Complexity Results for MPE in SPNs

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1 Approximating MPE is hard

In complexity theory, the class APX is the set of NP optimization problems that allow polynomial-time approximation algorithms with approximation ratio bounded by a constant. We show that the MPE problem in SPNs is not in APX unless $P = NP$.

The Maximum Independent Set (MAX-IS) problem is not in APX unless $P = NP$. Without loss of generality, we assume the graph of MAX-IS has no isolated vertex in the following discussion. We construct an auxiliary SPN to show that if MPE is in APX, then MAX-IS is also in APX.

Definition 1.1 (Auxiliary SPN)**.** Let graph $G(V, E)$ be an arbitrary instance of MAX-IS. The *auxiliary SPN* S corresponding to G is constructed as follows.

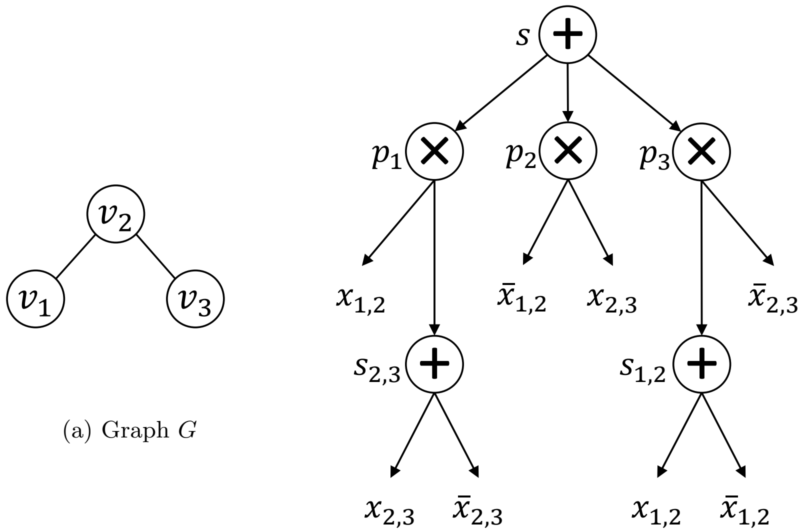
1. The root of S is a sum node s.
2. $\forall v_i \in V$, there is a product node p_i linked by s.
3. $\forall (v_i, v_j) \in E$, there is a Boolean variable $X_{i,j}$, with $x_{i,j}$ denoting $true$ linked by p_i and $\overline{x}_{i,j}$ denoting $false$ linked by p_j.
4. $\forall X_{i,j}, \forall p_k$, if $X_{i,j} \notin scope(p_k)$, we link p_k to $s_{i,j}$ where $s_{i,j}$ is a sum node linking to $x_{i,j}$ and $\overline{x}_{i,j}$.

The weight of every edge from sum nodes is 1.

Figure 1 shows an example of the definition 1.1.

Lemma 1.1. *Let graph $G(V, E)$ be an arbitrary instance of MAX-IS and let S be the corresponding auxiliary SPN.*

1. *Given any independent set of size k in G, we can find an $x \in val(X)$ such that $S(x) \geq k$.*
2. *Given any $x \in val(X)$ for which $S(x) = k$, we can find an independent set of size k in G.*



(a) Graph G

(b) Auxiliary SPN S

Figure 1: An instance of MAX-IS and the corresponding auxiliary SPN

Proof. **Part 1:** Let V' be the given independent set. $\forall (v_i, v_j) \in E$, let

$$X_{i,j} = \begin{cases} true, & \text{if } v_i \in V', \\ false, & \text{if } v_j \in V', \\ \text{arbitrary value}, & \text{otherwise.} \end{cases}$$

It is impossible that both $v_i, v_j \in V'$, since $(v_i, v_j) \in E$ and V' is an independent set. Thus $\forall v_i \in V', p_i(x) = 1$ where x is the corresponding assignment of X. Thus $S(x) \geq |V'| = k$.

Part 2: Let x be the given assignment of X. Since $p_i(x)$ is either 0 or 1 and $\sum p_i(x) = k$, there are k product nodes $p_i(x) = 1$. We construct an independent set $V' = \{v_i \in V | p_i(x) = 1\}$. So, $|V'| = k$. We show that V' is an independent set. $\forall v_i, v_j \in V'$, if $(v_i, v_j) \in E$, there exists a variable $X_{i,j}$ with $x_{i,j}$ linked by p_i and $\overline{x}_{i,j}$ linked by p_j. That means $p_i(x)$ and $p_j(x)$ cannot both be 1. This is conflict with $p_i(x) = 1$ and $p_j(x) = 1$. Therefore, V' is an independent set with size k. □

Corollary 1.1.1. *Let G be an arbitrary instance of MAX-IS and let S be the MPE instance of the corresponding auxiliary SPN. Let k^* and k'^* be the optimal solution of G and S. Then, $k^* = k'^*$.*

Lemma 1.2. *If there is an approximation algorithm for MPE with a constant ratio ρ in polynomial time, then there is an approximation algorithm for MAX-IS with the same constant ratio ρ in polynomial time.*

Proof. Let G be an arbitrary instance of MAX-IS and let S be the MPE instance of the corresponding auxiliary SPN. Let k^* be the optimal solution of S. Suppose we can approximate S with a bound $\rho = k/k^*$, i.e., we can find a solution $k = \rho \times k^*$ for S. Furthermore, we can find a solution $k' = k$ for G. The approximation ratio of G is $\rho' = k'/k^* = k/k^* = \rho$. Finally, all the above works are in polynomial time. $\square$

Corollary 1.2.1. *If MPE is in APX, then MAX-IS is also in APX.*

Theorem 1.3. *MPE is not in APX unless $P = NP$.*

Proof. Suppose $P \neq NP$. Suppose MPE is in APX. Thus MAX-IS is in APX. This is conflict with the fact MAX-IS is not in APX if $P \neq NP$. $\square$