Strange Attractors & Dynamical Systems

From Van der Pol Oscillators to Deterministic Chaos

Dynamical Systems Exploration

Mathematical Physics

December 9, 2025

Outline

1. What Are Strange Attractors?
2. Van der Pol Oscillator
3. The Lorenz System
4. Feigenbaum Cascade
5. Chaos Control at CERN
6. Conclusion

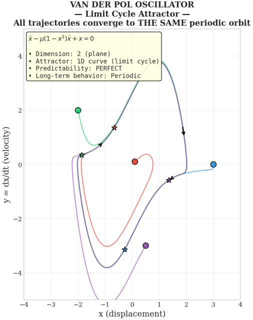
The Nature of Dynamical Systems

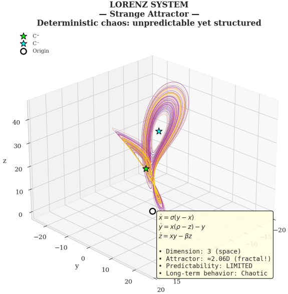
Key Question:

What happens to systems over long time periods?

Attractors are the geometric structures toward which dynamical systems evolve.

- **Fixed points** (stable equilibria)
- **Limit cycles** (periodic orbits)





From Order to Chaos

The Lyapunov Exponent

Measures the rate of separation of infinitesimally close trajectories:

$$\boxed{|\delta \mathbf{x}(t)| \sim |\delta \mathbf{x}_0| \, e^{\lambda t}}$$

- $\lambda < 0$: **Stable**
- $\lambda = 0$: Marginally stable
- $\lambda > 0$: **Chaotic!**

Deterministic Chaos

Simple equations can produce **unpredictable** behavior!

The Van der Pol Equation

Governing Equation (1927)

$$\boxed{\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0}$$

First-order system form:

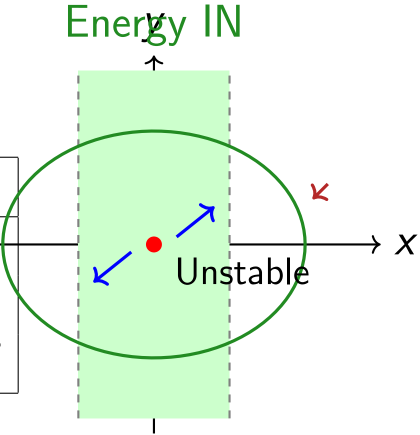
$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases}$$

Energy Pumping Mechanism

Effective damping coefficient:

$$\gamma_{\text{eff}}(x) = -\mu(1 - x^2)$$

Condition	Damping	Effect
$\lvert x\rvert < 1$	Negative	Energy IN
$\lvert x\rvert = 1$	Zero	Balance
$\lvert x\rvert > 1$	Positive	Energy OUT



Result: Self-regulating limit

Phase Plane Analysis: Nullclines

Nullclines are curves where $\dot{x} = 0$ or $\dot{y} = 0$:

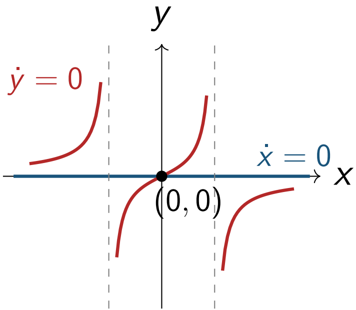
x-**nullcline** ($\dot{x} = 0$):

$$y = 0 \quad \text{(the } x\text{-axis)}$$

y-**nullcline** ($\dot{y} = 0$):

$$y = \frac{x}{\mu(1 - x^2)}$$

Equilibria occur at nullcline intersections.



