Chapter 10

The Dual Space and Duality

In this chapter all vector spaces are defined over an arbitrary field K. For the sake of concreteness, the reader may safely assume that $K = \mathbb{R}$.

10.1 The Dual Space E^* and Linear Forms

In Section 2.8 we defined linear forms, the dual space $E^* = \mathrm{Hom}(E, K)$ of a vector space E, and showed the existence of dual bases for vector spaces of finite dimension.

In this chapter we take a deeper look at the connection between a space E and its dual space E^*. As we will see shortly, every linear map $f\colon E \to F$ gives rise to a linear map $f^\top\colon F^* \to E^*$, and it turns out that in a suitable basis, the matrix of $f^\top$ is the transpose of the matrix of f. *Thus, the notion of dual space provides a conceptual explanation of the phenomena associated with transposition.*

But it does more, because it allows us to view a linear equation as an element of the dual space E^*, and thus to view subspaces of E as solutions of sets of linear equations and vice-versa. The relationship between subspaces and sets of linear forms is the essence of *duality*, a term which is often used loosely, but can be made precise as a bijection between the set of subspaces of a given vector space E and the set of subspaces of its dual E^*. In this correspondence, a subspace V of E yields the subspace V^0 of E^* consisting of all linear forms that vanish on V (that is, have the value zero for all input in V).

Consider the following set of two "linear equations" in $\mathbb{R}^3$,

$$\begin{aligned} x - y + z &= 0 \\ x - y - z &= 0, \end{aligned}$$

and let us find out what is their set V of common solutions $(x, y, z) \in \mathbb{R}^3$. By subtracting the second equation from the first, we get $2z = 0$, and by adding the two equations, we find that $2(x - y) = 0$, so the set V of solutions is given by

$$\begin{aligned} y &= x \\ z &= 0. \end{aligned}$$

This is a one dimensional subspace of $\mathbb{R}^3$. Geometrically, this is the line of equation $y = x$ in the plane $z = 0$ as illustrated by Figure 10.1.

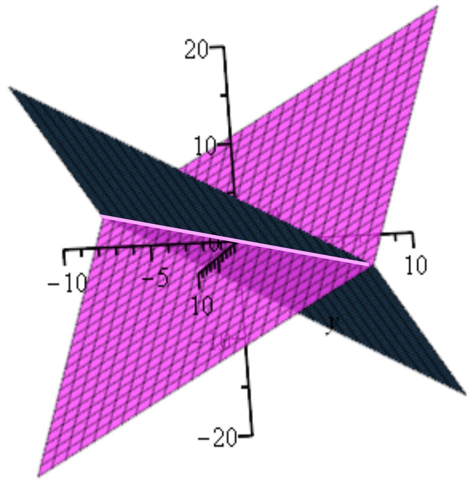

Figure 10.1: The intersection of the magenta plane $x - y + z = 0$ with the blue-gray plane $x - y - z = 0$ is the pink line $y = x$.

Now why did we say that the above equations are linear? Because as functions of (x, y, z), both maps $f_1 \colon (x, y, z) \mapsto x - y + z$ and $f_2 \colon (x, y, z) \mapsto x - y - z$ are linear. The set of all such linear functions from $\mathbb{R}^3$ to $\mathbb{R}$ is a vector space; we used this fact to form linear combinations of the "equations" f_1 and f_2. Observe that the dimension of the subspace V is 1. The ambient space has dimension $n = 3$ and there are two "independent" equations f_1, f_2, so it appears that the dimension $\dim(V)$ of the subspace V defined by m independent equations is

$$\dim(V) = n - m,$$

which is indeed a general fact (proven in Theorem 10.4).

More generally, in $\mathbb{R}^n$, a linear equation is determined by an n-tuple $(a_1, \ldots, a_n) \in \mathbb{R}^n$, and the solutions of this linear equation are given by the n-tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$a_1x_1 + \cdots + a_nx_n = 0;$$

these solutions constitute the kernel of the linear map $(x_1, \ldots, x_n) \mapsto a_1x_1 + \cdots + a_nx_n$. The above considerations assume that we are working in the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, but we can define "linear equations" independently of bases and in any dimension, by viewing them as elements of the vector space $\mathrm{Hom}(E, K)$ of linear maps from E to the field K.

Definition 10.1. Given a vector space E, the vector space $\mathrm{Hom}(E, K)$ of linear maps from E to the field K is called the *dual space (or dual)* of E. The space $\mathrm{Hom}(E, K)$ is also denoted by E^*, and the linear maps in E^* are called *the linear forms*, or *covectors*. The dual space E^{**} of the space E^* is called the *bidual* of E.

As a matter of notation, linear forms $f\colon E \to K$ will also be denoted by starred symbol, such as u^*, x^*, *etc.*

Given a vector space E and any basis $(u_i)_{i\in I}$ for E, we can associate to each u_i a linear form $u_i^* \in E^*$, and the u_i^* have some remarkable properties.

Definition 10.2. Given a vector space E and any basis $(u_i)_{i\in I}$ for E, by Proposition 2.17, for every $i \in I$, there is a unique linear form u_i^* such that

$$u_i^*(u_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for every $j \in I$. The linear form u_i^* is called the *coordinate form* of index i w.r.t. the basis $(u_i)_{i\in I}$.

The reason for the terminology *coordinate form* was explained in Section 2.8.

We proved in Theorem 2.20 that if $(u_1, \ldots, u_n)$ is a basis of E, then $(u_1^*, \ldots, u_n^*)$ is a basis of E^* called the *dual basis*.

If $(u_1, \ldots, u_n)$ is a basis of $\mathbb{R}^n$ (more generally K^n), it is possible to find explicitly the dual basis $(u_1^*, \ldots, u_n^*)$, where each u_i^* is represented by a row vector.

Example 10.1. For example, consider the columns of the Bézier matrix

$$B_4 = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, we have the basis

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} -3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \\ 0 \end{pmatrix} \qquad u_4 = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}.$$

Since the form u_1^* is defined by the conditions $u_1^*(u_1) = 1, u_1^*(u_2) = 0, u_1^*(u_3) = 0, u_1^*(u_4) = 0$, it is represented by a row vector $(\lambda_1 \;\; \lambda_2 \;\; \lambda_3 \;\; \lambda_4)$ such that

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

This implies that u_1^* is the first row of the inverse of B_4. Since

$$B_4^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the linear forms $(u_1^*, u_2^*, u_3^*, u_4^*)$ correspond to the rows of B_4^{-1}. In particular, u_1^* is represented by $(1\ 1\ 1\ 1)$.

The above method works for any n. Given any basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$, if P is the $n \times n$ matrix whose jth column is u_j, then the dual form u_i^* is given by the ith row of the matrix P^{-1}.

When E is of finite dimension n and $(u_1, \ldots, u_n)$ is a basis of E, by Theorem 10.4 (1), the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space E^*. Let us see how the coordinates of a linear form $\varphi^* \in E^*$ over the dual basis $(u_1^*, \ldots, u_n^*)$ vary under a change of basis.

Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two bases of E, and let $P = (a_{i\,j})$ be the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, so that

$$v_j = \sum_{i=1}^n a_{i\,j} u_i,$$

and let $P^{-1} = (b_{i\,j})$ be the inverse of P, so that

$$u_i = \sum_{j=1}^n b_{j\,i} v_j.$$

For fixed j, where $1 \leq j \leq n$, we want to find scalars $(c_i)_{i=1}^n$ such that

$$v_j^* = c_1 u_1^* + c_2 u_2^* + \cdots + c_n u_n^*.$$

To find each c_i, we evaluate the above expression at u_i. Since $u_i^*(u_j) = \delta_{i\,j}$ and $v_i^*(v_j) = \delta_{i\,j}$, we get

$$\begin{aligned} v_j^*(u_i) &= (c_1 u_1^* + c_2 u_2^* + \cdots + c_n u_n^*)(u_i) = c_i \\ v_j^*(u_i) &= v_j^*(\sum_{k=1}^n b_{k\,i} v_k) = b_{j\,i}, \end{aligned}$$

and thus

$$v_j^* = \sum_{i=1}^n b_{j\,i} u_i^*.$$

Similar calculations show that

$$u_i^* = \sum_{j=1}^n a_{i\,j} v_j^*.$$

This means that the change of basis from the dual basis $(u_1^*, \ldots, u_n^*)$ to the dual basis $(v_1^*, \ldots, v_n^*)$ is $(P^{-1})^\top$. Since

$$\varphi^* = \sum_{i=1}^n \varphi_i u_i^* = \sum_{i=1}^n \varphi_i \sum_{j=1}^n a_{ij} v_j^* = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} \varphi_i \right) v_j = \sum_{i=1}^n \varphi'_i v_i^*,$$

we get

$$\varphi'_j = \sum_{i=1}^n a_{i\,j} \varphi_i,$$

so the new coordinates φ'_j are expressed in terms of the old coordinates φ_i using the matrix $P^\top$. If we use the row vectors $(\varphi_1, \ldots, \varphi_n)$ and $(\varphi'_1, \ldots, \varphi'_n)$, we have

$$(\varphi'_1, \ldots, \varphi'_n) = (\varphi_1, \ldots, \varphi_n) P.$$

These facts are summarized in the following proposition.

