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The analysis of (n,k) nonlinear feedback shift registers via semi-tensor product

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Abstract: In this paper, the (n,k) nonlinear feedback shift register (NFSR) is regarded as a Boolean network (BN). Semi-tensor product (STP) of matrices is used to convert (n,k) NFSR into an equivalent algebraic equation. Based on STP of matrices, a novel way is proposed to study stability of (n,k) NFSR and the periodicity of (n,k) NFSR. Firstly, the stability of the (n,k) NFSR is investigated, and we propose an algorithm to judge the stability of an (n,k) NFSR. Secondly, we reveal relationship between the minimal period of output sequence of a cycle for an (n,k) NFSR and the length of the cycle. Thirdly, we investigate the period of an (n,k) NFSR. At last, the period of a composed (n,k) NFSR is investigated. Some existing methods can only be used to investigate the cycle of the (n,k) NFSR, while in this paper, we can simultaneously investigate stability of an (n,k) NFSR and the period of (n,k) NFSR and composed (n,k) NFSR by using the method of STP.

1 introduction

Information security is very important for our society. There are many confidential information about financial status, research, products for different organizations. In order to protect the confidential information from leakage, the stream cipher has been widely used in encryption [1]. The main building blocks of stream cipher is nonlinear feedback shift registers (NFSR). NFSR can produce pseudo-random sequences.

There are two types of NFSR: (1) Fibonacci NFSR; (2) Galois NFSR. Fibonacci NFSR is shown in Fig.1. Fibonacci NFSR consists of n binary storage elements from left to right as $n-1,n-2,...,1,0$. The value of the i-th $(0 \leq i \leq n-2)$ bit is updated by the value of $(i+1)$-th bit, and the value of $(n-1)$-th bit is updated by feedback function f which depends on values of bits of $0,1,...,n-1$.While the Galois NFSR shown in Fig.2 is different from Fibonacci NFSR. In Galois NFSR, the value of every bit is updated by its own feedback function which at most relate to every bit. In order to improve the speed of output sequence generation, E. Dubrova proposed a new type of NFSR named (n,k) NFSR in [2]. The period of Fibonacci NFSR is equal to the largest length of its state cycle. An (n,k) NFSR can be considered as a generalization of the Galois NFSR. In an (n,k) NFSR, every bit is updated by a feedback function, which is a nonlinear function relating to value of $(i+1)mod\ n$-th bit and up to values of $k-1$ other bits. E. Dubrova also provided a method to increase the period of output sequence of an (n,k) NFSR by composing m smaller (n_i,k_i) $(1 \leq i \leq m)$ NFSRs. But there is a weakness for the (n,k) NFSR. The period of an (n,k) NFSR is not necessary equal to the length of longest cycle of (n,k) NFSR, and this would cause some problems to observe the period of output sequences of an (n,k) NFSR. The advantages of (n,k) NFSR are shown as follows:

- (n,k) NFSR can potentially improve the speed of the generation of the pseudo-random sequence.
- (n,k) NFSR can generate the pseudo-random sequence with good statistical properties which can not be generated by Fibonacci NFSR.

In order to solve the above mentioned problems, in this paper, we provide a new method to investigate the period of output sequences of a cycle of an (n,k) NFSR by using the method of semi-tensor product (STP) of matrices. The method of STP was proposed by Cheng *et al.* in [3][4][5]. After the proposal of method of STP, it has been used in many fields, such as large-scale systems [6][7], graph coloring [8][9], Petri nets [10], Boolean networks (BNs) [11][12] and NFSR [13][14]. By using the method of STP, some fundamental problems of BNs have been solved, such as synchronization [15][16], stability [17][18][12], stabilization[19][20], controllability [21][22][23][24][25][30], observability [26][27] and so on. For example, in [24], Controllability of probabilistic Boolean control networks is investigated based on transition probability matrices. In [28], the pinning controllability of Boolean control networks has been studied. Based on the method of STP, many problems have been solved in finite-value system.

In this paper, we treat the (n,k) NFSR as a Boolean network (BN). The (n,k) NFSR can be transformed into a finite-value linear system by using the method of STP. In [13], Zhong *et al.* investigated stability of NFSR and the cycle of NFSR by using the method of STP. In [29], based on STP, Cheng *et al.* studied the stability of BNs. The method of STP is powerful to deal with the system with finite value. Based on the method of STP, the (n,k) NFSR can be an algebraic linear system. STP is another powerful method to investigate stability and period of (n,k) NFSR.

