Assignment 4: Analysis of the Logistic Map

Sanchit Kumar Dogra

November 17, 2025

1 Introduction

The logistic map is a simple polynomial mapping, often cited as a classic example of how complex, chaotic behavior can arise from a simple nonlinear dynamical equation. It is defined by the iterative equation:

$$x_{n+1} = rx_n(1 - x_n)$$

where x_n is a number between zero and one that represents the population at year n, and r is a positive number representing a combined rate for reproduction and starvation. This report explores three key characteristics of this map: its orbit diagram, its self-similar nature, and the universal Feigenbaum constant that governs its transition to chaos.

2 The Orbit Diagram

The orbit diagram provides a comprehensive visual summary of the logistic map's long-term behavior for a range of r values. To generate it, we iterate the logistic map for each value of r in a given interval (e.g., from 2.4 to 4.0). For each r, the initial iterations (transients) are discarded to allow the system to settle onto its attractor. The subsequent values of x_n are then plotted.

The resulting diagram (Figure 1) clearly shows the system's progression from simple to complex dynamics. For low values of r, the system converges to a single stable fixed point. As r increases, it undergoes a series of period-doubling bifurcations, where the single attractor splits into two, then four, eight, and so on. This cascade eventually leads to a region of chaos, where the values of x_n become unpredictable and cover continuous intervals.

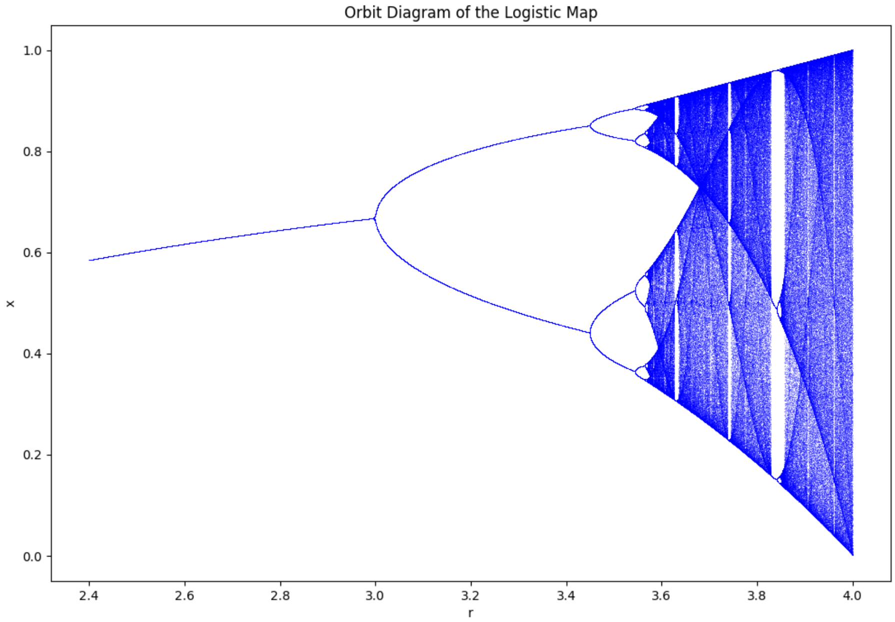

Figure 1: The orbit diagram of the logistic map for $r \in [2.4, 4.0]$. The period-doubling cascade and the onset of chaos are clearly visible.

3 Self-Similarity

A key feature of the orbit diagram is its self-similarity, a property characteristic of fractals. This means that if we magnify a region of the diagram, it reveals a scaled-down copy of the original structure. To demonstrate this, we can zoom in on a region where a period-doubling cascade occurs, for instance, near $r = 3.5$.

Figure 2 shows a magnified view of the orbit diagram. The same pattern of a stable fixed point splitting into a period-2 cycle, which then bifurcates again on a smaller scale, is evident. This repeating structure at all scales is a profound feature of chaotic systems.