Equilibrium Analysis: The Jacobian Matrix

General Jacobian

For $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$:

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\mu xy - 1 & \mu(1-x^2) \end{pmatrix}$$

At the origin $(x,y) = (0,0)$:

$$\boxed{J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}}$$

Eigenvalue Analysis: Stability Classification

Characteristic equation: $\lambda^2 - \mu\lambda + 1 = 0$

Eigenvalues

$$\boxed{\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}}$$

Discriminant: $\Delta = \mu^2 - 4$

μ **range**	**Eigenvalues**	**Stability**
$\mu = 0$	$\pm i$	Neutral center
$0 < \mu < 2$	Complex, Re > 0	Unstable spiral

The Poincaré-Bendixson Theorem

Question: If the origin is unstable, where do trajectories go?

Poincaré-Bendixson Theorem

If a trajectory in $\mathbb{R}^2$ enters a closed, bounded region R containing **no equilibria** and never leaves, then it must approach a **periodic orbit**.

Application to Van der Pol:

1. Origin is unstable $\Rightarrow$ trajectories spiral *outward*
2. Energy dissipation when $|x| > 1 \Rightarrow$ trajectories are *bounded*
3. Trajectories enter an annular region with no equilibria
4. $\therefore$ A **limit cycle** must exist!

Relaxation Oscillations: Large μ

When $\mu \gg 1$, the system exhibits **relaxation oscillations**:

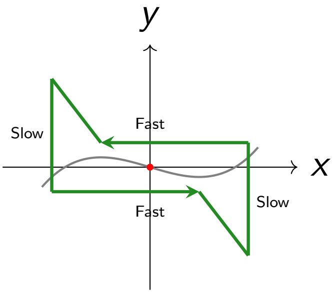
1. **Slow phase:** Trajectory creeps along the cubic nullcline
2. **Fast phase:** Rapid jump between branches

Period scaling:

$$T \approx (3 - 2\ln 2)\mu \approx 1.614\mu$$

Biological examples:

- Heartbeat (SA node)



Relaxation oscillation for $\mu \gg 1$

Heartbeat Modeling with Van der Pol

The Van der Pol oscillator models **cardiac rhythms**:

- Sinoatrial node acts as a *relaxation oscillator*
- Self-sustaining periodic signal
- Robust against perturbations

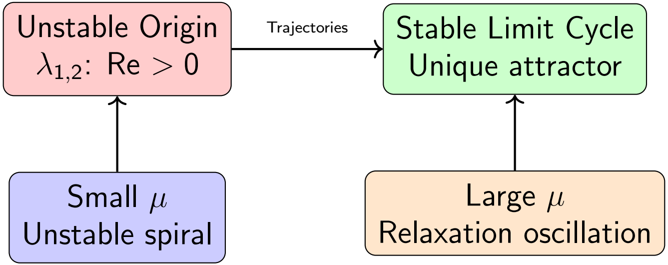
FitzHugh-Nagumo Model

$$\begin{cases} \dot{v} = v - \frac{v^3}{3} - w + I_{\text{ext}} \\ \dot{w} = \epsilon(v + a - bw) \end{cases}$$

Clinical connections:

Normal	Stable limit cycle (60-100 bpm)
Arrhythmia	Bifurcation from healthy cycle
Pacemaker	Artificial periodic forcing

Summary: Van der Pol Dynamics



Key Results

- **Unique equilibrium** at origin — always unstable for $\mu > 0$
- **Limit cycle** guaranteed by Poincaré-Bendixson theorem
- **Applications:** Electronics, cardiology, neuroscience

Lorenz Equations

The System (1963)

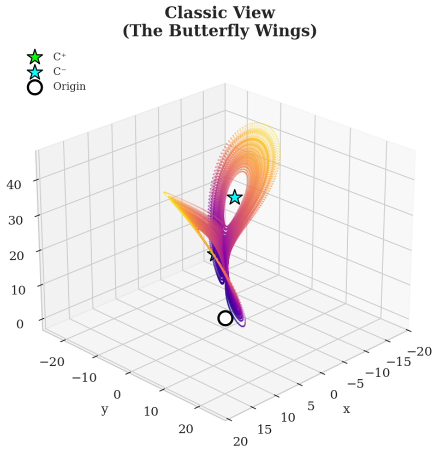
$$\boxed{\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases}}$$