In the following, we will first define the minimal period of output sequence of a cycle for an (n,k) NFSR to better study the output sequence of an (n,k) NFSR. The method of STP is used to investigate the stability of (n,k) NFSR. We focus on the stability of (n,k) NFSR, the period of output sequence of (n,k) NFSR, and the period of output sequence of (n,k) NFSR composed by m smaller (n_i,k_i), $1 \leq i \leq m$ NFSRs. The main contributions of this paper are listed as follows:

- We find out the relationship between the minimal period of output sequence of a cycle of an (n,k) NFSR and the length of the cycle.
- We propose an algorithm to judge the stability of an (n,k) NFSR.
- We propose an new algorithm to find all cycles of an (n,k) NFSR.
- We propose an algorithm to find the period of (n,k) NFSR and composed (n,k) NFSR.

The remainder of this paper is organized as follows. Section 2 gives some preliminaries on STP and (n,k) NFSR. Based on the

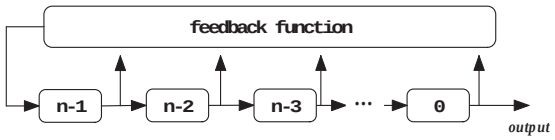

Fig. 1: The Fibonacci NLFSR

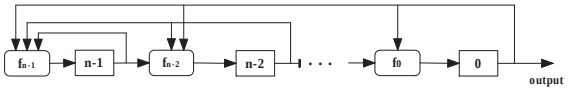

Fig. 2: The Fibonacci NLFSRs

method of STP, we obtain the multi-linear forms of (n,k) NFSR. Section 3 investigates the stability of (n,k) NFSR, the minimal period of output sequence of a cycle of an (n,k) NFSR, and the minimal period of output sequence of a cycle of an composed (n,k) NFSR. In Section 4, three examples are given to illustrate our theoretical results. At last, a conclusion is given.

2 preliminaries

In this section, we first review the STP of matrices briefly. Then the multi-linear form of Boolean function that is obtained by using the STP is recalled. Finally, we regard (n,k) NFSRs as a BN. Based on STP, (n,k) NFSRs are converted into a multi-linear form. We first give some notations which will be used in this paper.

- $\mathscr{D} = \{0,1\}$.
- I_n : the identity matrix of dimension n.
- δ_n^i : the i-th column of identity matrix I_n.
- $\Delta_{2^n} = \{\delta_{2^n}^i | i = 1,2,3,...,2^n\}$.
- $\mathscr{L}_{n\times m}$: the set of $n \times m$ matrices, whose column belong to Δ_n. For a matrix $L \in (L)_{n\times m}$, and $L = [\delta_n^{i_1}\ \delta_n^{i_2}\ ...\ \delta_n^{i_m}]$, we write $L = \delta_{2^n}[i_1\ i_2\ ...\ i_m]$ for simplicity.
- $col_i(L)$: the i-th column of matrix L.
- $col(L)$: the set of all column of matrix L.
- $\mathbb{R}$: the set of all real number.
- N : the set of all positive integers.
- $\star$: the least common multiple.
- $\diamond$: the greatest common divisor.
- $\times$: Cartesian product.
- $|S|$: the number of elements in set S.

2.1 Semi-tensor product of matrices

In this subsection, the definition of STP is given. The multi-linear form of nonlinear Boolean function is obtained by using the method of STP.

Definition 1. *[5] Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{p\times q}$. The STP of A and B is defined as:*

$$A \ltimes B = (A \otimes I_{\frac{l}{m}})(B \otimes I_{\frac{l}{p}}) \tag{1}$$

where l is the least common multiple of m and p.

Obviously, if $m = p$ in Definition 1, then the STP of A and B is reduced to their conventional matrix product AB.

We identify $\Delta_2 \sim \mathscr{D}$ i.e $(\delta_2^1 \sim 1, \delta_2^2 \sim 0)$, and $\delta_2^1(\delta_2^2)$ is called the vector form of logical value $1(0)$.

Lemma 1. *[5] Any Boolean function $f(x_1,x_2,...,x_n)$ with variables $x_1,x_2,...,x_n \in \Delta_2$ can be expressed as a multi-linear form:*

$$f(x_1,x_2,...,x_n) = Fx_1 \ltimes x_2 \ltimes ... \ltimes x_n. \tag{2}$$

where $F \in \mathscr{L}_{2\times 2^n}$ is called the structure matrix of f, and F can be uniquely expressed as

$$F = \begin{bmatrix} s_1 & s_2 & ... & s_{2^n} \\ 1-s_1 & 1-s_2 & ... & 1-s_{2^n} \end{bmatrix} \tag{3}$$

with $[s_1,s_2,...,s_{2^n}]$ being the truth table of f, arranged in the reverse alphabet order.

In the following, we omit the symbol $\ltimes$ for simplicity.