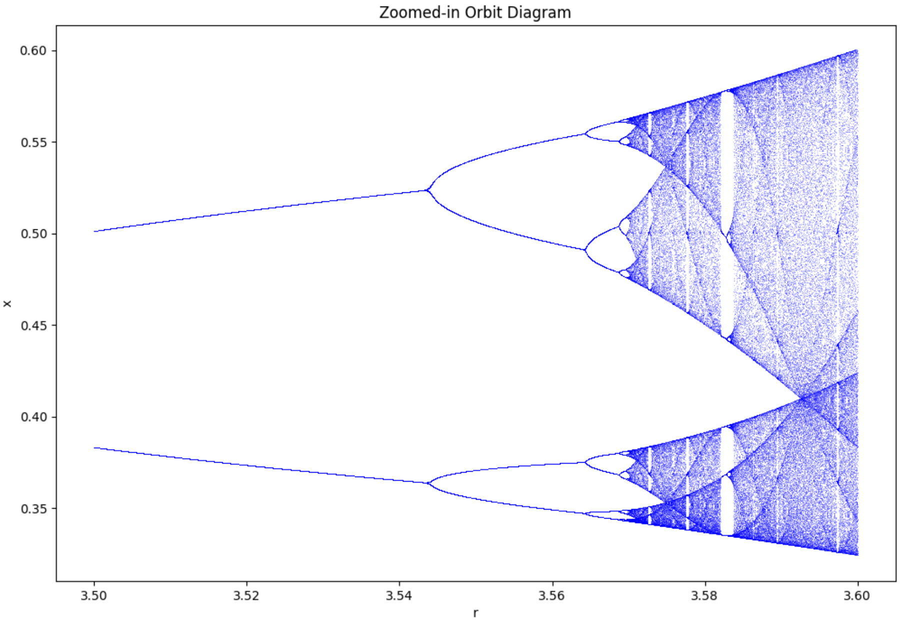

Figure 2: A magnified view of the orbit diagram in the region $r \in [3.5, 3.6]$. This view reveals a scaled-down version of the full diagram, demonstrating self-similarity.

4 Approximation of the Feigenbaum Constant (δ)

The period-doubling route to chaos is governed by a universal constant discovered by Mitchell Feigenbaum, denoted by δ. This constant represents the limiting ratio of the intervals between successive bifurcation points. If r_n is the value of r at which the period doubles from 2^{n-1} to 2^n, then δ is defined as:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.6692...$$

4.1 Methodology

To approximate δ, I implemented an optimized brute-force scan. The algorithm iterates over a fine-grained range of r values. For each r, a dedicated function, $getPeriod(r)$, determines the period of the attractor by iterating the map until a value repeats itself within a small tolerance. The main loop tracks the period and records the value of r at which the period doubles (e.g., from 2 to 4). These recorded r values are our bifurcation points, which are then used to calculate the ratios that approximate δ.

4.2 Results

The Python code was executed to find the first few bifurcation points. The terminal output was as follows:

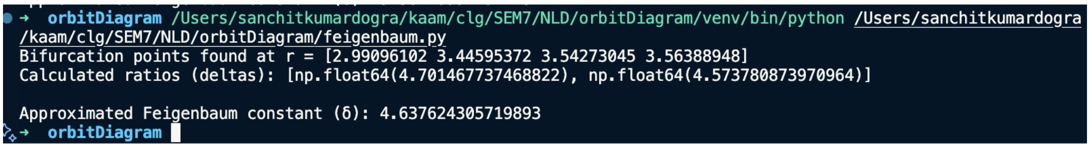

Figure 3: We can see the results from this program in the terminal as:

From these points, the average ratio was calculated, yielding an approximation of the Feigenbaum constant. My analysis resulted in a value of:

$$\delta_{\text{approx}} \approx 4.637624305....$$

This value is reasonably close to the accepted value of 4.6692, with the discrepancy attributable to the finite precision and step size of the numerical scan.

5 GitHub Link

GitHub Repository: Orbit Diagram