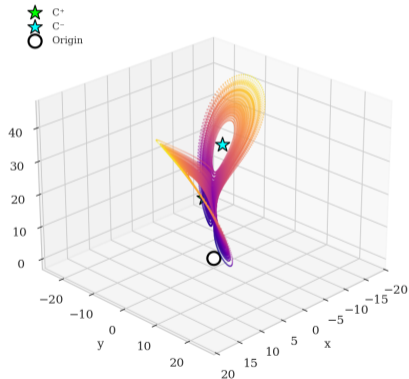
Side View (x-z plane)
(Showing vertical structure)

Top View
(z-axis pointing up)
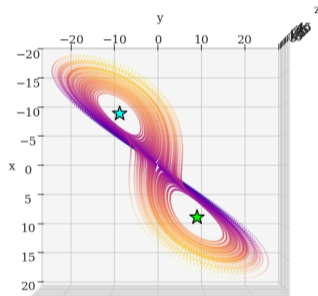
Rear View (y-z plane)
(Line plot showing flow)

The Lorenz Attractor
(Two trajectories, Δx₀ = 0.0001)
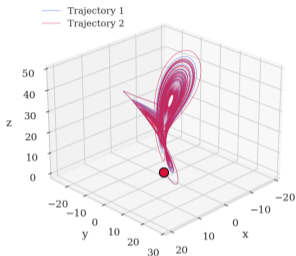
Exponential Divergence
(The Butterfly Effect)
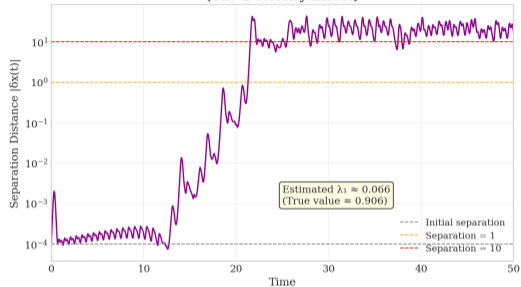
X-Coordinate vs Time
(Identical at first, then diverge)
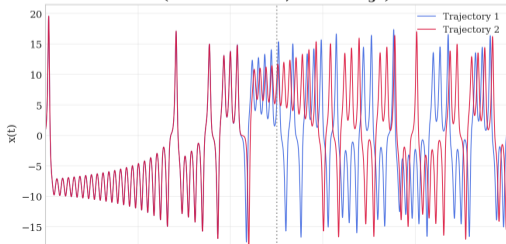
Difference in X-Coordinate
(Unpredictable after t ≈ 25)
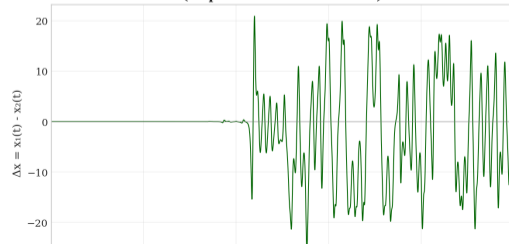
Fractal Structure of Strange Attractors

Properties:

- Non-integer (fractal) dimension
- Lorenz attractor: $D \approx 2.06$
- Self-similar at all scales
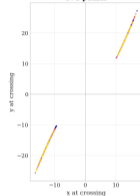- Infinite length, zero volume

Lyapunov spectrum:

- $\lambda_1 \approx +0.9$ (expansion)
- $\lambda_2 \approx 0$ (neutral)
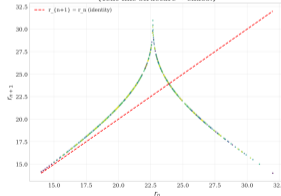- $\lambda_3 \approx -14.6$ (contraction)
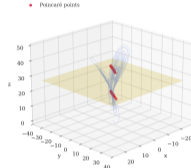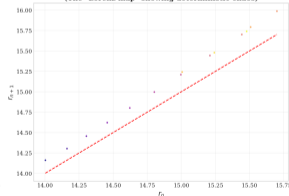
Period-Doubling Route to Chaos