Proposition 10.1. *Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two bases of E, and let $P = (a_{i\,j})$ be the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, so that*

$$v_j = \sum_{i=1}^n a_{i\,j} u_i.$$

Then the change of basis from the dual basis $(u_1^*, \ldots, u_n^*)$ to the dual basis $(v_1^*, \ldots, v_n^*)$ is $(P^{-1})^\top$, and for any linear form φ, the new coordinates φ'_j of φ are expressed in terms of the old coordinates φ_i of φ using the matrix $P^\top$; that is,

$$(\varphi'_1, \ldots, \varphi'_n) = (\varphi_1, \ldots, \varphi_n) P.$$

To best understand the preceding paragraph, recall Example 3.1, in which $E = \mathbb{R}^2$, $u_1 = (1, 0)$, $u_2 = (0, 1)$, and $v_1 = (1, 1)$, $v_2 = (-1, 1)$. Then P, the change of basis matrix from (u_1, u_2) to (v_1, v_2), is given by

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

with $(v_1, v_2) = (u_1, u_2)P$, and $(u_1, u_2) = (v_1, v_2)P^{-1}$, where

$$P^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Let (u_1^*, u_2^*) be the dual basis for (u_1, u_2) and (v_1^*, v_2^*) be the dual basis for (v_1, v_2). We claim that

$$(v_1^*, v_2^*) = (u_1^*, u_2^*) \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = (u_1^*, u_2^*)(P^{-1})^\top,$$

Indeed, since $v_1^* = c_1 u_1^* + c_2 u_2^*$ and $v_2^* = C_1 u_1^* + C_2 u_2^*$ we find that

$$\begin{aligned}
c_1 &= v_1^*(u_1) = v_1^*(1/2v_1 - 1/2v_2) = 1/2 \\
c_2 &= v_1^*(u_2) = v_1^*(1/2v_1 + 1/2v_2) = 1/2 \\
C_1 &= v_2^*(u_1) = v_2^*(1/2v_1 - 1/2v_2) = -1/2 \\
C_2 &= v_2^*(u_2) = v_1^*(1/2v_1 + 1/2v_2) = 1/2.
\end{aligned}$$

Furthermore, since $(u_1^*, u_2^*) = (v_1^*, v_2^*)P^\top$ (since $(v_1^*, v_2^*) = (u_1^*, u_2^*)(P^\top)^{-1}$), we find that

$$\begin{aligned}
\varphi^* &= \varphi_1 u_1^* + \varphi_2 u_2^* = \varphi_1(v_1^* - v_2^*) + \varphi_(v_1^* + v_2^*) \\
&= (\varphi_1 + \varphi_2)v_1^* + (-\varphi_1 + \varphi_2)v_2^* = \varphi_1' v_1^* + \varphi_2' v_2^*
\end{aligned}$$

Hence

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix},$$

where

$$P^\top = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Comparing with the change of basis

$$v_j = \sum_{i=1}^n a_{i\,j} u_i,$$

we note that this time, the coordinates (φ_i) of the linear form φ^* change in the *same direction* as the change of basis. For this reason, we say that the coordinates of linear forms are *covariant*. By abuse of language, it is often said that linear forms are *covariant*, which explains why the term *covector* is also used for a linear form.

Observe that if $(e_1, \ldots, e_n)$ is a basis of the vector space E, then, as a linear map from E to K, every linear form $f \in E^*$ is represented by a $1 \times n$ matrix, that is, by a *row vector*

$$(\lambda_1 \; \cdots \; \lambda_n),$$

with respect to the basis $(e_1, \ldots, e_n)$ of E, and 1 of K, where $f(e_i) = \lambda_i$. A vector $u = \sum_{i=1}^n u_i e_i \in E$ is represented by a $n \times 1$ matrix, that is, by a *column vector*

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$

and the action of f on u, namely $f(u)$, is represented by the matrix product

$$\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \lambda_1 u_1 + \cdots + \lambda_n u_n.$$

On the other hand, with respect to the dual basis $(e_1^*, \ldots, e_n^*)$ of E^*, the linear form f is represented by the column vector

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

Remark: In many texts using tensors, vectors are often indexed with lower indices. If so, it is more convenient to write the coordinates of a vector x over the basis $(u_1, \ldots, u_n)$ as (x^i), using an upper index, so that

$$x = \sum_{i=1}^n x^i u_i,$$

and in a change of basis, we have

$$v_j = \sum_{i=1}^n a_j^i u_i$$

and

$$x^i = \sum_{j=1}^n a_j^i x'^j.$$

Dually, linear forms are indexed with upper indices. Then it is more convenient to write the coordinates of a covector φ^* over the dual basis $(u^{*1}, \ldots, u^{*n})$ as (φ_i), using a lower index, so that

$$\varphi^* = \sum_{i=1}^n \varphi_i u^{*i}$$

and in a change of basis, we have

$$u^{*i} = \sum_{j=1}^n a_j^i v^{*j}$$

and

$$\varphi'_j = \sum_{i=1}^n a_j^i \varphi_i.$$

With these conventions, the index of summation appears once in upper position and once in lower position, and the summation sign can be safely omitted, a trick due to *Einstein*. For example, we can write

$$\varphi'_j = a_j^i \varphi_i$$

as an abbreviation for

$$\varphi'_j = \sum_{i=1}^n a_j^i \varphi_i.$$

For another example of the use of Einstein's notation, if the vectors $(v_1, \ldots, v_n)$ are linear combinations of the vectors $(u_1, \ldots, u_n)$, with

$$v_i = \sum_{j=1}^{n} a_{ij} u_j, \quad 1 \leq i \leq n,$$

then the above equations are written as

$$v_i = a_i^j u_j, \quad 1 \leq i \leq n.$$

Thus, in Einstein's notation, the $n \times n$ matrix (a_{ij}) is denoted by (a_i^j), a $(1,1)$-*tensor*.

Beware that some authors view a matrix as a mapping between *coordinates*, in which case the matrix (a_{ij}) is denoted by (a_j^i).

10.2 Pairing and Duality Between E and E^*

Given a linear form $u^* \in E^*$ and a vector $v \in E$, the result $u^*(v)$ of applying u^* to v is also denoted by $\langle u^*, v \rangle$. This defines a binary operation $\langle -, - \rangle \colon E^* \times E \to K$ satisfying the following properties:

$$\begin{aligned}
\langle u_1^* + u_2^*, v \rangle &= \langle u_1^*, v \rangle + \langle u_2^*, v \rangle \\
\langle u^*, v_1 + v_2 \rangle &= \langle u^*, v_1 \rangle + \langle u^*, v_2 \rangle \\
\langle \lambda u^*, v \rangle &= \lambda \langle u^*, v \rangle \\
\langle u^*, \lambda v \rangle &= \lambda \langle u^*, v \rangle.
\end{aligned}$$

The above identities mean that $\langle -, - \rangle$ is a *bilinear map*, since it is linear in each argument. It is often called the *canonical pairing* between E^* and E. In view of the above identities, given any fixed vector $v \in E$, the map $\mathrm{eval}_v \colon E^* \to K$ (*evaluation at* v) defined such that

$$\mathrm{eval}_v(u^*) = \langle u^*, v \rangle = u^*(v) \quad \text{for every } u^* \in E^*$$

is a linear map from E^* to K, that is, eval_v is a linear form in E^{**}. Again, from the above identities, the map $\mathrm{eval}_E \colon E \to E^{**}$, defined such that

$$\mathrm{eval}_E(v) = \mathrm{eval}_v \quad \text{for every } v \in E,$$

is a linear map. Observe that

$$\mathrm{eval}_E(v)(u^*) = \mathrm{eval}_v(u^*) = \langle u^*, v \rangle = u^*(v), \quad \text{for all } v \in E \text{ and all } u^* \in E^*.$$

We shall see that the map eval_E is injective, and that it is an isomorphism when E has finite dimension.

We now formalize the notion of the set V^0 of linear equations vanishing on all vectors in a given subspace $V \subseteq E$, and the notion of the set U^0 of common solutions of a given set $U \subseteq E^*$ of linear equations. The duality theorem (Theorem 10.4) shows that the dimensions of V and V^0, and the dimensions of U and U^0, are related in a crucial way. It also shows that, in finite dimension, the maps $V \mapsto V^0$ and $U \mapsto U^0$ are inverse bijections from subspaces of E to subspaces of E^*.

Definition 10.3. Given a vector space E and its dual E^*, we say that a vector $v \in E$ and a linear form $u^* \in E^*$ are *orthogonal* iff $\langle u^*, v\rangle = 0$. Given a subspace V of E and a subspace U of E^*, we say that *V and U are orthogonal* iff $\langle u^*, v\rangle = 0$ for every $u^* \in U$ and every $v \in V$. Given a subset V of E (resp. a subset U of E^*), the *orthogonal V^0 of V* is the subspace V^0 of E^* defined such that

$$V^0 = \{u^* \in E^* \mid \langle u^*, v\rangle = 0, \text{ for every } v \in V\}$$

(resp. the *orthogonal U^0 of U* is the subspace U^0 of E defined such that

$$U^0 = \{v \in E \mid \langle u^*, v\rangle = 0, \text{ for every } u^* \in U\}).$$

The subspace $V^0 \subseteq E^*$ is also called the *annihilator* of V. The subspace $U^0 \subseteq E$ annihilated by $U \subseteq E^*$ does not have a special name. It seems reasonable to call it the *linear subspace (or linear variety) defined by U*.

Informally, V^0 is the *set of linear equations that vanish on V*, and U^0 is *the set of common zeros of all linear equations in U*. We can also define V^0 by

$$V^0 = \{u^* \in E^* \mid V \subseteq \operatorname{Ker} u^*\}$$

and U^0 by

$$U^0 = \bigcap_{u^* \in U} \operatorname{Ker} u^*.$$

Observe that $E^0 = \{0\} = (0)$, and $\{0\}^0 = E^*$.

Proposition 10.2. *If $V_1 \subseteq V_2 \subseteq E$, then $V_2^0 \subseteq V_1^0 \subseteq E^*$, and if $U_1 \subseteq U_2 \subseteq E^*$, then $U_2^0 \subseteq U_1^0 \subseteq E$. See Figure 10.2.*

Proof. Indeed, if $V_1 \subseteq V_2 \subseteq E$, then for any $f^* \in V_2^0$ we have $f^*(v) = 0$ for all $v \in V_2$, and thus $f^*(v) = 0$ for all $v \in V_1$, so $f^* \in V_1^0$. Similarly, if $U_1 \subseteq U_2 \subseteq E^*$, then for any $v \in U_2^0$, we have $f^*(v) = 0$ for all $f^* \in U_2$, so $f^*(v) = 0$ for all $f^* \in U_1$, which means that $v \in U_1^0$. ☐

Here are some examples.