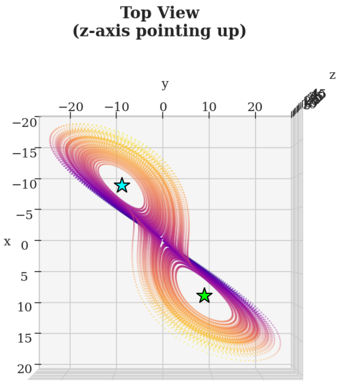
Standard parameters:

- $\sigma = 10$ (Prandtl number)

The Butterfly Attractor

THE LORENZ STRANGE ATTRACTOR
σ = 10, ρ = 28, β = 8/3

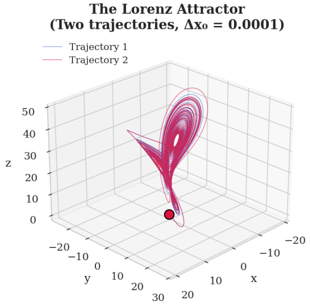


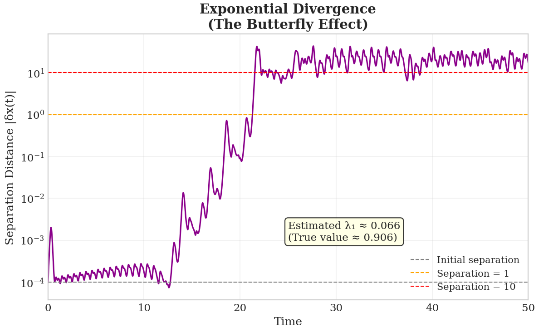


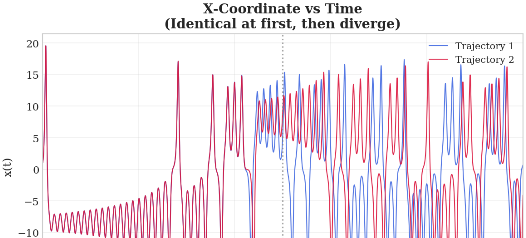
Side View (x-z plane)
(Showing vertical structure)

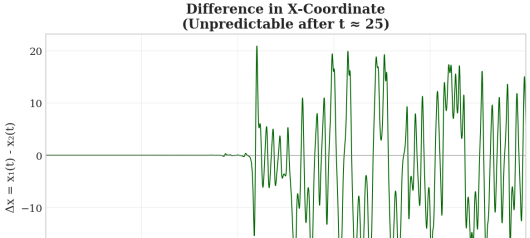
Rear View (y-z plane)
(Line plot showing flow)

Sensitive Dependence on Initial Conditions









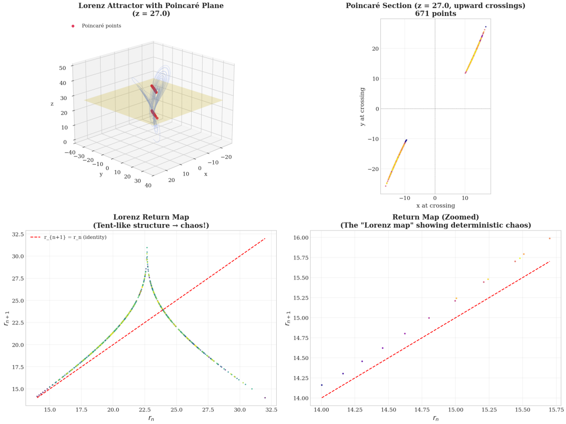
Fractal Structure of Strange Attractors

Properties:

- Non-integer (fractal) dimension
- Lorenz attractor: $D \approx 2.06$
- Self-similar at all scales
- Infinite length, zero volume

Lyapunov spectrum:

- $\lambda_1 \approx +0.9$ (expansion)
- $\lambda_2 \approx 0$ (neutral)
- $\lambda_3 \approx -14.6$ (contraction)



