📐 Mathematical Compendium

Strange Attractors & Dynamical Systems

Equations, Derivations, and Key Results

December 2025

Part I: Foundations

1. Sensitive Dependence on Initial Conditions

The hallmark of chaotic systems — two trajectories starting infinitesimally close diverge exponentially:

$$\boxed{|\delta \mathbf{x}(t)| \sim |\delta \mathbf{x}_0| \cdot e^{\lambda t}}$$

where $\lambda > 0$ is the **Lyapunov exponent** (measured in bits/time).

Part II: The Van der Pol Oscillator

2. Governing Equation

$$\boxed{\frac{d^2 x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0}$$

2.1 First-Order System

Introducing velocity $y = \dot{x}$:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases}$$

2.2 Damping Mechanism

Condition	Damping Sign	Effect
$\|x\| < 1$	**Negative**	Energy pumped IN
$\|x\| > 1$	**Positive**	Energy dissipated OUT

2.3 Jacobian Matrix

$$\mathbf{J}(x, y) = \begin{pmatrix} 0 & 1 \\ -2\mu xy - 1 & \mu(1 - x^2) \end{pmatrix}$$

At the origin $(0, 0)$:

$$\mathbf{J}_0 = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

2.4 Eigenvalue Analysis

Characteristic equation:

$$\lambda^2 - \mu\lambda + 1 = 0$$

Solutions:

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

2.5 Stability Classification

Parameter	Eigenvalues	Type	Behavior
$\mu = 0$	$\pm i$	Center	Harmonic oscillator
$0 < \mu < 2$	Complex, Re $>$0	Unstable spiral	Outward spiraling
$\mu \geq 2$	Real, both $>$ 0	Unstable node	Fast divergence

Conclusion: For $\mu > 0$, origin is unstable → **limit cycle exists** (Poincaré-Bendixson)

2.6 Relaxation Oscillation Period

For large μ:

$$T \approx (3 - 2\ln 2)\mu \approx 1.614\,\mu$$

Part III: The Lorenz System

3. Governing Equations

$$\boxed{\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned}}$$

3.1 Standard Parameters (Chaotic Regime)

Symbol	Name	Physical Meaning	Value
σ	Prandtl number	Momentum / thermal diffusivity	**10**
ρ	Rayleigh number	Temperature gradient strength	**28**
β	Geometric factor	Cell aspect ratio	**8/3**

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4. Equilibrium Analysis

4.1 Finding Fixed Points

Setting all derivatives to zero:

$$\sigma(y - x) = 0 \quad \Rightarrow \quad y = x$$
$$x(\rho - z) - y = 0 \quad \Rightarrow \quad x(\rho - z - 1) = 0$$
$$xy - \beta z = 0$$

Solution 1: Trivial equilibrium (no convection)

$$\boxed{O = (0, 0, 0)}$$

Solution 2: Non-trivial equilibria (for $\rho > 1$)

From $z = \rho - 1$ and $x^2 = \beta(\rho - 1)$:

$$\boxed{C^{\pm} = \left(\pm\sqrt{\beta(\rho - 1)},\ \pm\sqrt{\beta(\rho - 1)},\ \rho - 1\right)}$$

With standard parameters: $C^{+} = (8.485, 8.485, 27)$

4.2 The Jacobian Matrix

$$\mathbf{J} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}$$

4.3 Stability of the Origin

$$\mathbf{J}_O = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}$$

Eigenvalues:

- $\lambda_1 = -\beta = -2.667$ ✓
- $\lambda_2 \approx +11.83$ ✗
- $\lambda_3 \approx -22.83$ ✓

Result: Origin is a **saddle point** (2D stable, 1D unstable manifold)

4.4 Stability of $C^{\pm}$

At $C^+ = (8.485, 8.485, 27)$:

$$\mathbf{J}_{C^+} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -8.485 \\ 8.485 & 8.485 & -2.667 \end{pmatrix}$$

Characteristic polynomial:

$$\lambda^3 + (\sigma + \beta + 1)\lambda^2 + (\sigma + \rho)\beta\,\lambda + 2\sigma\beta(\rho - 1) = 0$$

Eigenvalues:

- $\lambda_1 \approx -13.85$ (real, stable direction)
- $\lambda_{2,3} \approx 0.094 \pm 10.19i$ (complex, **unstable spiral!**)

Part IV: The Hopf Bifurcation

5. Critical Rayleigh Number

The system transitions to chaos when $C^{\pm}$ lose stability:

$$\boxed{\rho_H = \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1} \approx 24.74}$$

5.1 Bifurcation Diagram

ρ Range	Equilibria	Behavior
$\rho < 1$	Only O (stable)	No convection
$1 < \rho < 24.74$	O unstable, $C^{\pm}$ stable	Steady convection
$\rho > 24.74$	All unstable	**Strange attractor**

5.2 Lyapunov Exponent

At $\rho = 28$:

$$\lambda_{\max} \approx 0.906 \text{ bits/time}$$

Prediction horizon:

$$T_{\text{pred}} \approx \frac{1}{\lambda_{\max}} \ln\left(\frac{\Delta_{\text{tol}}}{\Delta_0}\right)$$

Part V: Chaos Control

6. Control Methods

6.1 Parameter Adjustment

$$\rho < 24.74 \quad \Longrightarrow \quad \text{Chaos eliminated}$$

6.2 OGY Control (Ott-Grebogi-Yorke)

$$\delta\rho = -\mathbf{K}^T(\mathbf{x} - \mathbf{x}^*)$$

6.3 Pyragas Time-Delay Feedback

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + K\left[\mathbf{x}(t-\tau) - \mathbf{x}(t)\right]$$

Stabilizes unstable periodic orbits when τ matches their period.

Part VI: Universal Route to Chaos

7. The Logistic Map

$$x_{n+1} = r \cdot x_n \cdot (1 - x_n)$$

7.1 Period-Doubling Cascade

Parameter r	**Dynamics**
$r < 1$	Extinction ($x \to 0$)
$1 < r < 3$	Stable fixed point
$r \approx 3.0$	Period-2 bifurcation
$r \approx 3.45$	Period-4
$r \approx 3.54$	Period-8, 16, 32...
$r > 3.57$	**Chaos**

7.2 Feigenbaum's Universal Constant

$$\boxed{\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669201609\ldots}$$

This constant appears in **all** systems undergoing period-doubling — a remarkable universality!

Summary: Key Equations

System	Governing Equation
Van der Pol	$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$
Lorenz	$\dot{x} = \sigma(y - x),\ \dot{y} = x(\rho - z) - y,\ \dot{z} = xy - \beta z$
Lyapunov Divergence	$\|\delta\mathbf{x}(t)\| \sim \|\delta\mathbf{x}_0\| e^{\lambda t}$
Hopf Bifurcation	$\rho_H = \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1} \approx 24.74$
Feigenbaum Constant	$\delta = 4.669201609 \ldots$

References

1. **Van der Pol, B.** (1926). "On relaxation oscillations." *Phil. Mag.*
2. **Lorenz, E.N.** (1963). "Deterministic Nonperiodic Flow." *J. Atmos. Sci.* **20**, 130–141.
3. **Feigenbaum, M.J.** (1978). "Quantitative universality for a class of nonlinear transformations." *J. Stat. Phys.* **19**, 25–52.
4. **Strogatz, S.H.** (2015). *Nonlinear Dynamics and Chaos*, 2nd ed. Westview Press.

"Determinism ≠ Predictability"