2.2 (n,k) NFSR

An (n,k) NFSR consists of n binary memories device called bits. The output of an (n,k) NFSR is the value of the 0-th bit. An (n,k) NFSR is defined as follows. Let $x_i(t),\ 0 \le i \le n-1$ be variable representing the value of the i-th bit at time t. Let $f_i : \mathscr{D}^k \to \mathscr{D}$, $1 \le k \le n$, be the next value function of i-th bit. The Boolean function f_i depends on the bit $(i+1) mod\ n$ and up to $k-1$ other bits. Assume the indexes of other $k-1$ bits relating to function f_i are $i_1,i_2,...,i_{k-1}$, $i_j \in \{0,1,...,n-1\}$, $j \in 1,2,...,k-1$, then the value function of i-th bit can be expressed as follows:

$$x_i(t+1) = f_i(x_{(i+1)mod\ n}(t), x_{i_1}(t),...,x_{i_{k-1}}(t)). \tag{4}$$

Thus (n,k) NFSR can be described as a BN shown as follows:

$$\begin{cases} x_0(t+1) = f_0(x_1,x_{0_1},x_{0_2},...,x_{0_{k-1}}), \\ x_1(t+1) = f_1(x_2,x_{1_1},x_{1_2},...,x_{1_{k-1}}), \\ \vdots \\ x_{n-1} = f_{n-1}(x_0,x_{n-1_1},x_{n-1_2},...,x_{n-1_{k-1}}), \end{cases} \tag{5}$$

where $x_i(t) \in \mathscr{D},\ 0 \le i \le n-1,\ t = 1,2,....$ Let $x(t) = (x_0(t),x_1(t),...,x_{n-1}(t))$ denote the state of (n,k) NFSR and $f = [f_0,f_1,...,f_{n-1}]$ be the vectorial function. Thus system (8) can be expressed as follows:

$$x(t+1) = f(x(t)). \tag{6}$$

Definition 2. 1. *A state x_0 is called an equilibrium state of (n,k) NFSR (6), if $fx_0 = x_0$.*
2. *$x_0, f(x_0),...,f^p(x_0)$ is called a cycle of (n,k) NFSR (6) with length p, if $f^p(x_0) = x_0$, and the elements in set $\{x_0, f(x_0),...,f^p(x_0)\}$ are distinct.*

Let $R(x)$ denote the set of states which can reach state x. Let $R^k(x)$ denote the set of states which can reach state x after in k steps.

Next, we give the definition of *globally stable*, *locally stable* and the period of (n,k) NFSR.

Definition 3. *(n,k) NFSR (6) is called globally stable with respect to (w.r.t.) the equilibrium state x_0, if for any state $x \in \mathscr{D}^n$, there exists a positive integer N such that $f^N x = x_0$.*

Definition 4. *(n,k) NFSR (6) is called locally stable w.r.t. the equilibrium state x_0, if there exist some states $x \in \mathscr{D}^n \setminus x_0$, such that $f^N x = x_0$ for some positive integer N.*

By using Lemma 1, we can obtain that $x_i(t+1) = F_i \ltimes_{i=0}^{n-1} x_i(t)$, where $F_i \in \mathscr{L}_{2\times 2^n}$, then BN (8) can be converted into following system:

$$x(t+1) = Lx(t),\ t \in N, \tag{7}$$

where $x(t) = x_0(t) \ltimes x_1(t) \ltimes ... \ltimes x_{n-1}(t) \in \Delta_{2^n}$ is the state at time t, and $L \in \mathscr{L}_{2^n\times 2^n}$, $col_i(L) = \ltimes_{i=0}^{n-1} col_i(F_i)$.

Definition 5. *[2] The period of an (n,k) NFSR is the length of the longest cyclic output sequence produced.*

To better study the period of (n,k) NFSR, we propose the definition of minimal period of output sequence of a state in cycle of (n,k) NFSR.

Definition 6. *For arbitrary state x in a cycle $\mathscr{C}$ with length $l_{\mathscr{C}}$ of an (n,k) NFSR, the minimal period of output sequence $O_x = \{O_1, O_2, ...\}$ of x as an initial state is the minimal positive integer p_x such that $O_i = O_{i+kp_x}$, $k = 1,2,3,....$*

Remark 1. *For simplification, in the following, the output sequence of x means that the output sequence of x as an initial state, and x is in a cycle of (n,k) NFSR.*

In order to investigate the cycle of an (n,k) NFSR, we need the following definition.

Definition 7. *For an (n,k) NFSR, if the state x is shifted to the state y, then the state x is called the predecessor of state y, while state y is called the successor of state x. The state without predecessor is called a starting state.*

Let S denote the set of starting states of (n,k) NFSR (6).

3 main results

In this section, the stability of system (7) is firstly investigated. Then the period of output sequence of system (7) is studied. At last, we investigate the period of output sequence of a composed (n,k) NFSR.