The Logistic Map
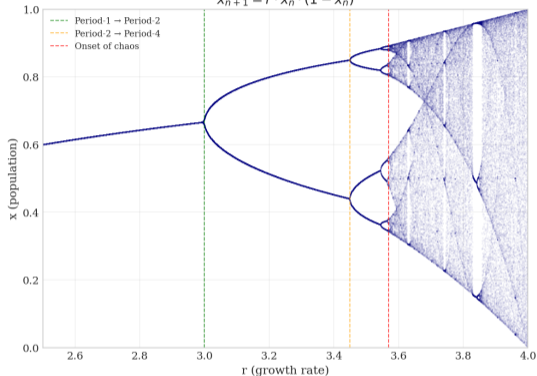
$$x_{n+1} = r \cdot x_n(1 - x_n)$$

As r increases:

1. $r < 3$: Single fixed point
2. $r = 3$: Period-2 cycle appears
3. $r \approx 3.45$: Period-4 cycle
4. $r \approx 3.54$: Period-8 cycle
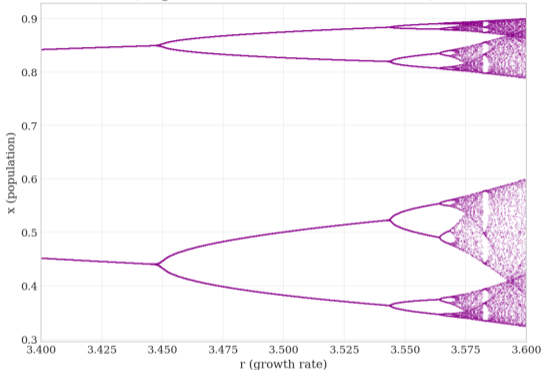5. $\vdots$
6. $r \approx 3.57$: Onset of chaos

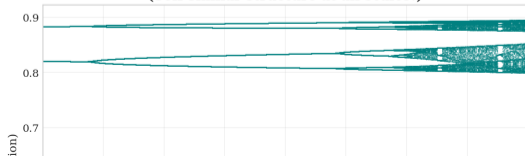# The Bifurcation Diagram

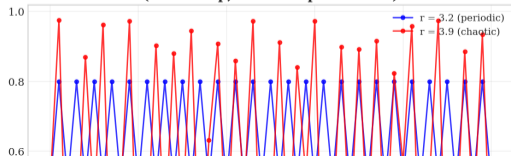**Logistic Map Bifurcation Diagram**
$x_{n+1} = r \cdot x_n \cdot (1 - x_n)$

- Period-1 → Period-2
- Period-2 → Period-4
- Onset of chaos

x (population)

r (growth rate)

Zoomed: Period-Doubling Cascade
(Feigenbaum's universal route to chaos)

x (population)

r (growth rate)

Even Deeper Zoom
(Self-similar structure at all scales!)

Time Series: Periodic vs Chaotic
(Same map, different parameter!)

- r = 3.2 (periodic)
- r = 3.9 (chaotic)

Feigenbaum's Universal Constant

The Feigenbaum Delta

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669201609\ldots$$

Remarkable universality:

- Same constant appears in *any* unimodal map
- Independent of the specific equation
- A deep mathematical invariant of chaos

Physical Significance

The ratio of successive bifurcation intervals approaches δ regardless of the system—from fluid dynamics to population models.

Summary: From Limit Cycles to Strange Attractors

Limit Cycles
- Periodic, predictable
- Van der Pol oscillator
- Heartbeat modeling
- $\lambda_{\max} = 0$

Strange Attractors
- Aperiodic, chaotic
- Lorenz system
- Fractal geometry
- $\lambda_{\max} > 0$

Key Takeaway

Deterministic systems can exhibit unpredictable behavior. Chaos is not randomness—it is *sensitive determinism*.

Further Reading

S.H. Strogatz, *Nonlinear Dynamics and Chaos*, Westview Press, 2015.

E.N. Lorenz, "Deterministic Nonperiodic Flow," *J. Atmos. Sci.* **20**, 130–141 (1963).

M.J. Feigenbaum, "Quantitative Universality for a Class of Nonlinear Transformations," *J. Stat. Phys.* **19**, 25–52 (1978).

B. van der Pol, "On Relaxation-Oscillations," *Phil. Mag.* **2**, 978–992 (1926).

Thank You

Questions?

"Chaos is found in greatest abundance wherever order is being sought. It always defeats order, because it is better organized."

— Terry Pratchett