Example 10.2. Let $E = \mathrm{M}_2(\mathbb{R})$, the space of real 2×2 matrices, and let V be the subspace of $\mathrm{M}_2(\mathbb{R})$ spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

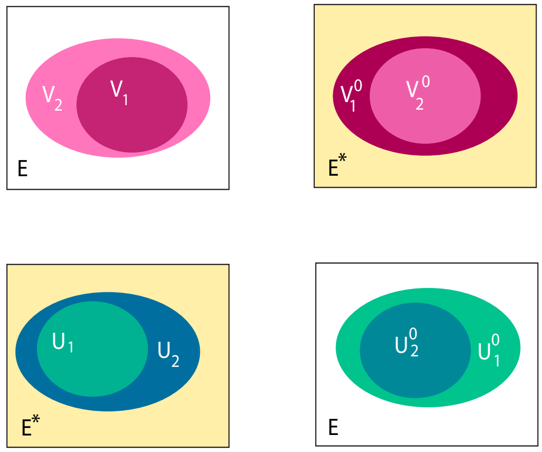

Figure 10.2: The top pair of figures schematically illustrates the relation if $V_1 \subseteq V_2 \subseteq E$, then $V_2^0 \subseteq V_1^0 \subseteq E^*$, while the bottom pair of figures illustrates the relationship if $U_1 \subseteq U_2 \subseteq E^*$, then $U_2^0 \subseteq U_1^0 \subseteq E$.

We check immediately that the subspace V consists of all matrices of the form

$$\begin{pmatrix} b & a \\ a & c \end{pmatrix},$$

that is, all symmetric matrices. The matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

in V satisfy the equation

$$a_{12} - a_{21} = 0,$$

and all scalar multiples of these equations, so V^0 is the subspace of E^* spanned by the linear form given by $u^*(a_{11}, a_{12}, a_{21}, a_{22}) = a_{12} - a_{21}$. By the duality theorem (Theorem 10.4) we have

$$\dim(V^0) = \dim(E) - \dim(V) = 4 - 3 = 1.$$

Example 10.3. The above example generalizes to $E = \mathrm{M}_n(\mathbb{R})$ for any $n \geq 1$, but this time, consider the space U of linear forms asserting that a matrix A is symmetric; these are the linear forms spanned by the $n(n-1)/2$ equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i < j \leq n;$$

Note there are no constraints on diagonal entries, and half of the equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i \neq j \leq n$$

are redundant. It is easy to check that the equations (linear forms) for which $i < j$ are linearly independent. To be more precise, let U be the space of linear forms in E^* spanned by the linear forms

$$u_{ij}^*(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) = a_{ij} - a_{ji}, \quad 1 \leq i < j \leq n.$$

The dimension of U is $n(n-1)/2$. Then the set U^0 of common solutions of these equations is the space $\mathbf{S}(n)$ of symmetric matrices. By the duality theorem (Theorem 10.4), this space has dimension

$$\frac{n(n+1)}{2} = n^2 - \frac{n(n-1)}{2}.$$

We leave it as an exercise to find a basis of $\mathbf{S}(n)$.

Example 10.4. If $E = \mathrm{M}_n(\mathbb{R})$, consider the subspace U of linear forms in E^* spanned by the linear forms

$$\begin{aligned} u_{ij}^*(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) &= a_{ij} + a_{ji}, \quad 1 \leq i < j \leq n \\ u_{ii}^*(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) &= a_{ii}, \quad 1 \leq i \leq n. \end{aligned}$$

It is easy to see that these linear forms are linearly independent, so $\dim(U) = n(n+1)/2$. The space U^0 of matrices $A \in \mathrm{M}_n(\mathbb{R})$ satifying all of the above equations is clearly the space $\mathbf{Skew}(n)$ of skew-symmetric matrices. By the duality theorem (Theorem 10.4), the dimension of U^0 is

$$\frac{n(n-1)}{2} = n^2 - \frac{n(n+1)}{2}.$$

We leave it as an exercise to find a basis of $\mathbf{Skew}(n)$.

Example 10.5. For yet another example with $E = \mathrm{M}_n(\mathbb{R})$, for any $A \in \mathrm{M}_n(\mathbb{R})$, consider the linear form in E^* given by

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

called the *trace* of A. The subspace U^0 of E consisting of all matrices A such that $\mathrm{tr}(A) = 0$ is a space of dimension $n^2 - 1$. We leave it as an exercise to find a basis of this space.

The dimension equations

$$\begin{aligned} \dim(V) + \dim(V^0) &= \dim(E) \\ \dim(U) + \dim(U^0) &= \dim(E) \end{aligned}$$

are always true (if E is finite-dimensional). This is part of the duality theorem (Theorem 10.4).

Remark: In contrast with the previous examples, given a matrix $A \in \mathrm{M}_n(\mathbb{R})$, the equations asserting that $A^\top A = I$ are not linear constraints. For example, for $n = 2$, we have

$$\begin{aligned} a_{11}^2 + a_{21}^2 &= 1 \\ a_{21}^2 + a_{22}^2 &= 1 \\ a_{11}a_{12} + a_{21}a_{22} &= 0. \end{aligned}$$

Remarks:

(1) The notation V^0 (resp. U^0) for the orthogonal of a subspace V of E (resp. a subspace U of E^*) is not universal. Other authors use the notation $V^\perp$ (resp. $U^\perp$). However, the notation $V^\perp$ is also used to denote the orthogonal complement of a subspace V with respect to an inner product on a space E, in which case $V^\perp$ is a subspace of E and not a subspace of E^* (see Chapter 11). To avoid confusion, we prefer using the notation V^0.

(2) Since linear forms can be viewed as linear equations (at least in finite dimension), given a subspace (or even a subset) U of E^*, we can define the set $\mathcal{Z}(U)$ of *common zeros* of the equations in U by

$$\mathcal{Z}(U) = \{v \in E \mid u^*(v) = 0, \text{ for all } u^* \in U\}.$$

Of course $\mathcal{Z}(U) = U^0$, but the notion $\mathcal{Z}(U)$ can be generalized to more general kinds of equations, namely polynomial equations. In this more general setting, U is a set of *polynomials* in n variables with coefficients in a field K (where $n = \dim(E)$). Sets of the form $\mathcal{Z}(U)$ are called *algebraic varieties*. Linear forms correspond to the special case where homogeneous polynomials of degree 1 are considered.

If V is a subset of E, it is natural to associate with V the *set of polynomials in* $K[X_1, \ldots, X_n]$ *that vanish on* V. This set, usually denoted $\mathcal{I}(V)$, has some special properties that make it an *ideal*. If V is a linear subspace of E, it is natural to restrict our attention to the space V^0 of linear forms that vanish on V, and in this case we identify $\mathcal{I}(V)$ and V^0 (although technically, $\mathcal{I}(V)$ is no longer an ideal).

For any arbitrary set of polynomials $U \subseteq K[X_1, \ldots, X_n]$ (resp. subset $V \subseteq E$), the relationship between $\mathcal{I}(\mathcal{Z}(U))$ and U (resp. $\mathcal{Z}(\mathcal{I}(V))$ and V) is generally not simple, even though we always have

$$U \subseteq \mathcal{I}(\mathcal{Z}(U)) \quad (\text{resp.} \quad V \subseteq \mathcal{Z}(\mathcal{I}(V))).$$

However, when the field K is algebraically closed, then $\mathcal{I}(\mathcal{Z}(U))$ is equal to the *radical* of the ideal U, a famous result due to Hilbert known as the *Nullstellensatz* (see Lang [41] or Dummit and Foote [19]). The study of algebraic varieties is the main subject of *algebraic geometry*, a beautiful but formidable subject. For a taste of algebraic geometry, see Lang [41] or Dummit and Foote [19].

The duality theorem (Theorem 10.4) shows that the situation is much simpler if we restrict our attention to linear subspaces; in this case

$$U = \mathcal{I}(\mathcal{Z}(U)) \quad \text{and} \quad V = \mathcal{Z}(\mathcal{I}(V)).$$

Proposition 10.3. *We have $V \subseteq V^{00}$ for every subspace V of E, and $U \subseteq U^{00}$ for every subspace U of E^*.*

Proof. Indeed, for any $v \in V$, to show that $v \in V^{00}$ we need to prove that $u^*(v) = 0$ for all $u^* \in V^0$. However, V^0 consists of all linear forms u^* such that $u^*(y) = 0$ for *all* $y \in V$; in particular, for a fixed $v \in V$, we have $u^*(v) = 0$ for all $u^* \in V^0$, as required.

Similarly, for any $u^* \in U$, to show that $u^* \in U^{00}$ we need to prove that $u^*(v) = 0$ for all $v \in U^0$. However, U^0 consists of all vectors v such that $f^*(v) = 0$ for *all* $f^* \in U$; in particular, for a fixed $u^* \in U$, we have $u^*(v) = 0$ for all $v \in U^0$, as required. □

We will see shortly that in finite dimension, we have $V = V^{00}$ and $U = U^{00}$.

10.3 The Duality Theorem and Some Consequences

Given a vector space E of dimension $n \geq 1$ and a subspace U of E, by Theorem 2.13, every basis $(u_1, \ldots, u_m)$ of U can be extended to a basis $(u_1, \ldots, u_n)$ of E. We have the following important theorem adapted from E. Artin [2] (Chapter 1).

Theorem 10.4. *(Duality theorem) Let E be a vector space of dimension n. The following properties hold:*

(a) For every basis $(u_1, \ldots, u_n)$ of E, the family of coordinate forms $(u_1^*, \ldots, u_n^*)$ is a basis of E^* (called the dual basis of $(u_1, \ldots, u_n)$).

(b) For every subspace V of E, we have $V^{00} = V$.

(c) For every pair of subspaces V and W of E such that $E = V \oplus W$, with V of dimension m, for every basis $(u_1, \ldots, u_n)$ of E such that $(u_1, \ldots, u_m)$ is a basis of V and $(u_{m+1}, \ldots, u_n)$ is a basis of W, the family $(u_1^*, \ldots, u_m^*)$ is a basis of the orthogonal W^0 of W in E^*, so that

$$\dim(W) + \dim(W^0) = \dim(E).$$

Furthermore, we have $W^{00} = W$.

(d) For every subspace U of E^*, we have

$$\dim(U) + \dim(U^0) = \dim(E),$$

where U^0 is the orthogonal of U in E, and $U^{00} = U$.

Proof. (a) This part was proven in Theorem 2.20.

(b) By Proposition 10.3 we have $V \subseteq V^{00}$. If $V \neq V^{00}$, then let $(u_1, \ldots, u_p)$ be a basis of V^{00} such that $(u_1, \ldots, u_m)$ is a basis of V, with $m < p$. Since $u_{m+1} \in V^{00}$, u_{m+1} is orthogonal to every linear form in V^0. By definition we have $u^*_{m+1}(u_i) = 0$ for all $i = 1, \ldots, m$, and thus $u^*_{m+1} \in V^0$. However, $u^*_{m+1}(u_{m+1}) = 1$, contradicting the fact that u_{m+1} is orthogonal to every linear form in V^0. Thus, $V = V^{00}$.