3.1 Stability of (n,k) NFSR

Clearly, in an (n,k) NFSR, the equilibrium state can be any state. Assume the equilibrium state of (n,k) NFSR is $\delta_{2^n}^i \sim (i_0, i_1, ..., i_{n-1})$. We can make a coordinate transformation

$$\begin{cases} y_j = \neg x_j, \ i_j = 1, \\ y_j = x_j, \ i_j = 0, \end{cases} \tag{8}$$

then system (7) is converted into a system with equilibrium state $\delta_{2^n}^{2^n} \sim (0,0,...,0)$ as follows:

$$y(t+1) = \bar{L}y(t), \tag{9}$$

where $y(t) = \ltimes_{i=0}^{n-1} y_i(t)$. Then equilibrium state of system (9) is $\delta_{2^n}^{2^n} \sim (0,0,...,0)$.

Since the form of system (7) is similar to the form of system in [13], we can obtain the same result about the global stability of system (7).

Theorem 1. *(n,k) NFSR (9) is globally stable with respect to state $\delta_{2^n}^{2^n}$ if and only if there exists a positive integer $1 \le N \le 2^n - 1$ such that $col(L^N) = \delta_{2^n}^{2^n}$.*

From Theorem 1, we can obtain the following corollary.

Corollary 1. *(n,k) NFSR (9) is globally stable with respect to $\delta_{2^n}^{2^n}$, if and only if $R(\delta_{2^n}^{2^n}) = \Delta_{2^n}$.*

Remark 2. *If an (n,k) NFSR is globally stable to $\delta_{2^n}^{2^n}$, then for any state, it can reach state $\delta_{2^n}^{2^n}$ at most 2^n steps.*

Theorem 2. *(n,k) NFSR (9) is locally stable with respect to state $\delta_{2^n}^{2^n}$ if and only if there exists state $x \in \Delta_{2^n} \setminus \delta_{2^n}^{2^n}$ and a positive integer $1 \le N \le 2^n - 1$ such that $L^N x = \delta_{2^n}^{2^n}$. (i. e there exist state $x \in \Delta_{2^n} \setminus \delta_{2^n}^{2^n}$ which can reach equilibrium state $\delta_{2^n}^{2^n}$.)*

In the next, we will give an algorithm to decide if an (n,k) NFSR is globally stable under the knowledge of state transition matrix of an (n,k) NFSR.

Algorithm 1 The global stability of an (n,k) NFSR.

```
initial set Δ = {1,2,...,2^n}
for i = 1 to 2^n do
    for j = 1 to 2^n do
        if col_j(L) = δ_{2^n}^{2^n} then
            R^i(δ_{2^n}^{2^n}) = R^i(δ_{2^n}^{2^n}) ∪ δ_{2^n}^{j}
        end if
    end for
    if R^{i-1}_{δ_{2^n}^{2^n}} = R^i_{δ_{2^n}^{2^n}} then
        num = i − 1
        break.
    end if
end for
if R^{num}_{δ_{2^n}^{2^n}} = Δ_{2^n} then
    the (n,k) NFSR is globally stable.
else
    the (n,k) NFSR is not globally stable.
end if
```

Lemma 2. *The set of starting states of an (n,k) NFSR (7), S is equal to $\Delta_{2^n} \setminus col(L)$.*

Proof: From the definition of starting state, we can obtain that for every starting state x, there does not exist state y, such that $Ly = x$, $Ly \in Col(L)$. Hence, we can conclude that the starting state $x \notin Col(L)$, and set $S = \Delta_{2^n} \setminus col(L)$. □

In the next, we will give an algorithm to find all cycles of (n,k) NFSR (7). We firstly give some notations in Algorithm 2. S_i denotes the i-th state in set S. V_i denotes the set of states which start from state S_i have been visited. V_i^k denotes the k-th element in set V_i. $\mathscr{C}_i$ denote the i-th cycle of (n,k) NFSR. In Algorithm 2, we will firstly find the cycles which start from starting states. Then the cycles without branches will be found.

Algorithm 2 All cycles of (n,k) NFSR (7)

```
Initialize set V_i = ∅
for i = 1 to |S| do
    V_i = S_i
    x = S_i
    for j = 1 to 2^n do
        x = Lx
        if x ∉ V_i then
            V_i = V_i ∪ x
            if i > 1 and x ∈ V_{i-1} ∪ V_{i-2} ∪ ... ∪ V_1 then
                continue
            end if
        else
            There exist V_i^k such that V_i^k equal to x
            The i-th cycle of (n,k) NFSR is C_i = {V_i^k, V_i^{k+1}, ..., V_i^{j-1}}.
            continue
        end if
    end for
end for
Re = Δ_{2^n} \ V_1 ∪ V_2 ∪ ... ∪ V_{|S|}
i = 0
while Re ≠ ∅ do
    i = i + 1
    Let x denote the first number of set Re, there exist L^k x = x.
    The |S| + i-th cycle is C_{|S|+i} = {x, Lx, ..., L^{k-1} x}
    Re = Re \ C_{|S|+i}
end while
```

3.2 The period of (n,k) NFSR

In this section, we firstly give the following lemma before investigating the period of (n,k) NFSR.