Period-Doubling Route to Chaos

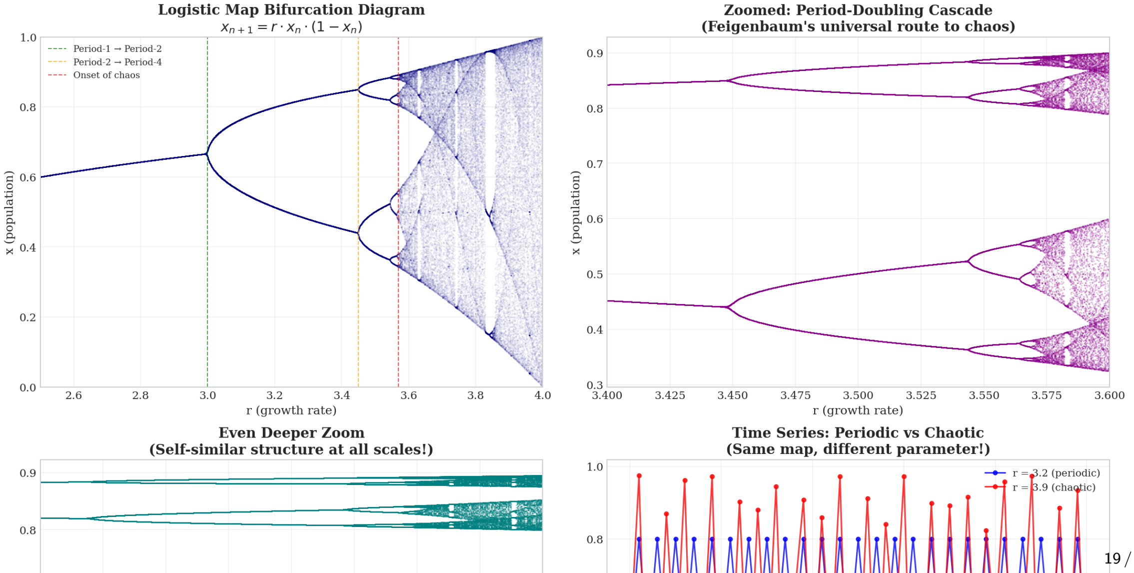
The Logistic Map

$$\boxed{x_{n+1} = r \cdot x_n(1 - x_n)}$$

As r increases:

1. $r < 3$: Single fixed point
2. $r = 3$: Period-2 cycle appears
3. $r \approx 3.45$: Period-4 cycle
4. $r \approx 3.54$: Period-8 cycle
5. $\vdots$

The Bifurcation Diagram



Feigenbaum's Universal Constant

The Feigenbaum Delta

$$\boxed{\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669201609\ldots}$$

Remarkable universality:

- Same constant appears in *any* unimodal map
- Independent of the specific equation
- A deep mathematical invariant of chaos

Physical Significance

The ratio of successive bifurcation intervals approaches δ regardless of

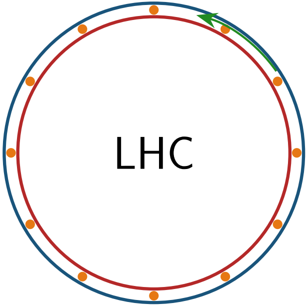
Why Chaos Threatens Particle Beams

The Challenge:

The LHC accelerates protons to **99.9999991%** the speed of light in a 27-km ring.

Sources of instability:

1. Magnet imperfections ($\sim 10^{-4}$)
2. Beam-beam interactions
3. Resonances ($Q = m/n$)



27 km ring

At 6.5 TeV, tiny errors grow **exponentially**

Particle Dynamics as a Nonlinear Map

One-Turn Map (Symplectic)

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix}}_{\text{Linear rotation}} \begin{pmatrix} x_n \\ p_n \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ K_2 x_n^2 + \cdot \end{pmatrix}}_{\text{Nonlinear kicks}}$$

- $Q \approx 64.31$: **Betatron tune** (oscillations per turn)
- K_2, K_3: Sextupole, octupole strengths
- This is essentially the **Hénon map**!