(c) Every linear form $f^* \in W^0$ is orthogonal to every u_j for $j = m+1, \ldots, n$, and thus, $f^*(u_j) = 0$ for $j = m+1, \ldots, n$. For such a linear form $f^* \in W^0$, let

$$g^* = f^*(u_1)u^*_1 + \cdots + f^*(u_m)u^*_m.$$

We have $g^*(u_i) = f^*(u_i)$, for every i, $1 \leq i \leq m$. Furthermore, by definition, g^* vanishes on all u_j with $j = m+1, \ldots, n$. Thus, f^* and g^* agree on the basis $(u_1, \ldots, u_n)$ of E, and so $g^* = f^*$. This shows that $(u^*_1, \ldots, u^*_m)$ generates W^0, and since it is also a linearly independent family, $(u^*_1, \ldots, u^*_m)$ is a basis of W^0. It is then obvious that $\dim(W) + \dim(W^0) = \dim(E)$, and by Part (b), we have $W^{00} = W$.

(d) The only remaining fact to prove is that $U^{00} = U$. Let $(f^*_1, \ldots, f^*_m)$ be a basis of U. Note that the map $h \colon E \to K^m$ defined such that

$$h(v) = (f^*_1(v), \ldots, f^*_m(v))$$

for every $v \in E$ is a linear map, and that its kernel $\operatorname{Ker} h$ is precisely U^0. Then by Proposition 5.8,

$$n = \dim(E) = \dim(\operatorname{Ker} h) + \dim(\operatorname{Im} h) \leq \dim(U^0) + m,$$

since $\dim(\operatorname{Im} h) \leq m$. Thus, $n - \dim(U^0) \leq m$. By (c), we have $\dim(U^0) + \dim(U^{00}) = \dim(E) = n$, so we get $\dim(U^{00}) \leq m$. However, by Proposition 10.3 it is clear that $U \subseteq U^{00}$, which implies $m = \dim(U) \leq \dim(U^{00})$, so $\dim(U) = \dim(U^{00}) = m$, and we must have $U = U^{00}$. ☐

Part (a) of Theorem 10.4 shows that

$$\dim(E) = \dim(E^*),$$

and if $(u_1, \ldots, u_n)$ is a basis of E, then $(u^*_1, \ldots, u^*_n)$ is a basis of the dual space E^* called the *dual basis* of $(u_1, \ldots, u_n)$.

Define the function $\mathcal{E}$ ($\mathcal{E}$ for equations) from subspaces of E to subspaces of E^* and the function $\mathcal{Z}$ ($\mathcal{Z}$ for zeros) from subspaces of E^* to subspaces of E by

$$\begin{aligned} \mathcal{E}(V) &= V^0, \quad V \subseteq E \\ \mathcal{Z}(U) &= U^0, \quad U \subseteq E^*. \end{aligned}$$

By Parts (c) and (d) of Theorem 10.4,

$$\begin{aligned} (\mathcal{Z} \circ \mathcal{E})(V) &= V^{00} = V \\ (\mathcal{E} \circ \mathcal{Z})(U) &= U^{00} = U, \end{aligned}$$

so $\mathcal{Z} \circ \mathcal{E} = \mathrm{id}$ and $\mathcal{E} \circ \mathcal{Z} = \mathrm{id}$, and the maps $\mathcal{E}$ and $\mathcal{Z}$ are inverse bijections. These maps set up a *duality* between subspaces of E and subspaces of E^*. In particular, every subspace $V \subseteq E$ of dimension m is the set of common zeros of the space of linear forms (equations) V^0, which has dimension $n - m$. This confirms the claim we made about the dimension of the subpsace defined by a set of linear equations.

One should be careful that this bijection does not hold if E has infinite dimension. Some restrictions on the dimensions of U and V are needed.

Remark: However, even if E is infinite-dimensional, the identity $V = V^{00}$ holds for every subspace V of E. The proof is basically the same but uses an infinite basis of V^{00} extending a basis of V.

We now discuss some applications of the duality theorem.

Problem 1 . Suppose that V is a subspace of $\mathbb{R}^n$ of dimension m and that $(v_1, \ldots, v_m)$ is a basis of V. The problem is to find a basis of V^0.

We first extend $(v_1, \ldots, v_m)$ to a basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$, and then by part (c) of Theorem 10.4, we know that $(v_{m+1}^*, \ldots, v_n^*)$ is a basis of V^0.

Example 10.6. For example, suppose that V is the subspace of $\mathbb{R}^4$ spanned by the two linearly independent vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$

the first two vectors of the Haar basis in $\mathbb{R}^4$. The four columns of the Haar matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

form a basis of $\mathbb{R}^4$, and the inverse of W is given by

$$W^{-1} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix}.$$

Since the dual basis $(v_1^*, v_2^*, v_3^*, v_4^*)$ is given by the rows of W^{-1}, the last two rows of W^{-1},

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix},$$

form a basis of V^0. We also obtain a basis by rescaling by the factor 1/2, so the linear forms given by the row vectors

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

form a basis of V^0, the space of linear forms (linear equations) that vanish on the subspace V.

The method that we described to find V^0 requires first extending a basis of V and then inverting a matrix, but there is a more direct method. Indeed, let A be the $n \times m$ matrix whose columns are the basis vectors $(v_1, \ldots, v_m)$ of V. Then a linear form u represented by a row vector belongs to V^0 iff $uv_i = 0$ for $i = 1, \ldots, m$ iff

$$uA = 0$$

iff

$$A^\top u^\top = 0.$$

Therefore, all we need to do is to find a basis of the nullspace of $A^\top$. This can be done quite effectively using the reduction of a matrix to reduced row echelon form (rref); see Section 7.10.

Example 10.7. For example, if we reconsider the previous example, $A^\top u^\top = 0$ becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the rref of $A^\top$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

the above system is equivalent to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ u_3 + u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the free variables are associated with u_2 and u_4. Thus to determine a basis for the kernel of $A^\top$, we set $u_2 = 1, u_4 = 0$ and $u_2 = 0, u_4 = 1$ and obtain a basis for V^0 as

$$\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}.$$

Problem 2 . Let us now consider the problem of finding a basis of the hyperplane H in $\mathbb{R}^n$ defined by the equation

$$c_1x_1 + \cdots + c_nx_n = 0.$$

More precisely, if $u^*(x_1, \ldots, x_n)$ is the linear form in $(\mathbb{R}^n)^*$ given by $u^*(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$, then the hyperplane H is the kernel of u^*. Of course we assume that some c_j is nonzero, in which case the linear form u^* spans a one-dimensional subspace U of $(\mathbb{R}^n)^*$, and $U^0 = H$ has dimension $n-1$.

Since u^* is not the linear form which is identically zero, there is a smallest positive index $j \leq n$ such that $c_j \neq 0$, so our linear form is really $u^*(x_1, \ldots, x_n) = c_jx_j + \cdots + c_nx_n$. We claim that the following $n-1$ vectors (in $\mathbb{R}^n$) form a basis of H:

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ j-1 \\ j \\ j+1 \\ j+2 \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n-1 \end{array} \\
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & -c_{j+1}/c_j & -c_{j+2}/c_j & \ldots & -c_n/c_j \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\end{array}.$$

Observe that the $(n-1)\times(n-1)$ matrix obtained by deleting row j is the identity matrix, so the columns of the above matrix are linearly independent. A simple calculation also shows that the linear form $u^*(x_1, \ldots, x_n) = c_jx_j + \cdots + c_nx_n$ vanishes on every column of the above matrix. For a concrete example in $\mathbb{R}^6$, if $u^*(x_1, \ldots, x_6) = x_3 + 2x_4 + 3x_5 + 4x_6$, we obtain the basis for the hyperplane H of equation

$$x_3 + 2x_4 + 3x_5 + 4x_6 = 0$$

given by the following matrix:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -2 & -3 & -4 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Problem 3 . Conversely, given a hyperplane H in $\mathbb{R}^n$ given as the span of $n-1$ linearly vectors $(u_1, \ldots, u_{n-1})$, it is possible using determinants to find a linear form $(\lambda_1, \ldots, \lambda_n)$ that vanishes on H.

In the case $n = 3$, we are looking for a row vector $(\lambda_1, \lambda_2, \lambda_3)$ such that if

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

are two linearly independent vectors, then

$$\begin{pmatrix} u_1 & u_2 & u_2 \\ v_1 & v_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and the cross-product $u \times v$ of u and v given by

$$u \times v = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

is a solution. In other words, the equation of the plane spanned by u and v is

$$(u_2v_3 - u_3v_2)x + (u_3v_1 - u_1v_3)y + (u_1v_2 - u_2v_1)z = 0.$$

Problem 4 . Here is another example illustrating the power of Theorem 10.4. Let $E = \mathrm{M}_n(\mathbb{R})$, and consider the equations asserting that the sum of the entries in every row of a matrix $A \in \mathrm{M}_n(\mathbb{R})$ is equal to the same number. We have $n - 1$ equations

$$\sum_{j=1}^{n} (a_{ij} - a_{i+1j}) = 0, \quad 1 \leq i \leq n - 1,$$

and it is easy to see that they are linearly independent. Therefore, the space U of linear forms in E^* spanned by the above linear forms (equations) has dimension $n - 1$, and the space U^0 of matrices satisfying all these equations has dimension $n^2 - n + 1$. It is not so obvious to find a basis for this space.

We will now pin down the relationship between a vector space E and its bidual E^{**}.