Since the number of states in an (n,k) NFSR is finite, we can obtain the following lemma.

Lemma 3. *For arbitrary initial state $\delta_{2^n}^i$ of an (n,k) NFSR, $\delta_{2^n}^i$ can always reach a cycle or an equilibrium point.*

From Lemma 3, in the sequel, we only need to investigate the output sequence of state within a cycle. Hence, in this paper, the minimal period of output sequence of a state in (n,k) NFSR means that the minimal period is only for the states within a cycle of (n,k) NFSR.

In [2], E. Dubrova *et al.* proposed that the minimal period of the output sequence of a state x in an (n,k) NFSR is not necessary equal to the length of the cycle which contains state x. However they only provide a method to compute the cycle of an (n,k) NFSR. Hence, in the sequel, we will provide Algorithm 3 to compute the minimal period of the output sequence of a state in an (n,k) NFSR. Before presenting Algorithm 3, we give a theorem to show the relationship between the period of output sequence of the state in a cycle and the length of the cycle of an (n,k) NFSR.

Theorem 3. *If the length of a cycle $\mathscr{C} = \{e_0, e_1, ..., e_{l-1}\}$ in (n,k) NFSR (7) is l, then the minimal period of output sequence of cycle $\mathscr{C}$ is one of the divisor of l, and the minimal period of output sequence of every state e_i, $0 \le i \le l-1$ in cycle $\mathscr{C}$ is equal to the minimal period of output sequence of cycle $\mathscr{C}$.*

Proof: Let $p_{\mathscr{C}}$ denote the minimal period of the output sequence of cycle $\mathscr{C}$, and p_i denote the minimal period of output sequence of initial state e_i, and l_i denote the output sequence of initial state e_i. Let $O_{\mathscr{C}} = \{O_{\mathscr{C}}^0, O_{\mathscr{C}}^1, ..., O_{\mathscr{C}}^{l-1}\}$ with length l denote the output sequence of cycle $\mathscr{C}$ which start at state e_0.

Since $\mathscr{C}$ is a cycle, l is one of period of the output sequence of cycle $\mathscr{C}$. Apparently, the multiple of l is the period of output sequence of cycle $\mathscr{C}$. We also can obtain that one of the divisor of l can be the period of the output sequence of cycle $\mathscr{C}$. Since $e_i \in \mathscr{C}$, p_i must be a divisors of l. The output sequence of e_i is equivalent to the output sequence of cycle $\mathscr{C}$ after i times shifting. Hence, the minimal period of output sequence e_i is equal to $p_{\mathscr{C}}$. □

From Theorem 3, we can know that the minimal period of every state in a cycle is the same, hence, the investigation of minimal period of output sequence of a state can be called the investigation of minimal period of a cycle of an (n,k) NFSR.

Now, based on Theorem 3, we give an algorithm to compute the minimal period of output sequence of an initial state of (n,k) NFSR (7). The following algorithm is based on the fact that all cycles of (n,k) NFSR (7) are known. Algorithm 3 can be used to find out the period of output sequence of given cycle. For the given cycle, the length of the cycle is denoted by l, and the period of the output sequence of the cycle is denoted by p. Algorithm 3 only can compute the minimal period of output sequence of a cycle. Thus if we want to compute the minimal period of all cycles, Algorithm 3 should be used repeatedly.

Thus the period of (n,k) NFSR (7) can be found by the following steps:

- Finding all cycles of the (n,k) NFSR (7).
- Finding the minimal periods of all cycles of the (n,k) NFSR (7) by using Algorithm 3.
- Finding the maximum value of minimal periods of all the cycles of (n,k) NFSR (7).

Algorithm 3 The period of output sequence of a cycle of (n,k) NFSR (7).

```
initial p = l
for i = 1 to l do
    flag=true
    if i is a divisor of l then
        I_1 is a subsequence which is formed by the 1-th element to
        i-th element of cycle C
        for j = 1 to l/p − 1 do
            I_2 is a subsequence which is formed by the j ∗ i-th
            element to (j + 1) ∗ i − 1-th element of cycle C.
            if I_1 ≠ I_2 then
                flag=false
            end if
        end for
    end if
    if flag=true then
        if i < p then
            p = i
        end if
    end if
end for
```

3.3 The period of synthesis of (n,k)-NFSR by composition

In order to increase the period of (n,k) NFSR, a method is provided in [2]. In [2], E. Dubrova *et al.* constructed an (n,k) NFSR with guaranteed long period by composing several smaller NFSR working in parallel and combining their output using operator $\oplus$.