The OGY Method: Taming Chaos

Key Insight (Ott, Grebogi, Yorke 1990)

Chaotic systems contain **infinitely many unstable periodic orbits**. With *tiny, well-timed perturbations*, you can stabilize any of them!

Near an unstable fixed point:

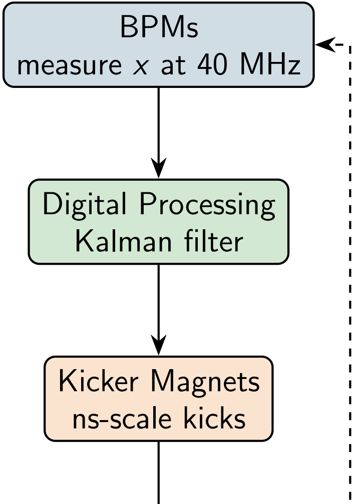
$$\delta \mathbf{x}_{n+1} = \mathbf{J} \cdot \delta \mathbf{x}_n$$

Eigenvalues: $|\lambda_s| < 1$ (stable) and $|\lambda_u| > 1$ (unstable)

Control law:

LHC Transverse Damper (ADT)

The Feedback Loop:



Performance Numbers:

Energy	6.5 TeV
Protons/bunch	$\sim 10^{11}$
Bunches	2,556
Rev. freq.	11.245 kHz
ADT bandwidth	20 MHz
Damping time	50-100 tur

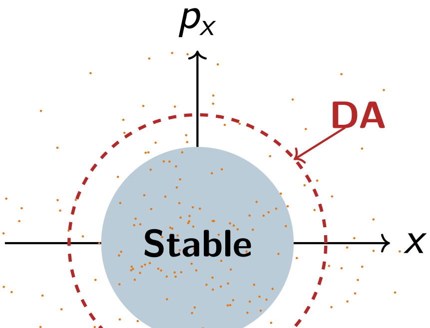
Without feedback:

Dynamic Aperture: Order vs Chaos Boundary

Dynamic Aperture (DA):

The region in phase space where particles remain stable for long times.

- **Inside DA**: Stable KAM tori
- **Boundary**: Fractal structure



Theory Meets Practice: Summary

Chaos Theory	**CERN Application**
Phase space	Track $(x, p_x, y, p_y, z, \delta)$
Strange attractor	Beam halo
Unstable fixed point	Design orbit
Sensitive dependence	Injection errors amplify
Lyapunov exponent	Tune stability margins
Poincaré section	Turn-by-turn BPM data
OGY control	Transverse damper
Dynamic aperture	Machine protection

Summary: From Limit Cycles to Chaos Control

Limit Cycles

- Periodic, predictable
- Van der Pol oscillator
- Heartbeat modeling
- $\lambda_{\text{max}} = 0$

Strange Attractors

- Aperiodic, chaotic
- Lorenz system
- Fractal geometry
- $\lambda_{\text{max}} > 0$

Chaos Control

- OGY method
- Tiny perturbations
- CERN beam control
- Stabilize UPOs

Key Equations Cheat Sheet

Van der Pol:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

Lorenz:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(\rho - z) - y, \quad \dot{z} = xy - \beta z$$

Lyapunov Exponent:

$$|\delta \mathbf{x}(t)| \sim |\delta \mathbf{x}_0| e^{\lambda t}$$

OGY Control:

Further Reading

- S.H. Strogatz, *Nonlinear Dynamics and Chaos*, Westview Press, 2015.
- E.N. Lorenz, “Deterministic Nonperiodic Flow,” *J. Atmos. Sci.* **20**, 130–141 (1963).
- E. Ott, C. Grebogi, J.A. Yorke, “Controlling Chaos,” *Phys. Rev. Lett.* **64**, 1196 (1990).
- M.J. Feigenbaum, “Quantitative Universality...” *J. Stat. Phys.* **19**, 25–52 (1978).

Thank You

Questions?

"The universe is not only queerer than we suppose,
but queerer than we can suppose."

— J.B.S. Haldane