10.4 The Bidual and Canonical Pairings

Proposition 10.5. *Let E be a vector space. The following properties hold:*

(a) The linear map $\mathrm{eval}_E \colon E \to E^{**}$ *defined such that*

$$\mathrm{eval}_E(v) = \mathrm{eval}_v \quad \textit{for all } v \in E,$$

that is, $\mathrm{eval}_E(v)(u^*) = \langle u^*, v \rangle = u^*(v)$ *for every $u^* \in E^*$, is injective.*

(b) *When E is of finite dimension n, the linear map* $\mathrm{eval}_E\colon E \to E^{**}$ *is an isomorphism (called the canonical isomorphism).*

Proof. (a) Let $(u_i)_{i\in I}$ be a basis of E, and let $v = \sum_{i\in I} v_i u_i$. If $\mathrm{eval}_E(v) = 0$, then in particular $\mathrm{eval}_E(v)(u_i^*) = 0$ for all u_i^*, and since

$$\mathrm{eval}_E(v)(u_i^*) = \langle u_i^*, v\rangle = v_i,$$

we have $v_i = 0$ for all $i \in I$, that is, $v = 0$, showing that $\mathrm{eval}_E\colon E \to E^{**}$ is injective.

If E is of finite dimension n, by Theorem 10.4, for every basis $(u_1, \ldots, u_n)$, the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space E^*, and thus the family $(u_1^{**}, \ldots, u_n^{**})$ is a basis of the bidual E^{**}. This shows that $\dim(E) = \dim(E^{**}) = n$, and since by Part (a), we know that $\mathrm{eval}_E\colon E \to E^{**}$ is injective, in fact, $\mathrm{eval}_E\colon E \to E^{**}$ is bijective (by Proposition 5.11). $\square$

When E is of finite dimension and $(u_1, \ldots, u_n)$ is a basis of E, in view of the canonical isomorphism $\mathrm{eval}_E\colon E \to E^{**}$, the basis $(u_1^{**}, \ldots, u_n^{**})$ of the bidual is *identified* with $(u_1, \ldots, u_n)$.

Proposition 10.5 can be reformulated very fruitfully in terms of pairings, a remarkably useful concept discovered by Pontrjagin in 1931 (adapted from E. Artin [2], Chapter 1). Given two vector spaces E and F over a field K, we say that a function $\varphi\colon E \times F \to K$ is *bilinear* if for every $v \in V$, the map $u \mapsto \varphi(u, v)$ (from E to K) is linear, and for every $u \in E$, the map $v \mapsto \varphi(u, v)$ (from F to K) is linear.

Definition 10.4. Given two vector spaces E and F over K, a *pairing between E and F* is a bilinear map $\varphi\colon E \times F \to K$. Such a pairing is *nondegenerate* iff

(1) for every $u \in E$, if $\varphi(u, v) = 0$ for all $v \in F$, then $u = 0$, and

(2) for every $v \in F$, if $\varphi(u, v) = 0$ for all $u \in E$, then $v = 0$.

A pairing $\varphi\colon E \times F \to K$ is often denoted by $\langle -, -\rangle\colon E \times F \to K$. For example, the map $\langle -, -\rangle\colon E^* \times E \to K$ defined earlier is a nondegenerate pairing (use the proof of (a) in Proposition 10.5). If $E = F$ and $K = \mathbb{R}$, any inner product on E is a nondegenerate pairing (because an inner product is positive definite); see Chapter 11. Other interesting nondegenerate pairings arise in exterior algebra and differential geometry.

Given a pairing $\varphi\colon E \times F \to K$, we can define two maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ as follows: For every $u \in E$, we define the linear form $l_\varphi(u)$ in F^* such that

$$l_\varphi(u)(y) = \varphi(u, y) \quad \text{for every } y \in F,$$

and for every $v \in F$, we define the linear form $r_\varphi(v)$ in E^* such that

$$r_\varphi(v)(x) = \varphi(x, v) \quad \text{for every } x \in E.$$

We have the following useful proposition.

Proposition 10.6. *Given two vector spaces E and F over K, for every nondegenerate pairing $\varphi\colon E \times F \to K$ between E and F, the maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are linear and injective. Furthermore, if E and F have finite dimension, then this dimension is the same and $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are bijections.*

Proof. The maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are linear because a pairing is bilinear. If $l_\varphi(u) = 0$ (the null form), then

$$l_\varphi(u)(v) = \varphi(u, v) = 0 \quad \text{for every } v \in F,$$

and since φ is nondegenerate, $u = 0$. Thus, $l_\varphi\colon E \to F^*$ is injective. Similarly, $r_\varphi\colon F \to E^*$ is injective. When F has finite dimension n, we have seen that F and F^* have the same dimension. Since $l_\varphi\colon E \to F^*$ is injective, we have $m = \dim(E) \leq \dim(F) = n$. The same argument applies to E, and thus $n = \dim(F) \leq \dim(E) = m$. But then, $\dim(E) = \dim(F)$, and $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are bijections. □

When E has finite dimension, the nondegenerate pairing $\langle -, - \rangle\colon E^* \times E \to K$ yields another proof of the existence of a natural isomorphism between E and E^{**}. When $E = F$, the nondegenerate pairing induced by an inner product on E yields a natural isomorphism between E and E^* (see Section 11.2).

We now show the relationship between hyperplanes and linear forms.

10.5 Hyperplanes and Linear Forms

Actually Proposition 10.7 below follows from Parts (c) and (d) of Theorem 10.4, but we feel that it is also interesting to give a more direct proof.

Proposition 10.7. *Let E be a vector space. The following properties hold:*

(a) Given any nonnull linear form $f^* \in E^*$, its kernel $H = \mathrm{Ker}\, f^*$ is a hyperplane.

(b) For any hyperplane H in E, there is a (nonnull) linear form $f^* \in E^*$ such that $H = \mathrm{Ker}\, f^*$.

(c) Given any hyperplane H in E and any (nonnull) linear form $f^* \in E^*$ such that $H = \mathrm{Ker}\, f^*$, for every linear form $g^* \in E^*$, $H = \mathrm{Ker}\, g^*$ iff $g^* = \lambda f^*$ for some $\lambda \neq 0$ in K.

Proof. (a) If $f^* \in E^*$ is nonnull, there is some vector $v_0 \in E$ such that $f^*(v_0) \neq 0$. Let $H = \mathrm{Ker}\, f^*$. For every $v \in E$, we have

$$f^*\left(v - \frac{f^*(v)}{f^*(v_0)} v_0\right) = f^*(v) - \frac{f^*(v)}{f^*(v_0)} f^*(v_0) = f^*(v) - f^*(v) = 0.$$

Thus,

$$v - \frac{f^*(v)}{f^*(v_0)} v_0 = h \in H,$$

and
$$v = h + \frac{f^*(v)}{f^*(v_0)} v_0,$$
that is, $E = H + Kv_0$. Also since $f^*(v_0) \neq 0$, we have $v_0 \notin H$, that is, $H \cap Kv_0 = 0$. Thus, $E = H \oplus Kv_0$, and H is a hyperplane.

(b) If H is a hyperplane, $E = H \oplus Kv_0$ for some $v_0 \notin H$. Then every $v \in E$ can be written in a unique way as $v = h + \lambda v_0$. Thus there is a well-defined function $f^* \colon E \to K$, such that, $f^*(v) = \lambda$, for every $v = h + \lambda v_0$. We leave as a simple exercise the verification that f^* is a linear form. Since $f^*(v_0) = 1$, the linear form f^* is nonnull. Also, by definition, it is clear that $\lambda = 0$ iff $v \in H$, that is, $\mathrm{Ker}\, f^* = H$.

(c) Let H be a hyperplane in E, and let $f^* \in E^*$ be any (nonnull) linear form such that $H = \mathrm{Ker}\, f^*$. Clearly, if $g^* = \lambda f^*$ for some $\lambda \neq 0$, then $H = \mathrm{Ker}\, g^*$. Conversely, assume that $H = \mathrm{Ker}\, g^*$ for some nonnull linear form g^*. From (a), we have $E = H \oplus Kv_0$, for some v_0 such that $f^*(v_0) \neq 0$ and $g^*(v_0) \neq 0$. Then observe that
$$g^* - \frac{g^*(v_0)}{f^*(v_0)} f^*$$
is a linear form that vanishes on H, since both f^* and g^* vanish on H, but also vanishes on Kv_0. Thus, $g^* = \lambda f^*$, with
$$\lambda = \frac{g^*(v_0)}{f^*(v_0)}.$$
$\square$

We leave as an exercise the fact that every subspace $V \neq E$ of a vector space E is the intersection of all hyperplanes that contain V. We now consider the notion of transpose of a linear map and of a matrix.

10.6 Transpose of a Linear Map and of a Matrix

Given a linear map $f \colon E \to F$, it is possible to define a map $f^\top \colon F^* \to E^*$ which has some interesting properties.

Definition 10.5. Given a linear map $f \colon E \to F$, the *transpose* $f^\top \colon F^* \to E^*$ *of* f is the linear map defined such that
$$f^\top(v^*) = v^* \circ f, \quad \text{for every } v^* \in F^*,$$
as shown in the diagram below:
$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "f^\top(v^*)"'] & F \arrow[d, "v^*"] \\ & K. \end{tikzcd}$$

Equivalently, the linear map $f^\top \colon F^* \to E^*$ is defined such that

$$\langle v^*, f(u)\rangle = \langle f^\top(v^*), u\rangle, \tag{$*$}$$

for all $u \in E$ and all $v^* \in F^*$.

It is easy to verify that the following properties hold:

$$\begin{aligned}(f+g)^\top &= f^\top + g^\top \\ (g \circ f)^\top &= f^\top \circ g^\top \\ \mathrm{id}_E^\top &= \mathrm{id}_{E^*}.\end{aligned}$$

Note the reversal of composition on the right-hand side of $(g \circ f)^\top = f^\top \circ g^\top$.

The equation $(g \circ f)^\top = f^\top \circ g^\top$ implies the following useful proposition.

Proposition 10.8. *If $f \colon E \to F$ is any linear map, then the following properties hold:*

(1) If f is injective, then $f^\top$ is surjective.

(2) If f is surjective, then $f^\top$ is injective.

Proof. If $f \colon E \to F$ is injective, then it has a retraction $r \colon F \to E$ such that $r \circ f = \mathrm{id}_E$, and if $f \colon E \to F$ is surjective, then it has a section $s \colon F \to E$ such that $f \circ s = \mathrm{id}_F$. Now if $f \colon E \to F$ is injective, then we have

$$(r \circ f)^\top = f^\top \circ r^\top = \mathrm{id}_{E^*},$$

which implies that $f^\top$ is surjective, and if f is surjective, then we have

$$(f \circ s)^\top = s^\top \circ f^\top = \mathrm{id}_{F^*},$$

which implies that $f^\top$ is injective. $\square$

The following proposition shows the relationship between orthogonality and transposition.