Let $N_1, N_2, ...N_m$ be $(n_1,k_1),(n_2,k_2),...,(n_m,k_m)$ NFSR respectively. Let R be an (n,k) NFSR composed by $N_1, N_2, ...N_m$, where $n = n_1 + n_2 + ... + n_m$, and $k = \max\{k_1, k_2, ..., k_m\}$. Let C_i denote the number of N_i's cycles. Let L_{ij} denote length of the j-th cycle $\mathscr{C}_{ij}$ of states in N_i, $i \in \{1,2,...,m\}$, $j \in \{1,2,...,C_i\}$.

The new composed (n,k) NFSR can be expressed as a BN as following:

$$\begin{cases} x_i(t+1) = f_i(x_{(i+1) mod\ n}, x_{i_1}, ..., x_{i_{k_1-1}}), \\ \qquad\qquad 0 \le i \le n_1, \\ x_i(t+1) = f_i(x_{(i+1) mod\ n}, x_{i_1}, ..., x_{i_{k_2-1}}), \\ \qquad\qquad n_1 \le i \le n_1 + n_2 - 1, \\ \qquad\qquad \vdots \\ x_i(t+1) = f_i(x_{(i+1) mod\ n}, x_{i_1}, ..., x_{i_{k_m-1}}), \\ \qquad\qquad \sum_{j=1}^{m-1} n_j \le i \le \sum_{j=1}^{m} n_j - 1, \end{cases} \tag{10}$$

where $x_i(t) \in \mathscr{D}$.

Let $y(t)$ denote the output of (n,k) NFSR at time t, it can be expressed as follows:

$$y(t) = x_0(t) \oplus x_{k_1}(t) \oplus x_{k_1+k_2} \oplus ... \oplus x_{\sum_{j=1}^{m-1} k_j}, \tag{11}$$

where $y(t) \in \mathscr{D}$.

By using Lemma 1, system (10) and equation (11) can be expressed as follows:

$$x(t+1) = L_c x(t), \tag{12}$$

with output as follows:

$$y(t) = Hx(t), \tag{13}$$

where $x(t) = \ltimes_{i=0}^{n-1} x_i(t) \in \Delta_{2^n}$, $y(t) \in \Delta$, $L_c \in \mathscr{L}_{2^n \times 2^n}$ is the state transition matrix of system (10), and $H \in \mathscr{L}_{2 \times 2^n}$ is the structure matrix of equation (11).

Lemma 4. *[2] The total number of cycles of states produced by composed (n,k) NFSR (10) is given by*

$$N=\sum_{\forall(i_1,\ldots,i_m)\in I}\prod_{j=2}^{m}(((L_{1i_1}\star L_{2i_2}\star L_{3i_3}\ldots\star L_{j-1i_{j-1}})\diamond L_{ji_j}, \quad (14)$$

where the set I_i represents indexes of cycles of states of the NFSR N_i, $I=I_1\times I_2\times\ldots\times I_m$.

Corollary 2. *For the new composed (n,k) NFSR (10), the cycle of states can be composed by $\mathscr{C}_{1i_1},\mathscr{C}_{2i_2},\ldots,\mathscr{C}_{mi_m}$, and the length of composed cycle is $lcm\{L_{1i_1},L_{2i_2},\ldots,L_{mi_m}\}$.*

Since the form of composed (n,k) NFSR (10) is the same as (n,k) NFSR (9), the only difference between (n,k) NFSR (9) and (n,k) NFSR (10) is the output of the systems. We can obtain the relationship between the minimal period of output sequence of cycle $\mathscr{C}$ and the length of the cycle $\mathscr{C}$.

Theorem 4. *If the length of a cycle $\mathscr{C}=\{e_0,e_1,\ldots,e_{l-1}\}$ of (n,k) NFSR (10) is l, then the minimal period of output sequence of cycle $\mathscr{C}$ is one of the divisor of l, and the minimal period of output sequence of every state e_i, $0\le i\le l-1$ in cycle $\mathscr{C}$ is equal to the minimal period of output sequence of cycle $\mathscr{C}$.*

Theorem 5. *For a cycle $\mathscr{C}$ of composed (n,k) NFSR (12), cycle $\mathscr{C}$ is composed by $\mathscr{C}_1$, $\mathscr{C}_2$,..., $\mathscr{C}_m$ in the $N_1,N_2,\ldots,N_m$ respectively, then the least common multiple of the minimal period of cycle $\mathscr{C}_1$, $\mathscr{C}_2$,..., $\mathscr{C}_m$ is a period of output sequence of the cycle $\mathscr{C}$.*

Remark 3. *As we know, any cycle in composed (n,k) NFSR (12) is composed by $\mathscr{C}_1$, $\mathscr{C}_2$,..., $\mathscr{C}_m$ in the $N_1,N_2,\ldots,N_m$ respectively, while more than one cycle can be composed by using these cycle $\mathscr{C}_1$, $\mathscr{C}_2$,..., $\mathscr{C}_m$ in $N_1,N_2,\ldots,N_m$ NFSRs respectively can compose more than one cycle in composed (n,k) NFSR (12). Hence, the minimal period of composed cycle can not be decided. We will discuss this issue in Example 3.*

The period of composed (n,k) NFSR (12) can be found in three steps like (n,k) NFSR.