Proposition 10.9. *Given a linear map $f \colon E \to F$, for any subspace V of E, we have*

$$f(V)^0 = (f^\top)^{-1}(V^0) = \{w^* \in F^* \mid f^\top(w^*) \in V^0\}.$$

As a consequence,

$$\mathrm{Ker}\, f^\top = (\mathrm{Im}\, f)^0.$$

We also have

$$\mathrm{Ker}\, f = (\mathrm{Im}\, f^\top)^0.$$

Proof. We have
$$\langle w^*, f(v)\rangle = \langle f^\top(w^*), v\rangle,$$
for all $v \in E$ and all $w^* \in F^*$, and thus, we have $\langle w^*, f(v)\rangle = 0$ for every $v \in V$, i.e. $w^* \in f(V)^0$ iff $\langle f^\top(w^*), v\rangle = 0$ for every $v \in V$ iff $f^\top(w^*) \in V^0$, i.e. $w^* \in (f^\top)^{-1}(V^0)$, proving that
$$f(V)^0 = (f^\top)^{-1}(V^0).$$

Since we already observed that $E^0 = (0)$, letting $V = E$ in the above identity we obtain that
$$\operatorname{Ker} f^\top = (\operatorname{Im} f)^0.$$

From the equation
$$\langle w^*, f(v)\rangle = \langle f^\top(w^*), v\rangle,$$
we deduce that $v \in (\operatorname{Im} f^\top)^0$ iff $\langle f^\top(w^*), v\rangle = 0$ for all $w^* \in F^*$ iff $\langle w^*, f(v)\rangle = 0$ for all $w^* \in F^*$. Assume that $v \in (\operatorname{Im} f^\top)^0$. If we pick a basis $(w_i)_{i\in I}$ of F, then we have the linear forms $w_i^* \colon F \to K$ such that $w_i^*(w_j) = \delta_{ij}$, and since we must have $\langle w_i^*, f(v)\rangle = 0$ for all $i \in I$ and $(w_i)_{i\in I}$ is a basis of F, we conclude that $f(v) = 0$, and thus $v \in \operatorname{Ker} f$ (this is because $\langle w_i^*, f(v)\rangle$ is the coefficient of $f(v)$ associated with the basis vector w_i). Conversely, if $v \in \operatorname{Ker} f$, then $\langle w^*, f(v)\rangle = 0$ for all $w^* \in F^*$, so we conclude that $v \in (\operatorname{Im} f^\top)^0$. Therefore, $v \in (\operatorname{Im} f^\top)^0$ iff $v \in \operatorname{Ker} f$; that is,
$$\operatorname{Ker} f = (\operatorname{Im} f^\top)^0,$$
as claimed. ∎

The following theorem shows the relationship between the rank of f and the rank of $f^\top$.

Theorem 10.10. *Given a linear map $f \colon E \to F$, the following properties hold.*

(a) The dual $(\operatorname{Im} f)^*$ of $\operatorname{Im} f$ is isomorphic to $\operatorname{Im} f^\top = f^\top(F^*)$; that is,
$$(\operatorname{Im} f)^* \cong \operatorname{Im} f^\top.$$

(b) If F is finite dimensional, then $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$.

Proof. (a) Consider the linear maps
$$E \xrightarrow{p} \operatorname{Im} f \xrightarrow{j} F,$$
where $E \xrightarrow{p} \operatorname{Im} f$ is the surjective map induced by $E \xrightarrow{f} F$, and $\operatorname{Im} f \xrightarrow{j} F$ is the injective inclusion map of $\operatorname{Im} f$ into F. By definition, $f = j \circ p$. To simplify the notation, let $I = \operatorname{Im} f$. By Proposition 10.8, since $E \xrightarrow{p} I$ is surjective, $I^* \xrightarrow{p^\top} E^*$ is injective, and since $\operatorname{Im} f \xrightarrow{j} F$ is injective, $F^* \xrightarrow{j^\top} I^*$ is surjective. Since $f = j \circ p$, we also have
$$f^\top = (j \circ p)^\top = p^\top \circ j^\top,$$

and since $F^* \xrightarrow{j^\top} I^*$ is surjective, and $I^* \xrightarrow{p^\top} E^*$ is injective, we have an isomorphism between $(\mathrm{Im}\, f)^*$ and $f^\top(F^*)$.

(b) We already noted that Part (a) of Theorem 10.4 shows that $\dim(F) = \dim(F^*)$, for every vector space F of finite dimension. Consequently, $\dim(\mathrm{Im}\, f) = \dim((\mathrm{Im}\, f)^*)$, and thus, by Part (a) we have $\mathrm{rk}(f) = \mathrm{rk}(f^\top)$.

Remark: When both E and F are finite-dimensional, there is also a simple proof of (b) that doesn't use the result of Part (a). By Theorem 10.4(c)

$$\dim(\mathrm{Im}\, f) + \dim((\mathrm{Im}\, f)^0) = \dim(F),$$

and by Theorem 5.8

$$\dim(\mathrm{Ker}\, f^\top) + \dim(\mathrm{Im}\, f^\top) = \dim(F^*).$$

Furthermore, by Proposition 10.9, we have

$$\mathrm{Ker}\, f^\top = (\mathrm{Im}\, f)^0,$$

and since F is finite-dimensional $\dim(F) = \dim(F^*)$, so we deduce

$$\dim(\mathrm{Im}\, f) + \dim((\mathrm{Im}\, f)^0) = \dim((\mathrm{Im}\, f)^0) + \dim(\mathrm{Im}\, f^\top),$$

which yields $\dim(\mathrm{Im}\, f) = \dim(\mathrm{Im}\, f^\top)$; that is, $\mathrm{rk}(f) = \mathrm{rk}(f^\top)$. $\square$

The following proposition can be shown, but it requires a generalization of the duality theorem, so its proof is omitted.

Proposition 10.11. *If $f\colon E \to F$ is any linear map, then the following identities hold:*

$$\begin{aligned}
\mathrm{Im}\, f^\top &= (\mathrm{Ker}\,(f))^0 \\
\mathrm{Ker}\,(f^\top) &= (\mathrm{Im}\, f)^0 \\
\mathrm{Im}\, f &= (\mathrm{Ker}\,(f^\top))^0 \\
\mathrm{Ker}\,(f) &= (\mathrm{Im}\, f^\top)^0.
\end{aligned}$$

Observe that the second and the fourth equation have already be proven in Proposition 10.9. Since for any subspace $V \subseteq E$, even infinite-dimensional, we have $V^{00} = V$, the third equation follows from the second equation by taking orthogonals. Actually, the fourth equation follows from the first also by taking orthogonals. Thus the only equation to be proven is the first equation. We will give a proof later in the case where E is finite-dimensional (see Proposition 10.18).

The following proposition shows the relationship between the matrix representing a linear map $f\colon E \to F$ and the matrix representing its transpose $f^\top\colon F^* \to E^*$.

Proposition 10.12. *Let E and F be two vector spaces, and let $(u_1, \ldots, u_n)$ be a basis for E and $(v_1, \ldots, v_m)$ be a basis for F. Given any linear map $f \colon E \to F$, if $M(f)$ is the $m \times n$-matrix representing f w.r.t. the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$, then the $n \times m$-matrix $M(f^\top)$ representing $f^\top \colon F^* \to E^*$ w.r.t. the dual bases $(v_1^*, \ldots, v_m^*)$ and $(u_1^*, \ldots, u_n^*)$ is the transpose $M(f)^\top$ of $M(f)$.*

Proof. Recall that the entry $a_{i\,j}$ in row i and column j of $M(f)$ is the i-th coordinate of $f(u_j)$ over the basis $(v_1, \ldots, v_m)$. By definition of v_i^*, we have $\langle v_i^*, f(u_j)\rangle = a_{i\,j}$. The entry $a_{j\,i}^\top$ in row j and column i of $M(f^\top)$ is the j-th coordinate of

$$f^\top(v_i^*) = a_{1\,i}^\top u_1^* + \cdots + a_{j\,i}^\top u_j^* + \cdots + a_{n\,i}^\top u_n^*$$

over the basis $(u_1^*, \ldots, u_n^*)$, which is just $a_{j\,i}^\top = f^\top(v_i^*)(u_j) = \langle f^\top(v_i^*), u_j\rangle$. Since

$$\langle v_i^*, f(u_j)\rangle = \langle f^\top(v_i^*), u_j\rangle,$$

we have $a_{i\,j} = a_{j\,i}^\top$, proving that $M(f^\top) = M(f)^\top$. □

We now can give a very short proof of the fact that the rank of a matrix is equal to the rank of its transpose.

Proposition 10.13. *Given an $m \times n$ matrix A over a field K, we have* $\mathrm{rk}(A) = \mathrm{rk}(A^\top)$.

Proof. The matrix A corresponds to a linear map $f \colon K^n \to K^m$, and by Theorem 10.10, $\mathrm{rk}(f) = \mathrm{rk}(f^\top)$. By Proposition 10.12, the linear map $f^\top$ corresponds to $A^\top$. Since $\mathrm{rk}(A) = \mathrm{rk}(f)$, and $\mathrm{rk}(A^\top) = \mathrm{rk}(f^\top)$, we conclude that $\mathrm{rk}(A) = \mathrm{rk}(A^\top)$. □

Thus, given an $m \times n$-matrix A, the maximum number of linearly independent columns is equal to the maximum number of linearly independent rows. There are other ways of proving this fact that do not involve the dual space, but instead some elementary transformations on rows and columns.

Proposition 10.13 immediately yields the following criterion for determining the rank of a matrix:

Proposition 10.14. *Given any $m \times n$ matrix A over a field K (typically $K = \mathbb{R}$ or $K = \mathbb{C}$), the rank of A is the maximum natural number r such that there is an invertible $r \times r$ submatrix of A obtained by selecting r rows and r columns of A.*

For example, the 3×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

has rank 2 iff one of the three 2×2 matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \quad \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

is invertible.

If we combine Proposition 6.12 with Proposition 10.14, we obtain the following criterion for finding the rank of a matrix.

Proposition 10.15. *Given any $m \times n$ matrix A over a field K (typically $K = \mathbb{R}$ or $K = \mathbb{C}$), the rank of A is the maximum natural number r such that there is an $r \times r$ submatrix B of A obtained by selecting r rows and r columns of A, such that* $\det(B) \neq 0$.

This is not a very efficient way of finding the rank of a matrix. We will see that there are better ways using various decompositions such as LU, QR, or SVD.

10.7 Properties of the Double Transpose

First we have the following property showing the naturality of the eval map.

Proposition 10.16. *For any linear map $f \colon E \to F$, we have*

$$f^{\top\top} \circ \mathrm{eval}_E = \mathrm{eval}_F \circ f,$$

or equivalently the following diagram commutes:

$$\begin{tikzcd}
E^{**} \arrow[r, "f^{\top\top}"] & F^{**} \\
E \arrow[u, "\mathrm{eval}_E"] \arrow[r, "f"'] & F. \arrow[u, "\mathrm{eval}_F"']
\end{tikzcd}$$

Proof. For every $u \in E$ and every $\varphi \in F^*$, we have

$$\begin{aligned}
(f^{\top\top} \circ \mathrm{eval}_E)(u)(\varphi) &= \langle f^{\top\top}(\mathrm{eval}_E(u)), \varphi \rangle \\
&= \langle \mathrm{eval}_E(u), f^{\top}(\varphi) \rangle \\
&= \langle f^{\top}(\varphi), u \rangle \\
&= \langle \varphi, f(u) \rangle \\
&= \langle \mathrm{eval}_F(f(u)), \varphi \rangle \\
&= \langle (\mathrm{eval}_F \circ f)(u), \varphi \rangle \\
&= (\mathrm{eval}_F \circ f)(u)(\varphi),
\end{aligned}$$

which proves that $f^{\top\top} \circ \mathrm{eval}_E = \mathrm{eval}_F \circ f$, as claimed. $\square$

If E and F are finite-dimensional, then eval_E and eval_F are isomorphisms, so Proposition 10.16 shows that

$$f^{\top\top} = \mathrm{eval}_F \circ f \circ \mathrm{eval}_E^{-1}. \qquad (*)$$

The above equation is often interpreted as follows: if we identify E with its bidual E^{**} and F with its bidual F^{**}, then $f^{\top\top} = f$. This is an abuse of notation; the rigorous statement is $(*)$.

As a corollary of Proposition 10.16, we obtain the following result.

Proposition 10.17. *If* $\dim(E)$ *is finite, then we have*

$$\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f)).$$

Proof. Indeed, if E is finite-dimensional, the map $\mathrm{eval}_E\colon E \to E^{**}$ is an isomorphism, so every $\varphi \in E^{**}$ is of the form $\varphi = \mathrm{eval}_E(u)$ for some $u \in E$, the map $\mathrm{eval}_F\colon F \to F^{**}$ is injective, and we have

$$\begin{aligned} f^{\top\top}(\varphi) = 0 \quad & \text{iff} \quad f^{\top\top}(\mathrm{eval}_E(u)) = 0 \\ & \text{iff} \quad \mathrm{eval}_F(f(u)) = 0 \\ & \text{iff} \quad f(u) = 0 \\ & \text{iff} \quad u \in \mathrm{Ker}\,(f) \\ & \text{iff} \quad \varphi \in \mathrm{eval}_E(\mathrm{Ker}\,(f)), \end{aligned}$$

which proves that $\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f))$. □

Remarks: If $\dim(E)$ is finite, following an argument of Dan Guralnik, the fact that $\mathrm{rk}(f) = \mathrm{rk}(f^{\top})$ can be proven using Proposition 10.17.

Proof. We know from Proposition 10.9 applied to $f^{\top}\colon F^* \to E^*$ that

$$\mathrm{Ker}\,(f^{\top\top}) = (\mathrm{Im}\, f^{\top})^0,$$

and we showed in Proposition 10.17 that

$$\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f)).$$

It follows (since eval_E is an isomorphism) that

$$\dim((\mathrm{Im}\, f^{\top})^0) = \dim(\mathrm{Ker}\,(f^{\top\top})) = \dim(\mathrm{Ker}\,(f)) = \dim(E) - \dim(\mathrm{Im}\, f),$$

and since

$$\dim(\mathrm{Im}\, f^{\top}) + \dim((\mathrm{Im}\, f^{\top})^0) = \dim(E),$$

we get

$$\dim(\mathrm{Im}\, f^{\top}) = \dim(\mathrm{Im}\, f).$$ □

As indicated by Dan Guralnik, if $\dim(E)$ is finite, the above result can be used to prove the following result.

Proposition 10.18. *If* $\dim(E)$ *is finite, then for any linear map* $f\colon E \to F$*, we have*

$$\mathrm{Im}\, f^{\top} = (\mathrm{Ker}\,(f))^0.$$

Proof. From

$$\langle f^\top(\varphi), u\rangle = \langle \varphi, f(u)\rangle$$

for all $\varphi \in F^*$ and all $u \in E$, we see that if $u \in \mathrm{Ker}\,(f)$, then $\langle f^\top(\varphi), u\rangle = \langle \varphi, 0\rangle = 0$, which means that $f^\top(\varphi) \in (\mathrm{Ker}\,(f))^0$, and thus, $\mathrm{Im}\, f^\top \subseteq (\mathrm{Ker}\,(f))^0$. For the converse, since $\dim(E)$ is finite, we have

$$\dim((\mathrm{Ker}\,(f))^0) = \dim(E) - \dim(\mathrm{Ker}\,(f)) = \dim(\mathrm{Im}\, f),$$

but we just proved that $\dim(\mathrm{Im}\, f^\top) = \dim(\mathrm{Im}\, f)$, so we get

$$\dim((\mathrm{Ker}\,(f))^0) = \dim(\mathrm{Im}\, f^\top),$$

and since $\mathrm{Im}\, f^\top \subseteq (\mathrm{Ker}\,(f))^0$, we obtain

$$\mathrm{Im}\, f^\top = (\mathrm{Ker}\,(f))^0,$$

as claimed. $\square$

Remarks:

1. By the duality theorem, since $(\mathrm{Ker}\,(f))^{00} = \mathrm{Ker}\,(f)$, the above equation yields another proof of the fact that
$$\mathrm{Ker}\,(f) = (\mathrm{Im}\, f^\top)^0,$$
when E is finite-dimensional.

2. The equation
$$\mathrm{Im}\, f^\top = (\mathrm{Ker}\,(f))^0$$
is actually valid even if when E if infinite-dimensional, but we will not prove this here.

10.8 The Four Fundamental Subspaces

Given a linear map $f\colon E \to F$ (where E and F are finite-dimensional), Proposition 10.9 revealed that the four spaces

$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^\top,\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^\top$$

play a special role. They are often called the *fundamental subspaces* associated with f. These spaces are related in an intimate manner, since Proposition 10.9 shows that

$$\begin{aligned}\mathrm{Ker}\, f &= (\mathrm{Im}\, f^\top)^0\\ \mathrm{Ker}\, f^\top &= (\mathrm{Im}\, f)^0,\end{aligned}$$

and Theorem 10.10 shows that

$$\mathrm{rk}(f) = \mathrm{rk}(f^\top).$$

It is instructive to translate these relations in terms of matrices (actually, certain linear algebra books make a big deal about this!). If $\dim(E) = n$ and $\dim(F) = m$, given any basis $(u_1, \ldots, u_n)$ of E and a basis $(v_1, \ldots, v_m)$ of F, we know that f is represented by an $m \times n$ matrix $A = (a_{i\,j})$, where the jth column of A is equal to $f(u_j)$ over the basis $(v_1, \ldots, v_m)$. Furthermore, the transpose map $f^\top$ is represented by the $n \times m$ matrix $A^\top$ (with respect to the dual bases). Consequently, the four fundamental spaces

$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^\top,\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^\top$$

correspond to

(1) The *column space* of A, denoted by $\mathrm{Im}\, A$ or $\mathcal{R}(A)$; this is the subspace of $\mathbb{R}^m$ spanned by the columns of A, which corresponds to the image $\mathrm{Im}\, f$ of f.

(2) The *kernel* or *nullspace* of A, denoted by $\mathrm{Ker}\, A$ or $\mathcal{N}(A)$; this is the subspace of $\mathbb{R}^n$ consisting of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

(3) The *row space* of A, denoted by $\mathrm{Im}\, A^\top$ or $\mathcal{R}(A^\top)$; this is the subspace of $\mathbb{R}^n$ spanned by the rows of A, or equivalently, spanned by the columns of $A^\top$, which corresponds to the image $\mathrm{Im}\, f^\top$ of $f^\top$.

(4) The *left kernel* or *left nullspace* of A denoted by $\mathrm{Ker}\, A^\top$ or $\mathcal{N}(A^\top)$; this is the kernel (nullspace) of $A^\top$, the subspace of $\mathbb{R}^m$ consisting of all vectors $y \in \mathbb{R}^m$ such that $A^\top y = 0$, or equivalently, $y^\top A = 0$.

Recall that the dimension r of $\mathrm{Im}\, f$, which is also equal to the dimension of the column space $\mathrm{Im}\, A = \mathcal{R}(A)$, is the *rank* of A (and f). Then, some our previous results can be reformulated as follows:

1. The column space $\mathcal{R}(A)$ of A has dimension r.
2. The nullspace $\mathcal{N}(A)$ of A has dimension $n - r$.
3. The row space $\mathcal{R}(A^\top)$ has dimension r.
4. The left nullspace $\mathcal{N}(A^\top)$ of A has dimension $m - r$.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part I* (see Strang [64]).

The two statements

$$\begin{aligned} \mathrm{Ker}\, f &= (\mathrm{Im}\, f^\top)^0 \\ \mathrm{Ker}\, f^\top &= (\mathrm{Im}\, f)^0 \end{aligned}$$

translate to

(1) The nullspace of A is the orthogonal of the row space of A.

(2) The left nullspace of A is the orthogonal of the column space of A.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part II* (see Strang [64]).