4 examples

In this section, we give three examples to illustrate our theoretical results.

Example 1. *Consider an $(3,3)$ NFSR with following equations:*

$$\begin{cases} x_0(t+1) &= (x_0(t)\wedge x_2(t))\vee(\neg x_0(t)\wedge x_1(t)),\\ x_1(t+1) &= (x_0(t)\wedge(\neg x_1(t)\vee\neg x_2(t)))\vee(\neg x_0(t)\\ &\quad\wedge(\neg x_1(t)\wedge x_2(t))),\\ x_2(t+1) &= (x_0(t)\wedge(\neg x_2(t)\wedge\neg x_1(t)))\vee(\neg x_0(t)\\ &\quad\wedge(\neg x_1(t)\vee\neg x_2(t))). \end{cases} \quad (15)$$

By using the method of STP, we can transfer system (15) into following multi-linear form:

$$x(t+1)=L_1x(t), \quad (16)$$

where $L_1=\delta_8[4\ 6\ 2\ 5\ 8\ 7\ 1\ 3]$, $x(t)=x_0(t)x_1(t)x_2(t)\in\Delta_{2^3}$.

In (n,k) NFSR (15), there is no equilibrium state, so it is not globally stable or locally stable.

We know that there is a cycle of (n,k) NFSR (15), and the cycle is $\delta_8^1\to\delta_8^4\to\delta_8^5\to\delta_8^8\to\delta_8^3\to\delta_8^2\to\delta_8^6\to\delta_8^7\to\delta_8^1$. The output sequence of the cycle is 11001100.... The period of this output sequence is 4, which is coincident with the result of Theorem 3. Hence, we can conclude that the period of (n,k) NFSR (15) is 3.

Example 2. *Consider an $(3,3)$ NFSR with following equations:*

$$\begin{cases} x_0(t+1) &= (x_0(t)\wedge\neg x_2(t))\vee(\neg x_0(t)\wedge x_1(t)),\\ x_1(t+1) &= (x_0(t)\wedge(\neg x_1(t)\vee x_2(t)))\vee(\neg x_0(t)\wedge\\ &\quad(x_1(t)\wedge x_2(t))),\\ x_2(t+1) &= (x_0(t)\wedge(x_1(t)\vee\neg x_2(t)))\vee(\neg x_0(t)\wedge\\ &\quad(\neg x_1(t)\wedge\neg x_2(t))). \end{cases} \quad (17)$$

By using the method of STP, we can turn system (15) into following multi-linear form:

$$x(t+1)=L_2x(t), \quad (18)$$

where $L_2=\delta_8[5\ 3\ 6\ 1\ 2\ 4\ 8\ 7]$, $x(t)=x_0(t)x_1(t)x_2(t)\in\Delta_{2^3}$. In (n,k) NFSR (17), there is no equilibrium state, so (n,k) NFSR (17) is not globally stable or locally stable.

We can know that there are two cycles denoted by $\mathscr{C}_{21},\mathscr{C}_{22}$. $\mathscr{C}_{21}=\delta_8^1\to\delta_8^5\to\delta_8^2\to\delta_8^3\to\delta_8^6\to\delta_8^4\to\delta_8^1$, $\mathscr{C}_{22}=\delta_8^7\to\delta_8^8\to\delta_8^7$. The output sequence of $\mathscr{C}_{21}$ is 101101101101.... The output sequence of $\mathscr{C}_{22}$ is 00.... Hence, we can know that the minimal period of output sequence of $\mathscr{C}_{21}$ is 3, and the minimal period of output sequence of $\mathscr{C}_{22}$ is 1. Hence, we can get that the period of (n,k) NFSR (17) is 3.