Since vectors are represented by column vectors and linear forms by row vectors (over a basis in E or F), a vector $x \in \mathbb{R}^n$ is orthogonal to a linear form y iff

$$yx = 0.$$

Then, a vector $x \in \mathbb{R}^n$ is orthogonal to the row space of A iff x is orthogonal to every row of A, namely $Ax = 0$, which is equivalent to the fact that x belong to the nullspace of A. Similarly, the column vector $y \in \mathbb{R}^m$ (representing a linear form over the dual basis of F^*) belongs to the nullspace of $A^\top$ iff $A^\top y = 0$, iff $y^\top A = 0$, which means that the linear form given by $y^\top$ (over the basis in F) is orthogonal to the column space of A.

Since (2) is equivalent to the fact that the column space of A is equal to the orthogonal of the left nullspace of A, we get the following criterion for the solvability of an equation of the form $Ax = b$:

The equation $Ax = b$ has a solution iff for all $y \in \mathbb{R}^m$, if $A^\top y = 0$, then $y^\top b = 0$.

Indeed, the condition on the right-hand side says that b is orthogonal to the left nullspace of A; that is, b belongs to the column space of A.

This criterion can be cheaper to check that checking directly that b is spanned by the columns of A. For example, if we consider the system

$$\begin{aligned} x_1 - x_2 &= b_1 \\ x_2 - x_3 &= b_2 \\ x_3 - x_1 &= b_3 \end{aligned}$$

which, in matrix form, is written $Ax = b$ as below:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we see that the rows of the matrix A add up to 0. In fact, it is easy to convince ourselves that the left nullspace of A is spanned by $y = (1, 1, 1)$, and so the system is solvable iff $y^\top b = 0$, namely

$$b_1 + b_2 + b_3 = 0.$$

Note that the above criterion can also be stated negatively as follows:

The equation $Ax = b$ has no solution iff there is some $y \in \mathbb{R}^m$ such that $A^\top y = 0$ and $y^\top b \neq 0$.

Since $A^\top y = 0$ iff $y^\top A = 0$, we can view $y^\top$ as a row vector representing a linear form, and $y^\top A = 0$ asserts that the linear form $y^\top$ vanishes on the columns $A^1, \ldots, A^n$ of A but does not vanish on b. Since the linear form $y^\top$ defines the hyperplane H of equation $y^\top z = 0$ (with $z \in \mathbb{R}^m$), geometrically the equation $Ax = b$ has no solution iff there is a hyperplane H containing $A^1, \ldots, A^n$ and not containing b.

10.9 Summary

The main concepts and results of this chapter are listed below:

- The *dual space* E^* and *linear forms* (*covector*). The *bidual* E^{**}.
- The *bilinear pairing* $\langle -, - \rangle \colon E^* \times E \to K$ (the *canonical pairing*).
- *Evaluation at* v: $\mathrm{eval}_v \colon E^* \to K$.
- The map $\mathrm{eval}_E \colon E \to E^{**}$.
- *Othogonality* between a subspace V of E and a subspace U of E^*; the *orthogonal* V^0 and the *orthogonal* U^0.
- *Coordinate forms*.
- The *Duality theorem* (Theorem 10.4).
- The *dual basis* of a basis.
- The isomorphism $\mathrm{eval}_E \colon E \to E^{**}$ when $\dim(E)$ is finite.
- *Pairing* between two vector spaces; *nondegenerate pairing*; Proposition 10.6.
- Hyperplanes and linear forms.
- The *transpose* $f^\top \colon F^* \to E^*$ of a linear map $f \colon E \to F$.
- The fundamental identities:
$$\mathrm{Ker}\, f^\top = (\mathrm{Im}\, f)^0 \quad \text{and} \quad \mathrm{Ker}\, f = (\mathrm{Im}\, f^\top)^0$$
(Proposition 10.9).
- If F is finite-dimensional, then
$$\mathrm{rk}(f) = \mathrm{rk}(f^\top).$$
(Theorem 10.10).
- The matrix of the transpose map $f^\top$ is equal to the transpose of the matrix of the map f (Proposition 10.12).
- For any $m \times n$ matrix A,
$$\mathrm{rk}(A) = \mathrm{rk}(A^\top).$$
- Characterization of the rank of a matrix in terms of a maximal invertible submatrix (Proposition 10.14).

- The *four fundamental subspaces*:

$$\operatorname{Im} f,\ \operatorname{Im} f^\top,\ \operatorname{Ker} f,\ \operatorname{Ker} f^\top.$$

- The *column space*, the *nullspace*, the *row space*, and the *left nullspace* (of a matrix).
- Criterion for the solvability of an equation of the form $Ax = b$ in terms of the left nullspace.

10.10 Problems

Problem 10.1. Prove the following properties of transposition:

$$\begin{aligned}(f+g)^\top &= f^\top + g^\top\\ (g\circ f)^\top &= f^\top \circ g^\top\\ \mathrm{id}_E^\top &= \mathrm{id}_{E^*}.\end{aligned}$$

Problem 10.2. Let $(u_1, \ldots, u_{n-1})$ be $n-1$ linearly independent vectors $u_i \in \mathbb{C}^n$. Prove that the hyperlane H spanned by $(u_1, \ldots, u_{n-1})$ is the nullspace of the linear form

$$x \mapsto \det(u_1, \ldots, u_{n-1}, x), \quad x \in \mathbb{C}^n.$$

Prove that if A is the $n \times n$ matrix whose columns are $(u_1, \ldots, u_{n-1}, x)$, and if $c_i = (-1)^{i+n}\det(A_{in})$ is the cofactor of $a_{in} = x_i$ for $i = 1, \ldots, n$, then H is defined by the equation

$$c_1x_1 + \cdots + c_nx_n = 0.$$

Problem 10.3. (1) Let $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the map defined by

$$\varphi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1y_1 + \cdots + x_ny_n.$$

Prove that φ is a bilinear nondegenerate pairing. Deduce that $(\mathbb{R}^n)^*$ is isomorphic to $\mathbb{R}^n$.
Prove that $\varphi(x, x) = 0$ iff $x = 0$.
(2) Let $\varphi_L\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be the map defined by

$$\varphi_L((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, , y_4)) = x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4.$$

Prove that φ is a bilinear nondegenerate pairing.
Show that there exist nonzero vectors $x \in \mathbb{R}^4$ such that $\varphi_L(x, x) = 0$.

Remark: The vector space $\mathbb{R}^4$ equipped with the above bilinear form called the *Lorentz form* is called *Minkowski space*.

Problem 10.4. Given any two subspaces V_1, V_2 of a finite-dimensional vector space E, prove that

$$\begin{aligned}(V_1 + V_2)^0 &= V_1^0 \cap V_2^0\\ (V_1 \cap V_2)^0 &= V_1^0 + V_2^0.\end{aligned}$$

Beware that in the second equation, V_1 and V_2 are subspaces of E, not E^*.

Hint. To prove the second equation, prove the inclusions $V_1^0 + V_2^0 \subseteq (V_1 \cap V_2)^0$ and $(V_1 \cap V_2)^0 \subseteq V_1^0 + V_2^0$. Proving the second inclusion is a little tricky. First, prove that we can pick a subspace W_1 of V_1 and a subspace W_2 of V_2 such that

1. V_1 is the direct sum $V_1 = (V_1 \cap V_2) \oplus W_1$.
2. V_2 is the direct sum $V_2 = (V_1 \cap V_2) \oplus W_2$.
3. $V_1 + V_2$ is the direct sum $V_1 + V_2 = (V_1 \cap V_2) \oplus W_1 \oplus W_2$.

Problem 10.5. (1) Let A be any $n \times n$ matrix such that the sum of the entries of every row of A is the same (say c_1), and the sum of entries of every column of A is the same (say c_2). Prove that $c_1 = c_2$.

(2) Prove that for any $n \geq 2$, the $2n - 2$ equations asserting that the sum of the entries of every row of A is the same, and the sum of entries of every column of A is the same are lineary independent. For example, when $n = 4$, we have the following 6 equations

$$\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0\\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0\\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0\\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0\\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0\\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0.
\end{aligned}$$

Hint. Group the equations as above; that is, first list the $n - 1$ equations relating the rows, and then list the $n - 1$ equations relating the columns. Prove that the first $n - 1$ equations are linearly independent, and that the last $n - 1$ equations are also linearly independent. Then, find a relationship between the two groups of equations that will allow you to prove that they span subspace V^r and V^c such that $V^r \cap V^c = (0)$.

(3) Now consider *magic squares*. Such matrices satisfy the two conditions about the sum of the entries in each row and in each column to be the same number, and also the additional two constraints that the main descending and the main ascending diagonals add up to this common number. Traditionally, it is also required that the entries in a magic square are positive integers, but we will consider generalized magic square with arbitrary real entries. For example, in the case $n = 4$, we have the following system of 8 equations:

$$\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0\\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0\\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0\\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0\\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0\\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0\\
a_{22} + a_{33} + a_{44} - a_{12} - a_{13} - a_{14} &= 0\\
a_{41} + a_{32} + a_{23} - a_{11} - a_{12} - a_{13} &= 0.
\end{aligned}$$

In general, the equation involving the descending diagonal is

$$a_{22} + a_{33} + \cdots + a_{nn} - a_{12} - a_{13} - \cdots - a_{1n} = 0 \tag{r}$$

and the equation involving the ascending diagonal is

$$a_{n1} + a_{n-12} + \cdots + a_{2n-1} - a_{11} - a_{12} - \cdots - a_{1n-1} = 0. \tag{c}$$

Prove that if $n \geq 3$, then the $2n$ equations asserting that a matrix is a generalized magic square are linearly independent.
Hint. Equations are really linear forms, so find some matrix annihilated by all equations except equation r, and some matrix annihilated by all equations except equation c.

Problem 10.6. Let $U_1, \ldots, U_p$ be some subspaces of a vector space E, and assume that they form a direct sum $U = U_1 \oplus \cdots \oplus U_p$. Let $j_i \colon U_i \to U_1 \oplus \cdots \oplus U_p$ be the canonical injections, and let $\pi_i \colon U_1^* \times \cdots \times U_p^* \to U_i^*$ be the canonical projections. Prove that there is an isomorphism f from $(U_1 \oplus \cdots \oplus U_p)^*$ to $U_1^* \times \cdots \times U_p^*$ such that

$$\pi_i \circ f = j_i^\top, \quad 1 \leq i \leq p.$$

Problem 10.7. Let U and V be two subspaces of a vector space E such that $E = U \oplus V$. Prove that

$$E^* = U^0 \oplus V^0.$$