Example 3. *Consider an $(3,3)$ NFSR with following equations:*

$$\begin{cases} x_0(t+1) &= (x_0(t)\wedge(x_1(t)\vee x_2(t)))\vee(\neg x_0(t)\wedge\\ &\quad((x_1(t)\wedge x_2(t))\vee(\neg x_1(t)\wedge\neg x_2(t)))),\\ x_1(t+1) &= (x_0(t)\wedge((x_1(t)\vee\neg x_2(t))\vee(\neg x_1(t)\wedge\\ &\quad x_2(t))))\vee(\neg x_0(t)\wedge(\neg x_1(t)\wedge x_2(t))),\\ x_2(t+1) &= (x_0(t)\wedge(\neg x_1(t)\vee x_2(t)))\vee(\neg x_0(t)\wedge\\ &\quad((x_1(t)\wedge x_2(t))\vee(\neg x_1(t)\wedge x_2(t))). \end{cases} \quad (19)$$

By using the method of STP, we can turn system (15) into following multi-linear form:

$$x(t+1)=L_3x(t), \quad (20)$$

where $L_3=\delta_8[3\ 6\ 1\ 7\ 2\ 8\ 5\ 4]$, $x(t)=x_0(t)x_1(t)x_2(t)\in\Delta_{2^3}$. We can observe that there are two cycles denoted by $\mathscr{C}_{31},\mathscr{C}_{32}$. $\mathscr{C}_{31}=\delta_8^5\to\delta_8^2\to\delta_8^6\to\delta_8^8\to\delta_8^4\to\delta_8^7\to\delta_8^5$. $\mathscr{C}_{32}=\delta_8^1\to\delta_8^3\to\delta_8^1$. The output sequence of $\mathscr{C}_{31}$ is 010010.... The output sequence of $\mathscr{C}_{32}$ is 111.... The minimal periods of $\mathscr{C}_{31}$ and $\mathscr{C}_{32}$ are 3 and 1 respectively. So we can obtain that the period of (n,k) NFSR (19) is 3.

Next, we composed $(3,3)$ NFSR (17) and (19) into an $(6,3)$ NFSR. We can obtain the state transition matrix of $(6,3)$ NFSR denoted by L_c. $L_c=\delta_{64}[35\ 38\ 33\ 39\ 34\ 40\ 37$ 36 19 22 17 23 18 24 21 20 43 46 41 47 42 48 45 44 3 6 1 7 2 8 5 4 11 14 9 15 10 16 13 12 27 30 25 31 26 32 29 28 59 62 57 63 58 64 61 60 51 54 49 55 50 56 53 52].

From Lemma 4, we can know that the number of cycle of (n,k) NFSR is $6+2+2+2=12$. We only pick 3 cycles denoted by $\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_3$. $\mathscr{C}_1$ is composed by $\mathscr{C}_{21}$ and $\mathscr{C}_{31}$ as $\mathscr{C}_1=\delta_{64}^5\to\delta_{64}^{34}\to\delta_{64}^{14}\to\delta_{64}^{24}\to\delta_{64}^{44}\to\delta_{64}^{31}\to\delta_{64}^5$. We can know the output sequence of $\mathscr{C}_1$ is 1111111... with period 1. $\mathscr{C}_2$ is composed by $\mathscr{C}_{21}$ and $\mathscr{C}_{32}$ and $\mathscr{C}_2=\delta_{64}^1\to\delta_{64}^{35}\to\delta_{64}^9\to\delta_{64}^{19}\to\delta_{64}^{41}\to\delta_{64}^{27}\to\delta_{64}^1$. And the output sequence of $\mathscr{C}_2$ is 010010, the minimal period of output sequence of $\mathscr{C}_2$ is 3. $\mathscr{C}_3$ is composed by $\mathscr{C}_{21}$ and $\mathscr{C}_{31}$. $\mathscr{C}_3=\delta_{64}^2\to\delta_{64}^{36}\to\delta_{64}^{16}\to\delta_{64}^{20}\to\delta_{64}^{47}\to\delta_{64}^{29}\to\delta_{64}^2$. The output sequence of $\mathscr{C}_3$ is 001001... with minimal period 3.

From above analysis, we can find that cycles $\mathscr{C}_1$ and $\mathscr{C}_3$ are all composed by cycles $\mathscr{C}_{21}$ and $\mathscr{C}_{31}$. But the output sequence of cycles $\mathscr{C}_1$ and $\mathscr{C}_3$ are different from each other, and the minimal periods of output sequences of cycles $\mathscr{C}_1$ and $\mathscr{C}_3$ are also different. The above analysis is coincident with our theoretical results in Theorem 4 and 5.

5 conclusion

This paper investigated the stability of (n,k) NFSR, the period of (n,k) NFSR and the period of composed (n,k) NFSR. Firstly, we treated the (n,k) NFSR as a BN. By using the method of STP, the (n,k) NFSR was transformed into a multi-linear system. Then an

algorithm was proposed to judge the stability of an (n,k) NFSR. After that, we proposed an algorithm to find the minimal period of output sequence of a cycle of an (n,k) NFSR. Then we revealed relationship between the minimal period of output sequence of an cycle and the length of the cycle in an composed (n,k) NFSR, and the period of composed (n,k) NFSR was investigated. Finally, three examples were given to verify the results obtained in this paper.

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